JOURNAL OF ALGEBRA **194,** 521–566 (1997)
ARTICLE NO. JA977038

The Simple Modules of Certain Generalized Crossed Products

V. Bavula*,†

Department of Mathematics, Kiev University, Vladimirskaya Str, 64, Kiev 252617, Ukraine

and

F. van Oystaeyen‡

University of Antwerp U.I.A., Department of Mathematics and Computer Science, Belgium

Communicated by Efim Zelmanov

Received October 11, 1996

We consider a Dedekind domain D and a $\mathbf{Z}$-graded ring $R = \bigoplus_{i\in\mathbf{Z}} R_i$ with $R_0 = D$ and each $R_i = Dv_i$ being a free D-module of rank 1. The structure of R is described by an automorphism of D and a generalized 2-cocycle $c: \mathbf{Z}\times\mathbf{Z}\to D$ not necessarily taking its values in the units of D. The aim of this paper is to classify the simple R-modules, say in the case where $c(i,j)\neq 0$ for all $i,j\in\mathbf{Z}$. We also deal with this problem in the $\mathbf{N}$-graded case. As a consequence we obtain a description of the simple modules of some classical algebras and of generalized Weyl algebras.

The $\mathbf{Z}$-graded ring studied here is of the following type: $A = \bigoplus_{i\in\mathbf{Z}} A_i$ with $A_i = A_0 v_i$ being a free left A_0-module of rank 1 and $v_0 = 1$. The multiplicative structure of A is given by: for $\alpha, \beta \in A_0$ and $i, j \in \mathbf{Z}$

$$\alpha v_i \beta v_j = \alpha\sigma^i(\beta)c(i,j)v_{i+j},$$

where σ an automorphism of A_0 and

$$c: \mathbf{Z}\times\mathbf{Z}\to Z(A_0), \qquad \text{the centre of } A_0,$$

*The first author was supported by an INTAS grant, research fellow at U.I.A.
† E-mail: sveta@kinz.kiev.ua.
‡ E-mail: francin@wins.uia.ac.be.

0021-8693/97 $25.00

is a 2-cocycle, i.e., it satisfies the condition

$$c(i,j)c(i+j,k) = \sigma^i(c(j,k))c(i,j+k),$$

for $i, j, k \in \mathbf{Z}$. Throughout the paper a "module" means a left module.

A nontrivial class of examples, the generalized Weyl algebras (GWA) introduced and studied extensively by the first author, is obtain by letting A be generated over A_0 by two indeterminates $X = v_1$ and $Y = v_{-1}$ satisfying the relations [Bav 1, 4]:

$$X\alpha = \sigma(\alpha)X \qquad \text{and} \qquad Y\alpha = \sigma^{-1}(\alpha)Y,$$
$$\forall \alpha \in D, YX = a, XY = \sigma(a) \tag{0.1}$$

for an automorphism σ of A_0 and $a \in Z(A_0)$. If $A_0 = D$ is commutative, then rings of this type have been considered in [Jor 1,2] resp. [Hod 1] in the case of a polynomial ring. We shall restrict attention to commutative D that are Dedekind domains. The class of graded rings considered contains, e.g., the first Weyl algebra A_1 as well as its quantum deformation $A_1(q)$, the quantum plane, the algebra of function of the quantum 2-dimensional sphere. All the infinite-dimensional prime factors of the following algebras are also of this type: the universal enveloping algebra $Usl(2)$ as well as its quantum version $U_q sl(2)$, Smith's algebra [Sm], Witten's first and Woronowicz's deformations [Za, Bav 5]; the quantum group $O_{q^2}(so(k,3))$ [Sm 1]; the quantum Heisenberg algebra [Ma, KS, Ros, Bav 5]; and more. It has been observed that all these algebras are GWA [Bav 5, 6]. Thus, as a consequence, for all these algebras we obtain a description of all the simple modules.

In [Bl 1–3], Block classified the simple modules over A_1 and $Usl(2)$ (up to irreducible elements of a certain Euclidean ring). Under suitable restrictions on D and ∂ it was also carried out for the differential operator ring $D[X;\partial]$ in [Bl 3] and for $K[[X]][\partial]$ in [EL]. In [Ros], Rosenberg described the simple modules over the quantum Weyl algebra $A_1(q)$. Before that only fragmentary results existed; e.g., [Bam; Di1, 2; AP1, 2; Le]. As a next step, in [Bav 2–4], the first author classified the simple modules of a GWA $D(\sigma, a)$ as defined in (1) in case $A_0 = D$ and all maximal ideals $\mathbf{p}$ of D are *linear*, i.e., $\mathbf{p} \neq \sigma^n(\mathbf{p})$ for all $n \neq 0 \in \mathbf{Z}$ (if $\mathbf{p} = \sigma^n(\mathbf{p})$ for some nonzero n then $\mathbf{p}$ is called *cyclic*). For a field K of characteristic zero $A_1(K)$ is in that class as well as all infinite-dimensional prime factors of $Usl(2)$ resp. $U_q sl(2)$; q is not a root of 1. On the other hand cyclic maximal ideals exist in case of the quantum Weyl algebra $A_1(q)$ and the quantum plane: $\Lambda \simeq K[H](\sigma, H)$ with $\sigma(H) = \lambda H$, $\lambda \neq 0, 1 \in K$, and $D = K[H]$ ($\Lambda = K\langle X, Y \mid XY = \lambda YX \rangle$). The present results will complete the picture.

For a ring A let $\hat{A}$ be the set of isomorphism classes of simple A-modules. The situation where $A_0 = D$ is a Dedekind domain and $c(i, j) \neq 0$ for $i, j \in \mathbf{Z}$ will be indicated by writing R instead of A (if it is not stated otherwise). Sometimes we want to allow some $c(i, j)$ to be zero and then this shall be stated explicitly.

Denote by k the field of fractions of D, i.e., $k = S^{-1}D$, $S := D \setminus \{0\}$. Then the localization $B = S^{-1}R$ of the ring R is the skew Laurent polynomial ring $B = k[X, X^{-1}; \sigma]$ with coefficients in the field k which contains $R(R \to B, r \to r/1)$. It is a Euclidean ring and, hence, a left and right principal ideal domain. By a classical theory the simple modules of B can be described in terms of factorization of elements of B. We recall [Jac] that a B-module M is simple if and only if $M \simeq B/Bb$ for some irreducible $b \in B$ (i.e., $b = ac$ implies a or c is a unit), and B-modules B/Ba and B/Bb are isomorphic if and only if a and b are similar, i.e., there exists $c \in B$ such that 1 is a greatest common right divisor of b and c and ac is a least common left multiple of b and c. If M is a simple R-module, then the localization $S^{-1}M = B \otimes_A M$ of M at S is either 0 or a (nonzero) simple B-module. In accordance with these two possibilities we say that M is *D-torsion* or *D-torsionfree*.

CONVENTION. If P is an isomorphism invariant property of simple R-modules then $\hat{R}(P) = \{[M] \in \hat{R},\ M \text{ has property } P\}$.

With this convention we may state

$$\hat{R} = \hat{R}(D - \text{torsion}) \cup \hat{R}(D - \text{torsionfree}).$$

An R-module M is *weight*, if M is semisimple as a D-module. One easily verifies (Section 5) that

$$\hat{R}(D - \text{torsion}) = \hat{R}(D - \text{weight}).$$

The terminology is inspired by the fact that for $Usl(2, \mathbf{C})$ these weight modules are precisely the weight modules in the usual sense. In Section 2 we describe $\hat{R}$ in case $D = K$ is a skew field (and we have four types of simple R-modules). In Section 3 we apply the foregoing in order to classify the simple weight R-modules in case D is a commutative ring. Section 4 specifies the foregoing in case of the GWA $D(\sigma, a)$. The core of this paper is the description of $\hat{R}(D - \text{torsionfree})$ over a Dedekind domain D and this is the topic of Section 5.

An R-module M is said to be an *R-socle* module if $\mathrm{Soc}_R(M)$ is nonzero. Put $S = D \setminus \{0\}$, $S^{-1}R = B$ and consider $\hat{B}(R - \text{socle})$. We say that $b \in B$ is R-socle if B/Bb is R-socle. First, in Lemma 5.1 we give a description of simple D-torsionfree R-modules in terms of simple R-socle

B-modules. In fact, *the canonical map*

$$S^{-1}: \hat{R}(D - \text{torsionfree}) \to \hat{B}(R - \text{socle}), \qquad [M] \to [S^{-1}M],$$

is bijective with inverse, given by $\text{Soc}:[N] \to [\text{Soc}_R(N)]$. *Each simple D-torsionfree R-module has the form $M_{\mathbf{m}} := R/R \cap B\mathbf{m}$ for some left maximal ideal $\mathbf{m}$ of B and M_{Bb} and M_{Bc} are isomorphic if and only if $B/\mathbf{m} \simeq B/\mathbf{n}$ ($\mathbf{m} = Bb$ and $\mathbf{n} = Bc$) as B-modules*; *i.e., the corresponding irreducible elements b and c are similar*.

For a GWA $R = D(\sigma; a \neq 0)$ such that all maximal ideals of D are linear, the first author's results [Bav 2, 4] yield that $\hat{B}(R - \text{socle}) = \hat{B} = k[X, X^{-1}; \sigma]\hat{}$. However, for more general GWA it need not be the case and in fact in most cases simple B-modules are not R-socle modules (Sect. 6). Theorem 5.14 establishes that $\hat{R}(D - \text{torsionfree})$ *is nonempty exactly if and only if only finitely many cyclic maximal ideals exist in D.*

Let G be a cyclic group generated by σ in $\text{Aut}(D)$; then G acts in the obvious way on $\text{Specm}(D) = \{\text{the maximal ideals of } D\}$. We say that an orbit $O(\mathbf{p}) = \{\sigma^i(\mathbf{p}), i \in \mathbf{Z}\}$ is *linear* resp. *cyclic* if $\mathbf{p}$ is linear resp. cyclic.

For $\alpha, \beta \in S$ we write $\alpha < \beta$ if there are no maximal ideals $\mathbf{p}$ and $\mathbf{q}$ of D belonging to the same linear orbit such that $\alpha \in \mathbf{p}$, $\beta \in \mathbf{q}$, and $\mathbf{p} = \sigma^i(\mathbf{q})$ for some $i \geq 0$. An element $b = v_{-m}\beta_{-m} + \cdots + \beta_0 \in R$ with $\beta_i \in D$, $\beta_{-m} \neq 0$, $\beta_0 \neq 0$, and $m > 0$, is called *l-normal* if $\beta_0 < \beta_{-m}$, and $\beta_0 < a := c(-1, 1)$.

In Proposition 5.6 we provide a procedure allowing us to make an element into an l-normal one.

To a cyclic orbit $O(\mathbf{p})$ we associate $\theta(O) = \sqcap_{i=0}^{n-1} \sigma^i(\mathbf{p})$ (where n is the first positive for which $\mathbf{p} = \sigma^n(\mathbf{p})$). Let b be irreducible in B; the cyclic orbit condition states:

- (CO) *for any cyclic orbit O, $R = R\theta(O) + R \cap Bb$.*

Theorem 5.13 provides a necessary and sufficient condition for $R/R \cap Bb$ to be an R-socle module (for b irreducible in B): in fact this happens exactly when (CO) holds and, moreover, if b is l-normal, then the R-module $R/R \cap Bb$ is simple. Note that condition (CO) has to be verified only for finitely many orbits (Theorem 5.14). Combining results of Section 5 leads to

THEOREM A. *Let R be as above with Dedekind D and $c(i, j) \neq 0$ for all $i, j \in \mathbf{Z}$, and let b be l-normal irreducible in B and* (CO) *holds. Then $R/R \cap Bb$ is a D-torsionfree simple R-module* $(= \text{Soc}_R(B/Bb))$. *Up to isomorphism every D-torsionfree simple R-module arises in this way, and from a b which is unique up to similarity.*

If $\mathcal{O}(\mathbf{p}) = \{\mathbf{p}\}$, i.e., $\mathbf{p} = \sigma(\mathbf{p})$, $c(-1,1) \notin \mathbf{p}$, $b \notin R\mathbf{p}$, then (CO) *holds for the orbit* $\mathcal{O}$ *if and only if exists exactly one* β_i*th such that* $\beta_i \notin \mathbf{p}$ (Proposition 5.16).

In Section 6 the foregoing is applied to classify the simple modules over the quantum Weyl algebra $A_1(q)$ over an algebraically closed field K. If $\operatorname{char} K = 0$, then $A_1(q)$ is a case where there is a unique cyclic maximal ideal which is σ-invariant and satisfies the assumption of Proposition 5.16. Of course, for A_1 there are no cyclic ideals at all and this shows that the presence of even a unique cyclic ideal may reduce substantially the set of simple modules.

In Section 7 we focus on another example given as a factor algebra V of the universal enveloping algebra of the Virasoro Lie algebra and describe $\hat{V}$, providing many simple modules of the Virasoro algebra.

In the positively graded case, $R = \bigoplus_{i \geq 0} R_i$, modifications are needed. In fact, Theorem 5.14 fails in the $\mathbf{N}$-graded case, e.g., $\hat{R}(D-\text{torsionfree})$ *may be nonempty but there are infinitely many cyclic maximal ideals in* D.

As in the $\mathbf{Z}$-graded case

$$\hat{R} = \hat{R}(\text{weight}) \cup \hat{R}(D - \text{torsionfree}).$$

The set of simple weight R-modules is described in Section 3 (for a commutative ring D) and the set $\hat{R}(D-\text{torsionfree})$ in Section 8. The localization $B = S^{-1}R$ of R at $S = D \setminus \{0\}$ is the skew polynomial ring $B = k[X;\sigma]$ with coefficients in the field k (B is Euclidean). The main result of Section 8 (Theorem 8.1) is:

THEOREM B. *Let R be $\mathbf{N}$-graded with Dedekind D and $c(i,j) \neq 0$ for all $i, j \in \mathbf{N}$, and let $f = a_0 + \cdots + a_n v_n \in R$ be irreducible in B, all $a_i \in D$. Then $R/R \cap Bf$ is a D-torsionfree simple R-module* ($= \operatorname{Soc}_R(B/Bf)$) *if and only if*

(CO) $R = R\theta(\mathcal{O}) + R \cap Bf$ *for all cyclic orbit* $\mathcal{O}$;

(LO) $R = \mathbf{p} + R^{(+)} + R \cap Bf$ *for all maximal linear ideal* $\mathbf{p}$ *such that* $a_0 \in \mathbf{p}$, *where* $R^{(+)} = \bigoplus_{i>0} R_i$.

Up to isomorphism every D-torsionfree simple R-module arises in this way, and from an f which is unique up to similarity.

The final section is devoted to the description of the simple modules of skew polynomial rings with coefficients in a Dedekind domain.

1. DEFINITIONS, GENERALIZED WEYL ALGEBRAS, EXAMPLES

Let R be a **Z**-graded ring as above with a *commutative* ring D. The map c is a "2-cocycle," i.e., it satisfies the associativity condition $(v_i v_j)v_k = v_i(v_j v_k)$ for $i, j, k \in \mathbf{Z}$, and this is equivalent to

$$c(i,j)c(i+j,k) = \sigma^i(c(j,k))c(i,j+k).$$

Every $R_i = v_i D$ is also a free right D-module of rank 1. We say that a maximal ideal $\mathbf{p}$ of D is of finite $(\sigma-)$order $n = \mathrm{ord}_\sigma \mathbf{p}$ if n is the natural integer minimal with respect to $\sigma^n(\mathbf{p}) = \mathbf{p}$.

Most natural examples of the ring R are generalized Weyl algebras (of degree 1; for a definition of the GWA of higher degree the reader is referred to [Bav 4, 5]). Let $A = D(\sigma, a)$ be a GWA. Setting $v_n = X^n$ $(n > 0)$, $v_n = Y^{-n}$ $(n < 0)$, $v_0 = 1$. It follows from the defining relations that

$$v_n v_m = (n,m)v_{n+m} = v_{n+m}\langle n,m\rangle$$

for some $(n,m) \in D$. If $n > 0$ and $m > 0$, then

$$n \geq m : (n,-m) = \sigma^n(a) \cdots \sigma^{n-m+1}(a),$$
$$(-n,m) = \sigma^{-n+1}(a) \cdots \sigma^{-n+m}(a),$$
$$n \leq m : (n,-m) = \sigma^n(a) \cdots \sigma(a), \qquad (-n,m) = \sigma^{-n+1}(a) \cdots a;$$

in other cases $(n,m) = 1$.

One of the main goals of this article is to give a classification (up to irreducible elements of some Euclidean ring) of the simple modules over a GWA $A = D(\sigma, a)$ with Dedekind D. This problem depends on whether there are maximal ideals of D of finite σ-order or not. If not (the relatively simple case) it was solved in [Bav 2–4]. The general case is considered in the present paper. Now we give examples of GWAs with Dedekind D for which the simple modules are described in this paper. Most important for us are those algebras where D has maximal ideals of finite σ-order.

It follows from the equality

$$\sigma^n(c(-n,n))v_n = v_n c(-n,n) = v_n v_{-n} v_n = c(n,-n)v_n$$

that $c(n,-n) = \sigma^n(c(-n,n))$; thus the subring $R_{\langle n\rangle}$ of R generated by D, v_{-n}, and v_n is the GWA

$$R_{\langle n\rangle} = D(\sigma^n, c(-n,n)).$$

Throughout the paper we denote by A the ring $R_{\langle 1\rangle}$, i.e.,

$$A := D\langle v_{-1} = Y, v_1 = X\rangle = D(\sigma, a := c(-1,1))$$

is a GWA which plays a crucial role in the representation theory of R. For each $i \geq 1$ there exists $\alpha_{\pm i} \in D$ such that $v^i_{\pm 1} = \alpha_{\pm i} v_{\pm i}$. It is clear that for every $i \geq 1$:

$$a\sigma^{-1}(a) \cdots \sigma^{-i+1}(a) = v^i_{-1} v^i_1 = \alpha_{-i}\sigma^{-i}(\alpha_i)c(-i,i); \qquad (1.1)$$

hence, if $\alpha_i \in \mathbf{p}$ for some maximal ideal $\mathbf{p}$ of D, then $a \in \sigma^j(\mathbf{p})$ for some $j \in \mathbf{Z}$. Multiplying $v^i_{\pm 1} v^j_{\pm 1}$ in two different ways we have (for some $k \in \mathbf{Z}$)

$$\begin{aligned} \ldots \alpha_k v_{\pm i \pm j} = v^i_{\pm 1} v^j_{\pm 1} &= \alpha_{\pm i}\sigma^{\pm i}(\alpha_{\pm j}) v_{\pm i} v_{\pm j} \\ &= \alpha_{\pm i}\sigma^{\pm i}(\alpha_{\pm j}) c(\pm i, \pm j) v_{\pm i \pm j}, \qquad (1.2) \end{aligned}$$

where three dots is a product of $\sigma^i(a)$th or 1. By (1.1), (1.2) we conclude that if $c(i,j) \in \mathbf{p}$ $(i, j \in \mathbf{Z})$ for some maximal ideal $\mathbf{p}$ of D, then $a \in \sigma^k(\mathbf{p})$ for some $k \in \mathbf{Z}$.

Throughout K is a field.

EXAMPLE 1. [Hod 1, Bav 1–4]. $A = K[H](\sigma, a)$, $\sigma(H) = H - 1$, where $D = K[H]$ is the polynomial ring over K. In [Hod 1] the analog of the Bernstein–Beilinson Theorem [BB] is established, the Grothendieck group $K_0(A)$ and the global dimension are calculated. In [Bav 4] the properties of these algebras (the global homological dimension, the Krull dimension, two-sided ideals, etc.) and the category of simple modules (the classification of simple modules, the finite dimension of the vector spaces Ext^i_A and Tor^A_i for simple modules, etc.) are studied.

The first Weyl algebra $A_1 = K\langle X, \partial \mid \partial X - X\partial = 1\rangle$ is isomorphic to the considered GWA:

$$A_1 \simeq K[H](\sigma, a = H), X \leftrightarrow X, \partial \leftrightarrow Y, \partial X \leftrightarrow H.$$

The other example is the fixed ring $F_m = A_1^G$ where the cyclic group G of order m acts by multiplication $(\partial \to \omega\partial, X \to \omega^{-1}X)$ on a primitive mth root of unity. Then

$$\begin{aligned} F_m &= K\langle \partial^m, \partial X, X^m\rangle \\ &\simeq K[H](\sigma, a = m^m H(H - 1/m)\cdots(H - (m-1)/m)), \\ &\qquad \partial^m \leftrightarrow Y, X^m \leftrightarrow X, \partial X/m \leftrightarrow H, \end{aligned}$$

is a GWA with $\sigma(H) = H - 1$, char $K = 0$ [Hod 1].

Let $U = Usl(2)$ be the universal enveloping algebra of the Lie algebra

$$sl(2) = \langle X, Y, H \mid [H, X] = X, [H, Y] = -Y, [X, Y] = 2H \rangle$$

over a field K, $C = H(H+1) + YX$ the *Casimir* element. Then

$$U \simeq K[H, C](\sigma, a = C - H(H+1)),$$

where $\sigma : H \to H - 1$, $C \to C$ [Bav 1].

For any $\lambda \in K$ the factor algebra $U(\lambda) := U/U(C - \lambda)$ is the GWA:

$$U(\lambda) \simeq K[H](\sigma, a = \lambda - H(H+1)).$$

EXAMPLE 2. $A = K[H](\sigma, a)$, $\sigma(H) = \lambda H$, $\lambda \neq 0 \in K$. Let $\Lambda = K\langle X, Y \mid XY = \lambda YX \rangle$ be the *quantum plane*. Then

$$\Lambda \simeq K[H](\sigma, a = H), X \leftrightarrow X, Y \leftrightarrow Y, YX \leftrightarrow H.$$

Let $A(S^2_\lambda) = K\langle X, Y, H \mid XH = \lambda HX, YH = \lambda^{-1}HY, \lambda^{1/2}YX = -(c - H)(d + H), \lambda^{1/2}XY = -(c - \lambda H)(d + \lambda H)\rangle$ be the *algebra of functions on the quantum* 2-*dimensional sphere*. Then

$$A(S^2_\lambda) \simeq K[H](\sigma, a = -\lambda^{-1/2}(c - H)(d + H)).$$

EXAMPLE 3. $A = K[H, H^{-1}](\sigma, a)$, $\sigma(H) = qH$, $q \neq 0 \in K$.

For $q, h = q - q^{-1} \in K = \mathbf{C}$ (the field of complex numbers), the algebra $U_q = U_q sl(2)$ is generated by X, Y, H_-, H_+ subject to the relations:

$$H_+H_- = H_-H_+ = 1, XH_\pm = q^{\pm 1}H_\pm X, YH_\pm = q^{\mp 1}H_\pm Y, [X, Y] = (H_+^2 - H_-^2)/h.$$

It follows from the relations that

$$U_q \simeq K[C, H, H^{-1}](\sigma, a = C + \{H^2/(q^2 - 1) - H^{-2}/(q^{-2} - 1)\}/2h),$$

where $\sigma(H) = qH$, $\sigma(C) = C$. The element C belongs to the center of U_9. For any $\lambda \in K$ the factor algebra $U_q(\lambda) := U_q/U_q(C - \lambda)$ is the GWA

$$U_q(\lambda) \simeq K[H, H^{-1}](\sigma, a = \lambda + \{H^2/(q^2 - 1) - H^{-2}/(q^{-2} - 1)\}/2h).$$

EXAMPLE 4. The *quantum Weyl* algebra $A_1(q) = \langle X, \partial \mid \partial X - qX\partial = 1\rangle$ of degree 1 over K $(q \neq 0 \in K)$ is the GWA

$$A_1(q) \simeq K[H](\sigma, a = H), X \leftrightarrow X, \partial \leftrightarrow Y, \partial X \leftrightarrow H,$$

with $\sigma : \sigma(H) = q^{-1}(H - 1)$.

Let Vir be the *Virasoro* Lie algebra, i.e., the infinite-dimensional vector space with basis e_i, $i \in \mathbf{Z}$, c, where the Lie algebra structure is defined by

$$[e_i, e_j] = (j - i)e_{i+j} + (j^3 - j)\delta_{i,-j}c/12$$

and c is the central element of Vir. Denote by V the image of the algebra homomorphism from the universal enveloping algebra $U(\text{Vir})$ to the localization $A_{1,(X)}$ of the first Weyl algebra A_1 at the multiplicatively closed subset $\{X^i, i \geq 0\}$ and defined as

$$U(\text{Vir}) \to V \subset A_{1,(X)}, \qquad e_i \to X^{i+1}\partial, \qquad c \to 0.$$

V is a homogeneous subalgebra of the $\mathbf{Z}$-graded algebra

$$A_{1,(X)} = K[H](\sigma, H)_{(X)} = K[H][X, X^{-1}; \sigma], \qquad \sigma(H) = H - 1.$$

Thus V is the example of the ring R with $D = K[H]$, the polynomial ring, and the automorphism $\sigma : H \to H - 1$. In fact, setting $v_i = X^{i+1}\partial$, $i \neq 0 \in \mathbf{Z}$, $v_0 = 1$, $H = \partial X$, we see that

$$V = \oplus_{i \in \mathbf{Z}} Dv_i$$

is $\mathbf{Z}$-graded with

$$v_i v_j = (H - i - 1)v_{i+j}, \qquad \text{if } i + j \neq 0$$

$$\left(v_i v_j = X^{i+1}\partial X^{j+1}\partial = X^i\sigma(H)X^{j+1}\partial = \sigma^{i+1}(H)X^{i+j+1}\partial = \sigma^{i+1}(H)v_{i+j}\right),$$

and

$$v_{-i}v_i = (H + i - 1)(H - 1), \qquad \text{if } i \neq 0 \in \mathbf{Z}.$$

Moreover, V is affine and generated over K by H, $v_{\pm 1}$, and $v_{\pm 2}$.

2. THE SIMPLE R-MODULES (D IS A SKEW FIELD)

Let K be a field not necessarily commutative and let $F = \bigoplus_{i \in \mathbf{Z}} F_i$ be a $\mathbf{Z}$-graded ring such that $F_0 = K$ and each nonzero component F_i is a 1-dimensional left and right vector space over K. We *aim to classify the simple F-modules*. If $F = K$, then $\hat{F} = \{K\}$, so we consider the case when $F \neq K$.

Denote by F^+, $F^{(+)}$, F^-, and $F^{(-)}$ the direct sum $\oplus F_i$ where i runs through $i \geq 0$, $i > 0$, $i \leq 0$, and $i < 0$, respectively. $F^{(\pm)}$ is an ideal of the ring $F^{\pm}$.

A ring R is called a *domain* if the product of nonzero elements is always nonzero, and an ideal I of R is *completely prime* if the factor ring R/I is a domain. An element $r \in R$ is *strongly nilpotent* if any sequence $r = r_0$, $r_1, \dots$ such that $r_{i+1} \in r_i R r_i$ is ultimately zero. The *prime radical* $N(R)$ (= the intersection of all prime ideals of R) is precisely the set of strongly nilpotent elements of R [MR1, 0.2.6].

The following proposition describes the prime radical of F.

PROPOSITION 2.1. *The prime radical $N(F)$ of F contains precisely all nilpotent elements of F and equals $N(F) = \sum\{F_i$ is nilpotent, i.e., $F_i^n = 0$ for some $n\}$. Denote by R the factor ring $F/N(F)$. Then $N(F)$ is not completely prime if and only if $R^{(+)} \neq 0$, $R^{(-)} \neq 0$, $R^{(+)}R^{(-)} = R^{(-)}R^{(+)} = 0$, both R^+ and R^- are domains.*

Proof. Denote by J the sum above. If $F_iF_jF_i \neq 0$ for some $i, j \in \mathbf{Z}$, then $F_iF_j = F_jF_i = F_{i+j}$. It follows that

$$(FF_iF)^n \subseteq FF_i^nF \qquad \text{for all } n \geq 1. \tag{2.1}$$

If F_i is nilpotent, then by (2.1) each element of F_i is strongly nilpotent and $J \subseteq N(F)$.

Suppose that $v \in F$ is nilpotent. Since F is $\mathbf{Z}$-graded we conclude that $v \in J$; then $N(F) \subseteq J$ and $N(F)$ contains precisely all nilpotent elements of F.

The ideal $N(F)$ is homogeneous with respect to the grading of F, so the factor ring $F/N(F)$ is also $\mathbf{Z}$-graded. Suppose that $N(F)$ is not completely prime. Then there exist nonzero homogeneous elements $u \in F_i + N(F)$ and $v \in F_j + N(F)$ of $F/N(F)$ such that $uv = 0$. Suppose that i and j have the same sign, say, $i, j \in \mathbf{N}$. Since u and v are not nilpotent, $\lambda u^j = v^i \neq 0$ for some $\lambda \in K$, but $v^{2i} = \lambda u^j v^i = 0$, a contradiction. So, the signs of i and j are different, say, $i > 0$ and $j < 0$; hence $R^{(+)} \neq 0$, $R^{(-)} \neq 0$, both R^+ and R^- are domains. It follows from $(vu)^2 = vuvu = 0$, that $uv = 0$ if and only if $vu = 0$. Let $x \in F_s + N(F)$ and $y \in F_{-t} + N(F)$ be nonzero elements of $F/N(F)$ for some $s, t > 0$. Then $x^i = \lambda u^s$ and $y^k = \mu v^t$, $k = -j$, for some nonzero scalars λ and μ; and $x^i y^k = y^k x^i = 0$. In fact, suppose the contrary, i.e., $xy = \nu yx \neq 0$ for some $\nu \neq 0 \in K$. Then for $m = \max\{i, k\} : 0 \neq (xy)^m = \gamma x^m y^m = 0$, a contradiction.

Under assumptions of the proposition it is clear that $N(F)$ is not completely prime. ∎

Proposition 2.1 shows that the prime radical $N(F)$ of F is homogeneous, the factor ring $F/N(F)$ is of the same type as F. The prime radical $N(F)$ acts trivially on every simple F-module M, i.e., $N(F)M = 0$, so

$$\hat{F} = (F/N(F))\hat{}$$

and until the end of this section (without loss of generality) we *suppose* that

$$N(F) = 0,$$

i.e., F is the *semiprime ring* and we have five types of the ring F:

- (0) $F = K$;
- $(+)$ $F = F^{+}$, $F \neq K$;
- $(-)$ $F = F^{-}$, $F \neq K$;
- $(-, +)$ $F \neq F^{+}$, F^{-}, and F is a domain;
- $(-, 0, +)$ $F \neq F^{+}$, F^{-}, and F is not a domain.

In the last case by Proposition 2.1 either $F^{(+)}$ or $F^{(-)}$ acts trivially on a simple F-module M; hence, $\hat{F} = \hat{F}^{-} \cup \hat{F}^{+}$ where $F^{\pm} \simeq F/F^{(\mp)}$. The cases $(+)$ and $(-)$ are symmetric in an obvious sense and the case (0) is trivial. So, *until the end of this section we assume that F belongs either to* $(+)$ *or to* $(-, +)$. In particular, F is domain.

The set $H(F)$ of all $i \in \mathbf{Z}$ such that $F_i \neq 0$ is called the (*graded*) *support* of F. The ring F is a domain and $F \neq K$, so $H(F)$ is a nonzero submonoid of $\mathbf{Z}$. Denote by $G(F) = n\mathbf{Z}$, $n > 0$, the subgroup of $\mathbf{Z}$ generated by $H(F)$ where $n = \mathrm{gr.ind}(F)$ is the *graded index* of F. Note that n may not belong to $H(F)$. Using the group automorphism $\mathrm{ref}\colon \mathbf{Z} \to \mathbf{Z}$, $i \to -i$ (the reflection) we attach to F the $\mathbf{Z}$-graded ring F^{ref} setting $(F^{\mathrm{ref}})_i = F_{-i}$ for all $i \in \mathbf{Z}$. Then $(F^{\mathrm{ref}})^{\pm} = F^{\mp}$.

LEMMA 2.2. *Let F be of type $(-, +)$. Then $H(F) = G(F)$ and $F = K\langle t, t^{-1}\rangle = K[t, t^{-1}; \sigma]$, the skew Laurent polynomial ring for some* (*any*) *$t \neq 0 \in F_n$, $n = \mathrm{gr.ind}(F)$ and the automorphism σ of K is defined by the rule $t\lambda = \sigma(\lambda)t$, $\lambda \in K$.*

Proof. Let m resp. n' be the maximal negative resp. minimal positive integer from $H(F)$. They exist since $F \neq F^{\pm}$. By the choice of m and n' we have $m + n' = 0$. The ring F is a domain, thus $H(F) = G(F) = n\mathbf{Z}$ and each nonzero element $t \in F^{n}$ is a unit. Using the grading of F we see that F is generated by K, t, and t^{-1} and is the skew polynomial ring as described above. ∎

Let F *be of type* $(+)$. Choose $0 < i$, $j \in H(F)$ such that $n := \mathrm{gr.ind}(F) = j - i$. All elements

$$i^2 + (k + li)n = lni + (i - k)i + kj, \qquad k = 0, \ldots, i - 1, l \geq 0,$$

belong to $H(F)$; thus the following definition is correct. The *starting point* $\mathrm{sp}(F)$ of F is the minimal nonzero $m \in H(F)$ such that $in \in H(F)$ for all $i \geq mn^{-1}$.

The localization $S^{-1}F$ of F at the multiplicatively closed set $S = \{\bigcup F_i\} \setminus \{0\}$ is the skew Laurent polynomial ring

$$S^{-1}F = E := K[t, t^{-1}; \sigma], \tag{2.2}$$

where $t = x^{-1}y$ for some nonzero $x \in F_i$ and $y \in F_j$ with $j - i = n$ and the automorphism σ of K is defined as follows, $t\lambda = \sigma(\lambda)t$, $\lambda \in K$. This ring is called *associated* with F. The ring of skew Laurent polynomials is Euclidean (the left and right division algorithms with remainder hold) with respect to the "length" function l given by $l(\alpha t^m + \cdots + \beta t^n) = n - m$, where α, $\beta \in K$ are nonzero and $m < \cdots < n$.

A simple F-module M is either S-torsion ($S^{-1}M = 0$) or S-torsionfree ($S^{-1}M \neq 0$):

$$\hat{F} = \hat{F}(S - \text{torsion}) \cup \hat{F}(S - \text{torsionfree}).$$

For i set $F_{<i} := \bigoplus_{j<i} F_j$.

LEMMA 2.3. *Let F be of type $(+)$. Then*

1. $\hat{F}$*(S − torsion)* $= \{K = F/F^{(+)}\}$;
2. *the map* $\hat{F}$*(S − torsionfree)* $\to \hat{E}$, $[M] \to [S^{-1}M]$, *is the* 1−1 *correspondence with inverse* $[N = E/Ef] \to [\mathrm{Soc}_F(N) = F/F \cap Ef = F_{<(m+dn)}/F_{<(m+dn)} \cap \sum_{0 \le i < mn^{-1}} F_{in}f]$, *where* $n = \mathrm{gr.ind}(F)$, $m = \mathrm{sp}(R)$, *d is the degree of the irreducible polynomial* $f = \alpha_0 + \cdots + \alpha_d t^d \in \mathrm{Irr}E$.

Proof. Part 1 is clear.

2. The map $[M] \to [S^{-1}M]$ is monic. It is easy to see that the module $M = F/F \cap Ef$ equals the factorspace above and is finite dimensional as the left K-module. Thus the socle $\mathrm{Soc}_F N$ is nonzero; hence, the map $[M] \to [S^{-1}M]$ is epic.

It remains to show that $\mathrm{Soc}_F N = M$ (where $M = F/F \cap Ef$). Suppose the contrary. Then $\mathrm{Soc}_F N = J/I$ for some left ideal J of F which strongly contains $I = F \cap Ef$ and $J \neq F$. Since $S^{-1}J = E$ (otherwise, $J = F \cap Ef$) and J/I is the simple F-module, we conclude that $J = Fu + I$ for some nonzero homogeneous $u \in F_i$. Then

$$J/I = Fu/Fu \cap Ef \simeq F/F \cap Efu^{-1} \simeq \omega(F)/\omega(F \cap Ef) \simeq^{\omega^{-1}} M,$$

the twisted F-module by the ring automorphism ω of F which is the restriction of the inner automorphism of E: $e \to ueu^{-1}$. Hence, M is simple, a contradiction. ∎

Let us introduce some definitions which will be used in later sections. Let F be of type other than $(-, +)$, i.e., F is not a skew Laurent polynomial ring. If $F^+ \neq K$ resp. $F^- \neq K$, then denote by E^+ resp. E^- the skew Laurent polynomial ring associated with F^+ resp. F^- and defined by (2.2). The simple F-module $K = F/(F^{(+)} + F^{(-)})$ is called *point* (if it exists); any other simple F-module is called *skew*.

Summing up we get a classification of the simple F-modules (up to irreducible elements of E).

THEOREM 2.4. *Let F be such as at the beginning of this section and $\overline{F} = F/N(F)$. Then according with the five possibilities above the set $\hat{F}$ of isoclasses of simple F-modules can be described as follows*:

1. *if $\overline{F}$ is of type* (0), *then* $\hat{F} = \{K\}$;
2. *if $\overline{F}$ is of type* $(+)$ (*resp*. $(-)$), *then* $\hat{F} = \{K\} \cup \hat{E}^+$ (*resp*. $\hat{F} = \{K\} \cup \hat{E}^-$);
3. *if $\overline{F}$ is of type* $(-, +)$, *then* $\hat{F} = \hat{E}$;
4. *if $\overline{F}$ is of type* $(-, 0, +)$, *then* $\hat{F} = \{K\} \cup \hat{E}^+ \cup \hat{E}^-$;

where E and $E^{\pm}$ are skew Laurent polynomial rings as above.

3. THE SIMPLE WEIGHT R-MODULES (D IS A COMMUTATIVE RING)

Throughout this section D is a *commutative* ring and a $\mathbf{Z}$-graded ring R is as before with an arbitrary cocycle c. In this section we classify the simple weight R-modules.

The cyclic group G, generated by σ, acts in an obvious way on the set $\mathrm{Specm}(D)$ of maximal ideals of D. An orbit $\mathcal{O}$ is *cyclic* of length n (respectively, *linear*) if it contains a finite (respectively, infinite) number $n = |\mathcal{O}|$ of elements. The set of all cyclic (linear) orbits is denoted by Cyc (Lin).

An R-module V is a *weight* module if ${}_DV$ is semisimple, i.e.,

$$V = \oplus_{\mathbf{p} \in \mathrm{Specm}(D)} V_{\mathbf{p}},$$

where $V_{\mathbf{p}} = \{v \in V : \mathbf{p}v = 0\} = \{$the sum of simple D-submodules which are isomorphic to ${}_D(D/\mathbf{p})\}$, the *component* of V of *weight* $\mathbf{p}$. The *support* $\mathrm{Supp}(V)$ of the weight module V is the set of maximal ideals $\mathbf{p}$ such that $V_{\mathbf{p}} \neq 0$. If $0 \to V_1 \to V \to V_2 \to 0$ is an exact sequence of weight modules, then $\mathrm{Supp}(V) = \mathrm{Supp}(V_1) \cup \mathrm{Supp}(V_2)$.

Since

$$v_i V_{\mathbf{p}} \subseteq V_{\sigma^i(\mathbf{p})} \qquad \text{for all } i \in \mathbf{Z},$$

each weight R-module V is decomposed into the direct sum of R-submodules

$$V = \oplus \{ V_{\mathcal{O}} \mid \mathcal{O} \text{ is an orbit} \},$$

where $V_{\mathcal{O}} = \oplus\{V_{\mathbf{p}} \mid \mathbf{p} \in \mathcal{O}\}$. Hence, the support of a simple weight module belongs exactly to one orbit. For a maximal ideal $\mathbf{p}$ of D the module

$$R(\mathbf{p}) := R/R\mathbf{p} \simeq R \otimes_D D/\mathbf{p} = \bigoplus_{i \in \mathbf{Z}} v_i \otimes D/\mathbf{p}$$

is a weight module $({}_D(v_i \otimes D/\mathbf{p}) \simeq D/\sigma^i(\mathbf{p}))$ with support $\mathcal{O}(\mathbf{p})$. If V is a simple weight R-module, then there is an R-module epimorphism $R(\mathbf{p}) \to V$ for any $\mathbf{p} \in \mathrm{Supp}(V)$. Denote by $\hat{R}(\text{weight})$ the set of isoclasses of simple weight R-modules. Then

$$\hat{R}(\text{weight}) = \hat{R}(\text{weight, linear}) \cup \hat{R}(\text{weight, cyclic}), \tag{3.1}$$

where $\hat{R}$(weight, linear resp. cyclic) is the set of isoclasses of simple weight R-modules with support from a linear resp. cyclic orbit.

$\hat{R}$(weight, linear). Let an orbit $\mathcal{O} = \mathcal{O}(\mathbf{p})$ be *linear*. Then each nonzero component $R(\mathbf{p})_{\mathbf{q}}$ of the R-module $R(\mathbf{p})$ is the simple D-module $D/\mathbf{q}$. Thus there exists a *largest submodule* $N(\mathbf{p})$ of $R(\mathbf{p})$ which is not equal to $R(\mathbf{p})$; it is the sum of all submodules which intersect $R(\mathbf{p})_{\mathbf{p}} = D/\mathbf{p}$ trivially. Denote the associated simple R-module by

$$L(\mathbf{p}) = R(\mathbf{p})/N(\mathbf{p}).$$

We conclude that

1. each simple weight R-module M with support from $\mathcal{O}$ is isomorphic to $L(\mathbf{p})$ for some (each) $\mathbf{p} \in \mathrm{Supp}\, M$;

2. R-modules $L(\mathbf{p})$ and $L(\mathbf{q})$ are isomorphic if and only if $\mathrm{Supp}\, L(\mathbf{p}) \cap \mathrm{Supp}\, L(\mathbf{q}) \neq \emptyset$ (then, of course, $\mathrm{Supp}\, L(\mathbf{p}) = \mathrm{Supp}\, L(\mathbf{q})$).

In other words, a simple weight R-module with support from a linear orbit is uniquely determined by its support. Thus, on the set $\mathrm{Specm.lin}(D)$ of maximal ideals which belong to linear orbits is defined the equivalence relation $\sim$: $\mathbf{p} \sim \mathbf{q}$ if and only if $\mathbf{p}, \mathbf{q} \in \mathrm{Supp}\, V$ for some $[V] \in \hat{R}$(weight, linear).

LEMMA 3.1. *Let an orbit $\mathcal{O}$ be linear. Then $\mathbf{p} \sim \mathbf{q} = \sigma^i(\mathbf{p})$, $i \in \mathbf{Z}$, if and only if $c(-i, i) \notin \mathbf{p}$.*

Proof. $\mathbf{p} \sim \mathbf{q}$ if and only if $L = L(\mathbf{p}) \simeq L(\mathbf{q})$, so $L_{\mathbf{p}} = v_{-i}L_{\mathbf{q}}$ and $L_{\mathbf{q}} = v_i L_{\mathbf{p}}$; hence,

$$L_{\mathbf{p}} = v_{-i}v_i L_{\mathbf{p}} = c(-i, i) L_{\mathbf{p}} \qquad \text{and} \qquad L_{\mathbf{q}} = v_i v_{-i} L_{\mathbf{q}} = c(i, -i) L_{\mathbf{q}}.$$

It is equivalent to $c(-i, i) \notin \mathbf{p}$ and $c(i, -i) \notin \mathbf{q}$. The last condition follows from the first since $c(i, -i) = \sigma^i(c(-i, i))$ and $\mathbf{q} = \sigma^i(\mathbf{p})$.

Conversely, suppose that $L_{\mathbf{q}} = 0$. Then $0 \neq c(-i, i)L_{\mathbf{p}} = v_{-i}v_i L_{\mathbf{p}} \subseteq v_{-i}L_{\mathbf{q}} = 0$, a contradiction. ∎

To sum up we prove the following theorem.

THEOREM 3.2. *The map*

$$\mathrm{Specm.lin}(D)/\sim \;\to \hat{R}(\mathit{weight}, \mathit{linear}), \qquad \Gamma \to [L(\Gamma)],$$

is bijective with inverse $[L] \to \mathrm{Supp}\, L$, *where*

$$L(\Gamma) := L(\mathbf{p}) = R/\big(R\mathbf{p} + \Sigma\{R_i \mid \sigma^i(\mathbf{p}) \notin \Gamma\}\big) \tag{3.2}$$

and $\mathbf{p} \in \Gamma$ (*any fixed*).

$\hat{R}$(weight, cyclic). For a natural number n, set

$$R_{[n]} = \bigoplus_{i \in \mathbf{Z}} R_{in},$$

the Veronese subring of R. The ring R considered as a left (or right) $R_{[n]}$-module is ($\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$)-graded,

$$R = \bigoplus_{\mathbf{i} \in \mathbf{Z}_n} R_{\mathbf{i}}, \qquad \text{where } R_{\mathbf{i}} = \bigoplus_{j \in \mathbf{Z}} R_{i+nj},\ \mathbf{i} = i + n\mathbf{Z}.$$

It means that $R_{[n]}R_{\mathbf{i}} \subseteq R_{\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{Z}_n$. Moreover, it is a $\mathbf{Z}_n$-graded ring (i.e., $R_{\mathbf{i}}R_{\mathbf{j}} \subseteq R_{\mathbf{i}+\mathbf{j}}$ for all $\mathbf{i}, \mathbf{j} \in \mathbf{Z}_n$).

Let $\mathcal{O}$ be a *cyclic* orbit which contains $n = |\mathcal{O}|$ elements. For a maximal ideal $\mathbf{p} \in \mathcal{O}$ ($\sigma^n(\mathbf{p}) = \mathbf{p}$), denote by

$$R_{[n], \mathbf{p}} := R_{[n]}/(\mathbf{p})$$

the factor ring of $R_{[n]}$ at the ideal $(\mathbf{p}) = \bigoplus_{i \in \mathbf{Z}}(\mathbf{p}v_{in} = v_{in}\mathbf{p})$ of $R_{[n]}$ generated by $\mathbf{p}$. This ring is of the type considered in the previous section and we keep the notation introduced there. Put

$$F_{[n], \mathbf{p}} := R_{[n], \mathbf{p}}/N\big(R_{[n], \mathbf{p}}\big),$$

where $N(R_{[n], \mathbf{p}})$ is the prime radical of $R_{[n], \mathbf{p}}$. If $F_{[n], \mathbf{p}}$ is of type $(+)$, $(-, 0, +)$ resp. $(-)$, $(-, 0, +)$ resp. $(-, +)$, denote by $E^+_{[n], \mathbf{p}}$ resp. $E^-_{[n], \mathbf{p}}$ resp. $E_{[n], \mathbf{p}}$ the skew Laurent polynomial rings associated with $F^+_{[n], \mathbf{p}}$ resp. $F^-_{[n], \mathbf{p}}$ resp. $F_{[n], \mathbf{p}}$ (as defined in the section above).

Let M be a simple R-module with support from $\mathcal{O}$. It follows from the weight decomposition

$$M = \bigoplus_{\mathbf{p} \in \mathcal{O}} M_{\mathbf{p}}$$

that $M_{\mathbf{p}}$ is the simple $F_{[n],\mathbf{p}}$-module (or $R_{[n],\mathbf{p}}$-module) for each $\mathbf{p} \in \mathrm{Supp}(M)$ and

$$R_{\mathbf{i}} M_{\mathbf{p}} \subseteq M_{\sigma^{i}(\mathbf{p})} \qquad \text{for all } \mathbf{i} \in \mathbf{Z}_n. \tag{3.3}$$

For a simple $F_{[n],\mathbf{p}}$-module V set

$$R(V) := R \otimes_{R_{[n]}} V.$$

This definition makes sense in view of the following natural ring epimorphisms: $R_{[n]} \to R_{[n],\mathbf{p}} \to F_{[n],\mathbf{p}}$. The R-module $R(V) = \bigoplus_{\mathbf{q} \in \mathcal{O}} R(V)_{\mathbf{q}}$ is a weight module with support from the orbit $\mathcal{O}$ and the $\mathbf{p}$th weight component $R(V)_p = V$ is the simple $F_{[n],\mathbf{p}}$-module. So there exists the *largest proper submodule* $N(V)$ of $R(V)$; $N(V)$ is the sum of all submodules of $R(V)$ which intersect trivially V. Denote by

$$L(V) := R(V)/N(V)$$

the associated simple R-module. We conclude that:

1. each simple weight R-module M with support from $\mathcal{O}$ is isomorphic to $L(V)$ for some simple $F_{[n],\mathbf{p}}$-module V $(= M_{\mathbf{p}})$, $\mathbf{p} \in \mathrm{Supp}\, M$;

2. two such modules are isomorphic, $L(V) \simeq L(U)$, if and only if there exists $\mathbf{p} \in \mathrm{Supp}(L(V)) \cap \mathrm{Supp}(L(U))$ such that $F_{[n],\mathbf{p}}$-modules $L(V)_{\mathbf{p}} \simeq L(U)_{\mathbf{p}}$ are isomorphic (then, of course, $\mathrm{Supp}\, L(\mathbf{p}) = \mathrm{Supp}\, L(\mathbf{q})$).

Let V be as above. Set

$$\begin{aligned} H(V) &= \{i \in n\mathbf{Z} : v_i V \neq 0\} \\ &= \{i \in n\mathbf{Z} : V \to V, v \to v_i v, \text{ defines a bijection}\}. \end{aligned}$$

By Theorem 2.4 $H(V)$ has one of the following types:

- (± 0) $H(V) = 0$, $[(0), (+), (-), (-, 0, +)]$;
- $(+, +)$ $H(V) = H(F^{+}_{[n],\mathbf{p}})$, $[(+), (-, 0, +)]$;
- $(-, -)$ $H(V) = H(F^{-}_{[n],\mathbf{p}})$, $[(-), (-, 0, +)]$;
- $(-, +)$ $H(V) = H(F_{[n],\mathbf{p}})$, $[(-, +)]$;

where in the last case $F_{[n],\mathbf{p}}$ is a skew Laurent polynomial ring and in brackets we point out the possible types for the ring $F_{[n],\mathbf{p}}$. Note that

$H(V) = 0$ if and only if V is the point module, i.e.,

$$V = F_{[n],\mathbf{p}} / \left(F_{[n],\mathbf{p}}^{(+)} + F_{[n],\mathbf{p}}^{(-)} \right) \simeq D/\mathbf{p}.$$

Set $TS := \{(\pm 0), (+,+), (-,-), (-,+)\}$, the type set. We say that V is of type $\mathbf{t} = tp(V)$, if $H(V)$ is.

Let $L(V)$ be as above. If $\mathbf{q} = \sigma^i(\mathbf{p}) \in \operatorname{Supp}(L(V))$, $0 \le i < n$, then

$$L(V)_{\mathbf{q}} = R_{\mathbf{i}} V = \sum_{j \in \mathbf{Z}} v_{i+nj} V;$$

hence, there exists $s \in i + n\mathbf{Z}$ such that $0 \neq v_s V \subseteq L(V)_{\mathbf{q}}$. Since $L(V)_{\mathbf{q}}$ is a simple $R_{[n],\mathbf{q}}$-module, $R_{-\mathbf{i}} = \sum_{j \in \mathbf{Z}} v_{-i+nj} D$ is a right $R_{[n]}$-module, and $R_{-\mathbf{i}} L(V)_{\mathbf{q}} = V$ we conclude that there exists $t \in -i + n\mathbf{Z}$ such that

$$0 \neq v_t v_s V = c(t,s) v_{t+s} V. \tag{3.4}$$

The action of $v_t v_s \in R_{[n]}$ on V is nonzero, thus it is bijective and the maps

$$V \to L(V)_{\mathbf{q}}, \qquad v \to v_s v, \qquad L(V)_{\mathbf{q}} \to V, \qquad u \to v_t u,$$

are bijective too. As a consequence, the following left vector space dimensions coincide:

$$\dim_{D/\mathbf{p}} V = \dim_{D/\mathbf{q}} L(V)_{\mathbf{q}}, \qquad \text{for all } \mathbf{q} \in \operatorname{Supp} L(V).$$

Moreover,

$$s + t + H(V) \subseteq H\left(L(V)_{\mathbf{q}}\right) \qquad \text{and} \qquad s + t + H\left(L(V)_{\mathbf{q}}\right) \subseteq H(V)$$
$$\text{for all } \mathbf{q} \in \operatorname{Supp} L(V). \tag{3.5}$$

It follows from (3.5) that all weight components $L(V)_{\mathbf{q}}$, $\mathbf{q} \in \operatorname{Supp} L(V)$, of the simple R-module $L(V)$ have the same type of $H(L(V)_{\mathbf{q}})$ which is denoted by $\operatorname{tp} L(V)$ and called the *type* of $L(V)$. We have the partition

$$\hat{R}(\text{weight}, \text{cyclic}) = \hat{R}(\pm 0) \cup \hat{R}(+,+) \cup \hat{R}(-,+) \cup \hat{R}(-,-), \tag{3.6}$$

where $\hat{R}(\pm 0)$ denote the set of isoclasses of simple R-modules of type (± 0), etc.

LEMMA 3.3. *Let $L(V)$, $\mathcal{O}$, n, and $\mathbf{p}$ be as above. Then $\mathbf{q} = \sigma^i(\mathbf{p}) \in$ $\operatorname{Supp} L(V)$ $(0 < i < n)$, if and only if the following holds*:

(sk) *there exists $s \in i + n\mathbf{Z}$ and $t \in -i + n\mathbf{Z}$ such that $s + t \in H(V)$ and $c(t,s) \notin \mathbf{p}$.*

Moreover, if V is point (sk) *is equivalent to*

(pnt) *there exists $s \in i + n\mathbf{Z}$ such that $c(-s,s) \notin \mathbf{p}$.*

Proof. ($\Rightarrow$) It follows from (3.4) and (3.5).

($\Leftarrow$) Suppose that $L(V)_{\mathbf{q}} = 0$ but such s and t exist. Then, on the one hand, $c(t,s)v_{t+s}V \neq 0$; on the other hand, $c(t,s)v_{t+s}V = v_t v_s V = 0$, since $v_s V \subseteq L(V)_{\mathbf{q}} = 0$, a contradiction.

If V is a point module, then $H(V) = 0$; hence, $s = -t$ and the result follows. ∎

Remark. If the type $\mathbf{t} = \text{tp}(V)$ of the module V is fixed, then (sk) *does not depend* on the module V (but does *only* on $\mathbf{t}$).

For a type $\mathbf{t} \in TS$, let Specm.$\mathbf{t}(D)$ be the set of maximal ideals $\mathbf{p}$ of D for which there exists a simple $F_{[n],\mathbf{p}}$-module V of type $\mathbf{t}$ (see the remark above); for instance, Specm.$(+,+)(D)$ consists of such $\mathbf{p}$ that the type of the ring $F_{[n],\mathbf{p}}$, $n = |O(\mathbf{p})|$ is either $(+)$ or $(-,0,+)$, etc.

On the set Specm.$\mathbf{t}(D)$ define the *equivalence relation* $\sim_{\mathbf{t}}$: $\mathbf{p} \sim_{\mathbf{t}} \mathbf{q}$ if and only if (sk) holds if and only if there exists a simple weight module of type $\mathbf{t}$ such that both ideals $\mathbf{p}$ and $\mathbf{q}$ belong to its support.

For each equivalence class Γ of Specm.$\mathbf{t}(D)/\sim_{\mathbf{t}}$ fix a representative $\mathbf{p} \in \Gamma$ and denote by $I(\Gamma, \mathbf{p})$ the set of $s \in \mathbf{Z}/n\mathbf{Z}$ such that (sk) holds (for some representative s' of the class s and some $t = t(s')$) and the map $I(\Gamma, \mathbf{p}) \to \Gamma$, $\mathbf{s} \to \sigma^s(\mathbf{p})$ is 1–1 correspondence.

$\hat{R}(\pm 0)$. In the case of $\mathbf{t} = (\pm 0)$ (i.e., $\sim_{(\pm 0)}$), for a given equivalence class the above simple weight module of type (± 0) is *unique*, since unique is the point $R_{[n],\mathbf{p}}$-module for each $\mathbf{p} \in$ Specm.$(\pm 0)(D)$. In other words, a simple R-module of type (± 0) is uniquely determined by its support, i.e., the map

$$\hat{R}(\pm 0) \to \text{Specm.}(\pm 0)(D)/\sim_{(\pm 0)}, \qquad [M] \to \text{Supp}\, M, \tag{3.7}$$

is bijective with inverse $\Gamma \to [L(D/\mathbf{p})]$, $\mathbf{p} \in \Gamma$, where

$$L(D/\mathbf{p}) = R(D/\mathbf{p})/N(D/\mathbf{p}) = \sum_{s \in I(\Gamma, \mathbf{p})} v_s \otimes_{R_{[n]}} D/\mathbf{p}. \tag{3.8}$$

$\hat{R}(\mathbf{t})$, $\mathbf{t} = (+,+)$, $(-,-)$, $(-,+)$. In the remaining cases, i.e., $\mathbf{t} = (+,+)$, $(-,-)$ or $(-,+)$, the above-mentioned simple weight module is not unique (as we shall see later), but according with the definition of $\sim_{\mathbf{t}}$ supports of two modules of this kind either coincide or their intersection is the empty set; hence, we have the following partition

$$\hat{R}(\mathbf{t}) = \sqcup \left\{ \hat{R}(\mathbf{t}, \Gamma) \mid \Gamma \in \text{Specm.}\mathbf{t}(D)/\sim_{\mathbf{t}} \right\}, \tag{3.9}$$

where $\hat{R}(\mathbf{t}, \Gamma)$ consists of the isoclasses of simple weight R-modules with support Γ.

Let $[L(V)] \in \hat{R}(\mathbf{t}, \Gamma)$, where V is the simple $F_{[n],\mathbf{p}}$-module of type $\mathbf{t}$ for some $\mathbf{p} \in \Gamma$. Denote by $E_{[n],\mathbf{p},\mathbf{t}}$ the skew Laurent polynomial ring $E^{+}_{[n],\mathbf{p}}$

resp. $E^{-}_{[n],\mathbf{p}}$ resp. $E_{[n],\mathbf{p}}$ for $\mathbf{t} = (+,+)$ resp. $(-,-)$ resp. $(-,+)$. Then

$$E_{[n],\mathbf{p},\mathbf{t}} \simeq D/\mathbf{p}[x, x^{-1}; \sigma^m],$$

where $m = m(\mathbf{p}, \mathbf{t})$ is the graded index of $F^{+}_{[n],\mathbf{p}}$ resp. $F^{-}_{[n],\mathbf{p}}$ resp. $F_{[n],\mathbf{p}}$. It follows from (3.5), that

$$m(\mathbf{p}, \mathbf{t}) = m(\mathbf{q}, \mathbf{t}), \qquad \text{if } \mathbf{p} \sim_{\mathbf{t}} \mathbf{q}; \tag{3.10}$$

denote by $m(\mathbf{t}, \Gamma) = m(\mathbf{p}, \mathbf{t})$ their common value.

If $\mathbf{p} \sim_{\mathbf{t}} \mathbf{q} = \sigma^i(\mathbf{p})$, $i \in \mathbf{Z}$, then the ring automorphism σ^i: $D/\mathbf{p} \to D/\mathbf{q}$, $u \to \sigma^i(u)$ is extended to the automorphism

$$\sigma^i : E_{[n],\mathbf{p},\mathbf{t}} \to E_{[n],\mathbf{q},\mathbf{t}}, \qquad x \to x. \tag{3.11}$$

Denote by $E(\mathbf{t}, \Gamma)$ the isoclass of the rings $E_{[n],\mathbf{p},\mathbf{t}}$, $\mathbf{p} \in \Gamma$.

The above module V is an $E_{[n],\mathbf{p},\mathbf{t}}$-module or $E(\mathbf{t}, \Gamma)$-module, since V is not of type (± 0). By Theorem 2.4 there is the bijective correspondence

$$\hat{R}(\mathbf{t}, \Gamma) \leftrightarrow \hat{E}_{[n],\mathbf{p},\mathbf{t}} \leftrightarrow \hat{E}(\mathbf{t}, \Gamma), \qquad [L(V)] \leftrightarrow [V], \tag{3.12}$$

where

$$L(V) = R(V)/N(V) = \sum_{s \in I(\Gamma, \mathbf{p})} v_s \otimes_{R_{[n]}} V. \tag{3.13}$$

THEOREM 3.4. *Keep the notations above. The map*

$$\left\{ \sqcup_{\mathbf{t} \neq (\pm 0) \in TS} \sqcup_{\Gamma \in \operatorname{Specm}.\mathbf{t}(D)/\sim_{\mathbf{t}}} \hat{E}(\mathbf{t}, \Gamma) \right\}$$
$$\sqcup \operatorname{Specm}.(\pm 0)(D)/\sim_{(\pm 0)} \to \hat{R}(\text{weight}, \text{cyclic}),$$

which is defined by (3.7), (3.12), *is bijective*; *the representatives of isoclasses of* $\hat{R}$(*weight*, *cyclic*) *are defined by* (3.8), (3.13).

***R* is *N*-Graded**. Let $R = \bigoplus_{i \geq 0} R_i$ be an $\mathbf{N}$-graded ring with $R_0 = D$ (not necessarily commutative), every $R_i = Dv_i$, $v_i \in R_i$, is a left free D-module of rank 1, $v_0 = 1$, there exists a ring automorphism σ of D such that

$$\alpha v_i \beta v_j = \alpha \sigma^i(\beta) c(i, j) v_{i+j}$$

for all α, $\beta \in D$ and $i, j \in \mathbf{N}$ and some map $c : \mathbf{N} \times \mathbf{N} \to Z(D)$, the centre of D.

Let D be *commutative*. The $\mathbf{N}$-graded ring R can be considered as $\mathbf{Z}$-graded if we put $R_i = 0$ for all negative i. This allows us to preserve all definitions, notations, and many results above up to substituting **N** for **Z**. We leave details for the reader. Now we are ready to describe the simple weight R-modules. In accordance with (3.1) we classify first the linear ones.

$\hat{R}$ (weight, linear), *R is* **N**-*graded*.

THEOREM 3.5. *Let R be* **N**-*graded. The map*

$$\mathrm{Specm.lin}(D)/\sim \;\to \hat{R}(\textit{weight}, \textit{linear}), \qquad \mathbf{p} \to \left[D/\mathbf{p} = R/(\mathbf{p} + R^{(+)})\right], \tag{3.14}$$

is bijective.

Proof. Let $[M] \in \hat{R}$(weight, linear). Then M is a factor module of $R(\mathbf{p}) := R/R\mathbf{p}$ for some (any) $\mathbf{p} \in \mathrm{Supp}\, M$. Since $\mathbf{p} + R^{(+)}$ is the largest proper submodule of $R(\mathbf{p})$, $M \simeq R/(\mathbf{p} + R^{(+)}) \simeq D/\mathbf{p}$. The rest is clear. ∎

$\hat{R}$(weight, cycle), *R is* **N**-*graded*. Let O be a *cyclic* orbit which contains $n = |O|$ elements and $\mathbf{p} \in O$. Let V be a simple $F_{[n], \mathbf{p}}$-module. Since R is **N**-graded, there are only two possibilities for the type of $F_{[n], \mathbf{p}}$ (namely, (0) and (+)) and $H(V)$:

- (± 0) $H(V) = 0$, $[(0), (+)]$;
- $(+, +)$ $H(V) = H(F^{+}_{[n], \mathbf{p}})$, $[(+)]$.

Then (3.6) is rewritten as

$$\hat{R}(\mathrm{weight}, \mathrm{cyclic}) = \hat{R}(\pm 0) \cup \hat{R}(+, +). \tag{3.6$'$}$$

LEMMA 3.6. *Let R be* **N**-*graded and let* $L(V)$, O, n, *and* $\mathbf{p}$ *be as above. Then* $\mathbf{q} = \sigma^i(\mathbf{p}) \in \mathrm{Supp}\, L(V)$, $(0 < i < n)$, *if and only if the following holds*:

(sk-N) *there exists* $s \in i + n\mathbf{N}$ *and* $t \in n - i + n\mathbf{N}$ *such that* $s + t \in H(V)$ *and* $c(t, s) \notin \mathbf{p}$. *Moreover, if V is point, then* (sk-N) *is never satisfied.*

Remark. If V is of type $(+, +)$, then (sk-N) *does not depend* on the module V.

Let $\sim_{\mathbf{t}}$ and $I(\Gamma, \mathbf{p})$ be the **N**-graded analogs of the same notions above (for (sk) is to be substituted for (sk-N)).

$\hat{R}(\pm 0)$, *R is* **N**-*graded*. It is clear that Specm.$(\pm 0)(D)$ equals the set Specm.cyc(D) of cyclic maximal ideals **p**. It follows from Lemma 3.6 that the equivalence relation $\sim_{(\pm)}$ is $=$, i.e., each cyclic maximal ideal is an equivalence class and vice versa. So, the map

$$\mathrm{Specm.cyc}(D) \to \hat{R}(\pm 0), \qquad \mathbf{p} \to \left[D/\mathbf{p} = R/(\mathbf{p} + R^{(+)})\right], \tag{3.15}$$

is bijective.

$\hat{R}(+,+)$, *R is* **N**-*graded*. Statements (3.9)–(3.13) are true for the **N**-graded ring R. Thus

$$\hat{R}(+,+) = \sqcup\left\{\hat{R}((+,+),\Gamma) \mid \Gamma \in \text{Specm.}(+,+)(D)/\sim_{(+,+)}\right\}, \tag{3.9′}$$

where $\hat{R}((+,+),\Gamma)$ consists of the isoclasses of simple weight R-modules with support Γ.

Let $[L(V)] \in \hat{R}((+,+),\Gamma)$, where V is the simple $F_{[n],\mathbf{p}}$-module of type $(+,+)$ for some $\mathbf{p} \in \Gamma$. Denote by $E_{[n],\mathbf{p},(+,+)}$ the skew Laurent polynomial ring $E^{+}_{[n],\mathbf{p}}$ and by $E((+,+),\Gamma)$ the isoclass of the ring $E_{[n],\mathbf{p},(+,+)}$, $\mathbf{p} \in \Gamma$. Then (3.12) and (3.13) are rewritten as

$$\hat{R}((+,+),\Gamma) \leftrightarrow \hat{E}_{[n],\mathbf{p},(+,+)} \leftrightarrow \hat{E}((+,+),\Gamma), \qquad [L(V)] \leftrightarrow [V], \tag{3.12′}$$

where

$$L(V) = R(V)/N(V) = \sum_{s \in I(\Gamma,\mathbf{p})} v_s \otimes_{R_{[n]}} V. \tag{3.13′}$$

THEOREM 3.7. *With notation as above. The map*

$$\text{Specm.cyc}(D) \sqcup \left(\sqcup_{\Gamma \in \text{Specm.}(+,+)(D)/\sim_{(+,+)}} \hat{E}((t,t),\Gamma)\right) \to \hat{R}(\text{weight}, \text{cyclic}),$$

which is defined by (3.15) *and* (3.12′), *is bijective*; *the representatives of isoclasses of* $\hat{R}$(*weight*, *cyclic*) *are defined by* (3.15), (3.13′).

4. THE SIMPLE WEIGHT MODULES OVER GENERALIZED WEYL ALGEBRAS

Let $R = A$ be a generalized Weyl algebra $A = D(\sigma, a)$ where D is a *commutative* ring. In this section, we apply the results of Section 3 in order to describe the simple A-modules.

An orbit $\mathcal{O}$ is called *degenerate* if it contains a *marked* ideal $\mathbf{p}$, i.e., one such that $a \in \mathbf{p}$. Denote by Cycd and Lind (Cycn and Linn) the set of all (non-) degenerate cyclic and linear orbits, respectively.

Every linear orbit $\mathcal{O}(\mathbf{p})$, $\mathbf{p} \in \mathcal{O}$, may be indentified by the map $\mathcal{O}(\mathbf{p}) \to \mathbf{Z}$, $\sigma^i(\mathbf{p}) \to i$, with the set of integers $\mathbf{Z}$. Therefore, definitions and notation used for $\mathbf{Z}$ (such as the order, the segment, the semiaxis, etc.) may be used for $\mathcal{O}(\mathbf{p})$. For example, $\sigma^i(\mathbf{p}) \le \sigma^j(\mathbf{p})$ iff $i \le j$; $(-\infty, \mathbf{p}] = \{\sigma^i(\mathbf{p}), i \le 0\}$. For a cyclic orbit $\mathcal{O}$ of length n we use only the definition of the segment

$(\mathbf{q}, \mathbf{p}] = \{\sigma^i(\mathbf{q}), 0 < i \leq j \leq n\}$ where j is the minimal positive integer such that $\sigma^j(\mathbf{q}) = \mathbf{p}$. Marked ideals $\mathbf{p}_1 < \cdots < \mathbf{p}_s$ (respectively, $\mathbf{p}_1$, $\mathbf{p}_2 = \sigma^{i(2)}(\mathbf{p}_1), \dots, \mathbf{p}_s = \sigma^{i(s)}(\mathbf{p}_1)$, $0 < i(2) < \cdots < i(s)) < n$ of a degenerate linear (respectively, cyclic) orbit $\mathcal{O}$ divide it into $s+1$ (respectively, s) parts,

$$\Gamma_1 = (-\infty, \mathbf{p}_1], \Gamma_2 = (\mathbf{p}_1, \mathbf{p}_2], \dots, \Gamma_{s+1} = (\mathbf{p}_s, \infty)$$

$$\left(\text{respectively, } \Gamma_1 = (\mathbf{p}_s, \mathbf{p}_1], \Gamma_2 = (\mathbf{p}_1, \mathbf{p}_2], \dots, \Gamma_s = (\mathbf{p}_{s-1}, \mathbf{p}_s]\right).$$

$\hat{A}$(weight, linear). Let $\sim$ be the equivalence relation defined in Section 3. Then $\mathbf{p} \sim \mathbf{q}$ if and only if $\mathbf{p}$ and $\mathbf{q}$ belong either to a nondegenerate orbit or to some Γ_i. By Theorem 3.2 we have.

COROLLARY 4.1. *The map*

$$\text{Specm.lin}(D)/\sim \,\to \hat{R}(\textit{weight}, \textit{linear}), \qquad \Gamma \to [L(\Gamma)],$$

is bijective with inverse $[L] \to \operatorname{Supp} L$, *where*

1. *if* $\Gamma \in$ Linn *is a nondegenerate orbit*, *then* $L(\Gamma) = A/A\mathbf{p}$, $\mathbf{p} \in \Gamma$;
2. *if* $\Gamma = (-\infty, \mathbf{p}]$, *then* $L(\Gamma) = A/A(\mathbf{p}, X)$;
3. *if* $\Gamma = (\sigma^{-n}(\mathbf{p}), \mathbf{p}]$, $n \in \mathbf{N}$, *then* $L(\Gamma) = A/A(\mathbf{p}, X, Y^n)$;
4. *if* $\Gamma = (\mathbf{p}, \infty)$, *then* $L(\Gamma) = A/A(\sigma(\mathbf{p}), Y)$.

$\hat{A}$(weight, cyclic). Let $\mathcal{O}$ be a cyclic orbit with n elements and $\mathbf{p} \in \mathcal{O}$. Then the ring $A_{[n], \mathbf{p}}$ is semiprime; thus $A_{[n], \mathbf{p}} = F_{[n], \mathbf{p}}$. Moreover,

$$F_{[n], \mathbf{p}} = D/\mathbf{p}\left(\sigma^n, \sigma^{-n+1}(a) \cdots \sigma(a)a + (\mathbf{p})\right)$$

is a generalized Weyl algebra; thus $F_{[n], \mathbf{p}} \simeq D/\mathbf{p}[X, X^{-1}; \sigma^n]$ is a skew Laurent polynomial ring, if $\mathcal{O}(\mathbf{p})$ is nondegenerate; and $F_{[n], \mathbf{p}} \simeq D/\mathbf{p}(\sigma^n, 0)$, if $\mathcal{O}(\mathbf{p})$ is degenerate. Hence, the type of $F_{[n], \mathbf{p}}$ *does not depend* on $\mathbf{p}$ but does on the orbit $\mathcal{O}(\mathbf{p})$ only and it is $(-, +)$ or $(-, 0, +)$, respectively, for a nondegenerate and degenerate orbit. Thus, the set Specm.$\mathbf{t}$ (D) is the union of cyclic orbits for each type $\mathbf{t}$; moreover,

$$\text{Specm.}\mathbf{t}(D) = CYCD, \qquad \text{for } \mathbf{t} = (\pm 0), (+, +), (-, -),$$

and

$$\text{Specm.}(-, +)(D) = CYCN,$$

where $CYCD$ and $CYCN$ are the set of all maximal ideals of D from degenerate and nondegenerate cyclic orbits, respectively. Using (3.6) we obtain four types of modules.

$\hat{A}(\pm 0)$. Specm.$(\pm 0)(D) = CYCD$ and $\mathbf{p} \sim_{(\pm 0)} \mathbf{q}$ if and only if both $\mathbf{p}$ and $\mathbf{q}$ belong to some set Γ_i. The following corollary follows from (3.7) and (3.8).

COROLLARY 4.2. *The map*

$$CYCD/\sim_{(\pm 0)} \to \hat{A}(\pm 0), \qquad \Gamma \to [L(\Gamma)],$$

is bijective with inverse $[M] \to \operatorname{Supp} M$, *where* $L(\Gamma) = A/A(\mathbf{p}, X, Y^n)$ *for* $\Gamma = (\sigma^{-n}(\mathbf{p}), \mathbf{p}]$, $n \in \mathbf{N}$.

$\hat{A}(\mathbf{t})$, $\mathbf{t} = (-,-), (+,+), (-,+)$. Let $\mathcal{O}$ be a cyclic orbit, $n = |\mathcal{O}|$, $\mathbf{p} \in \mathcal{O}$. Recall that each degenerate (resp. nondegenerate) cyclic orbit is an equivalence class under $\sim_{\mathbf{t}}$, $\mathbf{t} = (-,-), (+,+)$ (resp. $\mathbf{t} = (-,+)$). The ring $E(\mathbf{t}, \Gamma = \mathcal{O})$ is isomorphic to the skew Laurent polynomial ring

$$E(\mathbf{t}, \Gamma) \simeq D/\mathbf{p}[X, X^{-1}; \sigma^n]$$

for $\mathbf{p} \in \mathcal{O}$.

The following Corollary 4.3 follows from (3.12) and (3.13).

COROLLARY 4.3. *Let* $\mathbf{t} = (-,-), (+,+), (-,+)$ *and* $I = Cycd$, $Cycd$, *and* $Cycn$, *respectively*. *The map*

$$\sqcup_{\mathcal{O} \in I} \hat{E}(\mathbf{t}, \Gamma) \to \hat{A}(\mathbf{t}), \qquad [N] \to \left[A \otimes_{A_{[n]}} N\right], \qquad n = |\mathcal{O}|,$$

is bijective.

COROLLARY 4.4. *Let* $A = D(\sigma, a)$ *be a generalized Weyl algebra with commutative* D. *Then the simple weight* A-*modules are classified by Corollaries* 4.1–4.3.

Let $R = \bigoplus_{i \in \mathbf{Z}} R_i$ be a $\mathbf{Z}$-graded ring as in Section 3 with commutative $R_0 = D$. The subring A of R generated by D, v_{-1}, and v_1 is the generalized Weyl algebra

$$A = D(\sigma, a := c(-1, 1)). \tag{4.1}$$

Let M be a weight R-module with support from a linear orbit. It turns out that ${}_AM$ is the direct sum of nonisomorphic simple weight A-modules (Lemma 4.5).

Let $\mathcal{O}$ be an orbit. Two subsets I and J of $\mathcal{O}$ with $I \cap J = \emptyset$ are called *neighbouring* if $\sigma(I) \cap J \cup I \cup \sigma(J) \neq \emptyset$.

LEMMA 4.5. *Let $\mathcal{O}$ be a linear orbit and let M be a simple weight R-module with* $\operatorname{Supp} M \subset \mathcal{O}$. *If $\mathcal{O}$ is nondegenerate, then ${}_AM$ is simple.*

Suppose that $\mathcal{O}$ is degenerate.

1. *Then ${}_AM = \oplus N_i$ is the sum of nonisomorphic simple weight A-modules N_i with nonneighbouring supports (hence, $\operatorname{Ext}^1_A(N_i, N_j) = 0$ for all $i \neq j$).*

2. *If $\{\mathbf{p}_i, i \in I\}$ is the set of marked ideals of $\mathcal{O}$ (i.e., $a \in \mathbf{p}_i$), then* $\operatorname{Supp} M = U_i\{\Gamma(\mathbf{p}_i) \mid \mathbf{p}_i \sim \mathbf{p}\} \cup \{\Gamma(r) \mid r \sim p\}$ *where $\mathbf{p}$ is any (fixed) ideal from* $\operatorname{Supp} M$ *and $\Gamma(\mathbf{p}_i) := \{\mathbf{q}, \mathbf{q} \sim_{\mathrm{GWA}} \mathbf{p}_i\}$; here $\sim_{\mathrm{GWA}}$ is the equivalence relation from Corollary 4.1, and $r = \sigma(p_s)$ in case I contains a maximal (w.r.t. order) element, say p_s.*

3. *If the set I is finite containing an even number of elements, then there exists a simple weight R-module N of finite D-length $l_D(N) < \infty$, and* $\operatorname{Supp} N \subset \mathcal{O}$.

Proof. 1. The orbit $\mathcal{O}$ is linear, thus ${}_D M_{\mathbf{p}} \simeq D/\mathbf{p}$ for $\mathbf{p} \in \operatorname{Supp} M$ and there exists an epimorphism $A/A\mathbf{p} \to AM_{\mathbf{p}} \subseteq M$ of A-modules; hence $\Gamma(\mathbf{p}) \subseteq \operatorname{Supp} M$ for all $\mathbf{p} \in \operatorname{Supp} M$; moreover,

$$\operatorname{Supp} M = \bigcup\{\Gamma(\mathbf{p}) \mid \mathbf{p} \in \operatorname{Supp} M\}.$$

It follows from Lemma 3.1 that the support of M is the union of nonneighbouring equivalence classes $\Gamma(\mathbf{p})$ and the image $\operatorname{Im}(A/A\mathbf{p} \to M)$ is the simple weight A-module with support $\Gamma(\mathbf{p})$. By Corollary 4.1 ${}_AM = \oplus N_i$ is the sum of nonisomorphic simple weight A-modules with nonneighbouring supports.

Let $0 \to N_i \to N \to N_j \to 0$, $i \neq j$, be an exact sequence of A-modules. The supports of N_i and N_j are nonneighbouring; hence $L = \oplus\{N_{\mathbf{p}}, \mathbf{p} \in \operatorname{Supp} N_j\}$ is the submodule of N isomorphic to N_j (by Corollary 4.1); thus the sequence above splits, i.e., $\operatorname{Ext}^1_A(N_i, N_j) = 0$.

2. It follows from 1 and Lemma 3.1.

3. Let $\mathbf{p}_1 < \cdots < \mathbf{p}_s$ be the elements of I. By 1 there are at least two simple weight R-modules with support from $\mathcal{O}$. If there are no modules of finite D-length, then there exist exactly two equivalence classes of $\mathcal{O}$ (Corollary 4.1). One of these contains $\Gamma(\mathbf{p}_1)$, the other contains $\Gamma(\sigma(\mathbf{p}_s))$. This is possible only for odd s. ∎

5. THE SIMPLE D-TORSIONFREE R-MODULES

Let R be as in Section 1 with *Dedekind* $R_0 = D$ and $c(i, j) \neq 0$ for all $i, j \in \mathbf{Z}$.

Denote by k the field of fractions of D, i.e., $k = S^{-1}D$ where $S = D \setminus \{0\}$. The localization $B = S^{-1}R$ of the ring R at the multiplicatively closed set S is isomorphic to the skew Laurent polynomial ring

$$B \simeq k[X, X^{-1}; \sigma]$$

with coefficients in the field k. It is a Euclidean ring and, hence, a principal left and right ideal domain.

Let M be an R-module. Then $\text{tor}(M) = \{m \in M : \alpha m = 0 \text{ for some } \alpha \neq 0 \in D\}$ is the (D-torsion) submodule of M. So, every simple R-module M is either *D-torsion* ($M = \text{tor}(M) \Leftrightarrow S^{-1}M = 0$) or *$D$-torsionfree* ($\text{tor}(M) = 0 \Leftrightarrow S^{-1}M \neq 0$), i.e.,

$$\hat{R} = \hat{R}(D - \text{torsion}) \sqcup \hat{R}(D - \text{torsionfree}) \tag{5.1}$$

is a disjoint union.

Let $M \neq 0$ be a simple D-torsion R-module, and choose a nonzero $m \neq 0 \in M$ such that ${}_D Dm \simeq D/\mathbf{p}$ for some maximal ideal $\mathbf{p}$ of D. Then $R/R\mathbf{p} \to M = Rm$, $r + R\mathbf{p} \to rm$, is an epimorphism, and we conclude that M is a weight module, i.e.,

$$\hat{R}(D - \text{torsion}) = \hat{R}(D - \text{weight}), \tag{5.2}$$

and Theorems 3.2, 3.4 give the classification of $\hat{R}(D\text{-torsion})$.

So, to complete the classification of simple R-modules it remains to find $\hat{R}(D\text{-torsionfree})$ and this is our aim in this section.

An R-module M is called *R-socle* (or *socle*, for short) if the socle $\text{Soc}_R(M)$ (the sum of the simple R-submodules of M) is nonzero. Denote by $\hat{B}(R\text{-socle})$ the set of isoclasses of simple B-modules which are R-socle. A submodule N of M is called essential if it has nontrivial intersection with each nonzero submodule of M.

LEMMA 5.1. 1. *The canonical map*

$$S^{-1} : \hat{R}(D - \textit{torsionfree}) \to \hat{B}(R - \textit{socle}), \qquad [M] \to [S^{-1}M],$$

is a bijection with inverse $\text{Soc} : [N] \to [\text{Soc}_R(N)]$.

2. *Each simple D-torsionfree R-module has the form*

$$M_{\mathbf{m}} := R/R \cap \mathbf{m} \tag{5.3}$$

for some left maximal ideal $\mathbf{m}$ of the ring B. If $\mathbf{m} = Bb$ and $\mathbf{n} = Bc$ are left maximal ideals of B for some $b, c \in$ $\text{Irr}(B)$, then the R-modules $M_{\mathbf{m}}$ and $M_{\mathbf{n}}$ are isomorphic if and only if both b and c are similar (the B-modules $B/\mathbf{m}$ and $B/\mathbf{n}$ are isomorphic).

Proof. 1. It is easy to see that the maps S^{-1} and Soc are well defined. If $[M] \in \hat{R}(D\text{-torsionfree})$, then M is a simple essential R-submodule of $S^{-1}M$, so $\text{Soc}_R(S^{-1}M) = M$. Conversely, if $[N] \in \hat{B}(R\text{-socle})$, then $\text{Soc}_R(N)$ is nonzero; thus $S^{-1}(\text{Soc}_R(N)) = N$ (as a nonzero submodule of a simple module).

Part 2 follows directly from (5.3). ∎

The ring B is a principal left and right ideal domain. If a left maximal ideal $\mathbf{m}$ of B is generated by an (irreducible) element f, $\mathbf{m} = Bf$, then we very often abbreviate $M_{\mathbf{m}} = M_{Bf}$ to M_f.

The formula (5.3) yields the form of simple D-torsionfree R-modules. Unfortunately, in general, $M_{\mathbf{m}}$ need not be simple. If it is, then $\mathbf{m}$ (or an irreducible element $b \in \text{Irr}\, B$) is called *convenient*. The set of all convenient left maximal ideals of B (resp. all convenient irreducible elements) is denoted by $\text{Specm.con}(B)$ (resp. $\text{Irr.con}(B)$). Moreover, the R-module $M_{\mathbf{m}}$ may have the zero socle, i.e., $\text{Soc}_R(M_{\mathbf{m}}) = 0$. An ideal $\mathbf{m}$ is called a *socle* ideal if $\text{Soc}_R(M_{\mathbf{m}}) \neq 0$. Denote by $\text{Specm.soc}(B)$ resp. $\text{Sirr}_R(B)$ the set of all such ideals resp. elements b. Further a criterion will be given for $\mathbf{m}$ (or for b) to be a socle ideal (Theorem 5.13) as well as the procedure for converting a socle left maximal ideal (resp. an irreducible element b) into a convenient one (Lemma 5.6).

For $\mathbf{m} \in \text{Specm.soc}(B)$ denote by $J(\mathbf{m})$ the smallest of the left ideals of R that strictly contain $R \cap \mathbf{m}$. Then

$$J(\mathbf{m})/R \cap \mathbf{m} = \text{Soc}_R(M_{\mathbf{m}}).$$

Since $S^{-1}(J(\mathbf{m})/R \cap \mathbf{m}) = S^{-1}\,\text{Soc}_R(M_{\mathbf{m}})) = B/\mathbf{m}$, the ideal

$$\mathbf{a}(\mathbf{m}) := J(\mathbf{m}) \cap D \tag{5.4}$$

of D is nonzero.

The lemma below shows how a simple module is produced from a nonsimple socle module $M_{\mathbf{m}}$.

LEMMA 5.2. *Let* $\mathbf{m} \in \text{Specm.soc}(B)$ *and* $\alpha \neq 0 \in D$. *The following are equivalent.*

1. $\alpha \in \mathbf{a}(\mathbf{m})$;
2. $J(\mathbf{m}) = R \cap \mathbf{m} + R\alpha$;
3. $M_{\mathbf{m}\alpha^{-1}}$ *is a simple R-module.*

Proof. (1 ⇒ 2) Set $I(\alpha) = R \cap \mathbf{m} + R\alpha$. Then $I(\alpha) \subseteq J(\mathbf{m})$ and $S^{-1}I(\alpha) = B$, so $I(\alpha) \neq R \cap \mathbf{m}$; hence $I(\alpha) = J(\mathbf{m})$.

(2 ⇒ 1) We see that $\alpha \in J(\mathbf{m}) \cap D = \mathbf{a}(\mathbf{m})$.

(2 ⇒ 3) Follows from

$$\begin{aligned}\text{Soc}_R(M_{\mathbf{m}}) &= J(\mathbf{m})/R \cap \mathbf{m} = (R \cap \mathbf{m} + R\alpha)/R \cap \mathbf{m} \simeq R\alpha/R\alpha \cap \mathbf{m} \\ &\simeq R/R \cap \mathbf{m}\alpha^{-1} = M_{\mathbf{m}\alpha^{-1}}.\end{aligned}$$

(3 ⇒ 2) If $M_{\mathbf{m}\alpha^{-1}}$ is simple, then $J(\mathbf{m}) = R \cap \mathbf{m} + R\alpha$. As we have seen above $M_{\mathbf{m}\alpha^{-1}} \simeq (R \cap \mathbf{m} + R\alpha)/R \cap \mathbf{m}$. ∎

The lemma above shows that if $\mathbf{m}$ is a socle ideal, then $\mathbf{m}\alpha^{-1}$ is convenient for all nonzero $\alpha \in \mathbf{a}(\mathbf{m})$ and only for them.

LEMMA 5.3. *Let* $\mathbf{m}$ *be a left maximal ideal of* B. *Then* $M_{\mathbf{m}}$ *is a simple* R*-module if and only if* $\mathrm{Hom}_R(M_{\mathbf{m}}, N) = 0$ *for all simple* D*-torsion* R*-modules* N.

Proof. (⇒) Obvious.

(⇐) Suppose that $M_{\mathbf{m}}$ is nonsimple. Then $R \cap \mathbf{m}$ is strictly contained in some left maximal ideal J of the ring R and $S^{-1}J = B$ (otherwise, $S^{-1}J = \mathbf{m}$ and $J \subseteq R \cap S^{-1}J = R \cap \mathbf{m}$, a contradiction). So, $J \cap S \neq \emptyset$ and $N = R/J$ is a simple D-torsion R-module. Then $\mathrm{Hom}_R(M_{\mathbf{m}}, N) \neq 0$ since there exists a nonzero homomorphism $M_{\mathbf{m}} \to N$, $u + R \cap \mathbf{m} \to u + J$. ∎

Let us present a sufficient condition for $M_{\mathbf{m}}$ to be simple.

COROLLARY 5.4. *Let* $\mathbf{m}$ *be a left maximal ideal of the ring* B. *If the ideal* $R \cap \mathbf{m}$ *of* R *contains a nonzero element* v *that acts injectively on every simple* D*-torsion* R*-module, then the* R*-module* $M_{\mathbf{m}}$ *is simple*.

Proof. Let φ be a homomorphism from $M_{\mathbf{m}}$ to a simple D-torsion R-module N. Then $0 = \varphi(v) = v\varphi(1 + R \cap \mathbf{m})$, and by the definition of v, $\varphi(1 + R \cap \mathbf{m}) = 0$, and thus $\varphi = 0$. We complete the proof by applying Lemma 5.3. ∎

In the set S consider the relation $<$: $\alpha < \beta$ if and only if there are no maximal ideals $\mathbf{p}$ and $\mathbf{q}$ which belong to the same linear orbit, contain α and β respectively, and $\mathbf{p} \geq \mathbf{q}$, i.e., $\mathbf{p} = \sigma^i(\mathbf{q})$ for some nonnegative integer i.

DEFINITION. An element $b = v_{-m}\beta_{-m} + \cdots + \beta_0 \in R$, $\beta_i \in D$ $(i = -m, \dots, 0)$ of length $m > 0$ is called *l-normal* (or normal, for short) if $\beta_0 < \beta_{-m}$ and $\beta_0 < a$.

The first part of Lemma 5.5 follows immediately from the classification of simple D-torsion R-modules with supports from linear orbits (Theorem 3.1) and (1.1), (1.2).

LEMMA 5.5. 1. *Every l-normal element acts injectively on each simple* D*-torsion* R*-module with support from a linear orbit*.

2. *Let* $\mathbf{m}$ *be a left maximal ideal of* B *which contains an l-normal element*. *Then* $\mathrm{Hom}_R(M_{\mathbf{m}}, N) = 0$ *for simple* D*-torsion* R*-submodules* N *with supports from linear orbits*.

Proof. 1. Let M be a simple weight R-module with support from a linear orbit $\mathcal{O}$. As an A-module it is a direct sum $M = M_1 \oplus \cdots \oplus M_s$ of simple weight A-modules with support from $\mathcal{O}$ (Lemma 4.5); moreover, $\operatorname{Supp} M_i \cap \operatorname{Supp} M_j = \varnothing$ for all $i \neq j$. We may assume that $\mathbf{p} < \mathbf{q}$ for all $\mathbf{p} \in \operatorname{Supp} M_i$, $\mathbf{q} \in \operatorname{Supp} M_j$, if $i < j$.

Suppose that $bu = 0$ for some $u \neq 0 \in M$. Since $\beta_0 < a$, we conclude that $u \in M_1$ and $M_1 \simeq A/A(\mathbf{p}, v_1)$ has the support $(-\infty, \mathbf{p}]$ for some maximal ideal $\mathbf{p}$ of D such that $a \in \mathbf{p}$. The element v_{-1} acts injectively on M_1, and so does v_{-m} since (as we have seen early)

$$v_{-1}^m = \alpha_{-m} v_{-m} = v_{-m}\, \beta_{-m}$$

for some $\beta_{-m} = \sigma^m(\alpha_{-m}) \in D$ (an element $\alpha \in D$ acts injectively on M_1 if and only if it acts bijectively). Write $u = u_{\mathbf{q}} + \cdots + u_{\mathbf{r}}$ as the sum of weight components, i.e., $u_{\mathbf{q}} \in (M_1)_{\mathbf{q}}, \dots, u_{\mathbf{r}} \in (M_1)_{\mathbf{r}}$, such that $\mathbf{q} < \cdots < \mathbf{r}$. It follows from $bu = 0$ that $\beta_{-m} u_q = 0$ (v_{-m} acts injectively) and $\beta_0 u_{\mathbf{r}} = 0$; thus $\beta_{-m} \in \mathbf{q} < \mathbf{r} \ni \beta_0$ which contradicts $\beta_0 < \beta_{-m}$. Hence b acts injectively on M.

2. Repeat the arguments in the proof of Corollary 5.4. ∎

The proposition below shows that, in general, there are "many" l-normal elements and that there exists a canonical (and fully computable) procedure for transforming any element of positive length into an l-normal one.

PROPOSITION 5.6. *For any $b = v_{-m}\beta_{-m} + \cdots + \beta_0 \in R$, $\beta_i \in D$, $i = -m, \dots, 0$, of positive length $m > 0$ there exist nonzero $\alpha, \beta \in D$ such that $\beta b \alpha^{-1} \in R$ is l-normal. Moreover, it is enough to set*

$$\alpha = \prod \{\sigma^j(\beta_0) \mid -s \leq j \leq 0\}, \qquad \beta = \prod \{\sigma^j(\beta_0) \mid -m - s \leq j \leq -1\},$$

where $s \in \mathbf{N}$ such that $\sigma^{-s}(\beta_0) < \beta_{-m}$, $\sigma^{-s}(\beta_0) < a$, and $\sigma^{-s+m}(\beta_0) < \beta_0$.

Proof. A simple check shows that the element $c = \beta b \alpha^{-1} = v_{-m}\gamma_{-m} + \cdots + \gamma_0$ belongs to R, where

$$\gamma_0 = \prod \{\sigma^i(\beta_0) \mid -m - s < i < -S\}$$

and

$$\gamma_{-m} = \beta_{-m} \prod \{\sigma^{-i}(\beta_0) \mid 0 < i < m\}.$$

It follows directly from the explicit form of γ_0 and γ_{-m} that c is l-normal. ∎

Let V be a D-torsion R-module ($S^{-1}V = 0$). The ring D is Dedekind, so

$$V = \bigoplus_{\mathbf{p} \in \operatorname{Specm}(D)} V^{\mathbf{p}},$$

where $V^{\mathbf{p}} = \{v \in V : \mathbf{p}^i v = 0 \text{ for some } i = i(v)\}$ is a D-submodule. Define the *support* of M as $\operatorname{Supp} V = \{\mathbf{p} : V^{\mathbf{p}} \neq 0\}$. This definition extends the earlier one and inherits all its properties. Since

$$v_i V^{\mathbf{p}} \subseteq V^{\sigma^i(\mathbf{p})} \qquad \text{for all } i \in \mathbf{Z},$$

the module V decomposes into the direct sum of R-modules (the *orbit decomposition*)

$$V = \bigoplus \{V_O \mid O \text{ is an orbit}\},$$

where $V_O = \bigoplus_{\mathbf{p} \in O} V^{\mathbf{p}}$.

For $\alpha \neq 0 \in D$ denote by $[\alpha]$ the set of maximal ideals $\mathbf{p}$ of D such that $\alpha \in \sigma^i(\mathbf{p})$ for some $i \in \mathbf{Z}$. Then $[\alpha]$ is the finite union of orbits. For $b = \sum \beta_i v_i \in R$, $\beta_i \in D$, set $GCD(b) := \sum D\beta_i$.

LEMMA 5.7. *Let $\mathbf{m} = Bf$ be a left maximal ideal of B and $f \in R$.*

1. *Then the sequence of R-modules*

$$0 \to V_f := R \cap Bf/Rf \to R/Rf \to M_{\mathbf{m}} \to 0 \tag{5.5}$$

is exact and V_f is a D-torsion R-module.

2. *If D is a principal ideal domain and $f = a_0 + \cdots + a_{\pm n} v_{\pm n}$, $a_i \in D$, $a_0 \neq 0$, and $a_{\pm n} \neq 0$, $n > 0$, then*

$$\operatorname{Supp} V_f \subseteq [a_0] \cap \{[a_{\pm n}] \cup [a]\}. \tag{5.6}$$

Proof. 1. The exactness is clear and V_f is D-torsion since $S^{-1}V_f \simeq S^{-1}R \cap Bf/S^{-1}Rf \simeq Bf/Bf = 0$.

2. The support of the module V_f belongs to the union $U = \cup[GCD(bf)]$ over all $b \in R$ such that $GCD(b) = D$. We prove the statement for $(+)$ (then for $(-)$ it is clear by symmetry).

If $b = b_k v_k + \cdots + b_m v_m$, $b_i \in D$, $k < \cdots < m$, then the coefficient β_i of degree i $(k \leq i \leq m)$ of the element bf is equal to

$$\beta_i = b_i \sigma^i(a_0) + \sum_{j<i} \cdots b_j, \tag{5.7}$$

where here and later the three dots denote elements of D. Since $GCD(b) = D$, by (5.7) $U \subseteq [a_0]$.

Let $k \leq j \leq m$. Then

$$\beta_{n+j} = b_j \sigma^j(a_n) c(j, n) + \sum_{l>j} \cdots b_l. \tag{5.8}$$

Since $GCD(b) = D$ and by (1.1), (1.2) $\bigcup_{i,j \in \mathbf{Z}} [c(i,j)] \subseteq [a]$. It follows from (5.8) that $U \subseteq [a_n] \cup [a]$. ∎

A left ideal J of R is *D-artinian* if the set of left ideals $\{J + R\mathbf{a}\}$ is artinian (with respect to inclusion) where $\mathbf{a}$ runs through all nonzero ideals of D.

An element $r \in R$ is called *normal*, if $rR = Rr$. It is easy to see that

$$\alpha \in D \text{ is normal in } R \Leftrightarrow \sigma(\alpha) = \gamma\alpha \text{ for some unit } \gamma \in D \quad (5.9)$$

or, equivalently, $D\alpha$ is a σ-stable ideal of D $(\sigma(D\alpha) = D\alpha)$. If $\alpha \neq 0 \in D$ is normal in R, then the inner automorphism of B

$$\omega_\alpha : B \to B, \qquad b \to \alpha b \alpha^{-1},$$

is also the automorphism of R, since $\omega_\alpha(R) = R$.

Let $\mathcal{O} = \mathcal{O}(\mathbf{p})$ be a cyclic orbit, $n = |\mathcal{O}|$. Set

$$\theta(\mathcal{O}) = \prod_{i=0}^{n-1} \sigma^i(\mathbf{p}). \quad (5.10)$$

If D is a principal ideal domain (p.i.d.) and $p \in \mathbf{p}$ is a (fixed) generator of $\mathbf{p} = Dp$, then

$$\theta_{\mathcal{O}} = \prod_{i=0}^{n-1} \sigma^i(p) \quad (5.11)$$

is normal in R. The element $\theta_{\mathcal{O}}$ is uniquely determined up to units of D by the orbit $\mathcal{O}$, so choose one of them.

Let M be an R-module and τ a ring automorphism of R; denote by ${}^{\tau}M$ the twisted R-module: ${}^{\tau}M = M$ as an abelian group and the R-module structure is defined as follows: $rm = \tau(r)m$, where $m \in M$ and $r \in R$.

For $\alpha \neq 0 \in D$, $f \in B$, using the inclusion of R-modules

$$M_{f\alpha^{-1}} = R/R \cap Bf\alpha^{-1} \to M_f = R/R \cap Bf,$$

$$u + R \cap Bf\alpha^{-1} \to u\alpha + R \cap Bf, \quad (5.12)$$

we identify $M_{f\alpha^{-1}}$ with its image in M_f. So we have

$$M_{f\alpha^{-1}} \subseteq M_f \qquad \text{and} \qquad M_f/M_{f\alpha^{-1}} \simeq R/(R\alpha + R \cap Bf). \quad (5.13)$$

THEOREM 5.8. *Let D be a principal ideal domain. The following are equivalent.*

1. $f \in \mathrm{Sirr}_R B$;
2. *$R \cap Bf$ is D-artinian*;
3. *$R = R\theta_{\mathcal{O}} + R \cap Bf$ for every cyclic orbit $\mathcal{O}$*;
4. *$M_f = M_{f\theta_{\mathcal{O}}^{-1}}$ for every cyclic orbit $\mathcal{O}$.*

Moreover, if $f \in \mathrm{Sirr}_R B$ is l-normal, then M_f is simple.

Proof. (1 ⇒ 2) Since $f \in \mathrm{Sirr}_R B$, it follows from $R/R \cap Bf \subset B/Bf$ that $R/R \cap Bf \supset \mathrm{Soc}_R B/Bf = J/R \cap Bf$ for some left ideal J of R strictly containing $R \cap Bf$. Hence $J = R\mathbf{a} + R \cap Bf$ where $\mathbf{a} = J \cap D \neq 0 \subseteq D$ (see (5.4)). Let

$$J_1 \supseteq \cdots \supseteq J_n = R\mathbf{a}_n + R \cap Bf \supseteq \dots, \qquad \mathbf{a}_n \neq 0 \subseteq D,$$

be a decreasing chain of left ideals; we may suppose that $\mathbf{a}_n = J_n \cap D$. Each J_n contains J, the D-module $D/D\mathbf{a}$ is artinian, and thus the chain of D-modules

$$D\mathbf{a}_1 \supseteq \cdots \supseteq D\mathbf{a}_n \supseteq \cdots \supseteq D\mathbf{a}$$

must terminate as well as the chain above.

(2 ⇒ 1) Let $J = R\alpha + R \cap Bf$ be a minimal element in $\{R\beta + R \cap Bf, \beta \neq 0 \in D\}$. Every submodule of $R/R \cap Bf$ is essential, so $J/R \cap Bf$ is simple and equal to $\mathrm{Soc}_R B/Bf$.

(2 ⇒ 3) Set $\theta = \theta_O$. The chain of left ideals

$$J_1 \supseteq \cdots \supseteq J_i := R\theta^i + R \cap Bf \supseteq \cdots$$

must terminate, say $J_m = J_{m+1}$ or, equivalently,

$$\begin{aligned} 0 = J_m/J_{m+1} &\simeq R\theta^m/\big(R\theta^{m+1} + R\theta^m \cap Bf\big) \simeq R/(R\theta + R \cap Bf\theta^{-m}) \\ &= R/\omega^m(R\theta + R \cap Bf), \end{aligned} \tag{5.14}$$

i.e., $R = R\theta + R \cap Bf$, where $\omega = \omega_\theta$, the ring automorphism of R (as defined above).

(3 ⇔ 4) By (5.13) $M_{f\theta^{-1}} \subseteq M_f$ and

$$M_f/M_{f\theta^{-1}} \simeq R/(R\theta + R \cap Bf),$$

where we set $\theta = \theta_O$.

(4 ⇒ 1) Note that

$$M_{f\theta^{-m}} \simeq^{\omega_{\theta^m}^{-1}} M_f, \text{ the twisted module, } m \in \mathbf{Z}. \tag{5.15}$$

Namely,

$$M_{f\theta^{-m}} = R/R \cap Bf\theta^{-m} \simeq R/\omega_{\theta^m}(R \cap Bf) \simeq^{\omega_{\theta^m}^{-1}} (R/R \cap Bf) =^{\omega_{\theta^m}^{-1}} M_f.$$

Then statement 4 may be written as $M_f =^{\omega_\theta^{-1}} M_f$, $\theta = \theta_O$, for every cyclic orbit O. Without loss of generality we may suppose that f is l-normal. Let us show that M_f is simple. For this it is enough to prove that $M_f = M_{f\alpha^{-1}}$ for all $\alpha \neq 0 \in D$. Fix a nonzero α; since $M_{f\alpha^{-1}} \supseteq M_{f\beta^{-1}}$ for any β such that $\alpha \mid \beta$, we may suppose that $\alpha = p_1 \cdots p_s \theta_1 \cdots \theta_t$ where all p_i are irreducible and belong to linear orbits and $\theta_i = \theta_{O_i}$ for some cyclic orbit O_i. Set $\theta = \theta_1 \cdots \theta_t$. Using (5.15) and applying ω_θ to

$$M_f =^{\omega_\theta^{-1}} M_f \supseteq M_{f\alpha^{-1}} =^{\omega_\theta^{-1}} M_{f\beta^{-1}}, \qquad \beta = p_1 \cdots p_s,$$

we reduce our problem to the case $\alpha = \beta$.

Since $N := M_f / M_{f\alpha^{-1}} \simeq R/(R\alpha + R \cap Bf)$ there is a natural epimorphism $R/R\alpha \to N$. Suppose that $N \neq 0$. Then N maps onto a simple D-torsion R-module L with support from a linear orbit $O(p_i)$ for some i. Then $\ker f_L \neq 0$ where $f_L : L \to L$, $l \to fl$. But this contradicts Lemma 5.5, so $N = 0$. ∎

For an orbit O set $S_O = D \setminus \bigcup_{\mathbf{p} \in O} \mathbf{p}$, a multiplicatively closed set. Denote by $R_O = S_O^{-1}R$ (resp. $A_O = S_O^{-1}A$) the (two-sided) localization of R (resp. A) at S_O^{-1}. Then $A_O = D_O(\sigma, a)$ is a GWA with $D_O = S_O^{-1}D$. The rings R_O and A_O are subrings of B. The ring D_O is Dedekind. If the orbit O is cyclic, then D_O has a finite number of maximal ideals $\{S_O^{-1}\mathbf{p}, \mathbf{p} \in O\}$; thus D is a principal ideal domain (this fact will be used in the proof of Theorem 5.13).

LEMMA 5.9. *Let V be a nonzero D-torsion R-module.*

1. $O \cap \operatorname{Supp} V = \emptyset$ *if and only if* $S_O^{-1}V = 0$, *where* O *is an orbit*;
2. $\operatorname{Supp} V$ *does not contain a maximal ideal of finite* σ*-order if and only if* $S_O^{-1}V = 0$ *for every cyclic orbit* O.

Proof. If $V = \oplus V_O$ is the orbit decomposition, then the localization $S_O^{-1}V = V_O$ and $\operatorname{Supp} V_O \subseteq O$, if $V_O \neq 0$. The statements now follow easily. ∎

For a nonzero ideal $\mathbf{a}$ of D we set

$$O(\mathbf{a}) = \bigcup_{i=1}^{s} O(\mathbf{p}_i),$$

where $\mathbf{a} = \mathbf{p}_1 \cdots \mathbf{p}_s$ is the product of maximal ideals of D.

LEMMA 5.10. *Let $f \in R$ be irreducible in B. Then*

1. $R_O \cap Bf = R_O f$ *for every orbit* O *such that* $O \cap \operatorname{Supp} V_f = \emptyset$;
2. $R\mathbf{a} + R \cap Bf = R\mathbf{a} + Rf$ *for every ideal* $\mathbf{a}$ *of* D *such that* $O(\mathbf{a})$ *is an orbit and* $O(\mathbf{a}) \cap \operatorname{Supp} V_f = \emptyset$.

Proof. 1. It follows from $O \cap \operatorname{Supp} V_f = \emptyset$ that $(V_f)_O = 0$. Now applying the exact functor $(\ )_O$ to the exact sequence of R-modules $0 \to V_f \to R/Rf \to M_f \to 0$ we have $R_O f = R_O \cap Bf$.

2. The above equality holds if and only if the modules $(R\mathbf{a} + R \cap Bf)/R\mathbf{a}$ and $(R\mathbf{a} + Rf)/R\mathbf{a}$ are equal. These two modules can be considered as R_O-modules where $O = O(\mathbf{a})$, so the statement follows from 1. ∎

LEMMA 5.11. *$f \in \mathrm{Sirr}_R(B)$ if and only if $f \in \mathrm{Sirr}_{R_O}(B)$ for all cyclic orbit O.*

Proof. ($\Rightarrow$) $\mathrm{Soc}_{R_O}(B/Bf) = S_O^{-1}(\mathrm{Soc}_R(B/Bf))$.

($\Leftarrow$) Without loss of generality we may suppose that f is l-normal ($B/B\beta f\alpha^{-1} \simeq^{\omega_\alpha^{-1}} (B/Bf)$ for all nonzero $\alpha, \beta \in D$). We aim to show that M_f is simple or, equivalently, $M_f = M_{f\alpha^{-1}}$ for any $\alpha \neq 0 \in D$. Since D_O is a Dedekind domain with a finite number of maximal ideals, D_O is a principal ideal domain, so applying Theorem 5.8 we have $R_O = R_O\theta_O + R_O \cap Bf$ for all cyclic O, where $\theta = \theta_O$ as in Theorem 5.8. Applying S_O^{-1} to

$$N := M_f/M_{f\alpha^{-1}} \simeq R/(R\alpha + A \cap Bf)$$

we obtain

$$S_O^{-1}N = R_O/(R_O\alpha + R_O \cap Bf) = 0$$

since $R_O\alpha + R_O \cap Bf \supseteq R_O\theta^m + R_O \cap Bf$ for some integer $m \geq 1$. But the latter ideal equals R:

$$R_O\theta^m + R_O \cap Bf \supseteq (R_O\theta + R_O \cap Bf)^m = R^m = R.$$

Thus $S_O^{-1}N = 0$ for all cyclic O; by Lemma 5.9 Supp N does not contain a maximal ideal of finite order. If $N \neq 0$, then N maps onto a simple D-torsion R-module L with support from a linear orbit. Thus $\mathrm{Ker}\, f_L \neq 0$ where $f_L : L \to L$, $l \to fl$, which contradicts Lemma 5.5(2), so $N = 0$. ∎

LEMMA 5.12. *Let O be a cyclic orbit, $\theta = \prod_{\mathbf{p} \in O} \mathbf{p}$, and let I be a nonzero left ideal of R. Then $R = R\theta + I$ if and only if $R_O = R_O\theta + S_O^{-1}I$.*

Proof. ($\Rightarrow$) Applying S_O^{-1} to the left side we come to the right.

($\Leftarrow$) There exists $s \in S_O$ such that $s \in R\theta + I$. Since D is Dedekind by the choice of s: $Ds + D\theta = D$ hence $1 \in R\theta + I$. ∎

THEOREM 5.13 (Criterion for Socle Irreducibility). *Let D be a Dedekind domain. The following are equivalent*

1. $f \in \mathrm{Sirr}_R B$;
2. *$R \cap Bf$ is D-artinian*;
3. *$R = R\theta(O) + R \cap Bf$ for all cyclic orbit O where $\theta(O) := \prod_{\mathbf{p} \in O} \mathbf{p}$*;
4. *$\theta(O)^s \subseteq R\theta(O)^{s+1} + Rf$ for every cyclic orbit O and some $s = s(O) \geq 0$.*

Moreover, if $f \in \mathrm{Sirr}_R B$ is l-normal, then M_f is simple.

Proof. (1 ⇔ 2) Repeat the arguments of Theorem 5.8.

(2 ⇔ 3) It follows from Theorem 5.8, Lemma 5.11, and Lemma 5.12 where $I = R \cap Bf$. In fact, $f \in \mathrm{Sirr}_R B$ if and only if $f \in \mathrm{Sirr}_{R_O} B$ for every cyclic orbit O (Lemma 5.11) if and only if $R_O = R_O \theta(O) + S_O^{-1} I$ for every cyclic orbit O (Theorem 5.8 is applied for the Dedekind ring D_O which is a principal ideal domain) if and only if $R = R\theta(O) + I$ for all cyclic orbits O (Lemma 5.12).

(3 ⇔ 4) By Lemma 5.12 we may suppose that $D = D_O$, i.e., Specm $D = O$, a cyclic orbit. Then D is a principal ideal domain. Denote by θ a generator of the ideal $\theta(O)$ of D. Then $R = R\theta + R \cap Bf$ holds iff $1 = u\theta + \theta^{-s} gf$ for some $u, g \in R$ and $s \geq 0$ or, equivalently, $\theta^s = u\theta^{s+1} + gf$.

Suppose that M_f is not simple, by Lemma 5.3 there is a nonzero epimorphism φ from M_f to a simple D-torsion R-module N. f is l-normal, by Lemma 5.5 Supp N belongs to some cyclic orbit O. By Theorem 5.8 $S_O^{-1} M_f$ is a simple R_O-module and $S_O^{-1} N = N$ (see the proof of Lemma 5.9). The map $S_O^{-1}\varphi : S_O^{-1} M_f \to S_O^{-1} N$ is an isomorphism of modules, a contradiction (the first is not D-torsion, since $f \in \mathrm{Sirr}_R B$, but the second is). ∎

An element $r \in R$ is *left invertible* if $sr = 1$ for some $s \in R$.

THEOREM 5.14. *$\hat{R}$ (D-torsionfree) is empty if and only if there are infinitely many cyclic orbits.*

Proof. (⇐) Suppose there are infinitely many cyclic orbits. Let $f = \sum \alpha_i v_i \in R$, $\alpha_i \in D$, be an irreducible element in B. Choose a cyclic orbit O no element of which contains an α_i or a ($O \cap \{\bigcup_i O(\alpha_i) \cup O(a)\} = \emptyset$). We shall see that condition 3 of Theorem 5.13 does not hold.

Suppose the contrary, i.e., $R = R\theta + R \cap Rf$ where θ is as in Theorem 5.13. By Lemma 5.12 we may suppose that $D = D_O$. By the choice of O, it follows from Lemmas 5.7, 5.10 that $R = R\theta + Rf$; hence, $\bar{f} = f + R\theta$ is left invertible in the factor ring $\overline{R} = R/R\theta$ which is the skew Laurent polynomial ring

$$\overline{R} = D/(\theta)[X, X^{-1}; \sigma]$$

($O \cap O(a) = \emptyset$) with coefficients in the factor ring $D/(\theta)$. Moreover, if $\alpha_i \neq 0$, then $\alpha_i + (\theta)$ is a unit in $D/(\theta)$; hence, $\bar{f}$ could not be left invertible, a contradiction.

(⇒) Suppose that there are only a finite number of cyclic orbits. Then the intersection $I = \bigcap \mathbf{p}$ over all maximal ideals of finite σ-order is nonzero and contains a nonzero element α. Then $f = \alpha v_{-1} + 1 \in R$ is irreducible in B and l-normal. If O is cyclic, then $\alpha \in D\theta(O)$; thus $R = R\theta(O) + Rf \subseteq R\theta(O) + R \cap Bf$. In view of Theorem 5.13 M_f is a simple D-torsionfree R-module. ∎

LEMMA 5.15. *A sufficient condition for the canonical injection* $S^{-1}: \hat{R}(D\text{-torsionfree}) \to \hat{B}$, $[M] \to [S^{-1}M]$ *to be bijective is that there are no cyclic orbits* (*if* R *is a generalized Weyl algebra this condition is also necessary*). *If this condition holds, then every simple D-torsionfree R-module is isomorphic to some* $R/R \cap Bf$ *with* f *irreducible and l-normal, two of these R-modules being isomorphic if and only if the corresponding* f *are similar*.

Proof. If there are no cyclic orbits, it follows from Theorem 5.13.

Let R be a GWA and $\mathcal{O}$ a cyclic orbit. Without loss of generality we may suppose that Specm $D = \mathcal{O}$ (Lemma 5.12). Then D is a principal ideal domain. Let θ be a generator of the ideal $\theta(\mathcal{O})$ and $f = Y + 1 = v_{-1} + 1$, an irreducible element in B. Suppose that $\theta^s = u\theta^{s+1} + gf$ for some $u, g \in R$ and $s \geq 0$. If $s \geq 1$, then $\bar{g}\bar{f} = 0$ in the factor ring $R/R\theta$. Since R is a GWA, the element $\bar{f} = f + R\theta$ is not a zero divisor. Thus $s = 0$ and $\bar{g}\bar{f} = \bar{1}$ in $R/R\theta$; degree arguments shows that it is impossible. It means that condition 4 of Theorem 5.13 does not hold. ∎

PROPOSITION 5.16. *Let* $\mathbf{p}$ *be a* σ*-invariant maximal ideal of* D *such that* $a \notin \mathbf{p}$, $f = \sum \alpha_i v_i \in R \setminus R\mathbf{p}$, *all* $\alpha_i \in D$. *Then*

1. $R\mathbf{p} + R \cap Bf = R\mathbf{p} + Rf$;
2. $R = R\mathbf{p} + R \cap Bf$ *if and only if there is exactly one coefficient* α_i *such that* $\alpha_i \notin \mathbf{p}$.

Proof. 1. Since $a \notin \mathbf{p}$, the factor ring $R/R\mathbf{p}$ is the skew Laurent polynomial ring $D/\mathbf{p}[X, X^{-1}; \sigma]$ which is a domain. Thus, if $r \in R \setminus R\mathbf{p}$, then $rf \notin R\mathbf{p}$ (since $f \notin R\mathbf{p}$); hence 1 holds (see also Lemma 5.12).

2. By 1 $R = R\mathbf{p} + R \cap Bf$ if and only if $\bar{f} = f + R\mathbf{p}$ is an unit in $R/R\mathbf{p}$, so $\bar{f} = \alpha_i v_i + R\mathbf{p}$ for some i such that $\alpha_i \notin \mathbf{p}$. ∎

For $f \in B = \oplus B_i$ of length $l(f) > 0$ a natural number n is called *f-degenerate* if $f \in \sum_{j \in \mathbf{Z}} B_{i+nj}$ for some $i \in \mathbf{Z}$. The set of all such n is denoted by $f - \deg$. If n is f-degenerate, then any divisor of n is too.

LEMMA 5.17. *Let* $f = \sum_{i \in \mathbf{Z}} f_i \in B$, $f_i \in B_i$, *be of length* $l(f) > 0$. *Then* $f - \deg$ *consists of the divisors of* $dg(f) :=$ g.c.d.$\{i - j$, *for all* $i \neq j$ *such that* $f_i \neq 0$ *and* $f_j \neq 0\}$; *moreover* $dg(f) \leq l(f)$.

Proof. Denote by M the set of all i such that $f_i \neq 0$. $n \in f - \deg$ if and only if each $i \in M$ can be written as $i = l + nk_i$ for some $k_i \in \mathbf{Z}$ and fixed l, or, equivalently, $i - j = n(k_i - k_j) \in \mathbf{Z}$ for all $i, j \in M$ and the rest follows. ∎

PROPOSITION 5.18. *Let* $\mathcal{O} = \mathcal{O}(\mathbf{p}) \in Cycn$, $n = |\mathcal{O}|$, $\theta = \theta(\mathcal{O}) := \prod_{i=0}^{n-1} \sigma^i(\mathbf{p})$, *and* $f = \sum_{i \in \mathbf{Z}} \alpha_{ni} v_{ni} \in R_{[n]} \setminus R_{[n]}\sigma^j(\mathbf{p})$ *for all* $j = 0, \dots, n-1$ *where all* $\alpha_{ni} \in D$. *Then* $R = R\theta + R \cap Bf$ *if and only if for each* $i = 0, \dots, n-1$ *there is exactly one coefficient* α_{nj} *such that* $\alpha_{nj} \notin \sigma^i(\mathbf{p})$.

Proof. *Step* 1. The element $f \in R_{[n]}$ is homogeneous with respect to the $\mathbf{Z}_n$-grading, so the equality above holds if and only if

$$R_{[n]} = R_{[n]}\theta + R_{[n]} \cap B_{[n]} f, \tag{5.16}$$

where $B_{[n]} = k[X^n, X^{-n}; \sigma^n] = S^{-1} R_{[n]}$ is the localization of $R_{[n]}$ at $S = D \setminus \{0\}$.

Step 2. We claim that (5.16) is equivalent to

$$R_{[n]} = R_{[n]}\sigma^i(\mathbf{p}) + R_{[n]} \cap B_{[n]} f, \qquad \text{for all } i = 0, \dots, n-1. \tag{5.17}$$

Without loss of generality we may suppose that $D = D_{\mathcal{O}}$, i.e., Specm $D = \mathcal{O}$. Then D is a principal ideal domain, hence $\sigma^i(\mathbf{p}) = Dp_i$ for some $p_i \in \sigma^i(\mathbf{p})$. The ring $R_{[n]}$ is of type R with the automorphism σ^n. This automorphism σ^n has exactly n orbits in Specm D, each of them consists of one ideal $\{\sigma^i(\mathbf{p})\} = \mathcal{O}_i$, $i = 0, \dots, n-1$.

The right $R_{[n]}$-module $M = R_{[n]}/R_{[n]}\theta$ decomposes into the direct sum of weight submodules

$$M = \bigoplus_{i=0}^{n-1} M_{\mathcal{O}_i}, \qquad M_{\mathcal{O}_i} = q_i M = q_i R_{[n]}/R_{[n]}\theta, \qquad q_i = \prod_{j \neq i} p_j.$$

Now (5.16) is equivalent to $q_i R_{[n]} = R_{[n]}\theta + q_i(R_{[n]} \cap Bf)$ for all $i = 0, \dots, n-1$ or to

$$R_{[n]} = q_i^{-1} R_{[n]}\theta + R_{[n]} \cap Bf = R_{[n]} p_i + R_{[n]} \cap Bf \qquad \text{for all } i = 0, \dots, n-1.$$

Step 3. Since $f \in R_{[n]} \setminus R_{[n]}\sigma^j(\mathbf{p})$ for all $j = 0, \dots, n-1$, we finish the proof applying Proposition 5.16 to (5.17). ∎

6. THE SIMPLE MODULES OF THE QUANTUM WEYL ALGEBRA

Let K be a field assumed to be algebraically closed for simplicity.

The quantum Weyl algebra $A_1(q)$, $q \neq 0, 1 \in K$, is the generalized Weyl algebra

$$A_1(q) \simeq K[H](\sigma, a = H)$$

with $\sigma : H \to q^{-1}(H - 1)$ (see Example 4, Sect. 1).

Identify Specm $K[H]$ with K by the map $(H - \lambda) \to \lambda$. Then σ "acts" on K as $\sigma(\lambda) = q\lambda + 1$. Therefore, any orbit has the form $\mathcal{O}(\lambda) = \{\sigma^i(\lambda) = q^i\lambda + (q^i - 1)/(q-1), i \in \mathbf{Z}\}$; the element $\delta := (1-q)^{-1}$ is σ-invariant, i.e., $\mathcal{O}(\delta) = \{\delta\}$. Let 6 be the cyclic group generated by σ. Then the set of 6-orbits can be identified with $K/6$.

q is an nth root of 1 ($q^n = 1$). In this case $\sigma^n = 1$; hence all orbits are cyclic, and by Theorem 5.14 each simple $A_1(q)$-module is $K[H]$-torsion $\equiv$ weight.

All orbits but $\mathcal{O}(\delta)$ are of length n, $\mathcal{O}(0)$ is the unique degenerate orbit. We apply Corollaries 4.2 and 4.3.

$\hat{A}_1(q)(\pm 0)$ contains the unique class which corresponds to the module

$$L(\pm 0) = A_1(q)/A_1(q)(X, H, Y^n). \quad (6.1)$$

For $\mathbf{t} = (-,-), (+,+)$, the map

$$K^* := K \setminus \{0\} \to \hat{A}_1(q)(\mathbf{t}), \qquad \lambda \to [L(\mathbf{t}, \lambda)],$$

is bijective, where

$$L((-,-), \lambda) = A_1(q)/A_1(q)(X^n, H, Y^n - \lambda)$$

and

$$L((+,+), \lambda) = A_1(q)/A_1(q)(Y^n, H, X^n - \lambda). \quad (6.2)$$

Set $K^{**} = (K/6) \setminus \{\mathcal{O}(o), \mathcal{O}(\delta)\}$.
The map

$$K^{**} \times K^* \cup K^* \to \hat{A}_1(q)(-,+), \qquad (\mathcal{O}(\lambda), \mu) \to [L((-,+), \lambda, \mu)],$$
$$\mu \to [L((-,+), \mu)],$$

is bijective, where

$$L((-,+), \lambda, \mu) = A_1(q)/A_1(q)\big(H - \lambda, X^n - \mu, Y^n - \mu^{-1}\lambda(q\lambda + 1)$$
$$\times \cdots \big(q^{n-1}\lambda + (q^{n-1} - 1)/(q-1)\big)\big) \quad (6.3)$$

and

$$L((-,+), \mu) = A_1(q)/A_1(q)\big(H - \delta, X - \mu, Y - \delta\mu^{-1}\big). \quad (6.4)$$

COROLLARY 6.1. *If q is an nth root of* 1. *Then each simple $A_1(q)$-module is finite dimensional and weight and the set $\hat{A}_1(q)$ is described by* (6.1)–(6.4).

q is not a root of 1. $\hat{A}_1(q)$(weight). In this case there is the unique cyclic orbit $\mathcal{O}(\delta)$, $\delta = (1-q)^{-1}$, which is nondegenerate; and the unique degenerate orbit $\mathcal{O}(0)$. Thus

$$\hat{A}_1(q)(\text{weight}) = \hat{A}_1(q)(\text{weight, linear}) \sqcup \hat{A}_1(q)(-,+).$$

Specm.lin $K[H] = K \setminus \delta$ and two scalars λ and μ from $K \setminus \delta$ are equivalent in the sense of Corollary 4.1 iff λ, μ belong to an orbit $\mathcal{O} \neq \mathcal{O}(0)$ or to $\Gamma_1 := \{\sigma^i(0), i \leq 0\}$ or to $\Gamma_2 := \{\sigma^i(0), i > 0\}$.

By Corollary 4.1 the following map is bijective:

$$\begin{aligned}
(K \setminus \delta)/\sim &\to \hat{A}_1(q)(\text{weight}, \text{linear}),\\
\mathcal{O}(\lambda) &\to [L(\mathcal{O}(\lambda)) = A_1(q)/A_1(q)(H-\lambda)]\\
\Gamma_1 &\to [L(\Gamma_1) = A_1(q)/A_1(q)(H, X)],\\
\Gamma_2 &\to [L(\Gamma_2) = A_1(q)/A_1(q)(H-1, Y)].
\end{aligned} \tag{6.5}$$

Since $F_{[1],(H-\delta)} = E((-,+), \mathcal{O}(\lambda)) = A_1(q)/A_1(q)(H-\delta) \simeq K[X, X^{-1}]$, the Laurent polynomial ring, by Corollary 4.3 the map

$$\begin{aligned}
K^* &\to \hat{A}_1(q)(-,+),\\
\mu &\to [L((-,+),\lambda) = A_1(q)/A_1(q)(H-\delta, X-\mu, Y-\mu^{-1}\delta)],
\end{aligned} \tag{6.6}$$

is bijective. It is clear that each module in (6.6) is 1-dimensional.

COROLLARY 6.2. *If q is not a root of 1, then $\hat{A}_1(q)$ (weight) is described by* (6.5) *and* (6.6).

$\hat{A}_1(q)$(D-torsionfree), $D = K[H]$. We keep the notation of Section 5. The localization B of $A_1(q)$ at $S = K[H] \setminus \{0\}$ is the skew Laurent polynomial ring:

$$B = K(H)[X, X^{-1}; \sigma], \qquad \sigma(H) = q^{-1}(H-1),$$

with coefficients in the field $K(H)$ of rational functions.

An element $f = v_{-m}\beta_{-m} + \cdots + \beta_0 \in A_1(q)$ of length $m > 0$, all $\beta_i \in K[H]$, is l-normal if and only if the following condition holds:

(l-norm) every scalar $\sigma^i(0) = (q^i - 1)/(q-1)$, $i \geq 0$, is not a root of β_0 (i.e., $\beta_0 < a = H$) and if λ and $\mu = \sigma^j(\lambda) = q^j\lambda + (q^j-1)/(q-1)$ are roots of polynomials β_0 and β_{-m}, respectively, then $j > 0$ (i.e., $\beta_0 < \beta_{-m}$).

COROLLARY 6.3. *Suppose q is not a root of* 1 *and let M be a simple $K[H]$-torsionfree $A_1(q)$-module. Then $M \simeq A_1(q)/A_1(q) \cap Bf$ for an irreducible element $f = v_{-m}\beta_{-m} + \cdots + \beta_0$ of B (all $\beta_i \in K[H]$, $\beta_0 \neq 0$, $\beta_{-m} \neq 0$, $m > 0$) such that* (l-norm) *holds and all polynomials β_i but exactly one have the root $(1-q)^{-1}$. Two of these $A_1(q)$-modules are isomorphic if and only if the corresponding f are similar in B.*

Proof. It follows from Theorem 5.13 and Proposition 5.16.

7. SIMPLE MODULES OF THE VIRASORO ALGEBRA

Let K be an algebraically closed field of characteristic zero. Let V be the factor algebra of the universal enveloping algebra of the Virasoro algebra from Section 1 (we keep the notation introduced there). We shall describe $\hat{V}$.

The automorphism σ "acts" on the set of maximal ideals $K \equiv \text{Specm } D$ $(\lambda \to (H - \lambda))$, where $D = K[H]$ as $\sigma : \lambda \to \lambda + 1$. Thus an orbit equals $\lambda + \mathbf{Z}$, for some $\lambda \in K$ and is linear, hence V (weight) $= \hat{V}$(weight, linear). The generalized Weyl algebra (4.1)

$$A = D(\sigma ; a = v_{-1}v_1 = H(H-1)), \qquad \sigma(H) = H - 1,$$

is a subalgebra of V. The orbit $\mathbf{Z}$ is the unique degenerate orbit and it contains two "marked" ideals 0 and 1 (i.e., (H) and $(H-1)$). Applying Lemma 4.5 we see that $\Gamma_1 = \{1\}$ and $\Gamma_2 = \mathbf{Z} \setminus \{1\}$ are the equivalence classes of the orbit $\mathbf{Z}$ (in the sense of Lemma 3.1) and any other orbit is an equivalence class.

COROLLARY 7.1. *The map*

$$K/\sim \to \hat{V}(weight), \qquad \Gamma \to [L(\Gamma)],$$

is bijective with inverse $[L] \to \text{Supp } L$, *where*

1. $\Gamma_1 = \{1\}$, $L(\Gamma_1) = V/(D(H-1) + \sum_{i \neq 0 \in \mathbf{Z}} Dv_i) \simeq K$;
2. $\Gamma_2 = \mathbf{Z} \setminus \{1\}$, $L(\Gamma_2) = V/D(H, v_1)$;
3. $\Gamma = \lambda + \mathbf{Z} (\lambda \notin \mathbf{Z})$, $L(\Gamma) = V/V(H - \lambda)$.

Proof. It follows from Theorem 3.2 and Lemma 4.5. ∎

The localization $B = S^{-1}V$ of V at $S = D \setminus \{0\}$ is the skew Laurent polynomial ring

$$B = K(H)[X, X^{-1}; \sigma], \qquad \sigma(H) = H - 1,$$

with coefficients in the field $K(H)$ of rational functions. Since there are no cyclic orbits the following corollary follows from Lemma 5.1 and Theorem 5.13.

COROLLARY 7.2. *Let M be a simple $K[H]$-torsionfree V-module. Then $M \simeq V/V \cap Bf$ for an irreducible element $f = v_{-m}\beta_{-m} + \cdots + \beta_0$ of B (all $\beta_i \in K[H]$, $\beta_0 \neq 0$, $\beta_{-m} \neq 0$, $m > 0$) such that all roots of β_0 and all the differences $\lambda - \mu$ of the roots λ of β_0 and μ of β_{-m} are not nonnegative integers. Two of these V-modules are isomorphic if and only if the corresponding f are similar in B.*

For a polynomial $\alpha \in K[H]$ set $[\alpha] := \bigcup\{\lambda + \mathbf{Z}\}$ where λ runs through all roots of α.

LEMMA 7.3. *Let $p = \alpha v_1 + \beta$ or $p = \alpha v_{-1} + \beta$ where $\alpha \neq 0, \beta \neq 0 \in K[H]$ be such that β has no roots in $\mathbf{Z}$ and $[\alpha] \cap [\beta] = \emptyset$. Then V/Vp is a simple $K[H]$-torsionfree V-module.*

Proof. The element p is l-normal and irreducible in B, so by Theorem A the V-module $V/V \cap Bp$ is simple $K[H]$-torsionfree. By Lemma 5.7 and (5.6) $V/V \cap Bp = V/Vp$. ∎

The case of char $K = p > 0$ is left to the reader. Since $\sigma^p = 1$, by Theorem 5.14 every simple V-module is weight and finite dimensional and the set $\hat{V}$ can be easily described.

8. THE SIMPLE R-MODULES (R IS N-GRADED)

Let $R = \bigoplus_{i \geq 0} R_i$ be an $\mathbf{N}$-graded ring as at the end of Section 3 with *Dedekind* $R_0 = D$ and $c(i, j) \neq 0$ for all $i, j \in \mathbf{N}$.

One of the nontrivial examples of such a ring R is a skew polynomial ring $D[X; \sigma] = \oplus DX^i$, the ring generated by D and X subject to the relation: $Xd = \sigma(d)X$ for all $d \in D$.

The strategy for obtaining a classification of simple modules almost the same as for the $\mathbf{Z}$-graded case up to some not entirely innocent modifications. In fact Theorem 5.14 is not true for the $\mathbf{N}$-graded R and we construct a counterexample (Lemma 8.2).

Denote by k the field of fractions of D, i.e., $k = S^{-1}D$ where $S = D \setminus \{0\}$. The localization $B = S^{-1}R$ of the ring R at the multiplicatively closed set S is isomorphic to the skew polynomial ring

$$B \simeq k[X; \sigma] \tag{8.0}$$

with coefficients in the field k. It is a Euclidean ring and, hence, a principal left and right ideal domain. From the same arguments as used in (5.1) and (5.2) it follows that

$$\hat{R} = \hat{R}(D - \text{torsion}) \sqcup \hat{R}(D - \text{torsionfree}) \tag{8.1}$$

and

$$\hat{R}(D - \text{torsion}) = \hat{R}(\text{weight}). \tag{8.2}$$

Theorems 3.5, 3.7 yield the classification for $\hat{R}(D\text{-torsion})$. It remains to describe $\hat{R}(D\text{-torsionfree})$.

$\hat{R}$(D-torsionfree). We use the notation as in Section 5. It is easy to see that Lemma 5.1 is true for the **N**-graded ring R where $B = k[X;\sigma]$. So, *the canonical map*

$$S^{-1}: \hat{R}(D - \text{torsionfree}) \to \hat{B}(R - \text{socle}), \qquad [M] \to [S^{-1}M], \quad (8.3)$$

is bijective with inverse $\text{Soc}:[N] \to [\text{Soc}_R(N)]$. *Each simple D-torsionfree R-module has the form* $M_{\mathbf{m}} := R/R \cap B\mathbf{m}$ *for some left maximal ideal* $\mathbf{m}$ *of* B *and* $M_{(\mathbf{m}=Bb)} \simeq M_{(\mathbf{n}=Bc)}$ *are isomorphic if and only if* $B/\mathbf{m} \simeq B/\mathbf{n}$ *as B-modules, i.e., the corresponding irreducible elements b and c are similar.*

THEOREM 8.1. *Let R be* **N**-*graded with Dedekind D and* $c(i,j) \neq 0$ *for all* $i,j \in \mathbf{N}$, *and let* $f = a_0 + \cdots + a_n v_n \in R$ *be irreducible in* B, $a_i \in D$. *Then* $R/R \cap Bf$ *is a D-torsionfree simple R-module* $(= \text{Soc}_R(B/Bf))$ *if and only if*

(CO) $R = R\theta(O) + R \cap Bf$ *for all cyclic orbit* O, *where* $\theta(O) = \prod_{\mathbf{p} \in O} \mathbf{p}$;

(LO) $R = \mathbf{p} + R^{(+)} + R \cap Bf$ *for all maximal linear ideals* $\mathbf{p}$ (*such that* $a_0 \in \mathbf{p}$), *where* $R^{(+)} = \bigoplus_{i>0} R_i$.

Up to isomorphism every D-torsionfree simple R-module arises in this way, and from an f which is unique up to similarity.

Remark. Since $\mathbf{p} + R^{(+)} + R \cap Bf \supseteq \mathbf{p} + R^{(+)} + Rf \supseteq \mathbf{p} + Da_0 + R^{(+)}$, (LO) holds for all $\mathbf{p}$ such that $a_0 \notin \mathbf{p}$.

The *proof* is almost the same as in the **Z**-graded case (Theorems 5.8, 5.13). Using the localization $(\ )_O$ as in the proof of Theorem 5.13 we may suppose (if needed) that D is a principal ideal domain.

($\Rightarrow$) By (5.13) $M_{f\theta^{-1}} \subseteq M_f$, $\theta = \theta_O$, and

$$M_f/M_{f\theta^{-1}} \simeq R/(R\theta + R \cap Bf);$$

thus (CO) holds, where $M_f := M_{Bf}$.

Suppose that (LO) does not hold for some maximal linear ideal $\mathbf{p}$ of D. Since $\mathbf{p} + R^{(+)}$ is a maximal left ideal of R, we have $R \cap Bf \subseteq \mathbf{p} + R^{(+)}$ and M_f maps naturally onto the simple weight module $D/\mathbf{p} = R/(\mathbf{p} + R^{(+)})$, a contradiction.

($\Leftarrow$) It is enough to show that $M_f = M_{f\alpha^{-1}}$ for all $\alpha \neq 0 \in D$. Fix α; since $M_{f\alpha^{-1}} \supseteq M_{f\beta^{-1}}$ for any β such that $\alpha \mid \beta$, we may suppose that $\alpha = p_1 \cdots p_s \theta_1 \cdots \theta_t$ where all p_i are irreducible and belong to linear orbits and $\theta_i = \theta_{O_i}$ for some cyclic orbit O_i. The element $\theta = \theta_1 \cdots \theta_t \in R$ is normal, thus $\omega_\theta : R \to R$, $r \to \theta r \theta^{-1}$ is an automorphism of R. It follows from (CO) and (5.15) that $M_f =^{\omega_\theta^{-1}} M_f$. Applying ω_θ to

$$M_f =^{\omega_\theta^{-1}} M_f \supseteq M_{f\alpha^{-1}} =^{\omega_\theta^{-1}} M_{f\beta^{-1}}, \qquad \beta = p_1 \cdots p_s,$$

we see that one may suppose that $\alpha = \beta$.

Since $N := M_f/M_{f\alpha^{-1}} \simeq R/(R\alpha + R \cap Bf)$ there is a natural epimorphism $R/R\alpha \to N$. Suppose that $N \neq 0$. Then N maps onto a simple weight R-module L with support from a linear orbit $\mathcal{O}(p_i)$ for some i, i.e., $L = D/\mathbf{p} = R/(\mathbf{p} + R^{(+)})$ for some maximal ideal $\mathbf{p} \in \mathcal{O}(p_i)$. The annihilator $\operatorname{ann}_R L = \mathbf{p} + R^{(+)}$; thus $\mathbf{p} + R^{(+)} \subseteq R\alpha + R \cap Bf \neq R$ but that contradicts (LO). ∎

COUNTEREXAMPLE. Let $\{p_i, i \in \mathbf{N}\}$ be the set of all simple (irreducible) numbers of $\mathbf{N}$ and let $n : \mathbf{N} \to \mathbf{N}$, $i \to n(i)$ be a function with $n(i+j) > n(i) + n(j)$ for all $i, j \in \mathbf{N}$. Denote by R the subring of the polynomial ring $\mathbf{Q}[X]$ with rational coefficients generated by $\{v_i = q_i^{-1}x^i, i \geq 0\}$ where $q_i = (p_1 \cdots p_i)^{n(i)}$, $i \geq 1$, $q_0 = 1$. Then $R = \bigoplus_{i \geq 0} (R_i = \mathbf{Z}v_i)$ is $\mathbf{N}$-graded with $R_0 = \mathbf{Z}$, $\sigma = 1$ and

$$c(i,j) = q_{i+j}/q_i q_j \qquad \text{for all } i, j \in \mathbf{N}.$$

LEMMA 8.2. 1. *Let R be $\mathbf{N}$-graded such that there are infinitely many cyclic ideals which do not belong to $[c] := \bigcup [c(i,j)]$ where i, j run through $\mathbf{N}$. Then $\hat{R}(D$-torsionfree$)$ is empty.*

2. *Let R be as in the counterexample. Then there are infinitely many cyclic orbits but the set $\hat{R}(D$-torsionfree$)$ is nonempty.*

Proof. 1. Let S resp. T be the multiplicatively closed subset of D resp. R generated by $\sigma^k(c(i,j))$, $i, j \in \mathbf{N}$, $k \in \mathbf{Z}$ resp. and v_i, $i \in \mathbf{N}$. Then the localization $B = T^{-1}R$ of R at T is

$$B = S^{-1}D[X, X^{-1}; \sigma],$$

the skew Laurent polynomial ring with coefficients in the Dedekind ring $S^{-1}D$. The ring B is $\mathbf{Z}$-graded and $S^{-1}D$ has infinitely many cyclic maximal ideals, and by Theorem 5.14 $\hat{B}(S^{-1}D$-torsionfree$)$ is empty. It follows from the existence of the map

$$\hat{R}(D - \text{torsionfree}) \to \hat{B}(S^{-1}D - \text{torsionfree}), \qquad [M] \to [T^{-1}M],$$

that $\hat{R}(D$-torsionfree$)$ is empty too.

2. Let us show that the module $M = M_{B(1-X)} := R/R \cap B(1-X)$ is simple where $B = \mathbf{Q}[X]$, $X = v_1$ (if so, then $[M] \in \hat{R}(D$-torsionfree$)$).

The automorphism $\sigma = 1$ is trivial, so each orbit is cyclic and it is enough to show that (CO) holds (Theorem 8.1). By the choice of the function $n(i)$, p_i divides $c(j,k)$ if j or k is larger or equal to i. Thus $1 - X^{2^i} \in 1 + Rp_i$; on the other hand, $1 - X^{2^i} \in R(1-X)$, so $R = Rp_i + R(1-X)$ and (CO) holds. ∎

9. THE SIMPLE MODULES OF CERTAIN SKEW (LAURENT) POLYNOMIAL RINGS

Let $T = D[X; \sigma]$ (resp. $R = D[X, X^{-1}; \sigma]$) be a skew (resp. Laurent) polynomial ring over the *Dedekind* ring D. In this section the simple T- and R-modules are described. By (5.1), (5.2)

$$\hat{R} = \hat{R}(\text{weight}) \sqcup \hat{R}(D - \text{torsionfree})$$

and by (3.1)

$$\hat{R}(\text{weight}) = \hat{R}(\text{weight}, \text{linear}) \sqcup \hat{R}(\text{weight}, \text{cyclic}).$$

The same is true for T.

$\hat{R}$ (weight, linear). Since $c(i, j) = 1$ for all $i, j \in \mathbf{Z}$, any equivalence class in Specm.lin(D) is an orbit and vice versa. By Theorem 3.2 *the map*

$$\text{Lin} \to \hat{R}(\text{weight}, \text{linear}), \qquad \mathcal{O} \to [R/R\mathbf{p}], \qquad \mathbf{p} \in \mathcal{O}, \tag{9.1}$$

is bijective with inverse, $[L] \to \operatorname{Supp} L$.

$\hat{R}$ (weight, cyclic). Let $\mathcal{O}$ be a cyclic orbit, $n = |\mathcal{O}|$, and $\mathbf{p} \in \mathcal{O}$. Then

$$R_{[n], \mathbf{p}} = F_{[n], \mathbf{p}} = D/\mathbf{p}[x, x^{-1}; \sigma^n], \qquad x = X^n + R\mathbf{p},$$

is a skew Laurent polynomial ring, so by (3.6)

$$\hat{R}(\text{weight}, \text{cyclic}) = \hat{R}(-, +). \tag{9.2}$$

Specm.$(-, +)(D)$ is equal to the set CYC of all cyclic maximal ideals of D and two cyclic maximal ideals are equivalent, $\mathbf{p} \sim_{(-,+)} \mathbf{q}$, if and only if they belong to the same (cyclic) orbit; thus

$$\begin{aligned} \text{Specm.}(-, +)(D)/\sim_{(-,+)} &= CYC/\sim_{(-,+)} \\ &= Cyc \ \ (\text{the set of cyclic orbits}). \end{aligned}$$

Since any equivalence class Γ is a cyclic orbit $\Gamma = \mathcal{O}$ and vice versa,

$$E((-, +), \Gamma = \mathcal{O}) \simeq D/\mathbf{p}[x, x^{-1}; \sigma^n], \qquad n = |\mathcal{O}|, \qquad \mathbf{p} \in \mathcal{O},$$

a skew Laurent polynomial ring. By (3.9)

$$\hat{R}(-, +) = \sqcup \left\{ \hat{R}((-, +), \mathcal{O}) \mid \mathcal{O} \in Cyc \right\} \tag{9.3}$$

and by (3.12), (3.13) there is the bijective correspondence

$$\hat{R}((-, +), \mathcal{O}) \leftrightarrow \hat{E}((-, +), \mathcal{O}), \qquad [L(V)] \leftrightarrow [V], \tag{9.4}$$

$$L(V) = R \otimes_{R_{[n]}} V = \sum_{i=0}^{n-1} X^i \otimes_{R_{[n]}} V, \qquad n = |\mathcal{O}|. \tag{9.5}$$

LEMMA 9.1. *The map*

$$\mathrm{Lin} \sqcup \left(\sqcup_{\mathcal{O} \in Cyc} \hat{E}((-,+),\mathcal{O}) \right) \to \hat{R}(\text{weight})$$

defined by (9.1)–(9.5) *is bijective*.

$\hat{R}(D$-torsionfree). Since $c(-1, 1) = 1$, an element $b = X^{-m}\beta_{-m} + \cdots + \beta_0 \in R$, all $\beta_i \in D$, $\beta_{-m} \neq 0$, $\beta_0 \neq 0$, $m > 0$, is l-normal if and only if $\beta_0 < \beta_{-m}$. The next lemma follows from Theorem A.

LEMMA 9.2. *Let b be l-normal irreducible in B and assume that* (CO) *holds. Then $R/R \cap Bb$ is a D-torsionfree simple R-module. Up to isomorphism every D-torsionfree simple R-module arises in this way, and from a b which is unique up to similarity.*

Lemmas 9.1 and 9.2 describe $\hat{R}$.

We keep the notation of Section 3. By (3.14) and (3.15), the map

$$\mathrm{Specm}(D) \to \hat{T}\,(\text{weight, linear}) \sqcup \hat{T}(\pm 0), \qquad \mathbf{p} \to [D/\mathbf{p}], \quad (9.6)$$

is bijective.

$\hat{\mathbf{T}}(+,+)$. Specm.$(+,+)(D)$ equals to the set CYC and two cyclic maximal ideals are equivalent, $\mathbf{p} \sim_{(+,+)} \mathbf{q}$, iff they belong to the same (cyclic) orbit; thus

$$\mathrm{Specm.}(+,+)(D)/\sim_{(+,+)} = Cyc.$$

Since any equivalence class Γ is a cyclic orbit $\Gamma = \mathcal{O}$ and vice versa,

$$E((+,+),\Gamma = \mathcal{O}) \simeq D/\mathbf{p}[x, x^{-1}; \sigma^n], \qquad n = |\mathcal{O}|, \mathbf{p} \in \mathcal{O},$$

a skew Laurent polynomial ring. By (3.9′),

$$\hat{T}(+,+) = \sqcup \left\{ \hat{T}((+,+),\mathcal{O}) \mid \mathcal{O} \in Cyc \right\} \quad (9.7)$$

and by (3.12′), (3.13′) there is the bijective correspondence

$$\hat{T}((+,+),\mathcal{O}) \leftrightarrow \hat{E}((+,+),\mathcal{O}), \qquad [L(V)] \leftrightarrow [V], \quad (9.8)$$

$$L(V) = T \otimes_{T_{[n]}} V = \sum_{i=0}^{n-1} X^i \otimes_{T_{[n]}} V, \qquad n = |\mathcal{O}|. \quad (9.9)$$

LEMMA 9.3. *The map*

$$\mathrm{Specm}(D) \sqcup \left(\sqcup_{\mathcal{O} \in Cyc} \hat{E}((+,+),\mathcal{O}) \right) \to \hat{T}(\textit{weight}),$$

defined by (9.6)–(9.9) *is bijective*.

Lemma 9.3 and Theorem 8.1 decribe the simple T-modules.

In general Theorem 5.14 for **N**-graded rings is not true, but for skew polynomial rings it is.

LEMMA 9.4. *Let $T = D[X, \sigma]$ be as above. The $\hat{T}$(D-torsionfree) is empty if and only if there are infinitely many cyclic orbits.*

Proof. ($\Leftarrow$) Let M be a simple D-torsionfree T-module. The element X of T is normal, so the map X_M: $M \to M$, $m \to Xm$ is bijective and, hence, M is a simple D-torsionfree R-module. Now the result follows from Theorem 5.14.

($\Rightarrow$) Let α be as in the proof of Theorem 5.14. Then the element $f = 1 + \alpha X$ is irreducible in B, and, by Theorem 8.1, the T-module $T/T \cap Bf$ is simple D-torsionfree. ∎

REFERENCES

[AP1] D. Arnal and G. Pinczon, On algebraically irreducible representations of the Lie algebra $sl(2)$, *J. Math. Phys.* **15** (1974), 350–359.

[AP2] D. Arnal and G. Pinczon, Idéaux á gauche dans les quotients simple de l'algébre enveloppante de sl(2), *Bull. Soc. Math. France* **101** (1973), 381–395.

[Bam] K. S. Bamba, Sur le idéaux maximaux de l'algébre de Weyl A_1, *C. R. Acad. Sci. Paris Sér. I Math.* **283** (1976), 71–74.

[Bav1] V. V. Bavula, The finite-dimensionality of Ext^n's and Tor_n's of simple modules over a class of algebras, *Funktsional. Anal. i Prilozhen.* **25**, No. 3 (1991), 80–82.

[Bav2] V. V. Bavula, The simple $D[X, Y; \sigma, a]$-modules, *Ukrainian Math. J.* **44** (1992), 1628–1644.

[Bav3] V. V. Bavula, Generalized Weyl algebras, kernel and tensor-simple algebras, their simple modules, representations of algebras, *in* "CMS Conference Proceedings" (V. Dlab and H. Lenzing, Eds.), Vol. 14, pp. 83–106, 1993. [Sixth International Conference, August 19–22, 1992]

[Bav4] V. V. Bavula, Generalized Weyl algebras and their reprsentations, *Algebra i Analiz* **4**, No. 1 (1992), 75–97. [English trans. in *St. Petersburg Math. J* **4,** No. 1 (1993), 71–92]

[Bav5] V. V. Bavula, Tensor homological minimal algebras, global dimension of the tensor product of algebras and of generalized Weyl algebras, Bielefeld, preprint, 1994, *Bull. Sci. Math.*, **120** (1996), 293–335.

[Bav6] V. V. Bavula, Global dimension of generalized Weyl algebras, *in* "CMS Conference Proceedings" (R. Bautista, R. Martinez-Villa, and J. A. de la Pena, Eds.), Vol. **18,** pp. 81–107, 1996. [Proceedings of the 7th Int. Conf. on Represent. of Algebras, August 22–26, 1994]

[Bl 1] R. E. Block, Classification of the irreducible representations of $sl(2, \mathrm{C})$, *Bull. Amer. Math. Soc. (N.S.)* **1** (1979), 247–250.

[Bl 2] R. E. Block, The irreducible representations of the Weyl algebra A_1, *in* "Séminaire d'Algébre Paul Dubreil (Proceedings, Paris, 1977–1978)" (M. P. Malliavin, Ed.), Lecture Notes in Mathematics, Vol. 740, pp. 69–79, Springer-Verlag, Berlin/New York, 1979.

[Bl 3] R. E. Block, The irreducible representations of the Lie algebra $sl(2)$ and of the Weyl algebra, *Adv. Math.* **39** (1981), 69–110.
[Di 1] J. Dixmier, Represéntations irrèductibles des algébres de Lie nilpotentes, *An. Acad. Brasil. Ciênc.* **35** (1963), 491–519.
[Di 2] J. Dixmier, Sur les algébres de Weyl, *Bull. Soc. Math. France* **96** (1968), 209–242.
[EL] A. Van den Essen and A. Levelt, An explicit description of all simple $K[[X]][\partial]$-modules, *Contemp. Math.* **130** (1992), 121–131.
[Jac] N. Jacobson, "The Theory of Rings," Amer. Math. Soc., Providence, RI, 1943.
[Jor 1] D. A. Jordan, Krull and global dimension of certain iterated skew polynomial rings, *Contemp. Math.* **130** (1992), 201–213.
[Jor 2] D. A. Jordan, Primitivity in skew Laurent polynomial rings and related rings, *Math. Z.* **213** (1993), 353–371.
[Hod 1] T. J. Hodges, Noncommutative deformation of type-A Kleinian singularities, *J. Algebra* **161,** No. 2 (1993), 271–290.
[KS] E. E. Kirkman and L. W. Small, q-analogs of harmonic oscillators and related rings, Preprint, Wake Forest Univ. and Univ. of California, 1992.
[Le] F. W. Lemire, Existence of weight space decompositions for irreducible representations of simple Lie algebras, *Canad. Math. Bull.* **14** (1971), 113–115.
[Ma] M.-P. Malliavin, L'algèbre d'Heisenberg quantique, *C. R. Acad. Sci. Paris Sér. I Math.* **317** (1993), 1099–1102.
[MR] J. C. McConnell and J. C. Robson, Homomorphism and extensions of modules over certain polynomial rings, *J. Algebra* **26** (1973), 319–342.
[MR 1] J. C. McConnell and J. C. Robson, "Non-commutative Noethernian rings," Chichester, Wiley, 1987.
[NVO1] C. Naśtaśescu and F. Van Oystaeyen, "Graded Ring Theory," Library of Math., North-Holland, Amsterdam, 1982.
[NVO2] C. Naśtaśescu and F. Van Oystaeyen, "Graded Rings and Modules," Lecture Notes in Mathematics, Vol. 758, Springer-Verlag, Berlin.
[Ros] A. L. Rosenberg, The unitary irreducible representation of the quantum Heisenberg algebra, *Comm. Math. Phys.* **144** (1992), 41–51.
[Sm] S. P. Smith, A class of algebras similar to the enveloping algebra of $sl(2)$, *Trans. Amer. Math. Soc.* **322** (1990), 285–314.
[Sm 1] S. P. Smith, Quantum qroups: An introduction and survey for ring theoretists, *in* "Noncommutative Rings" (S. Montgomery and L. W. Small, Eds.), Publ. MSRI 24, pp. 131–178, Springer-Verlag, Berlin, 1992.
[Za] C. Zachos, Elementary paradigms of quantum algebras, *Contemp. Math.* **134** (1992), 351–377.