Stability of Central Finite Difference Schemes on Non-uniform Grids for the Black–Scholes PDE with Neumann Boundary Condition

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Abstract. This paper concerns the numerical solution of the Black–Scholes PDE with a Neumann boundary condition on the right boundary. We consider finite difference schemes for the semi-discretization, which leads to a system of ODEs with corresponding matrix M . In this paper stability bounds for $\exp(tM)$ ($t \geq 0$) are proved. A scaled version of the Euclidean norm, denoted by $\|\cdot\|_H$ is considered. The advection and diffusion term of the PDE are analyzed separately. It turns out that the Neumann boundary condition leads to a growth of $\|\exp(tM)\|_H$ with the number of grid points m for the pure advection problem.

Keywords: Black–Scholes partial differential equation, finite difference schemes, non-uniform grids, stability, Neumann boundary condition.

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INTRODUCTION

Let $u(s, t)$ denote the fair price of a European option if the underlying asset price is equal to s at time $T - t$. Financial option pricing theory yields that u satisfies the Black–Scholes partial differential equation (PDE) [1],

$$\frac{\partial u}{\partial t}(s, t) = \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2}(s, t) + rs \frac{\partial u}{\partial s}(s, t) - ru(s, t) \quad (s > L, 0 < t \leq T). \quad (1)$$

Here $r > 0$ and σ represent the interest rate and the volatility, respectively, which are given real constants. Secondly, $L \geq 0$ and $T > 0$ denote the given lower barrier and maturity time of the option. To render the numerical solution feasible, we restrict the spatial domain to a bounded set $[L, S]$ with fixed S taken sufficiently large. Initial and boundary conditions for (1) are determined by the specific option under consideration. For boundary conditions of Dirichlet type, stability estimates have been proved in [2]. In this paper we assume that $u(L, t)$ is given, i.e. a Dirichlet condition at the left boundary $s = L$ is provided. At the right boundary $s = S$ a *Neumann condition* is considered, i.e. $u_s(S, t)$ is given. This type of boundary conditions is frequently used in practical applications.

We consider a spatial discretization by finite difference (FD) schemes on non-uniform grids. We denote the grid points by $L = s_0 < s_1 < \dots < s_m < s_{m+1} = S$ and let $h_i = s_i - s_{i-1}$. Due to the Neumann boundary condition at $s = S$, an auxiliary point $s_{m+2} = s_{m+1} + h_{m+2}$ is useful, where it is natural to define $h_{m+2} := h_{m+1}$.

In this paper we consider the following FD schemes: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be any given smooth function and $1 \leq i \leq m+1$. Write $H_i = h_i + h_{i+1}$. For approximating the first derivative $f'(s_i)$ we deal with the central scheme

$$f'(s_i) \approx \frac{f(s_{i+1}) - f(s_{i-1}))}{H_i}, \quad (2)$$

which is called *method A* in [2, 5]. For approximating the second derivative $f''(s_i)$ we consider the central FD scheme

$$f''(s_i) \approx \frac{2}{h_i H_i} f(s_{i-1}) - \frac{2}{h_i h_{i+1}} f(s_i) + \frac{2}{h_{i+1} H_i} f(s_{i+1}). \quad (3)$$

Both schemes (2) and (3) have a second-order truncation error whenever the grid is smooth. For more information on smooth grids, we refer to [2].

This semi-discretization leads to an initial value problem for a system of ordinary differential equations (ODEs) of the form

$$U'(t) = MU(t) + b(t) \quad (0 \leq t \leq T), \quad U(0) = U_0. \quad (4)$$

Here M is a given constant $(m+1) \times (m+1)$ -matrix and $b(t)$ ($0 \leq t \leq T$) and U_0 are given $(m+1)$ -vectors. The vector U_0 is directly given by the initial condition and the vector function b is determined by the boundary conditions. For each $t > 0$, the entries of the solution vector $U(t)$ of (4) form approximations to the exact solution values $u(s, t)$ at the spatial grid points $s = s_1, s_2, \dots, s_{m+1}$. In this paper stability for the semi-discretization pertinent to the FD schemes (2) and (3) is analyzed. Accordingly, we are interested in rigorous bounds of the type $\|e^{tM}\| \leq Ke^{t\omega}$ for $t \geq 0$, where $\|\cdot\|$ denotes a given induced matrix norm and $\omega \in \mathbb{R}$ and $K \geq 1$ are constants, see e.g. [4]. To this purpose a scaled version of the Euclidean norm is considered. Let $|\cdot|_2$ be the Euclidean vector norm and let $\|\cdot\|_2$ denote the induced matrix norm. For vectors $x \in \mathbb{R}^{m+1}$ the scaled spectral norm is defined by

$$|x|_H = |H^{1/2}x|_2,$$

where H is a given scaling matrix. Our research revealed that

$$H = \frac{1}{2} \begin{pmatrix} H_1 & & & \\ & \ddots & & \\ & & H_m & \\ & & & h_{m+1} \end{pmatrix} \quad (5)$$

is a convenient choice. For matrices $M \in \mathbb{R}^{(m+1) \times (m+1)}$, the induced matrix norm is denoted by $\|M\|_H$.

STABILITY ESTIMATES IN THE NORM $\|\cdot\|_H$

In this section we prove rigorous stability estimates for the spatial discretization of the advection and diffusion term separately.

Advection

Spatial discretization of the advection term rsu_s with the FD scheme (2) leads to an $(m+1) \times (m+1)$ -matrix

$$M_a = r \cdot \begin{pmatrix} 0 & \frac{s_1}{H_1} & & \\ -\frac{s_2}{H_2} & 0 & \frac{s_2}{H_2} & \\ & \ddots & \ddots & \ddots \\ & & -\frac{s_m}{H_m} & 0 & \frac{s_m}{H_m} \\ & & & 0 & 0 \end{pmatrix}. \quad (6)$$

Note that the last row of this matrix contains only zeros, since $u_s(S, t)$ is given.

First we introduce the following lemma which will be useful to find an upper bound for $\|e^{tM_a}\|_H$. The proof is obtained by elementary calculus.

Lemma 1 Let $X \in \mathbb{R}^{m \times m}$ and $y \in \mathbb{R}^m$. It holds that

$$\left\| \begin{pmatrix} X & y \\ 0 & 1 \end{pmatrix} \right\|_2 \leq \sqrt{\|X\|_2^2 + (|y|_2 + 1)^2}.$$

This brings us to the stability theorem of this section.

Theorem 2 Consider the advection matrix M_a given by (6). It holds that

$$\|e^{tM_a}\|_H \leq \sqrt{e^{2t\omega} + \left(\frac{rs_m}{\sqrt{H_m h_{m+1}}} \cdot \frac{e^{t\omega} - 1}{\omega} + 1 \right)^2} \quad \text{with} \quad \omega = \frac{r}{1 + \sqrt{\varepsilon}}, \quad \varepsilon = \min_{2 \leq i \leq m} \frac{h_{i-1}h_{i+1}}{h_i^2}. \quad (7)$$

Proof Define $A \in \mathbb{R}^{m \times m}$ as the upper-left block of M_a , and $v \in \mathbb{R}^m$ as the first m elements of the last column of M_a , so that

$$M_a = \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix}. \quad (8)$$

It is readily shown that

$$e^{tM_d} = \left(\begin{array}{c|c} e^{tA} & \int_0^t e^{(t-\tau)A} d\tau \cdot v \\ \hline 0 & 1 \end{array} \right).$$

Analogously denote by $\tilde{H} \in \mathbb{R}^{m \times m}$ the upper-left block of H , where H is given by (5). Application of Lemma 1 yields that

$$\|e^{tM_d}\|_H \leq \sqrt{\left\| \tilde{H}^{1/2} e^{tA} \tilde{H}^{-1/2} \right\|_2^2 + \left(\left\| \sqrt{\frac{2}{h_{m+1}}} \cdot \tilde{H}^{1/2} \int_0^t e^{(t-\tau)A} d\tau \cdot v \right\|_2 + 1 \right)^2}. \quad (9)$$

In [2, Theorem 2.2] it has been proved that

$$\left\| \tilde{H}^{1/2} e^{tA} \tilde{H}^{-1/2} \right\|_2 \leq e^{t\omega} \quad (10)$$

with ω defined by (7). Next,

$$\begin{aligned} \left\| \sqrt{\frac{2}{h_{m+1}}} \cdot \tilde{H}^{1/2} \int_0^t e^{(t-\tau)A} d\tau \cdot v \right\|_2 &\leq \sqrt{\frac{2}{h_{m+1}}} \cdot \int_0^t \left\| \tilde{H}^{1/2} e^{(t-\tau)A} \tilde{H}^{-1/2} \right\|_2 d\tau \cdot \left\| \tilde{H}^{1/2} v \right\|_2 \\ &\leq \sqrt{\frac{2}{h_{m+1}}} \cdot \int_0^t e^{(t-\tau)\omega} d\tau \cdot \frac{rs_m}{\sqrt{2H_m}} = \frac{rs_m}{\sqrt{H_m h_{m+1}}} \cdot \frac{e^{t\omega} - 1}{\omega} \end{aligned} \quad (11)$$

Combining (9), (10) and (11) concludes the proof. □

Note that the upper bound in (7) is $\mathcal{O}(m)$ for smooth grids.

Diffusion

Spatial discretization of the diffusion term $\frac{1}{2}\sigma^2 s^2 u_{ss}$ with the FD scheme (3) and using

$$u(s_{m+2}, t) \approx u(s_m, t) + 2h_{m+1} \cdot u_s(s, t)$$

leads to an $(m+1) \times (m+1)$ -matrix

$$M_d = \sigma^2 \cdot \begin{pmatrix} -\frac{s_1^2}{h_1 h_2} & \frac{s_1^2}{h_2 H_1} & & & \\ \frac{s_2^2}{h_2 H_2} & -\frac{s_2^2}{h_2 h_3} & \frac{s_2^2}{h_3 H_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{s_m^2}{h_m H_m} & -\frac{s_m^2}{h_m h_{m+1}} & \frac{s_m^2}{h_{m+1} H_m} \\ & & \frac{s_{m+1}^2}{h_{m+1}^2} & -\frac{s_{m+1}^2}{h_{m+1}^2} & \end{pmatrix}. \quad (12)$$

In an analogous fashion to [2, Theorems 2.6, 2.8] the following useful stability result for the semidiscrete diffusion matrix can be proved.

Theorem 3 Consider the matrix M_d given by (12). Then

$$\begin{aligned} \|e^{tM_d}\|_H &\leq e^{t\omega} \quad (t \geq 0) \quad \text{with } \omega = \frac{1}{2}\sigma^2, \\ \|e^{tM_d}\|_H &\leq \sqrt{\frac{S}{s_1}} \quad (t \geq 0). \end{aligned}$$

FIGURE 1. Left: Graph of $\max\{\|e^{tM_a}\|_H : t = 0, 1, \dots, 100\}$ vs. $m = 10, 20, \dots, 1000$, where M_a is given by (6) and $r = 0.1$. Right: Graph of $\max\{\|e^{tM_d}\|_H : t = 0, 1, \dots, 100\}$ vs. $m = 10, 20, \dots, 1000$, where M_d is given by (12) and $\sigma = 0.3$. We consider both $L = 0$ (solid line) and $L = 10$ (dashed line).

NUMERICAL EXPERIMENT

In our numerical experiment we use a smooth, non-uniform spatial grid, which is described in detail in [2].

We numerically estimate the maximum of $\|e^{tM_a}\|_H$ and $\|e^{tM_d}\|_H$ over $t \geq 0$, where M_a and M_d are given by (6) and (12) respectively, for $m = 10, 20, \dots, 1000$. The results are displayed in Figure 1. Clearly, the maximum of $\|e^{tM_a}\|_H$ over $t \geq 0$ grows with the dimension m , which is in agreement with Theorem 2. Note that, although there is a growth with m , there still exists a moderate upper bound for the maximum of $\|e^{tM_a}\|_H$ up to $m = 1000$. Therefore we do not expect a reduction in the order of convergence to be observed in practice.

The maximum of $\|e^{tM_d}\|_H$ over $t \geq 0$ does not grow with the dimension m , which is in agreement with Theorem 3, and is furthermore of moderate size.

CONCLUSIONS AND FUTURE RESEARCH

In this paper it is shown that the Neumann boundary condition leads to an upper bound on $\|e^{tM_a}\|_H$ which grows with the dimension m . Although this theoretical upper bound gives a good indication of the actual behavior, it is seen from the numerical experiment that the actual upper bound is smaller than the theoretical upper bound (7). Therefore we conjecture that there exists a smaller theoretical upper bound, which is left for future research.

An alternative boundary condition which is widely used in financial applications is the *linear boundary condition*, i.e. $u_{ss}(S, t) = 0$. Stability and convergence results for this practical case will soon appear in [3].

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