ELSEVIER

Topology and its Applications 55 (1994) 163-183

TOPOLOGY AND ITS APPLICATIONS

Compactifications of products of metric spaces and their relations to Čech–Stone and Smirnov compactifications

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(Received 16 January 1992)
(Revised 29 December 1992)

Abstract

This paper is concerned with compactness and compactifications in the epireflective hull of the subcategory of extended pseudometric spaces of the category of approach spaces. Since in this context, compactness is epireflective, this provides a first characterization of a compactification which is the equivalent of the Čech–Stone compactification for topological spaces. An alternative construction of this compactification reveals that its topological coreflection is the Smirnov compactification of a canonically associated proximity relation. An interesting result which ensues is the extension of the Euclidean metric on the natural numbers to a distance on the underlying set of the Čech–Stone compactification which yields the Čech–Stone compactification as topological coreflection.

Key words: Approach space; Distance; Metric space; Uniform space; Proximity space; Compactification

AMS (MOS) Subj. Class.: 54D30, 54E35, 54E15, 54E05, 18B30

1. Introduction

This paper is a companion to our earlier work [8] in the sense that there we were concerned with completion of products of metric spaces and here we treat compactifications. However our motivation in the present paper is somewhat different. Whereas the completion of a metric space is uniquely defined and again a metric space, there are many different compactifications of metrizable spaces

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SSDI 0166-8641(93)E0042-M

none of which need to be metrizable in general. Especially the most important compactification, i.e., the Čech–Stone compactification of a noncompact metrizable space (or any noncompact space for that matter) never is metrizable. Moreover even if some compactification is metrizable (e.g. the Alexandroff compactification of $\mathbb{R}$) then unless the original metric was bounded it can never be extended to a compatible metric on the compactification. In this paper we show that by using the same technique as in our previous paper, i.e., embedding p-met$^{\infty}$ in the category AP of approach spaces and then considering its epireflective hull $\mathscr{M}$, these problems can be overcome to a large extent. In Sections 2 and 3 we recall the main concepts of the theory of approach spaces which we shall require in the sequel. We take the opportunity also to comment on the relation of approach spaces with topological spaces, uniform spaces and proximity spaces. In Section 3 we further give a number of preliminary results and in Section 4 we construct a compactification in $\mathscr{M}$ which is the categorical equivalent for approach spaces of the Čech–Stone compactification for topological spaces. Actually it coincides with the Čech–Stone compactification on TOP $\cap\, \mathscr{M}$, i.e., on all completely regular topological objects in AP. It also turns out that the topological coreflection of our compactification is the Smirnov compactification of an associated proximity. In Section 5, in the case of $\mathbb{N}$, when Smirnov and Čech–Stone compactifications coincide we show that our construction gives an extension of the usual metric on $\mathbb{N}$ to a distance on $\beta\mathbb{N}$ in such a way that its topological coreflection is the Čech–Stone topology.

2. Approach spaces

Approach spaces, which were introduced in [6], are spaces equipped with a structure which generalizes at the same time a topology (in the form of its associated closure operator) and a metric (in the form of its associated distance function between points and sets). They provide a natural setting for the solution of the problem that a product of metric spaces is not necessarily metrizable or that even when it is metrizable, then only exceptionally so by a metric which coincides with the original metrics on the component spaces. In order to explain this in detail we recall some concepts, and show how approach spaces are related with topological spaces and with extended pseudo-metric spaces.

Definitions 2.1. If X is a set, a map $\delta : X \times 2^X \to [0, \infty]$ is a *distance* on X if it satisfies

(D1) $\forall x \in X$: $\delta(x, \emptyset) = \infty$,

(D2) $\forall x \in X$, $\forall A \in 2^X$: $x \in A \Rightarrow \delta(x, A) = 0$,

(D3) $\forall x \in X$, for each finite family $(A_j)_{j \in J}$ of subsets of X: $\delta(x, \bigcup_{j \in J} A_j) = \min_{j \in J} \delta(x, A_j)$,

(D4) $\forall x \in X$, $\forall \varepsilon \geqslant 0$, $\forall A \in 2^X$: $\delta(x, A) \leqslant \delta(x, A^{(\varepsilon)}) + \varepsilon$, where

$$A^{(\varepsilon)} = \{x \in X \mid \delta(x, A) \leqslant \varepsilon\}.$$

The pair (X, δ) is called an *approach space*.

If (X, δ_X) and (Y, δ_Y) are approach spaces, a mapping $f: X \to Y$ is called a *contraction* if it fulfils

$$\delta_Y(f(x), f(A)) \leqslant \delta_X(x, A) \quad \forall x \in X, \forall A \in 2^X.$$

The category with objects all approach spaces and morphisms all contractions, denoted AP, is a topological construct. For details on topological categories and their importance in mathematics we refer to [1,4]. We recall that a category is called a construct if it is concrete over SET, i.e., if its objects are sets with a structure and its morphisms are certain functions between the underlying sets. A construct is called topological, if on any set there is only a set (as opposed to a proper class) of structures, if on any one-point set there is exactly one structure and if initial (and hence also final) structures exist. Since we shall often be dealing with subspaces and products, we need to describe initial structures in AP. For this it is useful to introduce the approach system, a concept which also axiomatizes AP.

A collection of lattice theoretical ideals $\Phi := (\Phi(x))_{x \in X}$ in $[0, \infty]^X$ is called an *approach system* if it satisfies

(A1) $\forall x \in X, \forall \phi \in \Phi(x)$: $\phi(x) = 0$,

(A2) $\forall x \in X, \forall \phi \in [0, \infty]^X$:

$$\left(\forall N \in \mathbb{R}_+^*, \forall \varepsilon \in \mathbb{R}_+^*, \exists \phi_\varepsilon^N \in \Phi(x) \text{ such that } \phi \wedge N \leqslant \phi_\varepsilon^N + \varepsilon \Rightarrow \phi \in \Phi(x)\right),$$

(A3) $\forall x \in X, \forall \phi \in \Phi(x), \forall N \in \mathbb{R}_+^*, \exists (\phi_x)_{x \in X} \in \prod_{x \in X} \Phi(x)$:

$$\forall z, y \in X: \quad \phi_x(z) + \phi_z(y) \geqslant \phi(y) \wedge N.$$

For $x \in X$, $\Phi(x)$ is called the ideal of local distances at x. Condition (A2) is technical and conceptually uninteresting, but (A1) and (A3) explain the terminology. Each $\phi \in \Phi(x)$ is a numerical function whose value $\phi(y)$ in any point $y \in X$ has to be interpreted as "the distance from x to y according to ϕ". Of course then (A1) has to hold. (A3) says that a triangle inequality, for all the local distances as a whole, also is fulfilled.

Approach systems and distances (as neighborhoods and closure operators in topology) are different, but equivalent, instances of the same structure. How we go from one to the other is given by the following formulas. If (X, δ) is an approach space, an approach system is associated with δ in the following way: for each $x \in X$, put

$$\Phi_\delta(x) := \left\{\phi: X \to [0, \infty] \mid \forall A \in 2^X: \inf_{a \in A} \phi(a) \leqslant \delta(x, A)\right\}.$$

Conversely, any approach system $\Phi = (\Phi(x))_{x \in X}$ on X determines a distance δ_Φ on X defined by

$$\delta_\Phi(x, A) := \sup_{\phi \in \Phi(x)} \inf_{a \in A} \phi(a) \quad \forall x \in X, \forall A \in 2^X.$$

Moreover we have that $\delta_{\Phi_\delta} = \delta$ and $\Phi_{\delta_\Phi} = \Phi$. Thus an approach space can be given as a set equipped with either a distance δ or an approach structure $(\Phi(x))_{x \in X}$.

A basis for an approach system $(\Phi(x))_{x \in X}$ on X is a family $\Lambda := (\Lambda(x))_{x \in X}$ such that $\forall x \in X$, $\Lambda(x) \subset \Phi(x)$ is a basis for an ideal in $[0, \infty]^X$ which generates $\Phi(x)$ in the sense that:

$$\forall \phi \in \Phi(x),\ \forall \varepsilon,\ N \in \mathbb{R}_+^*,\ \exists \lambda \in \Lambda(x):\quad \phi \wedge N \leqslant \lambda + \varepsilon.$$

Such a basis contains all the essential information about the approach structure: given a basis $(\Lambda(x))_{x \in X}$ for the approach system, we can recover not only the entire ideal of the local distances but also the associated distance δ by $\delta(x, A) = \sup_{\phi \in \Lambda(x)} \inf_{a \in A} \phi(a)$ for all $x \in X$ and for all $A \in 2^X$. There is also an easy characterization of morphisms in terms of the approach system: If $(\Phi_X(x))_{x \in X}$ is an approach system on X and $(\Phi_Y(y))_{y \in Y}$ is an approach system on Y with basis $(\Lambda_Y(y))_{y \in Y}$, then a mapping $f: X \to Y$ is a contraction if and only if

$$\forall x \in X \text{ and } \forall \phi \in \Phi_Y(f(x)):\quad \phi \circ f \in \Phi_X(x),$$

or equivalently, if

$$\forall x \in X \text{ and } \forall \lambda \in \Lambda_Y(f(x)):\quad \lambda \circ f \in \Phi_X(x).$$

In the following proposition, initial structures in AP are described by means of a basis for their approach system. If J is a set, $2^{(J)}$ denotes the set of finite subsets of J.

Proposition 2.2 [6]. *If* $(f_j : X \to (X_j, \Phi_j))_{j \in J}$ *is a source, and for each* $j \in J$, $(\Lambda_j(x))_{x \in X_j}$ *is a basis for the approach system* $(\Phi_j(x))_{x \in X_j}$, *then a basis for the approach system of the initial object* (X, Φ) *is given by* $(\Lambda(x))_{x \in X}$, *with*

$$\Lambda(x) := \left\{ \sup_{k \in K} \phi_k \circ f_k \mid K \in 2^{(J)},\ \phi_k \in \Lambda_k(f_k(x))\ \forall k \in K \right\} \quad \forall x \in X.$$

Theorem 2.3 [6]. TOP *is embedded as a concretely coreflective subcategory of* AP. *Given a space* $(X, \mathscr{T}) \in |\text{TOP}|$, *if we define a distance* $\delta_{\mathscr{T}}$ *on* X *by*:

$$\delta_{\mathscr{T}}(x, A) = \begin{cases} 0, & \text{if } x \in \overline{A}, \\ \infty, & \text{if } x \notin \overline{A}, \end{cases}$$

then the functor

$$\text{TOP} \to \text{AP},$$
$$(X, \mathscr{T}) \to (X, \delta_{\mathscr{T}}),$$
$$f \to f$$

is an embedding.

The TOP-*coreflection of a space* $(X, \delta) \in |\text{AP}|$ *is given by* $\text{id}_X : (X, \mathscr{T}_\delta) \to (X, \delta)$, *where* $\mathscr{T}_\delta$ *is the topology with closure operator* cl_δ *determined by*

$$\text{cl}_\delta(A) := \{x \in X \mid \delta(x, A) = 0\}.$$

Actually TOP is even embedded as a bireflective subcategory of AP (a property worth noting since most familiar categories as e.g. TOP and UNIF do not have simultaneously bireflectively and coreflectively embedded subcategories). Since, however, we shall not require the TOP-bireflection in this paper, we refrain from recalling it.

It is also useful to know that there is another description of the TOP-coreflection of an approach space, in terms of the approach system rather than in terms of the distance. If $(A(x))_{x\in X}$ is a basis for the approach system, the collection

$$\mathscr{N}(x)=\{(\phi<\varepsilon)\mid\phi\in A(x),\ \varepsilon\in\mathbb{R}_+^*\}$$

is a basis for the neighborhood filter of any $x\in X$ in the TOP-coreflection.

If X is a set, a map $d: X\times X\to[0,\infty]$ is an extended pseudo-metric (or shortly, an ∞-p-metric) if it fulfils

(1) $d(x, x)=0$, $\forall x\in X$,

(2) $d(x, y)=d(y, x)$, $\forall x, y\in X$,

(3) $d(x, y)\leqslant d(x, z)+d(z, y)$, $\forall x, y, z\in X$.

The pair (X, d) is an extended pseudo-metric space (or an ∞-p-metric space.) If (X, d) and (X', d') are ∞-p-metric spaces, a map $f: X\to X'$ is called nonexpansive if $d'(f(x), f(y))\leqslant d(x, y)$ holds for each pair of elements x and y of X. The category of ∞-p-metric spaces and nonexpansive maps is denoted by p-met$^\infty$.

Theorem 2.4 [6]. p-met$^\infty$ *is embedded as a concretely coreflective subcategory of* AP. *Given a space* $(X, d)\in|\text{p-met}^\infty|$, *if we define the distance* δ_d *on* X *by*

$$\delta_d(x, A):=\inf_{a\in A} d(x, a)$$

then the functor

$$\text{p-met}^\infty\to\text{AP},$$

$$(X, d)\to(X, \delta_d),$$

$$f\to f$$

is an embedding.

The p-met$^\infty$*-coreflection of a space* $(X, \delta)\in|\text{AP}|$ *is given by* $\text{id}_X:(X, d_\delta)\to(X, \delta)$, *where* d_δ *is the* ∞-p-*metric determined by*

$$d_\delta(x, y):=\delta(x, \{y\})\vee\delta(y, \{x\}).$$

Again it is useful to know that we can characterize, this time not the p-met$^\infty$-coreflection, but rather, the embedding of an ∞-p-metric space in terms of its approach system. If $(X, d)\in|\text{p-met}^\infty|$, the approach system associated with δ_d is given by

$$\Phi_d(x):=\{\lambda\leqslant d(x, \cdot)\},$$

in other words a basis for the approach system is given by $(\{d(x, \cdot)\})_{x\in X}$ [6].

As in the preceding two theorems we often use the same notation for a topological or a ∞-p-metric space and its embedding in AP. The following result gives internal characterizations of topological and metric objects in AP.

Theorem 2.5 [6]. *An approach space* (X, δ) *is topological (i.e., is in* TOP), *if and only if the following condition is fulfilled*:

$$\delta(X \times 2^X) = \{0, \infty\}$$

and is metric (i.e., is in p-met$^\infty$), *if and only if the following strengthening of* (D3) *holds*:

$\forall x \in X$, *for each family* $(A_j)_{j\in J}$ *of subsets of* X:

$$\delta\left(x, \bigcup_{j\in J} A_j\right) = \inf_{j\in J} \delta(x, A_j).$$

3. Products of metric spaces

Approach spaces are not only related to topological and metric spaces as we have seen in the foregoing section, but, equally important for the investigations of the present paper, also to uniform and proximity spaces. However, whereas approach spaces form an immediate generalization of both topological and metric spaces, their relationship to uniform and proximity spaces is considerably more intricate. We shall therefore, in this section, restrict ourselves to pointing out those relationships which are of immediate interest for the present paper.

Generally the product of a family of ∞-p-metric spaces in AP fails to be an ∞-p-metric space. If $(X_i, d_i)_{i\in I}$ is a family of nonempty ∞-p-metric spaces, let $X = \prod_{i\in I} X_i$ be the cartesian product, with $\mathrm{pr}_i : X \to X_i$ the canonical projections. The product of $(X_i, d_i)_{i\in I}$ is $(\mathrm{pr}_i : (X, \delta) \to (X_i, d_i))_{i\in I}$, where a basis for the approach system $(\Phi(x))_{x\in X}$ is given by

$$\Lambda(x) = \left\{\sup_{k\in K} d_k(\mathrm{pr}_k(x), \mathrm{pr}_k(\cdot)) \mid K \in 2^{(I)}\right\}.$$

Then for $x \in X$ and $A \subset X$, we have

$$\delta(x, A) = \sup_{K\in 2^{(I)}} \inf_{a\in A} \sup_{k\in K} d_k(\mathrm{pr}_k(x), \mathrm{pr}_k(a)).$$

Since the topological coreflection of (X_j, d_j) is the p-metrizable space $(X_j, \mathcal{T}_{d_j})$, and the topological coreflection of (X, δ) is the product of the spaces $(X_j, \mathcal{T}_{d_j})$, the problem that p-met$^\infty$ is not closed for the formation of products is solved by replacing ∞-p-metrics by approach structures; i.e., embedding p-met$^\infty$ in AP. For this reason approach spaces (X, δ_X) which are isomorphic with a subspace of a product of a family of ∞-p-metric spaces are of particular interest. They constitute $\mathcal{M}$, the epireflective hull of p-met$^\infty$ in AP, a subcategory which was already introduced in [8].

A family $\mathcal{D}$ of ∞-p-metrics on X generates (X, δ) when

(1) $\delta(x, A) = \sup_{d\in\mathcal{D}} \inf_{a\in A} d(x, a)$, $\forall x \in X$, $\forall A \in 2^X$.

(2) $\mathcal{D}$ is closed for the formation of finite suprema.

If a family $\mathscr{D}$ fulfilling (1) exists, the family consisting of finite suprema of elements of $\mathscr{D}$ generates (X, δ). Further note that if $\mathscr{D}$ generates (X, δ) then the family

$$\mathscr{D}' = \{d \wedge n \mid d \in \mathscr{D},\ n \in \mathbb{N}\}$$

also generates (X, δ), and $\mathscr{D}'$ consists of bounded pseudo-metrics.

Remark that in [3] uniform structures are introduced as families $\mathscr{D}$ of pseudo-metrics fulfilling property (2) above, and another property which implies that if $d \in \mathscr{D}$ and $\alpha > 0$ then also $\alpha d \in \mathscr{D}$. This shows at the same time a relationship with uniform structures and a fundamental difference, since it is this second property which destroys any numerical notion of distance.

Subsets of $\mathbb{R}$ will be regarded as (metric) approach spaces with the usual distance d_e. $\mathscr{G}^*(X)$ (or simply $\mathscr{G}^*$ when no confusion is possible) denotes the collection of all contractions from (X, δ) to $(\mathbb{R}, d_e)$ having a bounded image. A family $\mathscr{F} \subset \mathscr{G}^*(X)$ is said to generate (X, δ) when

$$(f : (X, \delta) \to (\mathbb{R}, d_e))_{f \in \mathscr{F}}$$

is an initial source. If $\mathscr{F} \subset \mathscr{G}^*(X)$ generates (X, δ) then so does the family

$$\mathscr{D}_{\mathscr{F}} := \left\{ \sup_{g \in A} | g(\cdot) - g(\cdot) | \,\middle|\, A \in 2^{(\mathscr{F})} \right\}$$

of pseudo-metrics. And if (X, δ) is generated by a family of bounded pseudo-metrics $\mathscr{D}$, then

$$\mathscr{G} := \{d(x, \cdot) \mid d \in \mathscr{D},\ x \in X\}$$

is a subfamily of $\mathscr{G}^*(X)$ which generates (X, δ). These considerations lead to the following characterization of $\mathscr{M}$-objects:

Proposition 3.1. *For an approach space (X, δ) the following are equivalent*:

(1) $(X, \delta) \in |\mathscr{M}|$.
(2) *There exists a family $\mathscr{D}$ of ∞-*p*-metrics which generates (X, δ).*
(3) *There exists a family $\mathscr{D}$ of bounded* p*-metrics which generates (X, δ).*
(4) *There exists a family $\mathscr{F} \subset \mathscr{G}^*(X)$ which generates (X, δ).*
(5) *$\mathscr{G}^*(X)$ generates (X, δ).*

Again we make the convention that in AP topological properties refer to the topological coreflection. The subcategory of $\mathscr{M}$ consisting of Hausdorff (or separated) objects is denoted $\mathscr{M}_2$.

A family $\mathscr{D}$ of ∞-p-metrics on X is said to separate points provided that for two distinct points x and y in X there is a $d \in \mathscr{D}$ such that $d(x, y) > 0$. A family $\mathscr{G}$ of real-valued contractions on (X, δ) is called point-separating if for two distinct points x and y in X there is a $g \in \mathscr{G}$ such that $g(x) \neq g(y)$.

Proposition 3.2. *For $(X, \delta) \in |\mathscr{M}|$, the following are equivalent*:

(1) $(X, \delta) \in |\mathscr{M}_2|$.
(2) *Each family $\mathscr{D}$ of pseudo-metrics which generates (X, δ) is point-separating.*

(3) *Each subfamily* $\mathscr{F}$ *of* $\mathscr{G}^*(X)$ *which generates* (X, δ) *is point-separating.*
(4) $\mathscr{G}^*(X)$ *is point-separating.*
(5) (X, δ) *is isomorphic to a subspace of a product of bounded subsets of* $\mathbb{R}$.

Proof. In order to prove that (4) implies (5), let $\mathbb{R}^{\mathscr{G}^*}$ be the product of $((\mathbb{R}_g, d_e))_{g \in \mathscr{G}^*(X)}$ where $\mathbb{R}_g = \mathbb{R}$ for each $g \in \mathscr{G}^*(X)$, and consider the map

$$e : X \to \mathbb{R}^{\mathscr{G}^*}$$

determined by $\mathrm{pr}_g \circ e = g$, for all $g \in \mathscr{G}^*(X)$. e is a monomorphism since $\mathscr{G}^*(X)$ separates points, and it is an embedding since $\mathscr{G}^*(X)$ generates (X, δ). The other implications are easy to verify. □

Proposition 3.3 [8]. (1) *A morphism in* $\mathscr{M}_2$ *is an epimorphism if and only if it has a dense image.*

(2) *A morphism in* $\mathscr{M}_2$ *is an extremal monomorphism if and only if it is an embedding with a closed image.*

Proposition 3.4. *For* $(X, \delta) \in |\mathscr{M}_2|$ *the following are equivalent:*
(1) (X, δ) *is isomorphic to a closed subspace of a product of compact subsets of* $(\mathbb{R}, d_e)$.
(2) (X, δ) *is compact.*

Proof. That (1) implies (2) is a consequence of the fact that an isomorphism of approach spaces induces a homeomorphism between the topological coreflections. For the proof that (2) implies (1) assume that (X, δ_X) is an $\mathscr{M}_2$-object with a compact topological coreflection. For each $g \in \mathscr{G}^*(X)$ let I_g be a compact subset of $\mathbb{R}$ containing the image of g. If e is the canonical embedding of (X, δ_X) in the product of the family $(I_g, d_e)_{g \in \mathscr{G}^*(X)}$ then $e(X)$ is compact, hence closed. □

Proposition 3.5. *The full and isomorphism closed subcategory of* $\mathscr{M}_2$ *whose objects are compact is an epireflective subcategory of* $\mathscr{M}_2$.

Proof. Indeed from Propositions 3.2 and 3.3 we may conclude that it is the epireflective hull in $\mathscr{M}_2$ of the family of bounded subsets of $\mathbb{R}$. □

If cREG denotes the category of completely regular topological spaces, and if $\boldsymbol{T}_{3\frac{1}{2}}$ denotes the category of the Tychonoff topological spaces, i.e., the subcategory of TOP whose objects are completely regular and T_1, then from the previous results we may conclude that $\mathscr{M}$ and $\mathscr{M}_2$ play a similar role in AP as cREG respectively $\boldsymbol{T}_{3\frac{1}{2}}$ do in TOP. Moreover, as is proven in [8], topological objects in $\mathscr{M}$ or $\mathscr{M}_2$ are exactly the completely regular spaces (respectively Tychonoff spaces). This again is an indication of the relation which exists between uniformities and approach structures in $\mathscr{M}$.

If $\mathscr{D}$ is a family of ∞-p-metrics which generates (X, δ) then $\mathscr{D}$ induces a uniformity $\mathscr{U}(\mathscr{D})$ on X which is compatible with the topological coreflection of

(X, δ). By way of this uniformity, each such family induces a proximity relation on X as well. In the future we will have more recourse to the proximity structure. For terminology and results we refer to [9].

The following result demonstrates the relationship which exists between approach structures in $\mathscr{M}$ and the proximity associated with it via $\mathscr{U}(\mathscr{D})$.

Proposition 3.6. *If (X, δ) is generated by a family $\mathscr{D}$ of pseudo-metrics then the relation $\vartriangle_{\mathscr{D}}$ on $2^X \times 2^X$ defined by*

$$A \vartriangle_{\mathscr{D}} B \quad \textit{if and only if} \quad \sup_{d \in \mathscr{D}} \inf_{a \in A} \inf_{b \in B} d(a, b) = 0$$

is a proximity relation on X which is compatible with the topological coreflection of (X, δ): For each $x \in X$ and each $A \subset X$ we have

$$A \vartriangle_{\mathscr{D}} \{x\} \quad \textit{if and only if} \quad \delta(x, A) = 0.$$

The proximity $\vartriangle_{\mathscr{D}}$ actually is the proximity induced by $\mathscr{U}(\mathscr{D})$. So given δ and $\mathscr{D}$ as above, the following are equivalent:

(1) $(X, \vartriangle_{\mathscr{D}})$ is a separated proximity space.

(2) $(X, \delta) \in |\mathscr{M}_2|$.

If $A \vartriangle_{\mathscr{D}} B$ then A and B are called $\mathscr{D}$-proximal. In the other case we write $A \not\vartriangle_{\mathscr{D}} B$.

Definition 3.7. For $(X, \delta) \in |\mathscr{M}_2|$, $\mathscr{D}_\delta^b$ denotes the collection of all bounded pseudo-metrics d on X such that

$$\inf_{a \in A} d(x, a) \leqslant \delta(x, A) \quad \forall x \in X, \forall A \in 2^X.$$

$\mathscr{D}_\delta^b$ is closed for finite suprema and generates (X, δ). Obviously each family $\mathscr{D}$ of bounded pseudo-metrics which generates (X, δ) is contained in $\mathscr{D}_\delta^b$. Among the uniformities and proximities on X which are associated with a generating family $\mathscr{D}$, we clearly have that $\mathscr{U}(\mathscr{D}_\delta^b)$ and $\vartriangle_{\mathscr{D}_\delta^b}$ are the finest.

Example 3.8. It is possible that two families $\mathscr{D}$ and $\mathscr{D}'$ of pseudo-metrics on a set X generate the same $\mathscr{M}$-object (X, δ) but induce different uniformities and proximities on X. Let $X = \mathbb{R}$ and $\mathscr{D} = \{d_e\}$. For $\alpha > 0$, define

$$g_\alpha : \mathbb{R} \to \mathbb{R}: \quad x \to \begin{cases} -\alpha, & \text{if } x \leqslant -\alpha; \\ x, & \text{if } -\alpha \leqslant x \leqslant \alpha; \\ \alpha, & \text{if } \alpha \leqslant x. \end{cases}$$

Then $(g_\alpha : (\mathbb{R}, d_e) \to (\mathbb{R}, d_e))_{\alpha > 0}$ is an initial source in AP. If we put

$$d_\alpha(x, y) = |g_\alpha(x) - g_\alpha(y)| \quad \forall x, y \in X \text{ and } \forall \alpha > 0$$

then obviously $\alpha \leqslant \beta$ implies $d_\alpha \leqslant d_\beta$, hence $\mathscr{D}' = \{d_\alpha | \alpha > 0\}$ is closed for finite suprema. Moreover $\mathscr{D}'$ generates $(\mathbb{R}, d_e)$.

The uniformity generated by $\mathscr{D}$ is the usual uniformity on $\mathbb{R}$, and is finer than $\mathscr{U}(\mathscr{D}')$ because $d_\alpha \leqslant d_e$, $\forall \alpha > 0$. For the same reason, $\vartriangle_{\mathscr{D}}$, the usual proximity relation on $\mathbb{R}$, is finer than $\triangle_{\mathscr{D}'}$. However if A is the collection of odd integers and B is the collection of even integers then

$$\sup_{\alpha>0} \inf_{a\in A} \inf_{b\in B} d_\alpha(a, b) = 0,$$

while A and B obviously are not proximal in the usual structure, so $\triangle_{\mathscr{D}}$ is strictly finer than $\triangle_{\mathscr{D}'}$, and $\mathscr{U}(\mathscr{D})$ is strictly finer than $\mathscr{U}(\mathscr{D}')$.

4. $\mathscr{M}_2$-compactifications

Definition 4.1. An *$\mathscr{M}_2$-compactification* of an approach space X is a pair (X', e) where X' is a compact $\mathscr{M}_2$-object, and $e: X \to X'$ is a dense embedding. In most cases it is clear from the context how X is embedded in X'; then we will refer to X' as a compactification.

Each $(X, \delta) \in |\mathscr{M}_2|$ has an $\mathscr{M}_2$-compactification: It suffices to consider a subset $\mathscr{G}$ of $\mathscr{G}^*(X)$ which generates (X, δ). For each $g \in \mathscr{G}$ let I_g be a bounded subset of $\mathbb{R}$ containing the image of g. If $(\mathrm{pr}_g : (X', \delta_{\mathscr{G}}) \to (I_g, d_e))_{g\in\mathscr{G}}$ is the product of the family $(I_g)_{g\in\mathscr{G}}$ then the evaluation map

$$e_{\mathscr{G}} : (X, \delta) \to (X', \delta_{\mathscr{G}}),$$
$$x \to (g(x))_{g\in\mathscr{G}}$$

is an embedding, and $((\overline{e_{\mathscr{G}}(X)}, \delta_{\mathscr{G}}), e_{\mathscr{G}})$ obviously is an $\mathscr{M}_2$-compactification of X. We denote this compactification $(\boldsymbol{e}_{\mathscr{G}}X, \delta_{\mathscr{G}})$, or shortly $\boldsymbol{e}_{\mathscr{G}}X$.

Now $(\boldsymbol{e}_{\mathscr{G}}X, \delta_{\mathscr{G}})$ is generated by the collection

$$\mathscr{D}^*_{\mathscr{G}} = \{d_\Lambda \mid \Lambda \in 2^{(\mathscr{G})}\},$$

where for each $\Lambda \in 2^{(\mathscr{G})}$, d_Λ is the bounded pseudo-metric defined by

$$d_\Lambda : \boldsymbol{e}_{\mathscr{G}}X \times \boldsymbol{e}_{\mathscr{G}}X \to \mathbb{R},$$
$$(p, q) \to d_\Lambda(p, q) = \sup_{g\in\Lambda} |\mathrm{pr}_g(p) - \mathrm{pr}_g(q)|.$$

Remark. If (X, δ) already is a compact $\mathscr{M}_2$-object and $\mathscr{G} \subset \mathscr{G}^*(X)$ generates (X, δ) then

$$e_{\mathscr{G}} : (X, \delta) \to (\boldsymbol{e}_{\mathscr{G}}X, \delta_{\mathscr{G}})$$

is an isomorphism.

Proposition 4.2. *Every $\mathscr{M}_2$-compactification of $(X, \delta) \in |\mathscr{M}_2|$ is of type $\boldsymbol{e}_{\mathscr{F}}X$ for some subfamily $\mathscr{F}$ of $\mathscr{G}^*$.*

Proof. Consider an $\mathscr{M}_2$-compactification (X', α) of X and let $\mathscr{G}'$ be any subfamily of $\mathscr{G}^*(X')$ for which

$$(g : (X', \delta_{X'}) \to (\mathbb{R}, d_e))_{g\in\mathscr{G}'}$$

is an initial source. Then $e_{\mathscr{G}'}:(X', \delta_{X'}) \to (e_{\mathscr{G}'}(X), \delta_{\mathscr{G}'})$ is an isomorphism. If we put

$$\mathscr{F} = \{g \circ \alpha \mid g \in \mathscr{G}'\}$$

then $\mathscr{F}$ separates points and generates (X, δ). Consider the $\mathscr{M}_2$-compactification $\boldsymbol{e}_{\mathscr{F}}X$. For each $x \in X$, we have $e_{\mathscr{G}'} \circ \alpha(x) = e_{\mathscr{F}}(x)$.

$$\begin{tikzcd}
e_{\mathscr{F}}(X) \arrow[rr, hook] & & e_{\mathscr{G}'}(X) \arrow[dd, "\mathrm{pr}_g"] \\
X \arrow[u, "e_{\mathscr{F}}"] \arrow[r, "\alpha"] & X' \arrow[ur, "e_{\mathscr{G}'}"] \arrow[dr, "g"'] & \\
& & \mathbb{R}
\end{tikzcd}$$

As $e_{\mathscr{G}'}$ is continuous, $X' = \overline{\alpha(X)}$ is mapped to $\overline{e_{\mathscr{F}}(X)} = \boldsymbol{e}_{\mathscr{F}}X$. Moreover $e_{\mathscr{G}'}: X' \to \boldsymbol{e}_{\mathscr{F}}X$ is onto, the image being a closed subset of $\boldsymbol{e}_{\mathscr{F}}X$ containing $e_{\mathscr{F}}(X)$. We conclude that X' is isomorphic with $\boldsymbol{e}_{\mathscr{F}}X$. □

Theorem 4.3. *Every $(X, \delta) \in |\mathscr{M}_2|$ has an $\mathscr{M}_2$-compactification $((X', \delta_{X'}), e')$ with the following equivalent properties:*

(1) *For every $f \in \mathscr{G}^*(X)$ there exists a unique $\bar{f} \in \mathscr{G}^*(X')$ such that $\bar{f} \circ e' = f$.*

(2) *For each contraction g from (X, δ) into a compact $\mathscr{M}_2$-space (Y, δ_Y) there exists a unique contraction $\bar{g}:(X', \delta_{X'}) \to (Y, \delta_Y)$ such that $\bar{g} \circ e' = g$.*

This compactification is essentially unique: Any other $\mathscr{M}_2$-compactification of (X, δ) with these properties is isomorphic with $(X', \delta_{X'})$ under an isomorphism which leaves X pointwise fixed.

Proof. If $(X, \delta_X) \in |\mathscr{M}_2|_2$ consider the $\mathscr{M}_2$-compactification $\boldsymbol{e}_{\mathscr{G}^*}X$.

For each $f \in \mathscr{G}^*$ let $\bar{f}: \boldsymbol{e}_{\mathscr{G}^*}X \to \mathbb{R}$ be the restriction of the canonical projection pr_f. Then $\bar{f} \circ e_{\mathscr{G}^*} = f$ holds.

If $h \in \mathscr{G}^*(\boldsymbol{e}_{\mathscr{G}^*}X)$ is another such extension then the set

$$\{x \in \boldsymbol{e}_{\mathscr{G}^*}X \mid \bar{f}(x) = h(x)\}$$

is a closed subspace of $\boldsymbol{e}_{\mathscr{G}^*}X$ containing $e_{\mathscr{G}^*}(X)$, hence $\bar{f} = h$ follows. This proves the existence of an $\mathscr{M}_2$-compactification satisfying (1).

Since $\mathscr{M}_2$ is generated by the bounded subsets of $\mathbb{R}$ (Proposition 2.5), equivalence of (1) and (2) can be proved by essentially the same argumentation as in TOP. (For example, see [3].)

Finally we show that such an $\mathscr{M}_2$-compactification is unique: If $((X_1, \delta_1), e_1)$ and $((X_2, \delta_2), e_2)$ are $\mathscr{M}_2$-compactifications of (X, δ) both fulfilling (2) then we can find contractions $\psi_2:(X_1, \delta_1) \to (X_2, \delta_2)$ and $\psi_1:(X_2, \delta_2) \to (X_1, \delta_1)$ such that $e_2 = \psi_2 \circ e_1$ and $e_1 = \psi_1 \circ e_2$.

$$\begin{tikzcd}
& (X, \delta) \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & \\
(X_1, \delta_1) \arrow[rr, shift left, "\psi_2"] & & (X_2, \delta_2) \arrow[ll, shift left, "\psi_1"]
\end{tikzcd}$$

As e_1 and e_2 are monomorphisms, it follows that

$$\psi_2 \circ \psi_1 = \mathrm{id}_{X_2},$$

and

$$\psi_1 \circ \psi_2 = \mathrm{id}_{X_1}. \qquad \square$$

The following characterization of $\boldsymbol{e}_{\mathscr{G}^*}X$ is a consequence of Propositions 3.4 and 3.5 and the foregoing theorem.

Corollary 4.4. *For* $(X, \delta) \in |\mathscr{M}_2|$, *the* $\mathscr{M}_2$*-compactification*

$$e_{\mathscr{G}^*} : (X, \delta) \to (\boldsymbol{e}_{\mathscr{G}^*}X, \delta_{\mathscr{G}^*})$$

is the epireflection of (X, δ) *in the subcategory of the compact* $\mathscr{M}_2$*-spaces.*

Remark. Another characterization of $(\boldsymbol{e}_{\mathscr{G}^*}X, \delta_{\mathscr{G}^*})$ is given as follow: If $((X_1, \delta_1), e_1)$ and $((X_2, \delta_2), e_2)$ are $\mathscr{M}_2$-compactifications of $(X, \delta) \in |\mathscr{M}_2|$ we say that (X_1, δ_1) is larger than (X_2, δ_2) if and only if there exists a contraction $g : (X_1, \delta_1) \to (X_2, \delta_2)$ such that $g \circ e_1 = e_2$. Then $((\boldsymbol{e}_{\mathscr{G}^*}X, \delta_{\mathscr{G}^*}), e_{\mathscr{G}^*})$ is the largest $\mathscr{M}_2$-compactification of (X, δ).

None of the previous characterizations give a concrete description of $(\boldsymbol{e}_{\mathscr{G}^*}X, \delta_{\mathscr{G}^*})$ in case (X, δ) is not compact. An alternate, more tangible, description of $\mathscr{M}_2$-compactifications in general will follow.

Proposition 4.5. *If* X *is a Tychonoff topological space then* $(\boldsymbol{e}_{\mathscr{G}^*}X, \delta_{\mathscr{G}^*})$ *is the Čech–Stone compactification of* X.

Proof. By Corollary 4.4 and the fact that the Čech–Stone compactification of a $\boldsymbol{T}_{3\frac{1}{2}}$ space is the reflection in the category of compact Hausdorff spaces, it is sufficient to prove that $(\boldsymbol{e}_{\mathscr{G}^*}X, \delta_{\mathscr{G}^*})$ is a topological object in $|\mathscr{M}_2|$.

If $z \in \boldsymbol{e}_{\mathscr{G}^*}X$ and $B \subset \boldsymbol{e}_{\mathscr{G}^*}X$ then

$$\delta_{\mathscr{G}^*}(z, B) = \sup_{\Lambda \in 2^{(\mathscr{G}^*)}} \inf_{b \in B} \sup_{g \in \Lambda} |\mathrm{pr}_g(z) - \mathrm{pr}_g(b)|.$$

Now $\mathscr{G}^* = C^*(X)$ so if $g \in \mathscr{G}^*$ and $\alpha \in \mathbb{R}$ then $\alpha g \in \mathscr{G}^*$. Moreover we have $\mathrm{pr}_{\alpha g} = \alpha \mathrm{pr}_g$, since both $\mathrm{pr}_{\alpha g}$ and $\alpha \mathrm{pr}_g$ are contractions from $\boldsymbol{e}_{\mathscr{G}^*}X$ to $\mathbb{R}$, extending $\alpha g \in \mathscr{G}^*$ over $\boldsymbol{e}_{\mathscr{G}^*}X$.

If $\delta_{\mathscr{G}^*}(z, B) > 0$ then $\inf_{b \in B} \sup_{g \in \Lambda} |\mathrm{pr}_g(z) - \mathrm{pr}_g(b)| > 0$ for some $\Lambda \in 2^{(\mathscr{G}^*)}$, and

$$\begin{aligned}\delta_{\mathscr{G}^*}(z, B) &\geqslant \sup_{\alpha \geqslant 1} \inf_{b \in B} \sup_{g \in \Lambda} |\mathrm{pr}_{\alpha g}(z) - \mathrm{pr}_{\alpha g}(b)| \\ &= \sup_{\alpha \geqslant 1} \inf_{b \in B} \sup_{g \in \Lambda} \alpha |\mathrm{pr}_g(z) - \mathrm{pr}_g(b)| = \infty.\end{aligned}$$

So $\delta_{\mathscr{G}^*}$ attains no other values than 0 and ∞, and we conclude that $\boldsymbol{e}_{\mathscr{G}^*}X$ is topological. $\square$

If $(X, \delta_X) \in |\mathscr{M}_2|$, the topological coreflection of $(e_{\mathscr{G}^*}X, \delta_{\mathscr{G}^*})$ is a compactification of the topological coreflection of (X, δ_X). Generally it is strictly coarser than the Stone–Čech compactification, reflecting the fact that the topological coreflector is not full, or that $\mathscr{G}^*(X)$ is a subclass of $C^*(X)$. Another description of $(e_{\mathscr{G}^*}X, \delta_{\mathscr{G}^*})$ will make clear that clusters (for a well-chosen proximity relation on X) rather than maximal z-filters represent the points of $e_{\mathscr{G}^*}X$.

It was shown in Proposition 4.2 that each $\mathscr{M}_2$-compactification of $(X, \delta) \in |\mathscr{M}_2|$ is isomorphic with $(e_{\mathscr{F}}X, \delta_{\mathscr{F}})$, for some $\mathscr{F} \subset \mathscr{G}^*(X)$ generating (X, δ). Then $(e_{\mathscr{F}}X, \delta_{\mathscr{F}})$ is generated by the collection of bounded pseudo-metrics

$$\mathscr{D}_{\mathscr{F}}^* := \left\{ \sup_{g \in A} |\mathrm{pr}_g(\cdot) - \mathrm{pr}_g(\cdot)| \,\middle|\, A \in 2^{(\mathscr{F})} \right\},$$

while (X, δ) itself is generated by $\mathscr{D}_{\mathscr{F}}$. Then it is clear that

$$e_{\mathscr{F}} : (X, \vartriangle_{\mathscr{D}_{\mathscr{F}}}) \to (e_{\mathscr{F}}X, \vartriangle_{\mathscr{D}_{\mathscr{F}}^*})$$

is an embedding of proximity spaces. Especially, if p is a cluster in $(X, \vartriangle_{\mathscr{D}_{\mathscr{F}}})$ then $\{e_{\mathscr{F}}(A) \mid A \in p\}$ generates a cluster in $(e_{\mathscr{F}}X, \vartriangle_{\mathscr{D}_{\mathscr{F}}^*})$.

As a compact Hausdorff space has only one compatible proximity, we have

$$\forall C_1, C_2 \subset e_{\mathscr{F}}X: \quad C_1 \vartriangle_{\mathscr{D}_{\mathscr{F}}^*} C_2 \Leftrightarrow \overline{C_1} \cap \overline{C_2} \neq \emptyset,$$

and each cluster in $(e_{\mathscr{F}}X, \vartriangle_{\mathscr{D}_{\mathscr{F}}^*})$ has a unique cluster point. In order to simplify the notations, in the sequel we will identify points of X with the corresponding points in $e_{\mathscr{F}}X$.

Given $(X, \delta) \in |\mathscr{M}_2|$ and $\mathscr{F} \subset \mathscr{G}^*(X)$ which generates (X, δ), let $\kappa_{\mathscr{F}}X$ denote the collection of clusters in $(X, \vartriangle_{\mathscr{D}_{\mathscr{F}}})$ and let

$$\begin{aligned} c_{\mathscr{F}} &: X \to \kappa_{\mathscr{F}}X, \\ &\quad x \to \{A \subset X \mid \delta(x, A) = 0\} \end{aligned}$$

denote the canonical injection. For $A \subset X$, $\emptyset \neq A$ we define

$$\begin{aligned} \hat{\delta}_A &: \kappa_{\mathscr{F}}X \to [0, \infty], \\ &\quad p \to \inf\{\varepsilon \geqslant 0 \mid \forall d \in \mathscr{D}_{\mathscr{F}}: A_d^{(\varepsilon)} \in p\}, \end{aligned}$$

where $A_d^{(\varepsilon)} = \{x \in X \mid \inf_{a \in A} d(x, a) \leqslant \varepsilon\}$, $\forall A \subset X$, $\forall d \in \mathscr{D}_{\mathscr{F}}$ and $\forall \varepsilon \geqslant 0$.

Then we define $\delta_{\mathscr{F}}^* : \kappa_{\mathscr{F}}X \times 2^{\kappa_{\mathscr{F}}X} \to [0, \infty]$ as follows:

(i) $\delta_{\mathscr{F}}^*(p, \emptyset) = \infty$, $\forall p \in \kappa_{\mathscr{F}}X$,

(ii) $\delta_{\mathscr{F}}^*(p, \mathscr{A}) = \sup\{\hat{\delta}_A(p) \mid A \text{ absorbs } \mathscr{A}\}$, $\forall p \in \kappa_{\mathscr{F}}X$, $\forall \mathscr{A} \subset \kappa_{\mathscr{F}}X$.

We recall that A is said to absorb $\mathscr{A}$ if for all $q \in \mathscr{A}$ we have that $A \in q$, i.e., if $A \in \bigcap_{q \in \mathscr{A}} q$.

Again we will identify the points of X with their images under $c_{\mathscr{F}}$. Further, in order to avoid confusion, if $A \subset X$ and $\varepsilon \in \mathbb{R}^+$ then $A^{(\varepsilon)}$, as usual, stands for $\{x \in X \mid \delta(x, A) \leqslant \varepsilon\}$, and we denote

$$A^{(\varepsilon)*} := \{p \in \kappa_{\mathscr{F}}X \mid \delta_{\mathscr{F}}^*(p, A) \leqslant \varepsilon\}.$$

Proposition 4.6. *If* $(X, \delta) \in |\mathscr{M}_2|$ *then*

(1) $(\kappa_{\mathscr{G}} X, \delta^*_{\mathscr{G}}) \in |\mathrm{AP}|$.

(2) $c_{\mathscr{G}} : (X, \delta) \to (\kappa_{\mathscr{G}} X, \delta^*_{\mathscr{G}})$ *is a dense embedding.*

(3) *The topological coreflection of* $(\kappa_{\mathscr{G}} X, \delta^*_{\mathscr{G}})$ *is homeomorphic with the Smirnov compactification of* $(X, \vartriangle_{\mathscr{D}_{\mathscr{G}}})$.

Proof. (1) First, note that for $A, B \subset X$ we have $A \in \bigcap \{q \mid q \in B\}$ if and only if $B \subset \overline{A}$. From this we may conclude that

$$\delta^*_{\mathscr{G}}(p, B) = \sup\left\{\hat{\delta}_A(p) \,\middle|\, A \in \bigcap_{q \in B} q\right\}$$
$$= \hat{\delta}_B(p).$$

It is clear that $\delta^*_{\mathscr{G}}$ satisfies (D1) and (D2).

Moreover, if $p \in \kappa_{\mathscr{G}} X$ and $\mathscr{A} \subset \mathscr{A}' \subset \kappa_{\mathscr{G}} X$ then $\delta^*_{\mathscr{G}}(p, \mathscr{A}') \leqslant \delta^*_{\mathscr{G}}(p, \mathscr{A})$ is easily seen. So in order to prove that $\delta^*_{\mathscr{G}}$ satisfies (D3) only one inequality remains to prove. Now for $A, B \subset X$ we have

$$\delta^*_{\mathscr{G}}(p, A \cup B) = \delta^*_{\mathscr{G}}(p, A) \wedge \delta^*_{\mathscr{G}}(p, B),$$

and the general case follows from this, because if A absorbs $\mathscr{A}$ and B absorbs $\mathscr{B}$ then it follows that $A \cup B$ absorbs $\mathscr{A} \cup \mathscr{B}$.

For the proof of (D4) we first make some observations.

Assertion 1. *For any* $\alpha, \beta \in \mathbb{R}^+$, $d \in \mathscr{D}_{\mathscr{G}}$ *and* $A \subset X$: $(A_d^{(\alpha)})_d^{(\beta)} \subset A_d^{(\alpha+\beta)}$.

This follows by straightforward verification.

Assertion 2. *For any* $A \subset X$ *and* $\varepsilon \in \mathbb{R}^+$: $A^{(\varepsilon)*} = \bigcap_{\varepsilon' > \varepsilon} \{q \mid \forall d \in \mathscr{D}_{\mathscr{G}}: A_d^{(\varepsilon')} \in q\}$.

This follows at once from the definition of $\delta^*_{\mathscr{G}}$ and $\hat{\delta}_A$.

Assertion 3. *If* A *absorbs* $\mathscr{A}$ *then for any* $\varepsilon \in \mathbb{R}^+$: $\mathscr{A}^{(\varepsilon)} \subset A^{(\varepsilon)*}$.

If $p \in \mathscr{A}^{(\varepsilon)}$ then it follows from Assertion 2 that for all $\varepsilon' > \varepsilon$, for all $d \in \mathscr{D}_{\mathscr{G}}$ and for any B which absorbs $\mathscr{A}$ we have $B_d^{(\varepsilon')} \in p$. Since A absorbs $\mathscr{A}$ a second application of Assertion 2 shows that $p \in A^{(\varepsilon)*}$.

Now first consider the case $p \in \kappa_{\mathscr{G}} X$ and $A \subset X$. From Assertion 1 and 2 we can then deduce that for any $\varepsilon \geqslant 0$:

$$\hat{\delta}_A(p) \leqslant \delta^*_{\mathscr{G}}(p, A^{(\varepsilon)*}) + \varepsilon.$$

For the general case let $\mathscr{A} \subset \kappa_{\mathscr{G}} X$ then from the foregoing inequality we obtain:

$$\delta^*_{\mathscr{G}}(p, \mathscr{A}) \leqslant \sup\{\delta^*_{\mathscr{G}}(p, A^{(\varepsilon)*}) \mid A \text{ absorbs } \mathscr{A}\} + \varepsilon,$$

and the result now follows by Assertion 3. This proves that $(\kappa_{\mathscr{G}} X, \delta^*_{\mathscr{G}})$ is indeed an approach space.

(2) That $c_{\mathscr{F}}$ is an embedding follows from the fact that if $x \in X$ and $A \subset X$ then

$$\delta^*_{\mathscr{F}}(x, A) = \inf\{\varepsilon \geqslant 0 \mid \forall d \in \mathscr{D}_{\mathscr{F}}\colon x \in A_d^{(\varepsilon)}\}$$
$$= \delta(x, A).$$

That the embedding is dense follows from the fact that X belongs to each cluster $p \in \kappa_{\mathscr{F}} X$.

(3) Given $p \in \kappa_{\mathscr{F}} X$ and $\mathscr{A} \subset \kappa_{\mathscr{F}} X$, p is in the closure of $\mathscr{A}$ for the Smirnov compactification if $A \in p$ for each A which absorbs $\mathscr{A}$, whereas by definition of $\delta^*_{\mathscr{F}}$, p is in the closure of $\mathscr{A}$ for the topological coreflection of $\delta^*_{\mathscr{F}}$ if $A_d^{(\varepsilon)} \in p$ for any $\varepsilon > 0$, $d \in \mathscr{D}_{\mathscr{F}}$ and A which absorbs $\mathscr{A}$. Clearly these two conditions are equivalent. $\square$

In order to prove that $e_{\mathscr{F}} X$ and $\kappa_{\mathscr{F}} X$ are isomorphic, we need a preliminary result which holds for $\mathscr{M}$-objects in general.

Lemma 4.7. *If $(X, \delta) \in |\mathscr{M}|$ and Y is a dense subspace of X then*

$$\delta(x, A) = \sup\{\delta(x, B) \mid B \subset Y \text{ and } A \subset \overline{B}\} \quad \text{for each } x \in X \text{ and } A \in 2^X.$$

Proof. It is trivial that $\delta(x, A) \geqslant \sup\{\delta(x, B) \mid B \subset Y,\ A \subset \overline{B}\}$.

Let $\mathscr{D}$ be any family generating (X, δ). If $\delta(x, A) > \varepsilon > 0$ then $d_0(x, A) > \varepsilon + \theta$ for some $d_0 \in \mathscr{D}$ and $\theta > 0$. Put $B = \{y \in Y \mid \inf_{a \in A} d_0(y, a) < \theta\}$. Then $A \subset \overline{B}$. Indeed if each $a \in A$, we have $\delta(a, Y) = 0$, so if $d \in \mathscr{D}$ and $\mu > 0$ we can find $y \in Y$ such that

$$d \vee d_0(a, y) < \theta \wedge \mu.$$

Then y belongs to B. Since further $\delta(x, B) \geqslant d_0(x, B) \geqslant \varepsilon$, this proves the other inequality. $\square$

We now define $\Psi : \kappa_{\mathscr{F}} X \to e_{\mathscr{F}} X$ to be the map which assigns to each cluster $p \in \kappa_{\mathscr{F}} X$ the unique adherence point of the unique cluster in $e_{\mathscr{F}} X$ which contains p.

Theorem 4.8. *If $(X, \delta) \in |\mathscr{M}_2|$ is generated by $\mathscr{F} \subset \mathscr{G}^*$, then*

$$\Psi : (\kappa_{\mathscr{F}} X, \delta^*_{\mathscr{F}}) \to (e_{\mathscr{F}} X, \delta_{\mathscr{F}})$$

is an isomorphism.

Proof. The map Ψ is well defined and bijective and we obtain a commutative diagram:

$$\begin{tikzcd}
 & (X, \delta) \arrow[dl, "c_{\mathscr{F}}"'] \arrow[dr, "e_{\mathscr{F}}"] & \\
(\kappa_{\mathscr{F}} X, \delta^*_{\mathscr{F}}) \arrow[rr, "\Psi"] & & (e_{\mathscr{F}} X, \delta_{\mathscr{F}})
\end{tikzcd}$$

First suppose that $A \subset X$ and $p \in \kappa_{\mathscr{F}} X$. Then

$$\begin{aligned}\delta_{\mathscr{F}}(\Psi(p), A)) &= \sup_{A \in 2^{(\mathscr{F})}} \inf_{a \in A} \sup_{f \in A} \left|\mathrm{pr}_f(\Psi(p)) - f(a)\right| \\ &= \sup_{d^* \in \mathscr{D}_{\mathscr{F}}^*} \inf_{a \in A} d^*(\Psi(p), a) \\ &= \sup_{d^* \in \mathscr{D}_{\mathscr{F}}^*} d^*(\Psi(p), A) \\ &= \inf\left\{\varepsilon \,\middle|\, \forall d^* \in \mathscr{D}_{\mathscr{F}}^*: \Psi(p) \in A_{d^*}^{(\varepsilon)}\right\} \\ &= \inf\left\{\varepsilon \,\middle|\, \forall d \in \mathscr{D}_{\mathscr{F}}: A_d^{(\varepsilon)} \in p\right\} \\ &= \delta_{\mathscr{F}}^*(p, A),\end{aligned}$$

where the equality prior to the last one follows from the fact that $\Psi(p) \in A_{d^*}^{(\varepsilon)}$ implies that for all $\varepsilon' > \varepsilon$: $A_d^{(\varepsilon')} \in p$; and similarly $A_d^{(\varepsilon)} \in p$ implies that for all $\varepsilon' > \varepsilon$: $\Psi(p) \in A_{d^*}^{(\varepsilon')}$.

Especially for $p \in \kappa_{\mathscr{F}} X$ and $A \subset X$ we have $p \in \overline{A}$ if and only if $\Psi(p) \in \overline{A}$, and for $A \subset X$ and $\mathscr{A} \subset \kappa_{\mathscr{F}} X$ it follows that A absorbs $\mathscr{A}$ if and only if $\Psi(\mathscr{A}) \subset \overline{A}$, by definition of the Smirnov compactification. So if $p \in \kappa_{\mathscr{F}} X$ and $\mathscr{A} \subset \kappa_{\mathscr{F}} X$ then by the definition of $\delta_{\mathscr{F}}^*$:

$$\begin{aligned}\delta_{\mathscr{F}}^*(p, \mathscr{A}) &= \sup\left\{\delta_{\mathscr{F}}^*(p, A) \,\middle|\, A \in \bigcap_{q \in \mathscr{A}} q\right\} \\ &= \sup\left\{\delta_{\mathscr{F}}(\Psi(p), A) \,\middle|\, \Psi(\mathscr{A}) \subset \overline{A}\right\} \\ &= \delta_{\mathscr{F}}(\Psi(p), \Psi(\mathscr{A})), \quad \text{by the foregoing lemma.}\end{aligned}$$

This proves that Ψ is an isomorphism. $\square$

The question we now ask is when there exists an $\mathscr{M}_2$-compactification such that the topological coreflection is the Čech–Stone compactification. The description of $\mathscr{M}_2$-compactifications in terms of clusters is useful in this context. It is clearly sufficient to restrict our attention to $e_{\mathscr{G}^*} X$.

Corollary 4.9. *For each $(X, \delta) \in |\mathscr{M}_2|$, the topological coreflection of $e_{\mathscr{G}^*} X$ is isomorphic with the Smirnov compactification of $(X, \vartriangle_{\mathscr{D}_\delta^b})$.*

Proof. We already know that the topological coreflection of $e_{\mathscr{G}^*} X$ is isomorphic with the Smirnov compactification of $(X, \vartriangle_{\mathscr{D}_{\mathscr{G}^*}})$. Obviously, $\vartriangle_{\mathscr{D}_\delta^b}$ is finer than $\vartriangle_{\mathscr{D}_{\mathscr{G}^*}}$, but in fact they coincide. For any A, $B \subset X$, we have that

$$A \vartriangle_{\mathscr{D}_{\mathscr{G}^*}} B \Rightarrow \inf_{a \in A} \inf_{b \in B} d(a, b) = 0 \quad \text{for each } d \in \mathscr{D}_\delta^b,$$

since $d(\cdot, A)$ belongs to $\mathscr{G}^*$. $\square$

If X is a Tychonoff space, subsets A and B of X are said to be completely separated if there exists a continuous function $f: X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. This holds if and only if A and B are contained in disjoint zerosets. The finest compatible proximity relation referred to as the fine proximity and denoted $\vartriangle_t$ is given by

$$A \not\vartriangle_t B \Leftrightarrow A \text{ and } B \text{ are completely separated.}$$

If X is normal, $A \vartriangle_t B$ is equivalent with $\bar{A} \cap \bar{B} \neq \emptyset$. If $Z(X)$ denotes the collection of zerosets of an arbitrary Tychonoff space X, then we have obviously

$$\forall A, B \in Z(X): A \vartriangle_t B \Leftrightarrow A \cap B \neq \emptyset,$$

and this characterizes the fine proximity. Indeed if $\vartriangle$ is any other compatible proximity with this property, then for $C, D \subset X$, $C \vartriangle D$ implies that for each pair of zerosets A and B containing C and D respectively, $A \vartriangle B$ holds. Then C and D are not completely separated, by the hypothesis, so $\vartriangle$ coincides with $\vartriangle_t$.

Proposition 4.10. *The topological coreflection of $e_{\mathscr{G}^*}X$ is isomorphic with the Čech–Stone compactification of the topological coreflection of (X, δ) if and only if $\vartriangle_{\mathscr{D}_{\mathscr{G}^*}}$ is the fine proximity.*

Proof. The Čech–Stone compactification βX is characterized by the property that disjoint zerosets in X have disjoint closures in βX.

Now we have the following equivalences:

$$\forall A, B \in Z(X): A \cap B = \emptyset \Rightarrow A \text{ and } B \text{ have disjoint closures in } e_{\mathscr{G}^*}X$$
$$\Leftrightarrow \forall A, B \in Z(X): A \cap B = \emptyset \Rightarrow \text{no } \vartriangle_{\mathscr{D}_{\mathscr{G}^*}} \text{ cluster contains at the same time } A \text{ and } B$$
$$\Leftrightarrow \forall A, B \in Z(X): A \cap B = \emptyset \Rightarrow A \not\vartriangle_{\mathscr{D}_{\mathscr{G}^*}} B$$
$$\Leftrightarrow \vartriangle_{\mathscr{D}_{\mathscr{G}^*}} = \vartriangle_t. \qquad \square$$

Remark. The finest compatible proximity relation $\vartriangle_t$ is generated by the family $\mathscr{D}_{C^*} = \{d_\Lambda \mid \Lambda \in 2^{(C^*)}\}$, and $\vartriangle_{\mathscr{D}_{\mathscr{G}}^*}$ or equivalently, $\vartriangle_{\mathscr{D}_\delta^b}$, is generated by $\mathscr{D}_{\mathscr{G}^*}$. These proximities coincide if and only if the corresponding totally bounded uniformities generated by $\mathscr{D}_{\mathscr{G}^*}$ and $\mathscr{D}_{C^*}$ are the same. Necessary and sufficient condition is that bounded continuous real-valued functions are uniformly continuous for the uniformity generated by $\mathscr{D}_{\mathscr{G}^*}$, i.e.,

$$\forall f \in C^*, \forall \varepsilon > 0, \exists d \in \mathscr{D}_\delta^b, \exists \theta > 0, \forall x, y \in X:$$
$$d(x, y) < \theta \Rightarrow |f(x) - f(y)| < \varepsilon,$$

or

$$\forall f \in C^*, \forall \varepsilon > 0, \exists \Lambda \in 2^{(\mathscr{G}^*)}, \exists \theta > 0, \forall x, y \in X:$$
$$\sup_{g \in \Lambda} |g(x) - g(y)| < \theta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Corollary 4.11. *For* $(X, d) \in |\text{p-met}^\infty|$ *the following are equivalent:*

(1) *The topological coreflection of* $e_{\mathscr{G}^*}X$ *is isomorphic with the Čech–Stone compactification of* (X, T_d).

(2) $\forall A, B \in 2^X$: $\inf_{a \in A} \inf_{b \in B} d(a, b) = 0 \Rightarrow \bar{A} \cap \bar{B} \neq \emptyset$.

5. The Čech–Stone compactification of $\mathbb{N}$

Consider the discrete space $\mathbb{N}$ of the positive integers with the Euclidean distance d_e. It follows from Corollary 4.11 that the compactification $(e_{\mathscr{G}^*}\mathbb{N}, \delta_{\mathscr{G}^*})$ is an extension of d_e to the Čech–Stone compactification $\beta\mathbb{N}$; it will be denoted $(\beta\mathbb{N}, \delta_\beta)$.

The underlying space of $\beta\mathbb{N}$ is the collection of ultrafilters on $\mathbb{N}$. The collection of nonconvergent ultrafilters, $\beta\mathbb{N} - \mathbb{N}$, will be denoted $\mathbb{N}^*$.

For $\mathscr{F} \in \beta\mathbb{N}$ and $A \subset \mathbb{N}$ we have

$$\begin{aligned}\delta_\beta(\mathscr{F}, A) &= \inf\{n \in \mathbb{N} \mid A^{(n)} \in \mathscr{F}\}\\ &= \sup_{F \in \mathscr{F}} \inf_{x \in F} \inf_{a \in A} |x - a|.\end{aligned}$$

Especially $\delta_\beta(n, A) = \inf_{a \in A}|n - a| = d_e(n, A)$, for each $n \in \mathbb{N}$ and $A \subset \mathbb{N}$.

If $\mathscr{F} \in \beta\mathbb{N}$ and $\mathscr{A} \subset \beta\mathbb{N}$ then

$$\begin{aligned}\delta_\beta(\mathscr{F}, \mathscr{A}) &= \sup\left\{\delta_\beta(\mathscr{F}, \mathscr{A}) \,\middle|\, A \subset \mathbb{N},\ A \in \bigcap_{\mathscr{U} \in \mathscr{A}} \mathscr{U}\right\}\\ &= \sup_{A \in \bigcap\{\mathscr{U} \mid \mathscr{U} \in \mathscr{A}\}} \sup_{F \in \mathscr{F}} \inf_{a \in A} \inf_{y \in F} |a - y|.\end{aligned}$$

The coreflection in p-met$^\infty$ is given by

$$d(\delta_\beta)(\mathscr{F}, \mathscr{G}) = \delta_\beta(\mathscr{F}, \{\mathscr{G}\}) = \inf\{n \in \mathbb{N} \mid \mathscr{G}^{(n)} \subset \mathscr{F}\},$$

where $\mathscr{G}^{(n)}$ is the filter generated by $\{G^{(n)} \mid G \in \mathscr{G}\}$. Actually $d(\delta_\beta)$ is an extended metric which induces the discrete topology on $\beta\mathbb{N}$.

For $\mathscr{F}, \mathscr{G} \in \mathbb{N}^*$ and $n, m \in \mathbb{N}$ we have

$$d(\delta_\beta)(n, m) = |n - m|,$$

$$d(\delta_\beta)(\mathscr{F}, \mathscr{G}) = \sup_{F \in \mathscr{F}} \sup_{G \in \mathscr{G}} \inf_{x \in F} \inf_{y \in G} |x - y|,$$

$$\begin{aligned}d(\delta_\beta)(n, \mathscr{F}) &= \sup_{F \in \mathscr{F}} \inf_{m \in F} |n - m|\\ &= \infty, \quad \text{since no finite set is contained in } \mathscr{F} \in \mathbb{N}^*.\end{aligned}$$

By the same argumentation we find that

$$\delta_\beta(\mathscr{F}, A) = \infty \quad \text{for each } \mathscr{F} \in \mathbb{N}^* \text{ and for each bounded subset } A \text{ of } \mathbb{N}.$$

For each $n \in \mathbb{N}$, the set $A_n = \{n, n+1, \dots\}$ belongs to each nonconvergent ultrafilter. As a consequence we have that points of $\mathbb{N}$ are extremely isolated in $\beta\mathbb{N}$:

$$\delta_\beta(k, \mathbb{N}^*) \geq \sup_{n \in \mathbb{N}} d_e(k, A_n) = \infty, \quad \text{for each } k \in \mathbb{N}.$$

Values of $\delta_\beta(n, \cdot)$ for $n \in \mathbb{N}$ are completely determined by this discussion.

It is easy to see that $\delta_\beta(\mathscr{F}, \cdot)$, with $\mathscr{F} \in \mathbb{N}^*$, attains other values than 0 and ∞. Indeed if A is a subset of $\mathbb{N}$ such that $A^{(k)} = \mathbb{N}$ for some $k \in \mathbb{N}$ then $\delta_\beta(\mathscr{F}, A) \leq k$ is obvious, and $\delta_\beta(\mathscr{F}, A) \geq 1$ if $A \notin \mathscr{F}$.

Actually $\delta_\beta(\mathscr{F}, \cdot)$ attains all values in $\mathbb{N}$. For each $k \in \mathbb{N}$ we have a translation

$$t_k : \mathbb{N} \to \mathbb{N},$$
$$n \to n + k.$$

The image of $\mathscr{F} \in \mathbb{N}^*$ generates a nonconvergent ultrafilter on $\mathbb{N}$ which will be denoted by $\mathscr{F} + k$. If $\mathscr{F} - k$ denotes the nonconvergent ultrafilter generated by $\{F - k \mid F \in \mathscr{F}\}$ where $F - k := \{x - k \mid x \in F \text{ and } x \geq k\}$ for each $F \in \mathscr{F}$, then we have $(\mathscr{F} + k) - k = \mathscr{F}$, and $(\mathscr{F} - k) + k = \mathscr{F}$, $\forall \mathscr{F} \in \mathbb{N}^*$. We will see that

$$\delta_\beta(\mathscr{F}, \{\mathscr{F} + k\}) = k, \quad \text{and} \quad \delta_\beta(\mathscr{F}, \{\mathscr{F} - k\}) = k.$$

We can prove even more.

Theorem 5.1. *If $\mathscr{F}, \mathscr{G} \in \mathbb{N}^*$ and $k \in \mathbb{N}$ then $\delta_\beta(\mathscr{F}, \{\mathscr{G}\}) = k \Leftrightarrow \mathscr{F} = \mathscr{G} \pm k$.*

Proof. Indeed $\delta_\beta(\mathscr{F}, \{\mathscr{F} + k\}) \leq k$ follows directly from the fact that $F + k \subset F^{(k)}$ for all $F \in \mathscr{F}$. For the converse inequality, note that each ultrafilter $\mathscr{F}$ on $\mathbb{N}$ contains precisely one of the restclasses Z_i, $i = 0, 1, \dots, 2k-1$, where $Z_i = \{2kn + i \mid n \in \mathbb{N}\}$.

If $\mathscr{F}$ contains Z_j and $j' \equiv j + k \pmod{2k}$ then $\mathscr{F} + k$ contains $Z_{j'}$, and

$$\delta_\beta(\mathscr{F}, \{\mathscr{F} + k\}) \geq \inf_{x \in Z_j} \inf_{y \in Z_j'} |x - y| = k.$$

This proves that $\delta_\beta(\mathscr{F}, \{\mathscr{F} + k\}) = k$. From this $\delta_\beta(\mathscr{F} - k, \{\mathscr{F}\}) = k$ follows at once, since $\delta_\beta(\mathscr{F}, \{\mathscr{F} - k\}) = \delta_\beta(\mathscr{F} + k, \{\mathscr{F}\}) = k$.

In order to prove necessity, assume that $\delta_\beta(\mathscr{F}, \{\mathscr{G}\}) \leq k < \infty$. Select i and j in $\{0, \dots, 2k\}$ such that the rest class $(2k+1)\mathbb{N} + i \in \mathscr{F}$ and $(2k+1)\mathbb{N} + j \in \mathscr{G}$. Then a base of $\mathscr{F}$ is given by $\mathscr{B}_{\mathscr{F}} = \{F \cap Z_i \mid F \in \mathscr{F}\}$, and $\mathscr{B}_{\mathscr{G}} := \{F \cap (2k+1)\mathbb{N} + j \mid F \in \mathscr{G}\}$ is a basis for $\mathscr{G}$. For each $A \in \mathscr{B}_{\mathscr{F}}$ we can find $B \in \mathscr{B}_{\mathscr{G}}$ such that $B \subset A^{(k)}$. Now there is a unique $l \in \{-k, \dots, -1, 0, 1, \dots, k\}$ such that $i + l \equiv j \pmod{2k+1}$. Then $B \subset \{a + l \mid a \in A\}$ holds, so $\mathscr{F} + l$ is contained in $\mathscr{G}$. It follows that $\mathscr{F} + l = \mathscr{G}$. Now by the first part of the proof, if $\delta_\beta(\mathscr{F}, \mathscr{G}) = k$ then $|l| = k$. $\square$

For example, if $m_2 : \mathbb{N} \to \mathbb{N} : n \to 2n$ and $\mathscr{F} \in \mathbb{N}^*$ then $\delta_\beta(\mathscr{F}, \{m_2(\mathscr{F})\}) = \infty$, for $m_2(\mathscr{F})$ is not a translation of $\mathscr{F}$.

The definition of $\mathscr{F} + k$ is equivalent with the one van Douwen [10] used to construct an extension of the usual addition in $\mathbb{N}$. This was done as follows. Fixing

$\mathcal{F} \in \mathbb{N}^*$ we have a contraction $f: \mathbb{N} \to \beta\mathbb{N}$: $k \to \mathcal{F} + k$ which actually is an embedding. So f has a unique extension $\tilde{f}: \beta\mathbb{N} \to \beta\mathbb{N}$: $\mathcal{G} \to \mathcal{F} + \mathcal{G}$. The binary operation $+$ on $\beta\mathbb{N}$ is an extension of ordinary addition in $\mathbb{N}$, which is associative. Considered as a function $\beta\mathbb{N} \times \beta\mathbb{N} \to \beta\mathbb{N}$ it is only left continuous, and no extension exists which is both left continuous and right continuous.

Van Douwen proved that $\mathbb{N}^* + \mathbb{N}^*$ is nowhere dense in $\mathbb{N}^*$. The context of $(\mathbb{N}, \delta_\beta)$ admits, after slight modifications of the proof, some extensions of this remarkable fact. For example: for all $n \in \mathbb{N}$, we have that $(\mathbb{N}^* + \mathbb{N}^*)^{(n)}$ is nowhere dense in $\mathbb{N}^*$. Actually the following is true:

Theorem 5.2. $\bigcup_{n \in \mathbb{N}} (\mathbb{N}^* + \mathbb{N}^*)^{(n)}$ *is nowhere dense in* $\mathbb{N}^*$.

Proof. A basis for the topology of $\mathbb{N}^*$ is given by $\{\bar{A} \mid A$ is an infinite subset of $\mathbb{N}\}$, where $\bar{A} = \{\mathcal{F} \in \mathbb{N}^* \mid A \in \mathcal{F}\}$.

If A is an infinite subset of $\mathbb{N}$, we can fix an increasing sequence $(s_n)_n$ in A such that

$$s_{n+1} > 2s_n + 3n \quad \forall n \in \mathbb{N}. \tag{1}$$

Put $S = \{s_n \mid n \in \mathbb{N}\}$ and $S_n = \{s_m \mid m \geqslant n\}$. If $n \in \mathbb{N}$ then $S^{(n)}$ belongs to no element of $\mathbb{N}^* + \mathbb{N}^*$. Indeed suppose that $S^{(n)} \in \mathcal{F} + \mathcal{G}$ for some $\mathcal{F}, \mathcal{G} \in \mathbb{N}^*$. Then necessarily $S_n^{(n)} \in \mathcal{F} + \mathcal{G}$, for $S^{(n)} - S_n^{(n)}$ is a finite set. Then we can find $B \in \mathcal{G}$ fulfilling

$$\forall b \in B, \exists F_b \in \mathcal{F}: \quad F_b + b \subset S_n^{(n)}.$$

First fix k and $l \in B$ such that

$$k + 2n < l. \tag{2}$$

Further let $C, D \in \mathcal{F}$ such that $C + k \subset S_n^{(n)}$ and $D + l \subset S_n^{(n)}$. As $C \cap D \in \mathcal{F}$ is an infinite set there is an $i \in C \cap D$ such that $i \geqslant l$. Then $i + k \in S_n^{(n)}$, so $s_m - n \leqslant i + k \leqslant s_m + n$ holds for some $m \geqslant n$. Using (2), we find that

$$s_m + n < i + l. \tag{3}$$

On the other hand we have $l \leqslant i \leqslant i + k \leqslant s_m + n$, so $i + l \leqslant 2i \leqslant 2s_m + 2n < s_{m+1} - n$, by (1). Together with (3) we obtain that $s_m + n < i + l < s_{m+1} - n$. So $i + l \notin S_n^{(n)}$, contradicting the fact that $i + l \in D + l$. It follows that $\mathbb{N} - S^{(n)}$ belongs to each element of $\mathbb{N}^* + \mathbb{N}^*$. As a consequence, if $\mathcal{F} \in \bigcup_{n \in \mathbb{N}} (\mathbb{N}^* + \mathbb{N}^*)^{(n)}$ then $S \notin \mathcal{F}$. In other words, S is an infinite subset of A such that

$$\bar{S} \cap \left(\bigcup_{n \in \mathbb{N}} (\mathbb{N}^* + \mathbb{N}^*)^{(n)} \right) = \emptyset.$$

This proves the statement. □

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