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# LINEARIZED TOPOLOGIES AND DEFORMATION THEORY

WENDY LOWEN

To my parents, with love,
and with the joy of sharing the love of mathematics.

Abstract. In this paper, for an underlying small category $\mathcal{U}$ endowed with a Grothendieck topology $\tau$, and a linear category $\mathfrak{a}$ which is graded over $\mathcal{U}$ in the sense of [13], we define a natural linear topology $\mathcal{T}_\tau$ on $\mathfrak{a}$, which we call the *linearized topology*. Grothendieck categories in (non-commutative) algebraic geometry can often be realized as linear sheaf categories over linearized topologies. With the eye on deformation theory, it is important to obtain such realizations in which the linear category contains a restricted amount of algebraic information. We prove several results on the relation between refinement (eliminating both objects, and, more surprisingly, morphisms) of the non-linear underlying site $(\mathcal{U}, \tau)$, and refinement of the linearized site $(\mathfrak{a}, \mathcal{T}_\tau)$. These results apply to several incarnations of (quasi-coherent) sheaf categories, leading to a description of the infinitesimal deformation theory of these categories in the sense of [17] which is entirely controlled by the Gerstenhaber deformation theory of the small linear category $\mathfrak{a}$, and the Grothendieck topology $\tau$ on $\mathcal{U}$. Our findings extend results from [17], [12] and [7] and recover the examples from [21], [20].

## 1. Introduction

In the 1960's, the Grothendieck school revolutionarized algebraic geometry by founding it on the theory of abelian categories, see [10], and on topos theory, see the SGA4 volumes, in particular [1]. The setup of scheme theory allows arbitrary commutative rings as buiding blocks, and is further centered around the concepts of (quasi-coherent) sheaves and sheaf cohomology. Schemes have underlying topological spaces, built from the Zariski topologies on the spectra of commutative rings. With the formulation of the Weil conjectures, it was realized that classical topological spaces and sheaf cohomology were insufficient, and it was the introduction of the more general étale Grothendieck topology, and corresponding étale cohomology, which eventually led to the proofs of the conjectures between 1960 and 1974. On the other hand, in 1962, in his thesis Gabriël developed localization theory in the context of abelian categories, involving, in the case of module categories over rings, the concept of a Gabriël filter on a ring. This notion can be recognized as a linear version of a Grothendieck topology, on a single object linear category, and can easily be extended to arbitrary small linear categories. In the famous Gabriël-Popescu theorem, it was proven that every Grothendieck abelian category can be realized as

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the localization of a module category. This gives Grothendieck categories the status of linear versions of Grothendieck topoi, the localizations of presheaf categories of sets which were characterized internally by Giraud's theorem. In both setups, the localizations can be realized as sheaf categories, and depend upon the choice of a suitable functor $\gamma : \mathfrak{a} \longrightarrow \mathcal{C}$ from a small (linear) category to a Grothendieck topos (or Grothendieck category), giving rise to a (linear) topology $\mathcal{T}$ on $\mathfrak{a}$ and an equivalence of catgeories $\mathcal{C} \cong \mathsf{Sh}(\mathfrak{a}, \mathcal{T})$. See [11] for a characterization of such functors $\gamma$.

Whereas the rings occuring in algebraic geometry are commutative, the theory of abelian categories and their localizations is not restricted to the commutative realm, and includes in particular module categories over non-commutative rings. For a commutative ring $A$, the module category $\mathsf{Mod}(A)$ is equivalent to the category of quasi-coherent sheaves on the spectrum $\mathrm{Spec}(A)$, and captures a lot of geometric information. With the development of so called non-commutative algebraic geometry by Artin, Tate, Stafford, Van den Bergh and others [3] [19], this observation is taken further and Grothendieck abelian categories are themselves considered as the main geometric objects. This is motivated by the fact that non-commutative rings typically have no well-behaved underlying "spectra" of points, whence one is forced to work in a point-free environment. Following this philosophy, one is primordially interested in Grothendieck categories which share a lot with the ones occuring in classical algebraic geometry. Examples are provided by deformations of commutative rings, with the Weyl algebra deforming the commutative polynomial algebra in two variables as prime example. Since, for instance, projective geometry involves more general quasi-coherent sheaf categories than module categories, in [17], Gerstenhaber's deformation theory of algebras was extended to a deformation theory for abelian categories. This theory allows to capture the important examples of non-commutative projective planes, quadrics and $\mathbb{P}_1$-bundles over commutative schemes from [21], [20], which motivated its development. Further, the theory leads to a description of non-commutative deformations of schemes in terms of twisted presheaves of non-commutative rings (see [12]).

Let $\mathcal{C}$ be a given Grothendieck category over a field $k$, and suppose we are interested in deformation in the direction of an Artin local $k$-algebra $R$. According to [17], a deformation is an $R$-linear Grothendieck abelian category which reduces to $\mathcal{C}$ upon restriction to $k$-linear objects. Now suppose we consider our favourite representation $\mathcal{C} \cong \mathsf{Sh}(\mathfrak{a}, \mathcal{T})$ as a sheaf category over a $k$-linear site $(\mathfrak{a}, \mathcal{T})$, corresponding to a functor $\gamma : \mathfrak{a} \longrightarrow \mathcal{C}$. Then, ideally, we would like to realize $\mathcal{D}$ as $\mathcal{D} \cong \mathsf{Sh}(\mathfrak{b}, \mathcal{S})$ for a $R$-linear site $(\mathfrak{b}, \mathcal{S})$ in which:

(A) $\mathfrak{b}$ is obtained as a linear (i.e., Gerstenhaber type) deformation of $\mathfrak{a}$;
(B) $\mathcal{S}$ is naturally an "$R$-linear variant" of $\mathcal{T}$.

In general, both requirements may fail. Whether or not we can realize (A) essentially depends on homological conditions involving the objects $\gamma(A) \in \mathcal{C}$, more precisely the vanishing of certain Ext groups between these objects. In order to realize (B), we first have to understand what an $R$-linear variant of a $k$-linear topology means. In a first approach, this could mean "a topology naturally induced by $\mathcal{T}$ along the map $\mathfrak{b} \longrightarrow \mathfrak{a}$". The drawback of this interpretation is that such a topology does not necessarily have an intrinsic "meaning" with respect to $\mathfrak{b}$. Let us look at the ideal case where $\mathcal{C} = \mathsf{Mod}(\mathfrak{a})$, the entire module category over $\mathfrak{a}$. A basic result from [17] states that there is a deformation equivalence

$$(1) \qquad \mathrm{Def}_{\mathrm{lin}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Mod}(\mathfrak{a})) : \mathfrak{b} \longrightarrow \mathsf{Mod}(\mathfrak{b})$$

from linear deformations of $\mathfrak{a}$ to abelian deformations of $\mathsf{Mod}(\mathfrak{a})$. Both $\mathsf{Mod}(\mathfrak{a})$ and $\mathsf{Mod}(\mathfrak{b})$ are sheaf categories with respect to the trivial topologies on $\mathfrak{a}$ and $\mathfrak{b}$

respectively. The trivial topology on $\mathfrak{b}$ happens to be the induced topology along $\mathfrak{b} \longrightarrow \mathfrak{a}$, but it moreover has an intrinsic description on $\mathfrak{b}$ regardless of $\mathfrak{a}$.

Our approach in this paper is to realize both (A) and (B) in the framework of map-graded categories from [13]. For a $k$-linear $\mathcal{U}$-graded category $\mathfrak{a}$ over a small category $\mathcal{U}$, we describe how to "linearize" a Grothendieck topology $\tau$ on $\mathcal{U}$ in order to obtain a linear topology $\mathcal{T}_\tau$ on $\mathfrak{a}$. In our applications to deformation theory, $\mathcal{U}$ will be the category associated to a poset $(\mathcal{U}, \sqsubseteq)$. To facilitate the discussion, we will continue the introduction in this setup. For a poset $(\mathcal{U}, \sqsubseteq)$, a $k$-linear $\mathcal{U}$-graded category can be described as a $k$-linear category $\mathfrak{a}$ with a map $f : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathcal{U})$ with fibers $\mathfrak{a}_U = f^{-1}(U)$, such that for $A \in \mathfrak{a}_U$, $B \in \mathfrak{a}_V$, $V \not\sqsubseteq U$ implies $\mathfrak{a}(B, A) = 0$. Hence, the only non-trivial algebraic information in the category $\mathfrak{a}$ naturally lives over the category $\mathcal{U}$. Now suppose $\tau$ is a Grothendieck topology on $\mathcal{U}$. To a sieve $R$ on $U$ and an object $A \in \mathfrak{a}_U$, we naturally associate a linear sieve $R^A$ on $A$ with, for $B \in \mathfrak{a}_V$,

$$R^A(B) = \begin{cases} \mathfrak{a}(B, A) & \text{if } (V \sqsubseteq U) \in R(V) \\ 0 & \text{otherwise.} \end{cases}$$

The linearized topology $\mathcal{T}_\tau$ on $\mathfrak{a}$ is by definition the smallest topology on $\mathfrak{a}$ which contains all the sieves $R^A$ for $R \in \tau(U)$ as covering sieves in $\mathcal{T}_\tau(A)$ (Definition 3.3). In §6.4, we generalize (1) and solve (B) by giving the following sheaf theoretic description of the map introduced in [17, §8]:

$$\mathrm{Def}_{\mathrm{lin}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\tau,\mathfrak{a}})) : \mathfrak{b} \longrightarrow \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\tau,\mathfrak{b}}). \tag{2}$$

We can now be more precise and reinforce (A) to the requirement that the map (2) is a bijection. Based upon [17, Thm. 8.14], we formulate conditions under which this is indeed the case (Theorem 6.10).

Let us give an example from [17], [12] (Example 6.14). Let $X$ be a scheme with structure sheaf $\mathcal{O}_X$. We endow $\mathcal{U} = \mathrm{open}(X)$ with the standard Grothendieck topology $\tau$ of coverings by unions of open subsets. By a linear version (see [13]) of the Grothendieck construction from [2], $\mathcal{O}_X$ gives rise to a $\mathcal{U}$-graded linear category $\mathfrak{o}$ with $\mathrm{Ob}(\mathfrak{o}) = \mathrm{Ob}(\mathcal{U})$ and $\mathfrak{o}(V, U) = \mathcal{O}_X(V)$ for $V \subseteq U$. For the linearized topology $\mathcal{T}_\tau$ on $\mathfrak{o}$, we obtain a realization of the category $\mathsf{Sh}(X, \mathcal{O}_X)$ of sheaves of $\mathcal{O}_X$-modules on $X$ as a sheaf category on the linearized site $(\mathfrak{o}, \mathcal{T}_\tau)$:

$$\mathsf{Sh}(X, \mathcal{O}_X) \cong \mathsf{Sh}(\mathfrak{o}, \mathcal{T}_\tau).$$

In order to realize (A) and arrive at a bijection like (1), one has to "refine" $\mathcal{U}$ to the full subcategory $\mathcal{U}' \subseteq \mathcal{U}$ containing only *affine* open subsets, with the induced topology $\tau'$ on $\mathcal{V}$. This refinement is such that for the naturally induced linear functor $\phi : \mathfrak{o}' \longrightarrow \mathfrak{o}$, we have $\mathsf{Sh}(\mathfrak{o}, \mathcal{T}_{\tau,\mathfrak{o}}) \cong \mathsf{Sh}(\mathfrak{o}', \mathcal{T}_{\tau',\mathfrak{o}'})$. The deformations of this abelian category are now obtained precisely by endowing the $R$-linear deformations $\bar{\mathfrak{o}}'$ of $\mathfrak{o}'$ with the corresponding $R$-linearized topologies $\mathcal{T}_{\tau',\bar{\mathfrak{o}}'}$, and taking sheaves in order to arrive at the deforming Grothendieck categories $\mathsf{Sh}(\bar{\mathfrak{o}}', \mathcal{T}_{\tau',\bar{\mathfrak{o}}'})$.

In this paper, we first investigate the following related general question: suppose $\mathfrak{a}$ is a $\mathcal{U}$-graded category over $(\mathcal{U}, \tau)$, $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ is a "refinement" functor, $\sigma = \varphi^{-1}\tau$ is the induced topology on $\mathcal{V}$ and $\phi : \mathfrak{a}^\varphi \longrightarrow \mathfrak{a}$ is the naturally induced linear functor. What are natural conditions under which $\phi$ is localizing, that is, we obtain an equivalence of categories

$$\mathsf{Sh}(\mathfrak{a}, \mathcal{T}_\tau) \cong \mathsf{Sh}(\mathfrak{a}^\varphi, \mathcal{T}_\sigma). \tag{3}$$

In Theorem 4.3, we show that if (like in the example) $\mathfrak{a}$ is fibered over $\mathcal{U}$, it is sufficient to require that $\varphi$ is localizaing, that is, $\varphi$ induces an equivalence of set-sheoretic sheaf categories $\mathsf{Sh}(\mathcal{U}, \tau) \cong \mathsf{Sh}(\mathcal{V}, \sigma)$. Further, we formulate a specific application of such refinements which moreover realize (A) to the context of prestacks (Theorem 6.13).

If $\mathfrak{a}$ is not fibered over $\mathcal{U}$, the situation turns out to be more subtle, and in §3.3, we formulate additional conditions upon $\varphi$, leading to the notion of a *stably localizing* functor. We prove that a stably localizing functor $\varphi$ gives rise to an equivalence (3) (Theorem 3.16). A striking setup in which stably localizing functors can be obtained, is at the other extreme from the example we gave before. Rather than restricting the object set of $\mathcal{U}$, we may keep te object set fixed and consider a subcategory $\mathcal{V} \subseteq \mathcal{U}$ in which certain morphisms are eliminated from $\mathcal{U}$. With the application to deformation theory in mind, these would typically consist of $V \sqsubseteq U$ for which the Ext vanishing from an object over $V$ to an object over $U$ may fail. In §3.4, we show that a stably localizing "refinement" functor $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ can be constructed by eliminating all morphisms occuring in an arbitrary choice of distinguished covering sieves $D_U \in \tau(U)$ for $U \in \mathcal{U}$ (Proposition 3.21).

In §5, we investigate this second type of refinement in the context of a particular topology, the *tails topology* (Theorem 5.8), and we formulate conditions for a "tails refinement" to realize (A) (Theorem 6.15). For a downwardly directed poset $(\mathcal{U}, \sqsubseteq)$, the tails topology tails on $\mathcal{U}$ consists of all non-empty sieves. It is naturally induced from the trivial topology on a one-morphism category, and the set-theoretic sheaf category is simply given by $\mathsf{Sh}(\mathcal{U}, \text{tails}) \cong \mathsf{Set}$. Never the less, linearized versions of this topology turn out to be potentially very interesting, as the following application, which was previously discussed in the survey [14] and includes the motivating examples from [21], [20], shows (Example 6.16). Let $X$ be a projective scheme with an ample invertible sheaf $\mathcal{L}$. Put $\mathcal{O}(n) = \mathcal{L}^n \in \mathsf{Qch}(X)$, the category of quasi-coherent sheaves on $X$. Consider the poset $(\mathbb{Z}, \geq)$ endowed with the tails topology tails. There is an associated $\mathbb{Z}$-graded category $\mathfrak{a}$ with $\mathfrak{a}(n, m) = \mathsf{Qch}(X)(\mathcal{O}(-n), \mathcal{O}(-m))$ for $n \geq m$ and

$$\mathsf{Qch}(X) \cong \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\text{tails}}).$$

Unfortunately, the cohomological conditions ensuring (A) are only fulfilled in the rather restrictive setup treated in [7]. On the other hand, the cohomological criterion of ampleness yields for every $m \in \mathbb{Z}$ a number $\nu(m) \geq m$ such that for every $n \geq \nu(m)$, we have

$$\text{Ext}^i(\mathcal{O}(-n), \mathcal{O}(-m)) = 0.$$

We can now effectively refine our tails site to $\mathbb{Z}'$ with $n \geq' m$ if and only if $n \geq \nu(m)$, and consider the induced $\mathbb{Z}'$-graded category $\mathfrak{a}'$. As soon as the (less restrictive) condition

$$H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0 \tag{4}$$

is fulfilled, we obtain a bijection (2) for $\mathfrak{a}'$ and the tails topology on $\mathbb{Z}'$ by "tracking the tails topology through the deformation process". Thus, for the class of projective schemes satisfying (4), all deformations can be described as "non-commutative projective schemes" over some deformed $\mathbb{Z}'$-graded category.

*Acknowledgement.* The unified approach to deformations using linearized topologies, which we develop in this paper, arose out of the analysis of two different approaches to non-commutative deformations of schemes: a local approach based upon prestacks, and a global approach based upon $\mathbb{Z}$-algebras. The author is deeply grateful to Michel Van den Bergh for sharing these, and many other, beautiful ideas.

## 2. Linear topologies

Let $k$ be a commutative ring. Basically, $k$-linear sites are $k$-linear versions of Grothendieck sites ([1], see also [5]). To obtain the precise definitions, one has to systematically replace the category $\mathsf{Set}$ by the category $\mathsf{Mod}(k)$ of $k$-modules.

It is known that in fact, it is more generally possible to replace the category $\mathsf{Set}$ by a symmetric monoidal category $\mathcal{C}$, leading to a theory of enriched sites and enriched sheaves [6]. In this paper, we are concerned both with the classical setup $\mathcal{C} = \mathsf{Set}$ and the setup $\mathcal{C} = \mathsf{Mod}(k)$. Hence, we recall the main relevant notions simultaneously in these two setups. To do so, we make the agreement that in what follows, $k \in \mathsf{CommRing} \cup \{\varnothing\}$, that is, either $k$ is a commutative ring or $k = \varnothing$, and for $k = \varnothing$, we specify the notions enriched over $\mathsf{Set}$. Thus, we put $\mathsf{Mod}(\varnothing) = \mathsf{Set}$.

2.1. **Linear sieves.** A *$k$-linear category* $\mathfrak{a}$ is by definition a category enriched over $\mathsf{Mod}(k)$, and a *$k$-linear functor* between $k$-linear categories is a functor enriched over $\mathsf{Mod}(k)$. Let $\mathfrak{a}$ be a small $k$-linear category and let

$$\mathsf{Mod}(\mathfrak{a}) = \mathsf{Mod}_k(\mathfrak{a}) = \mathsf{Fun}_k(\mathfrak{a}^{\mathrm{op}}, \mathsf{Mod}(k))$$

be the $k$-linear category of $k$-linear functors from $\mathfrak{a}^{\mathrm{op}}$ to $\mathsf{Mod}(k)$. In $\mathsf{Mod}(\mathfrak{a})$, we denote the product by $\prod$ and the coproduct by $\oplus$. We also denote the coproduct in $\mathsf{Set}$ by $\coprod$. Elements of $\mathsf{Mod}(\mathfrak{a})$ are called *$\mathfrak{a}$-modules* or simply *modules*. For an object $A \in \mathfrak{a}$, we consider the representable module

$$\mathfrak{a}(-, A) : \mathfrak{a}^{\mathrm{op}} \longrightarrow \mathsf{Mod}(k) : B \longmapsto \mathfrak{a}(B, A).$$

A *sieve on $A$* is by definition a subobject $R \subseteq \mathfrak{a}(-, A)$ in $\mathsf{Mod}(\mathfrak{a})$. Intuitively, $R$ corresponds to the following set $[R]$ of morphisms with codomain $A$:

$$[R] = \coprod_{A' \in \mathfrak{a}} R(A') \subseteq \coprod_{A' \in \mathfrak{a}} \mathfrak{a}(A', A).$$

We will often abusively denote $[R] = R$, in particular by $f \in R$ we mean $f \in [R]$. Let $F = (f_i : A_i \longrightarrow A)_{i \in I}$ be an arbitrary family of morphisms in $\mathfrak{a}$ with codomain $A$. There exists a smallest sieve $R \subseteq \mathfrak{a}(-, A)$ with $f_i \in R(A_i)$ for all $i \in I$. This sieve is called the *sieve generated by $F$* and is denoted by $\langle F \rangle = \langle f_i \rangle_i = \langle f_i \rangle$. For $k = \varnothing$, $\langle F \rangle(A')$ consists of all $f : A' \longrightarrow A$ which factor through some $f_i : A_i \longrightarrow A$ in $F$, that is, $f = f_i g$ for some morphism $g : A' \longrightarrow A_i$. For $k \neq \varnothing$, $\langle F \rangle(A')$ consists of all finite $k$-linear combinations

$$\sum_{j=1}^{n} f_{i_j} g_j$$

for $f_{i_j} : A_{i_j} \longrightarrow A$ in $F$ and $g_j : A' \longrightarrow A_{i_j}$ in $\mathfrak{a}$.

2.2. **Linear topologies.** Let $\mathfrak{a}$ be a small $k$-linear category. Before defining a $k$-linear topology on $\mathfrak{a}$, it is useful to introduce the corresponding notion "without axioms".

A *cover system* $\mathcal{T}$ on $\mathfrak{a}$ consists of specifying, for every $A \in \mathfrak{a}$, a collection $\mathcal{T}(A)$ of sieves on $A$, called *covering sieves on* $A$ or simply *covers of* $A$. An arbitrary collection of morphisms $(f_i : A_i \longrightarrow A)_{i \in I}$ is *covering* or *covers* $A$ provided that the generated sieve $\langle f_i \rangle_i$ is a cover.

Consider a sieve $R$ on $A$ and a morphism $f : B \longrightarrow A$ in $\mathfrak{a}$. The pullback $f^{-1}R$ of $R$ along $f$ is the sieve on $B$ obtained as the pullback

$$\begin{array}{ccc} \mathfrak{a}(-, B) & \xrightarrow{f-} & \mathfrak{a}(-, A) \\ \uparrow & & \uparrow \\ f^{-1}R & \longrightarrow & R \end{array}$$

in the category $\mathsf{Mod}(\mathfrak{a})$. Hence, $f^{-1}R(A')$ contains all $g : A' \longrightarrow B$ with $fg \in R(A')$. Similarly, for two sieves $R$ and $S$ on $A$, the *intersection* $R \cap S$ is the sieve on $A$ obtained as the pullback of $R \subseteq \mathfrak{a}(-, A)$ and $S \subseteq \mathfrak{a}(-, A)$.

Consider a sieve $R$ on $A$ and for every $f : A_f \longrightarrow A$ in $R$ a sieve $R_f$ on $A_f$. We define their *glueing* to be the sieve

$$R \circ (R_f)_{f\in R} = \langle gf\rangle_{f\in R, g\in R_f} \subseteq \mathfrak{a}(-, A).$$

For $k = \varnothing$, an element in $R \circ (R_f)_{f\in R}(B)$ can be written as $fg$ for $f : A_f \longrightarrow A$ in $R(A_f)$ and $g \in R_f(B)$. For $k \neq \varnothing$, an element in $R \circ (R_f)_{f\in R}(B)$ can be written as $\sum_{i=1}^n f_i g_i$ for $f_i : A_i \longrightarrow A$ in $R(A_i)$ and $g_i : B \longrightarrow A_i$ in $R_{f_i}(B)$.

We consider the following conditions for a cover system $\mathcal{T}$ on $\mathfrak{a}$:

- (Id) $\mathcal{T}$ satisfies the *identity axiom* if $\mathfrak{a}(-, A) \in \mathcal{T}(A)$ for every $A \in \mathfrak{a}$.
- (Pb) $\mathcal{T}$ satisfies the *pullback axiom* if for every $f : B \longrightarrow A$ in $\mathfrak{a}$ and $R \in \mathcal{T}(A)$, we have $f^{-1}R \in \mathcal{T}(B)$.
- (Glue) $\mathcal{T}$ satisfies the *glueing axiom* if $S \in \mathcal{T}(A)$ as soon as there exists an $R \in \mathcal{T}(A)$ and for every $f : A_f \longrightarrow A$ in $R(A)$ an $R_f \in \mathcal{T}(A_f)$ with $R_f \subseteq f^{-1}S$.
- (Glue') $\mathcal{T}$ is *closed under glueings* if for a cover $R \in \mathcal{T}(A)$ and for covers $R_f \in \mathcal{T}(A_f)$ for every $f : A_f \longrightarrow A$ in $R$, the glueing $R \circ (R_f)_{f\in R}$ is in $\mathcal{T}(A)$.
- (Glue") $\mathcal{T}$ *respects glueing of covering families* if for a covering family $f_i : A_i \longrightarrow A$ of $A$ and covering families $f_{ij} : A_{ij} \longrightarrow A_i$ of $A_i$, the collection of all compositions $f_i f_{ij} : A_{ij} \longrightarrow A$ is covering.
- (Up) $\mathcal{T}$ is *upclosed* if $R \in \mathcal{T}(A)$ and $R \subseteq S \subseteq \mathfrak{a}(-, A)$ implies $S \in \mathcal{T}(A)$.
- (Int) $\mathcal{T}$ is *closed under intersections* if $R, S \in \mathcal{T}(A)$ implies $R \cap S \in \mathcal{T}(A)$.
- (Loc) $\mathcal{T}$ is *localizing* if it satisfies (Id) and (Pb).
- (Top) $\mathcal{T}$ is a *topology* if it satisfies (Id), (Pb) and (Glue).

The following proposition shows that to define a topology, axioms (Id) and (Pb) can equivalently be combined with either (Glue') and (Up) or else with (Glue") and (Up).

**Proposition 2.1.** *Let $\mathcal{T}$ be a cover system on $\mathfrak{a}$.*

1. *If $\mathcal{T}$ satisfies (Id) and (Glue), then $\mathcal{T}$ satisfies (Up) and (Glue').*
2. *If $\mathcal{T}$ satisfies (Up) and (Glue'), then it satisfies (Glue).*
3. *If $\mathcal{T}$ is a topology then $\mathcal{T}$ satisfies (Int) and (Glue").*
4. *If $\mathcal{T}$ satisfies (Glue"), then it satisfies (Glue').*

*Proof.* (1) Suppose $\mathcal{T}$ satisfies (Id) and (Glue). Let us show that $\mathcal{T}$ satisfies (Up). Consider $R \subseteq S$ with $R \in \mathcal{T}$. Then for $f : B \longrightarrow A$ in $R(A)$, $\mathfrak{a}(-, B) = f^{-1}R \subseteq f^{-1}S$ and $\mathfrak{a}(-, B) \in \mathcal{T}(B)$. Let us show that $\mathcal{T}$ satisfies (Glue'). For a glueing $S = R \circ (R_f)_f$ of covers, we clearly have $R_f \subseteq f^{-1}S$.

(2) Suppose $\mathcal{T}$ satisfies (Glue') and (Up). Consider $S$, $R$, $R_f$ as stated in (Glue). By (Glue'), $R \circ (R_f)_f$ is a cover and since $R_f \subseteq f^{-1}S$, we have $R \circ (R_f)_f \subseteq S$. Thus, $S$ is a cover by (Up).

(3) Suppose $\mathcal{T}$ is a topology and consider $R, S \in \mathcal{T}(A)$. Then for $R \cap S$, we have for every $f \in R$ that $f^{-1}S$ is a cover by (Pb) and $f^{-1}S \subseteq f^{-1}(R \cap S)$. Hence $R \cap S$ is a cover by (Glue). Next we prove that (Glue") holds in case $k \neq \varnothing$. The case $k = \varnothing$ is similar. Consider a covering collection $f_i : A_i \longrightarrow A$ of $A$ and covering collections $f_{ij} : A_{ij} \longrightarrow A_i$ of $A_i$. Put $S = \langle f_i f_{ij}\rangle_{i,j}$ and $R = \langle f_i\rangle_i$. Consider an arbitrary element $f = \sum_{l=1}^n f_{i_l}\alpha_l$ in $R$. We put $R_f = \cap_{l=1}^n \alpha_l^{-1}\langle f_{i_l j}\rangle_j$, which is a cover. We claim that $R_f \subseteq f^{-1}S$. Indeed, for $\beta \in R_f$, we have that $f\beta = \sum_{l=1}^n f_{i_l}\alpha_l\beta = \sum_{l,l'} f_{i_l} f_{i_l j_{l'}}$ belongs to $S$.

(4) Clearly, a glueing $R \circ (R_f)_f$ as in (Glue') is a special case of a glueing of covering families, whence the result is covering. □

A cover system $\mathcal{B} \subseteq \mathcal{T}$ is a *basis* for $\mathcal{T}$ if for every $T \in \mathcal{T}(A)$, there is a $B \in \mathcal{B}(A)$ with $B \subseteq T$. For a cover system $\mathcal{B}$ on $\mathfrak{a}$ we define the *upclosure* $\mathcal{B}^{\mathrm{up}}$ to be the cover system on $\mathfrak{a}$ with

$$\mathcal{B}^{\mathrm{up}}(A) = \{R \subseteq \mathfrak{a}(-, A) \mid \exists\ S \in \mathcal{B}(A)\ \ S \subseteq R\}.$$

Clearly, $\mathcal{B}^{\mathrm{up}}$ is upclosed and $\mathcal{B}$ is a basis of $\mathcal{B}^{\mathrm{up}}$.

Note that all the listed properties (Id) to (Top) are stable under taking arbitrary intersections of cover systems. Consequently, for a given cover system $\mathcal{T}$, there exists a smallest cover system $\mathcal{T}'$ with $\mathcal{T} \subseteq \mathcal{T}'$ such that $\mathcal{T}'$ satisfies a particular combination of the properties in the list. In some cases, it is possible to give a more tangible description of such a smallest $\mathcal{T}'$.

For a cover system $\mathcal{L}$ on $\mathfrak{a}$, we define the cover system $\mathcal{L}^+$ on $\mathfrak{a}$ with

$$\mathcal{L}^+(A) = \{R \circ (R_f)_{f \in R} \mid R \in \mathcal{L}(A), R_f \in \mathcal{L}(A_f), f : A_f \longrightarrow A\}.$$

**Lemma 2.2.** *If $\mathcal{B}$ is a basis for $\mathcal{L}$, then $\mathcal{B}^+$ is a basis for $\mathcal{L}^+$.*

*Proof.* Consider a glueing $R \circ (R_f)_{f \in R}$ of $\mathcal{L}$-covers. Take $\mathcal{B}$-covers $B \subseteq R$, $B_f \subseteq R_f$. Clearly $B \circ (B_f)_{f \in B} \subseteq R \circ (R_f)_{f \in R}$. □

**Lemma 2.3.** *If $\mathcal{L}$ is localizing, then so is $(\mathcal{L}^+)^{\mathrm{up}}$.*

*Proof.* Consider a cover $R \circ (R_f)_{f \in R}$ of $A$ for $\mathcal{L}^+$ and a map $a : B \longrightarrow A$. Since $R \in \mathcal{L}(A)$, we have $a^{-1}R \in \mathcal{L}(B)$. For $g : B_g \longrightarrow B$ in $a^{-1}R$, we have $ag \in R$. Consider the glueing $a^{-1}R \circ (R_{ag})_{g \in a^{-1}R}$. Clearly $a^{-1}R \circ (R_{ag})_{g \in a^{-1}R} \subseteq a^{-1}(R \circ (R_f)_{f \in R})$. □

Let $\mathcal{B}$ and $\mathcal{L}$ be cover systems. We define the cover systems $\mathcal{B}^{\mathrm{glue}}$ and $\mathcal{L}^{\mathrm{upglue}}$ through transfinite induction. We put $\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{L}_0 = \mathcal{L}^{\mathrm{up}}$. For a successor ordinal $\alpha + 1$ we put $\mathcal{B}_{\alpha+1} = \mathcal{B}_\alpha^+$ and $\mathcal{L}_{\alpha+1} = (\mathcal{L}_\alpha^+)^{\mathrm{up}}$. For a limit ordinal $\beta$ we put $\mathcal{B}_\beta = \cup_{\alpha<\beta}\mathcal{B}_\alpha$ and $\mathcal{L}_\beta = \cup_{\alpha<\beta}\mathcal{L}_\alpha$. For cardinality reasons, the sequences of cover systems $\mathcal{B}_\alpha$ and $\mathcal{L}_\alpha$ become stationary. We define $\mathcal{B}^{\mathrm{glue}}$ and $\mathcal{L}^{\mathrm{upglue}}$ to be these stable values.

**Proposition 2.4.**
(1) *If $\mathcal{B}$ is a basis of $\mathcal{L}$, then $\mathcal{B}^{\mathrm{glue}}$ is a basis of $\mathcal{L}^{\mathrm{upglue}}$.*
(2) *$\mathcal{B}^{\mathrm{glue}}$ is the smallest cover system $\mathcal{B}'$ which is closed under glueings with $\mathcal{B} \subseteq \mathcal{B}'$.*
(3) *$\mathcal{L}^{\mathrm{upglue}}$ is the smallest upclosed cover system $\mathcal{L}'$ which is closed under glueings with $\mathcal{L} \subseteq \mathcal{L}'$.*
(4) *If $\mathcal{L}$ is localizing, then $\mathcal{L}^{\mathrm{upglue}}$ is the smallest topology $\mathcal{L}'$ with $\mathcal{L} \subseteq \mathcal{L}'$.*

*Proof.* (1) It suffices to show that $\mathcal{B}_\alpha$ is a basis for $\mathcal{L}_\alpha$. To begin with, $\mathcal{B}_0$ is a basis for $\mathcal{L}_0$. Suppose $\mathcal{B}_\alpha$ is a basis for $\mathcal{L}_\alpha$. Then by Lemma 2.2, $\mathcal{B}_{\alpha+1} = \mathcal{B}_\alpha^+$ is a basis for $\mathcal{L}_\alpha^+$ and hence also for $(\mathcal{L}_\alpha^+)^u = \mathcal{L}_{\alpha+1}$. For a limit ordinal $\beta$, if $\mathcal{B}_\alpha$ is a basis for $\mathcal{L}_\alpha$ for every $\alpha < \beta$, then $\mathcal{B}_\beta = \cup_{\alpha<\beta}\mathcal{B}_\alpha$ is a basis for $\cup_{\alpha<\beta}\mathcal{L}_\alpha = \mathcal{L}_\beta$. (2) and (3) are obvious. For (4), it suffices to note that by Lemma 2.3 and transfinite induction, $\mathcal{L}^{\mathrm{upglue}}$ is localizing. □

2.3. **Linear sites and sheaves.** The main interest in linear topologies on linear categories, lies in the fact that they allow the definition of well-behaved categories of sheaves. Let $\mathfrak{a}$ be a $k$-linear category and let $\mathcal{T}$ be a cover system on $\mathfrak{a}$. A module $M \in \mathsf{Mod}(\mathfrak{a})$ is a *sheaf* on $\mathfrak{a}$ if for every morphism $f : R \longrightarrow M$ in $\mathsf{Mod}(\mathfrak{a})$ with $R \in \mathcal{T}(A)$, there is a unique morphism $\xi : \mathfrak{a}(-, A) \longrightarrow M$ which restricts to $f$ on $R$, that is, the composition $\xi|_R : R \longrightarrow \mathfrak{a}(-, A) \longrightarrow M$ is equal to $f$. Let

$$\mathsf{Sh}(\mathfrak{a}, \mathcal{T}) \subseteq \mathsf{Mod}(\mathfrak{a})$$

be the full subcategory of sheaves on $\mathfrak{a}$. Recall that a *localization* of a category $\mathcal{C}$ is a fully faithful functor $\mathcal{C}' \longrightarrow \mathcal{C}$ which has an exact (that is, finite limit preserving) left adjoint. A *strict localization* of $\mathcal{C}$ is a full subcategory closed under adding isomorphic objects, for which the inclusion functor is a localization. A small $k$-linear category endowed with a $k$-linear topology $\mathcal{T}$ is called a *$k$-linear site*.

The following is well-known:

**Proposition 2.5.** *Let* $(\mathfrak{a}, \mathcal{T})$ *be a* $k$*-linear site. The category* $\mathsf{Sh}(\mathfrak{a}, \mathcal{T})$ *is a strict localization of* $\mathsf{Mod}(\mathfrak{a})$ *and the association* $\mathcal{T} \longmapsto \mathsf{Sh}(\mathfrak{a}, \mathcal{T})$ *defines a 1-1 correspondence between* $k$*-linear topologies on* $\mathfrak{a}$ *and strict localizations of* $\mathsf{Mod}(\mathfrak{a})$*.*

In order to construct the category of sheaves on a site, we do not need the entire topology:

**Proposition 2.6.** *Let* $\mathfrak{a}$ *be a small* $k$*-linear category with cover systems* $\mathcal{B} \subseteq \mathcal{L} \subseteq \mathcal{T}$ *such that* $\mathcal{L}$ *is localizing,* $\mathcal{B}$ *is a basis for* $\mathcal{L}$ *and* $\mathcal{T} = \mathcal{L}^{\text{upglue}}$*. We have*

$$\mathsf{Sh}(\mathfrak{a}, \mathcal{B}) = \mathsf{Sh}(\mathfrak{a}, \mathcal{L}) = \mathsf{Sh}(\mathfrak{a}, \mathcal{T}).$$

For $k \neq \varnothing$, we can further give an alternative description of the category of sheaves, based upon torsion modules. In this case, let $\mathcal{T}$ be a cover system on $\mathfrak{a}$. A module $M \in \mathsf{Mod}(\mathfrak{a})$ is *torsion* if for every morphism $\xi : \mathfrak{a}(-, A) \longrightarrow M$, there is a cover $R \in \mathcal{T}(A)$ on which $\xi$ vanishes, that is, the composition $\xi|_R : R \longrightarrow \mathfrak{a}(-, A) \longrightarrow M$ is equal to zero. Let $\mathsf{Tors}(\mathfrak{a}, \mathcal{T}) \subseteq \mathsf{Mod}(\mathfrak{a})$ denote the full subcategory of torsion modules. Cleary, if $\mathcal{B}$ is a basis for a cover system $\mathcal{T}$, we have $\mathsf{Tors}(\mathfrak{a}, \mathcal{B}) = \mathsf{Tors}(\mathfrak{a}, \mathcal{T})$.

Recall that a *localizing Serre subcategory* of a Grothendieck abelian category $\mathcal{C}$ is a full subcategory $\mathcal{S}$ closed under coproducts, subquotients and extensions. In this case the Gabriël quotient $\mathcal{C}/\mathcal{S}$ exists, and is equivalent to the right orthogonal

$$\mathcal{S}^{\perp} = \{C \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(S, C) = 0 = \mathrm{Ext}^1_{\mathcal{C}}(S, C) \text{ for all } S \in \mathcal{S}\}.$$

We have the following:

**Proposition 2.7.** *Let* $(\mathfrak{a}, \mathcal{T})$ *be a* $k$*-linear site for* $k \neq \varnothing$*. The category* $\mathsf{Tors}(\mathfrak{a}, \mathcal{T})$ *is a localizing Serre subcategory for which*

$$\mathsf{Sh}(\mathfrak{a}, \mathcal{T}) = \mathsf{Tors}(\mathfrak{a}, \mathcal{T})^{\perp}.$$

2.4. **The conditions (G), (F) and (FF).** For $k = \varnothing$, categories of sheaves on arbitrary small sites are characterized internally as so called Grothendieck topoi by Giraud's theorem [1]. For $k \neq \varnothing$, it follows from the Gabriël-Popescu theorem that the categories of $k$-linear sheaves are precisely the $k$-linear Grothendieck abelian categories. This leads to the point of view that Grothendieck abelian categories can be considered as "linear topoi". Following our earlier convention, by a *$k$-linear topos* we will mean a Grothendieck topos if $k = \varnothing$, and a $k$-linear Grothendieck abelian category if $k$ is a commutative ring.

For a given $k$-linear Grothendieck topos $\mathcal{C}$, one is interested in finding interesting representations of $\mathcal{C}$ as a category of sheaves. Such representations originate from $k$-linear functors $\gamma : \mathfrak{u} \longrightarrow \mathcal{C}$ with $\mathfrak{u}$ a small $k$-linear category, the functor $\gamma$ giving rise to a functor

$$\iota : \mathcal{C} \longrightarrow \mathsf{Mod}(\mathfrak{a}) : C \longmapsto \mathcal{C}(\phi(-), C)$$

with left adjoint $a : \mathsf{Mod}(\mathfrak{a}) \longrightarrow \mathcal{C}$ obtained as the colimit preserving extension of $\gamma$.

We recall the following theorem from [11], which is proven in loc. cit. for $k = \mathbb{Z}$. The proof for $k$ an arbitrary commutative ring is entirely similar, and the result for $k = \varnothing$ is proven along the same lines, generalizing the construction in Giraud's theorem. The case $k = \varnothing$ also follows from a suitable version of the "Lemme de Comparaison" [1].

We call a family of morphisms $(c_i : C_i \longrightarrow C)_{i \in I}$ in $\mathcal{C}$ *jointly epimorphic* if for $f, g : C \longrightarrow D$ in $\mathcal{C}$ we have $f = g$ if and only if $fc_i = gc_i$ for all $i \in I$. We define the cover system $\mathcal{T}$ on $\mathfrak{u}$ with $R \in \mathcal{T}(U)$ if and only if $(\gamma(f))_{f \in [R]}$ is jointly epimorphic. We say that the functor $\gamma$ satisfies

(G) and is called *generating* if the objects $\gamma(U)$ for $U \in \mathfrak{u}$ generate $\mathcal{C}$;

(F) if for every $c : \gamma(U) \longrightarrow \gamma(V)$ in $\mathcal{C}$ there exists a covering family $(f_i : U_i \longrightarrow U)_i$ in $\mathcal{U}$ with for all $i \in I$, $c\gamma(f_i) = \gamma(g_i)$ for some $g_i : U_i \longrightarrow V$.

(FF) if for every $f : U \longrightarrow V$ in $\mathfrak{u}$ with $\gamma(f) = 0$, there exists a covering family $(f_i : U_i \longrightarrow U)_i$ in $\mathcal{U}$ with $ff_i = 0$ for all $i \in I$.

*Remark* 2.8. (1) Note that it is equivalent to require, in the definitions of (F) and (FF), the existence of a covering sieve rather than a covering family, as is done in [11].

(2) The notation (F) stands for "full up to coverings" and (FF) stands for "faithful up to coverings".

**Theorem 2.9.** *The following are equivalent:*

(1) *The functor $\gamma$ satisfies (G), (F) and (FF);*

(2) *The functor $\iota$ is a localization.*

*In this case, $\mathcal{T}$ is a topology on $\mathfrak{u}$ and $\iota$ factors through an equivalence $\mathcal{C} \longrightarrow \mathsf{Sh}(\mathfrak{a}, \mathcal{T})$.*

**Definition 2.10.** The functor $\gamma : \mathfrak{u} \longrightarrow \mathcal{C}$ is called *localizing* if the equivalent conditions in Theorem 2.9 are fulfilled.

2.5. **Transfer of cover systems.** Let $\mathfrak{a}$, $\mathfrak{b}$ be small $k$-linear categories. Let $\mathrm{cov}(\mathfrak{a})$ denote the set of cover systems on $\mathfrak{a}$. Consider a $k$-linear functor $\varphi : \mathfrak{b} \longrightarrow \mathfrak{a}$ with colimit preserving extension $\hat{\varphi} : \mathsf{Mod}(\mathfrak{b}) \longrightarrow \mathsf{Mod}(\mathfrak{a})$. For a sieve $s : S \subseteq \mathfrak{b}(-, B)$, we define $\varphi S$ to be the image of the map $\hat{\varphi}(s) : \hat{\varphi}(S) \longrightarrow \hat{\varphi}(\mathfrak{b}(-, B)) = \mathfrak{a}(-, \varphi(B))$. For any collection of generators $b_i : B_i \longrightarrow B$ with $S = \langle b_i \rangle$, $\varphi S$ is the sieve of $\mathfrak{a}(-, \varphi(B))$ generated by the morphisms $\varphi(b_i)$.

We define natural maps between the sets of cover systems

$$\varphi : \mathrm{cov}(\mathfrak{b}) \longrightarrow \mathrm{cov}(\mathfrak{a}) : \mathcal{S} \longmapsto \varphi\mathcal{S} = \{\varphi S \mid S \in \mathcal{S}\}$$

and

$$\varphi^{-1} : \mathrm{cov}(\mathfrak{a}) \longrightarrow \mathrm{cov}(\mathfrak{b}) : \mathcal{T} \longmapsto \varphi^{-1}\mathcal{T} = \{S \mid \varphi S \in \mathcal{T}\}.$$

We obviously have $\varphi\varphi^{-1}\mathcal{T} \subseteq \mathcal{T}$ and $\mathcal{S} \subseteq \varphi^{-1}\varphi\mathcal{S}$. If we consider a second functor $\psi : \mathfrak{c} \longrightarrow \mathfrak{b}$ then for a sieve $T$ on $C \in \mathfrak{c}$, we have $(\varphi\psi)T = \varphi(\psi T)$. Consequently, for a cover system $\mathcal{T}$ on $\mathfrak{a}$, we have $(\varphi\psi)^{-1}\mathcal{T} = \psi^{-1}(\varphi^{-1}\mathcal{T})$.

**Definition 2.11.** Consider $\varphi : \mathfrak{b} \longrightarrow \mathfrak{a}$ as above and let $\mathcal{T}$ and $\mathcal{S}$ be cover systems on $\mathfrak{a}$ and $\mathfrak{b}$ respectively. For $B \in \mathfrak{b}$ with $A = \varphi(B)$, $\varphi$ is called *cover continuous in* $B$ if for every $R \in \mathcal{T}(A)$, there exist $S \in \mathcal{S}(B)$ with $\varphi S \subseteq R$. The functor $\varphi$ is called *cover continuous* if it is cover continuous in $B$ for every $B \in \mathfrak{b}$.

**Lemma 2.12.** *Consider $\varphi : \mathfrak{b} \longrightarrow \mathfrak{a}$ as above and let $\mathcal{T}$ and $\mathcal{S}$ be localizing cover systems on $\mathfrak{a}$ and $\mathfrak{b}$ respectively. Consider the associated topologies $\mathcal{T}^{\mathrm{upglue}}$ and $\mathcal{S}^{\mathrm{upglue}}$. If $\varphi$ is cover continuous with respect to $\mathcal{T}$ and $\mathcal{S}$, then $\varphi$ is also cover continuous with respect to $\mathcal{T}^{\mathrm{upglue}}$ and $\mathcal{S}^{\mathrm{upglue}}$.*

*Proof.* Without loss of generality, we may suppose that $\mathcal{S}$ is a topology. Let $\mathcal{R}$ be the largest cover system on $\mathfrak{a}$ for which $\varphi$ becomes cover continuous with respect to $\mathcal{S}$ on $\mathfrak{b}$ and $\mathcal{R}$ on $\mathfrak{a}$. Concretely, for $A \in \mathfrak{a}$, we have $R \in \mathcal{R}(A)$ if and only if for every $B \in \mathfrak{b}$ with $\varphi(B) = A$, there is a cover $S \in \mathcal{S}(B)$ with $\varphi S \subseteq R$. By the assumption, we have $\mathcal{T} \subseteq \mathcal{R}$. If we can show that $\mathcal{R}$ is closed under glueings, then by Proposition 2.4 (3), we also have $\mathcal{T}^{\mathrm{upglue}} \subseteq \mathcal{R}$ as desired. Hence, consider $R \in \mathcal{R}(A)$ and for every $f : A_f \longrightarrow A$ in $R$, $R_f \in \mathcal{R}(A_f)$. If $A$ is not in the image of $\varphi$, then every sieve on $A$ is in $\mathcal{R}$ so there is nothing to check. Suppose $A = \varphi(B)$ for some $B \in \mathfrak{b}$. Then there is an $S \in \mathcal{S}(B)$ with $\varphi S \subseteq R$. For every $g : B_g \longrightarrow B$ in $S$, we thus have $\varphi(g) : \varphi(B_g) \longrightarrow \varphi(B) = A$ in $R$. Further, for the cover $R_{\varphi(g)} \in \mathcal{R}(\varphi(B_g))$, there exists a cover $S_g \in \mathcal{S}(B_g)$ with $\varphi S_g \subseteq R_{\varphi(g)}$. Clearly, we have $\varphi(S \circ (S_g)) \subseteq R \circ (R_f)$. □

Consider $\varphi : \mathfrak{b} \longrightarrow \mathfrak{a}$ and suppose $\mathfrak{a}$ is endowed with a topology $\mathcal{T}$. Consider the canonical morphism $\mathfrak{a} \longrightarrow \mathsf{Sh}(\mathfrak{a}, \mathcal{T})$ obtained as the composition of the Yoneda embedding $\mathfrak{a} \longrightarrow \mathsf{Mod}(\mathfrak{a})$ with the sheafification $\mathsf{Mod}(\mathfrak{a}) \longrightarrow \mathsf{Sh}(\mathfrak{a}, \mathcal{T})$. Consider its composition $\phi : \mathfrak{b} \longrightarrow \mathsf{Sh}(\mathfrak{a})$ with $\varphi$. We can apply Theorem 2.9 to $\phi$.

Suppose $\mathfrak{a}$ is endowed with a cover system $\mathcal{T}$. We say that the functor $\varphi$ satisfies

(G) if for every $A \in \mathfrak{a}$ there is a covering family $(\varphi(B_i) \longrightarrow A)_i$ for $\mathcal{T}$.

Suppose $\mathfrak{b}$ is endowed with a cover system $\mathcal{S}$. We say that the functor $\varphi$ satisfies

(F) if for every $a : \varphi(B) \longrightarrow \varphi(B')$ in $\mathfrak{a}$ there exists a covering family $(f_i : B_i \longrightarrow B)_i$ for $\mathcal{S}$ with for all $i \in I$, $a\phi(f_i) = \phi(g_i)$ for some $g_i : B_i \longrightarrow B'$.

(FF) if for every $f : B \longrightarrow B'$ in $\mathfrak{a}$ with $\varphi(f) = 0$, there exists a covering family $(f_i : B_i \longrightarrow B)_i$ for $\mathcal{S}$ with $ff_i = 0$ for all $i \in I$.

The following result is a generalized “Lemme de comparaison” from [1]:

**Theorem 2.13.** *Let $\mathcal{T}$ be a topology on $\mathfrak{a}$ and consider the cover system $\varphi^{-1}\mathcal{T}$ on $\mathfrak{b}$. The following are equivalent:*

1. *The functor $\varphi$ satisfies (G), (F) and (FF) relative to $\mathcal{T}$ and $\varphi^{-1}\mathcal{T}$;*
2. *The cover system $\varphi^{-1}\mathcal{T}$ is a topology on $\mathfrak{b}$ and the forgetful functor $\mathsf{Mod}(\mathfrak{a}) \longrightarrow \mathsf{Mod}(\mathfrak{b})$ restricts to an equivalence $\mathsf{Sh}(\mathfrak{a}, \mathcal{T}) \longrightarrow \mathsf{Sh}(\mathfrak{b}, \varphi^{-1}\mathcal{T})$.*

**Definition 2.14.** The functor $\varphi$ is called *localizing* with respect to $\mathcal{T}$ if the equivalent conditions of Theorem 2.13 are fulfilled.

**Lemma 2.15.** *Let $\varphi : \mathfrak{b} \longrightarrow \mathfrak{a}$ be as above and let $\mathfrak{a}$ be endowed with a topology $\mathcal{T}$ and $\mathfrak{b}$ with the cover system $\varphi^{-1}\mathcal{T}$. If $\varphi$ satsifies (G) and (F), then $\varphi$ is cover continuous.*

*Proof.* Consider $B \in \mathfrak{b}$, $A = \varphi(B)$, and $R \in \mathcal{T}(A)$. Take arbitrary generators $R = \langle a_i \rangle$ for $a_i : A_i \longrightarrow A$. Since $\varphi$ satisfies (G), for every $i$, there is a covering family $a_{ij} : \varphi(B_{ij}) \longrightarrow A_i$. Since $\varphi$ satisfies (F), for every $a_i a_{ij} : \varphi(B_{ij}) \longrightarrow \varphi(B)$, there exists a covering family $b_{ijk} : B_{ijk} \longrightarrow B_{ij}$ and morphisms $c_{ijk} : B_{ijk} \longrightarrow B$ with $a_i a_{ij} \varphi(b_{ijk}) = \varphi(c_{ijk})$. By (Glue”), the morphisms $c_{ijk}$ constitute a covering family with $\langle \varphi(c_{ijk}) \rangle \subseteq R$. □

**Proposition 2.16.** *Let $\varphi : \mathfrak{b} \longrightarrow \mathfrak{a}$ be as above and let $\mathcal{T}$ and $\mathcal{S}$ be topologies on $\mathfrak{a}$ and $\mathfrak{b}$ respectively. If $\varphi$ is cover continuous and satisfies (F) and (FF) with respect to $\mathcal{S}$, then $\varphi^{-1}\mathcal{T} \subseteq \mathcal{S}$.*

*Proof.* Consider $B \in \mathfrak{b}$ and a sieve $R \subseteq \mathfrak{b}(-, B)$ with $\varphi R \in \mathcal{T}$. We are to show that $R \in \mathcal{S}$. Since $\varphi$ is cover continuous, there is a cover $S \in \mathcal{S}(B)$ with $\varphi S \subseteq \varphi R$. In particular, for every $s_i : B_i \longrightarrow B$ in $S(B')$, there exist finitely many morphisms $r_i^\alpha : B_i^\alpha \longrightarrow B$ in $R(B_i^\alpha)$ and morphism $g_i^\alpha : \varphi(B_i) \longrightarrow \varphi(B_i^\alpha)$ in $\mathfrak{a}$ with $\varphi(s_i) = \sum_{\alpha=1}^n \varphi(r_i^\alpha) g_i^\alpha$. Since $\varphi$ satisfies (F), taking the intersection of $n$ covers, there is a cover $S_i \in \mathcal{S}(B_i)$ with for every $s_{ij} : B_{ij} \longrightarrow B_i$ in $S_i(B_{ij})$, $g_i^\alpha \varphi(s_{ij}) = \varphi(f_{ij}^\alpha)$ for some $f_{ij}^\alpha$. hence $\varphi(s_i s_{ij}) = \varphi(\sum_{\alpha=1}^n r_i^\alpha f_{ij}^\alpha)$. Since $\varphi$ satisfies (FF), for every $i$, $j$ there is a further cover $S_{ij} \in \mathcal{S}(B_{ij})$ with for every $s_{ijk} : B_{ijk} \longrightarrow B_{ij}$ in $S_{ij}(B_{ijk})$, $s_i s_{ij} s_{ijk} = \sum_{\alpha=1}^n r_i^\alpha f_i^\alpha j h_{ijk}$ for some $h_{ijk}$. This shows that $S \circ (S_i) \circ (S_{ij}) \subseteq R$, which finishes the proof since $\mathcal{S}$ satisfies (Glue') and (Up). □

## 3. Linearized topologies

From now on, $k$ is a commutative ground ring. Let $\mathcal{U}$ be a small (non-linear) category and let $\mathfrak{a}$ be a $k$-linear $\mathcal{U}$-graded category in the sense of [13]. In this section, for a topology $\tau$ on $\mathcal{U}$, we define an associated *linearized topology* $\mathcal{T}_\tau$ on the $k$-linear category $\tilde{\mathfrak{a}}$ associated to $\mathfrak{a}$. If $\mathcal{T}_\tau$ equals a certain simpler localizing cover system $\mathcal{L}_\tau$, based upon [7, Thm. 2.8] we obtain a recognition result for the sheaf category over the linearized topology (Theorem 3.25). We investigate the following general question: suppose $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ is a localizing functor with respect ot $\tau$ (Definition 2.14), $\sigma = \varphi^{-1}\tau$ is the induced topology on $\mathcal{V}$ and $\phi : \mathfrak{a}^{\varphi} \longrightarrow \mathfrak{a}$ is the naturally induced linear functor, what are natural conditions under which $\phi$ is localizing? In particular, we introduce a class of *stably localizing* functors $\varphi$ (Definition 3.17) for which this is indeed the case (Theorem 3.16).

3.1. **Map-graded categories.** The setting for linearized topologies is that of map-graded categories in the sense of [13]. Map-graded categories have two levels, a lower level which is not enriched (hence, corresponding to the case $k = \varnothing$ before), and an upper level which is enriched over $k$ for a commutative ring $k$. If we would instead take $k = \varnothing$ for the upper level as well, a map-graded category would simply be a functor $\mathfrak{a} \longrightarrow \mathcal{U}$ in which $\mathcal{U}$ represents the lower level and $\mathfrak{a}$ represents the upper level. For $k$ a commutative ring, the correct notion is the following. Let $\mathcal{U}$ be a (non-linear) small category. A *$k$-linear $\mathcal{U}$-graded category* consists of the following data:

- For every object $U \in \mathcal{U}$, a set of objects $\mathfrak{a}_U$ "living over $U$".
- For every morphism $u : V \longrightarrow U$ in $\mathcal{U}$ and objects $A_V \in \mathfrak{a}_V$, $A_U \in \mathfrak{a}_U$, a $k$-module $\mathfrak{a}_u(A_V, A_U)$ of morphisms "living over $u$".
- For every $U \in \mathcal{U}$, $A \in \mathfrak{a}_U$, an identity morphism $1_A \in \mathfrak{a}_{1_U}(A, A)$.
- For $v : W \longrightarrow V$ and $u : V \longrightarrow U$ in $\mathcal{U}$ and $A_U \in \mathfrak{a}_U$, $A_V \in \mathfrak{a}_V$ and $A_W \in \mathfrak{a}_W$, a $k$-bilinear composition

$$\mathfrak{a}_u(A_V, A_U) \times \mathfrak{a}_v(A_W, A_V) \longrightarrow \mathfrak{a}_{uv}(A_W, A_U) : (a, b) \longmapsto ab.$$

These data have to satisfy the obvious identity and associativity axioms. Note however that $\mathfrak{a}$ is not itself a $k$-linear category, or even a category. On the other hand, we can associate a $k$-linear category $\tilde{\mathfrak{a}}$ to $\mathfrak{a}$ in the following way. We put $\mathrm{Ob}(\tilde{\mathfrak{a}}) = \coprod_{U \in \mathcal{U}} \mathfrak{a}_U$ and for objects $A_V \in \mathfrak{a}_V$, $A_U \in \mathfrak{a}_U$, we put

$$\tilde{\mathfrak{a}}(A_V, A_U) = \bigoplus_{u \in \mathcal{U}(V,U)} \mathfrak{a}_u(A_V, A_U).$$

Following the philosophy from [18] that $k$-linear categories can be viewed as $k$-algebras with several objects, $k$-linear $\mathcal{U}$-graded categories can be viewed as monoid-graded algebras with several objects (in which both the monoid and the algebra are allowed to have several objects).

3.2. **Linearized topologies.** Let $\mathfrak{a}$ be a $k$-linear $\mathcal{U}$-graded category with associated $k$-linear category $\tilde{\mathfrak{a}}$. Next we explain how to "linearize" a topology on $\mathcal{U}$ in order to obtain a $k$-linear topology on $\tilde{\mathfrak{a}}$. Consider a sieve $R = R^U \subseteq \mathcal{U}(-, U)$. For $A \in \mathfrak{a}_U$, $B \in \mathfrak{a}_V$, we consider the $k$-module

$$R^A(B) = \bigoplus_{f \in R^U(V)} \mathfrak{a}_f(B, A).$$

Clearly, an element $b \in \mathfrak{a}_g(B', B)$ for $g \in \mathcal{U}(V', V)$ induces a $k$-linear morphism $-b : R^A(B) \longrightarrow R^A(B')$ which sends $a \in \mathfrak{a}_f(B, A)$ to $ab \in \mathfrak{a}_{fg}(B', A)$, since $fg \in R^U(V')$. Consequently, an arbitrary element $\sum_{i=1}^n b_i \in \tilde{\mathfrak{a}}(B', B)$ induces $k$-linear morphism $\sum_{i=1}^n (-b_i) : R^A(B) \longrightarrow R^A(B')$, and we thus obtain a $k$-linear sieve $R^A \subseteq \tilde{\mathfrak{a}}(-, A)$.

Let $\tau$ be a cover system on $\mathcal{U}$. We define the cover system $\mathcal{B}_\tau$ on $\tilde{\mathfrak{a}}$ with, for $A \in \mathfrak{a}_U$:

$$\mathcal{B}_\tau(A) = \{R^A \mid R \in \tau(U)\}.$$

**Lemma 3.1.** (1) $\mathcal{U}(-, U)^A = \tilde{\mathfrak{a}}(-, A)$.
(2) $(R \cap S)^A = R^A \cap S^A$.

(3) *For* $a \in \mathfrak{a}_g(A', A)$ *and* $R \in \mathcal{T}(U)$, $(g^{-1}R)^{A'} \subseteq a^{-1}R^A$.
(4) *For* $a = \sum_i a_i \in \tilde{\mathfrak{a}}(A', A)$ *with* $a_i \in \mathfrak{a}_{g_i}(A', A)$ *and* $R \in \mathcal{T}(U)$, $(\cap_i g_i^{-1}R)^A \subseteq a^{-1}R^A$.

*Proof.* (1) $\mathcal{U}(-, U)^A(B) = \oplus_{f:V \longrightarrow U} \mathfrak{a}_f(B, A) = \tilde{\mathfrak{a}}(B, A)$. (2) $(R \cap S)^A(B) = \oplus_{f \in R(V) \cap S(V)} \mathfrak{a}_f(B, A) = \oplus_{f \in R(V)} \mathfrak{a}_f(B, A) \cap \oplus_{f \in S(V)} \mathfrak{a}_f(B, A)$. (3) Consider a morphism $b = \sum_i b_i : B \longrightarrow A'$ with $b_i \in \mathfrak{a}_{f_i}(B, A')$ with $gf_i \in R(U)$ for all $i$. Then $ab_i \in \mathfrak{a}_{gf_i}(B, A)$ do $ab_i \in R^A(B)$ as desired. (4) By (3), for every $i$ we have $(g_i^{-1}R)^{A'} \subseteq a_i^{-1}R^A$. Hence $(\cap_i g_i^{-1}R)^{A'} = \cap_i (g_i^{-1}R^{A'}) \subseteq \cap_i a_i^{-1}R^A \subseteq a^{-1}R^A$. □

We define the cover system $\mathcal{L}_\tau = \mathcal{B}_\tau^{\mathrm{up}}$ on $\mathfrak{a}$ with, for $A \in \mathfrak{a}_U$:

$$\mathcal{L}_\tau(A) = \{S \subseteq \tilde{\mathfrak{a}}(-, A) \mid \exists R \in \tau(U)\ R^A \subseteq S\}.$$

**Proposition 3.2.** *If* $\tau$ *satisfies the axioms (Id) and (Pb), then so does* $\mathcal{L}_\tau$. *We have*

*Proof.* This follows from Lemma 3.1 (1) and (4). □

**Definition 3.3.** Let $\mathfrak{a}$ be a $\mathcal{U}$-graded category and let $\tau$ be a topology on $\mathcal{U}$. The topology $\mathcal{T}_\tau$ on $\tilde{\mathfrak{a}}$ is the smallest topology containing $\mathcal{B}_\tau$. The category of *sheaves on* $\mathfrak{a}$ is by definition $\mathsf{Sh}(\mathfrak{a}, \tau) = \mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{T}_\tau)$ and the category of *torsion modules on* $\mathfrak{a}$ is $\mathsf{Tors}(\mathfrak{a}, \tau) = \mathsf{Tors}(\tilde{\mathfrak{a}}, \mathcal{T}_\tau)$.

**Proposition 3.4.** *Let* $\mathfrak{a}$ *be a* $\mathcal{U}$*-graded category and let* $\tau$ *be a topology on* $\mathcal{U}$. *We have* $\mathcal{T}_\tau = \mathcal{L}_\tau^{\mathrm{upglue}}$ *and* $\mathsf{Sh}(\mathfrak{a}, \tau) = \mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{L}_\tau) = \mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{B}_\tau)$.

*Proof.* This follows from Propositions 3.2 and 2.4(4). □

Unlike the category of sheaves $\mathsf{Sh}(\mathfrak{a}, \tau)$, in general the category $\mathsf{Tors}(\mathfrak{a}, \tau)$ of torsion modules cannot be defined directly in terms of $\tau$ and $\mathcal{B}_\tau$. However, in many cases, it is possible to give such a direct description.

**Proposition 3.5.** *Let* $\mathfrak{a}$ *be a* $\mathcal{U}$*-graded category and* $\tau$ *a topology on* $\mathcal{U}$. *If* $\mathcal{L}_\tau$ *satisfies the glueing axiom (Glue), then we have* $\mathcal{L}_\tau = \mathcal{T}_\tau$ *and A module* $M \in \mathsf{Mod}(\tilde{\mathfrak{a}})$ *is* $\tau$-torsion *if for every* $A \in \mathfrak{a}_U$ *and* $x \in M(A)$ *there is a cover* $R \in \tau(U)$ *such that for every* $f : V \longrightarrow U$ *in* $R$, $a : B \longrightarrow A$ *in* $\mathfrak{a}_f(B, A) \subseteq \tilde{\mathfrak{a}}(B, A)$, *we have* $M(a)(x) = 0$.

To end this section, we formulate conditions which ensure that $\mathcal{L}_\tau$ satisfies the glueing axiom, and hence $\mathcal{L}_\tau = \mathcal{T}_\tau$. Recall that a module $M \in \mathsf{Mod}(\tilde{\mathfrak{a}})$ is called finitely generated if there exists an epimorphism $\oplus_{i=1}^n \tilde{\mathfrak{a}}(-, A_i) \longrightarrow M$.

**Proposition 3.6.** *Let* $\mathfrak{a}$ *be a* $\mathcal{U}$*-graded category and* $\tau$ *a topology on* $\mathcal{U}$. *Suppose the following conditions hold:*

(1) *For every* $w : W \longrightarrow U$ *in* $\mathcal{U}$ *and cover* $R \in \tau(W)$ *there is a cover* $S \in \tau(U)$ *with* $w^{-1}S \subseteq R$.
(2) *There is a basis* $\beta$ *of* $\tau$ *such that for every* $R \in \beta(U)$ *and* $A \in \mathfrak{a}_U$ *the module* $R^A \in \mathsf{Mod}(\tilde{\mathfrak{a}})$ *is finitely generated.*

*Then* $\mathcal{L}_\tau$ *is a topology on* $\tilde{\mathfrak{a}}$.

*Proof.* By Proposition 3.2, $\mathcal{L}_\tau$ satisfies (Id) and (Pb). We show that it also satisfies (Glue). Consider $A \in \mathfrak{a}_U$ and $S \subseteq \tilde{\mathfrak{a}}(-, A)$ such that there is a $T \in \mathcal{L}_\tau(A)$ and for every $b : B \longrightarrow A$ in $T(B)$ a $T_b \in \mathcal{L}_\tau(B)$ with $T_b \subseteq b^{-1}S$. Clearly, we may suppose that $T$ and all the $T_b$ are in the basis of $\mathcal{L}_\tau$ consisting of covers $R^C$ for $R \in \beta(W)$, $C \in \mathfrak{a}_W$. So, suppose $T = R^A$ for $R \in \beta(U)$ and $T_b = (R_b)^{B_b}$ for $R_b \in \beta(U_b)$. Take finitely many generators $b_i : B_i \longrightarrow A$, $i : 1, \dots, n$ for $T$. We may suppose that $b_i \in \mathfrak{a}_{u_i}(U_i, U)$ for $u_i : U_i \longrightarrow U$ in $R(U_i)$. For every $R_{b_i}$, take a cover $R_i \in \tau(U)$ with $u_i^{-1}R_i \subseteq R_{b_i}$. Consider the cover $R' = R \cap \cap_{i=1}^n R_i$ in $\tau(U)$.

We claim that ${R'}^A \subseteq S$. To see this, consider $w : W \longrightarrow U$ in $R'$ and $c : C \longrightarrow A$ in $\mathfrak{a}_w(C, A)$. Since $c \in R^A$, we can write $c = \sum_{i=1}^n b_i c_i$ for $c_i \in \tilde{\mathfrak{a}}(C, B_i)$. Further, write $c_i = \sum_{j=1}^{n_i} c_{ij}$ for $c_{ij} \in \mathfrak{a}_{u_{ij}}(C, A)$. It follows that $u_i u_{ij} = w$ for all $i, j$. Since $u_i u_{ij} = w \in R_i$, it follows that $u_{ij} \in R_{b_i}$, and consequently $c_{ij} \in T_{b_i} \subseteq {b_i}^{-1} S$. It follows that $c \in S$, as required. □

The following notion will be important later on. Let $\mathfrak{a}$ be a $\mathcal{U}$-graded category and $\tau$ a topology on $\mathcal{U}$. We say that $\mathfrak{a}$ satisfies

(WG) if for every covering family $f_i : U_i \longrightarrow U$ with $R = \langle f_i \rangle \in \tau(U)$ and for every $A \in \mathfrak{a}_U$, the collection of morphisms $F = \coprod_i \coprod_{A' \in \mathfrak{a}_{U_i}} \mathfrak{a}_{f_i}(A', A)$ has $\langle F \rangle = R^A$.

3.3. **Comparison of linearized topologies.** Let $\mathcal{U}, \mathcal{V}$ be small categories, $\mathfrak{a}$ a small $\mathcal{U}$-graded category and

$$\varphi : \mathcal{V} \longrightarrow \mathcal{U}$$

a functor. According to [15], the category of map-graded categories is naturally fibered over the category of small categories, through the association $(\mathcal{U}, \mathfrak{a}) \longmapsto \mathcal{U}$. In particular, we can construct a canonical $\mathcal{V}$-graded category $\mathfrak{a}^{\varphi}$ as follows. For $V \in \mathfrak{v}$, we put $\mathfrak{a}^{\varphi}_V = \mathfrak{a}_{\varphi(V)}$. For $A \in \mathfrak{a}^{\varphi}_V$, $A' \in \mathfrak{a}^{\varphi}_{V'}$ and $v \in \mathcal{V}(V, V')$, we put $\mathfrak{a}^{\varphi}_v(A, A') = \mathfrak{a}_{\varphi(v)}(A, A')$. The identity maps $\mathfrak{a}^{\varphi}_v(A, A') \longrightarrow \mathfrak{a}_{\varphi(v)}(A, A')$ give rise to a graded functor $(\mathcal{V}, \mathfrak{a}^{\varphi}) \longrightarrow (\mathcal{U}, \mathfrak{a})$. We are interested in the induced linear functor

$$\phi : \tilde{\mathfrak{a}}^{\varphi} \longrightarrow \tilde{\mathfrak{a}} : A \longmapsto A$$

with

$$\tilde{\mathfrak{a}}^{\varphi}(A, A') = \bigoplus_{v \in \mathcal{V}(V, V')} \mathfrak{a}^{\varphi}_v(A, A') \longrightarrow \bigoplus_{u \in \mathcal{U}(\varphi(V), \varphi(V'))} \mathfrak{a}_u(A, A') = \tilde{\mathfrak{a}}(A, A')$$

determined by the identity maps $\mathfrak{a}^{\varphi}_v(A, A') \longrightarrow \mathfrak{a}_{\varphi(v)}(A, A')$.

**Proposition 3.7.** *Suppose $\mathcal{V}$ is endowed with a topology $\sigma$, and let $\tilde{\mathfrak{a}}^{\varphi}$ be endowed with the cover system $\mathcal{B}_{\sigma}$. If $\varphi$ satsifies (F) (resp. (FF)) with respect to $\sigma$, then $\phi$ satisfies (F) (resp. (FF)) with respect to $\mathcal{B}_{\sigma}$.*

*Proof.* We prove (F), the proof of (FF) is similar. Consider $A \in \mathfrak{a}^{\varphi}_V$, $A' \in \mathfrak{a}^{\varphi}_{V'}$ and a morphism $a \in \tilde{\mathfrak{a}}(A, A')$. We have morphims $u_i \in \mathcal{U}(\varphi(V), \varphi(V'))$ and $a_i \in \mathfrak{a}_{u_i}(A, A')$ with $a = \sum_{i=0}^n a_i$. Taking the intersection of $n$ covers, we obtain a cover $R \in \sigma(V)$ such that for every $f_j : V_j \longrightarrow V$ in $R$ and $i$, there exists $h_{ij} : V_j \longrightarrow V'$ with $u_i \varphi(f_j) = \varphi(h_{ij})$. Consider $R^A \in \mathcal{B}_{\sigma}(A)$. Now for fixed $j$ consider any finite collection of morphisms $g_k : V_j \longrightarrow V$ in $R(W)$ and $B \in \mathfrak{a}^{\varphi}_W$, and $b_k \in \mathfrak{a}^{\varphi}_{g_k}(B, A)$. For $b = \sum_{k=1}^m b_k \in R^A(B)$, we have $\phi(b) = b \in \tilde{\mathfrak{a}}(B, A)$ and $ab = \sum_{i,k} a_i b_k$ for $a_i b_k \in \tilde{\mathfrak{a}}_{\varphi(h_{ij})}(B, A') = \tilde{\mathfrak{a}}^{\varphi}_{h_{ij}}(B, A')$. □

**Lemma 3.8.** *Let $\mathcal{U}$ be endowed with a topology $\tau$, and $\mathcal{V}$ with the cover system $\sigma = \varphi^{-1}\tau$. Suppose $\sigma$ is a topology.*

1. *Let $\tilde{\mathfrak{a}}$ be endowed with the cover system $\mathcal{L}_{\tau}$ and $\tilde{\mathfrak{a}}^{\varphi}$ with the cover system $\mathcal{L}_{\sigma}$. If $\varphi$ is cover continuous in $U \in \mathcal{U}$, then for all $A \in \mathfrak{a}_U$, $\phi$ is cover continuous in $A$.*
2. *Let $\tilde{\mathfrak{a}}$ be endowed with the cover system $\mathcal{T}_{\tau}$ and $\tilde{\mathfrak{a}}^{\varphi}$ with the cover system $\mathcal{T}_{\sigma}$. If $\varphi$ is cover continuous, then $\phi$ is cover continuous.*

*Proof.* For $A \in \mathfrak{a}^{\varphi}_V = \mathfrak{a}_{\varphi(V)}$, consider the a cover $R \in \tau(\varphi(V))$ and the cover $R^A \in \mathcal{B}_{\tau}(A)$. By assumption, there is a sieve $S \subseteq \mathcal{V}(-, V)$ with $\varphi S \subseteq R$. Hence, for every $s : V' \longrightarrow V$ in $S(V')$, we have $\varphi(s) \in R(\varphi(V'))$. Consider $S^A \in \mathcal{B}_{\sigma}(A)$. Obviously, $\phi S^A \subseteq R^A$. This proves (1). (2) now follows from Lemma 2.12. □

**Proposition 3.9.** *Let $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ be as above and let $\mathcal{U}$ be endowed with a topology $\tau$. Suppose $\varphi$ satsifies the conditions (G), (F), (FF) with respect to $\tau$ and $\sigma = \varphi^{-1}\tau$. Then $\sigma$ is a topology, and we have $\phi^{-1}\mathcal{T}_\tau \subseteq \mathcal{T}_\sigma$.*

*Proof.* According to Proposition 2.16, it suffices that (a) $\phi$ is cover continuous with respect to $\mathcal{T}_\tau$ and $\mathcal{T}_\sigma$ and (b) $\phi$ satisfies (F) and (FF) with respect to $\mathcal{T}_\sigma$. Requirement (a) follows from Lemma 3.8 and requirement (b) follows from Proposition 3.7. □

In order to proceed, we need to impose further conditions relative to the topology $\tau$ on $\mathcal{U}$, either on the $\mathcal{U}$-graded category $\mathfrak{a}$, or on the functor $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$. We say that $\varphi$ satisfies

(SG1) if for every $U \in \mathcal{U}$, there is a cover $R \in \tau(U)$ such that for every $f : U' \longrightarrow U$ in $R(U')$, we have $U' = \varphi(V)$ for some $V \in \mathcal{V}$.

(SG2) if for every $V \in \mathcal{V}$ there is a cover $R \in \tau(\varphi(V))$ such that for every $f : U' \longrightarrow \varphi(V)$ in $R(U')$, we have $f = \varphi(g)$ for some $g : V' \longrightarrow V$ in $\mathcal{V}$.

**Definition 3.10.** A sieve $T$ on $V \in \mathcal{V}$ is called a *$\mathcal{U}$-sieve* if $[\varphi T] = \{\varphi(f) \mid f \in [T]\}$.

Hence, $T$ is a $\mathcal{U}$-sieve if for every $f : V' \longrightarrow V$ in $T(V')$ and for every $u : U \longrightarrow \varphi(V')$ in $\mathcal{U}$, we have $\varphi(f)u = \varphi(g)$ for some $g : V'' \longrightarrow V$ in $T(V'')$.

**Lemma 3.11.** *Let $R$ be an arbitrary sieve on $V \in \mathcal{V}$, and for every $f : V_f \longrightarrow V$ in $R$, let $T_f$ be a $\mathcal{U}$-sieve on $V_f$. The composition $R \circ (T_f)$ is a $\mathcal{U}$-sieve on $V$.*

*Proof.* An element of $R \circ (T_f)$ can be written as $fg$ for $f : V_f \longrightarrow V$ in $R(V_f)$ and $g : W_g \longrightarrow V_f$ in $T_f(W_g)$. For every $u : U \longrightarrow \varphi(W_g)$ in $\mathcal{U}$, we have $\varphi(fg)u = \varphi(f)\varphi(g)u = \varphi(f)\varphi(h)$ for some $h : Z_h \longrightarrow V_f$ in $T_f(Z_h)$. hence, $\varphi(fg)u = \varphi(fh)$ for $fh \in R \circ (T_f)(Z_h)$. □

**Proposition 3.12.** *The following are equivalent:*

1. *$\varphi$ satisfies (SG2).*
2. *for every $V \in \mathcal{V}$, there exists a $\mathcal{U}$-sieve $T_V \in \varphi^{-1}\tau(V)$.*
3. *$\varphi^{-1}\tau$ has a basis of $\mathcal{U}$-sieves.*

*Proof.* Suppose (1) holds, and for $V \in \mathcal{V}$, let $R \in \tau(\varphi(V))$ be as stated. We define the sieve $\bar{R}$ on $V$ with $\bar{R}(V') = \{g : V' \longrightarrow V \mid \varphi(g) \in R(\varphi(V'))$. We clearly have $\varphi\bar{R} = R \in \tau(\varphi(V))$, and $\bar{R}$ is a $\mathcal{U}$-sieve. Hence (2) holds. Suppose (2) holds. Let $R \in \varphi^{-1}\tau(V)$ be a cover on $V$. For every $f : V_f \longrightarrow V$ in $R$, let $T_f \in \varphi^{-1}\tau(V_f)$ be a $\mathcal{U}$-sieve. By Lemma 3.11, the composition $R \circ (T_f)$ is a $\mathcal{U}$-sieve, and it is a cover since $\tau$ satisfies (Glue"). Obviously, we have $R \circ (T_f) \subseteq R$. Hence (3) holds. Suppose (3) holds. For $V \in \mathcal{V}$, since $\mathcal{V}(-, V) \in \varphi^{-1}\mathcal{T}(V)$, there exists a $\mathcal{U}$-sieve $T_V \in \varphi^{-1}\tau(V)$ and (2) holds. Suppose (2) holds and for $V \in \mathcal{V}$, let $T_V \in \varphi^{-1}\tau(V)$ be a $\mathcal{U}$-sieve. Then $\varphi T_V \in \tau(\varphi(V))$ is as in (SG2) and (1) holds. □

**Proposition 3.13.** *Suppose $\varphi$ is injective on objects and satisfies (SG2) with respect to a topology $\tau$ on $\mathcal{U}$. Then $\varphi$ satsifies (F) with respect to $\varphi^{-1}\tau$.*

*Proof.* Consider $f : \varphi(V') \longrightarrow \varphi(V)$ in $\mathcal{U}$. Take $\mathcal{U}$-sieves $T \in \varphi^{-1}\tau(V)$ and $T' \in \varphi^1\tau(V')$. Consider the cover $S = \varphi T' \cap f^{-1}\varphi T \in \tau(\varphi(V'))$. Let $\bar{S}$ be the sieve on $V'$ with $\bar{S}(W) = \{v : W \longrightarrow V' \in \mathcal{V} \mid \varphi(v) \in S(\varphi(W))\}$. Then $\varphi\bar{S} = S$ since $R_{V'}$ is a $\mathcal{U}$-sieve, and hence $\bar{S} \in \varphi^{-1}\tau(V')$. For $v : W \longrightarrow V'$ in $\bar{S}(W)$, we thus have $f\varphi(v) = \varphi(v')$ for some $v' : W' \longrightarrow V$ in $T$. Since $\varphi$ is injective on objects, we have $W = W'$ which finishes the proof. □

**Proposition 3.14.** *Let $\mathcal{U}$ be endowed with a topology $\tau$, and suppose $\varphi$ satisfies (G). If either $\mathfrak{a}$ satsifies (WG), or else $\varphi$ satisfies (SG1), then $\phi$ satsifies (G) with respect to $\mathcal{B}_\tau$.*

*Proof.* Consider $U \in \mathcal{U}$ and $A \in \mathfrak{a}_U$. Suppose first that $\mathfrak{a}$ satsifies (WG). Since $\varphi$ satisfies (G), there is a covering family $f_i : \varphi(V_i) \longrightarrow U$ for $\tau$. For every $A' \in \mathfrak{a}_{\varphi(V_i)}$, we also have $A' \in \mathfrak{a}^{\varphi}_{V_i}$ and $A' = \phi(A')$. By (WG), we have $\langle \coprod_i \coprod_{A' \in \mathfrak{a}_{\varphi(V_i)}} \mathfrak{a}_{f_i}(A', A)\rangle = R^A \in \mathcal{B}_\tau$, as desired.

Next suppose that $\varphi$ satisfies (SG1). We thus have a cover $R \in \tau(U)$ consisting of morphism $\varphi(V) \longrightarrow U$. Then $R^A \in \mathcal{T}_\tau(A)$ is a cover with for $A' \in \mathfrak{a}_{U'}$, $R^A(A') = \oplus_{f: U' \longrightarrow U \in R(U')} \mathfrak{a}_f(A', A)$. Since the domain of every $f \in R(U)$ is given by $U' = \varphi(V)$ for some $V$, every morphism $a \in \mathfrak{a}_f(A', A)$ satisfies $A' = \phi(A')$ and $\langle \coprod_f \coprod_{A' \in \mathfrak{a}_{U'}} \mathfrak{a}_f(A', A)\rangle = R^A \in \mathcal{B}_\tau$, as desired. □

**Proposition 3.15.** *Let $\mathcal{U}$ be endowed with a topology $\tau$, and $\mathcal{V}$ with the cover system $\sigma = \varphi^{-1}\tau$. Suppose $\sigma$ is a topology. If either $\mathfrak{a}$ satisifies (WG), or else $\varphi$ satisfies (SG2), then we have $\mathcal{B}_\sigma \subseteq \phi^{-1}\mathcal{T}_\tau$.*

*Proof.* Consider $S \in \sigma(V)$ and $A \in \tilde{\mathfrak{a}}^{\varphi}_V = \tilde{\mathfrak{a}}_{\varphi(V)}$. By assumption, we have $\varphi S \in \tau(\varphi(V))$. We are to show that $\phi S^A \in \mathcal{T}_\tau(A)$. Suppose first that $\mathfrak{a}$ satsifies (WG). We have $\varphi S = \langle \varphi(g) \mid g \in [S]\rangle$ and thus

$$(\varphi S)^A = \langle \coprod_{g: V' \longrightarrow V \in [S]} \coprod_{A' \in \mathfrak{a}_{\varphi(V')}} \mathfrak{a}_{\varphi(g)}(A', A)\rangle = \phi S^A.$$

Next suppose that $\varphi$ satisfies (SG2). By Proposition 3.12, we thus have a cover $T \subseteq S$ with $T \in \sigma(V)$ and $[\varphi T] = \{\varphi(t) \mid t \in [T]\}$. Obviously, $T^A \subseteq S^A$ and $\phi T^A \subseteq \phi S^A$. Further, we have $\phi T^A = (\varphi T)^A \in \mathcal{B}_\tau$ whence $\phi S^A \in \mathcal{T}_\tau$. □

**Theorem 3.16.** *Let $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ be a functor between small categories, and suppose $\varphi$ is localizing with respect to a topology $\tau$ on $\mathcal{U}$. Let $\mathfrak{a}$ be a $\mathcal{U}$-graded category and let $\phi : \tilde{\mathfrak{a}}^{\varphi} \longrightarrow \tilde{\mathfrak{a}}$ be the induced linear functor. Let $\sigma = \varphi^{-1}\tau$ be the induced topology on $\mathcal{V}$ and let $\mathcal{T}_\tau$ and $\mathcal{T}_\sigma$ be the induced linearized topologies on $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{a}}^{\varphi}$ respectively. Supose further that either $\mathfrak{a}$ satisfies (WG), or else $\varphi$ satisfies (SG1) and (SG2). Then $\phi$ is localizing with respect to $\mathcal{T}_\tau$ and we have $\phi^{-1}\mathcal{T}_\tau = \mathcal{T}_\sigma$. In particular, the forgetful functor $\mathsf{Mod}(\tilde{\mathfrak{a}}) \longrightarrow \mathsf{Mod}(\tilde{\mathfrak{a}}^{\varphi})$ restricts to an equivalence*

$$\mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{T}_\tau) \longrightarrow \mathsf{Sh}(\tilde{\mathfrak{a}}^{\varphi}, \mathcal{T}_\sigma).$$

*Proof.* By Proposition 3.14, $\phi$ satisfies (G) with respect to $\mathcal{T}_\tau$, and by Proposition 3.7, $\phi$ satisfies (F) and (FF) with respect to $\mathcal{B}_\sigma$. By Proposition 3.15, we have $\mathcal{B}_\sigma \subseteq \phi^{-1}\mathcal{T}_\tau$, so $\phi$ also satisfies (F) and (FF) with respect to $\phi^{-1}\mathcal{T}_\tau$. By Theorem 2.9 applied to $\phi$, $\phi^{-1}\mathcal{T}_\tau$ is a topology. By Proposition 3.9, we now have $\mathcal{B}_\sigma \subseteq \phi^{-1}\mathcal{T}_\tau \subseteq \mathcal{T}_\sigma$ so since $\mathcal{T}_\sigma$ is the smallest topology containing $\mathcal{B}_\sigma$ according to Proposition 2.4, we conclude that $\phi^{-1}\mathcal{T}_\tau = \mathcal{T}_\sigma$. □

**Definition 3.17.** A functor $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ is called *stably localizing* with respect to a topology $\tau$ on $\mathcal{U}$ if it is localizing and satsifies (SG1) and (SG2) with respect to $\tau$.

3.4. **Refining sites.** Let $(\mathcal{U}, \tau)$ be a site, that is, $\mathcal{U}$ is a small category endowed with a topology $\tau$. In this section, we develop a technique for constructing a "refined site" from a suitable subcategory $\mathcal{V} \subseteq \mathcal{U}$ for which the inclusion functor $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ is stably localizing. First, we note the following:

**Lemma 3.18.** *Suppose $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ is faithful and bijective on objects. If $\varphi$ satisfies (SG2), it is stably localizing.*

*Proof.* Condition (FF) is fulfilled since $\varphi$ is faithful. Condition (SG1) is fulfilled since $\varphi$ is surjective on objects, and (G) follows from (SG1). By Proposition 3.13, condition (F) follows from (SG2). □

Let $\mathcal{V} \subseteq \mathcal{U}$ be a subcategory with $\mathrm{Ob}(\mathcal{V}) = \mathrm{Ob}(\mathcal{U})$, and let $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ be the inclusion functor. In this setup, we say that a sieve $R$ on $U \in \mathcal{U}$ is a *$\mathcal{V}$-sieve* if for every $u : U' \longrightarrow U$ in $R(U')$, we have $u \in \mathcal{V}$. For every $\mathcal{V}$-sieve $R$ on $U \in \mathcal{U}$, we let $\bar{R}$ be the sieve on $U \in \mathcal{V}$ with

$$\bar{R}(U') = \{v : U' \longrightarrow U \mid v \in R(U')\}.$$

We assume that $\varphi$ satisfies (SG2). Hence, for every $U \in \mathcal{U}$ there is a cover $R_U \in \tau(U)$ which is a $\mathcal{V}$-sieve. Since for an arbitrary $S \in \tau(U)$, $S \cap R_U$ is a $\mathcal{V}$-sieve, $\tau$ has a basis $\beta$ of $\mathcal{V}$-sieves. Let $\bar{\beta}$ be the cover system on $\mathcal{V}$ with

$$\bar{\beta}(V) = \{\bar{R} \mid R \in \beta(V)\}.$$

**Lemma 3.19.** *$\bar{\beta}$ is a basis of $\varphi^{-1}\tau$ consisting of $\mathcal{U}$-sieves.*

*Proof.* By Proposition 3.12, $\varphi^{-1}\tau$ has a basis $\beta'$ of $\mathcal{U}$-sieves. For $S \in \beta'(V)$, let $R \in \beta(V)$ be such that $R \subseteq \varphi S$. Consequently, we have $\bar{R} \subseteq S$. □

Now let $(\mathcal{U}, \tau)$ be a site. The ingredient we need for our construction of $\mathcal{V}$ is the choice, for every object $U \in \mathcal{U}$, of a distinguished cover $D_U \in \tau(U)$. We put $\mathrm{Ob}(\mathcal{V}) = \mathrm{Ob}(\mathcal{U})$ and for $U, U' \in \mathcal{U}$, we put

$$\mathcal{V}(U', U) = \begin{cases} D_U(U) \cup \{1_U\} & \text{if } U' = U \\ D_U(U') & \text{otherwise.} \end{cases} \tag{5}$$

**Lemma 3.20.** *With the higher definitions, $\mathcal{V}$ is a subcategory of $\mathcal{U}$.*

*Proof.* By construction, $\mathcal{V}$ contains the identity morphisms of $\mathcal{U}$, which act as identity morphisms for $\mathcal{V}$ as well. It remains to check that $\mathcal{V}$ is closed under the composition of two morphisms different from identities. Hence, consider $f : U' \longrightarrow U$ in $D_U(U')$ and $g : U'' \longrightarrow U' \in D_{U'}(U'')$. Since $D_U$ is a sieve, we have $fg \in D_U(U'')$ as desired. □

We denote the inclusion functor by $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$.

**Proposition 3.21.** *The functor $\varphi$ is stably localizing with respect to $\tau$ on $\mathcal{U}$.*

*Proof.* By Lemma 3.18, it suffices to show that $\varphi$ satsifies (SG2). By construction of $\mathcal{V}$, for $U \in \mathcal{U}$, $D_U \in \tau(U)$ is a $\mathcal{V}$-sieve. □

In the remainder of this section, we suppose $\mathcal{V} \subseteq \mathcal{U}$ is a subcategory with $\mathrm{Ob}(\mathcal{V}) = \mathrm{Ob}(\mathcal{U})$ for which the inclusion functor satisfies (SG2). Let $\mathfrak{a}$ be a $\mathcal{U}$-graded category and consider $\phi : \tilde{\mathfrak{a}}^{\varphi} \longrightarrow \tilde{\mathfrak{a}}$. Put $\sigma = \varphi^{-1}\tau$ on $\mathcal{V}$ and let $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{a}}^{\varphi}$ be endowed with $\mathcal{T}_\tau$ and $\mathcal{T}_\sigma$ respectively. By Theorem 3.16, we have $\mathcal{T}_\sigma = \phi^{-1}\mathcal{T}_\tau$ and $\phi$ is localizing. Next we list conditions which ensure that for the refined site $(\mathcal{V}, \sigma = \varphi^{-1}\tau)$, condition (2) in Proposition 3.6 is fulfilled.

**Proposition 3.22.** *Suppose the following conditions hold:*

(1) *There is a basis $\beta$ of $\tau$ such that for every $R \in \beta(U)$ and $A \in \mathfrak{a}_U$ the sieve $R^A \subseteq \tilde{\mathfrak{a}}(-, A)$ is finitely generated.*
(2) *For given $U \in \mathcal{U}$, $A \in \mathfrak{a}_U$, there are only finitely many couples $(u, B)$ with $u : V \longrightarrow U$ not in $\mathcal{V}$ and $B \in \mathfrak{a}_V$, and for every such couple the $k$-module $\mathfrak{a}_u(B, A)$ is finitely generated.*

*Then there is a basis $\beta'$ of $\varphi^{-1}\tau$ such that for every $R' \in \beta'(U)$ and $A \in \mathfrak{a}^{\varphi}_U$ the sieve ${R'}^A \subseteq \tilde{\mathfrak{a}}^{\varphi}(-, A)$ is finitely generated.*

*Proof.* For every $U \in \mathcal{U}$, let $R_U \in \tau(U)$ be a fixed $\mathcal{V}$-sieve. After changing from $\beta_0$ as given in (1) to the basis $\beta$ with $\beta_0(U) = \{R \cap R_U \mid R \in \beta(U)\}$, we may suppose that for $\beta$ in (1), $\beta(V)$ consists of $\mathcal{V}$-sieves. For every cover $R \in \beta(V)$, we let $\bar{R}$

be as before and by Lemma 3.19, we obtain the basis $\bar{\beta}$ of $\varphi^{-1}\tau$ consisting of $\mathcal{U}$-sieves. It remains to show that $\bar{R}^A \subseteq \tilde{\mathfrak{a}}^{\varphi}(-,A)$ is finitely generated in $\mathsf{Mod}(\tilde{\mathfrak{a}}^{\varphi})$ for $R \in \beta(V)$. Since $R^A \subseteq \tilde{\mathfrak{a}}(-,A)$ is finitely generated in $\mathsf{Mod}(\tilde{\mathfrak{a}})$, the result follows from Lemma 3.23 □

**Lemma 3.23.** *Suppose the conditions of Proposition 3.22 hold. For a $\mathcal{V}$-sieve $R$ on $V \in \mathcal{U}$, let $\bar{R}$ be the associated $\mathcal{U}$-sieve on $V \in \mathcal{V}$. If $R^A \subseteq \tilde{\mathfrak{a}}(-,A)$ is finitely generated in $\mathsf{Mod}(\tilde{\mathfrak{a}})$, then $\bar{R}^A \subseteq \tilde{\mathfrak{a}}^{\varphi}(-,A)$ is finitely generated in $\mathsf{Mod}(\tilde{\mathfrak{a}}^{\varphi})$.*

*Proof.* Take finitely many generators $a_i : A_i \longrightarrow A$ of $R^A$, with $a_i \in \mathfrak{a}_{v_i}(A_i, A)$ for $v_i : V_i \longrightarrow V$ in $R(V_i)$. For every $i$, there are finitely many couples $(u_{ij}, A_{ij})$ with $u_{ij} : V_{ij} \longrightarrow V_i$ not in $\mathcal{V}$ and $A_{ij} \in \mathfrak{a}_{V_{ij}}$, and in each case $\mathfrak{a}_{u_{ij}}(A_{ij}, A_i)$ is a finitely generated $k$-module. Let $a_{ijk} : A_{ij} \longrightarrow A_i$ be finitely many generators of this module. Consider all the $a_i : A_i \longrightarrow A$ together with al the compositons $a_i a_{ijk} : A_{ij} \longrightarrow A_i \longrightarrow A$. We claim that these morphisms together generate $\bar{R}^A$. It suffices to generate a morphism $b : B \longrightarrow A$ with $b \in \mathfrak{a}_w(W,V)$ for $w : W \longrightarrow V$ in $R(W)$. Since $b \in R^A(B)$, we can write $b = \sum_{l=1}^n a_{i_l} b_l$ for $b_l \in \mathfrak{a}_{u_l}(B, A_{i_l})$. If $u_l \in \mathcal{V}$, then $b_l \in \tilde{\mathfrak{a}}^{\varphi}$. If $u_l \notin \mathcal{V}$, then necessarily $u_l = u_{i_l j}$ and $B = A_{i_l j}$ for some $j$. Hence, we can write $b_l = \sum_{t=1}^m \kappa_t a_{i_l j k_t}$ with $\kappa_t \in k$ and $a_{i_l} b_l = \sum_{t=1}^m \kappa_t a_{i_l} a_{i_l j k_t}$. This proves the claim and finishes the proof. □

3.5. **A characterization.** If $\mathcal{L}_\tau = \mathcal{T}_\tau$ on a $\mathcal{U}$-graded category, it is easier to recognize the corresponding sheaf category. Let $\mathcal{C}$ be a Grothendieck category, let $(\mathcal{U}, \sqsubseteq)$ be a preordered set, let $\mathfrak{a}_U$ be sets for $U \in \mathcal{U}$, and consider a map $\gamma : \mathcal{U} \longrightarrow \mathrm{Ob}(\mathcal{C})$. We define the $\mathcal{U}$-graded category $\mathfrak{a}$ with

$$\mathfrak{a}_{\sqsubseteq}(V,U) = \mathcal{C}(\gamma(V), \gamma(U)).$$

Put $\mathfrak{u} = \tilde{\mathfrak{a}}$ and let $\gamma : \mathfrak{u} \longrightarrow \mathcal{C}$ be the canonical functor. Let $\tau$ be a topology on $\mathcal{U}$ for which $\mathcal{L}_\tau = \mathcal{T}_\tau$ on $\mathfrak{u}$. Following [7], we characterize when $\gamma$ induces an equivalence $\mathcal{C} \cong \mathsf{Sh}(\mathfrak{u}, \mathcal{T}_\tau)$.

**Definition 3.24.** (1) $\gamma$ is *$\tau$-full* if for every $c : \gamma(V) \longrightarrow \gamma(U)$ in $\mathcal{C}$ with $V \not\sqsubseteq U$, there is a $\tau$-cover $V_i \sqsubseteq V$ such that for every $c_i : \gamma(V_i) \longrightarrow \gamma(V)$ with $V_i \not\sqsubseteq U$ we have $cc_i = 0$.

(2) $\gamma$ is *$\tau$-projective* if for every $\mathcal{C}$-epimorphism $c : X \longrightarrow Y$ and morphism $y : \gamma(V) \longrightarrow Y$ with $V \in \mathcal{U}$, there is a $\tau$-cover $V_i \sqsubseteq V$ such that for every $c_i : \gamma(V_i) \longrightarrow \gamma(V)$, there is a $d_i : \gamma(V_i) \longrightarrow X$ with $yc_i = cd_i$.

(3) $\gamma$ is *$\tau$-finitely presented* if for every filtered colimit $\mathrm{colim}_\alpha X_\alpha$ in $\mathcal{C}$ the following conditions hold:
   (a) for every map $c : \gamma(V) \longrightarrow \mathrm{colim}_\alpha X_\alpha$ with $V \in \mathcal{U}$, there is a $\tau$-cover $V_i \sqsubseteq V$ such that for every $c_i : \gamma(V_i) \longrightarrow \gamma(V)$, there is an $\alpha_i$ and a $d_i : \gamma(V_i) \longrightarrow X_{\alpha_i}$ with $cc_i = s_{\alpha_i} d_i$ for $s_{\alpha_i} : X_{\alpha_i} \longrightarrow \mathrm{colim}_\alpha X_\alpha$.
   (b) for every map $c : \gamma(V) \longrightarrow X_\beta$ with $0 = s_\beta c : \gamma(V) \longrightarrow \mathrm{colim}_\alpha X_\alpha$, there is a cover $V_i \sqsubseteq V$ such that for every $c_i : \gamma(V_i) \longrightarrow \gamma(V)$ there is a $\beta'$ with $s_{\beta\beta'} cc_i = 0$ for $s_{\beta\beta'} : X_\beta \longrightarrow X_{\beta'}$.

(4) $\gamma$ is *$\tau$-ample* if for every $\tau$-cover $V_i \sqsubseteq V$ the canonical morphism

$$\bigoplus_{c \in \mathcal{C}(\gamma(V_{i_c}), \gamma(V))} \gamma(V_{i_c}) \longrightarrow \gamma(V)$$

is a $\mathcal{C}$-epimorphism.

The following theorem combines Theorem 2.9 with the requirement that the induced topology on $\mathfrak{u}$ coincides with $\mathcal{T}_\tau$.

**Theorem 3.25.** *Consider $\gamma : \mathfrak{u} \longrightarrow \mathcal{C}$ and $\tau$ as above. The following are equivalent:*

(1) *$\gamma$ induces an equivalence $\mathcal{C} \cong \mathsf{Sh}(\mathfrak{u}, \mathcal{T}_\tau)$.*

(2) $\gamma$ *is generating,* $\tau$*-full,* $\tau$*-projective,* $\tau$*-finitely presented and* $\tau$*-ample.*

*Proof.* This is a special case of [7, Thm. 2.8]. □

*Remarks* 3.26. (1) If all the objects $\gamma(U)$ for $U \in \mathcal{U}$ are projective (resp. finitely presented) in $\mathcal{C}$, then $\gamma$ is $\tau$-projective (resp. $\tau$-finitely presented).

(2) $\gamma$ is $\tau$-projective if and only if for every element $\xi \in \mathrm{Ext}^1_{\mathcal{C}}(\gamma(V), C)$ with $V \in \mathcal{U}$ and $C \in \mathcal{C}$, there is an $\tau$-cover $V_i \sqsubseteq V$ such that for every morphism $c : \gamma(V_i) \longrightarrow \gamma(V)$, the natural image of $\xi$ in $\mathrm{Ext}^1_{\mathcal{C}}(\gamma(V_i), C)$ is zero.

(3) The setup for Definition 3.24 and Theorem 3.25 can easily be extended to the case where we have a preordered set $(\mathcal{U}, \sqsubseteq)$, prescribed sets $\mathfrak{a}_U$ for $U \in \mathcal{U}$, and a map $\gamma : \coprod_{U\in\mathcal{U}} \mathfrak{a}_U \longrightarrow \mathrm{Ob}(\mathcal{C})$.

## 4. Fibered map-graded categories

In this section we apply the results from §3 to *fibered* $\mathcal{U}$-graded categories $\mathfrak{a}$. In this case, for a topology $\tau$ on $\mathcal{U}$, we always have $\mathcal{L}_\tau = \mathcal{T}_\tau$ on $\tilde{\mathfrak{a}}$ (Proposition 4.2) and for every localizing functor $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$, the induced $\phi : \tilde{\mathfrak{a}}^\varphi \longrightarrow \tilde{\mathfrak{a}}$ is localizing. We fomulate an application to pseudofunctors $\mathcal{A} : \mathcal{U}^{\mathrm{op}} \longrightarrow \mathsf{Cat}(k)$ (Theorem 4.6).

4.1. **Fibered map-graded categories.** Let $\mathcal{U}$ be a small category and let $\mathfrak{a}$ be a $\mathcal{U}$-graded category. In this section we recall some notions from [13], based upon the standard non-linear notions from [2]. Consider a morphism $u : U' \longrightarrow U$ in $\mathcal{U}$ and objects $A \in \mathfrak{a}_U$, $A' \in \mathfrak{a}_{U'}$. A morphism $f \in \mathfrak{a}_u(A', A)$ is called *cartesian* if for every $u' : U'' \longrightarrow U'$ in $\mathcal{U}$ and $A'' \in \mathfrak{a}_{U''}$, the canonical composition morphism

$$f- : \mathfrak{a}_{u'}(A'', A') \longrightarrow \mathfrak{a}_{uu'}(A'', A)$$

is an isomorphism of $k$-modules. The $\mathcal{U}$-graded category $\mathfrak{a}$ is called *fibered* provided that for every $u : U' \longrightarrow U$ in $\mathcal{U}$ and $A \in \mathfrak{a}_U$, there exist an object $u^*A \in \mathfrak{a}_{U'}$ and a cartesian morphism $\delta^{u,A} \in \mathfrak{a}_u(u^*A, A)$. If a cartesian morphism exists, it is unique up to an isomorphism $\rho \in \mathfrak{a}_{1_{U'}}(A'', A')$ where $A'$ and $A''$ are the two involved choices for $u^*A$. A composition of cartesian morphisms is readily seen to be cartesian.

By definition, a *prestack* $\mathcal{A}$ on $\mathcal{U}$ is a pseudofunctor $\mathcal{A} : \mathcal{U}^{\mathrm{op}} \longrightarrow \mathsf{Cat}(k)$ from $\mathcal{U}^{\mathrm{op}}$ to the 2-category of small $k$-linear categories and $k$-linear functors. In particular, $\mathcal{A}$ consists of $k$-linear categories $\mathcal{A}(U)$ for $U \in \mathcal{U}$, $k$-linear restriction functors $u^* : \mathcal{A}(U) \longrightarrow \mathcal{A}(V)$ for $u : V \longrightarrow U$ in $\mathcal{U}$, and natural isomorphisms $v^*u^* \cong (uv)^*$ for $u : V \longrightarrow U$ and $v : W \longrightarrow V$ in $\mathcal{U}$, satisfying a natural coherence condition for three composable morphisms in $\mathcal{U}$.

To the prestack $\mathcal{A}$, we associate a natural $\mathcal{U}$-graded catgeory $\mathfrak{a} = \mathcal{A}^\sharp$ with $\mathfrak{a}_U = \mathrm{Ob}(\mathcal{A}(U))$ and

$$\mathfrak{a}_u(B_V, A_U) = \mathfrak{a}(V)(B_V, u^*A_U)$$

for $u : V \longrightarrow U$ in $\mathcal{U}$ and $B_V \in \mathfrak{a}_V$, $A_U \in \mathfrak{a}_U$. For every $u : V \longrightarrow U$ in $\mathcal{U}$ and $A \in \mathfrak{a}_U$, the canonical morphism

$$\delta^{u,A} = 1_{u^*A} \in \mathcal{A}(V)(u^*A, u^*A) = \mathfrak{a}_u(u^*A, A)$$

is cartesian, whence $\mathfrak{a}$ is fibered. The association $\mathcal{A} \longmapsto \mathcal{A}^\sharp$ is a $k$-linear version of the Grothendieck construction which is part of the classical correspondence between pseudofunctors and fibered categories [2]. See [13] for further details in the linear setup.

4.2. **Linearized topologies.** Let $\mathcal{U}$ be a small category and let $\mathfrak{a}$ be a fibered $\mathcal{U}$-graded category. Let $\mathcal{U}$ be endowed with a topology $\tau$.

**Lemma 4.1.** *Consider $A \in \mathfrak{a}_U$ and $R \in \tau(U)$. For every $u : U' \longrightarrow U$ in $R(U')$, let $\delta^{u,A} \in \mathfrak{a}_u(u^*A, A)$ be a cartesian morphism. Suppose $R = \langle F \rangle$. We have*

$$R^A = \langle \delta^{f,A} \mid f \in F \rangle.$$

*In particular, $\mathfrak{a}$ satsifies (WG).*

*Proof.* It suffices to look at a morphism $a \in \mathfrak{a}_u(A', A)$ for arbitrary $u : U' \longrightarrow U$ and $A' \in \mathfrak{a}_{U'}$. We can write $u = fu'$ for $f : V \longrightarrow U$ in $F$ and $u' : U' \longrightarrow V$. Since $\delta^{f,A}$ is cartesian, there is a unique morphism $b \in \mathfrak{a}_{u'}(A', u^*A)$ with $a = \delta^{f,A}b$. □

**Proposition 4.2.** *Let $\mathcal{U}$ be a small category endowed with a topology $\tau$, and let $\mathfrak{a}$ be a fibered $\mathcal{U}$-graded category. The cover system $\mathcal{L}_\tau$ is a topology on $\tilde{\mathfrak{a}}$, that is, we have $\mathcal{L}_\tau = \mathcal{T}_\tau$.*

*Proof.* Since $\mathcal{L}_\tau$ satisfies (Loc) and (Up), it suffices to show that it satisfies (Glue'). For $A \in \mathfrak{a}_U$, consider $R^A$ for $R \in \tau(U)$. For every $f \in R^A(A_f)$ with $A_f \in \mathfrak{a}_{U_f}$, consider $R_f^{A_f}$ for $R_f \in \tau(U_f)$. For every $u : V \longrightarrow U$ in $R(V)$, $R^A$ contains the cartesian morphism $\delta^{u,A} : u^*A \longrightarrow A$. The cover $R_{\delta^{u,A}}^{u^*A}$ for $R_{\delta^{u,A}} \in \mathcal{T}(V)$ further contains the cartesian morphisms $\delta^{v,u^*A}$ for all $v : W \longrightarrow V$ in $R_{\delta^{u,A}}(W)$. Consequently, the composition $R^A \circ (R_f^{A_f})$ contains the cartesian compositions $\delta^{u,A}\delta^{v,u^*A}$ corresponding to the compositions $uv \in R \circ (R_u)$. Hence, by Lemma 4.1, $(R \circ (R_u))^A \subseteq R^A \circ (R_f^{A_f})$. □

4.3. **Comparison of linearized topologies.** Let $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ be a functor between small categories. Let $\mathfrak{a}$ be a fibered $\mathcal{U}$-graded category with induced $\mathcal{V}$-graded category $\mathfrak{a}^\varphi$. For $v : V' \longrightarrow V$ in $\mathcal{V}$ and $A \in \mathfrak{a}^\varphi_V = \mathfrak{a}_{\varphi(V)}$, a cartesian morphism

$$\delta^{\varphi(v),A} \in \mathfrak{a}_{\varphi(v)}(\varphi(v)^*A, A) = \mathfrak{a}^\varphi_v(\varphi(v)^*A, A)$$

for $\mathfrak{a}$ is readily seen to define a cartesian morphism

$$\delta^{v,A} = \delta^{\varphi(v),A} \in \mathfrak{a}^\varphi_v(\varphi(v)^*A, A)$$

for $\mathfrak{a}^\varphi$ as well. Hence, $\mathfrak{a}^\varphi$ is a fibered $\mathcal{V}$-graded category.

**Theorem 4.3.** *Let $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ be a functor between small categories, localizing with respect to a topology $\tau$ on $\mathcal{U}$. Let $\mathfrak{a}$ be a fibered $\mathcal{U}$-graded category and let $\phi : \tilde{\mathfrak{a}}^\varphi \longrightarrow \tilde{\mathfrak{a}}$ be the induced linear functor. Let $\sigma = \varphi^{-1}\tau$ be the induced topology on $\mathcal{V}$ and let $\mathcal{T}_\tau$ and $\mathcal{T}_\sigma$ be the induced linearized topologies on $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{a}}^\varphi$ respectively. Then $\phi$ is localizing and we have $\phi^{-1}\mathcal{T}_\tau = \mathcal{T}_\sigma$. In particular, the forgetful functor $\mathsf{Mod}(\tilde{\mathfrak{a}}) \longrightarrow \mathsf{Mod}(\tilde{\mathfrak{a}}^\varphi)$ restricts to an equivalence*

$$\mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{T}_\tau) \longrightarrow \mathsf{Sh}(\tilde{\mathfrak{a}}^\varphi, \mathcal{T}_\sigma).$$

*Proof.* Since by Lemma 4.1, $\mathfrak{a}$ satisfies (WG), the result is given by Theorem 3.16. □

4.4. **Sheaves of $k$-modules.** Let $(\mathcal{U}, \tau)$ be a small site. Let

$$\mathsf{Mod}_k(\mathcal{U}) = \mathsf{Fun}(\mathcal{U}^{\mathrm{op}}, \mathsf{Mod}(k)$$

be the category of presheaves of $k$-modules on $\mathcal{U}$, and let $\mathsf{Sh}_k(\mathcal{U}, \tau)$ be the full subcategory of sheaves of $k$-modules on $\mathcal{U}$. For every $U \in \mathcal{U}$, we obtain a slice site $(\mathcal{U}/U, \tau_U)$ in the obvious way. For $g : V \longrightarrow U$ in $\mathcal{U}$, the natural maps $g_* : \mathcal{U}/V \longmapsto \mathcal{U}/U$ and $i_U : \mathcal{U}/U \longrightarrow \mathcal{U} : g \longmapsto V$ induce functors $i_U^* : \mathsf{Mod}_k(\mathcal{U}) \longrightarrow \mathsf{Mod}_k(\mathcal{U}/U)$ and $g^* : \mathsf{Mod}_k(\mathcal{U}/U) \longrightarrow \mathsf{Mod}_k(\mathcal{U}/V)$ which restrict to

$$i_U^* : \mathsf{Sh}_k(\mathcal{U}, \tau) \longrightarrow \mathsf{Sh}_k(\mathcal{U}/U, \tau_U)$$

and

$$g^* : \mathsf{Sh}_k(\mathcal{U}/U, \tau_U) \longrightarrow \mathsf{Sh}_k(\mathcal{U}/V, \tau_V)$$

between categories of sheaves of $k$-modules.

**Proposition 4.4.** *The functors $i_U^*$ and $g^*$ define pseudofunctors $\mathcal{M}od_k(\mathcal{U})$ and $\mathcal{S}h_k(\mathcal{U}, \tau)$ of presheaves resp. sheaves of $k$-modules on $(\mathcal{U}, \tau)$ with*

$$\mathcal{M}od_k(\mathcal{U})(U) = \mathsf{Mod}_k(\mathcal{U}/U); \qquad \mathcal{S}h_k(\mathcal{U}, \tau)(U) = \mathsf{Sh}_k(\mathcal{U}/U, \tau_U).$$

*Proof.* This is a straightforward verification. □

4.5. **Sheaves of $\mathcal{A}$-modules.** Let $(\mathcal{U}, \tau)$ be as before and let $\mathcal{A} : \mathcal{U}^{\mathrm{op}} \longrightarrow \mathsf{Cat}(k)$ be a pseudofunctor. A presheaf of $\mathcal{A}$-modules is a morphism of pseudofunctors $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{M}od_k(\mathcal{U})$. A sheaf of $\mathcal{A}$-modules is a morphism of pseudofunctors $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{S}h_k(\mathcal{U}, \tau)$. We obtain abelian categories $\mathsf{Mod}(\mathcal{A})$ of presheaves of $\mathcal{A}$-modules and $\mathsf{Sh}(\mathcal{A}, \tau)$ of sheaves of $\mathcal{A}$-modules. For every $U \in \mathcal{U}$ we obtain the categories $\mathsf{Mod}(\mathcal{A}|_U)$ of presheves of $\mathcal{A}|_U$-modules and $\mathsf{Sh}(\mathcal{A}|_U, \tau_U)$ of sheaves of $\mathcal{A}|_U$-modules. For $g : V \longrightarrow U$ in $\mathcal{U}$, the natural maps $g_* : \mathcal{U}/V \longmapsto \mathcal{U}/U$ and $i_U : \mathcal{U}/U \longrightarrow \mathcal{U} : g \longmapsto V$ induce functors $i_U^* : \mathsf{Mod}(\mathcal{A}) \longrightarrow \mathsf{Mod}(\mathcal{A}|_U)$ and $g^* : \mathsf{Mod}(\mathcal{A}|_U) \longrightarrow \mathsf{Mod}(\mathcal{A}|_V)$ which restrict to

$$i_U^* : \mathsf{Sh}(\mathcal{A}, \tau) \longrightarrow \mathsf{Sh}(\mathcal{A}|_U, \tau_U)$$

and

$$g^* : \mathsf{Sh}(\mathcal{A}|_U, \tau_U) \longrightarrow \mathsf{Sh}(\mathcal{A}|_V, \tau_V)$$

between categories of sheaves of $k$-modules.

Now let $\mathfrak{a} = \mathcal{A}^\sharp$ be the associated $\mathcal{U}$-graded category of $\mathcal{A}$.

**Proposition 4.5.** *There are equivalences of categories $\mathsf{Mod}(\mathcal{A}) \cong \mathsf{Mod}(\tilde{\mathfrak{a}})$ and*

$$\mathsf{Sh}(\mathcal{A}, \tau) \cong \mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{T}_\tau).$$

*Proof.* We indicate how to define inverse equivalences

$$\varphi : \mathsf{Sh}(\mathcal{A}, \tau) \longrightarrow \mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{T}_\tau)$$

and

$$\psi : \mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{T}_\tau) \longrightarrow \mathsf{Sh}(\mathcal{A}, \tau).$$

Consider a sheaf $F : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{S}h(\mathcal{U}, \tau)$ on $\mathcal{A}$ with maps $F_U : \mathcal{A}(U)^{\mathrm{op}} \longrightarrow \mathsf{Sh}(\mathcal{U}/U, \tau_U)$. We define $\varphi F : \tilde{\mathfrak{a}}^{\mathrm{op}} \longrightarrow \mathsf{Mod}(k)$ by putting $\varphi F(A_U) = F_U(A_U)(1_U)$. For a morphism $x \in \mathfrak{a}_g(B_V, A_U) = \mathcal{A}(V)(B_V, g^* A_U)$ for $g : V \longrightarrow U$ we obtain

$$F_V(x)(1_V) : F_V(g^* A_U)(1_V) \longrightarrow F_V(B_V)(1_V).$$

Composing with the natural isomorphism $F_U(A_U)(g) \cong F_V(g^* A_U)(1_V)$ and

$$F_U(A_U)(g : g \longrightarrow 1_U) : F_U(A_U)(1_U) \longrightarrow F_U(A_U)(g),$$

we obtain the desired $\varphi F(x) : \varphi F(A_U) \longrightarrow \varphi F(B_V)$.

Conversely, let $M : \tilde{\mathfrak{a}} \longrightarrow \mathsf{Mod}(k)$ be a sheaf on $\tilde{\mathfrak{a}}$. We define $\psi M_U : \mathcal{A}(U) \longrightarrow \mathsf{Sh}(\mathcal{U}/U, \tau_U)$ by putting $\psi M_U(A_U)(g) = M(g^* A_U)$. For $g : V \longrightarrow U$, $f : W \longrightarrow U$, $h : g \longrightarrow f$ we have $g^* A_U \cong h^* f^* A_U$ yielding a morphism $\delta \in \mathfrak{a}_h(g^* A_u, f^* A_U)$ and $M(\delta) : M(f^* A_U) \longrightarrow M(g^* A_U)$. We obtain natural isomorphisms

$$g^* \psi M_U(A_U)(h) = M((gh)^* A_U) \cong M(h^* g^* A_U) = \psi M_V(g^* A_U)(h).$$

It is easily seen that the sheaf properties on both sides of the equivalence correspond. □

let $\mathcal{A} : \mathcal{U}^{\mathrm{op}} \longrightarrow \mathsf{Cat}(k)$ be a pseudofunctor with associated fibered $\mathcal{U}$-graded category $\mathfrak{a} = \mathcal{A}^\sharp$. For the composed pseudofunctor

$$\mathcal{B} = \mathcal{A}\varphi^{\mathrm{op}} : \mathcal{V}^{\mathrm{op}} \longrightarrow \mathcal{U}^{\mathrm{op}} \longrightarrow \mathsf{Cat}(k),$$

the associated fibered $\mathcal{V}$-graded category $\mathfrak{b} = \mathcal{B}^\sharp$ satisfies $\mathfrak{b} = \mathfrak{a}^\varphi$.

**Theorem 4.6.** *Let $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ be a functor between small categories, and suppose $\varphi$ is localizing with respect to a topology $\tau$ on $\mathcal{U}$. Put $\sigma = \varphi^{-1}\tau$ on $\mathcal{V}$. Let $\mathcal{A} : \mathcal{U}^{\mathrm{op}} \longrightarrow \mathsf{Cat}(k)$ be a pseudofunctor on $\mathcal{U}$ and let $\mathcal{B} = \mathcal{A}\varphi^{\mathrm{op}} : \mathcal{V} \longrightarrow \mathsf{Cat}(k)$ be the composed pseudofunctor on $\mathcal{V}$. There is a canonical equivalence of categories*

$$\mathsf{Sh}(\mathcal{A}, \tau) \longrightarrow \mathsf{Sh}(\mathcal{B}, \sigma).$$

*Proof.* This follows from Theorem 4.3 and Proposition 4.5. □

*Example* 4.7. Let $(X, \mathcal{O})$ be a ringed space, that is $X$ is a topological space and $\mathcal{O}$ is a sheaf of rings on $X$. Let $\mathcal{B} \subseteq \mathrm{open}(X)$ be a basis of the topological space $X$. We consider $\mathcal{B}$ as a category in the usual way, endowed with the topology $\tau_{\mathcal{B}}$ for which $(U_i \longrightarrow U)_i$ is covering if and only if $\cup_i U_i = U$. Now consider two bases $\mathcal{B}' \subseteq \mathcal{B}$, and consider the natural inclusion functor $\varphi : \mathcal{B}' \longrightarrow \mathcal{B}$. This functor is fully faithful hence satisfies the conditions (F) and (FF) with respect to $\tau_{\mathcal{B}'} = \varphi^{-1}\tau_{\mathcal{B}}$. By definition of a basis, $\varphi$ also satsifies (G) with respect to $\tau_{\mathcal{B}}$. Let $\mathsf{Sh}(X, \mathcal{O})$ be the classical category of sheaves of $\mathcal{O}$-modules on $X$, and let $\mathsf{Sh}(\mathcal{B}, \mathcal{O}) = \mathsf{Sh}(\mathcal{O}|_{\mathcal{B}}, \tau_{\mathcal{B}})$ be the category of sheaves of $\mathcal{O}|_{\mathcal{B}}$-modules on $(\mathcal{B}, \tau_{\mathcal{B}})$. We have canonical equivalences of categories

$$\mathsf{Sh}(X, \mathcal{O}) \longrightarrow \mathsf{Sh}(\mathcal{B}, \mathcal{O}) \longrightarrow \mathsf{Sh}(\mathcal{B}', \mathcal{O}).$$

## 5. Tails topologies

In this section, we apply the results from §3 to a particular *tails topology* tails which can be considered on a certain class of small categories $\mathcal{U}$ as naturally induced by the trivial topology on the single morphism category. We describe the refinement construction from §3.4 for tails topologies (Theorem 5.8), as well as natural conditions ensuring that $\mathcal{L}_{\mathrm{tails}} = \mathcal{T}_{\mathrm{tails}}$ (Definition 5.10). If this equality of cover systems holds, we further give a recognition result for the corresponding sheaf category (Theorem 5.20).

5.1. **Tails topologies.** Let $\mathcal{U}$ be a small (non-linear) category. Then $\mathcal{U}$ can be endowed with two extreme topologies.

The smallest topology on $\mathcal{U}$ is the *trivial* topology triv, for which

$$\mathrm{triv}(U) = \{\mathcal{U}(-, U)\}$$

for $U \in \mathcal{U}$. This is a topology with $\mathsf{Sh}(\mathcal{U}, \mathrm{triv}) = \mathsf{Mod}(\mathcal{U}) = \mathsf{Fun}(\mathcal{U}, \mathsf{Set})$.

The largest topology on $\mathcal{U}$ is the *discrete* topology disc, for which

$$\mathrm{disc}(U) = \{R \subseteq \mathcal{U}(-, U)\}$$

for $U \in \mathcal{U}$. In particular, the discrete topology has $\varnothing \in \mathrm{disc}(U)$ for $U \in \mathcal{U}$. Note that if $M \in \mathsf{Mod}(\mathcal{U})$ is a sheaf for a topology with $\varnothing \in \mathrm{disc}(U)$ for a certain $U \in \mathcal{U}$, we necessarily have $|M(U)| = 1$. Thus, for disc, up to isomorphism, the only sheaf $M$ is the constant sheaf with $M(U) = \{*\}$ for $U \in \mathcal{U}$. The category $\mathsf{Sh}(\mathcal{U}, \mathrm{disc})$ is equivalent to the category with a single morphism.

Our next aim is to introduce a topology on $\mathcal{U}$ which is as large as possible, but avoids the empty covers. Let $e$ be the category with a single object $\star$ and a single morphism $1 = 1_\star$. Let $e$ be endowed with the trivial topology triv. We have $\mathsf{Sh}(e, \mathrm{triv}) = \mathsf{Fun}(e, \mathsf{Set}) \cong \mathsf{Set}$. For an arbitrary small category $\mathcal{U}$, consider the unique functor $\varepsilon : \mathcal{U} \longrightarrow e$. We endow $\mathcal{U}$ with the *tails cover system* $\mathrm{tails} = \varepsilon^{-1}\mathrm{triv}$. Concretely, we have

$$\mathrm{tails}(U) = \{R \subseteq \mathcal{U}(-, U) \mid R \neq \varnothing\}.$$

Consider the forgetful functor

$$\varepsilon^* : \mathsf{Fun}(e, \mathsf{Set}) \longrightarrow \mathsf{Fun}(\mathcal{U}, \mathsf{Set}) : X \longmapsto c_X$$

which sends $X = (\star \longmapsto X)$ to the constant presheaf $c_X : U \longmapsto X$ on $\mathcal{U}$. As soon as $\mathcal{U}$ is connected (that is, every two objects in $\mathcal{U}$ can be joined by a zig-zag of morphisms in $\mathcal{U}$), every $c_X$ is a sheaf and we obtain a factorization of $\varepsilon^*$ through $\mathsf{Fun}(e, \mathsf{Set}) = \mathsf{Sh}(e, \mathrm{triv}) \longrightarrow \mathsf{Sh}(\mathcal{U}, \mathrm{tails})$. Consider the following conditions on the category $\mathcal{U}$:

(D0) Every couple of morphisms $V \longrightarrow U$ and $W \longrightarrow U$ fit into a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \uparrow & & \uparrow \\ Z & \longrightarrow & W \end{array}$$

(D1) Every couple of objects $V$ and $W$ fit into a diagram

$$\begin{array}{ccc} V & & \\ \uparrow & & \\ Z & \longrightarrow & W \end{array}$$

(D2) Every couple of morphisms $f, g : V \longrightarrow U$ fit into a commutative diagram

$$Z \longrightarrow V \underset{g}{\overset{f}{\rightrightarrows}} U$$

If $\mathcal{U}$ satisfies (D1) and (D2), we will call $\mathcal{U}$ *downwardly directed* (D). A category satisfying the dual condition of (D) is sometimes called *filtered* [5]. Note that (D) implies (D0). Obviously, the category associated to a downwardly directed poset is downwardly directed as a category.

**Lemma 5.1.** *The following are equivalent:*

1. *The category $\mathcal{U}$ satisfies (D0);*
2. *The cover system* tails *is a topology on $\mathcal{U}$.*

*Proof.* The cover system tails clearly satisfies (Id) and (Glue"). It satisfies (Pb) if and only if (D0) holds. □

**Proposition 5.2.** *The following are equivalent:*

1. *The category $\mathcal{U}$ is non-empty and downwardly directed;*
2. *The functor $\varepsilon$ is localizing.*

*In particular, in this case we have $\mathsf{Sh}(\mathcal{U}, \mathrm{tails}) \cong \mathsf{Set}$.*

*Proof.* By Definition 2.14 and Theorem 2.13, it suffices to show that $\varepsilon$ satisfies (G), (F) and (FF) if and only if $\mathcal{U}$ satsifies (D). First note that $\phi$ satisfies (G) if and only if $\mathcal{U}$ is non-empty. Next we rephrase (F). We may equivalently look at $\varphi : \mathcal{U} \longrightarrow e$ instead of $\phi$. Consider objects $V, W \in \mathcal{U}$ and the unique morphism $1_\star : \varphi(V) \longrightarrow \varphi(W)$. Condition (F) if fulfilled if and only if there exists a morphism $Z \longrightarrow V$ in $\mathcal{U}$ for which the composition $1_\star : \varphi(Z) \longrightarrow \varphi(V) \longrightarrow \varphi(W)$ is in the image of $\varphi$, that is, for which there exists a morphism $Z \longrightarrow W$ in $\mathcal{U}$. This is precisely condition (D1). Finally we rephrase (FF), again using $\varphi$ instead of $\phi$. Consider morphisms $f, g : V \longrightarrow U$ in $\mathcal{U}$ with $\varphi(f) = \varphi(g)$. Since $\varphi(f) = 1_\star$ for every morphism $f$, $f$ and $g$ are arbitrary morphism in $\mathcal{U}$. Condition (FF) is fulfilled if there exists a morphism $h : Z \longrightarrow V$ with $fh = gh$. This is precisely condition (D2). □

5.2. **Linearized tails topologies.** Let $\mathcal{U}$, $\mathcal{V}$ be small categories and let $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ be a functor. Let $\mathrm{tails}_{\mathcal{U}}$ and $\mathrm{tails}_{\mathcal{V}}$ be the tails cover systems on $\mathcal{U}$ and $\mathcal{V}$ respectively.

**Lemma 5.3.** *We have $\varphi^{-1}\mathrm{tails}_{\mathcal{U}} = \mathrm{tails}_{\mathcal{V}}$.*

*Proof.* This follows for instance from considering the composition of $\varphi$ with $\varepsilon : \mathcal{U} \longrightarrow e$. □

**Proposition 5.4.** *Suppose $\varphi$ satisfies is localizing. If $\mathcal{U}$ satisfies (D) (resp. (D0)), then so does $\mathcal{V}$.*

*Proof.* If $\mathcal{U}$ satisfies (D) and $\varphi$ satisfies (G), (F), (FF), then by Proposition 5.4, the composition $\varepsilon\varphi : \mathcal{V} \longrightarrow \mathcal{U} \longrightarrow e$ satsifies (G), (F) and (FF), that is, $\mathcal{V}$ satisfies (D). If $\mathcal{U}$ satsifies (D0), it is not hard to check directly that $\mathcal{V}$ also satisfies (D0), making consecutive use of the conditions (G), (F) and (FF). □

**Lemma 5.5.** *Suppose $\mathcal{U}$ satisfies (D0). Then for every $w : W \longrightarrow U$ in $\mathcal{U}$ and $\varnothing \neq R \subseteq \mathcal{U}(-, W)$ there is a $\varnothing \neq S \subseteq \mathcal{U}(-, U)$ with $w^{-1}S \subseteq R$.*

*Proof.* It suffices to define $S$ to be the image of $R \longrightarrow \mathcal{U}(-, W) \longrightarrow \mathcal{U}(-, U)$. By definition of the pullback, we have $w^{-1}S \subseteq R$. □

**Proposition 5.6.** *Suppose $\mathcal{U}$ satisfies (D0). Let $\mathfrak{a}$ be a $\mathcal{U}$-graded category. Suppose for every $\varnothing \neq S \subseteq \mathcal{U}(-, U)$ and $A \in \mathfrak{a}_U$, there is a $\varnothing \neq R \subseteq S$ such that the sieve $R^A \subseteq \tilde{\mathfrak{a}}(-, A)$ is finitely generated. Then $\mathcal{L}_{\mathrm{tails}} = \mathcal{T}_{\mathrm{tails}}$ on $\tilde{\mathfrak{a}}$.*

*Proof.* This follows from Proposition 3.6 and Lemma 5.5. □

In the remainder of this section, we suppose $\mathcal{V} \subseteq \mathcal{U}$ is a subcategory with $\mathrm{Ob}(\mathcal{V}) = \mathrm{Ob}(\mathcal{U})$. Suppose $\mathcal{U}$ satisfies (D0) and suppose the inclusion functor $\varphi$ satisfies (SG2), that is, for every $U \in \mathcal{U}$ there exists a non-empty $\mathcal{V}$-sieve on $U \in \mathcal{U}$. By Lemma 3.18, $\varphi$ is stably localizing and by Proposition 5.4, $\mathcal{V}$ satisfies (D0).

Let $\mathfrak{a}$ be a $\mathcal{U}$-graded category and consider $\phi : \tilde{\mathfrak{a}}^{\varphi} \longrightarrow \tilde{\mathfrak{a}}$. Let $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{a}}^{\varphi}$ be endowed with $\mathcal{T}_{\mathrm{tails}_{\mathcal{U}}}$ and $\mathcal{T}_{\mathrm{tails}_{\mathcal{V}}}$ respectively. By lemma 5.3 and Theorem 3.16, we have $\mathcal{T}_{\mathrm{tails}_{\mathcal{V}}} = \phi^{-1}\mathcal{T}_{\mathrm{tails}_{\mathcal{U}}}$ and $\phi$ is localizing. Next we list conditions which ensure that we have $\mathcal{L}_{\mathrm{tails}_{\mathcal{V}}} = \mathcal{T}_{\mathrm{tails}_{\mathcal{V}}}$ on $\tilde{\mathfrak{a}}^{\varphi}$.

**Proposition 5.7.** *Suppose the following conditions hold:*

1. *For every $\varnothing \neq S \subseteq \mathcal{U}(-, U)$ and $A \in \mathfrak{a}_U$, there is a $\varnothing \neq R \subseteq S$ such that the sieve $R^A \subseteq \tilde{\mathfrak{a}}(-, A)$ is finitely generated.*
2. *For given $U \in \mathcal{U}$, $A \in \mathfrak{a}_U$, there are only finitely many couples $(u, B)$ with $u : V \longrightarrow U$ not in $\mathcal{V}$ and $B \in \mathfrak{a}_V$, and for every such couple the $k$-module $\mathfrak{a}_u(B, A)$ is finitely generated.*

*Then there is a basis $\beta'$ of $\varphi^{-1}\tau$ such that for every $R' \in \beta'(U)$ and $A \in \mathfrak{a}^{\varphi}_U$ the sieve ${R'}^A \subseteq \tilde{\mathfrak{a}}^{\varphi}(-, A)$ is finitely generated.*

*Proof.* This follows from Propositions 3.6, 3.22 and Lemma 5.5. □

5.3. **Preorder tails topologies.** Let $(\mathcal{U}, \sqsubseteq)$ be a preorder considered as a category. Suppose $\mathcal{U}$ satisfies (D0), that is, if for $U, V \in \mathcal{U}$ there exists $W \in \mathcal{U}$ with $U \sqsubseteq W$ and $V \sqsubseteq W$, then there exists $Z \in \mathcal{U}$ with $Z \sqsubseteq U$ and $Z \sqsubseteq V$. Let $\mathfrak{a}$ be a $\mathcal{U}$-graded category with $\mathfrak{a}_U = \{U\}$.

For $V \sqsubseteq U$, consider the sieve $R_U^{\sqsubseteq V}$ on $U \in \mathcal{U}$ with

$$R_U^{\sqsubseteq V}(W) = \begin{cases} \mathcal{U}(W,U) = \{*\} & \text{if } W \sqsubseteq V \\ \varnothing & \text{otherwise} \end{cases}$$

and the sieve $\tilde{R}_U^{\sqsubseteq V}$ on $U \in \tilde{\mathfrak{a}}$ with

$$\tilde{R}_U^{\sqsubseteq V}(W) = \begin{cases} \mathfrak{a}_{\sqsubseteq}(W,U) & \text{if } W \sqsubseteq V \\ 0 & \text{otherwise.} \end{cases}$$

Consider the cover systems $\beta$ on $\mathcal{U}$ and $\tilde{\beta}$ on $\tilde{\mathfrak{a}}$ with

$$\beta(U) = \{R_U^{\sqsubseteq V} \mid V \sqsubseteq U\}; \qquad \tilde{\beta}(U) = \{\tilde{R}_U^{\sqsubseteq V} \mid V \sqsubseteq U\}.$$

Clearly, $\beta$ is a basis for the topology tails on $\mathcal{U}$ and $\tilde{\beta}$ is a basis for the cover system $\mathcal{L}_{\text{tails}}$ on $\tilde{\mathfrak{a}}$.

Let $\nu : \mathcal{U} \longrightarrow \mathcal{U}$ be a function with $\nu(U) \sqsubseteq U$ for all $U \in \mathcal{U}$. Define the relation $\sqsubseteq'$ on $\mathcal{U}$ by

$$V \sqsubseteq' U \iff [V \sqsubseteq \nu(U) \ \vee \ V = U]. \tag{6}$$

Let $\mathcal{U}'$ be the category associated to the new poset $(\mathcal{U}, \sqsubseteq')$ and let $\varphi : \mathcal{U}' \longrightarrow \mathcal{U}$ be the inclusion functor. Consider the resulting $\mathcal{U}'$-graded category $\mathfrak{a}^{\varphi}$. The associated $k$-linear category $\tilde{\mathfrak{a}}' = \tilde{\mathfrak{a}}^{\varphi}$ has

$$\tilde{\mathfrak{a}}'(V,U) = \begin{cases} \mathfrak{a}_{\sqsubseteq}(V,U) & \text{if } V \sqsubseteq \nu(U) \\ 0 & \text{otherwise.} \end{cases}$$

We obtain the naturally induced $k$-linear functor

$$\phi : \tilde{\mathfrak{a}}' \longrightarrow \tilde{\mathfrak{a}} : U \longmapsto U.$$

By Proposition 5.4, $\mathcal{U}'$ satisfies (D0) and we denote the tails topology on $\mathcal{U}'$ by tails$'$. Let $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{a}}'$ be endowed with the topologies $\mathcal{T}_{\text{tails}}$ and $\mathcal{T}_{\text{tails}'}$ respectively.

**Theorem 5.8.** *The functor $\varphi$ is stably localizing, we have $\phi^{-1}\mathcal{T}_{\text{tails}} = \mathcal{T}_{\text{tails}'}$ and the functor $\phi$ is localizing. In particular, the forgetful functor $\mathsf{Mod}(\tilde{\mathfrak{a}}) \longrightarrow \mathsf{Mod}(\tilde{\mathfrak{a}}^{\varphi})$ restricts to an equivalence*

$$\mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{T}_{\text{tails}}) \longrightarrow \mathsf{Sh}(\tilde{\mathfrak{a}}', \mathcal{T}_{\text{tails}'}).$$

*Proof.* This follows from Proposition 3.21 and Definition 3.17, Theorem 3.16. □

**Proposition 5.9.** *Suppose the following conditions hold:*

1. *For every $V \sqsubseteq U$, there exists $W \sqsubseteq V$ for which $\tilde{R}_U^{\sqsubseteq W}$ is finitely generated in $\mathsf{Mod}(\tilde{\mathfrak{a}})$.*
2. *For every $\nu(U) \sqsubset V \sqsubseteq U$ the $k$-module $\tilde{\mathfrak{a}}(V,U)$ is finitely generated.*

*Then $\mathcal{T}_{\text{tails}'} = \mathcal{L}_{\text{tails}'}$ on $\tilde{\mathfrak{a}}'$.*

*Proof.* This follows from Proposition 5.7. □

In the remainder of this subsection, for simplicity, we assume $(\mathcal{U}, \sqsubseteq)$ is a poset. The following definitions extend the ones from [7, §3].

**Definition 5.10.** (1) $\mathfrak{a}$ is *connected* if $\mathfrak{a}_{\sqsubseteq}(V,V) = k$ for $V \in \mathcal{U}$;

(2) $\mathfrak{a}$ is *locally finite* if the $k$-modules $\mathfrak{a}_{\sqsubseteq}(V,U)$ are finitely generated for $V \sqsubseteq U$ in $\mathcal{U}$.

(3) $\mathfrak{a}$ is *generated* by a collection of elements $X \subseteq \mathfrak{a}$ given by sets $X(V,U) \subseteq \mathfrak{a}_{\sqsubseteq}(V,U)$ for $V \sqsubseteq U$ if every element in $\mathfrak{a}$ can be written as a (finite) $k$-linear combination of (finite) products of elements in $X$ and elements $1_V \in \mathfrak{a}_{\sqsubseteq}(V,V)$.

(4) $\mathfrak{a}$ is *finitely generated* if it is generated by $X$ such that for all $U \in \mathcal{U}$, the set
$$X_U = \coprod_{V \in \mathcal{U}} X(V, U)$$
is finite.

For $V, U \in \mathcal{U}$, consider the *interval*
$$[V, U] = \{W \in \mathcal{U} \mid V \sqsubseteq W \sqsubseteq U\}.$$
We write $V \sqsubset_n U$ if $|[V, U]| = n$. We call $\mathcal{U}$ *interval finite* if for all $V, U \in \mathcal{U}$, $|[V, U]|$ is finite.

**Lemma 5.11.** *Suppose $\mathcal{U}$ is interval finite and suppose $\mathfrak{a}$ is finitely generated and connected. Then $\mathfrak{a}$ is locally finite.*

*Proof.* Similar to the proof of [7, Lem. 3.1]. □

**Proposition 5.12.** *Suppose $\mathcal{U}$ is interval finite and suppose $\mathfrak{a}$ connected. The following are equivalent:*

(1) *$\mathfrak{a}$ is finitely generated;*
(2) *the $\mathfrak{a}$-modules $\tilde{R}_U^{\sqsubseteq V}$ are finitely generated for all $V \sqsubseteq U$ in $\mathcal{U}$;*
(3) *the $\mathfrak{a}$-modules $\tilde{R}_U^{\sqsubseteq V}$ are finitely generated for all $V \sqsubset_2 U$ in $\mathcal{U}$;*

*Proof.* Similar to the proof of [7, Prop. 3.2]. □

**Proposition 5.13.** *Suppose for every $V \sqsubseteq U$, there is a $W \sqsubseteq V$ for which $\tilde{R}_U^{\sqsubseteq W}$ is finitely generated in $\mathsf{Mod}(\tilde{\mathfrak{a}})$. Then $\mathcal{T}_{\text{tails}} = \mathcal{L}_{\text{tails}}$ on $\tilde{\mathfrak{a}}$.*

*Proof.* This follows from Proposition 5.6. □

**Corollary 5.14.** *Suppose $\mathcal{U}$ is interval finite and $\mathfrak{a}$ is connected and finitely generated. Then $\mathcal{T}_{\text{tails}} = \mathcal{L}_{\text{tails}}$ on $\tilde{\mathfrak{a}}$ and $\mathcal{T}_{\text{tails}'} = \mathcal{L}_{\text{tails}'}$ on $\tilde{\mathfrak{a}}'$.*

*Proof.* This follows from Propositions 5.9 and 5.12 and Lemma 5.11. □

*Example* 5.15. Put $\mathcal{U} = (\mathbb{Z}, \geq)$. The category $\tilde{\mathfrak{a}}$ of a $\mathcal{U}$-graded category $\mathfrak{a}$ corresponds precisely to the notion of a positively graded $Z$-algebra from [4]. The resulting cover systems $\mathcal{L}_{\text{tails}}$ and $\mathcal{T}_{\text{tails}}$ on $\tilde{\mathfrak{a}}$ coincide with those defined in [7, §3.3]. Suppose $\mathfrak{a}$ is connected and finitely generated. Then by Corollary 5.14, we have $\mathcal{T}_{\text{tails}'} = \mathcal{L}_{\text{tails}'}$ for any choice of $\nu : \mathbb{Z} \longrightarrow \mathbb{Z}$ with $\nu(n) \geq n$.

*Example* 5.16. Let $A = (A_n)_{n \in \mathbb{N}}$ be a positively graded $k$-algebra. We obtain an associated $(\mathbb{Z}, \geq)$-graded category $\mathfrak{a}$ with $\mathfrak{a}(n, m) = A_{n-m}$, see [7, §3.1]. If $A$ is a connected (i.e $A_0 = k$) finitely generated graded algebra, then $\tilde{\mathfrak{a}}$ is connected and finitely generated by [7, Prop.3.3], and Example 5.15 applies.

5.4. **A characterization.** In this section, based upon §3.5, we discuss how sheaf categories over linearized tails topologies can be recognized. Let $\mathcal{C}$ be a Grothendieck category and let $(\mathcal{U}, \sqsubseteq)$ be a preordered set, considered as a category, satisfying (D0), with a map $\varphi : \mathcal{U} \longrightarrow \mathrm{Ob}(\mathcal{C})$. We define the $\mathcal{U}$-graded category $\mathfrak{a}$ with $\mathfrak{a}_U = \{U\}$ and
$$\mathfrak{a}_{\sqsubseteq}(V, U) = \mathcal{C}(\varphi(V), \varphi(U)).$$
Put $\mathfrak{u} = \tilde{\mathfrak{a}}$ and let $\varphi : \mathfrak{u} \longrightarrow \mathcal{C}$ be the canonical functor. Suppose the topology tails on $\mathcal{U}$ is such that $\mathcal{L}_{\text{tails}} = \mathcal{T}_{\text{tails}}$ on $\mathfrak{u}$. The following lemma generalizes [7, Lem. 3.13]:

**Lemma 5.17.** *The functor $\varphi$ is* tails*-full.*

*Proof.* It suffices to look at a map $c : \varphi(V) \longrightarrow \varphi(U)$ for $V \not\sqsubseteq U$. Since $\mathcal{U}$ is downwardly directed, we can take $W \in \mathcal{U}$ with $W \sqsubseteq V$ and $W \sqsubseteq U$. Then $W \sqsubseteq V$ generates a tails-cover of $V$. For every $Z \sqsubseteq W$ and $d \in \mathfrak{u}(Z, V) = \mathcal{C}(\varphi(Z), \varphi(V))$, we have $dc \in \mathfrak{u}(Z, U) = \mathcal{C}(\varphi(Z), \varphi(V))$ as desired. □

Let $\mathcal{C}$ be locally finitely presented with $\mathsf{Fp}(\mathcal{C})$ the set of finitely presented objects. Consider $\varphi : \mathcal{U} \longrightarrow \mathsf{Fp}(\mathcal{C})$.

**Definition 5.18.** We say that $\varphi$ is *ample* if for every $C \in \mathsf{Fp}(\mathcal{C})$ there is a $V_0 \in \mathcal{U}$ such that for every $V \sqsubseteq V_0$, there is an epimorphism in $\mathcal{C}$

$$\bigoplus_{i \in I} \varphi(V_i) \longrightarrow C$$

with $V_i \sqsubseteq V$ for all $i \in I$.

**Lemma 5.19.** *Let $\mathcal{C}$ and $\varphi : \mathcal{U} \longrightarrow \mathsf{Fp}(\mathcal{C})$ be as above. The following are equivalent:*

(1) *$\varphi$ is ample;*
(2) *$\varphi : \mathfrak{u} \longrightarrow \mathcal{C}$ satisfies (G) and is* tails-*ample.*

*Proof.* This can be proven along the lines of the proof of [7, Corollary 3.16]. □

We obtain the following generlization of [7, Corollary 3.16]:

**Theorem 5.20.** *Let $\mathcal{U}$ be a preorder satisfying (D0) and let $\mathcal{C}$ be a locally finitely presented Grothendieck category with $\mathsf{Fp}(\mathcal{C})$ the set of finitely presented objects. Consider a map $\varphi : \mathcal{U} \longrightarrow \mathsf{Fp}(\mathcal{C})$ and let the functor $\varphi : \mathfrak{u} \longrightarrow \mathcal{C}$ be as above. Suppose $\mathcal{L}_{\text{tails}} = \mathcal{T}_{\text{tails}}$ on $\mathfrak{u}$. The following are equivalent:*

(1) *$\varphi$ induces an equivalence $\mathcal{C} \cong \mathsf{Sh}(\mathfrak{u}, \mathcal{T}_{\text{tails}})$.*
(2) *$\varphi$ is ample and* tails-*projective.*

*Proof.* This follows from Theorem 3.25 and Lemmas 5.17 and 5.19. □

*Example* 5.21. Let $X$ be a projective scheme over a noetherian base ring $k$. The category $\mathcal{C} = \mathsf{Qch}(X)$ of quasi-coherent sheaves is locally finitely presented and has the category $\mathsf{coh}(X)$ of coherent sheaves as finitely presented objects. Recall that an invertible sheaf $\mathcal{L}$ on $X$ is called *ample* if for every coherent sheaf $M$, there is an $n_0$ such that for every $n \geq n_0$ there is an epimorphism

$$\oplus_i \mathcal{L}^{-n} \longrightarrow M.$$

Take $\mathcal{U} = (\mathbb{Z}, \geq)$ and consider $\varphi : \mathbb{Z} \longrightarrow \mathsf{Fp}(\mathcal{C}) : n \longmapsto \mathcal{L}^{-n}$. Then $\varphi$ is *ample* in the sense of Definition 5.18. Furthermore, by the cohomological criterion for ampleness, $\mathcal{L}$ is ample if and only if for every coherent sheaf $M$ there is an $n_0$ such that for each $i > 0$ and for each $n \geq n_0$,

$$\mathrm{Ext}^i(\mathcal{L}^{-n}, M) = 0.$$

Hence, by Remark 3.26(2), $\varphi$ is tails-projective and by Theorem 5.20, we have $\mathsf{Qch}(X) \cong \mathsf{Sh}(\mathfrak{u}, \mathcal{T}_{\text{tails}})$. Using the equivalence $\mathsf{Sh}(\mathfrak{u}, \mathcal{T}_{\text{tails}}) \cong \mathsf{Mod}(\mathfrak{u})/\mathsf{Tors}(\mathfrak{u}, \mathcal{T}_{\text{tails}})$, and the fact that the $\mathcal{U}$-graded category $\mathfrak{u}$ is naturally obtained from a $\mathbb{Z}$-graded algebra as in Example 5.16 (see for instance [7]), we recover Serre's original algebraic description of $\mathsf{Qch}(X)$.

## 6. Deformations of linearized sites

In this section, after recalling Gerstenhaber type algebraic deformation theory in the context of linear and map-graded categories, we recall the deformation theory for abelian and in particular Grothendieck categories from [17]. Our setup follows this reference, that is we deform along a ring map between coherent commutative rings $R \longrightarrow k$ with nilpotent kernel $I$. This includes the standard infinitesimal deformation setup where $R$ is an Artin local $k$-algebra. In (12) we describe a natural map

$$\theta : \mathrm{Def}_{\mathcal{U}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{T}_{\tau,\mathfrak{a}})) : \mathfrak{b} \longrightarrow \mathsf{Sh}(\tilde{\mathfrak{b}}, \mathcal{T}_{\tau,\mathfrak{b}}).$$

If $\mathcal{U}$ is given by a preordered set, then based upon [17, Thm. 8.14], we formulate conditions for $\theta$ to be a bijection (Theorem 6.10). We give applications to prestacks (Theorem 6.13) and tails topologies (Theorem 6.15).

6.1. **Algebraic deformations.** Every non-commutative algebraic deformation theory is somehow based upon the deformation theory of algebras due to Gerstenhaber [8, 9]. The fundamental notions are the following:

**Definition 6.1.** Let $A$ be a $k$-flat $k$-algebra. An $R$-*deformation* of $A$ is an $R$-flat $R$-algebra $B$ with an isomorphism $k \otimes_R B \cong A$ of $k$-algebras. An *equivalence* of $R$-deformations $B$ and $B'$ is an isomorphism $B \longrightarrow B'$ of $R$-algebras which reduces to the identity $1_A : A \longrightarrow A$ via the isomorphisms $k \otimes_R B \cong A$ and $k \otimes_R B' \cong A$. The set of $R$-deformations of $A$ up to equivalence of $R$-deformations is denoted by $\mathrm{Def}_{\mathrm{alg}}(A)$.

Inspired upon Definition 6.1, we define deformations of three types of algebraic objects. Each time, we have to specify how $R$-linear objects are reduced to $k$-linear objects, and what flatness for $k$-linear objects means. The notion of deformation and equivalence of deformations is then obtained in complete analogy with Definition 6.1:

(1) A $k$-linear category $\mathfrak{a}$ is $k$-flat is all the modules $\mathfrak{a}(A, A')$ are $k$-flat. The reduction $k \otimes_R \mathfrak{b}$ of an $R$-linear category is the category with the same object set $\mathrm{Ob}(k \otimes_R \mathfrak{b}) = \mathrm{Ob}(\mathfrak{b})$ and $(k \otimes_R \mathfrak{b})(B', B) = k \otimes_R \mathfrak{b}(B', B)$ for $B, B' \in \mathfrak{b}$. The set of $R$-deformations of $\mathfrak{a}$ up to equivalence of $R$-deformations is denoted $\mathrm{Def}_{\mathrm{lin}}(\mathfrak{a})$.
(2) A $k$-linear $\mathcal{U}$-graded category is $k$-flat if all the modules $\mathfrak{a}_u(A, A')$ are $k$-flat. The reduction $k \otimes_R \mathfrak{b}$ of an $R$-linear $\mathcal{U}$-graded category is the $k$-linear $\mathcal{U}$-graded category with the same object sets $(k \otimes_R \mathfrak{b})_U = \mathfrak{b}_U$ and $(k \otimes_R \mathfrak{b})_u(B', B) = k \otimes_R \mathfrak{b}_u(B', B)$ for $u : U' \longrightarrow U$, $B' \in \mathfrak{b}_{U'}$, $B \in \mathfrak{b}_U$. The set of $R$-deformations of $\mathfrak{a}$ up to equivalence of $R$-deformations is denoted $\mathrm{Def}_{\mathcal{U}}(\mathfrak{a})$.
(3) A pseudofunctor $\mathcal{A} : \mathcal{U}^{\mathrm{op}} \longrightarrow \mathsf{Cat}(k)$ is $k$-flat is all the categories $\mathcal{A}(U)$ are $k$-flat. The reduction $k \otimes_R \mathcal{B}$ of a pseudofunctor $\mathcal{B} : \mathcal{U}^{\mathrm{op}} \longrightarrow \mathsf{Cat}(R)$ is the pseudofunctor with $(k \otimes_R \mathcal{B})(U) = k \otimes_R \mathcal{B}(U)$. The set of $R$-deformations of $\mathcal{A}$ up to equivalence of $R$-deformations is denoted $\mathrm{Def}_{\mathrm{ps}}(\mathcal{A})$.

For a $\mathcal{U}$-graded category $\mathfrak{a}$, there is a canonical map

$$\alpha : \mathrm{Def}_{\mathcal{U}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{lin}}(\tilde{\mathfrak{a}}) : \mathfrak{b} \longrightarrow \tilde{\mathfrak{b}}. \tag{7}$$

**Proposition 6.2.** [12] [13] *For a fibered $\mathcal{U}$-graded category, every $\mathcal{U}$-graded deformation is fibered. For a pseudofunctor $\mathcal{A} : \mathcal{U}^{\mathrm{op}} \longrightarrow \mathsf{Cat}(k)$ with associated fibered $\mathcal{U}$-graded category $\mathcal{A}^\sharp$, we have a canonical bijection*

$$\beta : \mathrm{Def}_{\mathrm{ps}}(\mathcal{A}) \longrightarrow \mathrm{Def}_{\mathcal{U}}(\mathcal{A}^\sharp) : \mathcal{B} \longrightarrow \mathcal{B}^\sharp. \tag{8}$$

6.2. **Abelian categories.** Although abelian categories are specific linear categories, the notion of linear deformation of §6.1 is not appropriate for abelian categories. For an abelian $R$-category $\mathcal{B}$, we define the $k$-reduction to be the full (abelian!) subcategory

$$\mathcal{B}_k = \{B \in \mathcal{B} \mid IB = \mathrm{Im}(I \otimes_R B \longrightarrow B) = 0\}.$$

Furhermore, in [17, Definition 3.2], we introduce a notion of flatness for abelian categories which is such that a $k$-algebra $A$ is $k$-flat if and only if its module category $\mathsf{Mod}(A)$ is abelian flat.

**Definition 6.3.** [17] Let $\mathcal{A}$ be a flat abelian $k$-category. An *abelian $R$-deformation* of $\mathcal{A}$ is a flat abelian $R$-category $\mathcal{B}$ with an equivalence $\mathcal{A} \cong \mathcal{B}_k$. An *equivalence* of abelian $R$-deformations $\mathcal{B}$ and $\mathcal{B}'$ is an equivalence $\mathcal{B} \longrightarrow \mathcal{B}'$ of $R$-linear categories whose reduction is naturally isomorphic to the identity $1_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ via the equivalences $\mathcal{A} \cong \mathcal{B}_k$ and $\mathcal{A} \cong \mathcal{B}'_k$.

We have the following basic result:

**Proposition 6.4.** [17] *For a linear category $\mathfrak{a}$, there is a deformation equivalence*

$$\mu : \mathrm{Def}_{\mathrm{lin}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Mod}(\mathfrak{a})) : \mathfrak{b} \longrightarrow \mathsf{Mod}(\mathfrak{b}) \tag{9}$$

The main point in the proof is to associate a linear deformation of $\mathfrak{a}$ to a given abelian deformation $\mathcal{D}$ of $\mathcal{C} = \mathsf{Mod}(\mathfrak{a})$. Considering the objects $A \in \mathfrak{a}$ as objects of $\mathcal{C}$, we make essential use of the following two facts:

(1) $\mathrm{Ext}^1_{\mathcal{C}}(A, X \otimes_k A) = \mathrm{Ext}^2_{\mathcal{C}}(A, X \otimes_k A) = 0$ for all $A \in \mathfrak{a}$ and $X \in \mathsf{mod}(k)$ (in order to obtain unique flat lifts of the individual objects of $\mathfrak{a}$ along the left adjoint $k \otimes_R -$ of the embedding $\mathcal{C} \longrightarrow \mathcal{D}$);
(2) $\mathrm{Ext}^1_{\mathcal{C}}(A, X \otimes_k A') = 0$ for all $A, A' \in \mathfrak{a}$ and $X \in \mathsf{mod}(k)$ (in order to organize the lifted object as a linear deformation $\mathfrak{b} \subseteq \mathcal{D}$ of $\mathfrak{a}$).

Proposition 6.4 tells us that the abelian deformation theory of module categories is entirely controlled by Gerstenhaber's deformation theory for algebras.

For general abelian categories, an appropriate Hochschild complex controling abelian deformations was introduced and studied in [16].

6.3. **Grothendieck categories.** In [17, Theorem 6.29], it was proven that abelian deformations of Grothendieck categories remain Grothendieck. The proof is based upon the axiomatic definition of Grothendieck categories, and tells us nothing about what happens to a concrete representation of the original Grothendieck category as a sheaf category on a linear site. If we compare this result with Proposition 6.4 for module categories, clearly the latter is more precise.

Let $\gamma : \mathfrak{a} \longrightarrow \mathcal{C}$ be a $k$-linear functor from a small $k$-linear category $\mathfrak{a}$ to a Grothendieck category $\mathcal{C}$, which satisfies the conditions (G), (F), (FF). According to [17, §8], there is a canonical map

$$\lambda : \mathrm{Def}_{\mathrm{lin}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathcal{C}) \tag{10}$$

making use of the map $\mu$ from (9) and the fact that deformations can be induced upon localizations [17, §7]. In general, $\lambda$ will not be a bijection. Bijectivity of $\lambda$ will be further addressed in §6.5. In the remainder of this subsection, we will discuss concrete descriptions of $\lambda$.

First note that by Theorem 2.9, there is a canonical equivalence of categories $\mathcal{C} \cong \mathsf{Sh}(\mathfrak{a}, \mathcal{T})$ for an induced topology $\mathcal{T}$ on $\mathfrak{a}$. If we know $\lambda'$ associated to the canonical $\gamma' : \mathfrak{a} \longrightarrow \mathsf{Mod}(\mathfrak{a}) \longrightarrow \mathsf{Sh}(\mathfrak{a}, \mathcal{T})$, then $\lambda$ is obtained by composing $\lambda'$ with the canonical isomorphism

$$\eta : \mathrm{Def}_{\mathrm{ab}}(\mathsf{Sh}(\mathfrak{a}, \mathcal{T})) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathcal{C}). \tag{11}$$

Let $\mathfrak{a}$ be a $k$-linear category and let $\mathfrak{b}$ be a linear $R$-deformation with canonical map $\rho_{\mathfrak{b}} : \mathfrak{b} \longrightarrow \mathfrak{a}$. Consider the maps $\rho_{\mathfrak{b}} : \mathrm{cov}(\mathfrak{b}) \longrightarrow \mathrm{cov}(\mathfrak{a})$ and $\rho_{\mathfrak{b}}^{-1} : \mathrm{cov}(\mathfrak{a}) \longrightarrow \mathrm{cov}(\mathfrak{b})$ as in §2.5. The following result is based upon [17, Thm. 7.1] and can be found as [13, Prop. 3.16] and [7, Prop. 4.3]:

**Proposition 6.5.** *The given maps restrict to inverse bijections*

$$\rho_{\mathfrak{b}} : \mathrm{top}(\mathfrak{b}) \longrightarrow \mathrm{top}(\mathfrak{a})$$

*and*

$$\rho_{\mathfrak{b}}^{-1} : \mathrm{top}(\mathfrak{a}) \longrightarrow \mathrm{top}(\mathfrak{b})$$

*between the topologies on* $\mathfrak{a}$ *and on* $\mathfrak{b}$ *respectively.*

The following is a concrete description of the functor $\lambda$ from (10) in the given setup:

**Proposition 6.6.** *Let* $(\mathfrak{a}, \mathcal{T})$ *be a* $k$*-linear site. The canonical map* $\lambda$ *from* (10) *associated to* $\mathfrak{a} \longrightarrow \mathsf{Mod}(\mathfrak{a}) \longrightarrow \mathsf{Sh}(\mathfrak{a}, \mathcal{T})$ *is given by*

$$\lambda : \mathrm{Def}_{\mathrm{lin}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Sh}(\mathfrak{a}, \mathcal{T})) : \mathfrak{b} \longrightarrow \mathsf{Sh}(\mathfrak{b}, \rho_{\mathfrak{b}}^{-1}\mathcal{T}).$$

The main drawback in this description of $\lambda$, is the fact that although the topology $\rho_{\mathfrak{b}}^{-1}\mathcal{T}$ on $\mathfrak{b}$ is uniquely determined, it is not described intrinsically in terms of $\mathfrak{b}$, without reference to $\mathfrak{a}$. This is still in contrast with the map $\mu$ from (9), by which to a deformation $\mathfrak{b}$ of $\mathfrak{a}$, we associate the sheaf category with respect to the trivial topology on $\mathfrak{b}$. This will be remedied in §6.4 in the contexts of linearized topologies.

6.4. **Linearized sites.** Let $\mathcal{U}$ be a small category. Let $\mathfrak{a}$ be a $k$-linear $\mathcal{U}$-graded category and let $\mathfrak{b}$ be an $R$-deformation of $\mathfrak{a}$. Consequently, $\tilde{\mathfrak{b}}$ is an $R$-deformation of the $k$-linear category $\tilde{\mathfrak{a}}$ and we have a natural map $\rho = \rho_{\tilde{\mathfrak{b}}} : \tilde{\mathfrak{b}} \longrightarrow \tilde{\mathfrak{a}}$. Consider the maps $\rho : \mathrm{cov}(\tilde{\mathfrak{b}}) \longrightarrow \mathrm{cov}(\tilde{\mathfrak{a}})$ and $\rho^{-1} : \mathrm{cov}(\tilde{\mathfrak{a}}) \longrightarrow \mathrm{cov}(\tilde{\mathfrak{b}})$ as before.

**Proposition 6.7.** *Let* $\mathcal{U}$ *be endowed with a topology* $\tau$*. Let* $\tilde{\mathfrak{a}}$ *and* $\tilde{\mathfrak{b}}$ *be endowed with the linearized topologies* $\mathcal{T}_{\tau,\mathfrak{a}}$ *and* $\mathcal{T}_{\tau,\mathfrak{b}}$ *respectively. We have*

$$\rho(\mathcal{T}_{\tau,\mathfrak{b}}) = \mathcal{T}_{\tau,\mathfrak{a}}.$$

*Proof.* Consider $B \in \mathfrak{b}$ with $\rho(B) = A$, $A \in \mathfrak{a}_U$, and a cover $R \in \tau(U)$. From the description of $R^A \subseteq \tilde{\mathfrak{a}}(-, A)$ and $R^B \subseteq \tilde{\mathfrak{b}}(-, B)$ it is clear that $\rho(R^B) = R^A$. Consequently, $\rho(\mathcal{L}_{\tau,\mathfrak{b}}) \subseteq \mathcal{L}_{\tau,\mathfrak{a}}$. For an arbitrary sieve $R^A \subseteq T \subseteq \tilde{\mathfrak{a}}(-, A)$, consider the pullbacks $P$ and $P'$ of $T$ and $R^A$ respectively along $\tilde{\mathfrak{b}}(-, B)$. Then $P' = R^B$ and $\varphi(P) = T$, whence $\rho(\mathcal{L}_{\tau,\mathfrak{b}}) = \mathcal{L}_{\tau,\mathfrak{a}}$. Now consider the topology $\rho^{-1}\mathcal{T}_{\tau,\mathfrak{a}}$ on $\tilde{\mathfrak{b}}$ which corresponds to $\mathcal{T}_{\tau,\mathfrak{a}}$ under the bijection of Proposition 6.5. Since $\mathcal{L}_{\tau,\mathfrak{b}} \subseteq \rho^{-1}\mathcal{T}_{\tau,\mathfrak{a}}$, we have $\mathcal{T}_{\tau,\mathfrak{b}} \subseteq \rho^{-1}\mathcal{T}_{\tau,\mathfrak{a}}$. After taking $\rho$, it follows that $\mathcal{L}_{\tau,\mathfrak{a}} \subseteq \rho\mathcal{T}_{\tau,\mathfrak{b}} \subseteq \mathcal{T}_{\tau,\mathfrak{a}}$ and hence $\rho\mathcal{T}_{\tau,\mathfrak{b}} = \mathcal{T}_{\tau,\mathfrak{a}}$ as desired. □

**Proposition 6.8.** *Let* $(\mathcal{U}, \tau)$ *be a site and let* $\mathfrak{a}$ *be a* $\mathcal{U}$*-graded category. There is a canonical map*

$$\theta : \mathrm{Def}_{\mathcal{U}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{T}_{\tau,\mathfrak{a}})) : \mathfrak{b} \longrightarrow \mathsf{Sh}(\tilde{\mathfrak{b}}, \mathcal{T}_{\tau,\mathfrak{b}}). \tag{12}$$

*Proof.* This follows from Propositions 6.6 and 6.7. □

6.5. **Deformation equivalences.** Let $(\mathcal{U}, \sqsubseteq)$ be a preorder considered as a category and let $\mathfrak{a}$ be a $\mathcal{U}$-graded category with associated $k$-linear category $\tilde{\mathfrak{a}}$. Concretely, for $A \in \mathfrak{a}_U$, $B \in \mathfrak{a}_V$, we have

$$\tilde{\mathfrak{a}}(B, A) = \begin{cases} \mathfrak{a}_{\sqsubseteq}(B, A) & \text{if } V \sqsubseteq U; \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 6.9.** *The canonical map* $\alpha : \mathrm{Def}_{\mathcal{U}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{lin}}(\tilde{\mathfrak{a}})$ *from* (7) *is a bijection.*

*Proof.* It suffices to show that an arbitrary deformation $\mathfrak{c}$ of $\tilde{\mathfrak{a}}$ is in the image of te map. We may suppose that $k \otimes_R \mathfrak{c} = \tilde{\mathfrak{a}}$. Hence, for $A \in \mathfrak{a}_U$, $B \in \mathfrak{a}_V$ we have $k \otimes_R \mathfrak{c}(B, A) = 0$ for $V \not\sqsubseteq U$. By Nakayama, it follows that also $\mathfrak{c}(B, A) = 0$. □

Let $\tau$ be a topology on $\mathcal{U}$ with linearized topology $\mathcal{T}_\tau$ on $\tilde{\mathfrak{a}}$. Let $\mathcal{C}$ be a Grothendieck category and consider a linear functor

$$\gamma : \tilde{\mathfrak{a}} \longrightarrow \mathcal{C}.$$

Suppose that $\gamma$ induces an equivalence $\mathcal{C} \cong \mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{T}_\tau)$ (see §3.5).

We consider the following conditions.

(E0) An object $C \in \mathcal{C}$ satisfies (E0) if for all $X \in \mathsf{mod}(k)$, we have

$$\mathrm{Ext}^1_{\mathcal{C}}(C, X \otimes_k C) = \mathrm{Ext}^2_{\mathcal{C}}(C, X \otimes_k C) = 0.$$

(E1) A couple $(C, D)$ of objects in $\mathcal{C}$ satisfies (E1) if for all $X \in \mathsf{mod}(k)$, we have

$$\mathrm{Ext}^1_{\mathcal{C}}(C, X \otimes_k D) = 0.$$

(I) A couple $(B, A)$ of objects in $\tilde{\mathfrak{a}}$ satisfies (I) if

$$\gamma_{(B,A)} : \tilde{\mathfrak{a}}(B, A) \longrightarrow \mathcal{C}(\gamma(B), \gamma(A))$$

is an isomorphism.

The following is a direct application of [17, Thm. 8.14]:

**Theorem 6.10.** *Let $\gamma : \tilde{\mathfrak{a}} \longrightarrow \mathcal{C}$ be as before. Suppose:*

1. *for all $A \in \mathfrak{a}_U$, $\gamma(A)$ is $k$-flat and satisfies (E0);*
2. *for all $A \in \mathfrak{a}_U$, $B \in \mathfrak{b}_V$ for $V \sqsubseteq U$, the couple $(B, A)$ satisfies (I) and the couple $(\gamma(B), \gamma(A))$ satisfies (E2).*

*Then the canonical map*

$$\theta : \mathrm{Def}_{\mathcal{U}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{T}_{\tau,\mathfrak{a}})) : \mathfrak{b} \longrightarrow \mathsf{Sh}(\tilde{\mathfrak{b}}, \mathcal{T}_{\tau,\mathfrak{b}})$$

*from* (12) *is a bijection, from which the bijection $\lambda : \mathrm{Def}_{\mathrm{lin}}(\tilde{\mathfrak{a}}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathcal{C})$ from* (10) *is obtained as $\lambda = \eta\theta\alpha^{-1}$ for $\eta$ and $\alpha$ as in* (11) *and* (7) *respectively.*

If we analyze the previous theorem, we see that there are two possibly useful refinements of the site $(\mathcal{U}, \tau)$ with the eye on deformation theory. The first refinement involves condition (1), which only involves individual objects $A \in \mathfrak{a}_U$.

**Proposition 6.11.** *Let $\gamma : \tilde{\mathfrak{a}} \longrightarrow \mathcal{C}$ be as before. Suppose $\mathcal{V} \subseteq \mathcal{U}$ is a full subcategory such that either the inclusion $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ satsifies either (SG1) and (SG2), or else $\varphi$ satisfies (G) and $\mathfrak{a}$ satsifies (WG). Consider the $\mathcal{V}$-graded category $\mathfrak{a}' = \mathfrak{a}^\varphi$ and let $\mathcal{V}$ be endowed with the topology $\tau' = \varphi^{-1}\tau$. Then we canonically have $\mathcal{C} \cong \mathsf{Sh}(\tilde{\mathfrak{a}}', \mathcal{T}_{\tau',\mathfrak{a}})$.*

1. *If for all $V \in \mathcal{V}$ and $A \in \mathfrak{a}_V$, $\gamma(A)$ is $k$-flat and satisfies (E0), then $\gamma' : \tilde{\mathfrak{a}}' \longrightarrow \mathcal{C}$ satsifies condition (1) in Theorem 6.10.*
2. *If for all $A \in \mathfrak{a}_U$, $B \in \mathfrak{b}_V$ for $V \sqsubseteq U$ in $\mathcal{V}$, the couple $(B, A)$ satisfies (I) and the couple $(\gamma(B), \gamma(A))$ satisfies (E2), then $\gamma'$ satisfies condition (2) in Theorem 6.10.*

*Proof.* This follows from Theorem 3.16. □

The second refinement involves condition (2).

**Proposition 6.12.** *Let $\gamma : \tilde{\mathfrak{a}} \longrightarrow \mathcal{C}$ be as before. Suppose for every $U \in \mathcal{U}$, we have a cover $D_U \in \tau(U)$ such that for every $V \sqsubseteq U$ in $D_U(V)$, for every $A \in \mathfrak{a}_U$ and $B \in \mathfrak{a}_V$, the couple $(B, A)$ satsifies (I) and the couple $(\varphi(B), \varphi(A))$ satisfies (E1). Let $\mathcal{V} \subseteq \mathcal{U}$ be the subcategory constructed in §3.4(5). Consider the $\mathcal{V}$-graded category $\mathfrak{a}' = \mathfrak{a}^\varphi$ and let $\mathcal{V}$ be endowed with the topology $\tau' = \varphi^{-1}\tau$. Then we canonically have $\mathcal{C} \cong \mathsf{Sh}(\tilde{\mathfrak{a}}', \mathcal{T}_{\tau',\mathfrak{a}})$ and $\gamma' : \tilde{\mathfrak{a}}' \longrightarrow \mathcal{C}$ satsifies condition (2) in Theorem 6.10. Further, if $\gamma$ satisfies condition (1), then so does $\gamma'$.*

*Proof.* This follows from Proposition 3.21 and Theorem 3.16. □

We will illustrate Propositions 6.11 and 6.12 in the next two subsections.

6.6. **Prestacks.** Let $(\mathcal{U}, \sqsubseteq)$ be a preorder considered as a category and let $\mathcal{A} : \mathcal{U}^{\mathrm{op}} \longrightarrow \mathsf{Cat}(k)$ be a pseudofunctor on $\mathcal{U}$ with associated fibered $\mathcal{U}$-graded category $\mathfrak{a} = \mathcal{A}^\sharp$. We use the notations of §4.5. For every $A \in \mathcal{A}(U)$, we obtain a presheaf $\hat{A}^p \in \mathsf{Mod}(\mathcal{A}|_U)$ with

$$\hat{A}^p_V : \mathcal{A}(V) \longrightarrow \mathsf{Mod}_k(\mathcal{U}/V) : B \longmapsto \mathcal{H}om(B, A|_V)$$

and

$$\mathcal{H}om(B, A)(W) = \mathcal{A}(W)(B|_W, A|_W).$$

Let $\tau$ be a topology on $\mathcal{U}$. The pseudofunctor $\mathcal{A}$ is called a *prestack* if for every $A \in \mathcal{A}(U)$, $B \in \mathcal{A}(V)$, we have $\mathcal{H}om(B, A|_V) \in \mathsf{Sh}_k(\mathcal{U}/V)$ and hence $\hat{A}^p \in \mathsf{Sh}(\mathcal{A}|_U)$. In general, we let $\hat{A} \in \mathsf{Sh}(\mathcal{A}|_U)$ denote the sheafification of $\hat{A}^p$. Every restriction map $i_U^* : \mathsf{Mod}(\mathcal{A}) \longrightarrow \mathsf{Mod}(\mathcal{A}|_U)$ has a fully faithful exact left adjoint $i^p_{U,!} : \mathsf{Mod}(\mathcal{A}|_U) \longrightarrow \mathsf{Mod}(\mathcal{A})$ which is given by the "presheaf extension by zero", that is, for $F \in \mathsf{Mod}(\mathcal{A}|_U)$, for $A \in \mathcal{A}(Z)$ and $W \sqsubseteq Z$, we have

$$i^p_{U,!}F(A)(W) = \begin{cases} F(A|_W)(W) & \text{if } W \sqsubseteq U \\ 0 & \text{otherwise.} \end{cases}$$

Coposition with sheafification $\mathsf{Mod}(\mathcal{A}) \longrightarrow \mathsf{Sh}(\mathcal{A}, \tau)$ yields the fully faithful exact left adjoint $i_{U,!}$ of $i_U^* : \mathsf{Sh}(\mathcal{A}, \tau) \longrightarrow \mathsf{Sh}(\mathcal{A}|_U, \tau_U)$. Similarly, for $V \sqsubseteq U$, we obtain the fully faithful exact left adjoint $i^U_{V,!} : \mathsf{Sh}(\mathcal{A}|_V, \tau_V) \longrightarrow \mathsf{Sh}(\mathcal{A}|_U, \tau_U)$ of $i_V^{U,*} : \mathsf{Sh}(\mathcal{A}|_U, \tau_U) \longrightarrow \mathsf{Sh}(\mathcal{A}|_V, \tau_V)$. We naturally obtain a functor

$$\gamma : \tilde{\mathfrak{a}} \longrightarrow \mathsf{Sh}(\mathfrak{a}, \tau)$$

which sends $A \in \mathcal{A}(U)$ to $i_{U,!}(\hat{A})$ for $\hat{A} \in \mathsf{Sh}(\mathcal{A}|_U)$. For $V \sqsubseteq U$ and $B \in \mathcal{A}(V)$, $A \in \mathcal{A}(U)$, we have $\tilde{\mathfrak{a}}(B, A) = \mathcal{A}(V)(B, A|_V)$ and we obtain the canonical map

$$\gamma_{B,A} : \mathcal{A}(V)(B, A|_V) \longrightarrow \mathsf{Sh}(\mathcal{A}|_V, \tau_V)(\hat{B}, \hat{A}|_V) \longrightarrow \mathsf{Sh}(\mathcal{A}, \tau)(i_{V,!}\hat{B}, i_{U,!}\hat{A}).$$

The functor $\varphi$ induces the equivalence of categories from Proposition 4.5.

The following theorem generalizes and refines [17, Theorem 8.18] and [12, Theorem 3.22].

**Theorem 6.13.** *Let $\mathcal{A}$ be a prestack on $(\mathcal{U}, \tau)$ and let $\gamma$ be as above. Let $\mathcal{V} \subseteq \mathcal{U}$ be a full subcategory for which the inclusion $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ satsifies (G) and suppose for every $V \in \mathcal{V}$ and $A \in \mathcal{A}(V)$, the object $\hat{A} \in \mathsf{Sh}(\mathcal{A}|_V, \tau_V)$ is k-flat and for $X \in \mathsf{mod}(k)$, we have*

$$\mathrm{Ext}^1_{\mathsf{Sh}(\mathcal{A}|_V, \tau_V)}(\hat{A}, X \otimes_k \hat{A}) = \mathrm{Ext}^2_{\mathsf{Sh}(\mathcal{A}|_V, \tau_V)}(\hat{A}, X \otimes_k \hat{A}) = 0. \tag{13}$$

*Consider the topology $\tau' = \varphi^{-1}\tau$ on $\mathcal{V}$, and the prestack $\mathcal{A}' = \mathcal{A}\varphi$ on $\mathcal{V}$. Then there is a canonical equivalence $\mathsf{Sh}(\mathcal{A}, \tau) \cong \mathsf{Sh}(\mathcal{A}', \tau')$ and the canonical map*

$$\mathrm{Def}_{\mathrm{ps}}(\mathcal{A}') \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Sh}(\mathcal{A}', \tau')) : \mathcal{B} \longmapsto \mathsf{Sh}(\mathcal{B}, \tau')$$

*is a bijection.*

*Proof.* In order to apply Theorem 6.10, It suffices to check conditions (1) and (2) in Proposition 6.11. This can be done using the fact that $\mathcal{A}$ is a prestack and the functors $i_{U,!}$ and $i^U_{V,!}$ are fully faithful and exact. □

*Example* 6.14. [17, Theorem 8.18]. Let $(X, \mathcal{O})$ be a ringed space as in Example 4.7. Let $\mathcal{B} \subseteq \mathrm{open}(X)$ be a basis of $X$ such that for $U \in \mathcal{B}$ and $X \in \mathsf{mod}(k)$ we have

$$H^1(U, X \otimes_k \mathcal{O}|_U) = H^2(U, X \otimes_k \mathcal{O}|_U) = 0$$

for the sheaf cohomology groups. Let $\tau$ be the standard topology on open$(X)$ and $\tau|_{\mathcal{B}}$ its restriction to $\mathcal{B}$, and let $\mathcal{O}|_{\mathcal{B}}$ be the restriction of $\mathcal{O}$ to $\mathcal{B}$. We obtain a bijection

$$\mathrm{Def}_{\mathrm{ps}}(\mathcal{O}|_{\mathcal{B}}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Sh}(X, \mathcal{O})) : \mathcal{F} \longmapsto \mathsf{Sh}(\mathcal{F}, \tau|_{\mathcal{B}}).$$

6.7. **Tails topologies.** Let $\mathcal{C}$ be a Grothendieck category and let $(\mathcal{U}, \sqsubseteq)$ be a preordered set, considered as a category, satisfying (D0), with a map $\gamma : \mathcal{U} \longrightarrow \mathrm{Ob}(\mathcal{C})$. We define the $\mathcal{U}$-graded category $\mathfrak{a}$ with $\mathfrak{a}_U = \{U\}$ and

$$\mathfrak{a}_{\sqsubseteq}(V, U) = \mathcal{C}(\gamma(V), \gamma(U)).$$

Let $\gamma : \tilde{\mathfrak{a}} \longrightarrow \mathcal{C}$ be the canonical functor. Suppose $\gamma$ induces an equivalence $\mathcal{C} \cong \mathsf{Sh}(\tilde{\mathfrak{a}}, \mathcal{T}_{\mathrm{tails}})$ for the topology tails on $\mathcal{U}$ (see Theorem 5.20).

**Theorem 6.15.** *Let $\gamma : \tilde{\mathfrak{a}} \longrightarrow \mathcal{C}$ be as before. Suppose for all $U \in \mathcal{U}$, $\gamma(U)$ is $k$-flat and satsifies (E0). Suppose there is a function $\nu : \mathcal{U} \longrightarrow \mathcal{U}$ with $\nu(U) \sqsubseteq U$ such that for all $V \sqsubseteq \nu(U)$, the couple $(\gamma(V), \gamma(U))$ satisfies (E1). Let $\mathcal{U}' \subseteq \mathcal{U}$ be associated to the preorder* (6)*, and endowed with the tails topology* tails′*, and let $\mathfrak{a}'$ be the induced $\mathcal{U}'$-graded category as in §5.3. We canonically have $\mathcal{C} \cong \mathsf{Sh}(\tilde{\mathfrak{a}}', \mathcal{T}_{\mathrm{tails}'})$ and the canonical map*

$$\mathrm{Def}_{\mathcal{U}'}(\mathfrak{a}') \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Sh}(\tilde{\mathfrak{a}}', \mathcal{T}_{\mathrm{tails}'}) : \mathfrak{b} \longmapsto \mathsf{Sh}(\tilde{\mathfrak{b}}, \mathcal{T}_{\mathrm{tails}'})$$

*is a bijection.*

*Proof.* This follows from Proposition 6.12 and Theorem 6.10. □

*Example* 6.16. [14] Let $X$ be a projective scheme over a field $k$ with an ample invertible sheaf $\mathcal{L} \in \mathsf{Qch}(X)$ and consider $\mathcal{U} = (\mathbb{Z}, \geq)$, $\gamma : \mathbb{Z} \longrightarrow \mathsf{coh}(X) : n \longmapsto \mathcal{L}^{-n}$ as in Example 5.21. By the cohomological criterion of ampleness, for fixed $\mathcal{L}^{-m}$, there exists $\nu(m) \geq m$ such that for all $n \geq \nu(m)$, we have

$$\mathrm{Ext}^1(\mathcal{L}^{-n}, X \otimes_k \mathcal{L}^{-m}) = 0,$$

that is, the couple $(\mathcal{L}^{-n}, \mathcal{L}^{-m})$ satisfies (E1). If we further suppose that

$$H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0, \tag{14}$$

then every $\mathcal{L}^{-n}$ satsifies (E0) and Theorem 6.15 applies. Thus, for the class of projective schemes satisfying this restraint (14) on their cohomology, all deformations can be described as "non-commutative projective schemes" over some deformed $(\mathbb{Z}, \geq)$-category. This includes the original cases treated in [21] and [20]. See also [7] for the setup in which we can take $\nu(m) = m$.

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