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Krull Dimension of Generalized Weyl Algebras and Iterated Skew Polynomial Rings: Commutative Coefficients

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Many rings that have enjoyed growing interest in recent years, e.g., quantum enveloping algebras, quantum matrices, certain Witten-algebras, . . . , can be presented as generalized Weyl algebras. In the paper we develop techniques for calculating dimensions, here mainly the Krull dimension in the sense of Rentschler–Gabriel, of such generalized Weyl algebras and specify the results for some popular algebras including those mentioned above. © 1998 Academic Press

1. INTRODUCTION

Unless otherwise stated, by module we mean left module. We write $M < N$ to indicate that N is a proper (left) submodule of M. We use the term "Krull dimension" in the sense of [RG]; for a survey of the basic properties of this dimension we refer to [Go-Ro], [MR], [NVO1]. For a ring R, resp. an R-module M we let $\mathscr{K}(R)$, resp. $\mathscr{K}_R(M) = \mathscr{K}(M)$, denote the

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Krull dimension of the ring R, resp. of the module M. The centre of R will be denoted by $Z(R)$. To an automorphism σ of the ring R and an $a \in Z(R)$ we may associate a *generalized Weyl algebra* (GWA), $T = R(\sigma, a)$ (of degree 1) being the ring generated by R and indeterminates X and Y subject to the relations [Bav1]

$$X\alpha = \sigma(\alpha)X \quad \text{and}$$
$$Y\alpha = \sigma^{-1}(\alpha)Y, \text{ for all } \alpha \in R, YX = a, \text{ and } XY = \sigma(a).$$

Throughout the paper *T denotes a generalized Weyl algebra* $T = R(\sigma, a)$. In this paper we shall study the GWA defined over a commutative ring of coefficients. The GWA are by definition generalized crossed products (with respect to a cocycle for $\mathbf{Z}$ taking nonzero values that need however not be units in $Z(R)$). Many interesting algebras are GWA, e.g., the first Weyl algebra A_1, the universal enveloping algebra $U\,sl(2)$, and quantum deformations of it; Witten's, Woronowicz's deformations, the quantum Heisenberg algebra; etc. (see also Section 4).

A general fact concerning $\mathbf{Z}$-graded rings [NVO2], yields that when R is left Noetherian so is every $R(\sigma, a)$ as defined above (see [Bav2, Proposition 1.3]). For $a = 1$, $R(\sigma, 1)$ is the skew Laurent polynomials ring $S = R[X, X^{-1}; \sigma]$. In case R is left Noetherian $\mathscr{K}(R[X, X^{-1}; \sigma])$ is either $\mathscr{K}(R)$ or $\mathscr{K}(R) + 1$.

T. Hodges [Ho1] obtained (see also [MR, 6.9.22]):

THEOREM 1.1. *Let the commutative Noetherian ring R have finite Krull dimension*; *then*

$$\mathscr{K}(R[X, X^{-1}; \sigma]) = \sup\{\mathscr{K}(R), \operatorname{ht}\mathbf{q} + 1 \mid \mathbf{q} \text{ is a } \sigma\text{-semistable prime of } R\}$$

Here $\operatorname{ht}\mathbf{p}$ is the *height* of an ideal $\mathbf{p}$ and $\mathbf{p}$ is *σ-semistable*, if $\sigma^n(\mathbf{p}) = \mathbf{p}$ for some integer n. If there is no such n, the ideal is called *σ-unstable*.

In case of a left Noetherian coefficient ring a more general formula for $\mathscr{K}(S)$ was established by K. Goodearl and T. Lenagan [GL].

The main aim of the paper is to extend the foregoing to:

THEOREM 1.2. *Let R be a commutative Noetherian ring with* $\mathscr{K}(R) < \infty$ *and* $T = R(\sigma, a)$ *be a generalized Weyl algebra*. *Then*

$$\begin{aligned}\mathscr{K}(T) = \sup\{&\mathscr{K}(R), \operatorname{ht}\mathbf{p} + 1, \operatorname{ht}\mathbf{q} + 1 \mid \mathbf{p} \text{ is a } \sigma\text{-unstable prime ideal of } R \\ &\text{for which there exist infinitely many integers } i \text{ with } a \in \sigma^i(\mathbf{p}); \\ &\mathbf{q} \text{ is a } \sigma\text{-semistable prime ideal of } R\}.\end{aligned}$$

Note. The primes $\mathbf{p}$ and $\mathbf{q}$ in Theorem 1.2 may be supposed to be maximal of height $\mathscr{K}(R)$. In the GWA-case the complexity of the proof is enhanced by the more complicated structure of the induced modules in this case.

THEOREM 1.3 [Bav4, Theorem 3.7]. *Let R be a commutative Noetherian ring of global dimension $n < \infty$ and let a be regular. Suppose that* $\operatorname{gld} T < \infty$. *Then*

$$\operatorname{gld} T = \sup\{\operatorname{gld} R, \operatorname{ht}\mathbf{p} + 1, \operatorname{ht}\mathbf{q} + 1 | \mathbf{p} \textit{ is a } \sigma\textit{-unstable prime ideal of } R \textit{ for which there exist distinct integers } i \textit{ and } j \textit{ with } a \in \sigma^i(\mathbf{p}) \textit{ and } a \in \sigma^j(\mathbf{p}); \mathbf{q} \textit{ is a } \sigma\textit{-semistable prime ideal of } R\}.$$

When Theorem 1.2 is combined with Theorem 1.3, one sees that

$$\mathscr{K}(T) \le \operatorname{gld} T.$$

Let K be a field and $Usl(2)$ be the K-enveloping algebra of the Lie algebra

$$sl(2) = \langle X, Y, H | [H, X] = X, [H, Y] = -Y, [X, Y] = 2H \rangle.$$

The Casimir element $C = YX + H(H+1)$ is central in $Usl(2)$). As observed in [Bav1],

$$Usl(2) \simeq K[H, C](\sigma, a = C - H(H+1)),$$
$$X \to X, Y \to Y, H \to H,$$

σ given by $H \to H - 1, C \to C$. Applying Theorem 1.2 we obtain:

COROLLARY 1.4 ([Sm] for $K = \mathbf{C}$; see also [MR, 8.6.15]). *$\mathscr{K}(Usl(2))$ is either two when* $\operatorname{char} K = 0$, *or three when* $\operatorname{char} K = p > 0$.

Proof. If $\operatorname{char} K = 0$, then all maximal ideals of $K[H, C]$ are σ-unstable. if $I \subseteq \mathbf{Z}$ is infinite, then the K-subspace of $K[H, C]$ generated by $\{\sigma^i(a) = C - (H - i)(H - i + 1), i \in I\}$ contains 1; thus Theorem 1.2 implies that $\mathscr{K}(Usl(2)) = \mathscr{K}(K[H, C]) = 2$.

When $\operatorname{char} K = p > 0$, then $\sigma^p = 1$, thus $\mathscr{K}(Usl(2)) = 3$. ∎

The Krull dimension of $K[H](\sigma, a)$ with $\sigma(H) = H - 1$ was calculated in [Bav2] and, under some restriction on σ, the case with R a commutative principal ideal domain already appears in [Jo1].

Let us give a very brief description of the results of the paper.

In Section 2 we prove that if R is left Noetherian, then

$$\mathscr{K}(R) \le \mathscr{K}(T) \le \mathscr{K}(R) + 1 \qquad \text{(Proposition 2.2)}.$$

If, in addition, $\mathcal{K}(R) < \infty$ and no (nonzero) simple T-module has finite length over R, then $\mathcal{K}(T) = \mathcal{K}(R)$ (Theorem 2.3).

In Section 3 the structure of induced modules is clarified. The sufficiency of Theorem 1.2 is proved (Theorem 3.3). "All" subfactors of the finitely generated T-modules are described (Lemma 3.5); it is one of the keystones of the proof of Theorem 1.2.

In Section 4 the main result, Theorem 1.2, is proved.

In Section 5 we apply Theorems 1.2 and 1.7 for the computation of the Krull dimension of some popular algebras: $U_q sl(2)$, the *Smith's* and *Woronowicz's* deformations, the *first* and the *second Witten's* deformations, the quantum $\mathcal{O}_{q^2}(so(K,3))$, the coordinate ring $\mathcal{O}_q(M_2(K))$ of quantum 2×2 matrices, $\mathcal{O}_q(GL_2(K))$, $\mathcal{O}_q(SL_2(K))$; and some natural localizations of these.

2. SOME GENERAL FACTS

Let $T = R(\sigma, a)$ be a GWA. The algebra $T = \bigoplus_{n \in \mathbf{Z}} T_n$ is $\mathbf{Z}$-graded, where $T_n = Rv_n$ and

$$v_n = X^n \ (n > 0), \qquad v_n = Y^{-n} \ (n < 0), v_0 = 1.$$

Moreover,

$$v_n v_m = (n,m) v_{n+m}$$

for some $(n,m) \in R$. If $n > 0$ and $m > 0$, then

$$n \geq m: (n, -m) = \sigma^n(a) \dots \sigma^{n-m+1}(a),$$
$$(-n, m) = \sigma^{-n+1}(a) \dots \sigma^{-n+m}(a),$$
$$n \leq m: (n, -m) = \sigma^n(a) \dots \sigma(a), \qquad (-n, m) = \sigma^{-n+1}(a) \dots a,$$

and $(n,m) = 1$ in other cases.

The ring isomorphism

$$R(\sigma, a) \simeq R(\sigma^{-1}, \sigma(a)), \qquad X \leftrightarrow Y, Y \leftrightarrow X, r \leftrightarrow r, r \in R,$$

is called the $\pm$*-symmetry* (or the left-right symmetry).

PROPOSITION 2.1 [Bav2]. *If R is left (right) Noetherian, then so is T.*

The skew polynomial ring $R[X; \sigma]$ is the free left R-module generated by the symbols $1, X, X^2, \dots,$ given a ring structure by the relations $Xr = \sigma(r)X$ for $r \in R$, together with the usual multiplication in R. For a module M denote by $\mathcal{L}(M)$ the lattice of its submodules.

PROPOSITION 2.2. *Let R be a left Noetherian ring. Then*

$$\mathscr{K}(R) \leq \mathscr{K}(T) \leq \mathscr{K}(R) + 1.$$

Proof. The map $\mathscr{L}({}_R R) \to \mathscr{L}({}_T T)$, $I \to TI$, is injective, thus $\mathscr{K}(R) \leq \mathscr{K}(T)$.

On the other hand, the ring $T = \bigcup_{n \geq 0} T_n$ is filtered by $T_0 = R$ and $T_n = \sum_{i=1}^{n} RY^i + R + \sum_{i=1}^{n} RX^i$ for $n \geq 1$. The associated graded ring $\operatorname{gr} T = \oplus T_n / T_{n-1}$ contains two ideals I and J generated respectively by $x = X + T_0 \in T_1/T_0$ and $y = Y + T_0 \in T_1/T_0$. It is clear that $IJ = JI = 0$ and both factor-rings $\operatorname{gr} T/I \simeq R[Y; \sigma^{-1}]$ and $\operatorname{gr} T/J \simeq R[X; \sigma]$ are skew polynomial rings. Moreover, J can be considered as a $\operatorname{gr} T/I$-module. It follows from an exact sequence of $\operatorname{gr} T$-modules $0 \to J \to \operatorname{gr} T \to R[X; \sigma] \to 0$ and [MR, 6.5.6 and 6.5.4] that $\mathscr{K}(T) \leq \mathscr{K}(\operatorname{gr} T) \leq \max\{\mathscr{K}(R[X, \sigma]), \mathscr{K}(R[Y; \sigma^{-1}])\} = \mathscr{K}(R) + 1$. ▮

THEOREM 2.3. *Let R be a left Noetherian ring of finite Krull dimension. If no (nonzero) simple T-module has finite length over R, then $\mathscr{K}(T) = \mathscr{K}(R)$.*

Proof. The map $\mathscr{L}({}_T T) \to \mathscr{L}({}_{\operatorname{gr} T} \operatorname{gr} T)$, $I \to \operatorname{gr} I$, has the property that if ${}_T J > {}_T I$, then, by [MR, 6.6.5], $\mathscr{K}(\operatorname{gr} J / \operatorname{gr} I) \geq 1$. Hence, by [MR, 6.1.17], $1 + \mathscr{K}(T) \leq \mathscr{K}(\operatorname{gr} T) \leq \mathscr{K}(R) + 1$ (see the end of the proof of Proposition 2.2), so $\mathscr{K}(T) = \mathscr{K}(R)$. ▮

Given a ring R and an automorphism σ of R, an ideal I of R is called *σ-semistable* if $\sigma^n(I) = I$ for some $n \in \mathbf{Z}$. Otherwise, I is called *σ-unstable*.

COROLLARY 2.4. *Let R and T be as above. Suppose that R is a commutative ring such that each maximal ideal $\mathbf{p}$ of R is σ-unstable and if $a \in \mathbf{p}$, then $a \notin \sigma^i(\mathbf{p})$ for all $0 \neq i \in \mathbf{Z}$. Then $\mathscr{K}(T) = \mathscr{K}(R)$.*

Proof. There are no simple T-modules of finite R-length (see [Bav4, Theorem 3.1].

EXAMPLE. The (first) *Weyl* algebra $A_1 = \langle X, \partial \mid \partial X - X\partial = 1 \rangle$ is a GWA,

$$A_1 \simeq K[H](\sigma, a = H), \qquad X \leftrightarrow X, \partial \leftrightarrow Y, \sigma(H) = H - 1,$$

with $R = K[H]$, the polynomial ring in one variable H.

If $\mathbf{p}$ is a maximal ideal of $K[H]$ which contains $a = H$, then $\mathbf{p} = (H)$. If $\operatorname{char} K = 0$, then $\sigma^i(\mathbf{p}) = (H - i) \neq (H) = \mathbf{p}$ for all $i \neq 0$; thus by Corollary 2.4, $\mathscr{K}(A_1) = \mathscr{K}(K[H]) = 1$.

Given a ring R with an automorphism σ, the *skew Laurent polynomial* ring (or the *skew Laurent extension*) $R[X, X^{-1}; \sigma]$ is the free left R-module with basis $\{1, X, X^{-1}, \dots\}$, given a ring structure by the relation $X^{\pm 1}r = \sigma^{\pm 1}(r)X^{\pm 1}$ for $r \in R$, together with the usual multiplication in R. The skew Laurent polynomial ring is the generalized Weyl algebra, $R[X, X^{-1}; \sigma] = R(\sigma, 1)$.

If the element $a \in Z(R)$ is not nilpotent, then the (multiplicative) submonoid S of $R \setminus 0$ generated by all $\sigma^i(a)$, $i \in \mathbf{Z}$, satisfies the (left and right) Ore condition in T. In other words, there exists the (left and right) localization $S^{-1}T = T_S$ of the ring T at S. Moreover, $T_S \simeq R_S[X, X^{-1}; \sigma]$ is the skew Laurent polynomial ring. A T-module M contains the *S-torsion* (or the *a-torsion*, for short) submodule $\mathrm{tor}(M) := \{m \in M | sm = 0$ for some $s \in S\}$. A T-module M is called *a-torsion* if $M = \mathrm{tor}(M)$; and *a-torsionfree*, if $\mathrm{tor}(M) = 0$. If a is a nilpotent element, then, by definition, any T-module is a-torsion.

Let M be an R-module and $\tau \in \mathrm{Aut}(R)$. The *twisted* module ${}_{\tau}M$ as an abelian group coincides with M and the action of R on M is given as follows: $rm := \tau(r)m$. We write M_τ for some twisted module ${}_{\tau^i}M$, $i \in \mathbf{Z}$.

Given an R-module M, the nonzero elements of the induced module $T \otimes_R M$ can be written uniquely in the form

$$u = v_m \otimes u_m + v_{m+1} \otimes u_{m+1} + \cdots + v_n \otimes u_n,$$

where all $u_i \in M$ and $u_m, u_n \neq 0$. The u_i are called the *coefficients* of u and u_m resp. u_n is the $(-)$-*leading* resp. $(+)$-*leading* coefficient of u. The integer m resp. n is the $(-)$-degree resp. $(+)$-degree of u, denoted by $\deg_-(u)$ resp. $\deg_+(u)$. The non-negative integer $n - m$ is the length of u, denoted by $l(u)$. The element 0 is defined to have $\deg_{\pm}(0) = \mp\infty$, leading coefficients 0 and length $-\infty$.

There are doubly infinite filtrations

$$\cdots \subseteq U_{-1} \subseteq U_0 \subseteq U_1 \subseteq \cdots$$

and

$$\cdots \supseteq V_{-1} \supseteq V_0 \supseteq V_1 \supseteq \cdots$$

on $T \otimes_R M$ given by the R-submodules

$$U_n = \{u \in T \otimes_R M | \deg_+(u) \leq n\},$$

and

$$V_n = \{v \in T \otimes_R M | \deg_-(v) \geq n\},$$

$$U_n/U_{n-1} \simeq v_n \otimes M \simeq_{\sigma^{-n}} M \qquad \text{and} \qquad V_n/V_{n+1} \simeq v_n \otimes M \simeq_{\sigma^{-n}} M,$$

for all n.

For a T-submodule N of $T \otimes_R M$ denote by $\lambda_-(N)$ resp. $\lambda_+(N)$ the set of $(-)$-leading resp. $(+)$-leading coefficients of elements of N which have the non-positive $(-)$-degree resp. non-negative $(+)$-degree. The sets $\lambda_{\pm}(N)$ are R-submodules of M.

For $i > 0$ let N_{-i} resp. N_i be the sets of al $(-)$-leading resp. $(+)$-leading coefficients of V_{-i} resp. U_i and 0. Then

$$\cdots \geq N_{-i} \geq \cdots \geq N_{-1}, \qquad N_1 \leq \cdots \leq N_i \leq \cdots,$$

are chains of R-modules and

$$\lambda_{\pm}(N) = \bigcup_{i \geq 1} N_{\pm i}.$$

The most important case of an induced module is the ring T,

$${}_T T = \oplus_{i \in \mathbf{Z}} v_i R \simeq T \otimes_R R = \oplus_{i \in \mathbf{Z}} v_i \otimes R, \qquad v_i r \leftrightarrow v_i \otimes r,\ r \in R.$$

LEMMA 2.5. *Let R be a left Noetherian ring.*

(1) *If ${}_T I \leq T$, then $\lambda_{\pm}(I) = I_{\pm n}$ for some $n > 0$.*

(2) *If I, J are left ideals of T with $J \subseteq I$ and $\lambda_+(J) = \lambda_+(I)$, $\lambda_-(J) = \lambda_-(I)$, then ${}_R(I/J)$ is finitely generated.*

Proof. This is straightforward (see Proposition 2.1). ∎

LEMMA 2.6. *Let M be a-torsionfree R-module and N be a nonzero T-submodule of $T \otimes_R M$. Then there exists a nonzero R-submodule L of M such that $T \otimes_R L$ embeds in N.*

Proof. Choose a nonzero element $b = v_m \otimes r_m + v_{m+1} \otimes r_{m+1} + \cdots + v_n \otimes r_n \in N$ of minimal length $k = n - m$, where all $r_i \in M$ and $r_m, r_n \neq 0$. The module M is a-torsionfree, therefore the element $v_{-n} b$ is nonzero and of minimal length, so we may suppose $n = 0$. Set $I = \operatorname{ann}_R(r_0)$ $(Rr_0 \simeq R/I)$. Then $\operatorname{ann}_T(b) = TI$ and

$$T \otimes_R (R/I) \simeq T/TI \simeq Tb \leq N.$$

So it is enough to set $L = Rr_n$. ∎

Let M resp. N be an R-module with $aM = 0$ resp. $\sigma(a)N = 0$. The induced module $T \otimes_R M$ resp. $T \otimes_R N$ contains a submodule

$$\mathscr{L}(M)_+ := \sum_{i \geq 1} v_i \otimes M \qquad \text{resp. } \mathscr{L}(N)_- := \sum_{i \leq -1} v_i \otimes N.$$

Set

$$V_-(M) := T \otimes_R M / \mathscr{L}(M)_+ = \bigoplus_{i \leq 0} v_i \otimes M$$

$$\text{resp. } V_+(N) := T \otimes_R N / \mathscr{L}(N)_- = \bigoplus_{i \geq 0} v_i \otimes N.$$

A module N is a *subfactor* of a module M if there exist submodules $V \leq U$ in M such that $N = U/V$.

PROPOSITION 2.7. *Let M be a Noetherian R-module, and let I, J be T-submodules of $T \otimes_R M$ such that $I < J$. Suppose that N is a nonzero Noetherian R-module such that $T \otimes_R N$ resp. $V_+(N)$ resp. $V_-(N)$ is isomorphic to a T-submodule subfactor of J/I. Then there exists a nonzero subfactor L of $\lambda_\pm(J)/\lambda_\pm(I)$ resp. $\lambda_+(J)/\lambda_+(I)$ resp. $\lambda_-(J)/\lambda_-(I)$ such that $L \leq N_\sigma$.*

Proof. Let us consider the first case; the reader may verify the other statements. We may assume that $J/I \simeq T \otimes_R N$, in which case there exists a T-epimorphism $f: J \to T \otimes_R N$ with $\ker f = I$.

Let us consider the case $\lambda(J) := \lambda_+(J)$. The subring $S := \sum_{I \geq 0} Rv_i$ of T is the skew polynomial ring $S = R[X = v_1; \sigma]$. Set $\mathscr{M} = T \otimes_R M$ and $\mathscr{M}_{\geq n} = \sum_{i \geq n} v_i \otimes M$. Then $\mathscr{M}_{\geq n}$ is an S-module. For each n, $\mathscr{M}_{\geq n} > \mathscr{M}_{\geq n+1}$, $\mathscr{M} = \bigcup_{n \in \mathbf{Z}} \mathscr{M}_{\geq n}$. Set $\mathscr{N} = T \otimes_R N$ and define $\mathscr{N}_{\geq n}$ in the same way. For $n \geq 0$ set $J_n = J \cap \mathscr{M}_{\geq n}$. Then

$$(J \geq) J_0 \geq J_1 \geq \cdots \geq J_n \geq \cdots$$

in a descending chain of Noetherian S-submodules such that $\lambda(J) = \lambda(J_n)$ for all $n \geq 0$. In fact, M is a Noetherian R-module, so $S \otimes_R M$ is a Noetherian S-module. Since each J_n is an S-submodule of $S \otimes_R M$, J_n is a Noetherian S-module. The S-module J_0 is finitely generated, thus $f(J_0) \leq \mathscr{N}_{\geq -n}$ for some $n \geq 0$.

Set $P = Sv_n J_0 = v_n J_0$, which is an S-submodule of J_n, $\lambda(P) = \lambda(J_n) = \lambda(J)$, and $f(P) \subseteq v_n f(J_0) \subseteq v_n \mathscr{N}_{\geq -n} \subseteq \mathscr{N}_{\geq 0} = S \otimes_R N$. For $i \geq 0$ set $I_i := I \cap \mathscr{M}_i = I \cap J_i \subseteq J_i$. Then

$$(I \geq) I_0 \geq \cdots \geq I_i \geq \cdots$$

is a descending chain of Noetherian S-modules such that $\lambda(I) = \lambda(I_i)$ for all $i \geq 0$.

Set $Q = I_n \cap P$. It follows from $v_n I_0 \subseteq Q \subseteq I_n$ that

$$\lambda(I) = \lambda(v_n I_0) \subseteq \lambda(Q) \subseteq \lambda(I_n) = \lambda(I)$$

and finally $\lambda(Q) = \lambda(I)$. It is clear that the natural map

$$P/Q \to S \otimes_R N, \qquad u + Q \to f(u),$$

induced by f, is a monomorphism of S-modules. By [GL, Lemma 1.1], there exists a nonzero R-module D such that $D \leq N_\sigma$ and $S \otimes_R D$ embeds in P/Q. Applying [GL, Proposition 1.4] we obtain a nonzero subfactor L of $\lambda(P)/\lambda(Q) = \lambda(J)/\lambda(I)$ such that $L \leq D_\sigma$ and $L \leq N_\sigma$. Using the $\pm$-symmetry, the case $\lambda_-(J) \neq 0$ can be easily reduced to the case $\lambda_+(J) \neq 0$. ∎

LEMMA 2.8. *Let R be a commutative Noetherian ring and let $I \leq J$ be left ideals of T with $J/I \simeq T \otimes_R R/\mathbf{p}$ resp. $V_+(R/\mathbf{p})$ resp. $V_-(R/\mathbf{p})$, where $\mathbf{p}$ is a prime ideal of R. Then there exists $v \in \lambda_\pm(J)/\lambda_\pm(I)$ resp. $v \in \lambda_+(J)/\lambda_+(I)$ resp. $v \in \lambda_-(J)/\lambda_-(I)$ such that* $\operatorname{ann}_R(v) \leq \sigma^n(\mathbf{p})$ *for some $n = n_\pm$ resp. n_+ resp. n_-.*

Proof. Consider the first case for $(+)$, the other cases may be treated similarly. By Proposition 2.7 there exists a nonzero subfactor, say L, of $\lambda_+(J)/\lambda_+(I)$ such that L Is an R-submodule of ${}_{\sigma^{-n}}(R/\mathbf{p}) \simeq R/\sigma^n(\mathbf{p})$ for some n. Without loss of generality we may assume that L is cyclic, i.e., $L = Ru$ for some $u \in L$. It follows from $Ru \simeq R/\operatorname{ann}_R(u) \leq R/\sigma^n(\mathbf{p})$ that $\operatorname{ann}_R(u) = \sigma^n(\mathbf{p})$ ($\mathbf{p}$ is prime). Since L is a subfactor of $\lambda_+(J)/\lambda_+(I)$, $L \simeq V/U$ for some R-submodules $V \geq U \geq \lambda_+(I)$ of $\lambda_+(J)$. Consider the natural epimorphism $V/\lambda_+(I) \to V/U \simeq L$ and let v be an inverse image of $u \in L$. Then we have an epimorphism of R-modules $Rv \to Ru$, $v \to u$, thus $\operatorname{ann}_R(v) \subseteq \operatorname{ann}_R(u) = \sigma^n(\mathbf{p})$.

Using the $\pm$-symmetry the case $(-)$ is clear. ∎

A nonzero module M over an arbitrary ring R is *compressible* if for any nonzero submodule N of M there exists a monomorphism $M \to N$. If R is a commutative Noetherian ring and $\mathbf{p}$ is a prime ideal of R, then $R/\mathbf{p}$ is compressible. A nonzero module M is *α-critical*, for some ordinal α, provided $\mathscr{K}(M) = \alpha$ and $\mathscr{K}(M/N) < \alpha$ for each nonzero submodule N. A *critical* module is one which is α-critical for some α.

The next result provides examples of compressible T-modules.

LEMMA 2.9. *Let R be a ring and M be an a-torsionfree R-module. If ${}_RM$ is compressible, then $T \otimes_R M$ is a compressible T-module.*

Proof. Let V be any T-submodule of $T \otimes_R M$. By Lemma 2.6 there exists a nonzero R-submodule N of M such that $T \otimes_R N$ is a submodule of V. The module M is compressible so there exists a monomorphism $M \to N$ and the natural T-monomorphism $T \otimes_R M \to T \otimes_R N$ (T_R is flat). Thus $T \otimes_R M$ is a submodule of V; it means that $T \otimes_R M$ is compressible. ∎

COROLLARY 2.10. *If R is a commutative Noetherian ring and* $\mathbf{p} \in \operatorname{Spec} R$, *$R/\mathbf{p}$ is a-torsionfree, then $T \otimes_R R/\mathbf{p} \simeq T/T\mathbf{p}$ is compressible and critical.*

Let $S = R[X; \sigma]$ be a skew polynomial ring. Then the automorphism $\sigma \in \operatorname{Aut}(R)$ can be lifted from R to S as follows: $\sigma(X) = X$.

LEMMA 2.11. *Let R be a commutative Noetherian ring of finite Krull dimension,* $\mathbf{p} \in \operatorname{Spec} R$, $S = R[X; \sigma]$ *be a skew polynomial ring. Then $S \otimes_R R/\mathbf{p}$ is a critical S-module.*

Proof. The Krull dimension of the S-module $S \otimes_R R/\mathbf{p}$ is $n := \mathscr{K}(R/\mathbf{p}) + 1 < \infty$. If $S \otimes_R R/\mathbf{p}$ is not critical, then there exists a nonzero submodule, say N_1, such that $\mathscr{K}(S \otimes_R R/\mathbf{p}/N_1) = n$. By [GL, Lemma 1.1], N_1 contains a submodule $S \otimes_R (R/\mathbf{p})_\sigma$ which is isomorphic to a twisted S-module $(S \otimes_R R/\mathbf{p})_\sigma$. So, a fragment $S \otimes_R R/\mathbf{p} > N_1 > (S \otimes_R R/\mathbf{p})_\sigma$ can be completed to an infinite chain of submodules,

$$S \otimes_R R/\mathbf{p} > N_1 > (S \otimes_R R/\mathbf{p})_\sigma > N_2 > (S \otimes_R R/\mathbf{p})_\sigma > N_3 > (S \otimes_R R/\mathbf{p})_\sigma > \cdots$$

with $\mathscr{K}(N_i/N_{i+1}) = n = \mathscr{K}(S \otimes_R R/\mathbf{p})$, a contradiction. ∎

3. INDUCED MODULES, VERMA MODULES, AND THEIR FACTORS

Unless otherwise stated R will be a *commutative Noetherian* ring and Spec R be the set of prime ideals of R. It is easy to find a chain of ideals

$$R = I_0 > I_1 > \cdots > I_n = 0$$

such that $I_i/I_{i+1} \simeq R/\mathbf{p}_i$ with $\mathbf{p}_i \in \operatorname{Spec} R$. Applying $T \otimes_R -$ to the chain above we obtain a corresponding chain of left ideals

$$T = TI_0 > TI_1 > \cdots > TI_n = 0$$

(T_R is flat, even free) with $TI_i/TI_{i+1} \simeq T/T\mathbf{p}_i \simeq T \otimes_R R/\mathbf{p}_i$. Hence

$$\mathscr{K}(T) = \sup\{\mathscr{K}(T/T\mathbf{p})\},$$

where $\mathbf{p}$ runs through the (minimal) prime ideals of R. This explains why we focus on

$$T(\mathbf{p}) := T \otimes_R R/\mathbf{p}, \qquad \mathbf{p} \in \operatorname{Spec} R.$$

Using a natural isomorphism of T-modules

$$T(\mathbf{p}) \to T/T\mathbf{p}, \qquad v_i \otimes (r + \mathbf{p}) \to v_i r + T\mathbf{p}, r \in R,$$

we identify the module $T(\mathbf{p})$ with $T/T\mathbf{p}$.

Let M be an R-module. A prime ideal $\mathbf{p}$ of R is called an *associated prime ideal* of M if $\mathbf{p}$ is the annihilator $\operatorname{ann}_R(m) = \{r \in R | rm = 0\}$ of some $m \in M$. The set of all associated primes of M is written as $\operatorname{Ass}_R(M)$ or $\operatorname{Ass}(M)$. If M is finitely generated, then $\operatorname{Ass}(M)$ is finite.

The cyclic subgroup G of $\operatorname{Aut}(R)$, generated by σ, acts in an obvious way on Spec R. For $\mathbf{p} \in \operatorname{Spec} R$ set $\mathscr{O}(\mathbf{p}) = \{\sigma^i(\mathbf{p}), i \in \mathbf{Z}\}$ for the orbit of $\mathbf{p}$ under the action of G. The ideal $\mathbf{p}$ is called *marked* if $a \in \mathbf{p}$. Set $Mr(\mathscr{O})$ to be the set of all marked ideals from an orbit $\mathscr{O}$.

Set

$$St(\mathbf{p}) = \{i \in \mathbf{Z} | a \in \sigma^i(\mathbf{p})\},$$

$$Ho(\mathbf{p}) = \{j \in \mathbf{Z} | \sigma(a) \in \sigma^j(\mathbf{p})\} = \{i+1, i \in St(\mathbf{p})\}.$$

The elements of $St(\mathbf{p})$ and $H\mathbf{o}(\mathbf{p})$ are called *stars* and *holes*, respectively. These sets play a significant role in clarifying the module structure of $T(\mathbf{p})$ and Verma modules. Under the $(\pm)$-symmetry stars become holes and vice versa and there is the 1-1 correspondence,

$$R(\sigma, a) \simeq R(\sigma^{-1}, \sigma(a)), \qquad St(\mathbf{p}) \leftrightarrow Ho(\mathbf{p}),\, Ho(\mathbf{p}) \leftrightarrow St(\mathbf{p}).$$

So, when some statement is true for stars, the "dual" statement is true for holes.

Set $St_-(\mathbf{p}) = \{i \in St(\mathbf{p}), i < 0\}$ and $Ho_+(\mathbf{p}) = \{j \in Ho(\mathbf{p}), j > 0\}$. Let $s_-(\mathbf{p})$ be the largest element of $St_-(\mathbf{p})$ and $h_+(\mathbf{p})$ be the smallest element of $Ho_+(\mathbf{p})$ (if they exist).

The module $T(\mathbf{p}) = T \otimes_R R/\mathbf{p}$, considered as an R-module, is the direct sum of R-submodules,

$$T(\mathbf{p}) = \oplus_{i \in \mathbf{Z}} v_i \otimes R/\mathbf{p},$$

where ${}_R(v_i \otimes R/\mathbf{p}) \simeq {}_{\sigma^{-i}}(R/\mathbf{p}) \simeq R/\sigma^i(\mathbf{p})$ and $\operatorname{ann}_R(v_i \otimes R/\mathbf{p}) = \sigma^i(\mathbf{p})$. Thus $\operatorname{Ass}_R(T(\mathbf{p})) = \mathscr{O}(\mathbf{p})$.

The T-module $T(\mathbf{p})$ is $\mathbf{Z}$-graded,

$$T(\mathbf{p}) = \oplus_{i \in \mathbf{Z}} T(\mathbf{p})_i, \qquad T(\mathbf{p})_i := v_i \otimes R/\mathbf{p},$$

i.e.,

$$XT(\mathbf{p})_i \subseteq T(\mathbf{p})_{i+1} \qquad \text{and} \qquad YT(\mathbf{p})_i \subseteq T(\mathbf{p})_{i-1}.$$

Direct computation yields

$$Y(X^{n+1} \otimes R/\mathbf{p}) = a(X^n \otimes R/\mathbf{p}) \qquad \text{and}$$

$$X(Y^{m+1} \otimes R/\mathbf{p}) = \sigma(a)(Y^m \otimes R/\mathbf{p}),$$

and the maps

$$X|_{T(\mathbf{p})_i} : T(\mathbf{p})_i \to T(\mathbf{p})_{i+1}, \qquad u \to Xu,$$

$$Y|_{T(\mathbf{p})_i} : T(\mathbf{p})_i \to T(\mathbf{p})_{i-1}, \qquad u \to Yu,$$

are either zero or injective; and $X|_{T(\mathbf{p})_i}$ resp. $Y|_{T(\mathbf{p})_i}$ is zero iff i is a negative star resp. i a positive hole, i.e.,

$$\ker X|_{T(\mathbf{p})} = \oplus_{i \in St_-(\mathbf{p})} T(\mathbf{p})_i, \tag{3.1}$$

$$\ker Y|_{T(\mathbf{p})} = \oplus_{j \in Ho_+(\mathbf{p})} T(\mathbf{p})_j, \tag{3.2}$$

where $X|_{T(\mathbf{p})}: T(\mathbf{p}) \to T(\mathbf{p})$, $u \to Xu$, and $Y|_{T(\mathbf{p})}$ is defined in the same way. Now, it is clear that

$$\mathscr{L}(\mathbf{p}) := \sum_{i \leq s_-(\mathbf{p})} T(\mathbf{p})_i + \sum_{j \geq h_+(\mathbf{p})} T(\mathbf{p})_j$$

is the largest homogeneous (w.r. to the $\mathbf{Z}$-grading) submodule of $T(\mathbf{p})$ which intersects trivially $T(\mathbf{p})_0 = R/\mathbf{p}$. Denote by $\mathscr{L}(\mathbf{p})_-$ and $\mathscr{L}(\mathbf{p})_+$ the first and the second summand, respectively. When $St_-(\mathbf{p}) = \emptyset$ or $Ho_+(\mathbf{p}) = \emptyset$, we set $\mathscr{L}(\mathbf{p})_- = 0$ and $\mathscr{L}(\mathbf{p})_+ = 0$, respectively. The $\mathscr{L}(\mathbf{p})_\pm$ are T-submodules of $T(\mathbf{p})$. Set

$$L(\mathbf{p}) := T(\mathbf{p})/\mathscr{L}(\mathbf{p}) \simeq T/T(\mathbf{p}, Y^{-s_-(\mathbf{p})}, X^{h_+(\mathbf{p})}),$$

as an R-module

$$_R L(\mathbf{p}) = \bigoplus_{s_-(\mathbf{p}) < i < h_+(\mathbf{p})} v_i \otimes R/\mathbf{p}.$$

Let $M = \bigoplus_{i \in \mathbf{Z}} M_i$ be a $\mathbf{Z}$-graded module. Given a submodule N of M, then

$$N^{hom} := \bigoplus_{i \in \mathbf{Z}} N \cap M_i$$

is the largest homogeneous submodule of N.

Let N be a nonzero T-submodule of the module $L(\mathbf{p})$. Then $N^{hom} \neq 0$.

Suppose that $a \in \mathbf{p}$. A T-module

$$V_-(\mathbf{p}) := T(\mathbf{p})/\mathscr{L}(\mathbf{p})_+ = \bigoplus_{i \leq 0} T(\mathbf{p})_i$$

is called the $(-)$-*Verma* module. The dual notion

$$V_+(\sigma(\mathbf{p})) := T(\sigma(\mathbf{p}))/\mathscr{L}(\sigma(\mathbf{p}))_- = \bigoplus_{i \geq 0} T(\sigma(\mathbf{p}))_i$$

gives us the $(+)$-*Verma* module. The Verma modules can be also defined as

$$V_-(\mathbf{p}) \simeq T/T(\mathbf{p}, X) \qquad \text{and} \qquad V_+(\sigma(\mathbf{p})) \simeq T/T(\sigma(\mathbf{p}), Y).$$

Under the $(\pm)$-symmetry there is a 1-1 correspondence between the $(\pm)$-Verma modules and the $(\mp)$-Verma modules. Let

$$St_-(\mathbf{p}) = \{\cdots < i_2 < i_1 = s_-(\mathbf{p})\}, \tag{3.3}$$

$$Ho_+(\mathbf{p}) = \{h_+(\mathbf{p}) = J_1 < j_2, \dots\}. \tag{3.4}$$

Then there are two strictly descending chains (of Verma modules) of T-submodules of $T(\mathbf{p})$,

$$\cdots < V_-(i_n) < \cdots < V_-(i_1) < T(\mathbf{p}), \tag{3.5}$$

$$T(\mathbf{p}) > V_+(j_1) > \cdots > V_+(j_m) > \cdots, \tag{3.6}$$

where

$$V_-(i_n) = \oplus_{i \le i_n} T(\mathbf{p})_i \simeq V_-\big(\sigma^{i_n}(\mathbf{p})\big),$$

and

$$V_+(j_m) = \oplus_{j \ge j_m} T(\mathbf{p})_j \simeq V_+\big(\sigma^{j_m}(\mathbf{p})\big).$$

Let prime ideals $\mathbf{p}$ and $\mathbf{q} = \sigma^{-i}(\mathbf{p})$, $i > 0$, contain a. By (3.5), $V_-(\mathbf{q})$ is a submodule of $V_-(\mathbf{p})$. Set

$$L_-(q,p] := V_-(\mathbf{p})/V_-(\mathbf{q}) \simeq T/T\big(\mathbf{p}, X, Y^i\big) \simeq \oplus_{-i < j \le 0} v_j \otimes R/\mathbf{p}. \tag{3.7}$$

By (3.6), $V_+(\sigma(\mathbf{p}))$ is a submodule of $V_+(\sigma(\mathbf{q}))$. Set

$$\begin{aligned} L_+(\mathbf{q},\mathbf{p}] &:= V_+(\sigma(\mathbf{q}))/V_+(\sigma(\mathbf{p})) \simeq T/T\big(\sigma(\mathbf{q}), Y, X^i\big) \\ &\simeq \oplus_{0 \le j < i} v_j \otimes R/\sigma(\mathbf{q}). \end{aligned} \tag{3.8}$$

Then Ass $L_\pm(\mathbf{q},\mathbf{p}] = \{\sigma^j(\mathbf{p}), -i < j \le 0\}$.

It is easily seen that for a Noetherian R-module M the induced module $T \otimes_R M$ is a Noetherian T-module. Moreover,

$$\mathscr{K}({}_R M) \le \mathscr{K}(T \otimes_R M) \le \mathscr{K}({}_R M) + 1$$

(see the proof of Proposition 2.2).

LEMMA 3.1. *Let* $\mathbf{p} \in \operatorname{Spec} R$, $n = \mathscr{K}(R/\mathbf{p}) < \infty$. *Then*

(1) $n \le \mathscr{K}(T(\mathbf{p})) \le n + 1$; *if the set* $St(\mathbf{p})$ *is infinite, then* $\mathscr{K}(T(\mathbf{p})) = n + 1$;

(2) *if* $\mathbf{p}$ *and* $\mathbf{q} = \sigma^{-i}(\mathbf{p})$, $i > 0$, *contain* a, *then* $\mathscr{K}(L_\pm(\mathbf{q},\mathbf{p}]) = n$;

(3) *if* $a \in \mathbf{p}$, *then* $n \le \mathscr{K}(V) \le n + 1$, *where* $V = V_-(\mathbf{p})$ *or* $V_+(\sigma(\mathbf{p}))$. *If* $St_-(\mathbf{p})$ *resp.* $Ho_+(\mathbf{p})$ *is infinite, then* $\mathscr{K}(V_-(\mathbf{p})) = n + 1$ *resp.* $\mathscr{K}(V_+(\sigma(\mathbf{p}))) = n + 1$;

(4) *let* V *be one of the following* T*-modules*: $V_-(\mathbf{p})$ *and* $St_-(\mathbf{p}) = \emptyset$; $V_+(\mathbf{q})$ *and* $Ho_+(\mathbf{q}) = \emptyset$; $L_-(\sigma^i(\mathbf{p}),\mathbf{p}]$ *and* $i = s_-(\mathbf{p})$; $L_+(\mathbf{q},\sigma^j(\mathbf{q})]$ *and* $j = h_+(\mathbf{q})$. *If* N *is a nonzero* T*-submodule of* V, *then* $N^{hom} \ne 0$.

Proof. (2) Set $L = L_-(\mathbf{q},\mathbf{p}]$. The map

$$\mathscr{L}(R/\mathbf{p}) \to \mathscr{L}(L), \qquad I \to \sum_{-i<j\le 0} v_j \otimes I,$$

is injective, so $n \le \mathscr{K}(L)$.

On the other hand, $\mathscr{K}({}_T L) \leq \mathscr{K}({}_R L) = \mathscr{K}(\oplus_{-i<j\leq 0} R/\sigma^j(\mathbf{p})) = \max\{\mathscr{K}(R/\sigma^j(\mathbf{p}))\} = n$. By the $(\pm)$-symmetry, we conclude $\mathscr{K}(L_+(\mathbf{q}, \mathbf{p}]) = n$.

For (1) and (3), as we have seen $n \leq \mathscr{K}(T(\mathbf{p}) = T \otimes_R R/\mathbf{p}) \leq n+1$. The module $V_-(\mathbf{p})$ is a factormodule of $T(\mathbf{p})$, thus $\mathscr{K}(V_-(\mathbf{p})) \leq \mathscr{K}(T(\mathbf{p})) \leq n+1$. The map

$$\mathscr{L}(R/\mathbf{p}) \to \mathscr{L}(V_-(\mathbf{p})), \qquad I \to \sum_{i\leq 0} v_i \otimes I,$$

is injective, thus $n \leq \mathscr{K}(V_-(\mathbf{p}))$. By the $(\pm)$-symmetry $n \leq \mathscr{K}(V_+(\sigma(\mathbf{p}))) \leq n+1$.

The set $St(\mathbf{p})$ is infinite iff either $St_-(\mathbf{p})$ or $Ho_+(\mathbf{p})$ is infinite (or both). Set $V = T(\mathbf{p})$, $V_-(\mathbf{p})$, or $V_+(\sigma(\mathbf{p}))$. It follows from (3.5) and (3.6) that V has a strictly descending chain of Verma modules with factors of the type $L_\pm(\sigma^i(\mathbf{p}), \sigma^j(\mathbf{p})]$; by (2) the factors have the Krull dimension n, thus $\mathscr{K}(V) = n+1$.

(4) It follows from (3.1), (3.2) and (3.5), (3.6). ∎

Let $\mathbf{p}$ be a prime ideal of R. It is σ-semistable if $\sigma^n(\mathbf{p}) = \mathbf{p}$ for some $n \neq 0$, i.e., the orbit $\mathscr{O}(\mathbf{p})$ is finite. In case $\sigma(\mathbf{p}) = \mathbf{p}$ we say that $\mathbf{p}$ is σ-stable. Prime ideals which are not σ-stable are called σ-unstable. If $\mathbf{p}$ is a semistable prime ideal, then

$$\mathbf{p}^0 := \bigcap \sigma^i(\mathbf{p})$$

is the largest stable ideal inside $\mathbf{p}$.

LEMMA 3.2. *Let R be a commutative Noetherian ring and $\mathbf{p}$ a σ-semistable prime ideal such that $\mathbf{p}^0 = 0$.*

(1) *If a is a regular element of R, then $T = R(\sigma, a)$ is a prime ring;*

(2) $\mathscr{K}(T) = \mathscr{K}(T \otimes_R R/\mathbf{p})$.

Proof. (1) Suppose that J_1 and J_2 are ideals of T with $J_1 J_2 = 0$. Each $\lambda_\pm(J_i)$, $i = 1, 2$, is a σ-stable ideal of R. It follows from $J_1 J_2 = 0$ that $\lambda_\pm(J_1)\lambda_\pm(J_2) = 0$ and so $\lambda_+(J_i) \subseteq \mathbf{p}$ and $\lambda_-(J_j) \subseteq \mathbf{p}$ for some i, j. Then by [MR, 6.9.9 (i)], $\lambda_+(J_i) \subseteq \mathbf{p}^0 = 0$ and $\lambda_-(J_j) \subseteq \mathbf{p}^0 = 0$, thus $J_i = 0$ and $J_j = 0$ (a is regular).

(2) Let n be the minimal positive integer satisfying $\sigma^n(\mathbf{p}) = \mathbf{p}$. Consider the natural monomorphism $R = R/0 = R/\mathbf{p}^0 = R/\cap \sigma^i(\mathbf{p}) \to \prod_{i=0}^{n-1} R/\sigma^i(\mathbf{p})$. Tensoring it by $T \otimes_R -$ we have a natural T-monomorphism

$$T = T \otimes_R R \to \prod_{i=0}^{n-1} T \otimes_R R/\sigma^i(\mathbf{p}),$$

thus $\mathscr{K}(T) = \mathscr{K}(T \otimes_R R/\sigma^i(\mathbf{p}))$ for some i. Set $T_i = T \otimes_R R/\sigma^i(\mathbf{p})$.

Case (a). *The R-module $R/\mathbf{p}$ is a-torsionfree*. Then the T-submodule of T_0 generated by $X^i \otimes M$ is isomorphic to T_i, so $\mathscr{K}(T_0) \geq \mathscr{K}(T_i)$. By symmetry the opposite inequality is also true, thus $\mathscr{K}(T_0) = \mathscr{K}(T_i)$ for all i.

Case (b). *The R-module $R/\mathbf{p}$ is a-torsion*. Since the set $St(\mathbf{p})$ is infinite, by Lemma 3.1(1), we have

$$\mathscr{K}\big(T \otimes_R R/\sigma^i(\mathbf{p})\big) = \mathscr{K}\big(R/\sigma^i(\mathbf{p})\big) + 1 = \mathscr{K}(R/\mathbf{p}) + 1 = \mathscr{K}(T \otimes_R R/\mathbf{p}).$$

▮

Now we can establish one implication of Theorem 1.2 here (the easy one).

THEOREM 3.3. *Let R be a commutative Noetherian ring and $\mathbf{p}$ be a prime ideal such that the height $\operatorname{ht}\mathbf{p} = \mathscr{K}(R) < \infty$. If either $\mathbf{p}$ is σ-semistable or σ-unstable such that the set $St(\mathbf{p})$ is infinite, then $\mathscr{K}(T) = \mathscr{K}(R) + 1$.*

Proof. Let $S^{-1}R$ and $S^{-1}T$ be the localization of R and T with respect to $S := R \setminus \bigcup_{i \in \mathbf{Z}} \sigma^i(\mathbf{p})$. Since $\mathscr{K}(S^{-1}R) = \mathscr{K}(R)$ and $\mathscr{K}(S^{-1}T) \leq \mathscr{K}(T)$, it is enough to prove the theorem for the localized rings. So we may assume $R = S^{-1}R$.

Case (a). $\mathbf{p}$ is σ-semistable and $R/\mathbf{p}$ is a-torsionfree. The R-module $R/\mathbf{p}$ is a-torsionfree, so all $\{\sigma^i(a), i \in \mathbf{Z}\}$ belong to S and $T = R[X, X^{-1}; \sigma]$ is a skew Laurent polynomial ring. Now the result follows from [MR, 6.9.13].

Case (b). $\mathbf{p}$ is σ-semistable and $R/\mathbf{p}$ is a-torsion. Let M be a simple R-module, i.e., $M = R/\sigma^i(\mathbf{p})$ for some $i \in \mathbf{Z}$. Since $St(\sigma^i(\mathbf{p}))$ is an infinite set, by Lemma 3.1(1), $\mathscr{K}(T \otimes_R M) = 1 > 0$. Applying [MR, 6.1.17] to the lattice map $\mathscr{L}(R) \to \mathscr{L}(T)$, taking $\gamma = 1$ and $\delta = 0$, we conclude that $\mathscr{K}(T) \geq \mathscr{K}(R) + 1$. By Proposition 2.2 we have the opposite inequality.

Case (c). $\mathbf{p}$ is σ-unstable and the set $St(\mathbf{p})$ is infinite. The localization $R_{\mathbf{p}}$ of the ring R at $\mathbf{p}$ has Krull dimension n and has the unique simple module $R_{\mathbf{p}}/R_{\mathbf{p}}\mathbf{p} \simeq R/\mathbf{p}$. There exists an ordinal α of degree n such that α^{op} embeds in the poset $\mathscr{L}(R_{\mathbf{p}})$ of ideals of $R_{\mathbf{p}}$ (see [MR, 6.1.9-10] for details) and each simple factor of the image β of α^{op} is isomorphic to $R/\mathbf{p}$. Let γ be the inverse image of β (in $\mathscr{L}(R)$) under the map $R \to R_{\mathbf{p}}$, $r \to r/1$. Then to each simple factor from β corresponds a subfactor of R with socle isomorphic to $R/\mathbf{p}$. The set $St(\mathbf{p})$ is infinite so by Lemma 3.1(1), $\mathscr{K}(T \otimes_R R/\mathbf{p}) = 1$, hence $\mathscr{K}(T) \geq \operatorname{dev}\beta + 1 = \operatorname{dev}\alpha + 1 = n + 1$, i.e. $\mathscr{K}(T) = n + 1$ (dev is the deviation). ▮

LEMMA 3.4. *Let R be a commutative Noetherian ring, P a σ-unstable prime ideal of R, and I a left ideal of T such that $I > TP$. Then $I^{hom} > TP$. If, in addition, R/P is a-torsionfree, then $I \cap R > P$.*

Proof. Since $I > TP$, there exists a nonzero element $f = v_m a_m + \cdots + v_n a_n \in I$, $m < \cdots < n$, with $a_m \notin P$ and $a_n \notin P$. Choose f to be of minimal length $l = n - m$. We aim to show that $l = 0$. Suppose that $l > 0$. Since P is σ-unstable either $\sigma^l(P) \nsubseteq P$ or $P \nsubseteq \sigma^l(P)(\Leftrightarrow \sigma^{-l}(P) \nsubseteq P)$. In the former resp. latter case $p \in P$ with $\sigma^l(p) \notin P$ resp. $\sigma^{-l}(p) \notin P$. Then the element $g \in I \setminus TP$ below is nonzero and has length less than l,

$$g = fp - \sigma^n(p) f = v_m(p - \sigma^l(p)) a_m + \cdots + v_n(a_n p - p a_n),$$

resp.

$$g = fp - \sigma^m(p) f = v_m(a_m p - p a_m) + \cdots + v_n(p - \sigma^{-l}(p)) a_n.$$

So, $l = 0$ and $f = v_n a_n \in I \setminus TP$, i.e., $I^{hom} > TP$. Then $h := v_{-n} f = (-n, n) a_n \in I \cap R$. If the R-module R/P is a-torsionfree, then $(-n, n) \notin P$ and $h \notin P$, thus $I \cap R > P$. ∎

Let R be a commutative ring. Denote by $\mathscr{S}_R(0)$ resp. $\mathscr{S}_R(I)$ the set of all regular elements of R resp. R/I, where I is an ideal of R.

LEMMA 3.5. *Let R be a commutative Noetherian ring and M a finitely generated T-module. Then there exists a chain of T-submodules* $M = M_0 > \cdots > M_n = 0$ *such that to each i there corresponds a prime ideal* $\mathbf{p}_i$ *of R and the factor module* M_i/M_{i+1} *is one of the following types*:

(1) $T(\mathbf{p}_i)$;

(2) $V_-(\mathbf{p}_i)$;

(3) $L_-(\sigma^{-m}(\mathbf{p}_i), \mathbf{p}_i]$ *for some* $m > 0$ *such that* $a \notin \sigma^j(\mathbf{p})$ *for all* $-m < j < 0$;

(4) $V_+(\mathbf{p}_i)$;

(5) $V_-(\mathbf{p}_i)/N$ *is a proper factor module which is a torsionfree* $R/\mathbf{p}_i^0$*-module,* $\mathbf{p}_i$ *is σ-semistable, and* $N^{hom} = 0$;

(6) $V_+(\mathbf{p}_i)/N$ *is a proper factor module which is a torsionfree* $R/\mathbf{p}_i^0$*-module,* $\mathbf{p}_i$ *is σ-semistable, and* $N^{hom} = 0$;

(7) $T(\mathbf{p}_i)/N$ *is a proper factor module which is a torsionfree* $R/\mathbf{p}_i^0$*-module,* $\mathbf{p}_i$ *is σ-semistable,* $R/\mathbf{p}_i$ *is a-torsionfree, and* $N^{hom} = 0$;

(8) $V_-(\mathbf{p}_i)/N$ *is a proper factor module,* $\mathbf{p}_i$ *is σ-unstable with an infinite set* $St_-(\mathbf{p}_i)$, *and* $N^{hom} = 0$;

(9) $V_+(\mathbf{p}_i)/N$ *is a proper factor module,* $\mathbf{p}_i$ *is σ-unstable with an infinite set* $Ho_+(\mathbf{p}_i)$, *and* $N^{hom} = 0$.

Proof. The ring R is Noetherian, so is T (by Proposition 2.1). Then the T-module M is Noetherian, so it is enough to show that M has a submodule as required.

Choose a nonzero $m \in M$ to maximize $\mathbf{p} := \mathrm{ann}_R(m)$ and such that any ideal $\sigma^i(\mathbf{p})$, $i \in \mathbf{Z}$, is not contained in $\mathrm{ann}_R(m')$ for some nonzero $m' \in M$, So, $\mathbf{p}$ is a prime ideal of R and Tm is a homomorphic image of $T \otimes_R R/\mathbf{p}$. If $\mathbf{p}$ is σ-unstable and $R/\mathbf{p}$ is a-torsionfree, then $Tm \simeq T \otimes_R R/\mathbf{p}$, by Lemma 3.4 and the maximality of $\mathbf{p}$.

Thus we may suppose that $\mathbf{p}$ to be σ-semistable or a-torsionfree and Tm to be a proper homomorphic image of $T \otimes_R R/\mathbf{p} \simeq T/T\mathbf{p}$, i.e., $Tm \simeq T/I$ for some left ideal I of T such that $I > T\mathbf{p}$.

Case. $\mathbf{p}$ is σ-unstable and $\mathrm{R}/\mathbf{p}$ is a-torsion. Lemma 3.4 shows that $I^{hom} > T\mathbf{p}$. By the choice of $\mathbf{p}$ the ith graded component $(I^{hom})_i$ of the left ideal I^{hom} is equal either to Rv_i or to $v_i\mathbf{p}$. So, by (3.1), (3.2) and (3.5), (3.6) there are three possibilities:

- (a) $I^{hom} = \sum_{j \geq t} RX^j + T\mathbf{p}$, for some $t \in Ho_+(\mathbf{p})$;
- (b) $I^{hom} = \sum_{i \geq -s} RY^i + T\mathbf{p}$, for some $s \in St_-(\mathbf{p})$;
- (c) $I^{hom} = \sum_{i \geq -s} RY^i + \sum_{j \geq t} RX^j + T\mathbf{p}$, for some $s \in St_-(\mathbf{p})$ and $t \in Ho_+(\mathbf{p})$.

Let $St_-(\mathbf{p})$ and $Ho_+(\mathbf{p})$ be as in (3.3) and (3.4). The factor module T/I^{hom} is a $\mathbf{Z}$-graded T-module which maps naturally onto Tm, i.e., the following map is epic:

$$\varphi : T/I^{hom} \to T/I \simeq Tm, \qquad t + I^{hom} \to t + I.$$

By the choice of $\mathbf{p}$ each (nonzero) graded component of the module T/I^{hom} under the epimorphism above is canonically isomorphic to its image in Tm, thus

$$(\mathrm{Ker}\,\varphi)^{hom} = 0. \tag{3.9}$$

Case (a). If $t = h_+(\mathbf{p})$, then the T-submodule N of T/I^{hom} generated by $X^{t-1} + I^{hom}$ is isomorphic to $V = V_-(\sigma^{t-1}(\mathbf{p}))$, moreover it is a $\mathbf{Z}$-graded submodule. Then the image $\varphi(V)$ is nonzero. Let $\varphi(V) \simeq V/W$, where W is the kernel of the map $V \to \varphi(V)$, $v \to \varphi(v)$. By (3.9), $W^{hom} = 0$. If $W = 0$, then $\varphi(V) \simeq V$ (from (2)). So we may assume that $W \neq 0$. If $St_-(\mathbf{p})$ is an infinite set, then $\varphi(V)$ belongs to the class described in (8).

Let $St_-(\mathbf{p})$ be finite. Then $St_-(\sigma^t(\mathbf{p}))$ is finite as well and let j be the minimal element of it. By (3.5) the module V contains $V_-(\sigma^j(\mathbf{p}))$ which satisfies the conditions of Lemma 3.1(4), i.e., $St_-(\sigma^j(\mathbf{p}))$ is empty. It follows from Lemma 3.1(4) and (3.9) that $V_-(\sigma^j(\mathbf{p}))$ is isomorphic to its image under the map φ, so $\varphi(V_-(\sigma^j(\mathbf{p})))$ is a submodule of M from the class described in (2).

If $t \neq h_+(\mathbf{p})$, then $t = j_k$ for some $k > 1$. The T-submodule N of T/I^{hom} generated by $X^{t-1} + I^{hom}$ is isomorphic to $L = L_-(\sigma^{j_{k-1}-1}(\mathbf{p}), \sigma^{t-1}(\mathbf{p})]$ which satisfies the conditions of Lemma 3.1(4). Therefore the map $L \to \varphi(L)$, $l \to \varphi(l)$, is an isomorphism, thus $\varphi(L)$ belongs to the class described in (3).

Case (b). This is dual to the case (a); so we have modules from (3), (9) (which are dual to cases (2), (8)) or from the case dual to case (3); i.e., $M_i/M_{i+1} \simeq L_+(\sigma^{-m}(\mathbf{p}_i), \mathbf{p}_i] =: L$ for some $m > 0$ such that $a \notin \sigma^j(\mathbf{p})$ for all $m < j < 0$. But the submodule of L generated by $X^{m-1} + T(\sigma^{-m+1}(\mathbf{p}_i), Y, X^m)$ is isomorphic to $L_-(\sigma^{-m}(\mathbf{p}_i), \mathbf{p}_i]$ which belongs to case (3).

Case (c). If $s = s_-(\mathbf{p})$ and $t = h_+(\mathbf{p})$, then the T-submodule of N of T/I^{hom} generated by $X^{t-1} + I^{hom}$ is isomorphic to $L = L_-(\sigma^s(\mathbf{p}), \sigma^{t-1}(\mathbf{p})]$. By Lemma 3.1(4) and (3.9), $\varphi(N)$ is isomorphic to L (from the class described in (3)).

If $s \neq s_-(\mathbf{p})$ resp. $t \neq h_+(\mathbf{p})$, then $s = i_k$ for some $k > 1$ resp. $t = j_l$ for some $l > 1$. Then the T-submodule N of T/I^{hom} generated by $Y^{-i_{k-1}} + I^{hom}$ resp. $X^{t-1} + I^{hom}$ is isomorphic to $L = L_-(\sigma^s(\mathbf{p}), \sigma^{s_{k-1}}(\mathbf{p})]$ resp. to $L = L_-(\sigma^{j_{l-1}-1}(\mathbf{p}), \sigma^{t-1}(\mathbf{p})]$. Again in both cases, by Lemma 3.1(4) and (3.9), the image $\varphi(N)$ is isomorphic to L (from the class described in (3)).

Case. $\mathbf{p}$ is σ-semistable. We may suppose Tm to be a proper homomorphic image of $T \otimes_R R/\mathbf{p}$. Let n be the minimal positive integer with $\sigma^n(\mathbf{p}) = \mathbf{p}$. The ideal $\mathbf{p}^0 := \bigcap_{i=0}^{n-1} \sigma^i(\mathbf{p})$ is σ-stable, $\sigma(\mathbf{p}^0) = \mathbf{p}^0$. The module Tm is an $(R/\mathbf{p}^0)$-module. Let us show that Tm is a torsionfree $(R/\mathbf{p}^0)$-module. Note that $T\mathbf{p}^0 = \mathbf{p}^0 T$ is an ideal of T, so we may, by passing to a factor ring, take $\mathbf{p}^0$ to be zero.

Suppose that $rm' = 0$ for some $0 \neq m' \in Tm$ and $r \in \mathscr{S}_R(0)$. We may suppose that $\mathbf{q} := \text{ann}_R(m')$ is prime. It follows from [MR, 6.9.11] that $\{\sigma^i(\mathbf{p}), i = 0, \dots, n-1\}$ are the minimal prime ideals of R (note: $\mathbf{p}^0 = 0$), so $\mathscr{S}_R(0) = \bigcap_{i=0}^{n-1} \mathscr{S}_R(\sigma^i(\mathbf{p}))$. Thus $r \notin \sigma^i(\mathbf{p})$ for any i and $\mathbf{q}$ is not equal to any $\sigma^i(\mathbf{p})$, a minimal prime ideal. Therefore $\mathbf{q} > \sigma^i(\mathbf{p})$ for some i, which contradicts the choice of $\mathbf{p}$.

If $R/\mathbf{p}$ is a-torsionfree, then Tm is in the class described in (7).

So, it remains to consider the case where $R/\mathbf{p}$ is a-torsion. Then either $\ker X_M$ or $\ker Y_M$ is nonzero, where $X_M : M \to M$, $u \to Xu$ (the same for Y_M). Note that both kernels are R-submodules of M.

Case. $K := \ker X_M \neq 0$. Choose a nonzero $k \in K$ to maximize $\mathbf{q} := \text{ann}_R(u)$, $u \in K$. Then $\mathbf{q}$ is prime, so $\mathbf{q}$ contains some minimal prime ideal, say $\sigma^i(\mathbf{p})$, for some i. By the choice of $\mathbf{p}, \mathbf{q} = \sigma^i(\mathbf{p})$ and $a \in \mathbf{q}$ ($Xu = 0$, so $au = YXu = 0$, i.e., $a \in \mathbf{q}$). We see that Tk is a homomorphic image of $V_-(\mathbf{q})$. If $Tk \simeq V_-(\mathbf{q})$, then it belongs to the class described in (2).

So, suppose that Tk is a proper homomorphic image of $V_-(\mathbf{q})$ with the kernel N. If $N^{hom} = 0$, then Tk is as in (5). So, we may assume that $N^{hom} \neq 0$. Then, by the choice of $\mathbf{p}$, the jth graded component $(N^{hom})_j$ $(j \leq 0)$ of N^{hom} either coincides with $V_-(\mathbf{q})_j$ or equals 0. Then $N^{hom} = \sum_{j \leq p} V_-(\mathbf{q})_j$ for some $p \in St_-(\mathbf{q})$ and Tk is a homomorphic image of $V_-(\mathbf{q})/N^{hom} = L_-(\sigma^p(\mathbf{q}), \mathbf{q}] =: L$ with kernel U say, $Tk \simeq L/U$.

Since any nonzero submodule of L has the nonzero homogeneous part, we conclude that $U = 0$. But the module L contains a submodule from (3) (see the case (c) for details).

Case. $\ker Y_M \neq 0$ is dual to the case $\ker X_M \neq 0$; so we get modules from the classes described in (4) and (6). ∎

4. PROOF OF THE MAIN RESULT (THEOREM 1.2)

The next two Lemmas 4.1 and 4.3 may be viewed as an inductive step in the proof of Theorem 1.2.

LEMMA 4.1. *Let R be a commutative Noetherian ring of finite Krull dimension and $\mathbf{p}$ a σ-semistable prime ideal of R such that $\bigcap_{i \in \mathbf{Z}} \sigma^i(\mathbf{p}) = 0$, $a \notin \sigma^i(\mathbf{p})$ for any i and, for any prime ideal $\mathbf{q}$ with* $\operatorname{ht} q \geq 1$, $\mathscr{K}(T/T\mathbf{q}) \leq \mathscr{K}(R) - \operatorname{ht} \mathbf{q}$. *Then $\mathscr{K}(T) = \mathscr{K}(R)$.*

Proof. It is clear that $\mathbf{p}, \sigma(\mathbf{p}), \dots, \sigma^{m-1}(\mathbf{p})$ are all minimal prime ideals of R, where m is the minimal positive integer with $\sigma^m(\mathbf{p}) = \mathbf{p}$. By hypothesis $a \notin \bigcup_{i=0}^{m-1} \sigma^i(\mathbf{p})$, so a is a regular element of R. In fact, if $ab = 0$ for some $b \in R$, then $b \in \sigma^i(\mathbf{p})$ for any i ($\mathbf{p}$ is prime), so $b \in \bigcap_{i=0}^{m-1} \sigma^i(\mathbf{p}) = 0$ and $b = 0$.

Let $\mathscr{K}(R) = n$. By Proposition 2.2, $\mathscr{K}(T) = n$ or $n + 1$. We aim to show that the latter is impossible. Suppose that $\mathscr{K}(T) = n + 1$. Since a is a regular element of R, by Lemma 3.2, T is a prime ring and $\mathscr{K}(T \otimes_R R/\mathbf{p}) = n + 1$. Then $\mathscr{K}(T/T\mathbf{p}) = n + 1$, since $T \otimes_R R/\mathbf{p} \simeq T/T\mathbf{p}$. Then there exists a left ideal I of T such that $I > T\mathbf{p}$ and $\mathscr{K}(T/I) = n$. Since $a \notin \bigcup \sigma^i(\mathbf{p})$, $R/\mathbf{p}$ is a-torsionfree; by Lemma 2.10 the T-module $T/T\mathbf{p}$ is compressible, hence critical and hence uniform, thus $I/T\mathbf{p}$ is an essential submodule of $T/T\mathbf{p}$ and I is an essential submodule of T. Since $\mathscr{K}(T/I) = n$, there exists a chain of left submodules

$$T = J_0 > J_1 > \cdots > I$$

with $\mathscr{K}(J_i/J_{i+1}) = n - 1$ for infinitely many i, say for $i \in \Omega$. After refining this chain as in Lemma 3.5, we may assume that to each factor J_i/J_{i+1} then is associated a prime ideal $\mathbf{p}_i \in \operatorname{Spec} R$ and J_i/J_{i+1} has one of the properties (1)–(9) described in Lemma 3.5. Each J_i/J_{i+1} is a homomorphic image of $T/T\mathbf{p}_i$. If $i \in \Omega$, then $\mathcal{K}(T/T\mathbf{p}_i) \geq n-1$ and, by hypothesis, $\operatorname{ht}\mathbf{p}_i \leq 1$. Moreover, if $\operatorname{ht}\mathbf{p}_i = 1$, then $\mathcal{K}(T/T\mathbf{p}_i) = n-1$.

Set $\mathcal{S} = R \setminus \bigcup_{i=0}^{m-1}\sigma^i(\mathbf{p})$, the set of all regular elements of R. The ring R is semiprime, since the nil-radical $N(R)$ of R equals $\bigcap_{i=0}^{m-1}\sigma^i(\mathbf{p}) = 0$. By Goldie's theorem the ring $R_{\mathcal{S}}$ is a finite product of fields, so $R_{\mathcal{S}}$ is Artinian. As we know $a \in R$ is regular, so the localization ring $T_{\mathcal{S}} = R_{\mathcal{S}}[X, X^{-1}; \sigma]$ is a skew Laurent polynomial ring, $\mathcal{K}(T_{\mathcal{S}}) = 1$ and $T_{\mathcal{S}}$ is prime. Since I is an essential left ideal of T, $T_{\mathcal{S}}I$ is an essential left ideal of $T_{\mathcal{S}}$, then $T_{\mathcal{S}}/T_{\mathcal{S}}I$ must have finite length. So, for all sufficiently large i, $T_{\mathcal{S}} \otimes_T J_i/J_{i+1} = 0$. Let us show that for all sufficiently large $i \in \Omega$ all factors J_i/J_{i+1} are of types (1)–(4) described in Lemma 3.5 and $\operatorname{ht}\mathbf{p}_i = 1$.

If $\operatorname{ht}\mathbf{p}_i = 0$, i.e., $\mathbf{p}_i$ is a minimal prime of R, then the cases described in (8), (9) are evidently excluded and the modules from (1)–(7) are R-torsionfree (thus $T_{\mathcal{S}} \otimes_T J_i/J_{i+1} \neq 0$) and should be excluded as well.

If $\operatorname{ht}\mathbf{p}_i = 1$, then $\mathcal{K}(T/T\mathbf{p}_i) = n-1 = \mathcal{K}(J_i/J_{i+1})$. Suppose that J_i/J_{i+1} is of type (5), (6), (8), (9) in Lemma 3.5. Denote by V the module $V_+(\mathbf{p}_i)$ resp. $V_-(\mathbf{p}_i)$ which corresponds to these cases and by S the skew polynomial ring $R[v_1; \sigma]$ resp. $R[v_{-1}; \sigma^{-1}]$. By Lemmas 2.11, 3.1(1) we have

$$\begin{aligned}\mathcal{K}_T(J_i/J_{i+1}) &= \mathcal{K}_T(V/N) \leq \mathcal{K}_S(V/N) < \mathcal{K}_S(V) = \mathcal{K}(R/\mathbf{p}_i) + 1\\ &= \mathcal{K}_T(V) \leq \mathcal{K}_T(T(\mathbf{p}_i)),\end{aligned}$$

which contradicts $\mathcal{K}(J_i/J_{i+1}) = \mathcal{K}(T/T\mathbf{p}_i)$, where $\mathcal{K}_T$ and $\mathcal{K}_S$ are the Krull dimension as T- and S-module, respectively.

Suppose that J_i/J_{i+1} is of type (7) in Lemma 3.5. By Corollary 2.10 the T-module $T(\mathbf{p}_i)$ is critical, thus

$$\mathcal{K}(T(\mathbf{p}_i)) > \mathcal{K}(T(\mathbf{p}_i)/N = J_i/J_{i+1})$$

which contradicts $\mathcal{K}(T(\mathbf{p}_i)) = \mathcal{K}(J_i/J_{i+1})$.

We have shown that I is an essential left ideal of T, hence both leading ideals $\lambda_+(I)$ and $\lambda_-(I)$ are essential in R. If U is an ideal of R, then $I \cap TU \neq 0$ and, since a is regular, $0 \neq \lambda_\pm(I) \cap \lambda_\pm(TU) = \lambda_\pm(I) \cap U$.

Next suppose that i is such that $\operatorname{ht}\mathbf{p}_i = 1$ and J_i/J_{i+1} belongs to (1), (2), or (4) described in Lemma 3.5. We consider all three cases together. So, let J_i/J_{i+1} belong to (1) or (4) resp. (2). Using Lemma 2.8 we find $u_i \in \lambda_+(j_i)/\lambda_+(J_{i+1})$ resp. $u_i \in \lambda_-(J_i)/\lambda_-(J_{i+1})$ with $\operatorname{ann}_R(u_i) \subseteq \sigma^{n_i}(\mathbf{p}_i)$ for some n_i. Since $\lambda_\pm(I) \subseteq \bigcap_{i \geq 0}\lambda_\pm(J_i)$, $\lambda_+(I) \subseteq \operatorname{ann}_R(u_i)$ resp. $\lambda_-(I) \subseteq \operatorname{ann}_R(u_i)$. Thus $\sigma^{n_i}(\mathbf{p}_i)$ being a prime ideal of height 1 containing $\lambda_+(I)$ resp. $\lambda_-(I)$, must be a minimal prime of $\lambda_+(I)$ resp. $\lambda_-(I)$.

Let $\mathcal{F}$, resp. $\mathcal{G}$, be the complement in R of $\cup\mathbf{q}$ where $\mathbf{q}$ ranges over those height 1 primes of R which are minimal prime ideals of $\lambda_+(I)$, resp

$\lambda_-(I)$. Then $\mathscr{K}(R_{\mathscr{F}}) = \mathscr{K}(R_{\mathscr{G}}) = 1$. Since $R_{\mathscr{F}} I$ resp. $R_{\mathscr{G}} I$ is essential in $R_{\mathscr{F}}$ resp. $R_{\mathscr{G}}$, so $\mathscr{K}(R_{\mathscr{F}}/R_{\mathscr{F}}I) = \mathscr{K}(R_{\mathscr{G}}/R_{\mathscr{G}}I) = 0$. Thus there can only be finitely many i such that $R_{\mathscr{F}} \otimes_R \lambda_+(J_i)/\lambda_+(J_{i+1}) \neq 0$ resp. $R_{\mathscr{G}} \otimes_R \lambda_-(J_i)/\lambda_-(J_{i+1}) \neq 0$, therefore for all sufficiently large i from Ω the factors J_i/J_{i+1} belong to (3) described in Lemma 3.5.

If J_i/J_{i+1} is of type (3), then by Lemma 3.1(2), $\mathscr{K}(J_i/J_{i+1}) = \mathscr{K}(R/\mathbf{p}_i)$. On the other hand $\mathscr{K}(J_i/J_{i+1}) = n-1$, so $\mathscr{K}(R/\mathbf{p}_i) = n-1$, if the associated prime $\mathbf{p}_i$ is either σ-semistable or σ-unstable and there are infinitely many integers j with $a \in \sigma^j(\mathbf{p}_i)$, then by Lemma 3.5, $\mathscr{K}(T/T\mathbf{p}_i) = \mathscr{K}(R/\mathbf{p}_i) + 1 = n$, contradicting the hypothesis $\mathscr{K}(T/T\mathbf{p}_i) \leq n - \operatorname{ht}\mathbf{p}_i = n-1$ (since $\operatorname{ht}\mathbf{p}_i = 1$).

- (*) So, we may assume that for all sufficiently large $i \gg 0$ from Ω the factors $J_i/J_{i+1} \simeq L_-(\sigma^{-m_i}(\mathbf{p}_i), \mathbf{p}_i]$ have type (3), Lemma 3.5, and all associated primes $\mathbf{p}_i$ are σ-unstable with $\operatorname{ht}\mathbf{p}_i = 1$ and all $St(\mathbf{p}_i)$ are finite.

Let $\mathbf{b}$ be the set of minimal primes (in R) of Ra of height 1. Evidently, $\mathbf{b}$ is finite and there exists a natural q such that for all $i \geq q$ the associated prime ideals $\mathbf{p}_i$ belong to $\mathbf{b}$. Let $\mathscr{O}_1, \dots, \mathscr{O}_\alpha$ be the orbits of the elements $\mathbf{p}_i$. Each orbit $\mathscr{O}_j$ contains finitely many but not less than 2 ideals containing a, say, $\mathbf{q}$, $\sigma^{i_2}(\mathbf{q}), \dots, \sigma^{i_k}(\mathbf{q})$ for some $1 \leq i_2 < \cdots < i_k = i_k(j)$. Let $\operatorname{Int}\mathscr{O}_j = \{\sigma^j(\mathbf{q}) | 1 \leq j \leq i_k\}$. The set $\operatorname{Int} = \bigcup_{j=1}^{\alpha} \operatorname{Int}\mathscr{O}_j$ is finite. Denote by m the maximum of $\{i_k\}$.

For $j \geq i \geq q$, $\operatorname{Ass}_R(J_i/J_j) \subseteq \operatorname{Int}$, so, by the choice of m, the annihilator $\operatorname{ann}_T(J_i/J_j)$ contains all elements v_k, $|k| \geq m$. The (two-sided) ideal $\mathbf{a}$ of the ring T generated by X^m and Y^m is homogeneous (with respect to the grading of T), $\mathbf{a} = \sum_{i\in\mathbf{Z}} v_i \mathbf{a}_i$ where each $\mathbf{a}_i$ is an ideal of R and $\mathbf{a}_k = R$ for all $|k| \geq m$. Moreover,

$$\mathbf{a}_{-m+1} \supseteq \mathbf{a}_{-m+2} \supseteq \cdots \supseteq \mathbf{a}_0 \subseteq \mathbf{a}_1 \subseteq \cdots \subseteq \mathbf{a}_{m-1},$$

and $\mathbf{a}_0 \supseteq RY^mX^m + RX^mY^m = R\sigma^{-m+1}(a)\cdots a + R\sigma(a)\cdots\sigma^m(a)$. The factor ring $T/\mathbf{a} = \bigoplus_{-m<i<m} v_i R/\mathbf{a}_i$ is a finitely generated $R/\mathbf{a}_0$-module, so $\mathscr{K}(T/\mathbf{a}) \leq \mathscr{K}(R/\mathbf{a}_0) \leq n-1$ (a is regular). On the other hand, $J_i/J_{i+1} \simeq L_-(\sigma^{-m_i}(\mathbf{p}_i), \mathbf{p}_i]$ is a $T/\mathbf{a}$-module of Krull dimension $n-1$, thus $\mathscr{K}(T/\mathbf{a}) = n-1$. The ring $R/\mathbf{a}$ is Noetherian, so its nil radical N is nilpotent, say $N^{s+1} = 0$ but $N^s \neq 0$ for some $s \geq 1$. By Goldie's theorem the full quotient ring Q of the semiprime ring $\overline{T} = (T/\mathbf{a})/N$ is a semisimple (Artinian) ring. Evidently, $\mathscr{K}(\overline{T}) = \mathscr{K}(T/\mathbf{a}) = n-1$. Let l be the length of the Q-module Q. Fix $i \geq q$; up to the numeration, we may assume that $i = q = 1$. Let g be the maximum of number of generators of the left ideals $I_1, NI_1, \dots, N^sI_1$. Choose $j = (gl+1)(s+1)+1$.

We have two descending chains of left ideals of the ring T: $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_j$ and $I_1 = J_1 \supseteq I_2 = NI_1 + J_j \supseteq \cdots \supseteq I_{s+1} = N^sI_1 + J_j \supseteq I_{s+2} = J_j$. The refinements $\{J_{ki} := (J_k \cap I_i + J_{k+1})/(J_k \cap I_{i+1} + J_{k+1})\}$ and $\{I_{ik} :=$

$(I_i \cap J_k + I_{i+1})/(I_i \cap J_{k+1} + I_{i+1})\}$ of the first and the second chain respectively have isomorphic factors:

$$J_{ki} \simeq J_k \cap I_i/(J_k \cap I_{i+1} + J_{k+1} \cap I_i) \simeq I_{ik}.$$

Each factor J_i/J_{i+1} has Krull dimension $n-1$, so there are at least $j-1$ factors from $\{I_{ik}\}$ which have Krull dimension $n-1$, hence there is at least one factor, say I_k/I_{k+1}, which contains at least $(j-1)(s+1)^{-1}$ refined subfactors $\{I_{kj}\}$ of Krull dimension $n-1$. Each such subfactor under the localization yields a nonzero Q-module $Q \otimes_{\overline{T}} I_{kj}$. So, on the one hand, the length $l_Q(Q \otimes_{\overline{T}}(I_k/I_{k+1})) \geq (j-1)(s+1)^{-1}$. On the other, $l_Q(Q \otimes_{\overline{T}}(I_k/I_{k+1})) \leq gl$, which contradicts the choice of j. ∎

LEMMA 4.2. *Let R be a commutative Noetherian ring ring with $\mathscr{K}(R) < \infty$.*

(1) *Let $\mathbf{p}$ be a σ-unstable prime ideal such that the set $St(\mathbf{p})$ is finite and non-empty. Then*

$$\mathscr{K}(T \otimes_R R/\mathbf{p}) = \max\left\{\mathscr{K}\left(V_-\left(\sigma^i(\mathbf{p})\right)\right), \mathscr{K}\left(V_+\left(\sigma^{j+1}(\mathbf{p})\right)\right)\right\},$$

where i resp. j the minimal resp. maximal element of $St(\mathbf{p})$.

(2) *Let $\mathbf{p}$ be a prime ideal of R such that $St(\mathbf{p}) = \emptyset$. Then $\mathscr{K}(T \otimes_R R/\sigma^l(\mathbf{p})) = \mathscr{K}(T \otimes_R R/\mathbf{p})$ for any $l \in \mathbf{Z}$.*

Proof. (1) Let $i = i_1 < \cdots < i_s = j$ be the elements of $St(\mathbf{p})$, $s > 0$. Partition the set of all integers $\mathbf{Z}$ into $s+1$ intervals

$$\Gamma_1 = (-\infty, i_1], \Gamma_2 = (i_1, i_2], \dots, \Gamma_s = (i_{s-1}, i_s], \Gamma_{s+1} = (i_s, \infty).$$

If $0, l \in \Gamma_k$ for some k, then the following maps are monomorphisms of T-modules:

$$T(\mathbf{p}) \to T\left(\sigma^l(\mathbf{p})\right), \qquad 1 + T\mathbf{p} \to v_{-l} + T\sigma^l(\mathbf{p}), \tag{4.1}$$

$$T\left(\sigma^l(\mathbf{p})\right) \to T(\mathbf{p}), \qquad 1 + T\sigma^l(\mathbf{p}) \to v_l + T\mathbf{p}. \tag{4.2}$$

Therefore, if $0, l \in \Gamma_k$, then

$$\mathscr{K}(T(\mathbf{p})) = \mathscr{K}\left(T\left(\sigma^l(\mathbf{p})\right)\right). \tag{4.3}$$

There are three possibilities: $0 \in \Gamma_1$; $0 \in \Gamma_k$ for some $k = 2, \dots, s$; $0 \in \Gamma_{s+1}$. In view of (4.3) these possibilities reduce to the cases: $i = 0$; $i_k = 0$ for some $k = 2, \dots, s$; $j = -1$.

Let $i = 0$. From (3.5), (3.6), and (3.8) it follows that the following sequences of T-modules are exact,

$$0 \to V_+(\sigma(\mathbf{p})) \to^{\alpha} T(\mathbf{p}) \to^{\beta} V_-(\mathbf{p}) \to 0,$$

where $\alpha : 1 + T(\sigma(\mathbf{p}), Y) \to X + T\mathbf{p}$, $\beta : 1 + T\mathbf{p} \to 1 + T(\mathbf{p}, X)$;

$$0 \to V_+\big(\sigma^{j+1}(\mathbf{p})\big) \to V_+\big(\sigma(\mathbf{p})\big) \to L_+\big(\mathbf{p}, \sigma^j(\mathbf{p})\big] \to 0.$$

Now the result follows immediately from Lemma 3.1.

Let $i_k = 0$ for some $k = 2, \dots, s$. From (3.5) and (3.6) it follows that $V := V_-(\sigma^i(\mathbf{p})) \oplus V_+(\sigma^{j+1}(\mathbf{p}))$ is a T-submodule of $T(\mathbf{p})$ such that the factor-module

$$_R(L := T(\mathbf{p})/V) \simeq \oplus_{i<k\le j} R/\sigma^k(\mathbf{p}).$$

Since $\mathscr{K}(_T L) \le \mathscr{K}(_R L) = \mathscr{K}(R/\mathbf{p})$, the result is evident (Lemma 3.1).

Let $j = -1$. This case is dual to the first. It follows from (3.5), (3.6), and (3.7) that the following sequences of T-modules are exact,

$$0 \to V_-\big(\sigma^{-1}(\mathbf{p})\big) \to^{\delta} T(\mathbf{p}) \to^{\gamma} V_+(\mathbf{p}) \to 0,$$

where $\delta : 1 + T(\sigma^{-1}(\mathbf{p}), X) \to Y + T\mathbf{p}$, $\gamma : 1 + T\mathbf{p} \to 1 + T(\mathbf{p}, Y)$;

$$0 \to V_-\big(\sigma^i(\mathbf{p})\big) \to V_-\big(\sigma^{-1}(\mathbf{p})\big) \to L_-\big(\sigma^i(\mathbf{p}), \sigma^{-1}(\mathbf{p})\big] \to 0.$$

The result follows in view of Lemma 3.1.

(2) The set $St(\mathbf{p})$ is empty, therefore for any $l \in \mathbf{Z}$ there exist the monomorphisms (4.1) and (4.2), so $\mathscr{K}(T(\sigma^l(\mathbf{p}))) = \mathscr{K}(T(\mathbf{p}))$. ∎

LEMMA 4.3. *Let R be a commutative Noetherian ring with $\mathscr{K}(R) < \infty$ and let $\mathbf{p}$ be a σ-unstable prime ideal of R such that, for each prime $\mathbf{q}$ with $\mathbf{q} \supset \mathbf{p}$, $\mathscr{K}(T/T\mathbf{q}) \le \mathscr{K}(R) - \operatorname{ht}\mathbf{q}$. Then $\mathscr{K}(T/T\mathbf{p}) \le \mathscr{K}(R) - \operatorname{ht}\mathbf{p}$.*

Proof. Let I be a left ideal of T which strictly contains $T\mathbf{p}$. Let $R/\mathbf{p}$ be a-torsionfree. Then, by Lemma 3.4, $I \cap R > \mathbf{p}$. There is a finite chain of ideals $R = U_0 \supset U_1 \supset \cdots \supset U_n = I \cap R$ with $U_i/U_{i+1} \simeq R/\mathbf{p}_i$, where $\mathbf{p}_i \in \operatorname{Spec} R$ and $\mathbf{p}_i \supset \mathbf{p}$. Tensoring the chain above by $T \otimes_R -$ we obtain a chain

$$T = TU_0 \supset TU_1 \supset \cdots \supset TU_n = T(I \cap R),$$

with $TU_i/TU_{i+1} \simeq T \otimes_R U_i/U_{i+1} \simeq T \otimes_R R/\mathbf{p}_i \simeq T/T\mathbf{p}_i$.

Now we arrive at

$$\begin{aligned}\mathscr{K}(T/I) &\le \mathscr{K}(T/T(I \cap R)) = \max\{\mathscr{K}(T/T\mathbf{p}_i)\}\\ &\le \max\{\mathscr{K}(R) - \operatorname{ht}\mathbf{p}_i\} < \mathscr{K}(R) - \operatorname{ht}\mathbf{p}.\end{aligned}$$

Since the above inequality holds for all $I \supset T\mathbf{p}$, we may conclude that $\mathscr{K}(T/T\mathbf{p}) \le \mathscr{K}(R/\mathbf{p}) - \operatorname{ht}\mathbf{p}$.

It remains to consider the case where $R/\mathbf{p}$ is a-torsion. The inequality $\mathscr{K}(R/\mathbf{p}) \le \mathscr{K}(R) - \operatorname{ht}\mathbf{p}$ holds for any prime ideals $\mathbf{p}$ of R. If, in our case, $\mathscr{K}(R/\mathbf{p}) < \mathscr{K}(R) - \operatorname{ht}\mathbf{p}$, then we know that $\mathscr{K}(T/T\mathbf{p}) < \mathscr{K}(R/\mathbf{p}) + 1$; so

$$\mathscr{K}(T/T\mathbf{p}) \le \mathscr{K}(R) - \operatorname{ht}\mathbf{p} - 1 + 1 = \mathscr{K}(R) - \operatorname{ht}\mathbf{p}$$

as desired. Thus we may assume that

$$\mathscr{K}(R/\mathbf{p}) = \mathscr{K}(R) - \operatorname{ht}\mathbf{p}. \tag{4.4}$$

Let us show that the set $St(\mathbf{p})$ is finite. Since $R/\mathbf{p}$ is a-torsion, $St(\mathbf{p})$ is non-empty. Suppose the contrary, i.e., $|ST(\mathbf{p})|$ is infinite. It follows from (4.4) that there exists a maximal idea, say $\mathbf{m}$, of R with $\operatorname{ht}\mathbf{m} = \mathscr{K}(R)$ and $\mathbf{m} \supset \mathbf{p}$. Then $St(\mathbf{m}) \supset St(\mathbf{p})$, so $St(\mathbf{m})$ is infinite and by Lemma 3.1(1) $\mathscr{K}(T/T\mathbf{m}) = 1$ which contradicts the hypothesis, since $\mathscr{K}(T/T\mathbf{m}) \le \mathscr{K}(R) - \operatorname{ht}\mathbf{m} = 0$.

So, the set $St(\mathbf{p})$ is finite and let $i_1 < \cdots < i_s$ be the elements of $St(\mathbf{p})$. It follows from Lemma 4.2 that

$$\mathscr{K}(T/T\mathbf{p}) = \max\{\mathscr{K}(V_-), \mathscr{K}(V_+)\},$$

where $V_- = V_-(\sigma^{i_1}(\mathbf{p}))$ and $V_+ = V_+(\sigma^{i_s+1}(\mathbf{p}))$. Without loss of generality we may assume that $\mathscr{K}(T/T\mathbf{p}) = \mathscr{K}(V_-)$. Let I be a left ideal of T such that $I > T(\sigma^{i_1}(\mathbf{p}), X)$. Since i_1 is the minimal element of $St(\mathbf{p})$, $I \cap R > \mathbf{p}$. Now we continue as above. Choose a finite chain of ideals $R = U_0 \supset U_1 \supset \cdots \supset U_n = I \cap R$ with $U_i/U_{i+1} \simeq R/\mathbf{p}_i$, where $\mathbf{p}_i \in \operatorname{Spec} R$ and $\mathbf{p}_i \supset \sigma^{i_1}(\mathbf{p})$. Then we obtain the chain of T-submodules,

$$T = TU_0 \supset TU_1 \supset \cdots \supset TU_n = T(I \cap R)$$

with $TU_i/TU_{i+1} \simeq T/T\mathbf{p}_i$; and

$$T = W_0 \supset W_1 \supset \cdots \supset W_i := TU_i + TX \supset \cdots \supset W_n = T(I \cap R) + TX.$$

For each i we have the natural epimorphism $TU_i/TU_{i+1} \to W_i/W_{i+1}$, thus $\mathscr{K}(W_i/W_{i-1}) \le \mathscr{K}(T/T\mathbf{p}_i)$, consequently

$$\begin{aligned} \mathscr{K}(V_-/I) &\le \max\{\mathscr{K}(W_i/W_{i-1})\} \le \max\{\mathscr{K}(T/T\mathbf{p}_i)\} \\ &\le \max\{\mathscr{K}(R) - \operatorname{ht}\mathbf{p}_i\} < \mathscr{K}(R) - \operatorname{ht}\mathbf{p}. \end{aligned}$$

Since the inequality above holds for any $I \supset T(\mathbf{p}, X)$, we have $\mathscr{K}(V_-) \le \mathscr{K}(R) - \operatorname{ht}\mathbf{p}$ but $\mathscr{K}(T/T\mathbf{p}) = \mathscr{K}(V_-)$ and the claims have been proved. ∎

Proof of Theorem 1.2. Theorem 3.3 reduces the problem to showing that if $\mathscr{K}(T) = \mathscr{K}(R) + 1$, then there exists either a σ-unstable prime ideal $\mathbf{p}$ of R with $\operatorname{ht}\mathbf{p} = \mathscr{K}(R)$ and the set $St(\mathbf{p})$ is infinite; or a σ-semistable prime ideal $\mathbf{q}$ of R with $\operatorname{ht}\mathbf{q} = \mathscr{K}(R)$.

We have shown at the beginning of Section 3 that

$$\mathscr{K}(T) = \sup\{\mathscr{K}(T/T\mathbf{p}) | \mathbf{p} \text{ is a minimal prime ideal of } R\}. \tag{4.5}$$

There are only finitely many minimal prime ideals, thus each minimal prime is σ-semistable. Let $\mathbf{p}$ be a minimal prime ideal of R. Then the ideal $\mathbf{p}^0 = \bigcap \sigma^i(\mathbf{p})$ is σ-stable $(\sigma(\mathbf{p}^0) = \mathbf{p}^0)$ and $T\mathbf{p}^0 = \mathbf{p}^0 T$ is an ideal of T. The factor ring $T/T\mathbf{p}^0$ is isomorphic to the GWA $R/\mathbf{p}^0(\sigma, a + \mathbf{p}^0)$, where $\sigma(r + \mathbf{p}^0) = \sigma(r) + \mathbf{p}^0$. Since $T/T\mathbf{p}$ is a homomorphic image of $T/T\mathbf{p}^0$, without loss of generality we may assume that $\mathbf{p}^0 = 0$. Then, as we have seen above, $\mathbf{p}, \sigma(\mathbf{p}), \dots, \sigma^{m-1}(\mathbf{p})$ are the minimal prime ideals of R.

If a is non-regular in R, then $a \in \sigma^i(\mathbf{p})$ for some i. By Theorem 3.3, $\mathscr{K}(T/T\sigma^i(\mathbf{p})) = \mathscr{K}(R/\sigma^i(\mathbf{p})) + 1 = \mathscr{K}(R) + 1$.

So, we may assume that a is regular, i.e., $a \notin \bigcup \sigma^i(\mathbf{p})$. We use induction on $n = \mathscr{K}(R)$. If $n = 0$, then all prime ideals are minimal prime and, hence σ-semistable. The element a is unit, therefore $T \simeq R[X, X^{-1}; \sigma]$ is the skew Laurent polynomial ring with $\mathscr{K}(T) = 1$.

Suppose that $n > 0$ and R has no σ-semistable prime ideal with $\mathbf{q}$ with $\operatorname{ht}\mathbf{q} = n$, yet $\mathscr{K}(T) = n + 1$. We aim to find a σ-unstable prime ideal P with $\operatorname{ht} P = n$ and with the infinite set $St(P)$. Since a is regular, $R/\mathbf{p}$ is a-torsionfree, by Lemma 4.2(2), $\mathscr{K}(T/T\mathbf{p}) = \mathscr{K}(T/T\sigma^i(\mathbf{p}))$ for any i. Using (4.5) we conclude that

$$\mathscr{K}(T) = \mathscr{K}\big(T/T\sigma^i(\mathbf{p})\big) \qquad \text{for any } i. \tag{4.6}$$

Since $\mathscr{K}(T) > \mathscr{K}(R)$, then $\mathscr{K}(T/T\mathbf{p}) > \mathscr{K}(R) = \mathscr{K}(R) - \operatorname{ht}\mathbf{p}$. Choose a prime ideal, say $\mathbf{p}_1$, maximal with respect to having $\mathscr{K}(T/T\mathbf{p}_1) > \mathscr{K}(R) - \operatorname{ht}\mathbf{p}_1$ and then having maximal height as much. It follows from Lemma 4.3 that $\mathbf{p}_1$ is σ-semistable. Then $\mathbf{p}_1$ cannot be a minimal prime, since otherwise the hypotheses of Lemma 4.1 are satisfied and thus $\mathscr{K}(T) = \mathscr{K}(R)$, a contradiction.

So, we may suppose $\operatorname{ht}\mathbf{p}_1 \geq 1$. Since $\mathscr{K}(R/\mathbf{p}_1^0) < n$, by induction

$$\mathscr{K}\big(T/T\mathbf{p}_1^0\big) = \sup\big\{\mathscr{K}\big(R/\mathbf{p}_1^0\big), \operatorname{ht}\big(P/\mathbf{p}_1^0\big) + 1, \operatorname{ht}\big(Q/\mathbf{p}_1^0\big) + 1\big\},$$

where P is a σ-unstable prime for which there exist finitely many i with $a \in \sigma^i(P)$, and $P \supseteq \mathbf{p}_1^0$; Q is a σ-semistable prime with $Q \supseteq \mathbf{p}_1^0$.

Since $\mathrm{ht}\, Q \geq \mathrm{ht}(Q/\mathbf{p}_1^0) + \mathrm{ht}\,\mathbf{p}_1$ and $\mathscr{K}(R/\mathbf{p}_1^0) + \mathrm{ht}\,\mathbf{p}_1 \leq n$, we have

$$\begin{aligned}\sup\{\mathscr{K}(R/\mathbf{p}_1^0), \mathrm{ht}(Q/\mathbf{p}_1^0) + 1\} &\leq \sup\{n - \mathrm{ht}\,\mathbf{p}_1, \mathrm{ht}\, Q - \mathrm{ht}\,\mathbf{p}_1 + 1\}\\ &\leq \sup\{n - \mathrm{ht}\,\mathbf{p}_1, (n-1) - \mathrm{ht}\,\mathbf{p}_1 + 1\} = n - \mathrm{ht}\,\mathbf{p}_1.\end{aligned}$$

On the other hand,

$$n - \mathrm{ht}\,\mathbf{p}_1 < \mathscr{K}(T/T\mathbf{p}_1) \leq \mathscr{K}(T/T\mathbf{p}_1^0), \tag{4.7}$$

therefore, $\mathscr{K}(T/T\mathbf{p}_1^0) = \mathrm{ht}(P/\mathbf{p}_1^0) + 1$ for some σ-unstable prime ideal P, containing $\mathbf{p}_1^0$, and with infinitely many i: $a \in \sigma^i(P)$. By (4.7), $n - \mathrm{ht}\,\mathbf{p}_1 + 1 \leq \mathrm{ht}(P/\mathbf{p}_1^0) + 1$. Then $n \leq \mathrm{ht}(P/\mathbf{p}_1^0) + \mathrm{ht}\,\mathbf{p}_1 \leq \mathrm{ht}\, P \leq n$, i.e., $P = n$ and P is a σ-unstable prime ideal with infinitely many i: $a \in \sigma^i(P)$. ∎

Krull Dimension of Iterated Skew Polynomial Rings

Consider a ring R, $\sigma \in \mathrm{Aut}(R)$ and $b, \rho \in Z(R)$ with ρ being a σ-stable unit, i.e., $\sigma(\rho) = \rho$. Form $E = R\langle \sigma; b, \rho\rangle$ by adjoining symbols X and Y to R subjected to the relations

$$Xr = \sigma(r)X, \qquad Yr = \sigma^{-1}(r)Y, \text{ for all } r \in R;\ XY - \rho YX = b.$$

In case $R = K[H]$, $\rho = 1$, and σ defined by $\sigma(H) = H - 1$, then E appeared in [Sm, Bav1], and for $b = 2H$, E is nothing but $Usl(2)$.

We may view E as the iterated skew polynomial ring $E = R[Y; \sigma^{-1}][X; \sigma, \partial]$, where ∂ is a σ-derivation of $R[Y; \sigma^{-1}]$ and $\partial R = 0$, $\partial Y = b$; moreover σ is extended from R to $R[Y; \sigma^{-1}]$ by $\sigma(Y) = \rho Y$.

For R left Noetherian it is known [MR, 6.5.4] that

$$\mathscr{K}(R\langle \sigma; b, \rho\rangle) = \mathscr{K}(R) + 1 \text{ or } \mathscr{K}(R) + 2.$$

It is not trivial to decide which of the two cases actually happens; let us recall the following particular case.

The rings of this type E are in the scope of this paper because of, as the lemma below shows, the rings $E = D\langle \sigma; b, \rho\rangle$ are generalized Weyl algebras.

LEMMA 4.4 [Bav4]. *Let R be a ring; then $R\langle \sigma; b, \rho\rangle \simeq R[H](\sigma, H)$ and σ is extended from R to $R[H]$ by $\sigma(H) = \rho H + b$.* ∎

THEOREM 4.5. *Let R be a commutative Noetherian ring, $\mathscr{K}(R) < \infty$, $\sigma \in \mathrm{Aut}\, R$, $b \in R$, $\rho \in R$ be a unit such that $\sigma(\rho) = \rho$. Let $R[H]$ be a polynomial ring in a variable H; the automorphism σ is extended from R to $R[H]$ by the rule $\sigma(H) = \rho H + b$.*

Then

$$\mathscr{K}(R\langle\sigma : b, \rho\rangle) = 1 + \sup\{\mathscr{K}(R), \operatorname{ht}\mathbf{p} + 1, \operatorname{ht} Q | \mathbf{p} \textit{ is a prime ideal of } R$$

for which there are infinitely many integers $0 \neq n \in \mathbf{Z}$ *such that the inclusion* $I(n)$ *holds*: $n > 0$,

$$I(n): \sum_{i=1}^{n} \rho^{-1}\sigma^{i-1}(b) \in \mathbf{p};\ I(-n): \sum_{i=1}^{n} \rho^{i-1}\sigma^{-i}(b) \in \mathbf{p};$$

Q *is a* σ*-semistable ideal of the polynomial ring* $R[H]\}$.

Proof. Note that the ideals $\mathbf{p}$ and $\mathbf{q}$ in Theorem 1.2 may be supposed to be maximal of height $\mathscr{K}(R)$ and that the hypothesis that $\mathbf{p}$ is σ-semistable can be lifted.

By Lemma 4.4 the iterated polynomial ring $E = R\langle\sigma; b, \rho\rangle$ is isomorphic to GWA

$$E \simeq R[H](\sigma, H), \qquad \sigma(H) = \rho H + b.$$

Set (H) for the ideal of the polynomial ring $R[H]$ generated by H. The map $\mathbf{p} \to \mathbf{p} + (H)$ is a 1-1 correspondence between Spec R and the set of all prime ideals of $R[H]$ which contain H. Let P be a maximal ideal of $R[H]$ of height $\mathscr{K}(R[H]) = \mathscr{K}(R) + 1$ for which there exist infinitely many i with $H \in \sigma^i(P)$. Fix one such, say J, i.e., $H \in \sigma^j(P)$. Then $\sigma^j(P) = \mathbf{p} + (H)$ for some maximal ideal $\mathbf{p}$ of R of height $\operatorname{ht}\mathbf{p} = \mathscr{K}(R)$. Then $H \in \sigma^i(P) = \sigma^{i-j}(\sigma^j(P)) = \sigma^{i-j}(\mathbf{p}) + (\sigma^{i-j}(H))$. The automorphism σ of $R[H]$ preserves the natural filtration on $R[H]$ by the degree of H, thus

$$H \in \sigma^i(P) \qquad \text{iff } H \in \sigma^{i-j}(\mathbf{p}) + \sigma^{i-j}(H)R. \tag{4.8}$$

An easy induction argument shows that for $n > 0$,

$$\sigma^n(H) = \rho^n\left(H + \sum_{i=1}^{n} \rho^{-i}\sigma^{i-1}(b)\right),$$

$$\sigma^{-n}(H) = \rho^{-n}\left(H - \sum_{i=1}^{n} \rho^{i-1}\sigma^{-i}(b)\right).$$

Therefore, (4.8) holds iff $I(i-j)$ does. Now the result follows from Theorem 1.2 and the remark at the beginning of the proof. ∎

5. KRULL DIMENSION OF SOME QUANTUM-TYPE ALGEBRAS

Let R be a ring, $\sigma \in \mathrm{Aut}(R)$, b and ρ be elements of the centre $Z(R)$ of R such that ρ is a unit, and $\sigma(\rho) = \rho$. Set $E = R\langle \sigma; b, \rho \rangle$ for the iterated skew polynomial ring (see Section 4).

COROLLARY 5.1 [Bav3, Bav5]. *If $b = \rho\alpha - \sigma(\alpha)$ for some $\alpha \in Z(R)$, then*

$$D\langle \sigma; b, \rho \rangle \simeq D[C](\sigma, a = \rho^{-1}C - \alpha), \qquad \sigma(C) = \rho C.$$

Let K be a field, $D = K[H]$ a polynomial ring, $\sigma \in \mathrm{Aut}_K(D)$ is a K-automorphism defined by $\sigma(H) = H - 1, b \in D$. The *Smith's* deformation [Sm2, Bav1].

$$\Lambda(b) = K[H]\langle \sigma; b, \rho = 1 \rangle,$$

is a GWA (Lemma 4.4). Moreover, if char $K = 0$, then in view of Corollary 5.1,

$$\Lambda(b) \cong K[H, C](\sigma, a = C - \alpha), \qquad \sigma : H \to H - 1, C \to C,$$

where $\alpha \in K[H]$ is a solution of the equation $\alpha - \sigma(\alpha) = b$ (which exists since the map $1 - \sigma\colon\ K[H] \to K[H],\ \alpha(H) \to \alpha(H) - \alpha(H-1)$, is surjective).

By Theorems 1.2 and 4.5 if (char $K = p > 0$, then $\sigma^p = 1$, where σ is as in Lemma 4.4)

$$\mathscr{K}(\Lambda(b)) = \begin{cases} 2, & \textit{if } \text{char } K = 0 \text{ and } b \neq 0; \\ 3, & \text{otherwise}. \end{cases}$$

For the *quantum* $U_q sl(2)$, for $q, h = q - q^{-1} \in K = \mathbf{C}$, $q^2 \neq 0, 1$, the algebra $U_q = U_1 sl(2)$ is generated by X, Y, H_-, H_+ subject to the relations

$$H_+H_- = H_-H_+ = 1, \qquad XH_\pm = q^{\pm 1}H_\pm X,$$

$$YH_\pm = q^{\mp 1}H_\pm Y, \qquad [X, Y] = (H_+^2 - H_-^2)/h.$$

This algebra is a GWA [Bav6],

$$U_q \simeq K[C, H, H^{-1}]\big(\sigma, a = C + \{H^2/(q^2 - 1) - H^{-2}/(q^2 - 1)\}/2h\big),$$

where $K[C, H, H^{-1}] = K[C][H, H^{-1}]$ is the Laurent polynomial ring with coefficients in the polynomial ring $K[C]$; $\sigma(H) = qH$, $\sigma(C) = C$. It is clear that $\mathscr{K}(K[C, H, H^{-1}]) = 2$. It follows from Theorem 1.2 that [Jo1]

$$\mathscr{K}(U_q sl(2)) = \begin{cases} 2, & q \text{ is not a root of } 1; \\ 3, & \text{otherwise}. \end{cases}$$

The *Woronowicz's deformation* V is generated by V_0, V_-, V_+ subject to the relations ($s \in K, s^4 \neq 1$)

$$[V_0, V_+]_{s^2} \equiv s^2 V_0 V_+ - s^{-2} V_+ V_0 = V_+, \qquad [V_-, V_0]_{s^2} = V_-,$$
$$[V_+, V_-]_{1/s} \equiv s^{-1} V_+ V_- - s V_- V_+ = V_0.$$

The algebra V is a GWA [Bav4]

$$V \simeq K[H, Z](\sigma, a = Z + \alpha H + \beta),$$
$$V_+ \leftrightarrow X, V_- \leftrightarrow Y, \qquad V_0 \leftrightarrow H - s^2/(1 - s^4),$$

where $\sigma : H \to s^4 H, Z \to s^2 Z$; $\alpha = -1/s(1 - s^2)$ and $\beta = s/(1 - s^4)$. Applying Theorem 1.2 we conclude that $\mathscr{K}(V) = 3$. Let $S^{-1}V$ be the localization of V at $S = \{H^i Z^j\}$. Then

$$\mathscr{K}(S^{-1}V) = \begin{cases} 2, & s \text{ is not a root of } 1; \\ 3, & \text{otherwise}. \end{cases}$$

The *Witten's first deformation* E is the algebra generated by E_0, E_-, E_+ subject to the relations

$$[E_0, E_+]_p \equiv p E_0 E_+ - p^{-1} E_+ E_0 = E_+, \qquad [E_-, E_0]_p = E_-,$$
$$[E_+, E_-] = E_0 - (p - 1/p) E_0^2,$$

where $p \neq 0, \pm 1, \pm i \in K$. The algebra E is a GWA [Bav4]

$$E \simeq K[C, H]\big(\sigma, a = C - H(H + 1)/(p + p^{-1})\big),$$
$$E_+ \leftrightarrow X, E_- \leftrightarrow Y, E_0 \leftrightarrow pH,$$

where $\sigma : C \to C, H \to p^2(H - 1)$. Set $\lambda = 1/(1 - \rho^{-2})$. It follows from direct calculations that $\sigma(H - \lambda) = \rho^2(H - \lambda)$, therefore the maximal ideal $CD + (H - \lambda)D$ of $D = K[C, H]$ is σ-stable, so by Theorem 1.2

$$\mathscr{K}(E) = 3.$$

Set $h = H - \lambda$. Then $K[C, H] = K[C, h]$ and $\sigma(h) = \rho^2 h$. A multiplicatively closed set $S = \{h^i, i \geq 0\}$ satisfies the Ore condition and the (two-sided) localization $F = S^{-1}E$ of E at S is a GWA

$$F \simeq K[C, h, h^{-1}]\big(\sigma, a = C - (h + \lambda)(h + \lambda + 1)/(\rho + \rho^{-1})\big),$$

where $\sigma : C \to C, h \to \rho^2 h$. Now there is a σ-semistable maximal ideal in $K[C, h, h^{-1}]$ iff ρ is a root of 1. By Theorem 1.2

$$\mathscr{K}(F) = \begin{cases} 2, & \rho \text{ is not a root of } 1; \\ 3, & \text{otherwise}. \end{cases}$$

The *Witten's second deformation* [Za] W is the algebra generated by W_0, W_-, W_+,

$$[W_0, W_+]_r = W_+, \qquad [W_-, W_0]_r = W_-, \qquad [W_+, W_-]_{1/r^2} = W_0,$$

where $r \neq 0, \pm 1, \pm i \in K$. The algebra W is the GWA [Bav4]

$$W \simeq K[C, H](\sigma, a = C - \alpha),$$

$$W_+ \leftrightarrow X, W_- \leftrightarrow Y, W_0 \leftrightarrow H - r/(1 - r^2),$$

where $\sigma : C \to r^4 C, H \to r^2 H$ is the automorphism of the polynomial ring $K[C, H]$ in two variables C and H, $\alpha = (H - r/(1 - r^2))(H - r^3/(1 - r^2))/r^2(r + r^{-1})$. By Theorem 1.2, $\mathscr{K}(W) = 3$. Let $S^{-1}W$ be the localization of W at $S = \{C^i H^j\}$. Then

$$\mathscr{K}(S^{-1}W) = \begin{cases} 2, & r \text{ is not a root of } 1; \\ 3, & \text{otherwise}. \end{cases}$$

The coordinate ring $\mathscr{M} = \mathscr{O}_q(M_2(K))$ of *quantum* 2×2 matrices is the algebra generated by elements a, b, c, d subject to the relations ($q \in K, q^4 \neq 0, 1$)

$$ab = q^2 ba, \qquad ac = q^2 ca, \qquad db = q^{-2} bd, \qquad dc = q^{-2} cd,$$

$$bc = cb, \qquad ad - da = (q^2 - q^{-2}) bc.$$

The quantum determinant $C = ad - q^2 bc = da - q^{-2} bc$ belongs to the centre of $\mathscr{O}_q(M_2(K))$. The ring $\mathscr{O}_q(M_2(K))$ is a GWA [Bav3]

$$\mathscr{O}_q(M_2(K)) \simeq K[b, c, C](\sigma, C + q^{-2} bc),$$

$$a \leftrightarrow X, d \leftrightarrow Y, b \leftrightarrow b, c \leftrightarrow c, da - q^2 bc \leftrightarrow C,$$

where $K[b, c, C]$ is the polynomial ring in 3 variables; $\sigma(b) = q^2 b$, $\sigma(c) = q^2 c$, $\sigma(C) = C$. The maximal ideal $\mathbf{p} = Db + Dc + DC$ is σ-stable and has $\operatorname{ht} \mathbf{p} = 3 = \mathscr{K}(D)$, by Theorem 1.2

$$\mathscr{K}(\mathscr{O}_q(M_2(K))) = 4.$$

Let $\mathscr{M}_{(b)}$ resp. $\mathscr{M}_{(c)}$ be the localization of $\mathscr{M}$ at $\{b^i, i \in \mathbf{Z}\}$ resp. $\{c^i, i \in \mathbf{Z}\}$. Then

$$\mathscr{M}_{(b)} \simeq K[b, b^{-1}, c, C](\sigma, a = C + q^{-2} bc)$$

and, by the $b - c$-symmetry of the defining relations, $\mathscr{M}_{(b)} \simeq \mathscr{M}_{(c)}$. All maximal ideals $\mathbf{p}(\lambda) = D(b - \lambda) + Dc + DC, 0 \neq \lambda \in K$, contain a and

$\operatorname{ht}\mathbf{p}(\lambda) = 3 = \mathcal{K}(K[b, b^{-1}, c, C])$. Fix λ. Then all $\sigma^i(\mathbf{p}(\lambda)) = \mathbf{p}(q^{-2i}\lambda)$ contain a as well, by Theorem 1.2

$$\mathcal{K}(\mathcal{M}_{(b)}) = \mathcal{K}(\mathcal{M}_{(c)}) = 4.$$

Let $\mathcal{M}_{(b,c)}$ be the localization of $\mathcal{M}$ at $\{b^i c^j, i, j \in \mathbf{Z}\}$. It is clear that

$$\mathcal{M}_{(b,c)} \simeq K[b, b^{-1}, c, c^{-1}, C](\sigma, a = C + q^{-2}bc).$$

By Theorem 1.2,

$$\mathcal{K}(\mathcal{M}_{(b,c)}) = \begin{cases} 3, & q \text{ is not a root of } 1; \\ 4, & \text{otherwise}. \end{cases}$$

The coordinate ring $\mathcal{G} = \mathcal{O}_q(GL_2(K))$ of quantum $GL(2, K)$ is obtained from $\mathcal{M}$ by localizing at $\{C^i, i \geq 0\}$. Then

$$\mathcal{O}_q(GL_2(K)) \simeq K[b, c, C, C^{-1}](\sigma, C + q^{-2}bc).$$

By Theorem 1.2, $\mathcal{K}(\mathcal{O}_q(GL_2(K))) = 4$. Let $S = \{b^i c^j\}$. Then

$$\mathcal{K}(S^{-1}\mathcal{O}_q(GL_2(K))) = \begin{cases} 3, & q \text{ is not a root of } 1; \\ 4, & \text{otherwise}. \end{cases}$$

The ring $\mathcal{S} = \mathcal{M}/\mathcal{M}(C-1)$ is the coordinate ring $\mathcal{O}_q(SL_2(K))$ of quantum $SL(2, K)$ and

$$\mathcal{O}_q(SL_2(K)) \simeq K[b, c](\sigma, 1 + q^{-2}bc), \qquad \sigma(b) = q^2 b, \sigma(c) = q^2 c.$$

The maximal ideal $\mathbf{p} = Db + Dc$ of $D = K[b, c]$ is σ-stable and $\operatorname{ht}\mathbf{p} = 2 = \mathcal{K}(D)$, by Theorem 1.2

$$\mathcal{K}(\mathcal{O}_q(SL_2(K))) = 3.$$

Let A be the localization of $\mathcal{O}_q(SL_2(K))$ at $\{b^i, i \geq 0\}$; then

$$A \simeq K[b, b^{-1}, c](\sigma, 1 + q^{-2}bc),$$

by Theorem 1.2

$$\mathcal{K}(A) = \begin{cases} 2, & q \text{ is not a root of } 1; \\ 3, & \text{otherwise}. \end{cases}$$

The *quantum group* $\mathcal{O}_{q^2}(so(k,3)) = K[H]\langle \sigma; b = (q-q^{-1})H, \rho = 1\rangle$, $\sigma(H) = q^2 H$, $q \in K$ [Sm1], by Lemma 5.1 is isomorphic to a GWA:

$$\mathcal{O}_{q^2}(so(k,3)) = K[H,C](\sigma, a = C + H^2/q(1+q^2)),$$
$$\sigma(H) = q^2 H, \qquad \sigma(C) = C.$$

By Theorem 1.2, $\mathscr{K}(\mathcal{O}_{q^2}(so(k,3))) = 3$. Let $S^{-1} = \{C^i H^j\}$. Then

$$\mathscr{K}\left(S^{-1}\mathcal{O}_{q^2}(so(k,3))\right) = \begin{cases} 2, & q \text{ is not a root of } 1; \\ 3, & \text{otherwise}. \end{cases}$$

The *quantum Heisenberg* algebra [KS, Ma, Ros] is

$$\mathscr{H}_q = K\langle X, Y, H \mid XH = q^2 HX, YH = q^{-2} HY, XY - q^{-2} YX = q^{-1} H\rangle,$$
$$q \in K, q^4 \neq 1.$$

In [Bav3] it is proved that

$$\mathscr{H}_q \simeq K[H,C](\sigma; a = q^2(C - H/q(1-q^4)),$$
$$\sigma(H) = q^2 H, \sigma(C) = q^{-2} C.$$

The maximal ideal $\mathbf{p} = DH + DC$ of $D = K[H,C]$ is σ-stable and $\operatorname{ht}\mathbf{p} = 2 = \mathscr{K}(D)$, by Theorem 1.2,

$$\mathscr{K}(\mathscr{H}_q) = 3.$$

Let $\mathscr{H}_{q,(H)}$ resp. $\mathscr{H}_{q,(C)}$ be the localization of $\mathscr{H}_q$ at $\{H^i, i \geq 0\}$ resp. $\{C^i, i \geq 0\}$. Then

$$\mathscr{H}_{q,(H)} \simeq K[H, H^{-1}, C](\sigma, a) \qquad \text{and} \qquad \mathscr{H}_{q,(C)} \simeq K[H, C, C^{-1}](\sigma, a).$$

By Theorem 1.2

$$\mathscr{K}(\mathscr{H}_{q,(H)}) = \mathscr{K}(\mathscr{H}_{q,(C)}) = \begin{cases} 2, & q \text{ is not a root of } 1; \\ 3, & \text{otherwise}. \end{cases}$$

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