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Moves on Pseudoline Diagrams

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Pseudoline diagrams are simple arrangements of pseudolines in the affine plane where at each vertex one indicates which pseudoline *crosses over* the other. They naturally appear as projections of configurations of lines in 3-space. Motivated by isotopies of these spatial configurations, we define an equivalence for diagrams, generated by two types of moves. We encode diagrams by words and consider the associated word problem. Furthermore, in this combinational framework we obtain two new results on the *chirality* of triples of lines, which are useful in the study of isotopies of line configurations in 3-space.

1. INTRODUCTION

The motivation for this paper is the isotopy problem of line configurations in 3-space. A *rigid isotopy* of a configuration $\mathscr{L}$ of n mutually skew lines in $\mathbb{R}^3$ is a homotopy of the ambient space, $\{h_t; t \in [0, 1]\}$, such that, for each $t \in [0, 1]$ the image $h_t(\mathscr{L})$ is still a configuration of skew lines. The ultimate goal would be to classify such configurations up to rigid isotopy. In particular, we look for techniques which are able to distinguish the isotopy type of one configuration from another.

As in recent developments in the theory of *knots* and *links*, we have chosen here a 'diagram approach'. If we consider an appropriate (central or parallel) projection of $\mathbb{R}^3$ upon a plane for a given configuration of skew lines, we obtain a *simple arrangement* of lines in the affine plane (no pair of lines is parallel, and no triple of lines is concurrent). Furthermore, if this plane is given some orientation, we can determine for each double point of the projection which line is 'crossing over' the other. The projection of a line configuration augmented with this crossing information is called its *planar layout,* and is visually represented as in Figure 3 below, in the same spirit as knot diagrams or link diagrams. Of course, the planar layout of a configuration of lines 'moves along' if one performs a rigid isotopy. The ways in which the combinatorial type of a planar layout changes during an isotopy can be restricted to two types of local diagram moves (and finite sequences of these), namely $\|$-moves and $*$-moves (see Figure 6 below) [2]. However, these moves have a purely combinatorial description, and there might be geometric obstacles to realizing a given sequence of moves by a spatial rigid isotopy of lines. So, in order to create a framework comfortable enough to focus on the combinatorial behavior of these moves, and to disregard the more involved geometric constraints for the moment, we introduce *pseudoline diagrams.* To construct these diagrams one takes a simple arrangement of pseudolines in the affine plane $\mathbb{R}^2$, and then one adds 'depth information' at the crossings and obtains a weaving pattern upon the underlying arrangement. Now we can consider a topological relaxation of the original rigid isotopy problem of spatial line configurations: we face the problem of whether two given pseudoline diagrams are equivalent under $\|$-moves and $*$-moves.

In Section 2 we repeat the relevant definitions and facts of pseudoline arrangements. For our purposes, the most convenient way to encode such arrangements (in the affine

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plane!) is by means of *vertex orders*. These are the orders in which the intersections (vertices) of an arrangement appear when its pseudolines are drawn all 'monotone' with respect to a given direction in the plane. We characterize those vertex orders coming from arrangements of pseudolines. Furthermore, we define two 'word moves' on vertex orders, giving rise to equivalence classes that are in one-to-one correspondence to isomorphism classes of arrangements. On the way, we give a new proof for a theorem of Goodman and Pollack, saying that every arrangement of pseudolines is isomorphic to a 'wiring diagram arrangement', but without making use of Levi's Enlargement Lemma.

In Section 3 we introduce the leading concepts in the paper, namely pseudoline diagrams and *chirality*. The latter is a sign which is assigned to three woven pseudolines (see Figure 5 below). It is the combinatorial generalization of the product of the three linking numbers of three disjoint lines in $\mathbb{RP}^3$ with arbitrary orientations, a well-known isotopy invariant for lines [16]. In [9] it is shown that the chiralities of a given line configuration do not determine its rigid isotopy type. In Section 4, we recall the most important facts from the area of line configurations and rigid isotopy, which are necessary to motivate the reader for moves on pseudoline diagrams.

In Section 5 we add weaving information to vertex orders and obtain *diagram words* which encode pseudoline diagrams. We define four types of moves on diagram words which generate the equivalence of diagrams. In fact, these four types of word moves are an extension of the two types of moves on vertex orders considered before. This enables us to prove the decidability of the equivalence of pseudoline diagrams under diagram moves.

In Section 6 we enlarge our set of admissible diagram moves by replacing $*$-moves by arbitrary 'triangle moves'. We prove that the new equivalence classes exactly match the chirality classes (diagrams with the same chiral signature), which gives us a better understanding of how far chirality is from being a complete invariant. In Section 7 we consider a special class of diagrams, called *stacks.* These are pseudoline diagrams where the under–over relation between each pair of pseudolines gives rise to a total order on the lines. We prove that the chiralities are a complete invariant for the equivalence of stacks. We regard these two theorems as the most important results of this paper.

Although the study of moves on pseudoline diagrams has been exclusively motivated by the isotopy problem for line configurations, it happens to be useful for the *realizability problem* for line diagrams as well. In Section 8 we illustrate this lucky fact by an example.

2. Affine Arrangements of Pseudolines, Intersection Patterns and Vertex Orders

In this section we recall some facts of arrangements of pseudolines. For more details we refer to the relevant chapters of [1] or [5]. Most authors consider such arrangements in a projective setting and in close connection with 'oriented matroids'. A pseudoline in the real projective plane is a simple closed curve L in $\mathbb{P}^2$ such that $\mathbb{P}^2 \backslash L$ is connected. An arrangement of pseudolines in $\mathbb{P}^2$ is a finite collection of pseudolines, $\mathcal{P} = (L_e)_{e \in E}$, such that:

(1) $\cap \mathcal{P} = \varnothing$;

(2) if $e \neq f$ then L_e intersects L_f in exactly one point.

Two such arrangements are called *isomorphic* if they induce isomorphic cell complexes of $\mathbb{P}^2$. Equivalently, two arrangements of pseudolines are isomorphic if they can be mapped to each other under a self-homeomorphism of $\mathbb{P}^2$. However, for our purposes it is necessary to consider pseudolines in the affine plane, exclusively.

An *affine pseudoline* is the image of a straight line under a self-homeomorphism of $\mathbb{R}^2$. Furthermore, we require that an affine pseudoline has a *projective continuation.* An *arrangement of affine pseudolines* is a finite collection of affine pseudolines, $\mathcal{H} = \{h_1, \ldots, h_n\}$, such that each pair has exactly one point in common, where they *transversally* cross. $\mathcal{H}$ is called *simple* if no point of the plane belongs to more than two pseudolines of $\mathcal{H}$. By default, an 'arrangement' will always mean a simple arrangement of labeled affine pseudolines. We will always assume that $n \geq 2$.

One can always extend an arrangement $\mathcal{H}$ to an arrangement $\mathcal{H}^*$ of pseudolines in $\mathbb{P}^2$, called the *projective completion of* $\mathcal{H}$ [1]. If we fix the 'line at infinity' in $\mathbb{P}^2$, L_∞, then we can define two labeled arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$ to be *isomorphic* or *of the same combinatorial type,* denoted by $\mathcal{H}_1 \cong \mathcal{H}_2$, if $(\mathcal{H}_1^*, L_\infty)$ and $(\mathcal{H}_2^*, L_\infty)$ are isomorphic labeled arrangements in $\mathbb{P}^2$. Equivalently, $\mathcal{H}_1 \cong \mathcal{H}_2$ iff they can be mapped to each other by a self-homeomorphism of the affine plane which respects the labels. $\mathcal{H}$ is called *stretchable* if it is isomorphic to an arrangement of straight lines. Levi [7] observed the existence of non-stretchable pseudoline arrangements, while it was shown by Ringel [15] that there are non-stretchable simple arrangements as well. The smallest such example has size $n = 9$. It is known that every arrangement of at most eight pseudolines is stretchable [4]. However, it is not hard to prove, by induction on the number of pseudolines, that every arrangement of pseudolines is isomorphic to an arrangement having polygonal pseudolines [5]. For the convenience of the following sections, we will now derive some equivalent ways in which to encode isomorphism classes of arrangements. Although most of the theorems in this section are probably well known, we chose to present them with complete proofs anyway, because we do not know of any systematic description elsewhere in the literature.

We can equip each pseudoline h_i of an arrangement $\mathcal{H}$ with an orientation $\overline{h_i}$. The order in which $\overline{h_i}$ intersects the other pseudolines gives rise to a total order on $\{1, \ldots, n\} \backslash \{i\}$, denoted by $I(\overline{h_i})$ and called the *intersection sequence* of $\overline{h_i}$. In general, an abstract *intersection pattern* is an array $(I_1, \ldots, I_n)$, where each I_i is a total order on $\{1, \ldots, n\} \backslash \{i\}$. We will often confuse total orders on finite sets with ordered sequences. Two intersection patterns $(I_1, \ldots, I_n)$ and $(I'_1, \ldots, I'_n)$ are said to be *reorientations* of each other if they can be obtained from each other by global reversals of some of the I_k or I'_k. In this terminology, we have associated a reorientation class of intersection patterns with each labeled arrangement. Let us detect the properties which characterize those intersection patterns that come from arrangements. Observe that each intersection pattern determines a relation R on the unordered pairs $\{i, j\}$ as follows. For ease of notation, we introduce the symbols $a_{ij} = a_{ji}$ for these pairs $\{i, j\}$. Now let $\mathcal{I} = (I_1, \ldots, I_n)$ be an intersection pattern: then we define its associated relation by

$$R(\mathcal{I}) = \{(a_{ij}, a_{ik}) \mid j < k \text{ in } I_i\}.$$

Theorem 1. *If $\mathcal{H}$ is a simple arrangement of pseudolines in $\mathbb{R}^2$, then we can choose orientations for the members of $\mathcal{H}$ such that the relation $R(\mathcal{I})$ has no cycles.*

Proof. First we draw a simple closed curve Γ in the affine plane such that it (tranversally) intersects each member of $\mathcal{H}$ exactly twice and such that it contains all intersections of $\mathcal{H}$ in its interior.† To this end, we may first have to replace $\mathcal{H}$ by an isomorphic arrangement of polygonal pseudolines. Let $\{a_i, b_i\}$ denote the intersection

† Here, and elsewhere in this paper, the 'interior' of a simple closed curve in the affine plane means the unique bounded component of its complement. Its 'exterior' is the other (unbounded) component (Jordan's Curve Theorem).

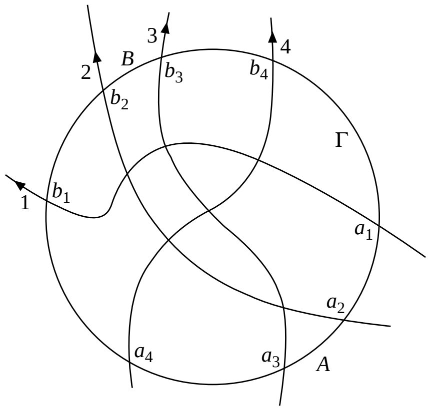

FIGURE 1. An oriented arrangement of pseudolines, leading to the monotone intersection pattern $\mathcal{I} = \{(432), (341), (241), (231)\}$. Notice that (a_{23}, a_{13}) and (a_{13}, a_{12}) both belong to $R(\mathcal{I})$, such that in the transitive closure $a_{23} < a_{13} < a_{12}$. This means that $2 \in \sigma\{1, 3\}$. Observe that a_2 separates $\{a_1, a_3\}$ on A.

of the pseudoline h_i with Γ, and write r_i for the portion of h_i between a_i and b_i (Figure 1).

Suppose first that $\{a_j, b_j\}$ does not separate $\{a_i, b_i\}$ on Γ, which means that there exist connected arcs A_i and A_j of Γ such that A_i connects a_i and b_i, A_j connects a_j and b_j, and $A_i \cap A_j = \varnothing$. Clearly, $A_i \cup r_i$ is a closed, simple curve. Recall that r_i and r_j have exactly one point in common, at which they cross. But this contradicts Jordan's Curve Theorem, because r_j connects two points in the exterior of $A_i \cup r_i$ and $r_j \cap A_i = \varnothing$. Consequently, there are two closed connected arcs A and B of Γ, $A \cap B = \varnothing$ such that, for all $i = 1, \ldots, n$,

$$|r_i \cap A| = |r_i \cap B| = 1.$$

Assume that $\{a_i\} = h_i \cap A$ and $\{b_i\} = h_i \cap B$, $i = 1, \ldots, n$. The orientations on h_i that are defined by $a_i < b_i$, finally, can be shown to satisfy the 'no cycle condition' for $R = R(\mathcal{I})$. □

REMARK. An orientation of (the elements of) $\mathcal{H}$ the corresponding intersection pattern $\mathcal{I}$ of which induces a cycle-free relation, $R(\mathcal{I})$, is called a *monotone orientation.* We will use the adjective *monotone* for intersection patterns $\mathcal{I}$ the relation $R(\mathcal{I})$ of which has no cycles, as well as for reorientation classes that contain such patterns. See Figure 1 for an illustration.

The counterclockwise orientation of Γ (proof of Theorem 1) induces total order on both $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$, yielding twice the same permutation of $(1, 2, \ldots, n)$. Since A is not uniquely determined, for it can start at any arbitrary point of Γ, this permutation is only determined up to circular shifts. The resulting circular order is called the *base order* of $\mathcal{H}$, denoted by $\beta(\mathcal{H})$. From the proof of Theorem 1 it follows that a given linear representative b of the base order always determines a monotone orientation for the pseudolines of $\mathcal{H}$. Since the parts of A and B may be switched, the reversal of all these orientations yields a second 'good' choice that is determined by b. In Theorem 2 we will see that every monotone orientation of $\mathcal{H}$ comes from a linear representative of the base order.

The circular base order $\beta(\mathcal{H})$ is unfortunately not invariant under isomorphism of labeled arrangements. This is a natural place to consider isotopy classes of such arrangements. Two arrangements of affine pseudolines are called *isotopic* if they can be reached from each other by a continuous deformation of the whole arrangement such that, at any time, it remains an arrangement of pseudolines of the same combinatorial type. For instance, if $\mathcal{H}'$ is obtained from $\mathcal{H}$ by a reflection w.r.t. some straight line, then $\mathcal{H}' \cong \mathcal{H}$, but $\mathcal{H}'$ is not isotopic to $\mathcal{H}$. Observe that $\mathcal{H}'$ has the reversed base order of $\mathcal{H}$. It is a result of Ringel that combinatorial types of arrangements fall apart into exactly two isotopy classes [15]. The base order is an invariant for these smaller classes.

If a relation R has no cycles then its (reflexive and) transitive closure is a partial order. In case $R = R(\mathcal{I})$ is induced by a monotone $\mathcal{I}$, this construction results into a partial order $<$ of $\mathcal{A} = \{a_{ij} = a_{ji} \mid \{i, j\} \subset \{1, \ldots, n\}\}$. Let $\{i, j, k\} \subset \{1, \ldots, n\}$: then k is called a *separator* of $\{i, j\}$ if a_{ij} 'separates' a_{ik} and a_{jk} w.r.t. $<$; that is, $a_{ik} < a_{ij} < a_{jk}$ or $a_{jk} < a_{ij} < a_{ik}$. Furthermore, we put $\sigma\{i, j\}$ to be the set of separators of $\{i, j\}$, and define $[i, j] = \sigma\{i, j\} \cup \{i, j\}$. Notice that from the definition it follows that $\{i, j\} \cap \sigma\{i, j\} = \varnothing$, and that for each $\{i, j, k\} \subset \{1, \ldots, n\}$ exactly one of the following three possibilities occurs:

$$k \in \sigma\{i, j\} \qquad \text{or} \qquad j \in \sigma\{i, k\} \qquad \text{or} \qquad i \in \sigma\{i, k\}.$$

We say that a monotone intersection pattern $\mathcal{I}$ has the *filter property* if

$$k \in \sigma\{i, j\} \Rightarrow [i, k] \cup [k, j] = [i, j]. \tag{FP}$$

Notice that this is equivalent to

$$k \in \sigma\{i, j\} \Rightarrow \sigma\{i, k\} \cup \{k\} \cup \sigma\{k, j\} = \sigma\{i, j\}.$$

A reorientation class of intersection patterns is said to have the filter property if it has a monotone member that satisfies (FP).

THEOREM 2. *Every monotone orientation of a given arrangement $\mathcal{H}$ is determined by some linear representative b of the circular order $\beta(\mathcal{H})$. Consequently, an intersection pattern $\mathcal{I}$ that is induced by a monotone orientation of $\mathcal{H}$ always satisfies* (*FP*).

PROOF. Let Γ be a closed simple curve as in the proof of Theorem 1. Furthermore, let $\{\overline{h_1}, \ldots, \overline{h_n}\}$ be a monotone orientation for $\mathcal{H}$, and say that Γ intersects the pseudolines h_i in a_i and b_i, respectively, with $a_i < b_i$ w.r.t. the orientation $\overline{h_i}$. Now suppose that some pair of $\{a_1, \ldots, a_n\}$ is separated by a pair of $\{b_1, \ldots, b_n\}$; say, we meet $(a_1 b_2 a_3 b_4)$ in this order during a (counterclockwise) traversal of Γ (Figure 2). We have already noted in the proof of Theorem 1 that $\{b_1, b_3\}$ cannot separate $\{a_1, a_3\}$. This implies that one element of $\{b_2, b_4\}$ is in a different component of $\Gamma \backslash \{a_1, a_3\}$ as b_1 and b_3, b_2 say. If s_{ij} denotes the intersection of h_i and h_j, and if $\pi_i(x, y)$ denotes the connected portion of h_i between two points x and y of h_i, then h_2 intersects either $\pi_1(a_1, s_{13})$ or $\pi_3(a_3, s_{13})$, but not both (Jordan's Curve Theorem). W.l.o.g., we may

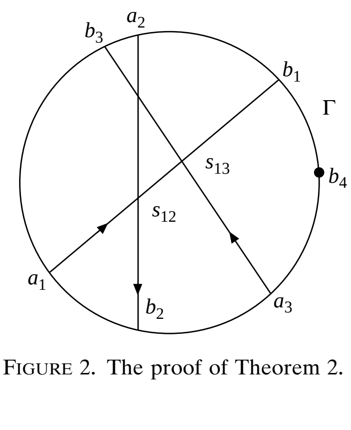

FIGURE 2. The proof of Theorem 2.

assume that $s_{12} \in \pi_1(a_1, s_{13})$. But then the simple closed curve $\pi_1(s_{12}, s_{13}) \cup \pi_3(s_{13}, s_{23}) \cup \pi_2(s_{23}, s_{12})$ is a direct cycle, which contradicts the monotone orientation of $\mathcal{H}$. We conclude that there exists a connected arc A on Γ such that $A \cap h_i = \{a_i\}$, for all $i = 1, \ldots, n$, of which counterclockwise orientation yields a linear representative b of $\beta(\mathcal{H})$. So, indeed, the given monotone orientation is determined by b. Furthermore, by virtue of Jordan's Curve Theorem, a_k separates a_i and a_j on A iff $k \in \sigma\{i, j\}$ (Figure 1). The fact that the filter property trivially holds for linear orders in particular for the points on the oriented arc A, proves the second claim. □

THEOREM 3. *If $\mathcal{I}$ is a monotone intersection pattern with the filter property then there exist exactly two total orders on $\{1, \ldots, n\}$, completely opposite to each other, such that the separators of each pair $\{i, j\}$ w.r.t. these orders are given by $\sigma\{i, j\}$.*

PROOF. The proof proceeds by induction on n. Notice that the assertion becomes trivial when $n = 2$. So suppose that $n \geqslant 3$.

First, we show that there is a unique pair $\{b, e\} \subset \{1, \ldots, n\}$ such that $[b, e] = \{1, \ldots, n\}$. The existence of such an extremal pair follows from the fact that $k \notin [i, j]$ always implies that $[i, j] \subset [i, k]$ or $[i, j] \subset [j, k]$. Furthermore, if $\{i, j\} \subset [b, e]$ then, due to (FP), $\sigma\{i, j\} \subset \sigma\{b, e\} = \{1, \ldots, n\} \backslash \{b, e\}$. So if $[i, j] = \{1, \ldots, n\}$, then $\{i, j\} = \{b, e\}$.

Now we show that there exists a unique $p \in \sigma\{b, e\}$ such that $[b, p] = \{1, \ldots, n\} \backslash \{e\}$. Indeed, let $q \in \sigma\{b, e\} \backslash \{p\}$. Clearly, $b \notin \sigma\{p, q\}$, since $\sigma\{p, q\} \subset \sigma\{b, e\}$. So this claim follows from the observation that, if $q \notin \sigma\{b, p\}$, then $p \in \sigma\{b, q\}$. To obtain the uniqueness of p, one can use a similar argument as for the uniqueness of $\{b, e\}$.

Next, we observe that $\sigma\{p, e\} = \varnothing$. Indeed, if $\sigma\{p, e\}$ contained i, then i would belong to $\sigma\{b, p\} \cap \sigma\{p, e\} (i \neq b$ and $i \neq p)$. This would imply that $p \notin \sigma\{b, i\}$ and that $p \notin \sigma\{e, i\}$. We conclude that $p \notin \sigma\{e, i\} \cup \{i\} \cup \sigma\{b, i\}$, which contradicts (FP).

So let us now assume by induction that there exist exactly two total orders on $\{1, \ldots, n\} \backslash \{e\}$, both consistent with the separator sets of $\mathcal{I}$. Clearly, the minimum and maximum of these orders must be b and p. Furthermore, there are only two candidates left for total orders on the whole $\{1, \ldots, n\}$ that are consistent with the $\sigma\{i, j\}$; namely, $t_1 = (b \cdots pe)$ and $t_2 = (ep \cdots b)$. We still have to check each $\sigma\{i, e\}$. Observe that $e \notin \sigma\{i, p\} \subset \sigma\{b, e\}$, and that $i \notin \sigma\{p, e\} = \varnothing$, whence $p \in \sigma\{i, e\}$. Furthermore, by virtue of (FP), $(\sigma\{p, e\} = \varnothing)$:

$$\sigma\{i, e\} = \sigma\{i, p\} \cup \{p\}.$$

So the induction hypothesis applies for $\sigma\{i, p\}$, and hence $\sigma\{i, e\}$ exactly consists of the elements between i and e in t_1 or t_2. □

It is instructive to consider the case in which $\mathcal{I}$ is an intersection pattern that is obtained from an arrangement $\mathcal{H}$ as in Theorem 1. In the proof of Theorem 2 we have observed that the linear representative b of $\mathcal{B}(\mathcal{H})$ that is used to construct $\mathcal{I}$ is an example of one of the two total orders of Theorem 3 that is consistent with the separator sets of $\mathcal{I}$. Furthermore, if $\mathcal{H}'$ is the reflection of $\mathcal{H}$ w.r.t. some line, then we can use the opposite order $\bar{b}$ to represent the new base order, giving rise, however, to the same intersection pattern. By Theorem 3, b and $\bar{b}$ are the only total orders that are consistent with the separator sets of $\mathcal{I}$.

It will be more convenient to represent the transitive closure of a cycle-free $R(\mathcal{I})$ by some linear order extension or, rather, by an *exhaustive word* w on the alphabet $\mathcal{A}$; that is, a word which uses each symbol exactly once. So, w is just a linear sequence on the symbols a_{ij}, representing a linear order on $\mathcal{A}$ which has the transitive closure of

$R(\mathscr{I})$ as subset. We say that w has the filter property if $\mathscr{I}$ has, and in this case we call w a *vertex order.*

Now select one of the permutations that are given by Theorem 3. The given word w can now be considered as a *sequence of switches,* where the symbol a_{ij} corresponds to the transposition of i and j in the current permutation. More precisely, w yields a sequence of permutations, starting with the 'base order', and obtaining successive permutations by performing successive switches. Observe that the final permutation is opposite to the initial permutation, for w is an exhaustive word. Such a sequence of permutations is called *allowable* if each switch only transposes adjacent elements of the current permutation.

THEOREM 4. *A vertex order w always determines two allowable sequences of permutations, corresponding to the two base permutations of Theorem* 3.

PROOF. If k separates i and j in a base permutation, then exactly one of $\{a_{ik}, a_{jk}\}$ precedes a_{ij} in w, due to Theorem 3. This implies that i and j are adjacent at the moment at which the transposition (ij) must be carried out. □

EXAMPLES. 1. $w = a_{34}a_{14}a_{24}a_{23}a_{13}a_{12}$ is not a vertex order. Indeed, $\sigma\{1, 4\} = \{3\}$ but $\sigma\{1, 3\} = \{2\}$ can never be contained in $[1, 4] = \{1, 3, 4\}$.

2. $w = a_{14}a_{13}a_{34}a_{12}a_{23}a_{24}$ is also not a vertex order, but now by the fact that $\sigma\{1, 4\} \cup \{4\} \cup \sigma\{4, 2\} = \{4\}$ does not fill up the whole $\sigma\{1, 2\} = \{3, 4\}$.

3. $w = a_{23}a_{13}a_{12}a_{14}a_{24}a_{34}$ is a valid vertex order. The two corresponding base orders are (1234) and (4321). One of the two resulting allowable sequences of permutations is:

(1234)
(1324)
(3124)
(3214)
(3241)
(3421)
(4321)

An allowable sequence of permutations can always be pictured as a *wiring diagram* [4]. Formally, we represent the base permutation by n points on the x-axis, $(1, 0), \ldots, (n, 0)$ with appropriate labels. Similarly, we represent the permutation after the ith switch (that is, the $(i + 1)$th permutation of the sequence) on the line $y = i$ by labeling the points $(1, i), \ldots, (n, i)$. The corresponding wiring diagram is obtained by connecting the equally labeled points at two successive levels $y = i$ and $y = i + 1$ by straight segments. So a wiring diagram consists of n piecewise linear curves $s_1, \ldots, s_n$, such that s_j connects the point $(p, 0)$ with label j with the point $(q, \binom{n}{2})$ with label j. Observe that $q = n + 1 - p$. Because the given sequence is allowable, eacn pair of strings intersects exactly once. We can turn each string s_j into a pseudoline h_j by 'gluing' two vertical rays at its end points. The resulting simple arrangements of affine pseudolines $\mathscr{H}$ is called a *wiring diagram arrangement.*

Recall that we can associate a unique reorientation class of intersection patterns, monotone and filtered, with a given arrangement $\mathscr{H}$. Using the previous remarks, we can make the following stronger statement.

THEOREM 5. *There is a one-to-one correspondence between isomorphism classes of arrangements and monotone reorientation classes of intersection patterns which satisfy (FP).*

PROOF. First, we take two isomorphic arrangements $\mathscr{H}$ and $\mathscr{H}'$. Endow the members of $\mathscr{H}$ with arbitrary orientations, and let $\mathscr{I}$ be the corresponding intersection pattern. If φ is a self-homeomorphism of $\mathbb{R}^2$ that maps $h_i \in \mathscr{H}$ on $h_i' \in \mathscr{H}'$, then φ induces orientations for the members of $\mathscr{H}'$ as well. Since the resulting $\mathscr{I}'$ clearly equals $\mathscr{I}$, we see that $\mathscr{H}$ and $\mathscr{H}'$ determine the same reorientation class of intersection patterns.

Conversely, let $\mathscr{I}$ be a monotone intersection pattern with the filter property. Let w be a vertex order that linearly extends the transitive closure of $R(\mathscr{I})$. Then w yields an allowable sequence of permutations, which, in turn, determines a wiring diagram arrangement $\mathscr{H}$. Choose the order in which the x-axis meets the pseudolines of $\mathscr{H}$ as linear representative of $\beta(\mathscr{H})$. Clearly, if we consider the induced orientations, then the corresponding intersection pattern equals $\mathscr{I}$. So each monotone $\mathscr{I}$ satisfying (FP) comes from an arrangement $\mathscr{H}$. Now suppose that $\mathscr{H}'$ is another arrangement leading to the intersection pattern $\mathscr{I}$. By Theorem 2, one of the two linear orders determined by $\mathscr{I}$ (Theorem 3) is a linear representative b for the base order $\beta(\mathscr{H}')$. Clearly, this order b, or its global reversal, together with the intersection pattern

$$(I(\overline{h_1}), \ldots, I(\overline{h_n}))$$

completely determines the cell decomposition of $\mathbb{P}^2$ induced by the projective completion $(\mathscr{H}')^*$ of $\mathscr{H}'$. We conclude that $\mathscr{H}' \cong \mathscr{H}$. Finally, note that each reorientation of $\mathscr{I}$ is still an intersection pattern for $\mathscr{H}$, adapting the orientations of the pseudolines. □

Observe that, on the way, in the previous arguments we provided for an alternative proof for the following result of Goodman and Pollack, without making use of *Levi's enlargement lemma* [1].

COROLLARY 6. *Each arrangement of pseudolines is isomorphic to a wiring diagram arrangement.*

REMARK. We can even replace 'isomorphic' by 'isotopic'. Indeed, the two allowable sequences of permutations that are determined by a vertex order yield two wiring diagram arrangements in different isotopy classes.

In order to describe the set of different vertex orders that are determined by the same simple arrangement of labeled pseudolines, we introduce two types of moves on the class of exhaustive words. If w is an exhaustive word on the alphabet $\mathscr{A}$ and if s is a substring of w consisting of $n-1$ symbols where the same label is involved, then we say that s is a *line* of w. Furthermore, by $\bar{v}$ we will mean the reversal of the substring v. We define two types of moves:

1: $v_1 s v_2 \to v_1 \bar{s} v_2$ if s is a line;

2: $v_1 a_{ij} a_{kl} v_2 \to v_1 a_{kl} a_{ij} v_2$ if $\{i, j\} \cap \{k, l\} = \varnothing$.

Let us call two exhaustive words *equivalent*, denoted by $w \sim w'$, if we can transform one into the other by a finite sequence of moves of type 1 and 2. Now we have arrived at the most elegant way to encode combinatorial types of (labeled) arrangements (cf. exercise 6.12 in [1]).

THEOREM 7. *There is a one-to-one correspondence between combinatorial types of simple arrangements of labeled pseudolines in* $\mathbb{R}^2$ *and equivalence classes of vertex orders.*

Before we prove this theorem, we prefer to present a lemma first, which has applications in other sections as well. It is in fact a restatement that the symmetric group is generated by 'adjacent' transpositions. If π and π' are two linear orders on $\{1, \ldots, n\}$, then π and π' are sets of ordered pairs, and hence we can consider their symmetric difference as sets, $\pi\Delta\pi' = (\pi\backslash\pi') \cup (\pi'\backslash\pi)$. To avoid duplicates we now define a set $\Delta(\pi, \pi')$ of unordered pairs:

$$\{x, y\} \in \Delta(\pi, \pi') \Leftrightarrow (x, y) \in \pi\Delta\pi',$$
$$\Leftrightarrow (y, x) \in \pi\Delta\pi'.$$

LEMMA 8. *If* π *and* π' *are two linear orders on* $\{1, \ldots, n\}$ *with* $\pi \neq \pi'$, *then there exists at least one pair* $\{i, j\}$ *in* $\Delta(\pi, \pi')$ *such that i and j are adjacent in* π.

PROOF. Take $\{i, j\}$ in $\Delta(\pi, \pi')$, such that its 'π-distance' $\delta(i, j)$ is minimal. We assume that $(i, j) \in \pi$ and that $(j, i) \in \pi'$. If $\delta(i, j) = 0$, then $\{i, j\}$ is already an adjacent pair. If $\delta(i, j) > 0$, then there is a k such that (i, k) and (k, j) are both in π. However, since π' is a total order, we either have $(k, i) \in \pi'$ or $(j, k) \in \pi'$. We conclude that either $\{i, k\}$ or $\{j, k\}$ belongs to $\Delta(\pi, \pi')$, which contradicts the minimality of $\delta(i, j)$. □

PROOF OF THEOREM 7. First, let us take two isomorphic arrangements $\mathscr{H}$ and $\mathscr{H}'$. Let $\mathscr{I}$ and $\mathscr{I}'$, respectively, be two corresponding monotone intersection patterns (Theorem 1), and take two arbitrary corresponding vertex orders, w and w'. By Theorem 2 we know that both $\mathscr{I}$ and $\mathscr{I}'$ satisfy (FP), and by Theorem 5 we know that they belong to the same reorientation class. Let b and b' be the linear representatives of $\beta(\mathscr{H})$ and $\beta(\mathscr{H}')$, respectively, that determine the chosen intersection pattens. Since b and b' are only determined up to global reversal, and since the base orders of $\mathscr{H}$ and $\mathscr{H}'$ are either equal or opposite, we may assume that b' can be obtained from b by circular shifts. W.l.o.g., we may assume that $b = (1\ 2 \cdots n)$ and that $b'' = (2 \cdots n\ 1)$ is the first shift in the transformation of shifts from b to b'. Let $\mathscr{I}''$ be the pattern that is induced by b''. Clearly, $\mathscr{I}''$ is obtained from $\mathscr{I}$ by globally reversing $I(\overline{h_1})$. note that '1' does not separate any pair in b, and hence no symbol a_{ij} in w separates a_{1i} from a_{1j}. This means that w can be transformed to a word v by merely applying moves of type 2, with the property that all symbols in which '1' is involved are consecutive, and hence form a line s of v. Since moves of type 2 do not affect the transitive closure of $R(\mathscr{I})$, v is still a vertex order corresponding to $\mathscr{I}$. Next, we perform a move of type 1 on v by reversing the line s, and obtain a word w'' associated with $\mathscr{I}''$. By induction on the minimal number of circular shifts to obtain b' from b, we obtain a vertex order $v' \sim w$ which linearly extends the transitive closure of $R(\mathscr{I}')$. However, w' is a linear extension of the same partial order, and so, due to Lemma 8, w' can be obtained from v' by swapping adjacent pairs $\{a_{ij}, a_{pq}\}$ that are not compared in that partial order; that is, $\{i, j\} \cap \{p, q\} = \varnothing$.

Conversely, let w be a vertex order on $\mathscr{A}$. Then w determines two opposite base permutations b and $\bar{b}$, which uniquely determine a monotone intersection pattern $\mathscr{I}$

which, in turn, uniquely determines the combinatorial type of an arrangement $\mathscr{H}$ (Theorem 5). A move of type 2 does not modify $\mathscr{I}$, so it still yields a vertex order that corresponds to the same combinatorial type of $\mathscr{H}$. Furthermore, if s is a line of w w.r.t. i, then i does not belong to any $\sigma\{p, q\}$ and, hence, i is extreme in b (or $\bar{b}$). A move of type 1 that reverses s still gives a vertex order for $\mathscr{H}$, but now we have reversed the orientation of h_i or, equivalently, we have used a circular shift of b w.r.t. i to construct the intersection pattern. We conclude that the combinatorial type of the associated arrangement of a vertex order is invariant under moves of type 1 and 2. □

3. Pseudoline Diagrams and Chirality

The study of moves on pseudoline diagrams aims to provide a combinatorial tool for classifying line configurations in $\mathbb{RP}^3$ up to rigid isotopy. For more details on this rather new research area, we refer to [2], [8], [9], [13], [16], [17] and so on.

Let $\mathscr{L} = \{L_1, \ldots, L_n\}$ be a configuration of n mutually skew lines in $\mathbb{R}^3$. We can always choose an affine reference frame such that the projection of the members of $\mathscr{L}$ upon the x–y plane along the z-axis gives a simple arrangement of lines $\mathscr{H}$ in $\mathbb{R}^2$. Let $a_{ij} = (x, y)$ be the intersection of h_i and h_j and let $(x, y, z_i) \in L_i$ and $(x, y, z_j) \in L_j$ be the two preimages; then we say that h_i *crosses over* (resp., *under*) h_j if $z_i > z_j$ (resp., $z_i < z_j$) and we define the *crossing function* under$(i, j) = -1$ (resp., $+1$). The pair $D(\mathscr{L}) = (\mathscr{H}, \text{under})$ is called the *planar layout* of $\mathscr{L}$. A drawing of $D(\mathscr{L})$ is a drawing of $\mathscr{H}$ in which we omit a small open segment around each intersection on the under-crossing line (Figure 3). By choosing an arbitrary crossing function for some simple arrangement $\mathscr{H}$ we can also consider abstract *line diagrams,* which turn out to form a much larger class than the planar layouts (see [11–15] and [18]). In this paper we will ascend to one higher level of abstraction by superimposing a 'weaving pattern' (crossing function) upon arrangements of pseudolines. However, this abstraction will appear to be natural within the framework of diagram moves.

A *pseudoline diagram* $\mathscr{D}$ is a pair $(\mathscr{H}, \text{under})$ consisting of a simple arrangement $\mathscr{H}$ of labeled affine pseudolines and a *crossing function* under, which is an antisymmetric function on ordered pairs (i, j), $i \neq j$, taking values in $\{-1, +1\}$:

$$\text{under}(i, j) = -\text{under}(j, i) \in \{-1, +1\}.$$

Again, when under$(i, j) = +1$, then, we say that h_i *crosses under* h_j or that h_j *crosses over* h_i.

Now suppose that the pseudolines of a diagram $\bar{\mathscr{D}}$ are oriented in some arbitrary fashion. For each pair of oriented pseudolines $\{\bar{h}_i, \bar{h}_j\}$ we define the *linking number* link(i, k) as follows: if the over-crossing pseudoline crosses the under-crossing pseudoline from the right-hand side to the left-hand side, then we put link$(i, j) = +1$,

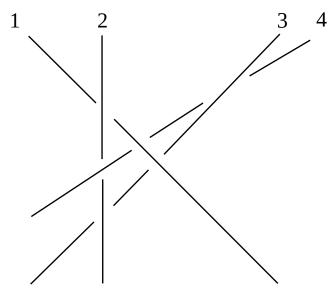

Figure 3. The planar layout of four mutually skew lines in $\mathbb{R}^3$.

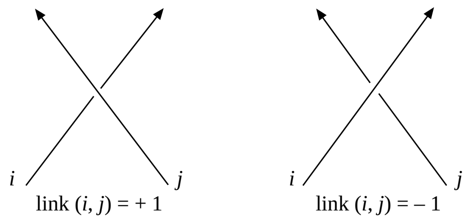

FIGURE 4. The linking number of two oriented skew lines in $\mathbb{R}^3$, or two oriented pseudolines in a diagram.

else we define $\text{link}(i, j) = -1$ (Figure 4).† Similarly to line configurations, we can use linking numbers to define the 'unoriented' magnitude

$$\chi_{ijk} = \text{link}(i, j) \times \text{link}(i, k) \times \text{link}(j, k),$$

called the *chirality* of the three pseudolines $\{h_i, h_j, h_k\}$ of $\mathscr{D}$. Observe that χ_{ijk} is indeed invariant under reorientations of the involved pseudolines. If we arrange the ordered triples (ijk), $i<j<k$ lexicographically, then we can collect all chiral signatures of a diagram into a sign sequence

$$\chi(\mathscr{D}) = (\chi_{ijk})_{i<j<k} \in \{-1, +1\}^{\binom{n}{3}}$$

The concept of chirality was introduced by O. Ya. Viro in [16] for three lines in $\mathbb{RP}^3$, using the notion of linking number of two non-intersecting oriented lines directly in an oriented $\mathbb{RP}^3$. It seems that his sign convention is opposite to ours.

The following theorems are known for line configurations, and easily transfer to pseudoline diagrams (see [2] and [17] for the geometric versions of Theorem 9 and Theorem 10, respectively). We state them without proofs. Three pseudolines h_1, h_2 and h_3 of a diagram $\mathscr{D}$ are called a *stack* if the relation $\{(i, j); \text{under}(i, j) = +1\}$ is a total order on $\{1, 2, 3\}$, and a *vortex* otherwise. If the lines of a vortex appear to spiral down into the page in a counterclockwise orientation then we will use the specification *CCW vortex*; otherwise we call it a *CW vortex.* A stack is *CCW* (resp., *CW*) if going around the triangular region from the upper to the lower line corresponds to a counterclockwise (resp. clockwise) turn (Figure 5).

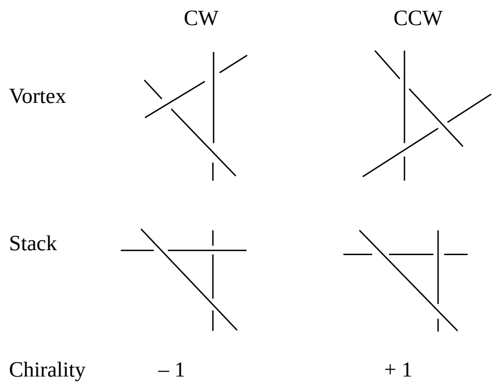

FIGURE 5. The chirality of three (pseudo)lines.

† At first sight our definition of linking number seems to be oppsoite to the common sign convention of topologists. However, if $\{\overline{h_i}, \overline{h_j}\}$ is the projection of two skew oriented lines in $\mathbb{R}^3$, with projective completion $\{\overline{L_i}, \overline{L_j}\} \subset \mathbb{RP}^3$, then we can apply the Klein Map as described in [2] and obtain two oriented circles $\{\overline{C_i}, \overline{C_j}\}$ in $\mathbb{R}^3$ with topological linking number equal to link (i, j) (using the standard orientation of $\mathbb{R}^3$).

THEOREM 9. *A triple of pseudolines in a diagram* $\mathscr{D}$ *has chiral signaiture* $+1$ (*resp.,* -1) *iff it is a CCW* (*resp., CW*) *vortex or a CCW* (*resp., CW*) *stack.*

THEOREM 10. *For any four pseudolines* $\{h_1, h_2, h_3, h_4\}$ *of a diagram, the product*

$$\chi_{123}\chi_{124}\chi_{134}\chi_{234}$$

is positive.

Two pseudoline diagrams $\mathscr{D} = (\mathscr{H}, \text{under})$ and $\mathscr{D}' = (\mathscr{H}', \text{under}')$ are called *combinatorially equivalent,* denoted by $\mathscr{D} \cong \mathscr{D}'$, if under = under′ and $\mathscr{H}$ is isotopic to $\mathscr{H}'$. Alternatively, $\mathscr{D} \cong \mathscr{D}'$ iff:
(1) $\mathscr{H} \cong \mathscr{H}'$,
(2) under = under′,
(3) $\chi_{ijk} = \chi'_{ijk}$ for some $\{i, j, k\} \subset \{1, \dots, n\}$.
A *combinatorial pseudoline diagram* is a class of pseudoline diagrams under combinatorial equivalence. We can immediately verify that chirality is defined for combinatorial diagrams.

THEOREM 11. *If* $\mathscr{D} \cong \mathscr{D}'$, *then* $\chi(\mathscr{D}) = \chi(\mathscr{D}')$.

So, once we have fixed the combinatorial type of the underlying arrangement of pseudolines and once we have chosen the crossing function, all chiral signatures are determined by the choice of one chiral signature. We have exactly two combinatorial diagrams with the same underlying arrangement and the same crossing function, having completely opposite chiral signatures. We say that they are a pair of *mirror images.* For a given combinatorial pseudoline diagram we can find (a geometric representation of) its mirror image by reflecting the arrangement of pseudolines w.r.t. some straight line in the plane and by maintaining the crossing function.

4. MOVES ON PSEUDOLINE DIAGRAMS

The reason why we generalize planar layouts to pseudoline diagrams is that the latter naturally arise under the action of *diagram moves* [2]. Moreover, in the framework of pseudolines we find ourselves in a more convenient position to give formal definitions for the two types of diagram moves. To this end, we first introduce some terminology. Let $\mathscr{H}$ be a simple arrangement of pseudolines. Let $\{h_i, h_j, h_k\} \subset \mathscr{H}$ be an (*affine*) *mutation*; that is, they bound a simplicial region of $\mathbb{R}^2 \backslash \mathscr{H}$. Then i and j are always consecutive in $I(\overline{h_k})$, and so are i and k in $I(\overline{h_j})$ and j and k in $I(\overline{h_i})$; moreover, this is a sufficient condition for $\{h_i, h_j, h_k\}$ to be a mutation. Two pseudolines h_i and h_j are said to form a *wedge* if they both contribute to some unbounded face of $\mathscr{H}$ that, moreover, has only one vertex. Observe that this unique vertex has to be the intersection of h_i and h_j. The characterizing property for a wedge in terms of the intersection sequences is simply that i is either the first or the last element in $I(\overline{h_j})$ and so is j in $I(\overline{h_i})$.

Now let $\mathscr{H}$ and $\mathscr{H}'$ be two arrangements of n pseudolines such that $\{h_i, h_j, h_k\}$ and $\{h'_i, h'_j, h'_k\}$ are mutations in their corresponding arrangements. Moreover, suppose that we can choose orientations for each $p \in \{1, \dots, n\} \backslash \{i, j, k\}$ such that $I(\overline{h_p}) = I(\overline{h'_p})$. Then we can equip the remaining two times three lines with orientations such that $I(\overline{h_i})\backslash\{j,k\} = I(\overline{h_i'})\backslash\{i,k\}$, $I(\overline{h_j})\backslash\{i,k\} = I(\overline{h_j'})\backslash\{i,k\}$ and $I(\overline{h_k})\backslash\{i,j\} = I(\overline{h_k'})\backslash\{i,j\}$. Furthermore, these orientations have the following property:

$$I(\overline{h_i}) = I(\overline{h_i'}) \Leftrightarrow I(\overline{h_j}) = I(\overline{h_j'}) \Leftrightarrow I(\overline{h_k}) = I(\overline{h_k'}) \Leftrightarrow \mathcal{H} \cong \mathcal{H}'.$$

In the case in which $\mathcal{H}'$ is not isomorphic to $\mathcal{H}$, we say that $\mathcal{H}$ and $\mathcal{H}'$ are *triangle-related* (w.r.t. $\{i, j, k\}$).

Furthermore, let $\{h_i, h_j\}$ and $\{h_i', h_j'\}$ be wedges of $\mathcal{H}$ and $\mathcal{H}'$, respectively. Suppose that, for all p in $\{1, \ldots, n\}\backslash\{i, j\}$, we can find orientations such that $I(\overline{h_p}) = I(\overline{h_p'})$. Then there exist orientations for $\{h_i, h_j, h_i', h_j'\}$ such that $I(\overline{h_i})\backslash\{j\} = I(\overline{h_i'})\backslash\{j\}$ and $I(\overline{h_j})\backslash\{i\} = I(\overline{h_j'})\backslash\{i\}$. Again, we see that for these orientations

$$I(\overline{h_i}) = I(\overline{h_i'}) \Leftrightarrow I(\overline{h_j}) = I(\overline{h_j'}) \Leftrightarrow \mathcal{H} \cong \mathcal{H}'.$$

In the case in which $I(\overline{h_i}) \neq I(\overline{h_i'})$, we call $\mathcal{H}$ and $\mathcal{H}'$ *wedge-related* (w.r.t. $\{i, j\}$). Let $\mathcal{D} = (\mathcal{H}, \text{under})$ and $\mathcal{D}' = (\mathcal{H}', \text{under}')$ be two labeled pseudoline diagrams of size n.

1. We say that $\mathcal{D}$ and $\mathcal{D}'$ differ by a $\|$-*move*, denoted by $\mathcal{D} \xrightarrow{\|} \mathcal{D}'$, if there is a pair $\{i, j\} \subset \{1, \ldots, n\}$ such that (Figure 6):
(a) $\mathcal{H}$ and $\mathcal{H}'$ are wedge-related w.r.t. $\{i, j\}$;
(b) $\text{under}(i, j) = -\text{under}'(i, j)$;
(c) $\text{under}(p, q) = \text{under}'(p, q)$ if $\{p, q\} \neq \{i, j\}$.

2. We say that $\mathcal{D}$ and $\mathcal{D}'$ differ by a $*$-*move*, denoted by $\mathcal{D} \xrightarrow{*} \mathcal{D}'$, if there is a triple $\{i, j, k\} \subset \{1, \ldots, n\}$ such that (Figure 6):
(a) $\mathcal{H}$ and $\mathcal{H}'$ are triangle-related w.r.t. $\{i, j, k\}$;
(b) $\{h_i, h_j, h_k\}$ is a stack in $\mathcal{D}$;
(c) $\text{under} = \text{under}'$.

Finally, two (combinatorial) pseudoline diagrams are *equivalent*, denoted by $\mathcal{D} \sim \mathcal{D}'$, if there exists a finite sequence of (combinatorial) pseudoline diagrams $\mathcal{D}_0, \ldots, \mathcal{D}_t$ such that:
(1) $\mathcal{D}_0 = \mathcal{D}$;
(2) $\mathcal{D}_t = \mathcal{D}'$;
(3) for all $i = 1, \ldots, t-1$: $\mathcal{D}_i \xrightarrow{\|} \mathcal{D}_{i+1}$ or $\mathcal{D}_i \xrightarrow{*} \mathcal{D}_{i+1}$.

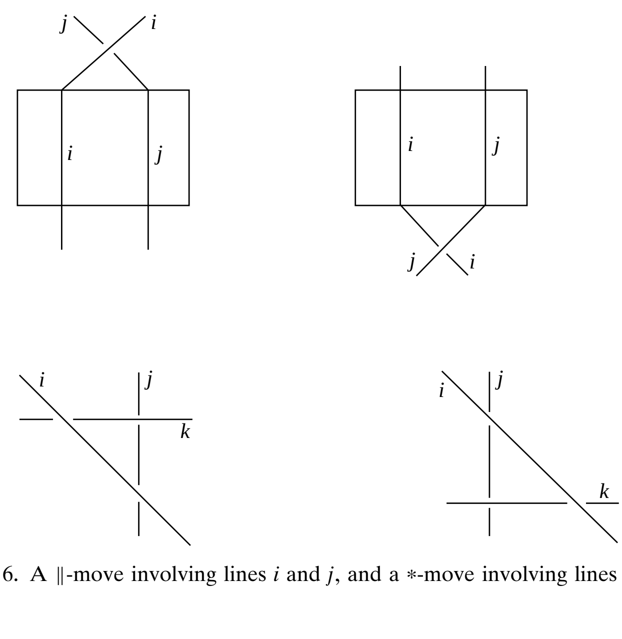

FIGURE 6. A $\|$-move involving lines i and j, and a $*$-move involving lines i, j and k.

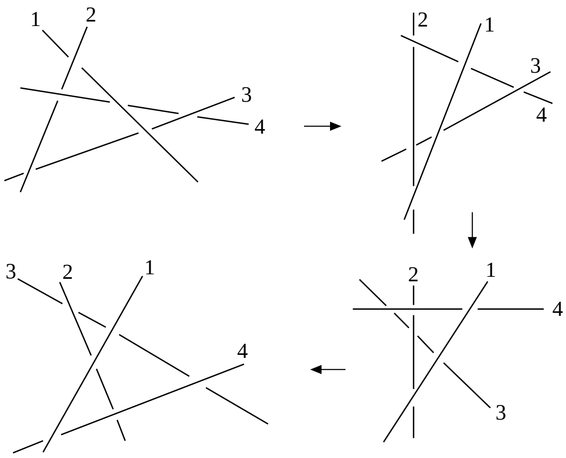

FIGURE 7. The equivalence of diagrams. The first two steps correspond to two $\|$-moves. The bottom operation embraces three $*$-moves.

Two equivalent diagrams and the connecting move sequence are illustrated in Figure 7. It should be pointed out that the $\|$-move first appeared in [3] in the isotopy problem for links in $\mathbb{RP}^3$, while the $*$-move is essentially the third Reidemeister move in Knot Theory. The definition of equivalence of pseudoline diagrams is motivated by the isotopy problem for line configurations in 3-space. Two configurations $\mathscr{L}_0$ and $\mathscr{L}_1$ of pairwise disjoint lines in $\mathbb{RP}^3$ are *rigidly isotopic* if there exists a continuously parametrized family $(H_t)_{0\leq t\leq 1}$ of homeomorphisms of $\mathbb{RP}^3$ such that $H_0 = \mathrm{id}$, $H_1(\mathscr{L}_0) = \mathscr{L}_1$ and $H_t(\mathscr{L}_0)$ is a configuration of (disjoint) lines for all t. If we fix a plane at infinity, $\mathscr{L}_0$ and $\mathscr{L}_1$ become configurations of mutually skew lines in $\mathbb{R}^3$, having well-defined planar layouts, and w.l.o.g. we may restrict outselves to rigid isotopies in $\mathbb{R}^3$. The following theorem, proved in [2], connects isotopy of lines with diagram moves.

THEOREM 12. *If $\mathscr{L}_1$ and $\mathscr{L}_2$ are two configurations of n mutually skew lines in $\mathbb{R}^3$ and if $\mathscr{L}_1$ is rigidly isotopic to $\mathscr{L}_2$, then $D(\mathscr{L}_1) \sim D(\mathscr{L}_2)$.*

Furthermore, it is not hard to see that the intermediate diagrams $\mathscr{D}_i$ which connect $D(\mathscr{L}_1)$ and $D(\mathscr{L}_2)$ by diagram moves can be chosen such that they are all stretchable. In this case we call both line diagrams *rigidly equivalent.* Since an application of $\|$-moves or $*$-moves on a stack or wedge of $\mathscr{D}$ modifies the combinatorial type of the underlying arrangement, those moves are not always well-defined if we restrict to straight line diagrams. Indeed, diagram moves may destroy stretchability. In order to avoid the difficult problem of deciding the stretchability of the intermediate diagrams, we prefer the concept of equivalence of pseudoline diagrams rather than rigid equivalence of line diagrams, and this in fact motivated us to introduce them. We are aware of the fact that this relaxation from the geometric situation of lines in 3-space towards the combinatorial model of pseudoline diagrams may cause a loss of information. However, a negative anwer for the equivalence of two line diagrams always implies a negative answer for the isotopy problem of two lifted configurations in 3-space.

Although the equivalence of pseudoline diagrams will turn out to be decideable (Section 5), we do not yet know of an efficient algorithm. Therefore, the search of invariants under diagram moves, maybe not complete but easy to compute, is still in order. We have already encountered such an invariant, namely chirality.

THEOREM 13. *Let $\mathcal{D}$ and $\mathcal{D}'$ be two pseudoline diagrams. If $\mathcal{D} \sim \mathcal{D}'$, then $\chi(\mathcal{D}) = \chi(\mathcal{D}')$.*

PROOF. The proof follows immediately from Theorem 9, that the chiral signatures are not affected by the two types of diagram moves. □

It is well known that chirality is an invariant for line isotopy, and this in fact was the motivation in [16] to introduce it. In the same pioneering paper it was observed that the chiral signatures completely characterize the rigid isotopy classes of configurations of at most five lines. In general, however, chirality is not a complete invariant. This has been discovered by V. Mazurovskii for $n = 6$ in [9] (see also [8] and [10]).

5. DIAGRAM WORDS

In Section 2 we have elaborated how to encode combinatorial types of simple arrangements of affine pseudolines by means of vertex orders. If we take the order of $\{i, j\}$ in the symbols a_{ij} into account, so that now $a_{ij} \neq a_{ji}$, then we can include crossing information as well. In this section we will encode equivalence classes of pseudoline diagrams by orbits of such 'woven vertex orders' under four types of moves. This leads to a principal (but not practical) manner in which to generate all diagrams of an equivalence class, and hence to decide the equivalence of pseodoline diagrams.

Let W_n be the set of all words (strings) over the alphabet A_n of all ordered pairs of $\{1, \ldots, n\}$, where we represent (i, j) by the symbol s_{ij} and the unordered pair $\{i, j\}$ by $p(s_{ij})$. Let $\mathcal{D} = (\mathcal{H}, \text{under})$ be a pseudoline diagram with $|\mathcal{H}| = n$. We define a subset $s(\mathcal{D})$ of the alphabet A_n by

$$s_{ij} \in s(\mathcal{D}) \Leftrightarrow \text{under}(i, j) = -1.$$

Observe that $|A_n\| = n(n-1)$ and $|s(D)| = |A_n|/2$. We can associate with each exhausted word w on $s(\mathcal{D})$ (using each symbol exactly once) a unique exhausted word $\underline{w}$ on $\mathcal{A}$ by replacing each s_{ij} by a_{ij} (Section 2). A *diagram word* for $\mathcal{D}$ is an exhausted word w on $s(\mathcal{D})$ with the property that $\underline{w}$ is a vertex order for $\mathcal{H}$. An *oriented word* w^* of a pseudoline diagram $\mathcal{D}$ is a pair (w, χ), where w is a word for $\mathcal{D}$ and χ is the chirality of a specified triple of lines, $\chi = \chi_{123}$ say. In Theorem 3 we have seen that each vertex order determines exactly two base orders, which are opposite to each other; or, rather, it determines two linear representatives of these base orders. However, as w contains crossing information, the orientation of a diagram word rules out one of these two orders, by virtue of the CW–CCW rule of chirality (Theorem 9).

We intend to define four types of moves on W_n, such that the resulting orbits encode equivalence classes of pseudoline diagrams. A substring s of a word w such that $\underline{s}$ is a line of the underlying word $\underline{w}$ (Section 2) is called a *line* of w as well. Furthermore, a substring consisting of three consecutive symbols that correspond to the three crossings of a stack is called a *stack* of w. With or without subscript, we use w and v for (possible empty) strings in W_n, and s for symbols in A_n. Again, $\bar{v}$ denotes the reversal of the string v:

$$\begin{array}{lll} m_1\colon & s_{ij}w \leftrightarrow_1 ws_{ji} & \\ m_2\colon & w_1vw_2 \rightarrow_2 w_1\bar{v}w_2 & \text{if } v \text{ is a stack,} \\ m_3\colon & w_1vw_2 \rightarrow_3 w_1\bar{v}w_2 & \text{if } v \text{ is a line,} \\ m_4\colon & w_1s_1s_2w_2 \rightarrow_4 w_1s_2s_1w_2 & \text{if } p(s_1) \cap p(s_2) = \emptyset. \end{array}$$

These moves can be easily extended to act on oriented words by insisting that the orientation χ remains unmodified. The reflexive, symmetric and transitive closure of these four moves determines an equivalence relation $\sim$ on words. Observe that m_1 and m_2 are merely the translations of the $\|$-moves and $*$-moves, respectively, in the language of words. Moves m_3 and m_4 are included to connect two possible words for the same diagram, and are just the two types of moves that generate the orbit of vertex orders corresponding to the same isomorphism class of arrangements.

THEOREM 14. *Let w_1^* and w_2^* be two oriented words for the diagrams $\mathscr{D}_1 = (\mathscr{H}_1, \text{under}_1)$ and $\mathscr{D}_2 = (\mathscr{H}_2, \text{under}_2)$, respectively. If $w_1^* \sim w_2^*$ then $\mathscr{D}_1 \sim \mathscr{D}_2$.*

PROOF. We treat the four types of word moves separately. First, let $w_1^* \rightarrow_3 w^*$ or $w_1^* \rightarrow_4 w^*$. This means that the underlying vertex orders, $\underline{w_1}$ and $\underline{w}$ are equivalent. This implies that w, too, is a diagram word for $\mathscr{D}_1$ (Theorem 7). Next, suppose that $w_1^* \rightarrow_1 w^*$ or $w_1^* \rightarrow_2 w^*$. if we perform the corresponding diagram moves on $\mathscr{D}_1$, then we obtain a diagram $\mathscr{D} \sim \mathscr{D}_1$, such that w^* is an oriented word for $\mathscr{D}$. Consequently, by induction on the length of the minimal sequence of word moves that connects w_1 with w_2, we obtain that w_2^* is an oriented word for a diagram $\mathscr{D}' = (\mathscr{H}', \text{under}')$ with $\mathscr{D}' \sim \mathscr{D}_1$. Since w_2 is also a word for $\mathscr{D}_2$, the arrangements $\mathscr{H}'$ and $\mathscr{H}_2$ have the same vertex order $\underline{w_2}$. From Theorem 7 it follows that $\mathscr{H}' \cong \mathscr{H}_2$. Furthermore, since w_2^* also determines the crossing function and the chirality, $\mathscr{D}_2$ and $\mathscr{D}'$ are combinatorially equivalent. □

To prove the converse statement we prefer to introduce some terminology. Suppose that i and j are adjacent in some linear representative b of the base order of a given arrangement $\mathscr{H}$. We endow the pseudolines of $\mathscr{H}$ with the canonical orientations w.r.t. b. The adjacency of i and j implies that

$$k < j \text{ in } I(\overline{h_i}) \Leftrightarrow k < i \text{ in } I(\overline{h_j}).$$

This enables us to define the set of *lower transversals* of $\{i, j\}$ as a subset $LT(i, j)$ of $\{1, \ldots, n\}$ by

$$k \in LT(i, j) \Leftrightarrow k < j \text{ in } I(\overline{h_i}) \; 41$$
$$\Leftrightarrow k < i \text{ in } I(\overline{h_j}).$$

The set of *upper transversals* of $\{i, j\}$, denoted by $UT(i, j)$, can be defined analogously. Since we only regard simple arrangements,

$$LT(i, j) \cup UT(i, j) = \{1, \ldots, n\} \setminus \{i, j\}.$$

THEOREM 15. *If $\mathscr{D}_1$ and $\mathscr{D}_2$ are two equivalent diagrams, and if w_1^* and w_2^* are two oriented words of the respective diagrams, then $w_1^* \sim w_2^*$.*

PROOF. First, suppose that $\mathscr{D}_1'$ is a diagram that is obtained from $\mathscr{D}_1$ by a $\|$-move w.r.t. $\{i, j\}$. Notice that i and j are adjacent in the base order of $\mathscr{H}_1$. If, however, they are not adjacent in the linear representative b_1 which is consistent with w_1, then i and j

must be extremal in b_1. From the proof of Theorem 7, we learn that we can pass to a word w_1' from w_1 by means of moves of type m_3 and m_4, with the property that w_1' corresponds to a linear base order which is a circular shift of b_1. So we may assume that i and j are adjacent in b_1. Furthermore, since $\{h_i, h_j\}$ forms a wedge, either $LT(i, j) = \emptyset$ or $UT(i, j) = \emptyset$. Consequently, if s is a symbol in w_1 with $p(s) = \{i, j\}$, then no symbol t before (resp., after) s in w_1 contains i or j in $p(t)$, and hence w_1 is equivalent (under m_4) to a word having s in front (resp., at the back). This enables us to perform an m_1-move relative to s, and to arrive at a word w_1' for $\mathscr{D}_1'$.

Next, suppose that $\mathscr{D}_1'$ is obtained from $\mathscr{D}_1$ by one $*$-move w.r.t. $\{i, j, k\}$. Let s_1, s_2 and s_3 be the symbols in w_1 with $p(s_1) = \{i, j\}$, $p(s_2) = \{i, k\}$ and $p(s_3) = \{j, k\}$, and assume that $s_1 < s_2 < s_3$ $\underline{\text{in}}$ w_1. If t separates s_1 and s_2 in w_1, then $i \notin p(t)$ as j and k are consecutive in $I(\overline{h_j})$. So we can assume that s_1 and s_2 are adjacent in w_1, modulo m_4-moves that switch s_1 with the intermediate symbols. Similarly, we can *move* s_3 next to s_2. Now we can apply an m_2-move relative to $s_1s_2s_3$, yielding a word w_1' that represents $\mathscr{D}_1'$.

We conclude, by induction on the number of diagram moves connecting $\mathscr{D}_1$ with $\mathscr{D}_2$, that w_1 is equivalent to a word w_2' for the diagram $\mathscr{D}_2$. In Theorem 7 we have proved that two vertex orders $\underline{w_2'}$ and $\underline{w_2}$ for the same arrangement $\mathscr{H}_2$ must be equivalent, and so $w_2 \sim w_2'$. Finally, as diagram moves preserve chiralities, $w_1^* \sim w_2^*$. □

COROLLARY 16. *The equivalence of pseudoline diagrams is decidable.*

PROOF. Notice that the four types of moves which generate $\sim$ on words do not affect the length of the words. Since we have only a finite number of words over A_n with a given length, we can generate the set of all words that are equivalent to a given word, and so $\sim$ is easy to decide for diagram words. If two words of two given diagrams are not equivalent, then the diagrams themselves are not equivalent. In the other case we still have to check the chirality of some fixed triple for both diagrams (the orientation of the words), to be sure that they are not each other's mirror image. □

EXAMPLE. Let us show how word moves encode the diagram moves of Figure 7. The word $w = s_{23}s_{13}s_{34}s_{14}s_{42}s_{21}$ represents the initial diagram w.r.t. $b = (3214)$, while $w' = s_{14}s_{42}s_{12}s_{43}s_{13}s_{23}$ is a word for the final diagram w.r.t. $b' = (4123)$. Behold

$$\begin{aligned} w &\xrightarrow{1} s_{12}s_{23}s_{13}s_{34}s_{14}s_{43} \xrightarrow{3} s_{12}s_{34}s_{13}s_{23}s_{14}s_{42} \\ &\xrightarrow{4} s_{34}s_{12}s_{13}s_{23}s_{14}s_{42} \xrightarrow{1} s_{12}s_{13}s_{23}s_{14}s_{42}s_{43} \\ &\xrightarrow{4} s_{12}s_{13}s_{14}s_{23}s_{42}s_{43} \xrightarrow{2} s_{12}s_{13}s_{14}s_{43}s_{42}s_{23} \\ &\xrightarrow{2} s_{12}s_{43}s_{14}s_{13}s_{42}s_{23} \xrightarrow{4} s_{12}s_{43}s_{14}s_{42}s_{13}s_{23} \\ &\xrightarrow{3} s_{12}s_{42}s_{14}s_{43}s_{13}s_{23} \xrightarrow{2} w' \end{aligned}$$

6. GENERALIZED TRIANGLE MOVES

We have already mentioned that chirality classes are strictly larger than equivalence classes of pseudoline diagrams (for $n \geqslant 6$). In order to understand better the weakness and strength of the chiral invariant, it is natural to enlarge the set of diagram moves until the corresponding equivalence classes match the chirality classes. The first candidate for such an additional move is the *generalized triangle move,* which allows

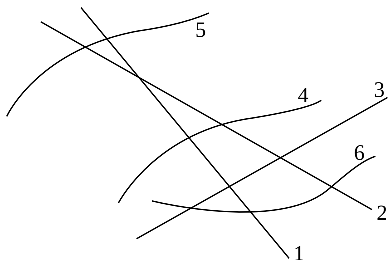

FIGURE 8. $\{h_1, h_2, h_3\}$ is a parallel triangle with respect to $\{1, 2\}$. Observe that h_4 is a separator, while h_5 and h_6 are not.

the switching of each mutation of the underlying arrangement, also when the involved triangle is a vortex.† Formally, we say that $\mathscr{D} = (\mathscr{H}, \text{under})$ and $\mathscr{D}' = (\mathscr{H}', \text{under}')$ *differ by a* Δ-*move*, denoted by $\mathscr{D} \xrightarrow{\Delta} \mathscr{D}'$, if

(1) $\mathscr{H}$ and $\mathscr{H}'$ are triangle-related;

(2) under = under$'$.

We will write $\sim'$ for the reflexive, symmetric and transitive closures of $\xrightarrow{\parallel}$ and $\xrightarrow{\Delta}$. The reason why we have generalized $*$-moves to Δ-moves is that the chiralities still remain invariant under $\sim'$. Surprisingly, it turns out that conversely the chiralities determine the diagram classes under $\sim'$. For the benefit of the proof, we first state some technical intermediate results. If $\{h_i, h_j, h_k\}$ is a triple of pseudolines in $\mathscr{D}$ such that $\{i, j\}$ is consecutive in $I(\overline{h_k})$, then we say that $\{h_i, h_j, h_k\}$ is a *parallel triangle* of $\mathscr{H}$ w.r.t. $\{i, j\}$. Clearly, p separates i and k in $I(\overline{h_j})$ iff p separates j and k in $I(\overline{h_i})$. In this case, we call h_p (or just p) a *separator* of the involved parallel triangle (Figure 8).

LEMMA 17. *Let $\mathscr{D} = (\mathscr{H}, \text{under})$ be a pseudoline diagram, and let $T = \{h_i, h_j, h_k\}$ be a parallel triangle of $\mathscr{H}$ w.r.t. $\{i, j\}$. Then there exists a diagram $\mathscr{D}' = (\mathscr{H}', \text{under}')$ such that*:

(1) *$\{i, j, k\}$ is a mutation in $\mathscr{H}'$;*

(2) *$\mathscr{D}'$ can be obtained from $\mathscr{D}$ by applying only Δ-moves where the involved switched triangles are bounded by triples out of $\{h_i, h_j\} \cup \{$separators of $T\}$.*

PROOF. Let $\sigma(T)$ denote the number of separators of T. The proof goes by induction on $\sigma(T)$. If $\sigma(T) = 0$, then we can take $\mathscr{D}' = \mathscr{D}$ and the proof is done; so suppose that $\sigma(T) > 0$. If we orient $\overline{h_i}$ and $\overline{h_j}$ such that $j < k$ in $I(\overline{h_i})$ and $i < k$ in $I(\overline{h_j})$, then the open segments $\rho_i = (j, k) \subset I(\overline{h_i})$ and $\rho_j = (i, k) \subset I(\overline{h_j})$ are two linear orders on the separators of T.

If $\rho_i = \rho_j$, then $\min(\rho_i) = \min(\rho_j) = p$, and hence $\{i, j, p\}$ is a mutation. A generalized triangle move w.r.t. $\{i, j, p\}$ gives a diagram $\mathscr{D}'$, where $\{h_i', h_j', h_k'\}$ still bound a parallel triangle T' with $\sigma(T') = \sigma(T) - 1$ (p is no longer a separator), and hence induction applies. So suppose that $\rho_i \neq \rho_j$. By Lemma 8 there exists a pair $\{p, q\} \in \Delta(\rho_i, \rho_j)$ that is adjacent in ρ_i. This means that $\{h_i, h_p, h_q\}$ bound a parallel triangle T' w.r.t. $\{p, q\}$. Furthermore, since $\{p, q\} \in \Delta(\rho_i, \rho_j)$ the intersection of h_p and h_q must be in the 'interior' of T. Consequently, each separator of T' is a separator of T, whence $\sigma(T') \leqslant \sigma(T) - 2$ (p and q are not separators of T'). By induction, we can transform $\mathscr{D}$ to $\mathscr{D}'$ purely by $\triangle$-moves such that $\{p, q, i\}$ is a mutation of $\mathscr{H}'$. Moreover, we assume

† I was kindly informed by the referee that generalized triangle moves were first considered by S. V. Matveev in *Russian Math. Surveys,* **42**(2) (1987).

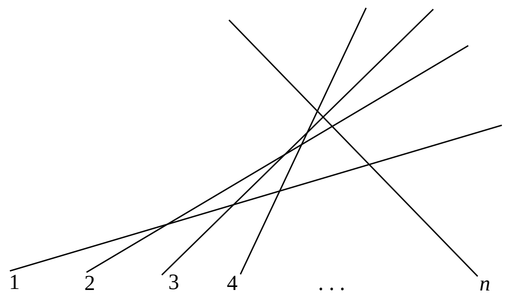

FIGURE 9. A well-based arrangement with respect to $(1\ 2 \cdots n)$.

by induction that all triangles involved in the transformation sequence from $\mathscr{D}$ to $\mathscr{D}'$ are bounded by triples out of $\{p, q\} \cup \{x \mid x \text{ is a separator of } T'\}$. This implies that j and k are never involved and hence that $\{i, j, k\}$ has the same separators in $\mathscr{D}'$ as in $\mathscr{D}$. Next, we perform $\mathscr{D}' \overset{\triangle}{\Rightarrow} \mathscr{D}''$ w.r.t. $\{p, q, i\}$ such that $\{h_i'', h_j'', h_k''\}$ bound a triangle T'' in $\mathscr{H}''$ that no longer contains the intersection of h_p and h_q, whence $\Delta(\rho_i'', \rho_j'') = \Delta(\rho_i, \rho_j) \setminus \{\{p, q\}\}$. By induction on $|\Delta(\rho_i, \rho_j)|$ we conclude that $\mathscr{D}$ can be transformed to a diagram $\mathscr{D}'$ with $\Delta(\rho_i', \rho_j') = \varnothing$; that is, $\rho_i' = \rho_j'$; and hence to a diagram where $\{i, j, k\}$ is a mutation. Observe that during this transformation we only used generalized triangles moves in which only i, j and the separators of T are involved. □

Let b be some linear representative of the circular base order of $\mathscr{H}$. Let us endow each pseudoline with the orientation induced by b. Then we call $\mathscr{D}$ *well-based* w.r.t. b if, for all $i \in \{1, \ldots, n\}$ (Figure 9),

$$p < q \text{ in } I(\overline{h_i}) \Leftrightarrow p <_b q.$$

LEMMA 18. *For each diagram* $\mathscr{D} = (\mathscr{H}, \text{under})$, *and for each linear base order b of* $\mathscr{H}$, *there exists a well-based diagram* $\bar{\mathscr{D}}$ *w.r.t. b that can be reached from* $\mathscr{D}$ *by means of* $\triangle$*-moves only.*

PROOF. W.l.o.g., we may assume that $b = (1, 2 \cdots n)$. Let ρ_i be the totally ordered sequence $b \setminus \{i\}$. If $I(\overline{h_1}) \neq \rho_1$ then, by Lemma 8, there exists a pair $\{p, q\} \in \Delta(I(\overline{h_1}), \rho_1)$ that is adjacent in $I(\overline{h_1})$. Consequently, $\{h_1, h_p, h_q\}$ forms a parallel triangle w.r.t. $\{p, q\}$. By Lemma 17, we can transform $\mathscr{D}$ to a diagram $\mathscr{D}'$ such that $\{h_1', h_p', h_q'\}$ is a mutation, using $\triangle$-moves only, without involving $\overline{h_1}$ in any such move. So, $I(\overline{h_1'}) = I(\overline{h_1})$, and a $\triangle$-move w.r.t. $\{1, p, q\}$ decreases $|\Delta(I(\overline{h_1}), \rho_1)|$ by one. We conclude that we can transform $\mathscr{D}$ to a diagram $\mathscr{D}_1$ with $I(\overline{h_1}) = \rho_1$, by applying only $\triangle$-moves.

Now suppose by induction that we have transformed $\mathscr{D}$ by means of a sequence of $\triangle$-moves to a diagram $\mathscr{D}_{i-1}$ $(i > 1)$ such that $I(\overline{h_p}) = \rho_p$ for each $1 \leqslant p < i$. This implies that in $\mathscr{D}_{i-1}$ for all $j \geqslant i$:

$$p < q < i \text{ in } I(\overline{h_j}) \Leftrightarrow p <_b q.$$

Now compare $\pi_{i-1} = \{x \in I(\overline{h_{i-1}}) = \rho_{i-1} \mid x > i\}$ with the sequence $\pi_i = \{x \in I(\overline{h_i}) \mid x > i - 1\}$. Again we select a pair $\{p, q\} \in \Delta(\pi_{i-1}, \pi_i)$ which is adjacent in π_i, such that Lemma 17 applies, transforming $\{i, p, q\}$ to a mutation, such that a $\triangle$-move w.r.t. $\{i, p, q\}$ can be performed. By induction on $|\Delta(\pi_{i-1}, \pi_i)|$, we come up with $\mathscr{D}_i$, where $\pi_i = \pi_{i-1}$ and hence $I(\overline{h_i}) = \rho_i$. □

Now suppose that i and j are adjacent in some linear representative b of the base order of the underlying arrangement $\mathscr{H}$ of $\mathscr{D}$. Recall that for this situation we can consider the sets of lower and upper transversals of $\{i, j\}$ (Section 5), denoted by $LT(i, j)$ and $UT(i, j)$, respectively.

LEMMA 19. *Let i and j be adjacent in b. We can always transform $\mathscr{D}$ to $\mathscr{D}'$, using $\triangle$-moves only, such that $|LT(i, j)| = 0$ in $\mathscr{H}'$. Furthermore, we may assume that the triangles involved in the $\triangle$-moves are bounded by elements of $\{i, j\} \cup LT(i, j)$ only.*

PROOF. We proceed by induction on $\lambda(i, j) = |LT(i, j)|$. Suppose that $\lambda(i, j) > 0$. It suffices to show that we can transform $\mathscr{D}$ to $\mathscr{D}'$, using $\triangle$-moves only, such that $\lambda'(i, j) < \lambda(i, j)$. The orders $I(\overline{h_i})$ and $I(\overline{h_j})$ induce two linear orders ρ_i and ρ_j, respectively, on $LT(i, j)$. If $\Delta(\rho_i, \rho_j) = \emptyset$, then $\max(\rho_i) = \max(\rho_j) = p$, and so we can perform a $\triangle$-move w.r.t. $\{i, j, p\}$. In the resulting diagram $p \notin LT(i, j)$ and hence $\lambda'(i, j) = \lambda(i, j) - 1$. If, however, $|\Delta(\rho_i, \rho_j)| > 0$ then, by Lemma 8, there is a pair $\{p, q\} \in \Delta(\rho_i, \rho_j)$ which appears consecutively in ρ_i. This implies that the pseudolines $\{h_i, h_p, h_q\}$ bound a parallel triangle T. By Lemma 17, we can reach a diagram $\mathscr{D}'$ by means of $\triangle$-moves where $\{i, p, q\}$ is a mutation. We can even choose the triangle moves between $\mathscr{D}$ and $\mathscr{D}'$ such that only separators of T of p and q are involved, and so we do not modify $LT(i, j)$. If we now transform $\mathscr{D}' \overset{\triangle}{\Rightarrow} \mathscr{D}''$ w.r.t. $\{i, p, q\}$, then $\Delta(\rho_i'', \rho_j'') = \Delta(\rho_i, \rho_j) \backslash \{\{p, q\}\}$, and induction on $|\Delta(\rho_i, \rho_j)|$ applies. Observe that we have only switched triangles that are exclusively bounded by elements of $\{i, j\} \cup LT(i, j)$. $\square$

Next, we arrive at one of the main results of this paper.

THEOREM 20. $\mathscr{D} \sim' \mathscr{D}' \Leftrightarrow \chi(\mathscr{D}) = \chi(\mathscr{D}')$.

PROOF. If $\mathscr{D} \sim' \mathscr{D}'$, then it follows immediately that $\chi(\mathscr{D}) = \chi(\mathscr{D}')$ by checking that $\triangle$ does not modify the orientation (CCW or CW) of a stack or vortex.

Conversely, let $\chi(\mathscr{D}) = \chi(\mathscr{D}')$. By Lemma 19, we may assume that $\lambda(i, j) = |LT(i, j)| = 0$. Notice that we have only used generalized triangle moves to accomplish this, and hence the base order has not been affected. Now we can perfrom a $\|$-move w.r.t. $\{i, j\}$, which switches i and j in the base order. We conclude that there is a $\mathscr{D}_1 \sim' \mathscr{D}$ the base order of which can be represented by $b_1 = (1\ 2 \cdots n)$. Consider now an arbitrary pseudoline h_i of $\mathscr{D}_1$. By the previous procedure we can successively transpose the pairs $\{i, i+1\}, \{i, i+2\}, \ldots, \{i, n\}, \{i, 1\}, \ldots, \{i, i-1\}$ in the base order. At the end, we come up with a diagram $\mathscr{D}_2$ the base order of which can still be represented by $b_2 = (1\ 2 \cdots n)$, but such that

$$\text{under}_2(i, x) \neq \text{under}_1(i, x) \qquad \text{if } x \in \{1, \ldots, n\} \backslash \{i\},$$

$$\text{under}_2(p, q) = \text{under}_1(p, q) \qquad \text{if } i \notin \{p, q\}.$$

We conclude that for each diagram $\mathscr{D}$ there is a diagram $\mathscr{D}^*$ such that:
(1) $\mathscr{D}^* \sim' \mathscr{D}$;
(2) the base order of $\mathscr{D}^*$ has a linear representative equal to $(1\ 2 \cdots n)$;
(3) under*$(1, \imath) = +1$ for all $i \in \{2, \ldots, n\}$.
Condition (3) is due to the fact that if under*$(1, i) = -1$, then we can always switch all the crossings of i without violating (1) or (2).

Due to Lemma 18, we can transform $\mathscr{D}^*$ to a well-based diagram $\bar{\mathscr{D}}$ w.r.t. $(1\ 2 \cdots n)$ by using only $\triangle$-moves, and hence $\bar{\mathscr{D}}$ still obeys the conditions (1), (2) and (3). Similarly, we can bring the second diagram $\mathscr{D}'$ in such a normal form $\bar{\mathscr{D}}'$. Since both normal forms are well-based and have the same base order, it follows that their underlying arrangements $\bar{\mathscr{H}}$ and $\overline{\mathscr{H}'}$ are combinatorially equivalent. Furthermore, we observe that

$$\chi(\bar{\mathscr{D}}) = \chi(\mathscr{D}) = \chi(\mathscr{D}') = \chi(\overline{\mathscr{D}'}).$$

And hence, as $\overline{\text{under}}(1, i) = \overline{\text{under}'}(1, i)$ for all i, the crossing functions must completely match

$$\overline{\text{under}} = \overline{\text{under}'}.$$

We conclude that $\bar{\mathscr{D}} = \overline{\mathscr{D}'}$ and so that $\mathscr{D} \sim' \mathscr{D}'$. □

Theorem 20 might appear as very artificial to the reader, since generalized triangle-moves no longer have any geometric meaning. However, this result can be very useful when one wants to compare new invariants with the chirality-invariant. Indeed, if one has designed an invariant for diagram moves which also remains invariant under generalized triangle-moves, then it is also an invariant for chirality classes. This means that the new invariant certainly is inferior to chirality (and hence worthless).

Remark. V. Mazurovskii kindly informed me that Theorem 20 has its analogue in Knot Theory. Indeed, M. Hitoshi and N. Yasutaka proved that generalized triangle-moves generate *link-homology* [6].

Another related result is the theorem of Ringel [15], stating that every pair of combinatorial types of simple arrangements of n pseudolines (in $\mathbb{P}^2$) can be connected by a sequence of *triangle-switches* which, in fact, are exactly the moves on the underlying arrangement when applying $\triangle$-moves and, if we add the line at infinity, $\|$-moves. We did not appeal to this result, since we had to distinguish the switches of affine triangles and non-affine triangles very carefully, on account of the superimposed weaving under. For a proof of Ringel's theorem, making use of *Coxeter relations* and *reduced decompositions* in the symmetric group S_n, see [1]. As a matter of fact, the arguments of this section yield a generalization of this result for labeled arrangements.

7. Stacks and Chirality

In this section we investigate a special class of diagrams. A *stack* is a pseudoline diagram with the property that its crossing function under determines a total order on $\{1, \ldots, n\}$. More precisely, this order is defined by

$$i < j \Leftrightarrow \text{under}(i, j) = +1.$$

Consequently, stacks are diagrams that are most easy to draw. In the case of a line diagram, when $\mathscr{H}$ is stretchable, we can always regard a stack as the planar layout of n horizontally stacked lines in $\mathbb{R}^3$.†

The linear order λ on the (pseudo)lines of a stack is called its *stack order.* We will represent λ by an ordered sequence; that is, a permutation of $(1, 2, \ldots, n)$,

$$\lambda = (\lambda(1), \ldots, \lambda(n)),$$

where $h_{\lambda(n)}$ is the line of $\mathscr{H}$ that lies 'on top' of the stack. Apart from λ, we can also consider the circular base order of $\mathscr{H}$, called κ in the case of a stack. If $\mathscr{H}$ consists of

† Such configurations are of the same rigid isotopy type as so-called 'join-configurations' introduced by O. Ya. Viro [8, 9, 16, 17] or 'spindles' [2].

straight lines, then κ reflects the slope order. Labeled stacks are easily given by the (κ, λ)-code, a $2 \times n$ matrix the first row of which is some linear representative of κ, $(\kappa(1), \ldots, \kappa(n))$, and the second row of which is exactly the λ-sequence. For unlabeled configurations we can always choose $\kappa = (1, \ldots, n)$. The following three results are again combinatorial analogues for well-known geometric facts for spatial line configurations (cf. [2]).

Theorem 21. *Two stacks with the same (κ, λ)-representation are equivalent as pseudoline diagrams.*

Proof. Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be two stacks with the same (κ, λ)-code. Then we can choose twice the same linear representatve b for their base orders. By virtue of Lemma 18, we can transform each diagram $\mathscr{D}$ to a well-based diagram $\bar{\mathscr{D}}$ w.r.t. b by means of (generalized) triangle moves. These triangle moves modify neither the base order nor the crossing function. So, if $\mathscr{D}$ is a stack, then each intermediate diagram between $\mathscr{D}$ and $\bar{\mathscr{D}}$ is still a stack, whence the involved triangle moves are valid $*$-moves. We conclude that $\mathscr{D}_1 \sim \overline{\mathscr{D}_1}$ and $\mathscr{D}_2 \sim \overline{\mathscr{D}_2}$, where both $\overline{\mathscr{D}_1}$ and $\overline{\mathscr{D}_2}$ are well-based w.r.t. b. By the definition of a well-based diagram, b completely determines the corresponding intersection pattern. Furthermore, $\overline{\mathscr{D}_1}$ and $\overline{\mathscr{D}_2}$ have the same crossing function, given by λ, and they are not each other's mirror image for they have the same base order. We conclude that $\overline{\mathscr{D}_1}$ and $\overline{\mathscr{D}_2}$ are combinatorially equivalent. □

Because the (κ, λ)-representation determines the stack up to diagram equivalence, it must determine the chiralities as well (Theorem 13). In order to have a convenient formulation, we define λ_{ijk} as the order in which the triple $\{i, j, k\}$ occurs in λ. As usual, we can consider λ_{ijk} as the permutation reaching the involved order by acting on the sequence (i, j, k). We take the convention that the sign of a permutation is $+1$ if it is an even permutation, and that its sign is -1 if it is an odd permutation. Notice that κ_{ijk} depends on the chosen representative for κ, but $\operatorname{sign}(\kappa_{ijk})$, however, is well-defined.

Theorem 22. *If $\mathscr{D}$ is an (κ, λ)-stack, then $\chi(\mathscr{D})$ is given by the formula*

$$\chi_{ijk} = \operatorname{sign}(\lambda_{ijk})\operatorname{sign}(\kappa_{ijk}).$$

Proof. This formula is a restatement of the CW–CCW rule for chiralities in the case of stacks. □

Example. For the stack

$$\begin{pmatrix} 1 & 3 & 4 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

we obtain $\chi_{134} = \chi_{234} = +1$ and $\chi_{123} = \chi_{124} = -1$.

In the remainder of this section we prove perhaps the most important result of this paper (Theorem 24). We will show that two stacks with the same chiralities are always equivalent as pseudoline diagrams. This definitely closes the problem of diagram equivalence for stacks. Indeed, the most obvious invariant, namely chirality, turns out to be complete in this case, yielding an $O(n^2)$ algorithm to decide the equivalence of stacks. In personal communication, V. Mazurovskii claims to have proved the geometric analogue of this results, in joint work with S. Khashin.† This would mean that two 'join configurations' are rigidly isotopic as soon as they have equal chirality.

LEMMA 23. *If $\mathscr{D}$ is an (κ, λ)-stack and if λ' is obtained from λ by circular shifts, so that λ and λ' determine the same circular order, then $\mathscr{D}$ is equivalent to the (κ, λ')-stack.*

PROOF. First we notice that we can always find a linear representative b for the base order κ such that $b(1) = \lambda(1)$. Next, by Lemma 18 and the proof of Theorem 21, we see that $\mathscr{D} \sim \bar{\mathscr{D}}$, where $\bar{\mathscr{D}}$ is well-based w.r.t. b. By a successive application of $n-1$ $\|$-moves, we reverse all crossings of the pseudoline $h_{\lambda(1)}$. Observe that the resulting diagram is still a stack, the stack order of which is just a circular shift of λ; indeed, $\lambda(1)$ is now at the top. Furthermore, each $\|$-move transposes $\lambda(1)$ with its adjacent element in b, so at the end we come up with a circular shift of b as well, which still represents the same circular base order κ. □

THEOREM 24. *Two stacks $\mathscr{D}$ and $\mathscr{D}'$ are equivalent iff they have the same chiralities.*

PROOF. If $\mathscr{D}$ and $\mathscr{D}'$ are equivalent, then we have already observed that they have the same chiralities (Theorem 13). Conversely, if they are in the same chirality class then we first replace both diagrams by equivalent diagrams such that $\lambda(1) = \lambda'(1) = 1$, using Lemma 23. Furthermore, we let b and b' represent κ and κ', respectively, such that $b(1) = b'(1) = 1$. We now use

$$\chi_{1ij} = \operatorname{sign}(\lambda_{1ij})\operatorname{sign}(\kappa_{1ij}),$$

and the similar formula for χ'_{1ij}. Since both chiral signatures are equal for all i and j, we conclude that

$$\begin{aligned} \{i, j\} \in \Delta(b, b') &\Leftrightarrow \operatorname{sign}(\kappa_{1ij})\operatorname{sign}(\kappa'_{1ij}) < 0 \\ &\Leftrightarrow \operatorname{sign}(\lambda_{1ij})\operatorname{sign}(\lambda'_{1ij}) < 0 \\ &\Leftrightarrow \{i, j\} \in \Delta(\lambda, \lambda'), \end{aligned}$$

whence $\Delta(b, b') = \Delta(\lambda, \lambda')$.

Now suppose for the moment that $\mathscr{D} = (\mathscr{H}, \text{under})$ is some diagram, not necessarily a stack, and let b represent its base order and take some other total order b', arbitrarily. Endow $\mathscr{H}$ with the monotone orientation induced by b, and let $\mathscr{I}$ be the associated intersection pattern. Recall that $R(\mathscr{I})$ induces a total order $\leqslant$ on the alphabet a_{ij}, after the transitive closure (Section 2). Furthermore, we put $M = \Delta(b, b')$ and let $V(M)$ denote all diagram crossings that are dominated by some $\{i, j\} \in M$ relative to $\leqslant$. We claim that if $\mathscr{D}$ has the property that all triangles (not necessarily faces) vertices of which exclusively belong to $V(M)$ are stacks, then $\mathscr{D}$ is equivalent to some diagram $\mathscr{D}_* = (\mathscr{H}_*, \text{under}_*)$ such that b' represents the base order of $\mathscr{H}_*$. Indeed, if $M = \varnothing$, then $b' = b$ and we can take $\mathscr{D}_* = \mathscr{D}$. So let us suppose that $|M| > 0$. By Lemma 8 there exists a pair $\{i, j\} \in M$ such that i and j are adjacent in b. Now we can consider the set of lower traversals $LT(i, j)$ as in Lemma 19. By virtue of this lemma, we can transform $\mathscr{D}$ to a diagram in which $LT(i, j) = \varnothing$ by means of (generalized) triangle moves, where only i, j or members of $LT(i, j)$ are involved. Recall that we

† *Note added in proof*:
S. Khashin and V. Muzurovskii, Stable Equivalence of Real Projective Configurations, to appear in 'Advances in Soviet Mathematics'.

have assumed that there are no vortices among these triangles, and hence only $*$-moves have been used. Once $LT(i, j) = \emptyset$ we can perfrom a $\|$-move w.r.t. $\{i, j\}$, leading to a switch of this pair in the base order. So, if $\mathscr{D}''$ denotes the new diagram then we can represent its base order by b'' with $M'' = \Delta(b'', b') = M \setminus \{\{i, j\}\}$. Furthermore, we have only modified under(i, j) in the crossing function, and $V(M'') \subset V(M) \setminus \{a_{ij}\}$, implying that there are still no vortices on $V(M'')$. So the claim follows by induction on $|M|$.

Of course, since $\mathscr{D}$ is a stack, the initial condition on $V(M)$ trivially holds. So $\mathscr{D} \sim \mathscr{D}_*$, where b' represents the base order of $\mathscr{H}_*$. Furthermore, in the process from $\mathscr{D}$ to $\mathscr{D}_*$ we have changed the crossing function exactly for $\{i, j\} \in M = \Delta(b, b')$. But $M = \Delta(\lambda, \lambda')$ as well, and hence under$_*$ is given by the total order λ' which implies that $\mathscr{D}_*$ is a stack. Because both $\mathscr{D}_*$ and $\mathscr{D}'$ are represented by (κ', λ'), they must be equivalent, by Theorem 21. □

8. Application

As described in Section 4, the most important application of moves on pseudoline diagrams is what motivated us to study them: the isotopy of configurations of mutually skew lines in 3-space. If two configurations are isotopic, then their planar layouts are equivalent under diagram moves. So, invariants for diagram moves provide us with invariants of rigid isotopy of line configurations. Examples are ‘chirality’ and the ‘Kauffman polynomial’ for pseudoline diagrams [2, 3, 8]. One should be aware that diagram equivalence is a relaxation of (rigid) line isotopy to a combinatorial framework. However, until now we did not know of any example of non-isotopic line configurations with equivalent planar layouts.

Another area of application of diagram equivalence is the realizability of *weaving patterns*; that is, to recognize planar layouts among all possible line diagrams. We illustrate this by the *free line principle.* A *free line* in a line diagram $\mathscr{D} = (\mathscr{H}, \text{under})$ is a member h of $\mathscr{H}$ with at most one change of crossing signs. More precisely, either h crosses over all the other lines of $\mathscr{H}$, or h crosses under all these lines, or there is a point x on h that separates the over-crossings from the under-crossings. It is not hard to prove that every configuration $\mathscr{L}$ in $\mathbb{R}^3$ is isotopic to a configuration $\mathscr{L}'$ such that its layout $D(\mathscr{L})$ has at least one free line [13]. Consequently, every realizable diagram must be equivalent to a diagram with a free line. For instance, let $\mathscr{D} = (\mathscr{H}, \text{under})$ be a line diagram where the over- and under-crossings alternate on each line of $\mathscr{H}$, introduced in [11] and called a *complete weaving pattern.* Furthermore, let $n = |\mathscr{H}|$ be an even number different from 2. Obviously, $\mathscr{D}$ contains no free line ($n > 3$). Since each triangular face of $\mathscr{D}$ (mutation) must be a vortex, no $*$-move can be performed on $\mathscr{D}$. Furthermore, since the intersection sequence of each line has odd length (n is even), each diagram $\mathscr{D}'$ that differs from $\mathscr{D}$ by a $\|$-move is still a complete weaving pattern. By induction on the length of the move sequence, we see that each diagram that is equivalent to $\mathscr{D}$ completely alternates. By the ‘free line principle’ we must conclude that $\mathscr{D}$ is not realizable. This result has already been obtained in [11], even for general $n \geqslant 4$, but it is the first time that diagram moves have been used in realizability arguments.

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