Continuous One-counter Automata

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We study the reachability problem for continuous one-counter automata, COCA for short. In such automata, transitions are guarded by upper- and lower-bound tests against the counter value. Additionally, the counter updates associated with taking transitions can be (non-deterministically) scaled down by a nonzero factor between zero and one. Our three main results are as follows: we prove (1) that the reachability problem for COCA with global upper- and lower-bound tests is in NC$^2$; (2) that, in general, the problem is decidable in polynomial time; and (3) that it is NP-complete for COCA with parametric counter updates and bound tests.

CCS Concepts: • Theory of computation → Logic and verification; Automata over infinite objects; Problems, reductions and completeness;

Additional Key Words and Phrases: Counter automata, reachability problem, parametric automata, continuous relaxation


1 INTRODUCTION

Counter machines form a fundamental computational model that captures the behavior of infinite-state systems. Unfortunately, their central decision problems, like the (configuration) reachability problem, are undecidable as the model is Turing-complete [20, 21]. To circumvent this issue, numerous restrictions of counter machines have been studied in the literature. For instance, vector addition systems with states (VASS) arise from restricting the types of tests that can be used to
guard transitions [11, 17–19]. One-counter automata are yet another well-studied model [6, 12, 13], in this case arising from the restriction to a single counter, hence the name.

We consider one-counter automata that can use tests of the form “≤ c” and “≥ d”—where c and d are constants—to guard their transitions. As a natural extension of finite-state automata, one-counter automata allow for better conservative approximations of classical static-analysis problems like instruction reachability (see, e.g., program graphs as defined in [3]). They also enable the verification of programs with lists [7] and XML-stream validation [10]. Furthermore, their reachability problem seems intrinsically connected to that of timed automata (TA). The reachability problem for two-clock TA is known to be logspace-equivalent to the same problem for succinct one-counter automata (SOCA), that is, where constants used in counter updates and tests are encoded in binary [14]. An analog of this connection holds when SOCA are enriched with parameters that can be used on updates: reachability for two-parametric-clock TA reduces to the (existential) reachability problem for parametric one-counter automata [9]. Interestingly, Alur et al. [1] observe that the former subsumes a long-standing open problem of Ibarra et al. [15] concerning “simple programs.”

All of the above connections from interesting problems to reachability for SOCA and parametric SOCA indicate that efficient algorithms for the problem are very much desirable. Unfortunately, it is known that reachability for SOCA (with upper- and lower-bound tests) is PSPACE-complete [12]. For parametric SOCA, the situation is even worse as the general problem is not even known to be decidable. In this work, we study continuous relaxations of these problems and show that their complexities belong in tractable complexity classes. We thus give the first efficient conservative approximation for the reachability problem for SOCA and parametric SOCA.

The continuous relaxations. We observe that the model considered by Fearnley and Jurdziński [12] is not precisely our SOCA; rather, they consider bounded 1-dimensional VASS. In such VASS, the counter is not allowed to take negative values. Additionally, it is not allowed to take values greater than some global upper bound. Note that inequality tests against constants can be added to such VASS as “syntactic sugar” since these can be implemented making use of the upper and lower bounds. These observations allow us to adapt Blondin and Haase’s definition of continuous VASS [4] to introduce globally guarded continuous one-counter automata (GG-COCA) that have global upper- and lower-bound tests: Transitions are allowed to be “partially taken” in the sense that the respective counter updates can be scaled by some factor α ∈ (0, 1).

In contrast to the situation in the discrete world, because of the continuous semantics, adding arbitrary upper- and lower-bound tests to COCA does result in the more expressive model of locally guarded COCA, or simply COCA for short. Importantly, COCA are a “tighter” relaxation of SOCA than GG-COCA (via the translation to bounded 1-VASS). Finally, we also study the reachability problem for parametric COCA. These are COCA where counter updates can be variables x ∈ X whose values range over the rationals; bound tests can also be against variables from X. The resulting model can be seen as a continuous relaxation of Ibarra’s simple programs [9, 15].

Contributions. Our main contributions are threefold (see Theorem 1). First, we show that the reachability problem for GG-COCA is in NC². This closes a complexity gap for continuous VASS, where reachability is known to be NP-hard when at least two counters are available [4, Lemma 4.13]. Thus, our result shows that an improvement of this lower bound to the case of one counter is unlikely. Second, we give a polynomial-time algorithm for the same problem for COCA. Finally, we show that the reachability problem for parametric COCA is NP-complete. The last upper bound improves on the conference version of this work [5], where we showed that the reachability problem for parametric COCA lies in the polynomial hierarchy.
On the way, we prove that the reachability problem for GG-COCA enriched with equality tests is in NC\(^2\), and that the reachability problem for parametric COCA where only counter updates are allowed to be parametric is equivalent to the integer-valuation restriction of the problem.

*Other related work.* To complete a full circle of connections between timed and counter automata, we note that the closest model to ours is that of one-clock TA. The value of the clock in such automata evolves (continuously) at a fixed positive rate and can be reset by some transitions. COCA can simulate clock delays using +1 self-loops and resets using −1 self-loops and bound tests “\(\leq\)” and “\(\geq\).” Our model thus generalizes one-clock TA.

The reachability problem for (non-parametric) one-clock TA is NL-complete [16]. The NL membership proof from [16] relies on the fact that clock delays can always occur and do so without changing the state. This allows partitioning the counter values into regions. Only the current region is important, not the precise clock value, as the next region can always be reached by letting time pass. This does not hold in the more general framework of COCA. Here, we need to know how far away the next region is, since the counter value in COCA cannot necessarily be incremented at will in every state. Hence, the proof does not directly extend to COCA.

The reachability problem for parametric one-clock TA with integer-valued parameters is known to lie in NEXP [9]. Since non-parametric clocks can be removed at the cost of an exponential blow-up [1], it is also argued in [9] that the problem belongs to N2EXP if an arbitrary number of non-parametric clocks is allowed [9]. For the latter problem, the authors also prove that it is NEXP-hard. Our NP upper bound for update-parametric COCA with integer-valued parameters improves the latter two bounds.

*Organization.* In Section 2 we introduce the basic notation and models. Then, in Sections 3, 4, and 5 we prove membership in NC\(^2\), membership in polynomial time, and NP-completeness for reachability of GG-COCA, COCA, and parametric COCA, respectively. Finally, we conclude in Section 6.

2 PRELIMINARIES

We write \(\mathbb{Q}_{\geq 0}\) for the set of non-negative rationals, and \(\mathbb{Q}_{> 0}\) for the set of positive rationals. We use symbols “[” and “]” for closed intervals, and “(“ and “)” for open intervals of rational numbers. For example, \([a, b)\) denotes \(\{q \in \mathbb{Q} \mid a \leq q < b\}\). Intervals do not have to be bounded; e.g., we allow \([3, +\infty)\). We denote the set of all intervals over \(\mathbb{Q}\) by \(\Gamma\). We write \(\overline{X} \subseteq \mathbb{Q}\) to denote the closure of a set \(X \subseteq \mathbb{Q}\), i.e., \(X\) enlarged with its limit points. For example, \((3, 5) = [3, 5], [1, 4) \cup (4, 5) = [1, 5],\) and \([2, 3) \cup (4, 5) = [2, 3] \cup [4, 5]\). Note that \((-\infty, +\infty)\) remains \((-\infty, +\infty)\). Throughout the article, numbers are encoded in binary and we assume intervals to be encoded as pairs of endpoints, together with binary flags indicating whether the endpoints are contained or not. We present rational numbers as quotients of integers. We assume the sizes of formulas to be the number of symbols needed to write them when numbers are encoded in binary.

2.1 One-counter Automata

A GG-COCA is a triple \(\mathcal{V} = (Q, T, \tau)\), where \(Q\) and \(T \subseteq Q \times Q \times Q\) are finite sets of states and transitions, and \(\tau \in \Gamma\). A configuration of \(\mathcal{V}\) is a pair \((q, a) \in Q \times \mathbb{Q}\), denoted \(q(a)\). A run from \(p(a)\) to \(q(b)\) in \(\mathcal{V}\) is a sequence \(a_1t_1 \cdots a_nt_n\), where \(a_i \in (0, 1]\) and \(t_i = (q_{i-1}, z_i, q_i) \in T\), for which there exist configurations \(q_0(a_0), \ldots, q_n(a_n)\) such that \(q_0(a_0) = p(a), q_n(a_n) = q(b)\) and \(a_i = a_{i-1} + a_i \cdot z_i\) for all \(i \in [1, \ldots, n]\). We say that such a run is admissible if \(a_0, \ldots, a_n \in \tau\). For readers familiar with one-counter automata, note that the model of one-counter nets is obtained by setting \(\tau = [0, +\infty)\) and \(a_i = 1\) for all \(i\).

ACM Transactions on Computational Logic, Vol. 24, No. 1, Article 3. Publication date: January 2023.
A (locally guarded) COCA is a triple $\mathcal{W} = (Q, T, \tau)$, where $(Q, T)$ is as for a GG-COCA and $\tau : Q \to \Gamma$ assigns intervals to states. Configurations and runs of $\mathcal{W}$ are defined as for a GG-COCA. A run is admissible if each of its configurations $q_i(a_i)$ satisfies $a_i \in \tau(q_i)$. Hence, a GG-COCA can be seen as a COCA where $\tau(q)$ is the same for all $q \in Q$.

The set $\Gamma_X$ of parameterized intervals over a set $X$ is the set of intervals whose endpoints belong either to $\mathbb{Q} \cup \{ -\infty, +\infty \}$ or $X$. A parametric COCA is a tuple $\mathcal{P} = (Q, T, \tau, X)$, where $Q, X,$ and $T \subseteq \mathbb{Q} \times (\mathbb{Q} \cup X) \times \mathbb{Q}$ are finite sets of states, parameters, and transitions, and where $\tau : Q \to \Gamma_X$. A valuation of $X$ is a function $\mu : X \to \mathbb{Q}$. We write $\mathcal{P}^\mu = (Q, T^\mu, \tau^\mu)$ to denote the COCA obtained from $\mathcal{P}$ by replacing each parameter $x \in X$ occurring in $T$ or $\tau$, with $\mu(x)$. We say that there is a run from $p(a)$ to $q(b)$ in $\mathcal{P}$ if there exists a valuation $\mu$ such that $\mathcal{P}^\mu$ has a run from $p(a)$ to $q(b)$. In particular, $\mathcal{P}$ is a COCA if $X = \emptyset$. Otherwise, the notion of a run only makes sense w.r.t. a valuation $\mu$, i.e., in the COCA $\mathcal{P}^\mu$.

In summary, we deal with three increasingly richer models: GG-COCA $\subseteq$ COCA $\subseteq$ parametric COCA. In all variants, the size of the automaton is $(|Q| + |T|) \cdot s$, where $s$ is the maximal number of bits required to encode a number in $T$ or $\tau$.

### 2.2 Runs, Paths, Cycles, and Linear Path Schemes

Let $\mathcal{W} = (Q, T, \tau)$ be a COCA. We write $\text{Paths}_{p, q} \subseteq T^*$ to denote the set of paths from state $p \in Q$ to state $q \in Q$ in the graph induced by $T$. Let $\pi = t_1 \cdots t_n \in T^*$ be a path. We write $\pi_i = t_i$ to denote the $i$th element of $\pi$. Let $\rho = \alpha t_1 \cdots \alpha t_n$ be a run where each $t_i = (q_{i-1}, z_i, q_i)$. The underlying path of $\rho$ is $p(\rho) := t_1 \cdots t_n \in \text{Paths}_{q_0, q_n}$. We further define $\rho[i..j] := \alpha t_i \cdots \alpha t_j$, $t_i := \rho[i..i]$, $\text{in}(\rho) := q_0$, out($\rho$) := $q_n$, and $\Delta(\rho) := \sum_{i=1}^n \alpha_i z_i$. By convention, $\rho[i..j] := \epsilon$ if $j < i$, and $\Delta(\epsilon) := 0$. We write $p(a) \rightarrow_{\rho} q(b)$ to denote the fact that $\rho$ is admissible from $p(a)$ to $q(b)$. Since states $p$ and $q$ are determined by $\rho$, we may omit them and simply write $a \rightarrow_{\rho} b$. For every $\beta \in (0, 1]$, we define $\beta \rho := (\beta \alpha_1) t_1 \cdots (\beta \alpha_n) t_n$. Note that $\beta \rho$ is a run, but it is not necessarily admissible, even if $\rho$ is.

For a path $\pi = t_1 \cdots t_n$, we denote as $|\pi| := n$ the length of $\pi$.

Let $\pi = t_1 \cdots t_n \in \text{Paths}_{p, q}$ be such that each $t_i = (q_{i-1}, z_i, q_i)$. We say that $\pi$ is a cycle if $p = q$, and simple if $\pi$ does not repeat any state. Let $\Delta(\pi) := z_1 + \ldots + z_n$, $\Delta^+ (\pi) := \sum_{i=1}^n \max(0, z_i)$ and $\Delta^- (\pi) := \sum_{i=1}^n \min(0, z_i)$, with $\Delta(\epsilon) = \Delta^+(\epsilon) = \Delta^-(\epsilon) := 0$. In particular, $\Delta^- (\pi) \leq 0 \leq \Delta^+(\pi)$. Moreover, scaling the positive or negative transitions of a path $\pi$ arbitrarily close to zero yields a run of effect arbitrarily close to $\Delta^-(\pi)$ or $\Delta^+(\pi)$.

We say that a linear path scheme is a regular expression of the form

$$L = \sigma_0 \theta_0^* \sigma_1 \theta_1^* \cdots \sigma_{k-1} \theta_{k-1}^* \sigma_k,$$

where each $\sigma_i$ is a path, each $\theta_j$ is a cycle, and $\theta_0 \sigma_0 \sigma_1 \theta_1 \cdots \sigma_{k-1} \theta_{k-1} \sigma_k$ is a path. For ease of notation, we denote by a regular expression $L$ also the language described by $L$. The size of $L$ is defined as $|L| := |\sigma_0 \theta_0 \sigma_1 \theta_1 \cdots \sigma_{k-1} \theta_{k-1} \sigma_k|$, that is, the length of the underlying path of $L$.

We write $p(a) \rightarrow_{\pi} q(b)$ to denote the existence of a run $\rho$ such that $p(a) \rightarrow_{\rho} q(b)$ and path($\rho$) = $\pi$. As for runs, we may omit states and simply write $a \rightarrow_{\pi} b$. The reachability function given by $\pi$ is defined as $\text{Post}_{\pi}(a) := \{ b \in Q \mid p(a) \rightarrow_{\pi} q(b) \}$. We generalize this notion to sets of paths and numbers:

$$\text{Post}_{S}(A) := \bigcup_{\pi \in S} \bigcup_{a \in A} \text{Post}_{\pi}(a).$$

Note that linear path schemes express sets of paths, so this notion also naturally extends to them. If $S = \text{Paths}_{p, q}$, then we write $\text{Post}_{p, q}(a)$ and $\text{Post}_{p, q}(A)$. For example, for the COCA of Figure 1,
Fig. 1. A COCA; each state $s$ is labeled with the interval $\tau(s)$.

the following holds:

$$\text{Post}_{p,q}(a) = \begin{cases} (10, 18) \cup [19, 100) & \text{if } a = 15, \\ (a - 5, a + 3) & \text{if } a \in (-5, 15), \\ \emptyset & \text{otherwise.} \end{cases}$$

Finally, we define the set of starting points as $\text{enab}(\pi) := \{a \in \mathbb{Q} \mid \text{Post}_{\pi}(a) \neq \emptyset\}$ and $\text{enab}(S) := \bigcup_{\pi \in S} \text{enab}(\pi)$.

2.3 Our Contribution

In this work, we study the reachability problem that asks the following question: given a GG-COCA or a COCA $W$ with configurations $p(a)$ and $q(b)$, is there an admissible run from $p(a)$ to $q(b)$? In other words, by abbreviating “Paths$_{p,q}$” with “*,” the problem asks whether $p(a) \to_* q(b)$ holds. For parametric COCAs, the (existential) reachability problem asks whether $p(a) \to_* q(b)$ for some parameter valuation.

We will establish the following complexity results:

**Theorem 1.** The reachability problem is

1. in NC$^2$ for GG-COCAs;
2. in P for COCAs; and
3. NP-complete for parametric COCAs.

Recall that NC is the class of problems solvable in polylogarithmic parallel time, i.e., $\text{NC} = \bigcup_{i \geq 0} \text{NC}^i$, where $\text{NC}^i$ is the class of problems decidable by logspace-uniform families of circuits of polynomial size, depth $O(\log^i n)$, and bounded fan-in (e.g., see [2, 22] for a thorough definition). It is well known that $\text{NL} \subseteq \text{NC}^2 \subseteq \text{P}$. We also refer to the functional variant of $\text{NC}^i$ as $\text{NC}^f$.

The two first results of Theorem 1 are obtained by characterizing reachability functions and by showing how to efficiently compute a representation in terms of a small number of intervals. More precisely, we show that for every $a \in \mathbb{Q}$:

1. if $W$ is a GG-COCA, then $\text{Post}_{p,q}(a)$ consists of at most two intervals whose representations are computable in NC$^2$ (Corollary 7);
2. if $W$ is a COCA, then $\text{Post}_{p,q}(a)$ is made of $|W|^{O(1)}$ intervals, and a representation of $\text{Post}_{p,q}(a)$ is computable in polynomial time (Lemma 21).

To derive the third result, i.e., NP-completeness, we need to look more carefully at the second result. It turns out that the reachability question for COCA can be reduced to linear path schemes (Lemma 32). This allows us to reduce reachability of parametric COCA to determining the satisfiability of an existential linear arithmetic formula.
3 GLOBALLY GUARDED COCA REACHABILITY

In this section, we prove that the reachability problem for GG-COCAs belongs in NC$^2$ by showing how to compute a representation of Post$_{p,q}(a)$ from some $a$. As a first step, we describe how to utilize graph reachability to check whether Post$_{p,q}(a)$ is nonempty in Section 3.1. In Section 3.2, we observe that due to the fact that COCAs allow continuous scaling factors when firing transitions, the closure Post$_{p,q}(a)$ is an interval $[b, c]$ that differs from Post$_{p,q}(a)$ in at most the three points $a$, $b$, and $c$. We utilize results on computing shortest and longest paths in graphs to identify the endpoints $b$ and $c$ in Section 3.3. In Section 3.4, we reduce checks for membership of $a$, $b$, and $c$ in Post$_{p,q}(a)$ to graph reachability queries, using similar techniques as in Section 3.1, and thus obtain an NC$^2$ algorithm for deciding reachability in GG-COCAs. Finally, we generalize the achieved results to GG-COCAs with equality tests in Section 3.5 by using the fact that equality tests are passed by a single configuration to construct a graph where nodes represent equality tests and edges correspond to reachability in the GG-COCA.

In the remainder, we fix a GG-COCA $\mathcal{V} = (Q, T, \tau)$ where $T$ and $\tau$ use numbers from $\mathbb{Z}$ rather than $\mathbb{Q}$. This assumption merely simplifies the presentation. Indeed, for any $\lambda \in \mathbb{Q}_{\geq 0}, p(a) \rightarrow \lambda, q(b)$ holds in $\mathcal{V}$ if $p(\lambda a) \rightarrow \tau, q(\lambda b)$ holds after all numbers are multiplied by $\lambda$. Hence, $\lambda$ could be precomputed in NC$^2$ as the product of all denominators of fractions occurring within $T$ and $\tau$.

3.1 Testing Emptiness

We first aim to show that deciding whether Post$_{p,q}(a) \neq \emptyset$ can be checked in NC$^2$. To this end, we first state some simple graph properties checkable in NC$^2$. For a path $\pi$, let us write first($\pi$) (resp. last($\pi$)) to denote the first (resp. last) index such that $\Delta(\pi_i) \neq 0$ if there are any, and first($\pi$) = last($\pi$) := $\infty$ if none exist.

This lemma follows from standard results on NC$^2$:

**Lemma 2.** Let $\mathcal{V} = (Q, E, \tau)$ be a COCA whose weights are encoded in binary, and let $p, q \in Q$ be nodes. Deciding whether $S = \emptyset$ is in NC$^2$, where $S$ is the set of paths $\pi \in \text{Paths}_{p,q}$ that satisfy a fixed subset of these conditions$^2$:

(a) $\Delta^+(\pi) \neq 0$ (resp. $\Delta^-(\pi) \neq 0$);
(b) $\Delta^+(\pi) = 0$ (resp. $\Delta^-(\pi) = 0$);
(c) $\Delta(\text{first}(\pi)) < 0$ (resp. $\Delta(\text{first}(\pi)) > 0$);
(d) $\Delta(\text{last}(\pi)) < 0$ (resp. $\Delta(\text{last}(\pi)) > 0$).

Furthermore, for any such set $S$, the following value can be computed in NC$^2$: $\text{opt}\{w(\pi) \mid \pi \in S \text{ and } |\pi| \leq |Q|\}$, where $\text{opt} \in \{\min, \max\}$ and $w \in \{\Delta^+, \Delta^-, \Delta\}$.

**Proof.** For each condition, we focus only on one of the two stated cases, while the other case will follow similarly. Each time, we will give a graph $H$ such that reachability from $p$ to $q$ in the multi-graph $G = (Q, E)$ via a path that satisfies the condition is equivalent to graph reachability between two fixed nodes of $H$, which can be decided in NL $\subseteq$ NC$^2$.

(a) Let us define $H$ as joining $G$ with a modified copy $\overline{G}$ that remains in $\overline{G}$ with nonpositive edges and moves into $G$ with positive edges. More formally, let $H := (Q', E')$, where $Q' := Q \cup \{\overline{q} \mid q \in Q\}$. Further, $E' := E \cup \{(\overline{p}, z, \overline{q}) \mid (p, z, q) \in E, z \leq 0\} \cup \{(\overline{p}, z, q) \mid (p, z, q) \in E, z > 0\}$. It is easy to see that any path $\overline{\pi}$ from $\overline{p}$ to $q$ in $H$ corresponds to a path $\pi$ from $p$ to $q$ in $G$ such that $\Delta^+(\pi) \neq 0$.

(b) We set $H := (Q, E')$, where $E' := \{(p', z, q') \in E \mid z \leq 0\}$. Clearly, reachability from $p$ to $q$ in $H$ is equivalent to reachability from $p$ to $q$ in $G$ via a path $\pi$ with $\Delta^+(\pi) = 0$.

1 The set of conditions may be empty, in which case $S = \text{Paths}_{p,q}$.
(c) We again define $H$ as joining $G$ with a modified copy $\overline{G}$. In $\overline{G}$, we omit all positive edges, while edges with weight zero remain in $\overline{G}$ and negative edges lead to $G$. Formally, let $H := (Q', E')$, where $Q' := Q \cup \{ \overline{q} \mid q \in Q \}$ and $E' := E \cup \{(p, 0, \overline{q}) \mid (p, 0, q) \in E \} \cup \{(\overline{p}, z, q) \mid (p, z, q) \in E \land z < 0 \}$. A path $\overline{p}$ from $p$ to $q$ in $H$ corresponds to a path $\pi$ from $p$ to $q$ in $G$ such that $\Delta(\text{first}(\pi)) < 0$.

(d) We again join $G$ with a modified copy $\overline{G}$. Now, $\overline{G}$ omits all positive edges, and for each negative edge in $G$, we add a copy that leads from $G$ to $\overline{G}$. We define $H := (Q', E')$ with $Q' := Q \cup \{ \overline{q} \mid q \in Q \}$ and $E' := E \cup \{(p, z, \overline{q}) \mid (p, z, q) \in E \land z < 0 \} \cup \{(\overline{p}, z, q) \mid (p, z, q) \in E \land z \leq 0 \}$. A path $\overline{p}$ from $p$ to $q$ in $H$ corresponds to a path $\pi$ from $p$ to $q$ in $G$ such that $\Delta(\text{last}(\pi)) < 0$.

Note that for each condition, we construct a graph $H$ from a given input graph $G$. To require many conditions at once, we simply apply the transformations for each condition sequentially and obtain a graph $H'$ such that paths of $H'$ satisfy all imposed conditions and correspond to paths in the original graph $G$. Observe that $H'$ is of polynomial size.

Finally, let us argue that the following value can be computed in NC$^2$: $\text{opt}(w(\pi) \mid \pi \in S \text{ and } |\pi| \leq |Q|)$, where $w \in \{\Delta^+, \Delta^-\}$. Let us first deal with $w = \Delta^+$, for which it suffices to treat edges with negative weight as having zero weight.

The problem of finding a shortest weighted path in a graph with edges of nonnegative weights is in NC$^2$ (e.g., see [8, Example 12.4]). The procedure relies on the fact that there must be an acyclic shortest path, and hence that it suffices to consider paths of length at most $|Q|$. We can easily adapt the standard procedure for maximization: instead of successively minimizing among weighted paths of length $1, 2, 4, 8, \ldots, |Q|$, one maximizes among those paths. It follows that the claim holds for $\text{opt} \in \{\min, \max\}$. The case of $\Delta^-$ can be handled similarly by flipping the signs of weights and the optimization type (max/min).

Now, let us give necessary and sufficient conditions for enabledness of a path from a value. Intuitively, these conditions stem from the observation that effects may be scaled arbitrarily small, yet it is still impossible to apply a transition with negative effect when starting at $\inf \tau$, and similar for transitions with positive effect when starting at $\sup \tau$.

**Lemma 3.** Let $a \in \mathbb{Z}$, $p, q \in Q$, and $\pi \in \text{Paths}_{p,q}$. We have $a \in \text{enab}(\pi)$ iff $a \in \tau$ and any of these conditions hold:

1. $a \notin [\inf \tau, \sup \tau]$;
2. $a = \inf \tau = \sup \tau$ and $\text{first}(\pi) = \infty$;
3. $a = \inf \tau < \sup \tau$, $\text{first}(\pi) \neq \infty \implies \Delta(\pi_{\text{first}(\pi)}) > 0$;
4. $a = \sup \tau > \inf \tau$, $\text{first}(\pi) \neq \infty \implies \Delta(\pi_{\text{first}(\pi)}) < 0$.

**Proof.** Having $a \in \tau$ is obviously necessary, so we assume it holds throughout the proof.

$\Leftarrow$ We proceed by induction on $|\pi|$. If $|\pi| = 0$, then the claim is trivial as the empty path is admissible from $a$. Assume $|\pi| = n > 0$ and $\pi$ satisfies a condition. Let $t := \pi_1$ and $\sigma := \pi[2..n]$. If (a) holds, then $a \rightarrow_{\beta, t} a'$ for some $a' \in \tau \setminus [\inf \tau, \sup \tau]$ and sufficiently small $\beta \in (0, 1]$. If (b) holds, then $a \rightarrow_{\tau} a' = a$ as $\Delta(t) = 0$. If (c) or (d) holds, then either $a \rightarrow_{\tau} a' = a$ if $\Delta(t) = 0$, or $a \rightarrow_{\beta, t} a'$ for some $a' \in \tau \setminus [\inf \tau, \sup \tau]$ and sufficiently small $\beta \in (0, 1]$ otherwise. In all cases, $\sigma$ satisfies one of the conditions w.r.t. value $a'$. Thus, were done by the induction hypothesis.

$\Rightarrow$ Toward a contradiction, let us assume that $a \in \text{enab}(\pi)$ and that no condition is satisfied. If $a = \inf \tau = \sup \tau$ and $\text{first}(\pi) \neq \infty$, then there is obviously a contradiction. Otherwise, either (i) $a = \inf \tau$ and $\Delta(\pi_{\text{first}(\pi)}) < 0$ or (ii) $a = \sup \tau$ and $\Delta(\pi_{\text{first}(\pi)}) > 0$. We only consider (ii) as (i) is symmetric. Let $a \rightarrow_{\pi[1..\text{first}(\pi)\rightarrow \pi]} a'$. We have $a' = a$ by definition of $\text{first}(\cdot)$. Moreover,

Note that finding a maximal simple path is NP-complete, but this is not a problem since we allow non-simple paths.
a' + β · Δ(π_{first}(π)) > a' = a = sup τ for any β ∈ (0, 1]. Since exceeding sup τ is forbidden, we obtain the contradiction a ∉ enab(π[1..first(π)]) ⊇ enab(π).

**Corollary 4.** Given a ∈ Z and p, q ∈ Q, deciding whether a ∈ enab(Paths_{p,q}), or equivalently Post_{p,q}(a) ≠ ∅, is in NC^2.

**Proof.** We report “empty” if a ∉ τ. Otherwise, let

\[ S_0 := \{ π \in \text{Paths}_{p,q} | Δ^+(π) = Δ^-(π) = 0 \}, \]

\[ S_+ := \{ π \in \text{Paths}_{p,q} | \text{first}(π) ≠ ∞ \implies Δ(π_{first}(π)) > 0 \}, \]

\[ S_- := \{ π \in \text{Paths}_{p,q} | \text{first}(π) ≠ ∞ \implies Δ(π_{first}(π)) < 0 \}. \]

By Lemma 3, it suffices if one of the following holds:

(a) a ∉ [inf τ, sup τ] and Paths_{p,q} ≠ ∅;
(b) a = inf τ = sup τ and S₀ ≠ ∅;
(c) a = inf τ < sup τ and S₊ ≠ ∅;
(d) a = sup τ > sup τ and S₋ ≠ ∅.

All of the above can be checked in NC^2 by Lemma 2.

**3.2 Characterization of Reachability Sets**

As a step toward computing a representation of Post_{p,q}(a), we characterize Post_{p,q}(a) in terms of its closure. To this end, we note that admissible runs remain admissible whenever they are scaled down. Consequently, Post_{p,q}(a) is a closed interval that differs from Post_{p,q}(a) in at most three points.

**Proposition 5 (Adapted from [4, Lemma 4.2(c)]).** Let β ∈ (0, 1] and let ρ be an admissible run from configuration p(a). It is the case that run ρβ is also admissible from p(a).

**Proof.** We will show that for all i ∈ {0, . . . , |ρ|}, either a ≤ a + Δ(βρ[1..i]) ≤ a + Δ(ρ[1..i]) or a ≥ a + Δ(βρ[1..i]) ≥ a + Δ(ρ[1..i]). Since a + Δ(ρ[1..i]) ∈ τ holds by the admissibility of ρ from a, it follows that a + Δ(βρ[1..i]) ∈ τ, and so that ρβ is admissible.

By definition, we have a + Δ(ρ[1..i]) = a + βΔ(ρ[1..i]). Additionally, β ∈ (0, 1]. Hence, if Δ(ρ[1..i]) ≥ 0, then we have a + Δ(ρ[1..i]) ≥ a + Δ(βρ[1..i]) ≥ a. If Δ(ρ[1..i]) < 0, then a + Δ(βρ[1..i]) ≤ a + Δ(βρ[1..i]) < a, so we are done.

**Lemma 6.** For every b ∈ Post_{p,q}(a), it is the case that (a, b) ⊆ Post_{p,q}(a) and (b, a) ⊆ Post_{p,q}(a).

**Proof.** We only prove (a, b) ⊆ Post_{p,q}(a) as the other inclusion is symmetric. We assume that a < b as we are otherwise done. Let c ∈ (a, b). Since b ∈ Post_{p,q}(a), there exists b′ ∈ Post_{p,q}(a) such that b′ ∈ [c, b]. Let ρ be an admissible run from p(a) to q(b′). By definition, Δ(ρ) = b′ − a. Let β := (c − a)/(b′ − a) ∈ (0, 1]. By Proposition 5, ρβ is admissible from p(a). Since Δ(βρ) = c − a, this concludes the proof of the main claim.

**Corollary 7.** Set Post_{p,q}(a) is a closed interval. Moreover,

\[ \text{Post}_{p,q}(a) \setminus \text{Post}_{p,q}(a) \subseteq \{ \inf \text{Post}_{p,q}(a), a, \sup \text{Post}_{p,q}(a) \}. \]

**Proof.** Let b := inf Post_{p,q}(a) and c := sup Post_{p,q}(a). For the sake of contradiction, suppose there is some v ∈ Post_{p,q}(a) \ Post_{p,q}(a) such that v ∉ [b, a, c]. By Lemma 6, we have (a, b) ∪ (a, c) ∪ (b, a) ∪ (c, a) ⊆ Post_{p,q}(a) ⊆ Post_{p,q}(a). Since v ∈ (b, c) \ {a}, we obtain v ∈ Post_{p,q}(a), which is a contradiction.
3.3 Identifying the Endpoints

We now show that a representation of the interval $\overline{\text{Post}_{p,q}(a)}$ can be obtained by identifying its endpoints in NC$^2$. Some simple observations follow from Proposition 5 and Lemma 6:

**Proposition 8.** The following statements hold:

(a) If $\inf \tau \notin [-\infty, \infty]$ and $\overline{\text{Post}_{p,q}(\inf \tau)} \neq \emptyset$, then $\inf \overline{\text{Post}_{p,q}(\inf \tau)} = \inf \tau$.

(b) If $\sup \tau \notin [-\infty, \infty]$ and $\overline{\text{Post}_{p,q}(\sup \tau)} \neq \emptyset$, then $\sup \overline{\text{Post}_{p,q}(\sup \tau)} = \sup \tau$.

(c) Let $v \in \tau \setminus (\inf \tau, \sup \tau)$ and let $\beta$ be a run. There exists $\varepsilon \in (0, 1]$ such that for all $\beta \in (0, \varepsilon]$ there exists $v_{\beta} > 0$ such that $v \rightarrow_{\beta} v_{\beta}$. Moreover, $\lim_{\beta \to 0} v_{\beta} = v$.

**Proof.**

(a) Let $a \equiv \inf \tau$ and $c \equiv \sup \overline{\text{Post}_{p,q}(a)}$. By Lemma 6, we have $(a, c) \subseteq \overline{\text{Post}_{p,q}(a)}$. Since the latter is closed by definition, we have $\inf \overline{\text{Post}_{p,q}(a)} = \inf \tau$.

(b) The proof is symmetric to (a).

(c) Since $v \notin (\inf \tau, \sup \tau)$, there is a small enough $\varepsilon \in (0, 1]$ such that $v + \varepsilon \cdot \Delta([1..i]) \in \tau$ for all $i \in \{1, \ldots, |\tau|\}$. By definition, $\varepsilon \rho$ is admissible from $v$. Let $v_{\beta} \equiv v + \varepsilon \cdot \Delta(\rho)$. By Proposition 5, $v \rightarrow_{\beta} v_{\beta}$ is admissible for all $\beta \in (0, \varepsilon]$. Moreover, $\lim_{\beta \to 0} v_{\beta} = \lim_{\beta \to 0} v + \varepsilon \cdot \Delta(\rho) = v$. $\square$

The two forthcoming lemmas characterize the endpoints of $\overline{\text{Post}_{p,q}(a)}$ through so-called admissible cycles. We say that a cycle $\theta$ is $(a, p, q)$-admissible if its first transition $t$ satisfies $\Delta(t) \neq 0$, $a \in \text{enab(Paths}_{p,q}(t))$, and $\text{Paths}_{p,q}(t) \neq \emptyset$. We say that such an admissible cycle is positive if $\Delta(t) > 0$, and negative if $\Delta(t) < 0$. Such cycles can be iterated to approach the endpoints of $\tau$, by scaling all transitions but $t$ arbitrarily close to zero.

**Lemma 9.** If $\overline{\text{Post}_{p,q}(a)} \neq \emptyset$ and $V$ has an $(a, p, q)$-admissible cycle $\theta$, then the following holds:

(a) $\inf \overline{\text{Post}_{p,q}(a)} = \inf \tau$, if $\theta$ is negative;

(b) $\sup \overline{\text{Post}_{p,q}(a)} = \sup \tau$, if $\theta$ is positive.

**Proof.** Let $\theta = t \pi$, where $t$ is the first transition of $\theta$ and $\pi$ is the remaining path. Let $r \equiv \text{in}(t)$. Since $a \in \text{enab(Paths}_{p,q}(r))$, there is an admissible run $\rho_1$ from $p(a)$ that ends in state $r$. Similarly, since $\text{Paths}_{p,q}(q) \neq \emptyset$, there is a run $\rho_2$ from $r$ to $q$.

We only show (b) as (a) is symmetric. We assume that $a < \sup \tau$, as otherwise we are done by Proposition 8(b). We make a case distinction on whether $\sup \tau = \infty$.

**Case** $\sup \tau \neq \infty$. We must show that we can reach values arbitrarily close to $\sup \tau$, i.e., that for every $\varepsilon \in (0, 1]$, there exists a value $b \in [\sup \tau - \varepsilon, \sup \tau]$ and a run $p(a) \rightarrow_{\rho_2} q(b)$.

By Proposition 5 and $a < \sup \tau$, we have $p(a) \rightarrow_{t(\rho_2)} r(a')$ for some $a' < \sup \tau$. Let

$$m \equiv \sum_{i=1}^{|\pi|} |\Delta(\pi_i)|, \alpha_t \equiv \frac{\varepsilon}{4|\Delta(t)|} \text{ and } \alpha_\pi \equiv \frac{\varepsilon}{4m + 1}.$$

Let $\rho_2 \equiv \alpha_t t \alpha_\pi \pi$. By definition, we have $\Delta(\alpha_t t) = \varepsilon/4$ and $|\Delta(\alpha_\pi \pi [1..i])| < \varepsilon/4$ for all $i \in \{1, \ldots, |\pi|\}$. Consequently, it is the case that $\Delta(\rho_2 [1..i]) \in (0, \varepsilon/2)$ for all $i \in \{2, \ldots, |\rho_2|\}$.

Hence, there exists $k \geq 0$ such that $\rho_2^k$ is admissible from $a'$ and $\sup \tau - \varepsilon/2 \leq a' + \Delta(\rho_2^k) < \sup \tau$.

Therefore, we have

$$r(a') \rightarrow_{\rho_2^k} r(b') \text{ where } b' \in [\sup \tau - \varepsilon/2, \sup \tau].$$

By Proposition 8(c), we can scale the run $\rho_3$ so that it is admissible from $r(b')$ and reaches a value arbitrarily close to $b'$ in state $q$. More formally, there exists $\beta \in (0, 1]$ such that

$$r(b') \rightarrow_{\beta \rho_1} q(b) \text{ where } b \in [b' - \varepsilon/2, \sup \tau].$$
We are done since \( p(a) \rightarrow (1/2)p_i \rightarrow (1/2)p_i a \rightarrow \rho^* \rightarrow b \rightarrow \rho^* q(b) \) and \( b \in [\sup \tau - \varepsilon, \sup \tau] \).

Case \( \sup \tau = \infty \). We must show that we can reach arbitrarily large values. Let \( b \geq a \). For all \( \ell \geq 0 \), the run \( \rho^{\ell}_b := (1/2)p_1 \rho^{\ell}_2 a \) is admissible from \( a \), and such that \( \Delta(\rho^{\ell}_b) > 0 \). Thus, there exists \( \ell \geq 0 \) such that \( \Delta(\rho^{\ell}_b) \geq (b-a) + \Delta^-(\rho_3) \). We are done since

\[
a \rightarrow \rho^{\ell}_b a \rightarrow \rho^* b'' \rightarrow \rho^* a \rightarrow \rho^* b''',\quad \text{where} \quad b'' \geq a + \Delta^-(\rho_3) \quad \text{and} \quad b''' \geq b.
\]

\[\square\]

**Lemma 10.** Let \( \text{Post}_{p,q}(a) \neq \emptyset \), \( b := \inf \text{Post}_{p,q}(a) \) and \( c := \sup \text{Post}_{p,q}(a) \). If \( \mathcal{V} \) has no \((a, p, q)\)-admissible cycle that is

(a) negative, then \( b \neq -\infty \) and \( b = \max(\inf \tau, a + \min(\Delta^-(\pi) \mid \pi \in \text{Paths}_{p,q}, a \in \text{enab}(\pi))) \);

(b) positive, then \( c \neq +\infty \) and \( c = \min(\inf \tau, a + \max(\Delta^+(\pi) \mid \pi \in \text{Paths}_{p,q}, a \in \text{enab}(\pi))) \).

**Proof.** We only prove (b) as (a) is symmetric. Assume \( \mathcal{V} \) has no positive \((a, p, q)\)-admissible cycle. Let \( D^+ := \{\Delta^+(\pi) \mid \pi \in \text{Paths}_{p,q}, a \in \text{enab}(\pi)\} \). We show that \( \max D^+ \) is well defined. For the sake of contradiction, suppose that \( D^+ \) is infinite. By a pigeonhole argument, we obtain a run \( \rho \) admissible from \( a \) and such that \( \rho \) contains at least two occurrences of a transition \( t \) with \( \Delta(t) > 0 \). Let \( \text{path}(\rho) = t\pi t'\pi'' \), where \( \pi, \pi', \pi'' \) are paths. The cycle \( 0 := t\pi' \) is a positive admissible cycle, which yields a contradiction.

Note that \( c \leq \min(\sup \tau, a + \max D^+) \), so \( c \neq +\infty \). It remains to show that \( c = \min(\sup \tau, a + \max D^+) \). Let \( \pi \in \text{Paths}_{p,q} \) be such that \( a \in \text{enab}(\pi) \) and \( \Delta^+(\pi) = \max D^+ \). By definition, there exists a run \( \rho = a^\varepsilon \alpha_1^\varepsilon \cdots a_n^\varepsilon \) admissible from \( p(a) \) and such that \( \text{path}(\rho) = \pi \). Since \( a \in \tau \), there exists \( \lambda = \min(a, \{0, 1\}) \) such that \( a + \lambda \cdot \max D^+ = \min(\sup \tau, a + \max D^+) \). For all \( \varepsilon \in (0, 1) \), let \( \rho^\varepsilon := a^\varepsilon \alpha_1^\varepsilon \cdots a_n^\varepsilon \) be the run such that

\[
\alpha_i^\varepsilon := \begin{cases} (1 - \varepsilon) \cdot \lambda & \text{if } \Delta(t_i) \geq 0, \\ \varepsilon \cdot (1/|\Delta(t_i)|) \cdot (1/n) & \text{otherwise.} \end{cases}
\]

Informally, if we were allowed to scale transitions by \( 0 \), then we would be done by using \( p_0 \) from \( a \), as it would never decrease and reach exactly \( a + \lambda \cdot \max D^+ = \min(\sup \tau, a + \max D^+) \).

Formally, we choose a small \( \varepsilon \in (0, 1) \) as follows. If \( a > \inf \tau \), then we pick \( \varepsilon \) so that \( a - \varepsilon \geq \inf \tau \). Otherwise, we pick \( \varepsilon \) so that \( (1 - \varepsilon) \cdot \lambda \geq \varepsilon \). We claim that the run \( \rho^\varepsilon \) is admissible from \( a \) for every \( \delta \in (0, \varepsilon] \). First note that the top guard is never exceeded since \( a + \Delta^+(\rho^\varepsilon) = a + (1 - \delta) \cdot \lambda \cdot \max D^+ \leq a + \lambda \cdot \max D^+ \leq \sup \tau \). Let us now consider the bottom guard.

If \( a > \inf \tau \), then \( a + \Delta^-(\rho^\varepsilon) \geq a - \delta \geq a - \varepsilon \geq \inf \tau \). Otherwise, if \( a = \inf \tau \), then either \( \Delta^-(\rho^\varepsilon) = \Delta^+(\rho^\varepsilon) = 0 \), in which case admissibility is trivial, or the first transition \( t_i \) such that \( \Delta(t_i) \neq 0 \) is such that \( \Delta(t_i) \geq 1 \). In that case, the following holds for every \( j \geq i \):

\[
a + \Delta(\rho^\varepsilon[1..j]) = a + (1 - \delta) \cdot \lambda \cdot \Delta(t_i) + \Delta^-(\rho^\varepsilon[i + 1..j]) \geq a + (1 - \varepsilon) \cdot \lambda + \Delta^-(\rho^\varepsilon[i + 1..j]) \geq a + (1 - \varepsilon) \cdot \lambda - \varepsilon \geq a = \inf \tau.
\]

This shows the admissibility of \( \rho^\varepsilon \). Thus, for all \( \delta \in (0, \varepsilon] \), we have \( a \rightarrow \rho^\varepsilon \rightarrow a_\delta \), where \( a_\delta \geq \min(\sup \tau, a + \max D^+) - \delta \cdot \lambda \cdot \max D^+ \). We are done since \( \lim_{\delta \to 0} a_\delta = \min(\sup \tau, a + \max D^+) \). \[\square\]

In the forthcoming propositions, we show how the previous characterizations can be turned into \( \text{NC}^2 \) procedures.

**Lemma 11.** On input \( a \in \mathbb{Z} \) and \( p, q \in Q \), deciding if \( \mathcal{V} \) has a positive or negative \((a, p, q)\)-admissible cycle is in \( \text{NC}^2 \).
PROOF. We consider the positive case; the negative one is symmetric. Testing whether there is a positive \((a, p, q)-\text{admissible cycle beginning with a transition} t\) with \(\Delta(t) > 0\) amounts to testing whether (i) \(a \in \text{enab}(\text{Paths}_{p, \text{in}}(t))\), (ii) \(\text{Paths}_{\text{in}}(t) \neq \emptyset\), and (iii) \(\text{Paths}_{\text{out}}(t, \text{in}(t)) \neq \emptyset\). Condition (i) can be checked in \(\text{NC}^2\) by Corollary 4. Conditions (ii) and (iii) are graph reachability queries, which can be tested in \(\text{NL} \subseteq \text{NC}^2\). There are at most \(|T|\) transitions with positive effect, so the conditions can be tested in parallel for each \(t \in T\). \(\square\)

**Proposition 12.** On input \(a \in \mathbb{Z}\) and states \(p, q\), the values \(\inf \text{Post}_{p, q}(a)\) and \(\sup \text{Post}_{p, q}(a)\) can be computed in \(\text{NC}^2\).

**Proof.** We explain how to compute \(c := \sup \text{Post}_{p, q}(a)\) in \(\text{NC}^2\). The procedure for \(\inf \text{Post}_{p, q}(a)\) is symmetric. By Corollary 4, testing whether \(\text{Post}_{p, q}(a) = \emptyset\) is in \(\text{NC}^2\). If it holds, then trivially \(c = -\infty\). Otherwise, assume that \(\text{Post}_{p, q}(a) \neq \emptyset\), and hence \(a \in \tau\). Additionally, if \(\beta = \sup \tau\), then \(c = \sup \beta\) by Proposition 8(b). So, we assume \(a < \sup \beta\).

By Lemma 11, it can be decided in \(\text{NC}^2\) whether there exists a positive \((a, p, q)-\text{admissible cycle. If such a cycle exists, then} c = \sup \beta\) by Lemma 9. Otherwise, by Lemma 10, we have \(c = \min(\sup \beta, a + \max D^+)\), where \(D^+ := \{\Delta^+(\pi) \mid \pi \in \text{Paths}_{p, q}, a \in \text{enab}(\pi)\}\).

Consider a path \(\pi \in \text{Paths}_{p, q}\) that satisfies \(a \in \text{enab}(\pi)\) and which can be decomposed as \(\pi = \sigma \theta a^\tau\), where \(\theta = \tau\) is a cycle. As \(\theta\) is \((a, p, q)-\text{admissible by definition, it cannot contain a positive transition. Otherwise,} V\) would admit a positive \((a, p, q)-\text{admissible cycle, which is a contradiction. Hence,} \Delta^+(\pi) = \Delta^+(\sigma \sigma^\tau\). Recall that \(a \neq \sup \tau\). Then \(a \in \text{enab}(\sigma \sigma^\tau)\) follows from Lemma 3(a) or (c).

The above shows that there is a simple path \(\pi_{\text{max}}\) such that \(\max D^+ = \Delta^+(\pi_{\text{max}})\) and \(a \in \text{enab}(\pi_{\text{max}})\). Therefore, to obtain \max D^+, it is sufficient to compute \(\max E^+\), where \(E^+ := \{\Delta^+(\pi) \mid \pi \in \text{Paths}_{p, q}, a \in \text{enab}(\pi), |\pi| \leq |Q|\}\).

Now, let us make a case distinction based on whether \(a = \inf \tau\). Assume this is true. By Lemma 3, \(\max E^+ = \max\{\Delta^+(\pi) \mid \pi \in \text{Paths}_{p, q}, \text{first}(\pi) = \infty, |\pi| \leq |Q|\} \cup\{\Delta^+(\pi) \mid \pi \in \text{Paths}_{p, q}, \Delta(\pi_{\text{first}(\pi)}) > 0, |\pi| \leq |Q|\}\). By Lemma 2, the two sets that are joined in the expression can be computed in \(\text{NC}^2\). If \(a \neq \inf \tau\), then we have \(a \in \tau \setminus \text{(inf } \tau, \sup \tau)\), so by Lemma 3(a) all paths are admissible from \(a\), and hence \(\max E^+ = \max\{\Delta^+(\pi) \mid \pi \in \text{Paths}_{p, q} \wedge |\pi| \leq |Q|\}\). This value can be computed in \(\text{NC}^2\) by Lemma 2. \(\square\)

### 3.4 Computing the Representation

To obtain a representation of \(\text{Post}_{p, q}(a)\), it remains to explain how to check in \(\text{NC}^2\) which of the three limit elements \(\inf \text{Post}_{p, q}(a)\), \(\sup \text{Post}_{p, q}(a)\), and \(a\) belong to \(\text{Post}_{p, q}(a)\). Intuitively, for each of these three elements, membership is equivalent to the existence of paths satisfying some conditions.

**Proposition 13.** Let \(\text{Post}_{p, q}(a) \neq \emptyset\). It is the case that \(a \in \text{Post}_{p, q}(a)\) iff at least one of these conditions holds:

(a) there exists a path \(\pi \in \text{Paths}_{p, q}\) whose transitions are all zero, i.e., \(\Delta(\pi) = \Delta^+(\pi) = \Delta^-(\pi) = 0\);

(b) there exist \(\pi \in \text{Paths}_{p, q}\) and indices \(i, j\) such that \(\Delta(\pi_i) > 0\) and \(\Delta(\pi_j) < 0\). If \(a = \inf \tau\), then we also require that \(\Delta(\pi_k) = 0\) for all \(k < i\) and \(k > j\). Similarly, if \(a = \sup \tau\), then we also require \(\Delta(\pi_k) = 0\) for all \(k < j\) and \(k > i\).

**Proof.** If (a) holds, then trivially \(a \in \text{Post}_{p, q}(a)\). Assume (b) holds. Let \(\rho := \pi t_1 \ldots t_n\), where \(\pi = t_1 \ldots t_n\). Suppose \(a \notin \{\inf \tau, \sup \tau\}\). By Proposition 8(c), for all \(\beta\) small enough, it is the case
that \( a \to_{p} \alpha \beta \), where \(|a - a\beta| < 1/2\). Let \( p(a) = q_{0}(a_{0}), \ldots, q_{n}(a_{n}) = q(a_{\beta}) \) be the sequence of configurations witnessing \( a \to_{p} \alpha \beta \). Since \( n \) is fixed, we can choose \( \beta < 1/2 \) small enough so that \(|a_{i} - a| < 1/2\) for all \( i \). If \( a_{\beta} > a \), then we enlarge the coefficient of \( t_{j} \) to \( \alpha_{j} > \beta \) so that \((\alpha_{j} - \beta) \cdot (\Delta(t_{j}) = a - a_{\beta} \). By the choice of \( \beta \), we get an admissible run \( p' := \beta t_{1} \ldots \beta t_{j-1} \alpha_{j} t_{j} \beta t_{j+1} \ldots \beta t_{n} \) that satisfies \( a \to_{p'} a \). If \( a_{\beta} < a \), then we proceed analogously with index \( i \).

It remains to prove the case where \( a = \inf \tau \); the case where \( a = \sup \tau \) is symmetric. By assumption, we have \( \Delta(t_{k}) = 0 \) for all \( k < i \) and \( k > j \). For the sake of simplicity, assume \( \Delta(t_{1}) > 0 \) and \( \Delta(t_{n}) < 0 \). Let \( \alpha_{1} \in (0, 1) \) be such that \( \alpha_{1} \cdot \Delta(t_{1}) < 1/2 \). Let \( \rho_{1} := 1t_{2} \ldots t_{n-1} \). By Proposition 8(c), there exists \( \beta \in (0, 1) \) such that \( \inf \tau \to_{\alpha_{1} t_{1} \beta \rho_{1}} \delta \), where \( \delta < 1 \). Since \( \Delta(t_{1}) > 0 \), there exists \( \delta_{0} \in (0, 1) \) such that \( \alpha_{n} \cdot \Delta(t_{n}) = -\delta \). Thus, we have \( p(\inf \tau) \to_{\alpha_{1} t_{1} \beta \rho_{1} a_{n} t_{n}} q(\inf \tau) \).

\[ \Rightarrow \] Let \( p(a) \to_{p} q(a) \) and \( \pi := \text{path}(\rho) \). Suppose \( \pi \) does not hold. If all transitions of \( \pi \) are positive, then we would obtain the contradiction \( p(a) \to_{p} q(a') \) with \( a' > a \). Similarly, all transitions cannot be negative. For the specific case where \( a = \inf \tau \), observe that if the first nonzero transition is negative, then \( \rho \) cannot be admissible. Similarly, if the last nonzero transition is positive, then \( p(\inf \tau) \to_{p} q(\delta) \) for some \( \delta \) greater than \( \inf \tau \). The reasoning for the case \( a = \sup \tau \) is symmetric.

It is easy to see that both conditions of the prior proposition can be checked in \( \text{NC}^{2} \) by Lemma 2.

We introduce and prove a similar characterization of \( \text{sup Post}_{p,q}(a) \in \text{Post}_{p,q}(a) \):

Proposition 14. Let \( \text{Post}_{p,q}(a) \neq 0 \), \( b := \inf \text{Post}_{p,q}(a) \) and \( c := \sup \text{Post}_{p,q}(a) \). If \( b < a < c \) and \( c \in \tau \), then \( c \in \text{Post}_{p,q}(a) \) if and only if there is a state \( r \) and a path \( \sigma \in \text{Paths}_{r,q} \) that satisfy \( \Delta^{+}(\sigma) > 0 \), \( \Delta^{-}(\sigma) = 0 \), and either of the following:

(i) there exists a path \( \sigma' \in \text{Paths}_{p,r} \) such that \( |\sigma|, |\sigma'| \leq |Q| \), \( \Delta^{-}(\sigma') = 0 \) and \( \Delta^{+}(\pi) \geq c - a \), where \( \pi := \sigma' \sigma \);

(ii) there exists a path \( \sigma' \in \text{Paths}_{p,r} \) such that \( |\sigma|, |\sigma'| \leq |Q| \) and \( \Delta^{+}(\pi) > c - a \), where \( \pi := \sigma' \sigma \);

(iii) there is a positive \( (a, p, r) \)-admissible cycle \( \theta \).

Proof. \( \Rightarrow \) Assume \( c \in \text{Post}_{p,q}(a) \). There is a run \( \rho \) such that \( p(a) \to_{p} q(c) \). Let \( \rho' \) be the run obtained from \( \rho \) by repeatedly removing a cycle \( \theta \) with \( \Delta^{+}(\theta) = 0 \), until no further possible. Let \( \pi := \text{path}(\rho') \). We have \( \Delta^{+}(\pi) \geq \Delta(\rho') \geq \Delta(\rho) = c - a \). Since \( c > a \), there is a maximal index \( i \) such that \( \Delta(\pi_{i}) > 0 \). Let \( r := \text{in}(\pi_{i}) \), \( \sigma' := \pi_{i+1} \cdots \pi_{n} \) and \( \sigma := \pi_{i} \cdots |\pi| \). Note that \( \sigma' \in \text{Paths}_{p,r} \) and \( \sigma \in \text{Paths}_{r,q} \). Moreover, \( \Delta^{+}(\sigma) > 0 \) holds by maximality of \( i \). It must also be the case that \( \Delta^{-}(\sigma) = 0 \). Indeed, otherwise the last nonzero transition \( t_{i} \) of \( \sigma \), and consequently of \( \rho \), would be negative. Hence, this would contradict \( c = \text{sup Post}_{p,r}(a) \) as we could reach values arbitrarily close to \( c + \varepsilon \) for some \( \varepsilon > 0 \) by scaling \( t \) arbitrarily close to zero. Observe that if \( \Delta^{+}(\pi) = c - a \), then \( \Delta(\rho) = \Delta^{+}(\pi) = c - a \), which implies \( \Delta^{-}(\rho) = \Delta^{-}(\pi) = 0 \). Therefore, if \( |\sigma|, |\sigma'| \leq |Q| \), we have shown (i) or (ii).

Otherwise, \( \pi \) is a non-simple path. So, by our past cycle, \( \pi \) contains a cycle \( \theta \) with \( \Delta^{+}(\theta) > 0 \). Let us reorder \( \theta \) and \( \theta' \) so that the first transition \( \theta' \) satisfies \( \Delta(t_{0}) > 0 \). We have \( a \in \text{enab} \left( \text{Paths}_{p,q}(\text{in}(t_{0})) \right) \) as state \( t_{0} \) occurs on the original run \( \rho \) that leads to state \( q \). Moreover, \( \text{Paths}_{\text{in}(t),r} \neq 0 \) holds by maximality of \( i \). Thus, \( \theta' \) is a positive \( (a, p, r) \)-admissible cycle. Hence, we have shown (iii) holds.

\( \Leftarrow \) If (i) holds, then \( \Delta^{+}(\pi) = c - a \) or \( \Delta^{+}(\pi) > c - a \). The latter case is subsumed by (ii), and in the former case we are done as \( a \to_{\pi} c \) due to \( \Delta^{-}(\pi) = 0 \). If (iii) holds, then since \( \theta \) is a positive \( (a, p, r) \)-admissible cycle—and hence \((a, p, q)\)-admissible—Lemma 9(b) yields \( \sup \text{Post}_{p,q}(a) = c = \sup \tau \). Thus, there exists \( \varepsilon \in [0, 1] \) such that \( c - \varepsilon \in \text{Post}_{p,q}(a) \). Note that \( \Delta^{+}(\sigma) > 0 \) implies \( \Delta^{+}(\sigma) \geq 1 \), since we assume transition effects to be integral. By \( \Delta^{+}(\sigma) \geq 1 \) and \( \Delta^{-}(\sigma) = 0 \), we have

\[ p(a) \to_{r} (c - \varepsilon) \to_{\beta \rho_{a}} q(c) \text{ where } \beta := \varepsilon/\Delta^{+}(\sigma). \]
If (ii) holds, then we proceed as follows. Recall that $b < a < c$. Therefore, $a \not\in \{\inf \tau, \sup \tau\}$, since $\inf \tau \leq b$ and $c \leq \sup \tau$ by definition of $b$ and $c$. Due to $a \not\in \{\inf \tau, \sup \tau\}$, we can scale the negative transitions of $\sigma'$ arbitrarily close to zero and scale its positive transitions so that either $a + \Delta^+(\sigma') - \varepsilon \in \text{Post}_{p,r}(a)$ or $\tau - \varepsilon \in \text{Post}_{p,r}(a)$ for some $\varepsilon \in (0, 1]$. Since $\Delta^-(\sigma) \geq c - a - \Delta^+(\sigma') + 1$, $\Delta^+(\sigma) \geq 1$, and $\Delta^-(\sigma') = 0$, we can derive either $c \in \text{Post}_{r,q}(a)$ or $\tau \in \text{Post}_{r,q}(a)$. As the latter means $c = \sup \tau$, we are done proving the claim.

Finally, these characterizations allow us to conclude our first major result.

**Proposition 15.** On input $a \in \mathbb{Z}$ and $p, q \in Q$, computing $\text{Post}_{p,q}(a) \cap \{\inf \text{Post}_{p,q}(a), \sup \text{Post}_{p,q}(a), a\}$ is in NC$^2$.

**Proof.** By Corollary 4, $\text{Post}_{p,q}(a) \neq \emptyset$ can be tested in NC$^2$. Thus, we assume that it is nonempty. By Proposition 13, it holds that $a \in \text{Post}_{p,q}(a)$ iff at least one of these conditions holds:

- (a) there exists a path $\pi \in \text{Paths}_{p,q}$ whose transitions are all zero, i.e., $\Delta(\pi) = \Delta^+(\pi) = \Delta^-(\pi) = 0$;
- (b) there exist $\pi \in \text{Paths}_{p,q}$ and $i, j$ such that $\Delta(\pi_i) > 0$ and $\Delta(\pi_j) < 0$. If $a = \inf \tau$, then we also require $\Delta(\pi_k) = 0$ for all $k < i$ and $k > j$. Similarly, if $a = \sup \tau$, then we also require $\Delta(\pi_k) = 0$ for all $k < j$ and $k > i$.

Note that (b) is trivially unsatisfiable if $a = \inf \tau = \sup \tau$.

It remains to argue that both conditions can be checked in NC$^2$. We can check condition (a) in NC$^2$ via Lemma 2(b). If $a \not\in \{\inf \tau, \sup \tau\}$, then condition (b) can be checked in NC$^2$ via Lemma 2(a). If $a \in \{\inf \tau, \sup \tau\}$, then condition (b) can be checked in NC$^2$ via Lemma 2(d) and Lemma 2(c).

Let $b := \inf \text{Post}_{p,q}(a)$ and $c := \sup \text{Post}_{p,q}(a)$, which can be computed in NC$^2$ by Proposition 12. We check whether $b = c$. If it is, we return $\{b, c\}$ since $\text{Post}_{p,q}(a) \neq \emptyset$. Otherwise, we explain how to check whether $c \in \text{Post}_{p,q}(a)$; the case of $b$ can be handled symmetrically. We assume that $b < a < c$, as the first half of the proof handles the case $a \in \{b, c\}$ when checking membership of $a$.

If $c \not\in \tau$, then $c \not\in \text{Post}_{p,q}(a)$. Otherwise, by Proposition 14, it holds that $c \in \text{Post}_{p,q}(a)$ iff there is a state $r \in Q$ and a path $\sigma \in \text{Paths}_{r,q}$ that satisfy $\Delta^+(\sigma) > 0, \Delta^-(\sigma) = 0$ and either of the following holds:

- (i) there exists a path $\sigma' \in \text{Paths}_{p,r}$ such that $|\sigma|, |\sigma'| \leq |Q|, \Delta^-(\sigma') = 0$ and $\Delta^+(\pi) \geq c - a$, where $\pi := \sigma' \sigma$;
- (ii) there exists a path $\sigma' \in \text{Paths}_{p,r}$ such that $|\sigma|, |\sigma'| \leq |Q|$ and $\Delta^+(\pi) > c - a$, where $\pi := \sigma' \sigma$;
- (iii) there is a positive $(a, p, r)$-admissible cycle $\theta$.

It remains to show that the conditions can be tested in NC$^2$. There are $|Q|$ choices for state $r$, so we can test the conditions for all choices in parallel. Let $S := \{\sigma \in \text{Paths}_{r,q} \mid \Delta^+(\sigma) > 0 \text{ and } \Delta^-(\sigma) = 0\}$. We first check whether $S \neq \emptyset$, which can be done in NC$^2$ by Lemma 2(a)-(b). Moreover, condition (iii) can be checked in NC$^2$ by Lemma 11. We proceed as follows to check (i). Let

$$W := \{\Delta^+(\sigma) \mid \sigma \in S, |\sigma| \leq |Q|\},$$

$$W' := \{\Delta^+(\sigma') \mid \sigma' \in \text{Paths}_{p,r}, |\sigma'| \leq |Q|, \Delta^-(\sigma') = 0\}.$$

By Lemma 2, we can compute $m := \max W + \max W'$ in NC$^2$ and check that $m \geq c - a$. Lastly, we define $W'' := \{\Delta^+(\sigma') \mid \sigma' \in \text{Paths}_{p,r}, |\sigma'| \leq |Q|\}$ and test whether $\max W + \max W'' > c - a$ to verify condition (ii).

\[\Box\]
THEOREM 16. Given $a, a' \in \mathbb{Z}$ and $p, q \in Q$, the following can be done in NC$^2$: obtaining a representation of $\text{Post}_{p,q}(a)$ and testing whether $a' \in \text{Post}_{p,q}(a)$.

PROOF. By Proposition 12, we can compute $b := \inf \text{Post}_{p,q}(a)$ and $c := \sup \text{Post}_{p,q}(a)$ in NC$^2$. By Corollary 7 and Proposition 15, the set $S := \text{Post}_{p,q}(a) \setminus \text{Post}_{p,q}(a)$, of size at most three, can be computed in NC$^2$. By Corollary 7, this yields the representation $\text{Post}_{p,q}(a) = [b, c] \setminus S$. Thus, $a' \in \text{Post}_{p,q}(a)$ iff $b \leq a' \leq c$ and $a' \notin S$. □

3.5 Equality Tests

A GG-COCA with equality tests is a tuple $\mathcal{V} = (Q, T, \tau, \phi)$, where $(Q, T, \tau)$ is a GG-COCA and $\phi: Q \to \{(z, z) \mid z \in Q\} \cup \{(-\infty, \infty)\}$. We say that a run of $\mathcal{V}$ is admissible if each of its configurations $q(a)$ satisfies $a \in \tau \cap \phi(q)$.

Using the previous results, we can extend the NC$^2$ membership of the reachability problem $p(a) \rightarrow_q q(b)$ to GG-COCA with equality tests. The proof relies on the fact that each equality test is passed by exactly one configuration. For this reason, we can construct a reachability graph between equality tests using a quadratic number of GG-COCA reachability queries.

Let us assume that $p$ has no incoming edges; if it does, we can simply add new initial state $p'$ and a single transition $(p', 0, p)$. Similarly, we can assume $q$ has no outgoing edges.

We will reason about reachability in $\mathcal{V}$, where we avoid all equality tests. For every pair of states $p', q' \in Q$, let us define the GG-COCA $\mathcal{V}_{p',q}' := (Q_{p',q}', T_{p',q}')$, where $Q_{p',q}' := \{s \in Q \mid \phi(s) = Q\} \cup \{p', q'\}$. We treat $p'$ as a dedicated input state, and $q'$ as a dedicated output state. That is,

$$T_{p',q}' := \{t \in T \mid \text{in}(t) \in Q_{p',q}' \setminus \{q'\}, \text{out}(t) \in Q_{p',q}' \setminus \{p'\}\}.$$

Let us define a graph $\mathcal{G} := (V, E)$, where $V := \{(p(a), q(b)) \cup \{z, z\} \mid r \in Q, \phi(r) = (z, z), z \in \tau\}$. If $a \notin \tau \cap \phi(p)$ or $b \notin \tau \cap \phi(q)$, then we trivially conclude that $p(a)$ cannot reach $q(b)$. Hence, $|V| \leq |Q|$ holds. Intuitively, the nodes of $\mathcal{G}$ correspond to the initial and final configurations plus, for each equality test, the configuration that passes this test. Let us define $E := \{(p'(x), q'(y)) \mid p'(x) \rightarrow_q q'(y) \in \mathcal{V}_{p',q}'\}$.

LEMMA 17. It is the case that $p(a) \rightarrow_q q(b)$ in $\mathcal{V}$ if and only if there is a path from $p(a)$ to $q(b)$ in $\mathcal{G}$.

PROOF. We show the “if” direction first. Assume that $p(a) \rightarrow_{\rho} q(b)$ for some run $\rho$. Without loss of generality, we assume that no configuration repeats when starting at $p(a)$ with $\rho$; otherwise, we can simply shorten $\rho$. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the unique maximal decomposition of $\rho$ into runs such that $\phi(\text{out}(\sigma_i)) \neq Q$ for all $1 \leq i < n$. For ease of notation, let $q_i := \text{out}(\sigma_i)$. It holds that $p(a) \rightarrow_{\sigma_1} q_1(a_1) \rightarrow_{\sigma_2} q_2(a_2) \cdots \rightarrow_{\sigma_n} q_n(b)$, where $\phi(q_i) = [a_i, a_i]$ for all $i$. Since $\sigma_1, \sigma_2, \ldots, \sigma_n$ is the maximal decomposition, $\phi(\text{out}((\sigma_i)_j)) = Q$ holds for all $j < |\sigma_i|$

Additionally, recall that $p$ has no incoming edges and $q$ has no outgoing edges. Hence, the following holds:

$$p(a) \rightarrow_{\sigma_1} q_1(a_1) \in \mathcal{V}_{p,q,1},$$
$$q_{i-1}(a_{i-1}) \rightarrow_{\sigma_i} q_i(a_i) \in \mathcal{V}_{q_{i-1},q_i} \text{ for all } 1 \leq i < n,$$
$$q_{n-1}(a_{n-1}) \rightarrow_{\sigma_n} q_n(b) \in \mathcal{V}_{q_{n-1},q}.$$

We are done as $p(a)q_1(a_1) \cdots q_{n-1}(a_{n-1})q(b)$ is a path of $\mathcal{G}$.

It remains to show the “only if” direction. Suppose there is a path $p(a)q_1(a_1) \cdots q_{n-1}(a_{n-1})q(b)$ in $\mathcal{G}$. Note that if $p'(a') \rightarrow_q q'(b')$ in $\mathcal{V}_{p',q'}$, then by definition we also have $p'(a') \rightarrow_q q'(b')$ in $\mathcal{V}$.
The reachability problem for GG-COCA with equality tests is in NC\(^2\).

**Theorem 18.** The reachability problem for GG-COCA with equality tests is in NC\(^2\).

**Proof.** Let us first argue that the graph \(G\) can be constructed in NC\(^2\). There are at most \(|Q|\) nodes in \(G\), and hence at most \(|Q|^2\) edges. Note that an edge \((p'(x), q'(y))\) is present iff \(p'(x) \rightarrow^* q'(y)\) in \(V'_{p', q'}\), which can be decided in NC\(^2\) by Theorem 16, as \(V'_{p', q'}\) is a GG-COCA. By running these \(|Q|^2\) queries in parallel, it follows that \(G\) can be obtained in NC\(^2\).

Once the graph \(G\) has been constructed, by Lemma 17, it suffices to test reachability from \(p(a)\) to \(q(b)\) in \(G\). Since graph reachability is in NL \(\subseteq\) NC\(^2\), we are done. \(\square\)

### 4 COCA REACHABILITY

We now turn to the reachability problem \(p(a) \rightarrow^* q(b)\) for a COCA \(W = (Q, T, \tau)\). In contrast to GG-COCA, the set \(\text{Post}_{p, q}(a)\) does not necessarily admit a decomposition into a constant number of intervals. Nevertheless, we show that it can always be decomposed into a polynomial number of intervals with respect to the number of states (see Section 4.1). Then, we present a formalization of the natural forward computation one would employ to obtain under-approximations of the reachability function (see Section 4.2), which can be efficiently stored due to the aforementioned fact. Finally, in an approach reminiscent of the Bellman-Ford algorithm, we introduce a way of “accelerating” our forward computation of under-approximations so as to reach a fixed point in finite time. Concretely, in Section 4.3, we give sufficient (and efficient-to-check) conditions for the existence of certain cycles. We then propose an acceleration scheme based on those cycles. Our polynomial-time algorithm is summarized in Section 4.4.

Throughout this section, we write \(I(R)\) to denote the unique decomposition of a set \(R \subseteq Q\) into maximal disjoint nonempty intervals. For example, \(I([3, 4] \cup (4, 5) \cup (5, +\infty)) = \{[3, 5], (5, +\infty)\}\) and \(I(\emptyset) = \emptyset\).

#### 4.1 Controlling the Number of Intervals

We will prove that for every \(k\) the set \(\{b \in Q \mid p(a) \rightarrow^* q(b), |p| = k\}\) decomposes into at most \(|Q|^{O(1)}\) intervals. (Note that the bound is independent of \(k\).) To do so, we will bound the size of the decomposition of sets obtained by updating \(A = [a, a]\) with operations that suffice to implement continuous counter updates and guard tests. More precisely, these are Minkowski sums, intersections (with elements of \(L = \{\tau(q) \mid q \in Q\}\)), and unions (with sets constructed similarly). For technical reasons, we also consider a fourth operation.

Let us fix a bounded interval \(A \in \Gamma\) and \(L \subseteq \Gamma\). We write \(P_L := \{\inf I, \sup I \mid I \in L\}\) to denote the set of endpoints within \(L\). Further, for two sets \(B, B' \subseteq Q\), we write \(B + B' = \{b + b' \mid b \in B, b' \in B'\}\) to mean the Minkowski sum of \(B\) and \(B'\). We define the MIUN-closure (short for Minkowski sum, Intersection, Union, and New), of interval \(A\) w.r.t. \(L\), as the smallest collection \(C \subseteq 2^L\) such that \(A \in C\) and

- **M:** if \(B \in C\) and \(z \in Q_{>0}\), then \(B + (0, z), B + [-z, 0] \in C\);
- **I:** if \(B \in C\) and \(L \in L\), then \(B \cap L \in C\);
- **U:** if \(B, B' \in C\), then \(B \cup B' \in C\);
- **N:** if \(B \in C\) and \(I \in \Gamma\) s.t. \(I \cap P_L \neq \emptyset\), then \(B \cup I \in C\).

The forthcoming lemma forms the basis of our bound. It is based on so-called indicator functions that give us, for every interval \(I\), the set of endpoints of \(L\) and \(A\) that belong to the closure of \(I\). As we will see later, the set of endpoints needed to analyze COCA is small. Furthermore, all
MIUN-operations are set endpoints such that sets of D decompose into intervals whose closure contains at least one such endpoint.

More formally, for all $B \subseteq C$, let $\phi_B : I(B) \to 2^{P_L \cup P_A}$ be the function defined as $\phi_B(I) := \overline{I} \cap (P_L \cup P_A)$, where $P_A = \{\inf A, \sup A\}$.

**Lemma 19.** We have $\phi_B(I) \neq \emptyset$ for all $B \subseteq C$ and $I \in I(B)$.

**Proof.** We proceed by induction on the definition of MIUN-closures. We define $C_0 := \{A\}$ and $C_{i+1}$ as $C_i$ extended with all sets obtained by applying the MIUN-operations applied to any $B, B' \in C_i$. We will show that the lemma holds for all $C_i$, which will conclude the proof since $C = \bigcup_{i \in \mathbb{N}} C_i$.

We have $I(A) = \{A\}$ and the claim holds since $P_A \subseteq \phi_A(A)$. For the induction step, we suppose the claim holds for $C_i$. We have to prove that for all $C \subseteq C_{i+1}$ and all $I \in I(C)$ it holds that $\phi_C(I) \neq \emptyset$. Notice that this is trivial if $C$ is obtained from $B \in C_i$ by application of the New operation.

First, we consider the Minkowski sum. Consider some $B \subseteq C_i$ with the function $\phi_B$ and let $I := ((0, z], [-z, 0])$ for some $z \in \mathbb{Q}_{>0}$. Let $C := B + I$. For all $J \in I(C)$ there exists $K_B \in I(B)$ such that $K_B \subseteq J$. Thus, $\phi_B(K_B) \subseteq \phi_C(J)$ and the claim holds by the inductive hypothesis for $\phi_B$.

Second, we consider intersections. We only deal with intervals of the form $[\ell, +\infty)$, $(\ell, +\infty)$, $(-\infty, \ell)$, or $(-\infty, \ell]$, since intersection with any interval can be expressed by at most two consecutive intersections with intervals of this form. Let $B \in C_i$ and $L \subseteq \mathbb{L}$. Suppose that $\overline{L} = [\ell, +\infty)$ and let $C := B \cap L$. Recall that $\ell \in P_L$. Observe that $I(B)$ contains at most two intervals $I$ such that $\ell \in \overline{I}$. If such an $I$ exists, then $\ell \in \phi_C(I \cap \overline{L})$. For all other intervals $J \in I(B)$, we have that $J \cap L$ is either $J$ or $\emptyset$. If the intersection is nonempty, then $\phi_C(J) = \phi_B(J)$ and the claim holds by inductive hypothesis. If $\overline{L}$ is instead of the form $(-\infty, \ell]$, then we proceed similarly.

Finally, we consider unions. Let $B, B' \subseteq C_i$ and $I \in I(B \cup B')$. By definition, there exists $J \subseteq I(B) \cup I(B')$ with $J \subseteq I$. Therefore, either $\phi_B(J)$ or $\phi_B'(J)$ is nonempty and contained in $\phi_{B \cup B'}(I)$. 

**Lemma 20.** For every set $B \subseteq \mathbb{Q}$ and every pairwise distinct interval $I_1, I_2, I_3 \in I(B)$, it is the case that $\overline{I_1} \cap \overline{I_2} \cap \overline{I_3} = \emptyset$.

A point can belong to at most one interval among disjoint intervals. Moreover, a point can belong to at most two closures; e.g., consider $[0, 1)$ and $(1, 2]$. This is no longer possible for three intervals due to maximality of intervals in $I(B)$. Thus, the proof of Lemma 20 follows from a simple case analysis.

Now, we show that if $\mathbb{L}$ is finite, and then there is a polynomial bound on the number of intervals within the decomposition of any set from the MIUN-closure $C$. More formally:

**Lemma 21.** If $\mathbb{L}$ is finite, then $I(B)$ consists of at most $4(|\mathbb{L}| + 1)$ intervals, for every $B \subseteq C$.

**Proof.** By Lemma 20, there are at most two pairwise disjoint intervals that share a point in their closure. By Lemma 19, the indicator function guarantees that $\overline{I} \cap (P_L \cup P_A) \neq \emptyset$ for all $J \in I(B)$. Thus, $I(B)$ has at most $2(|\mathbb{L}| + 2)$ intervals. Otherwise, by the pigeonhole principle, a point of $P_L \cup P_A$ would belong to at least three closures of intervals from $I(B)$.

### 4.2 Approximations of the Reachability Function

It will be convenient to manipulate mappings from states to (under-approximations of) their reachability functions. We consider the mappings $\mathcal{R}_Q := \{S : Q \to 2^S\}$. An example of such a mapping is $\text{Reach}_{p_0}^a$, defined as $\text{Reach}_{p_0}^a(q) := \text{Post}_{p_0}^a(q)$. Given $S, S' \in \mathcal{R}_Q$, we write $S \preceq S'$ if $S(q) \subseteq S'(q)$ for all $q \in Q$. We seek to define a sequence of mappings $S_0 \preceq S_1 \preceq \cdots$ such that $S_n = \text{Reach}_{p_0}^a$ for some $n \in \mathbb{N}$. 

ACM Transactions on Computational Logic, Vol. 24, No. 1, Article 3. Publication date: January 2023.
For all states \( q \in Q \), we define the successor mapping-update function \( \text{Succ}_q : \mathcal{R}_Q \to 2^Q \) as follows:

\[
\text{Succ}_q(S) := S(q)
\cup \bigcup \{(S(r) + (0, z)) \cap \tau(q) \mid (r, z, q) \in T, z > 0\}
\cup \bigcup \{(S(r) + [z, 0)) \cap \tau(q) \mid (r, z, q) \in T, z < 0\}
\cup \bigcup \{(S(r) \cap \tau(q) \mid (r, 0, q) \in T\}.
\]

Let \( \text{Succ} : \mathcal{R}_Q \to \mathcal{R}_Q \) be defined as \( \text{Succ}(S)(q) := \text{Succ}_q(S) \). Below, we state the key property enjoyed by \( \text{Succ} \). In words, its \( i \)-fold composition coincides with the set of configurations reachable via runs of length at most \( i \). It can easily be proven by induction on the definition of \( \text{Succ} \).

**Lemma 22.** Let \( S_0 \in \mathcal{R}_Q \) and \( S_i := \text{Succ}(S_{i-1}) \) for all \( i \geq 1 \). The following holds:

\[
S_i(q) = \bigcup_{p \in Q} \{ b \in Q \mid a \in S_0(p), p(a) \to_p q(b) \text{ and } |p| \leq i\}.
\]

Now we can state a proposition that shows how the previous section relates to these definitions. Let us fix a configuration \( p(a) \). We will focus on the MIUN-closure \( C \) of \( A := [a, a] \) with respect to \( \mathcal{L} := \{ \tau(q) \mid q \in Q \} \). We say that a mapping \( S \in \mathcal{R}_Q \) is \( C \)-valid if \( S(q) \in C \) for all \( q \in Q \).

**Proposition 23.** Let \( S \in \mathcal{R}_Q \) be a \( C \)-valid mapping. We have \( S \preceq \text{Succ}(S) \), and \( \text{Succ}(S) \) is a \( C \)-valid mapping. Moreover, for every \( q \in Q \), if \( b \in \text{Succ}(S)(q) \setminus S(q) \), then there exists \( r(c) \) such that \( c \in S(r) \) and \( r(c) \to_t q(b) \) for some transition \( t \).

**Proof.** From the definition of \( \text{Succ} \), we have that the following holds for all \( q \in Q \). If \( b \in \text{Succ}(S)(q) \setminus S(q) \), then there exists \( r(c) \) such that \( c \in S(r) \) and \( r(c) \to_t q(b) \) for some transition \( t \). Hence, \( S \preceq \text{Succ}(S) \) follows directly from the definition of \( \text{Succ} \). To prove that \( \text{Succ}(S)(q) \in C \), it suffices to observe that \( \text{Succ}_q(S) \) is defined using Minkowski sums, intersections, and unions, which are building blocks of MIUN-closures.

### 4.3 Accelerations

Unfortunately, applying \( \text{Succ} \) might not give us \( \text{Reach}_{p(a)} \) in a small or even finite number of steps, e.g., if \( \text{Reach}_{p(a)}(q) \) is unbounded for some \( q \in Q \). We introduce another operation on mappings to resolve this. We start by defining some special form of cycles.

Let us fix a mapping \( S_0 \in \mathcal{R}_Q \) and let \( S_{i+1} := \text{Succ}(S_i) \) for every \( i \geq 0 \). We say that a run \( \rho = \alpha_1 t_1 \cdots \alpha_n t_n \) is a *positively expanding cycle* from \( S_0 \) if it is admissible and there exist configurations \( p_0(a_0), p_1(a_1), \ldots, p_n(a_n) \) such that

1. \( p_0 = p_n \) and \( \Delta(\rho) > 0 \);
2. \( a_q \in S_0(p_0) \) and \( p_0(a_0) \to_{\rho[\ldots]} p_1(a_1) \) for all \( i \geq 1 \); and
3. \( a_i \in S_i(p_i) \setminus S_{i-1}(p_i) \) for all \( i \geq 1 \).

Moreover, letting \( I_0, \ldots, I_n \) be the sequence of intervals such that \( a_i \in I_i \in I(S_i(p_i)) \) for all \( i \in \{0, \ldots, n\} \), we require

1. \( I_0 \subseteq I_n \);
2. for all \( i \geq 1 \), there is a unique interval \( I'_i \in I(S_{i-1}(p_i)) \) such that \( I'_i \subseteq I_i \); and
3. \( a_i \geq \sup(I'_i) \) for every \( i \geq 1 \).

Intuitively, the third condition states that each \( a_i \) is a "new" value, and the fifth and sixth conditions state that \( a_i \) expands some interval toward the top.
For example, consider a state \( q \in Q \) with guard \( \tau(q) = [0, \infty) \), a self-loop \( \rho = (q, 1, q) \), and a mapping \( S_0 \) s.t. \( S_0(q) = \{0\} \). The sequence of configurations we get from following \( \rho \) is \( q(0)q(1) \). By applying Succ, we get that \( S_1(q) = [0, 1] \). It is easy to see that the first two conditions are met.

For the third condition, we have that \( 1 \in [0, 1] \setminus \{0\} \). According to our definition, we let \( I_0 = [0, 0] \) and \( I_1 = [0, 1] \). Since \( I_0 \subseteq I_1 \), conditions four and five are met, where \( I_1' = I_0 \). Finally, we have that \( 1 > \sup(I_0) \). Thus, \( \rho \) is a positively expanding cycle.

Similarly, we say that \( \rho \) is a negatively expanding cycle from \( S_0 \) if in the first item we replace \( \Delta(\rho) > 0 \) with \( \Delta(\rho) < 0 \), and in the last item we replace \( a_i \geq \sup(I_i') \) with \( a_i \leq \inf(I_i') \).

The following property follows from the definitions:

**Lemma 24.** It holds that \( a_0, a_n \in I_n \subseteq \text{Reach}_{p_0(a_0)}(p_0) \).

It transpires that the Succ function always yields expanding cycles after a polynomial number of applications. The proof of this claim relies on our bounds for interval decompositions of sets from the MIUN-closure. We will also need to define a measure on the mappings from states to interval decompositions, which progresses with an increasing number applications of Succ, and finally leads us to find an expanding cycle.

Let \( C \) be the MIUN-closure of \( A \) w.r.t. \( L \), and let \( B, B' \in C \) be such that \( B \subseteq B' \). We say that \( B' \) is a progressing extension of \( B \) if

1. there is \( I' \in I(B') \) such that \( B \cap I' = \emptyset \);
2. or if there are \( I \in I(B) \) and \( I' \in I(B') \) such that \( I \subseteq I' \) and at least one of the following holds:
   - \( \phi_B(I') \setminus \phi_B(I) \neq \emptyset \), or
   - there exists \( \ell \in \phi_B(I) \) such that \( \ell \notin I \) and \( \ell \in I' \).

See Figure 2 for a pictorial description of progressing extensions. Observe that in case (3) we necessarily have that \( \ell \in I \).

Let us prove a bound on the number of progressing extensions in any \( \subseteq \)-increasing sequence.

**Lemma 25.** Let \( B_0, B_1, B_2, \ldots \in C \) be a sequence such that \( B_i \subseteq B_{i+1} \) for all \( i \in \mathbb{N} \). The set of \( i \in \mathbb{N} \) such that \( B_{i+1} \) is a progressing extension of \( B_i \) has cardinality at most \( |L|^{O(1)} \).

**Proof.** Let \( P := P_L \cup P_A \). First, observe that (3) can happen only if there exists some \( \ell \in P \) such that \( \ell \in B_{i+1} \setminus B_i \), and thus at most \( |P| \) times.

Let \( \phi_{B_i}(I(B_i)) \subseteq P \) be the image of all intervals of \( I(B_i) \). Note that \( \phi_B(I(B_i)) \subseteq \phi_{B_{i+1}}(I(B_{i+1})) \). A strict inclusion can happen at most \( |P| \) times. Thus, we can assume that \( \phi_{B_i}(I(B_i)) = \phi_{B_{i+1}}(I(B_{i+1})) \). Note that (1) can happen at most \( |P| \) times due to Lemma 20 and because \( \phi_{B_{i+1}}(I') \neq \emptyset \) for all \( I' \in I(B_{i+1}) \). Indeed, for all \( \ell \in P \), Lemma 20 tells us there are no pairwise distinct intervals \( I_1, I_2, I_3 \in I(B_{i+1}) \) such that \( \ell \in \phi_{B_{i+1}}(I_1) \), \( \ell \in \phi_{B_{i+1}}(I_2) \), and \( \ell \in \phi_{B_{i+1}}(I_3) \).

Fig. 2. Left: A set \( B \) such that \( I(B) = ((-\infty, 3], [4, 5)) \). Right: Example of the three possible types of progressing extensions of \( B \). Dashed lines denote open interval borders; \( \ell_1 = 4, \ell_2 = 5, \) and \( \ell_3 = 6 \) denote values in \( P_L \cup P_A \).
Now, assume that (1) and (3) are not the case and that \( \phi_B(I(B_i)) = \phi_{B_{i+1}}(I(B_{i+1})) \). Since (1) does not hold, we have \(|I(B_{i+1})| \leq |I(B_i)|\). Note that a strict inequality can happen at most \(|P|\) times, so we can assume that \(|I(B_{i+1})| = |I(B_i)|\). We define a function \( f : I(B_i) \rightarrow I(B_{i+1}) \). Recall that \( B_i \subseteq B_{i+1} \). So, for every \( I \in \tilde{I}(B_i) \) there exists a unique \( f(I) \in I(B_{i+1}) \) such that \( I \subseteq f(I) \).

Uniqueness follows from maximality of intervals within \( \tilde{I}(B_{i+1}) \). Then, we have that \( \tilde{I} \subseteq f(I) \) and therefore \( \tilde{I} \cap P \subseteq \tilde{f}(I) \cap P \). It follows that \( \phi_{B_i}(I) \subseteq \phi_{B_{i+1}}(f(I)) \). Since (1) does not hold, the function \( f \) is a surjection. Moreover, \( \phi_{B_i}(I) = \phi_{B_{i+1}}(f(I)) \). Thus, (2) can happen at most \(|P|\) times by Lemma 20.

Now, let us show an interesting property of extensions that are not progressing:

**Lemma 26.** Let \( B, B' \in C \) be such that \( B \subseteq B' \), where \( B' \) is not a progressing extension of \( B \). There is a bijection \( f : I(B) \rightarrow I(B') \) s.t. \( \phi_B(I) = \phi_B(f(I)) \) for all \( I \in I(B) \).

**Proof.** Since \( B \subseteq B' \), for all \( I \in I(B) \) there is a unique \( f(I) \in I(B') \) such that \( I \subseteq f(I) \). We show that if \( f \) is not a bijection, then it will contradict that \( B \subseteq B' \) is not a progressing extension.

First, we prove that \( f \) is an injection. Suppose this is not the case and that \( f(I) = f(J) \) for some \( I, J \in I(B) \). Then \( \phi_B(I) = \phi_B(J) \). If there exists \( \ell \in f(\phi_B(I) \cap \phi_B(J)) \), then \( \ell \neq I, \ell \neq J \), and \( \ell \notin f(\phi_B(I)) \). This is a contradiction because the extension is progressing due to (3). Otherwise, there is \( \ell \notin f(\phi_B(I)) \). Since \( \phi_B(I) \subseteq \phi_B(I) \cap \phi_B(J) \cup \phi_B(f(I)) \), we get a contradiction because the extension is progressing due to (2).

Now, we prove that \( f \) is a surjection. If we suppose this does not hold, then there is an interval \( I' \in I(B') \) such that \( f^{-1}(I') = \emptyset \). Thus, \( I' \cap B = \emptyset \), which is a contradiction because this would mean the extension is progressing due to (1).

To conclude, we note that for all \( I \in I(B) \), since \( I \subseteq f(I) \), we have that \( \tilde{I} \subseteq \tilde{f}(I) \). It follows that \( \tilde{I} \cap (P_L \cup P_A) \subseteq \tilde{f}(I) \cap (P_L \cup P_A) \). Hence, by definition of the indicator functions, we have that \( \phi_B(I) \subseteq \phi_{B'}(f(I)) \). If the inclusion is strict for some \( I \), then we get a contradiction because the extension is progressing due to (2). \( \Box \)

Now, we are able to prove the following proposition. The computational part is a simple backward construction of a run containing a cycle.

**Proposition 27.** Let \( S_0 \in \mathcal{R}_Q \) let \( S_{i+1} := \text{Succ}(S_i) \) for all \( i \in \mathbb{N} \). For some \( n \) polynomially bounded in \(|Q|\), at least one of the following holds:

- \( S_n = S_{n+1} \),
- there is a positively expanding cycle \( \rho \) from \( S_n \), or
- there is a negatively expanding cycle \( \rho \) from \( S_n \).

Moreover, it can be determined in time \(|Q|^{O(1)}\) whether the second or third case hold, and then \( \rho \) and its witnessing configurations can be computed in time \(|Q|^{O(1)}\).

**Proof.** The value of the polynomially bounded number \( n \) will be determined by the proof. By definition, we have \( S_i \leq S_{i+1} \) for all \( i \in \mathbb{N} \). By Lemma 25, there is a polynomial number of indices \( i \) (w.r.t. \(|Q|\)) such that \( S_i(q) \subseteq S_{i+1}(q) \) is a progressing extension for some \( q \in Q \). Thus, there exists an index \( j \) polynomial in \(|Q|\) such that the extensions \( S_{j}(q) \subseteq S_{j+1}(q) \) are not progressing for all \( j \leq i \leq j + k \), where \( k \) is sufficiently large (but polynomially bounded in \(|Q|\)). To simplify the notation we will assume that \( j = 0 \) and consider \( S_0 \leq S_1 \leq \cdots \leq S_k \).

By Lemma 26, there is a bijection \( f_i : I(S_i(q)) \rightarrow I(S_{i+1}(q)) \) for every \( q \in Q \) and \( 0 \leq i < k \). Thus, for every \( q \in Q \), the sets \( I(S_i(q)) \), \( \cdots I(S_k(q)) \) have the same number \( m_q \) of intervals. By Lemma 21, \( m_q \) is polynomially bounded in \(|Q|\). Let us denote the intervals \( I_{q,1}^{q,1}, \ldots, I_{m_q}^{q,1} \), where
We claim the following holds: $v_i \geq \sup \left( t_{j_i, i-1}^{q_i} \right)$ for all $0 < i \leq k$.

Let us argue that Equation (1) allows us to conclude. For a large enough $k$, we can find an infix $\rho := q_a(v_a), \ldots, q_b(v_b)$ such that $a < b$, $q_a = q_b$, and $j_a = j_b$. The following inequalities thus hold:

$$v_b \geq \sup \left( t_{j_b, b-1}^{q_b} \right) \geq \sup \left( t_{j_a, a}^{q_a} \right) \geq v_a.$$  

Since $v_a \in I_{j_a, a}^{q_a}$ and $v_b \notin I_{j_b, b}^{q_b}$, we obtain $\Delta(\rho) > 0$. The remaining conditions of the positively expanding cycle follow directly from the definition.

It remains to prove Equation (1). We proceed by induction, going from $i = k$ down to $i = 1$. The base case follows by assumption. For the inductive step, toward a contradiction, suppose that $v_i < \sup \left( t_{j_i, i-1}^{q_i} \right)$. Observe that, by construction, $v_i \notin I_{j_i, i-1}^{q_i}$. Hence, it must also be the case that $v_i \leq \inf \left( I_{j_i, i-1}^{q_i} \right)$. Recall that active($q_i, i-1, j_i$) = active($q_i, i, j_i$). Since $v_i \rightarrow a_{i+1, i+1}$, $v_{i+1}$, there exists $\overline{v} \in I_{j_i, i-1}^{q_i}$ and $\beta \in (0, 1]$ such that $\overline{v} \rightarrow \beta \gamma t_{i+1} \overline{w}$ for some $\overline{w} \in S_i(q_{i+1})$.

We show that $v_{i+1} < \overline{w}$ holds for all choices of $\beta$. Note that if there exists some $\beta$ such that $v_{i+1} = \overline{w}$, then $v_{i+1} \in S_i(q_{i+1})$ and we get a contradiction with the definition of $v_{i+1}$. It follows that either $v_{i+1} > \overline{w}$ for all choices of $\beta$ or $v_{i+1} < \overline{w}$ for all choices of $\beta$. Let $t_{i+1} = (q_i, z_{i+1}, q_{i+1})$. Since $v_i \leq \inf \left( t_{j_i, i-1}^{q_i} \right)$ and $\overline{v} \in I_{j_i, i-1}^{q_i}$, we have that

$$v_{i+1} = v_i + a_{i+1, i+1} z_{i+1} \leq \overline{v} + a_{i+1, i+1} z_{i+1}.$$  

Thus, it must be the case that $v_{i+1} < \overline{w}$ for all choices of $\beta$.

We now prove that $\overline{w} \in I_{j_{i+1}, i+1}^{q_{i+1}}$. Recall that $v_i, \overline{v} \in I_{j_i, i-1}^{q_i}$ and thus $[v_i, \overline{v}] \subseteq I_{j_i, i-1}^{q_i}$. Also, $[v_i + a_{i+1, i+1} z_{i+1}, \overline{v} + \beta z_{i+1}] = [v_{i+1}, \overline{w}]$. Hence, for every $v_{i+1} \leq w' \leq \overline{w}$ there exists $v_i \leq v' \leq \overline{v}$ and $\gamma \in (0, 1]$ such that $v' \rightarrow \gamma t_{i+1} w'$. Thus, $\overline{w}$ and $v_{i+1}$ belong to the same interval in $S_{i+1}(q_{i+1})$ as required.

Since $\overline{w} \in I_{j_{i+1}, i+1}^{q_{i+1}}$ and $\overline{w} \in S_i(q_{i+1})$, we have $\overline{w} \in I_{j_{i+1}, i+1}^{q_{i+1}}$. We have reached a contradiction, since by the inductive hypothesis we have

$$v_{i+1} \geq \sup \left( t_{j_{i+1}, i+1}^{q_{i+1}} \right).$$
Observe that since Succ is easily computable in polynomial time, one can also find $q_0(v_0), \ldots, q_k(v_k)$ and their corresponding intervals in polynomial time. □

We are ready to define the acceleration operation. Let $\rho$ be a positively or negatively expanding cycle from $S \in R_\varnothing$ and let $p_0(a_0), p_1(a_1), \ldots, p_n(a_n)$ be the configurations witnessing the run. Let $I_0, \ldots, I_n$ be the intervals given by the definition of expanding cycles. If $\rho$ is positively expanding, then we define $\delta^+_i := \sup (\tau(p_i) - a_i)$ for all $i \in \{1, \ldots, n\}$. If $\rho$ is a negatively expanding cycle, then we define $\delta^-_i := a_i - \inf \tau(p_i)$. Let $j \in \{1, \ldots, n\}$ be such that $\delta^+_j = \min\{\delta^+_i \mid 1 \leq i \leq n\}$ or $\delta^-_j = \min\{\delta^-_i \mid 1 \leq i \leq n\}$.

We define $\text{Acc}$ so that, given $\rho$ and the mapping $S$, it outputs a new mapping $\text{Acc}(S, \rho) = S'$. If $\rho$ is positively expanding from $S$, then $S'(q) := S(q)$ for all $q \neq p_j$ and

$$S'(p_j) := S(p_j) \cup I_j \cup (\tau(p_j) \cap [a_j, +\infty)).$$

Recall that $a_j \in I_j$, and $a_j \in \tau(p_j)$ since $a_0 \rightarrow \rho_{[1..j]} a_j$, so $K$ is an interval. Also, since $\rho$ is positively expanding and $j \geq 1$, we have $a_j \notin S(p_j)$ and $S'(p_j) \setminus S(p_j) \neq \emptyset$. Similarly, if $\rho$ is negatively expanding, then $S'(q) := S(q)$ for all $q \neq p_j$ and

$$S'(p_j) := S(p_j) \cup I_j \cup ((-\infty, a_j] \cap \tau(p_j)).$$

**Lemma 28.** Let $S \in R_\varnothing$ be a $\mathcal{C}$-valid mapping such that $S \leq \text{Reach}_{\text{p}(a)}$, and let $\rho$ be a positively or negatively expanding cycle from $S$. If $S' = \text{Acc}(S, \rho)$, then $S \leq S'$, $S'$ is a $\mathcal{C}$-valid mapping, and $S' \leq \text{Reach}_{\text{p}(a)}$. Moreover, for every $q \in Q$, if $b \in \text{Acc}(S, \rho)(q) \setminus S(q)$, then there exists $\tau(c)$ such that $c \in S(r)$ and $r(c) \rightarrow \pi q(b)$, where $\pi \in \rho^*$.

**Proof.** We have $S \leq S'$ directly from the definition of $\text{Acc}$. Similarly, $S'$ is $\mathcal{C}$-valid because the operation to define $S'(p_j)$ is the “New” operation since the closure of the added interval always contains one of the endpoints from $\tau(p_j)$. It remains to prove that $S' \leq \text{Reach}_{\text{p}(a)}$.

We assume that $\Delta(\rho) > 0$; the proof is similar for the other case. Let $\rho = \alpha_1 t_1 \cdots \alpha_n t_n$. Let $p_0(a_0), p_1(a_1), \ldots, p_n(a_n)$ and $I_0, \ldots, I_n$ be the configurations and intervals given by the definition of positively minimizing cycles. Let $j$ be an index minimizing $\delta^+_j$. We must prove that $S'(p_j) \leq \text{Reach}_{\text{p}(a)}$.

Since $S \leq \text{Reach}_{\text{p}(a)}$ and $a_0 \in S(p_0)$ by definition of positively expanding cycles, it suffices to show that for every $b \in S'(p_j) \setminus S(p_j)$ there is an admissible run from $p_0(a_0)$ to $p_j(b)$.

If $\delta^+_j = +\infty$, then sup $\tau(p_j) = +\infty$ for all $i \in \{1, \ldots, n\}$. Since $\Delta(\rho) > 0$, for all $\alpha, \beta \in (0, 1]$ and $m \in \mathbb{N}$ the run $\rho' := (\beta \rho)^m \alpha \rho[1..j]$ is admissible from any $p_n(a')$ with $a' \geq a_n$ to state $p_j$. Note that $\Delta(\rho') = m \beta \Delta(\rho) + \alpha \Delta(\rho[1..j])$, which can be any positive rational number by properly choosing $\alpha$, $\beta$, and $m$. Thus, $b \in \text{Reach}_{\text{p}(a)}(p_j)$ for every $b > a_n$.

It remains to consider the case $a_j \leq a_n$ to prove the claim for every $b \in [a_j, a_n]$. Let $\epsilon \in (0, a_n - a_j]$. Since $a_0 \leq a_0 + \epsilon \leq a_n$, we have $a_0 + \epsilon \in I_n \subseteq \text{Reach}_{\text{p}(a)}(p_0)$, where the latter follows from Lemma 24. Note that $\rho$ is admissible from all $p_n(a')$ with $a' \geq a_n$, and hence from $a_n + \epsilon$.

Thus, $p(a) \rightarrow_s p_0(a_0 + \epsilon) \rightarrow_\rho p_n(a_n + \epsilon)$, and $p_0(a_0 + \epsilon) \rightarrow_\rho p[1..j] p_j(a_j + \epsilon)$. This shows that $b \in \text{Reach}_{\text{p}(a)}(p_j)$ for all $b \in [a_j, a_n]$.

Now, suppose $\delta^+_j < +\infty$. If $a_j = \sup (\tau(p_j))$, then we are done because $a_j \in I_j \subseteq \text{Reach}_{\text{p}(a)}(p_j)$. Otherwise, let $b \in [a_j, +\infty) \cap \tau(p_j)$. We need to prove that $b \in \text{Reach}_{\text{p}(a)}(p_j)$. Note that, by definition, we have $0 \leq b - a_j \leq \delta^+_j$.

Let $m \in \mathbb{N}$ and $c \in Q_{\geq 0}$ be the unique numbers that satisfy $b - a_j = m \Delta(\rho) + c$ and $c < \Delta(\rho)$. Since $a_0 \leq a_0 + c \leq a_0 + \Delta(\rho) = a_n$, then by Lemma 24 we conclude that $a_0 + c \in \text{Reach}_{\text{p}(a)}(p_0)$. Notice that $a_j + c + m \Delta(\rho) = b$. It thus remains to prove that $p_0(a_0 + c) \rightarrow_\rho p[1..j] p_j(b)$. We prove something stronger, namely that $p_0(a_0 + c) \rightarrow_{\rho^m} p_n(a_n + b - a_j)$. Since $\Delta(\rho) > 0$, for the bottom
guards it suffices to check whether the configurations are large enough when $\rho$ is applied the first time. Indeed, since $\rho$ is admissible from $p_0(a_0)$, we get $a_i + c + \Delta(p_i) \geq a_i + \Delta(p_i) \geq \inf \tau(q_i)$. Similarly, for the top guards, it suffices to check whether the configurations are small enough when $\rho$ is applied last.

Indeed, since $b - a_j \leq \delta^+_j$, we have $a_i + c + m\Delta(p) = a_i + b - a_j \leq a_i + \delta^+_i \leq a_i + \delta^+_i = \sup \tau(p_i)$.

If $\sup \tau(p_j) \notin \tau(p_j)$, then $b - a_j < \delta^+_j$ and the previous inequalities are strict.

\[ \Delta(\rho) \geq 1 \]

\section{Polynomial Time Algorithm}

We summarize how to obtain the polynomial time algorithm for deciding $p(a) \rightarrow q(b)$. We begin with the mapping $R_0 \in R_\mathcal{Q}$ defined as $R_0(p) := [a, a]$ and $R_0(r) := \emptyset$ for every $r \neq p$. Clearly, $R_0 \leq \text{Reach}_{p(a)}$. The next mappings $R_1, R_2, \ldots$ are defined as follows. Suppose we have defined $R_0, \ldots, R_i$. Let $S^i_0 := R_i$ and $S^i_{j+1} := \text{Succ}(S^i_j)$ for all $j \geq 0$.

By Proposition 27, we will either find an expanding cycle $\rho$ from some $S^i_n$, where $n$ is bounded polynomially, or we will find some $S^i_n = S^i_{n+1}$, again for $n$ bounded polynomially. If there is an expanding cycle—a fact that, by Proposition 27, we can check in polynomial time—then we define $R_{i+1} := S^i_j$ for $1 \leq j < n$ and $R_{i+n} := \text{Acc}(R_{i+n-1}, \rho)$. Otherwise, we define $R_{i+1} := S^i_j$ for $j \in \{1, \ldots, n\}$ and the algorithm returns $R_{i+n}$. By Lemmas 22 and 28, we have $R_i \leq \text{Reach}_{p(a)}$ for all defined $R_i$. Hence, if the algorithm terminates, then by Lemma 22 it returns $\text{Reach}_{p(a)}$.

The rest of this section is devoted to proving that the above-described algorithm has a polynomial worst-case running time. It suffices to argue that expanding cycles can only be found some polynomial number of times.

\begin{proposition}
The algorithm computes a representation of $\text{Reach}_{p(a)}$ in time $|\mathcal{Q}|^{O(1)}$.
\end{proposition}

By Proposition 27, it suffices to show that $\text{Acc}$ can be applied at most polynomially many times. We show that accelerating leads to a progressing extension.

\begin{lemma}
Let $\rho$ be an expanding cycle from $R$ and let $R' := \text{Acc}(R, \rho)$. There is some state $p_j$ such that $R(p_j) \subseteq R'(p_j)$ is a progressing extension.
\end{lemma}

\begin{proof}
Let $p_j \in \mathcal{Q}$ be such that $R'(p_j) = R(p_j) \cup I_j \cup J$ for some interval $J$. In the proof, we write $\alpha(t_i, p_i(a_i))$, and $I_i$, as in the definition of expanding cycles. We will assume that $\rho$ is positively expanding; the other case is similar. Thus, $J \in \{(a_j, g), [a_j, g), [a_j, +\infty)\}$, where $g := \sup \tau(p_j)$.

Recall that by definition, $a_j \notin R(p_j)$ and $a_j \in I_j \in \text{Succ}^k(R(p_j))$ for some $k$. Thus, $I_j \cup J$ is an interval.

If $(I_j \cup J) \cap R(p_j) = \emptyset$, then $R'(p_j)$ is a progressing extension due to (1). For the remaining case, let $b \in (I_j \cup J) \cap R(p_j)$ and $K \in \mathcal{I}(R(p_j))$ be such that $b \in K$. Note that $K \cup I_j \cup J$ is an interval. If $b < a_j$, then, because $a_j \notin R(p_j)$, either $a \notin K$ or $K$ has an upper bound if $J = [a_j, +\infty)$. Thus, $R'(p_j)$ is a progressing extension due to (2) or (3). Finally, suppose that $b > a_j$. By definition of $\rho$, there is a unique interval $I'_j \in \text{Succ}^{k-1}(R(p_j))$ such that $I'_j \subseteq I_j$. Moreover, $a_j \notin I'_j$ and $a_j \geq \sup I'_j$. Thus, $K \cap I'_j$ is empty and $g \notin I'_j$ or $I'_j$ has an upper bound if $J = [a_j, +\infty)$. Since $K \cup I_j \cup J \in \Gamma, R'(p_j)$ is a progressing extension due to (2) or (3).

\end{proof}

\section{Parametric COCA Reachability}

This section is dedicated to characterizing the complexity of existential reachability in parametric COCAs. We will show the following result:

\begin{theorem}
The existential reachability problem for parametric COCAs is NP-complete.
\end{theorem}

NP-membership will be based on results from Section 4 and heavily rely on the fact that reachability in COCAs can be expressed as reachability in one of many linear path schemes (see
Lemma 32), and that the reachability relation of such path schemes can be expressed as a small existential linear formula. In Section 5.1, we describe a straightforward formula for describing the reachability relation in simple paths. In Section 5.2, we similarly give a formula for the reachability relation of simple cycles, which requires a more careful analysis of the behavior of cycles in CO-CAs (see particularly Lemma 38). We conclude the proof of NP-membership in Section 5.3, where the full formula is assembled. We briefly switch to the setting where only updates can be parameterized in Section 5.4. We show that in this setting, any rational valuation of the parameters can be turned into an integral valuation of the parameters while maintaining reachability. Finally, we conclude the proof of NP-completeness in Section 5.5 by showing that existential reachability in parametric COCA is NP-hard, even when they are acyclic and parameters occur only on updates or only on guards. To do so, we give a reduction from 3-SAT.

For the rest of this section, let \( V = (Q, T, \tau, X) \) be a parametric COCA, and let \( a, b \in \mathbb{Z} \). We will show that existential reachability from \( p(a) \) to \( q(b) \) can be witnessed by an existential linear formula \( \varphi \) of polynomial size. Membership in NP will follow from the fact that we can both guess \( \varphi \) and check satisfiability of \( \varphi \) in NP. This formula will be obtained based on the following corollary, which can be derived from the last section:

**Lemma 32.** If \( b \in \text{Post}_n p(a) \), then there exists a path \( \pi \) from some linear path scheme \( \sigma_0 \theta_0 \sigma_1 \theta_1 \cdots \sigma_k \) such that \( p(a) \rightarrow^{\pi} q(b) \), where

- \( k \leq |Q|^{O(1)} \),
- \( |\sigma_i| \leq |Q|^{O(1)} \) for each \( 0 \leq i \leq k \), and
- \( |\theta_j| \leq |Q|^{O(1)} \) for each \( 0 \leq j < k \).

**Proof.** The proof is by induction on the definition of the sequence of \( R_i \)'s. We argue that, for all \( q \in Q \), if \( b \in R_i(q) \), then there is a path \( \pi \) of the claimed form such that \( p(a) \rightarrow^{\pi} q(b) \). Note that this is sufficient in view of Proposition 29.

For the induction, let us first focus on the last two items from the claim. The base case is trivial and the inductive step is a straightforward application of Proposition 23 and Lemma 28 (with Proposition 27 giving us the required polynomial bounds). Finally, we still need to argue that \( k \) is also polynomial. However, this follows from our induction on the sequence of \( R_i \)'s together with the polynomial bound on the length of the sequence established in Lemmas 30 and 25. \( \square \)

In order to exploit Lemma 32, we will further need the forthcoming technical lemmas.

**Lemma 33.** Let \( t \in T \), \( \alpha, \beta \in (0, 1) \) and \( a, a', b, b' \in \mathbb{Q} \) be such that \( a' = a + \alpha \Delta(t) \) and \( b' = b + \beta \Delta(t) \). If \( a \leq b \) and \( a' \geq b' \), then \( b' = a + \alpha' \Delta(t) \) and \( a' = b + \beta' \Delta(t) \) for some \( \alpha', \beta' \in (0, 1) \).

**Proof.** The claim is trivial whenever \( a = b \) or \( a' = b' \); hence assume \( a < b \) and \( a' > b' \). Note that \( \Delta(t) \neq 0 \), as we would otherwise derive the contradiction \( a < b = b' < a' = a \). Let

\[
\alpha' := \frac{b' - a}{\Delta(t)} \quad \text{and} \quad \beta' := \frac{a' - b}{\Delta(t)}.
\]

We have \( a + \alpha' \Delta(t) = b' \) and \( b + \beta' \Delta(t) = a' \) as desired. It remains to show that \( \alpha', \beta' \in (0, 1) \). From \( a' > b' \), we have

\[
\alpha' = (b' - a)/\Delta(t) < (a' - a)/\Delta(t) = \alpha \leq 1,
\]

\[
\beta' = (a' - b)/\Delta(t) > (b' - b)/\Delta(t) = \beta > 0.
\]

From \( a < b \), we symmetrically derive \( \alpha' > 0 \) and \( \beta' \leq 1 \). \( \square \)

**Corollary 34.** Let \( a, a', b, b' \in \mathbb{Q} \), and let \( \pi = t_1 \ldots t_n \) be a path such that \( a \rightarrow^{\pi} a' \) and \( b \rightarrow^{\pi} b' \). If \( a \leq b \) and \( a' \geq b' \), then \( a \rightarrow^{\pi} b' \) and \( b \rightarrow^{\pi} a' \).
Proof. Let \( \sigma_1, \sigma_2 \) be runs such that \( a \rightarrow_{\sigma_1} a' \) and \( b \rightarrow_{\sigma_2} b' \). Let us define \( a_i := a + \Delta(\sigma_1[i..]) \) and \( b_i := b + \Delta(\sigma_2[i..]) \). In particular, \( a_0 = a, b_0 = b, a_n = a' \), and \( b_n = b' \). By assumption, it is the case that \( a_0 \leq b_0 \) and \( a_n \geq b_n \). Clearly, if \( a_0 = b_0 \), then \( a_0 \rightarrow_{\sigma_1} b' \) and we are done. So we can make the stronger assumption that \( a_0 < b_0 \).

Let \( i \) be the first index such that \( a_i \geq b_i \). We have \( i > 0 \), as \( a_0 < b_0 \). By minimality of \( i \), it holds that \( a_{i-1} < b_{i-1} \). Thus, by Lemma 33, it holds that \( a_{i-1} \rightarrow t_i \ b_i \) and \( b_{i-1} \rightarrow t_i \ a_i \). Hence, we are done since \( a \rightarrow_{\sigma_1[i..]} a_{i-1} \rightarrow t_i \ b_i \rightarrow_{\sigma_2[i+1..n]} b' \) and \( b \rightarrow_{\sigma_1[i..]} b_{i-1} \rightarrow t_i \ a_i \rightarrow_{\sigma_2[i+1..n]} a' \). \( \Box \)

Lemma 35. Let \( \pi = t_1 \cdots t_n \) be a path, and let \( a, b, a', b' \in \mathbb{Q} \) be such that \( b \leq a < b' \leq a' \). If \( a \rightarrow_{\pi} a' \) and \( b \rightarrow_{\pi} b' \), then \( a \rightarrow_{\pi} b' \).

Proof. If \( b' = a' \), we are done. So assume \( b' < a' \). Let \( \sigma_1, \sigma_2 \) be runs with \( \text{path}(\sigma_1) = \text{path}(\sigma_2) = \pi \) such that \( a \rightarrow_{\sigma_1} a' \) and \( b \rightarrow_{\sigma_2} b' \). Let us define \( \beta := (b' - a)/(a' - a) \). As \( b' < a' \), it holds that \( \beta \in (0, 1) \). So we can define a new run \( \sigma'_1 := \beta \sigma_1 \). Clearly, \( \sigma'_1 \) has the right effect to go from \( a \) to \( b' \):

\[
a + \Delta(\sigma'_1) = a + \frac{b' - a}{a' - a} \Delta(\sigma_1) = a + \frac{b' - a}{a' - a} (a' - a) = b'.
\]

It remains to show that \( \sigma'_1 \) is an admissible run from \( a \). Let \( a_i := a + \Delta(\sigma_1[i..]), a'_i := a + \Delta(\sigma'_1[i..]) \) and \( b_i := b + \Delta(\sigma_2[i..]) \). If there exists \( i \) such that \( a'_i \leq b_i \), then by Lemma 33, it follows that \( a \rightarrow_{\pi[i..]} b_i \rightarrow_{\sigma_2[i+1..n]} b' \), and so we are done. Thus, let us assume that \( a'_i > b_i \) for all \( i \). By the definition of \( \sigma'_1 \), it is the case that \( a_i \geq a'_i \). So, for all \( i \), we have \( a_i \geq a'_i \geq b_i \). Since \( \sigma_1 \) and \( \sigma_2 \) are respectively admissible runs from \( a \) and \( b \), it follows that \( \sigma'_1 \) is an admissible run from \( a \). Thus, \( a \rightarrow_{\sigma'_1} b' \), and hence we are done. \( \Box \)

For completeness, we note that we can also state and prove an equivalent formulation of the lemma when \( b \geq a > b' \geq a' \).

### 5.1 Formula for \( \sigma \)

Let \( I \in \Gamma_X \), let \( a \) be a number, and let \( \mu \) be a valuation of \( X \). Recall that \( \Gamma_X \) is the set of all intervals with endpoints from \( \mathbb{Q} \cup (-\infty, +\infty) \cup X \). We write \( I^\mu \) to mean \( I \) where each parameter \( x \in X \) is replaced with \( \mu(x) \). We define a formula \( \phi_{\in I} \( a, \mu \) \) that is satisfied if and only if \( a \in I \) under \( \mu \):

\[
\phi_{\in I}(a, \mu) := (\inf I^\mu \leq_{\inf} a <_{\sup} \sup I^\mu),
\]

where \( \leq_{\inf} := \leq \) if the lower end of \( I \) is open, and \( \leq_{\inf} := \leq \) otherwise. Similarly, we define \( <_{\sup} := < \) if the upper end of \( I \) is open, and \( <_{\sup} := \leq \) otherwise. For example, we have \( \phi_{\in (4,9)}(a, \mu) = 4 \leq a < 9 \). Note that this also generalizes to the presence of parameters in the endpoints of \( I \). For example, if \( x, y \in X \) are parameters, then \( \phi_{\in (x,y)}(a, \mu) = \mu(x) < a \leq \mu(y) \).

Let \( t = (q_0, z, q_1) \in T \) be a transition. We give a formula \( \phi_t(a, b, \mu) \) that is satisfied iff \( b \in \text{Post}_t(a) \):

\[
\phi_t(a, b, \mu) := [\phi_{\in (q_0)}(a, \mu)] \land [z^\mu = 0 \rightarrow a = b] \land
[z^\mu < 0 \rightarrow (a + z^\mu \leq b < a)] \land [z^\mu > 0 \rightarrow (a + z^\mu \geq b > a)] \land \phi_{\in (q_1)}(b, \mu).
\]

Clearly, we can generalize the formula \( \phi_t(a, b, \mu) \) to work for paths instead of transitions. Let \( \pi = t_1 t_2 \cdots t_n \). We give a formula \( \phi_{\pi}(a, b, \mu) \) as follows:

\[
\phi_{\pi}(a, b, \mu) := \exists a_0, a_1, \ldots, a_n : (a_0 = a) \land (a_n = b) \land \bigwedge_{1 \leq i \leq n} \phi_{t_i}(a_{i-1}, a_i, \mu).
\]

The following is straightforward:

**Lemma 36.** It is the case that \( \phi_{\pi}(a, b, \mu) \) holds iff \( b \in \text{Post}_{\pi}(a) \). Furthermore, the size of \( \phi_{\pi} \) is linear in \( |\pi| \) and the sum of non-parametric endpoints and updates along \( \pi \).
\subsection{Formula for $\theta^*$}

We define a formula $\phi_{\theta^*}(a, b, \mu)$ as follows:

$$\phi_{\theta^*}(a, b, \mu) := (a = b) \lor \phi_{\theta^*}(a, b, \mu).$$

Intuitively, we differentiate two cases: To reach $b$ from $a$, we iterate $\theta$

- zero times: $a = b$,
- one or more times: $\phi_{\theta^*}(a, b, \mu)$.

It remains to give a formula for the case where we take $\theta$ one or more times. Formula $\phi_{\theta^*}(a, b, \mu)$ is split into two cases, based on whether $a < b$ or $b < a$. Note that we do not need to handle the case $a = b$, as this is trivially included in the case where we iterate zero times. So, we define

$$\phi_{\theta^*}(a, b, \mu) := [a \neq b] \land [a < b \rightarrow \phi_{\theta^*}^+(a, b, \mu)] \land [a > b \rightarrow \phi_{\theta^*}^-(a, b, \mu)].$$

Formally, we require that if $a < b$, then $\phi_{\theta^*}^+(a, b, \mu)$ is satisfied iff $b \in \text{Post}_{\theta^*}(a) \cup \text{Post}_{\theta}(a)$. The requirement for $\phi_{\theta^*}^-(a, b, \mu)$, assuming $a > b$, is symmetric.

Let us give a necessary and sufficient condition for the membership of a value $b'$ in $\text{Post}_{\theta^*(\theta^*)}(a) \setminus \text{Post}_{\theta^*}(a)$. We first proceed to give a series of technical lemmas.

\textbf{Lemma 37.} Let $a < b$. If $a \rightarrow_{(\theta^*)^*} b$, then there exists $n \geq 1$ and values $a_0, \ldots, a_n$ such that

$$a_0 \rightarrow_{\theta^*} a_1 \rightarrow_{\theta^*} \cdots \rightarrow_{\theta^*} a_n \text{ and } a_i < b \text{ for all } i.$$

\textbf{Proof.} Let $\pi \in \theta^+$ be a path such that $a \rightarrow_{\pi} b$. Let $n$ be the number of times $\pi$ iterates $\theta$, that is, $n = |\pi|/|\theta|$. By definition of $\pi$, $n$ is a natural number. Let $a_i = a + \Delta(\pi[1..(i \cdot |\theta|)])$. We can assume that $a_i \neq b$ for all $i$; otherwise we can easily shorten $\pi$.

For the sake of contradiction, assume there exists some $i$ such that $a_i > b$. Clearly, if such an $i$ does not exist, we are done. So let $i$ be the smallest such index. Note that $0 < i < n$, since $a < b$. By definition, $a_{i-1} \rightarrow_{\theta} a_i$. Further, by minimality of $i$, it holds that $a_{i-1} < b$. Recall that we have $a_{n-1} \rightarrow_{\theta} b$. In the following, we show that $a_{i-1} \rightarrow_{\theta^*} b$, which finishes the proof by minimality of $i$.

We distinguish three cases based on the order of $a_{n-1}$ and $a_{i-1}$. First, if $a_{n-1} = a_{i-1}$, then we are clearly done. If $a_{n-1} < a_{i-1}$, then together with the fact that $b < a$, we invoke Lemma 35 to derive that $a_{i-1} \rightarrow_{\theta^*} b$. Lastly, if $a_{n-1} > a_{i-1}$, then together with the fact that $b < a_i$, it follows by Corollary 34 that $a_{i-1} \rightarrow_{\theta^*} b$.

Now, we can prove a useful lemma that allows us to assume that iterations of the cycle behave in a monotonic manner:

\textbf{Lemma 38.} Let $a < b$. If $a \rightarrow_{(\theta^*)^*} b$, then there exists $n \geq 1$ and values $a_0, \ldots, a_n$ such that

$$a_0 \rightarrow_{\theta^*} a_1 \rightarrow_{\theta^*} \cdots \rightarrow_{\theta^*} a_n \text{ and } a = a_0 < a_1 < \cdots < a_n = b.$$

\textbf{Proof.} Let $\pi \in (\theta^*)^+$ such that $a \rightarrow_{\pi} b$. Let $n$ be the number of times $\pi$ iterates $\theta$. Note that $n \geq 1$, since $\pi$ iterates $\theta$ at least once by membership in $(\theta^*)^+$. Let $a_0 := a$, $a_n := b$ and let $a_i$ be the value reached after the $i$th iteration of $\theta$ when following $\pi$. We have $a_i \rightarrow_{\pi[1..(\theta)]} a_{i+1}$ for all $0 \leq i < n$. By Lemma 37, we can assume that $a_i \leq b$ for all $i$. Note that we can assume that $a_i \neq a_j$ for all $i, j$; otherwise we can trivially shorten $\pi$.

If $a_0 < a_1 < \cdots < a_n$, then we are done. So assume there exists $i$ such that $a_i > a_{i+1}$. Let $i$ be the smallest such index. Note that $i < n-1$, as we assume $a_i < b = a_n$ for all $i$. Let $j > i$ be the smallest index such that $a_j > a_i$. Note that such an index must exist, since $a_n > a_i$. Recall that $a_i \rightarrow_{\theta^*} a_{i+1}$ and $a_{j-1} \rightarrow_{\theta^*} a_j$. Further, by minimality of our choice of $j$, $a_{j-1} < a_j$ and $a_j > a_{i+1}$. Then, we can invoke Corollary 34, from which it follows that $a_i \rightarrow_{\theta^*} a_j$. Thus, we can shorten $\pi$ by going
directly from $a_i$ to $a_j$. Note that $a_i < a_j$. It is easy to see that we can iteratively repeat this process to remove all occurrences where $a_i > a_{i+1}$. Thus, the statement follows.

Now, we are ready to state the necessary and sufficient condition:

**Lemma 39.** Let $b' > a$. It is the case that $b' \in \text{Post}_{\Theta^\nu}(a) \cup \text{Post}_{\Theta^\nu}(a)$ if and only if $a \rightarrow_{\Theta^\nu} b'$

(1) there exists $a' > a$ such that $a \rightarrow_{\Theta^\nu} a'$,

(2) there exists $b < b'$ such that $b \rightarrow_{\Theta^\nu} b'$.

**Proof.** $\Leftarrow$ If $a \rightarrow_{\Theta^\nu} b'$, then we are done. So assume $a \not \rightarrow_{\Theta^\nu} b$ and that (1) and (2) both hold. That is, $a \rightarrow_{\sigma_1\sigma_2} a'$ and $b \rightarrow_{\sigma_1'\sigma_2'} b'$ with $a < a'$, $b < b'$, and $\text{path}(\sigma_1) = \text{path}(\sigma_2) = \text{path}(\sigma_2') = \text{path}(\sigma_2') = \emptyset$. Let $\sigma := \sigma_1\sigma_2$ and $\sigma' := \sigma_1'\sigma_2'$. We define $a_i := a + \Delta(\sigma[1..i])$ and $b_i := b + \Delta(\sigma'[1..i])$ as the counter values along runs $a \rightarrow_{\sigma} a'$ and $b \rightarrow_{\sigma'} b'$, respectively. In particular, $a_0 = a$ and $b_0 = b$. Further, $a_{2n} = a'$ and $b_{2n} = b'$. We differentiate cases based on the order of $a$ and $b$.

Case $b < a$: Assume there exists $i$ such that $b_i \geq a_i$. By Corollary 34, it follows that $a \rightarrow_{\text{path}(\sigma[1..i])} b_i$. By definition of $\sigma'$, we have $b_i \rightarrow_{\sigma'[1+1..2n]} b'$. Thus, $a \rightarrow_{\Theta^\nu} b'$, and we are done. So it remains to handle the case where $b_i < a_i$ for all $i$. Recall that we assume $b' > a$. Thus, we have $b' > a$, $b \leq a$, and $b' \leq a'$. So, we can invoke Lemma 35, from which it follows that $a \rightarrow_{\Theta^\nu} b'$ as desired. See Figures 3(a) and 3(b) for illustrations of both subcases.

Case $b > a$: This is similar to the previous case. If there exists $i$ such that $b_i \leq a_i$, then we derive $a \rightarrow_{\Theta^\nu} b'$ from Corollary 34. Thus, assume that $b_i > a_i$ for all $i$. Intuitively, we now iterate $\sigma$ to obtain larger and larger configurations, until we find a configuration that exceeds the respective configuration on the run from $b$. See Figure 3(c) for a sketch of this case.

Formally, let $a_{i,0} := a_0 + j \cdot \Delta(\sigma)$, and $a_{j,i} := a_{j,0} + \Delta(\sigma[1..i])$. Recall that $b_i = b + \Delta(\sigma'[1..i])$. Let $j$ be the first index such that there exists $i$ such that $a_{j,i} \geq b_i$, and let $i$ be the smallest value for which this holds for that choice of $j$. Note that such a $j$ must exist, since $\Delta(\sigma) > 0$, as $a < a_n$. Furthermore, note that by minimality of $j$, it follows that $i > 0$, as $a_{j,0} = a_{j-1,n}$.

We argue that $\sigma' \sigma[1..i-1]$ is an admissible run from $a$ to $a_{j,i-1}$. We first show that for all $j', i'$ that $a_{0,i'} \leq a_{j',i'}$. It is clear that $a_{j',i'} = j' \cdot \Delta(\sigma) + \Delta(\sigma[1..i']) = j' \cdot \Delta(\sigma) + a_{0,i'} > a_{0,i'}$ because $\Delta(\sigma) > 0$. Further, for $j' < j$ and for all $i'$, it holds that $a_{j',i'} < b_i'$ by minimality of $j$. In a similar manner, $a_{j,i'} < b_i'$ holds for all $i' < i$ by minimality of $j$ and $i$. Therefore, we have

$$a_{0,i'} < a_{j',i'} < b_i' \quad \text{for all} \quad j' < j \quad \text{and} \quad i' \quad \text{and} \quad a_{0,i'} < a_{j,i'} < b_i' \quad \text{for all} \quad i' < i.$$

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ACM Transactions on Computational Logic, Vol. 24, No. 1, Article 3. Publication date: January 2023.
Note the subtle difference between the statements, as the latter statement is concerned with the parts of the run from \( a_{i,0} \) to \( a_{i,i} \), and the former statement is concerned with the parts of the run from \( a_{0,0} \) to \( a_{1,0} \), from \( a_{1,0} \) to \( a_{1,i} \), and so on.

Recall that run \( \sigma \) from \( a \) gives rise to values \( a_{0,i} \), and that run \( \sigma' \) from \( b \) gives rise to values \( b_{i} \). Thus, admissibility of \( \sigma' \sigma[1..i-1] \) from \( a \) follows from admissibility of \( \sigma \) from \( a \) and of \( \sigma' \) from \( b \). So we have \( a \rightarrow_{\sigma[1..i-1]} a_{j,i} \). We further have \( a_{j,i} \geq b_{i} \) and \( a_{j,i-1} < b_{i-1} \) by choice of \( j,i \). By definition of \( a_{j,i} \) and \( b_{i} \), there exist \( \alpha, \beta \) such that \( a_{j,i} = a_{j,i-1} + \alpha \Delta(t_{i}) \) and \( b_{i} = b_{i-1} + \beta \Delta(t_{i}) \). By Lemma 33, there thus exists \( \alpha' \) such that \( a_{j,i-1} + \alpha' \Delta(t_{i}) = b_{i} \). So we derive that \( a_{j,i-1} \rightarrow_{\mu} b_{i} \).

Thus, we finally derive that

\[
a \rightarrow_{\sigma[1..i-1]} a_{j,i-1} \rightarrow_{\mu} b_{i} \rightarrow_{\sigma[i+1..2n]} b'.
\]

Hence, \( a \rightarrow_{\text{path}(\sigma)^{j+1}} b' \), and by definition of \( \sigma \), \( a \rightarrow_{\text{path}(\sigma)^{j+1}} b \). So \( b \in \text{Post}_{\mu}(\text{path}(\sigma)^{j+1}) \). By assumption, \( a \not\rightarrow \theta b \), so \( b \in \text{Post}_{\theta}(\theta b)^{j+1} \cap \text{Post}_{\theta}(\text{path}(\sigma)^{j+1}) \).

If \( b \in \text{Post}_{\theta}(\theta b)^{j+1} \) \( \rightarrow \) it holds that \( b' \in \text{Post}_{\theta}(\theta b)^{j+1}(a) \cup \text{Post}_{\theta}(\text{path}(\sigma)^{j+1})(a) \). Thus, \( a \rightarrow_{\theta b} b' \) or \( a \rightarrow_{\theta \theta} b' \). If \( a \not\rightarrow_{\theta} b' \), then clearly we are done. So assume \( a \rightarrow_{\theta} b' \). By Lemma 38, there exist values \( a_{0}, \ldots, a_{n} \) such that \( a_{0} \rightarrow_{\theta} a_{1} \rightarrow_{\theta} a_{2} \cdots \rightarrow_{\theta} a_{n} \) and \( a = a_{0} < a_{1} < \cdots < a_{n} = b' \). Note that because \( a \not\rightarrow_{\theta} b' \), the sequence must have at least three elements. So take the first three elements \( a_{0}, a_{1}, a_{2} \). Since \( a_{0} < a_{1} < a_{2} \), there exists \( a_{0} \rightarrow_{\theta} a_{1} \rightarrow_{\theta} a_{2} \). We have \( a \rightarrow_{\theta^2} a_{2} \), and so (1) is satisfied. By a similar argument, \( a_{n-2} < a_{n-1} < a_{n} \) and \( a_{n-2} \rightarrow_{\theta} a_{n-1} \rightarrow_{\theta} a_{n} = b, \) and hence (2) is satisfied.

Now we are ready to define and prove the correctness of \( \phi_{L}^{+}(a, b, \mu) \). Let

\[
\phi_{L}^{+}(a, b, \mu) := (a > b) \land [\phi_{\theta}(a, b, \mu) \lor \exists a' > a, \exists b' < b : \phi_{\theta\theta}(a, a', \mu) \land \phi_{\theta\theta}(b', b, \mu)].
\]

Lemma 40. It is the case that \( \phi_{L}^{+}(a, b, \mu) \) is satisfied iff \( a < b \) and \( b \in \text{Post}_{\theta}(\theta b)^{j+1}(a) \cup \text{Post}_{\theta}(\theta b)^{j+1}(a) \).

Proof. It follows immediately from the sufficient and necessary condition of Lemma 39.

Corollary 41. It is the case that \( \phi_{\theta}(a, b, \mu) \) is satisfied iff \( b \in \text{Post}(\theta b)^{j+1}(a) \). Furthermore, the size of \( \phi_{\theta} \) is linear in the length of \( \theta \) and nonparametric updates and endpoints on \( \theta \).

5.3 Formula for a Linear Path Scheme

Lemma 42. For every linear path scheme \( L = \sigma_{0} \theta_{0}^{*} \sigma_{1} \theta_{1}^{*} \cdots \sigma_{k} \), there exists a formula of existential linear arithmetic \( \phi_{L}(a, b, \mu) \) such that \( \phi_{L}(a, b, \mu) \) holds iff \( b \in \text{Post}_{\mu}(\text{path}(\sigma)^{j+1})(a) \), and \( \phi_{L} \) has size polynomial in \( |L| \).

Proof. We define the following formula whose correctness follows Lemma 36 and Corollary 41:

\[
\phi_{L}(a, b, \mu) := \exists a_{0}, a'_{0}, \ldots, a_{k}, a'_{k} : (a_{0} = a) \land (a'_{k} = b) \land \bigwedge_{0 \leq i \leq k} \phi_{a_{i}}(a_{i}, a'_{i}, \mu) \land \bigwedge_{0 \leq i < k} \phi_{a_{i}'}(a'_{i}, a_{i+1}, \mu).
\]

Theorem 43. The existential reachability problem for parametric COCA is in NP.

Proof. By Lemma 32, \( b \in \text{Post}_{P}(\theta b)(a) \) iff there exists a linear path scheme \( L = \sigma_{0} \theta_{0}^{*} \sigma_{1} \theta_{1}^{*} \cdots \sigma_{k} \) of polynomial size such that \( b \in \text{Post}_{L}(a) \). By Lemma 42, from any linear path scheme \( L \), we can construct, in polynomial time, an existential linear formula \( \phi_{L} \) such that \( \phi_{L}(a, b, \mu) \) holds iff \( b \in \text{Post}_{\mu}(\text{path}(\sigma)^{j+1})(a) \). By quantifying existentially over \( \mu \), we obtain a formula \( \phi_{L}^{3}(a, b) := \exists \mu : \phi_{L}(a, b, \mu) \), which is satisfied iff \( b \in \text{Post}_{\mu}(a) \).

Thus, to determine whether \( p(a) \rightarrow q(b) \), we (1) guess a linear path scheme \( L \), (2) construct \( \phi_{L}^{3} \), and (3) check whether \( \phi_{L}^{3}(a, b) \) holds. Since (3) can be achieved in NP [23], we are done.
5.4 Integer Valuations

We briefly consider parametric COCAs where only updates can be parameterized. In this setting, a rational valuation that witnesses reachability can be turned into an integer valuation witnessing reachability. This follows by rescaling the factors of the witnessing run so that it remains admissible.

Lemma 44. Let \( \mu \) be a valuation under which \( p(a) \rightarrow_s q(b) \). For any valuation \( \mu' \) such that \( \mu'(x) = \lambda \mu(x) \) with \( \lambda \in \mathbb{N}_{\geq 1} \), it is the case that \( p(a) \rightarrow_s q(b) \) under \( \mu' \).

Proof. Let \( \lambda \in \mathbb{N}_{\geq 1} \) and let \( \mu' \) be defined w.r.t. \( \mu \) and \( \lambda \). Let \( s(v) \rightarrow_{at} s'(v') \) be consecutive configurations from the run \( \rho \) witnessing \( p(a) \rightarrow_{\rho} q(b) \) under valuation \( \mu \). If \( \Delta(t) \in \mathbb{Q} \), then no rescaling is needed as the update of \( t \) is nonparametric. Otherwise, we have \( v' - v = \alpha \cdot \mu(\Delta(t)) \).

Now, consider a rational valuation \( \mu \) witnessing \( p(a) \rightarrow_s q(b) \). Since \( \mu \) is rational, each parameter value \( \mu(x) \) can be represented as a fraction \( a_x/b_x \). By Lemma 44, we know that valuation \( \mu'(x) = \lambda \mu(x) \), where \( \lambda := \prod_{x \in X} a_x/b_x \), also witnesses reachability. Moreover, it is integral, hence:

Corollary 45. The (existential) reachability problem for parametric COCAs, where valuations must be integral, is equivalent to the rational variant if guards are nonparametric.

5.5 Hardness

To conclude our treatment of parametric COCAs, we establish NP-hardness of the reachability problem, even for the special case of acyclic COCAs.

Theorem 46. The reachability problem for acyclic parametric COCAs is NP-hard, even when parameters occur only on updates or only on guards.

Proof. The reductions for the case where parameters are only on updates and only on guards are very similar, differing only in the construction of certain gadgets that otherwise have the same purpose, so we present these two reductions in parallel.

We give a reduction from 3-SAT. Let \( \varphi = \bigwedge_{1 \leq j \leq m} C_j \) be a 3-CNF formula over variables \( X = \{x_1, \ldots, x_n\} \).

Let us give two acyclic parametric COCAs \( P \) and \( P' \), both with parameters \( X \). Each one will guess an assignment to \( X \) and check whether it satisfies \( \varphi \). Additionally, \( P \) uses parameters only on guards; \( P' \), only on updates. We sketch the constructions in the following.

The first part is done by sequentially composing \( n \) copies of the gadget depicted at the top of Figure 4, for \( P \) on the left-hand side, and for \( P' \) on the right-hand side. The gadgets function as follows: (1) state \( p_1 \) is entered with counter value 0, (2) the counter is set to \( x_i \), (3) membership of the counter value in \( \{0, 1\} \) is checked, and (4) the counter is reset to zero upon leaving to \( q_i \). The only way to traverse the chain of \( n \) such gadgets from \( p_1 \) to \( q_n \) is to have \( x_i \in \{0, 1\} \) for each \( x_i \in X \).

The second part is achieved by chaining a gadget for each clause similar to the one depicted on the bottom of Figure 4 for \( C_j = (x_1 \lor x_2 \lor -x_3) \). The left-hand side depicts the gadget for \( P \), and the right-hand side depicts it for \( P' \). In words, it (1) enters state \( r_j \) with the counter value set to 0, (2) nondeterministically picks a variable \( x_i \) of some literal of \( C_j \) and increments the counter by \( x_i \), (3) checks whether the counter holds the right value w.r.t. the literal polarity, and (4) resets the counter to zero upon leaving to state \( s_j \). Thus, the chain of gadgets can be traversed from \( r_1 \) to \( s_m \) iff \( \varphi \) is satisfied by the assignment.
Altogether, these statements are equivalent: (1) formula $\varphi$ is satisfiable, (2) there exists a valuation $\mu : X \to \mathbb{Q}$ such that $p_1(0) \rightarrow_{s} s_m(0)$ holds in $P^\mu$, and (3) there exists a valuation $\mu' : X \to \mathbb{Q}$ such that $p_1(0) \rightarrow_{s} s_m(0)$ holds in $P^{\mu'}$.

Thus, together with Theorem 43, NP-completeness of the existential reachability problem in parametric COCAs follows.

6 CONCLUSION

In this work, we have introduced globally guarded COCA and COCA as over-approximations of SOCA, and we have given efficient algorithms for their reachability problems. For both models, the only hardness result we are aware of is the NL-hardness that follows trivially from the directed-graph reachability problem. Giving tighter hardness results for the complexity of reachability in GG-COCA and COCA seems desirable yet challenging. In particular, one goal could be to show an equivalence with other problems known to be in NC$^2$ or P-hard. However, it seems necessary to encode information in the counter that, intuitively, encodes more precise information than that the counter value is in a certain range. If only information of this sort is available, then the problem is equivalent to reachability in one-clock timed automata, which is in NL [16]. However, the continuous semantics seem to make encoding more precise information nontrivial. Consequently, it remains open whether our algorithms are computationally optimal.

For parametric COCA, we have shown that the reachability problem is NP-complete when numbers are encoded in binary. Our construction for establishing NP-hardness works regardless of whether numbers are encoded in unary or in binary, but it works only when there is an arbitrary number of parameters. For a fixed number of parameters, including no parameters at all, the only hardness result is the trivial NL-hardness bound derived from graph reachability. The complexity results are summarized in Table 1.
Table 1. Overview of the Complexity for COCA Reachability, Depending on Whether There Are Zero, a Fixed Number, or an Arbitrary Number of Parameters, and Whether Numbers Are Encoded in Unary or in Binary

<table>
<thead>
<tr>
<th></th>
<th>Unary</th>
<th>Binary</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>NL-complete</td>
<td>∈ P</td>
</tr>
<tr>
<td>Fixed</td>
<td>NL-complete</td>
<td>∈ NP</td>
</tr>
<tr>
<td>Arbitrary</td>
<td>NP-complete</td>
<td>NP-complete</td>
</tr>
</tbody>
</table>

REFERENCES


Received 1 November 2021; revised 25 March 2022; accepted 7 June 2022