Path integrals as a tool for pricing interest rate contingent claims: the case of reflecting and absorbing boundaries.

Marc Decamps[1], Ann De Schepper[2], Marc Goovaerts[1,3]

Abstract

Common interest rate models are faced with the problem of volatilities vanishing for spot rates in the vicinity of zero. A possible answer to this difficulty can be given by the introduction of a reflecting boundary at zero, at the same time guaranteeing the spot rate to be non-negative, which is needed in order to avoid the possibility of arbitrage. In the present paper, we obtain closed form expressions for transition probabilities and for prices of general interest-rate contingent claims by means of path integrals, when the spot rate process is modelled by means of a general diffusion with a reflecting or absorbing boundary. We also show how to derive accurate closed form approximations in case the path integrals are not analytically computable.

1 Introduction

A problem which is often encountered when modeling the term structure of interest rates, is the fact that ordinary one-factor models partly fail in capturing the empirical behaviour of spot rates.

To start with, nominal interest rates should be non-negative, in order to preclude arbitrage. Indeed, if a bond e.g. would show a negative rate of return, selling the bond at time zero, keeping the money and simply bying it at maturity would lead to a riskless profit. Although this condition seems to be a natural one, the first term structure models (e.g. Merton [35] and Vasicek [42]) were Gaussian models, and thus they do not meet this qualification. The Cox-Ingersoll-Ross model on the other hand does satisfy the non-negativity.
A second desirable characteristic is the mean-reverting property, which holds for the Vasicek model and again for the CIR model; it is more or less established for more recent models, e.g. the broad class of models related to the CIR model.

[1] University of Leuven, Belgium
[2] University of Antwerp, Belgium
[3] University of Amsterdam, the Netherlands

The most difficult requirement however involves the volatility of the interest rate. Most models (including CIR) deal with a volatility which vanishes when the interest rate reaches zero, and thus they do not correspond to reality when used to describe low interest rates. The short term Japanese interest rates, particularly after 1991, represent a nice example of this problem. Indeed, during this period, the Japanese spot rate remained at a very low level, but with a rather high volatility[1]. Figure 1 illustrates this statement.

A possible solution to the problem of this last requirement, consists of combining a model satisfying the first two conditions (such as the CIR family) with the introduction on the interest rate dynamics of a reflecting or absorbing boundary. This is exactly what we want to consider in this paper.
This idea is not completely new. Indeed, in a paper of 1997, Goldstein and Keirstead (see [16]) already suggested to act in this way. They also proved that absorbing and reflecting spot rates are consistent with HJM models and thus preclude arbitrage. Following Black [6] and Harrison and Taksar [23], they argue that the most efficient way of keeping interest rates positive, is to impose a reflecting boundary at zero. Referring to Ait-Sahalia [2] and Stanton [41], they also mention empirical motivation for this choice (e.g. the US spot rate in the period before 1995). Absorbing boundaries also assure interest rates to be positive, but they are less realistic than reflecting boundaries in that the interest rate remains at the absorbing level once it has reached it. In addition, it has the nice property to generate an upper bound to the reflecting boundary case.

In the present paper, we assume that the term structure can be described starting from a single exogeneous factor which is the spot rate. In absence of any reflecting or absorbing boundary (at zero or at another level), the spot rate is then defined as the solution of the diffusion equation

$$dr_t = \mu(r_t;\theta)dt + \sigma(r_t;\theta)dW_t \tag{1}$$

with $W = \{W_t : t \geq 0\}$ a standard Brownian motion.
Now, if we impose a reflecting boundary at a level a, this generates an additional term in the drift, proportional to a Dirac delta function. Stochastic processes whose drift is a generalized function, are solutions of diffusion equations involving their (symmetric) local time, see e.g. Le Gall [32]. The symmetric local time of the process $r = \{r_t : t \geq 0\}$ in a measures the time spent in the vicinity of the boundary, and it can be expressed as

$$L_t^a(r) = \frac{1}{2}\left(L_t^{a-}(r) + dL_t^{a+}(r)\right),$$

where the left-local time L_t^{a-} is defined by

$$L_t^{a-}(r) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t 1_{]a-\epsilon,a]}(r_s)\, d\langle r,r\rangle_s$$

[1] We will use this Japanese example later on for the numerical illustration of our results.

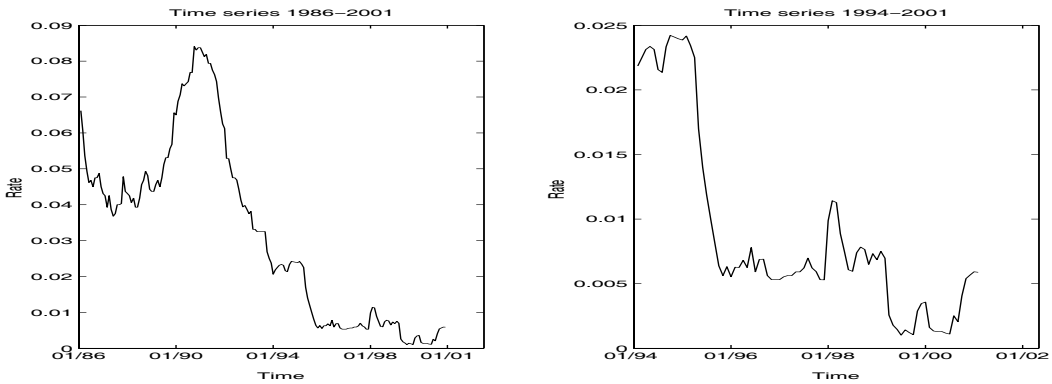

Figure 1: Japanese yen, three-month interbank rate

and the right-local time L_t^{a+} by

$$L_t^{a+}(r) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t 1_{[a,a+\epsilon[}(r_s)\ d\langle r, r\rangle_s\ .$$

If left- and right-local time are equal, the local time is continuous in a. However, if the process is reflected in a (with $r_0 \geq a$), the local time is discontinuous in a. Indeed, in this case $L_t^{a-}(r) = 0$ and as a consequence $L_t^a(r) = \frac{1}{2} L_t^{a+}(r)$. Therefore, in the case of a reflecting boundary, equation (1) has to be modified into

$$dr_t = \mu(r_t;\theta)dt + \sigma(r_t;\theta)dW_t + \frac{1}{2} dL_t^{a+}(r), \tag{2}$$

see also [28, 39]. We will return to this local time immediately. Note that the case $r_0 \leq a$ is completely analogous.
In case an absorbing boundary is imposed, the diffusion equation (1) is not altered until the first time the process hits the barrier.

Diffusion processes can be related to quantum mechanical particles with convenient Lagrangians as explained in detail e.g. in [18]. As a consequence, the transition probability of the spot rate describing its time-evolution can be formulated by means of a path integral. Interest rate contingent claims are financial contracts that guarantee the holder a continuous payment $c(r_u, u)$ during the contract and a payoff $h(r_T)$ at maturity date. The price of a contingent claim is the expectation under some risk neutral measure of the discounted payments and is solution of a partial differential equation with terminal condition determined by the payoff of the claim. The kernel of the pricing equation is the particular solution for which $h(r_T) = \delta(r_T - \hat{r})$ and can be interpreted as the price of basic securities, or Arrow-Debreu securities. In this paper, we derive a path integral representation for the prices of the Arrow-Debreu securities in case an absorbing or reflecting boundary is imposed at a certain level of the spot rate.

When a barrier is imposed on the spot rate process, the pricing equation is subject to specific boundary conditions, see [16]. Goldstein and Keirstead [16] consider the associated Sturm-Liouville problem and obtain eigenfunction expansions for the prices of the Arrow-Debreu securities when closed form expressions exist for the no boundary case. The aim of this paper is to show how path integral representations can be obtained for transition probabilities, prices of Arrow-Debreu securities and of more general interest rate contingent claims. As we will show immediately, the Laplace transforms of path integrals for the boundary cases directly follow from the Laplace transforms of the ordinary, no boundary case. As a consequence, the problem of the pricing of general contingent claims in the presence of a reflecting or absorbing boundary, can be solved in the Laplace domain as soon as it is solved for the ordinary, no boundary case. Series expansions can be recovered by an inversion of the Laplace transform. The advantage of our approach compared with the methodology of Goldstein and Keirstead [16], has to be found in the fact that the path integral representations can be used to derive approximations also when no analytical solution exists.

In the recent research in the econophysics field, the use of path integrals for the calculation of security and option prices is widespread. Without claiming any exhaustiveness, we refer e.g. to [4, 8, 9, 30, 33, 34, 36, 37, 38, 40].

When path integrals are used for the derivation of closed form expressions, the introduction of an absorbing or reflecting boundary at a level a both cause an extra term in the potential of the path integral. For the absorbing case, we need to add $\beta\ \delta(x-a)$ (see [21]), for the reflecting case this is $\beta\ \delta'(x-a)$ (see [22]), both with β going to infinity. In addition, due to the presence of the local time in the diffusion equation for the reflecting boundary case (see equation (2)), the potential then also contains a component proportional to $\delta(x-a+)$. We must be aware of the importance and logic of the $a+$. The $a+$ appears due to the discontinuity of the local time in a, and this is nicely consistent with the discontinuity of the path integral with potential $\delta'(x-a)$. Indeed, the path integral with potential $\delta(x-a)+\beta\ \delta'(x-a)$ is not the same as a path integral with potential $\delta(x-a+)+\beta\ \delta'(x-a)$, the latter one corresponding to the correct formulation. More details can be found in section 2.
Note that in probability theory, local time is treated as a random field, while in the path integral formalism, expansions of the path integral are used. The conclusions, however, are the same.

As mentioned above, our approach allows to write exact closed forms for transition probabilities and security pricing for any process, not only for the common models. Although this is right, in many cases the results are not analytically tractable. Therefore, we are also interested in closed form approximations; these are particularly useful in case the exact result deals with a path integral which is not computable. As we will explain in detail, our approximation is based on a combination of an upper and a lower bound for the path integals in question. For the upper bound, we rely on the methodology of comonotonic risks (see [11, 12]); a lower bound can be constructed by means of an application of the inequality of Jensen. The numerical examples at the end of the paper show the accuracy of this type of approximation.

The paper is organized as follows. We start in section 2 with a brief overview of path integrals especially in the context of δ and δ' functions. In section 3, we present closed form results for the transition probabilities for general spot rate processes in the presence of reflecting or absorbing boundaries, and we generalize these results to prices for Arrow-Debreu securities in section 4. Section 5 is meant to describe how closed form approximations can be obtained when exact calculation is not possible. Finally, everything is numerically illustrated in section 6.

2 About δ and δ' theory

As explained in the introduction, we aim at calculating transition probabilities and prices of interest rate contingent claims, when the spot rate is modelled by means of a general diffusion process subject to an absorbing or a reflecting boundary. Our methodology consists of writing the quantities under investigation by means of path integrals, followed by the construction of an approximation for these path integrals. Therefore, in this section, we give a very brief overview about the concept of path integrals and Green's functions, especially when considered in relation to delta functions. These last results will be needed when working with the situations of an absorbing or reflecting boundary. For proofs and more details about both concepts, we refer to [19, 20, 21, 22].

2.1 About path integrals and Green's functions

The Feynman path integral[2]

$$I_{[V(x)]}(t_0, x_0; t, x_t) = \int_{(t_0,x_0)}^{(t,x_t)} Dx(s) e^{-\frac{1}{2}\int_{t_0}^{t} \dot{x}^2(s)ds - \int_{t_0}^{t} V(x(s))ds} \tag{3}$$

is the solution of the forward partial differential equation

$$\frac{\partial I}{\partial t} = \frac{1}{2}\frac{\partial^2 I}{\partial x_t^2} - V(x_t) \cdot I \tag{4}$$

and of the backward partial differential equation

$$-\frac{\partial I}{\partial t_0} = \frac{1}{2}\frac{\partial^2 I}{\partial x_0^2} - V(x_0) \cdot I \tag{5}$$

with conditions

$$\begin{aligned} &\lim_{x_t \to +\infty} I_{[V(x)]}(t_0, x_0; t, x_t) = 0 \\ &\lim_{x_t \to -\infty} I_{[V(x)]}(t_0, x_0; t, x_t) = 0 \\ &I_{[V(x)]}(t_0, x_0; t_0, x_t) = \delta(x_t - x_0). \end{aligned} \tag{6}$$

This can be demonstrated in a straightforward way by means of an expansion of the path integral (3).

[2]Note that Feynman (see [13]) used an imaginary argument of the exponential function, whereas in this paper, we use a real argument but keep the formalism.

The Green's function

$$G_{[V(x)]}(0, x_0; s, x_t) = \int_0^{+\infty} e^{-st} I_{[V(x)]}(0, x_0; t, x_t) dt \ , \tag{7}$$

which is the Laplace transform of the previous path integral, is the solution of the forward partial differential equation

$$s \cdot G = \frac{1}{2} \frac{\partial^2 G}{\partial x_t^2} - V(x_t) \cdot G + \delta(x_t - x_0) \tag{8}$$

with conditions

$$\begin{array}{l} \lim_{x_t \to +\infty} G_{[V(x)]}(0, x_0; s, x_t) = 0 \\ \lim_{x_t \to -\infty} G_{[V(x)]}(0, x_0; s, x_t) = 0. \end{array} \tag{9}$$

2.2 Green's functions for potentials with δ and δ'

In many applications, the boundary conditions (6) at infinity appear like an idealization and one may be willing to incorporate at $x = a$ an impermeable barrier to the potential and specify the behaviour of the particle while attaining this limit. Dirichlet and Neumann problems have been treated in the path integral formalism using infinitely repulsive δ and δ' perturbations.

The Green's function of the one dimensional δ-potential can be solved as done in [19] by summing all the terms of the perturbation expansion. The Green's function of the path integral $I_{[V(x)+\beta\delta(x-a)]}$ can be expressed by means of the regular Green's function

$$\begin{array}{l} G_{[V(x)+\beta\delta(x-a)]}(0, x_0; s, x_t) \\ \quad = G_{[V(x)]}(0, x_0; s, x_t) - \frac{G_{[V(x)]}(0,x_0;s,a) G_{[V(x)]}(0,a;s,x_t)}{G_{[V(x)]}(0,a;s,a)+1/\beta}. \end{array} \tag{10}$$

The δ'-potential expansion leads to a summation of strongly interrelated terms without straightforward solution. Grosche [20, 21] proposes a specific regularization procedure and obtains the following expression for the Green's function of the path integral $I_{[V(x)+\beta\delta'(x-a)]}$:

$$\begin{array}{l} G_{[V(x)+\beta\delta'(x-a)]}(0, x_0; s, x_t) \\ \quad = G_{[V(x)]}(0, x_0; s, x_t) - \frac{\frac{\partial G_{[V(x)]}}{\partial x_t}(0,x_0;s,a) \frac{\partial G_{[V(x)]}}{\partial x_0}(0,a;s,x_t)}{\frac{\partial^2 G_{[V(x)]}}{\partial x_0 \partial x_t}(0,a;s,a)-1/\beta}. \end{array} \tag{11}$$

For our purpose, the limiting situations are very interesting. In this respect, we use the notations

$$G^D_{[V(x)]}(0, x_0; s, x_t) = \lim_{\beta \to +\infty} G_{[V(x)+\beta\delta(x-a)]}(0, x_0; s, x_t) \tag{12}$$

and

$$G^N_{[V(x)]}(0, x_0; s, x_t) = \lim_{\beta \to +\infty} G_{[V(x)+\beta\delta'(x-a)]}(0, x_0; s, x_t). \tag{13}$$

The choice of the notation is justified if we look at the boundary conditions. Indeed, for the first function we have $\lim_{x_t \to a} G^D_{[V(x)]}(0, x_0; s, x_t) = 0$, which corresponds with the Dirichlet solution of the corresponding Schrödinger equation. For the second function, the Neumann solution arises, since in that case $\lim_{x_t \to a} \frac{\partial}{\partial x_t} G^N_{[V(x)]}(0, x_0; s, x_t) = 0$. In this paper, we show how the combination of δ and δ'-potentials can be used to implement more general Robin conditions in the path integral formalism. Note that even in the simplest case where $V \equiv 0$, the Green's function of (11) exhibits a discontinuity at $x = a$. As a consequence, the Green's functions $G_{[\delta(x-a-)+\beta\delta'(x-a)]}$ and $G_{[\delta(x-a+)+\beta\delta'(x-a)]}$ are not the same.

In the special case where $V \equiv 0$, we get the absorbing and reflecting Brownian motion, for which the results are well known:

$$I^D_{(0)}(0, x_0; s, x_t) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{1}{2t}(x_t - x_0)^2} - e^{-\frac{1}{2t}(|x_t - a| + |a - x_0|)^2} \right) \tag{14}$$

and

$$I^N_{(0)}(0, x_0; s, x_t) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{1}{2t}(x_t - x_0)^2} + e^{-\frac{1}{2t}(|x_t - a| + |a - x_0|)^2} \right), \tag{15}$$

with $x_0 \geq a$ and $x_t \geq a$.

3 Transition probabilities

In single factor models, we assume that all the information about the term structure of interest rate is summarized by a unique diffusion process which is usually the spot rate. Diffusion processes can be related to quantum mechanical particles with convenient Lagrangians through the Feynman *sum over all paths* method. In this section, we show how transition probabilities of general spot rate processes can be expressed by means of path integrals, in case an absorbing or a reflecting boundary is imposed.

3.1 Definition

Consider a time-homogeneous diffusion process $X = \{X_t : t \geq 0\}$ for which $\mu(x; \theta)$ and $\sigma^2(x; \theta)$ are the instantaneous mean and variance per unit of time change depending on some parameter vector θ. If x_0 denotes the state variable at time t_0 and x_t at a latter time t, then the transition probability for this process X, defined as

$$p(0, x_0; t, x_t) = \frac{d}{dx_t} \operatorname{Prob}\left[X_t \leq x_t | X_0 = x_0\right], \tag{16}$$

satisfies the forward Fokker Planck equation

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x_t^2} \left(\sigma^2(x_t; \theta) \cdot p \right) - \frac{\partial}{\partial x_t} \left(\mu(x_t; \theta) \cdot p \right) \tag{17}$$

with $p(0, x_0; 0, x_t) = \delta(x_t - x_0)$ and with boundary conditions $\lim_{x_t \to \pm\infty} \; p(0, x_0; t, x_t) = 0$ (see [10]). If the diffusion is restricted to one barrier, the solution of the Kolmogorov equation is subject to a specific boundary condition at the location of the barrier.

Boundaries imposed by conditions on the Kolmogorov equation as absorption and reflection are called regular. Singularities of the infinitesimal mean $\mu(x; \theta)$ and variance $\sigma^2(x; \theta)$ can confine the process into a restricted domain and preclude us from imposing explicit boundaries. In such cases, the boundary is implicit. Detailed discussion of the problem can be found in [24]. In this paper, we assume that the barrier is located at an *attainable* point where an explicit boundary can be imposed.

3.2 Transition probabilities and path integrals

Stochastic processes X defined as above are solutions of a diffusion equation and their transition probability can be expressed in a closed form by means of a path integral. In order not to complicate the formulas, we will assume that the volatility is equal to one. This can be done without loss of generality, see remark below.

If the function $V : (\mathbb{R}; \Theta) \to \mathbb{R}$ is defined by

$$V(x; \theta) = \frac{1}{2}\mu^2(x; \theta) + \frac{1}{2}\frac{\partial \mu(x; \theta)}{\partial x}, \tag{18}$$

the following results hold :

(i) no boundary

The transition probability of the process X, solution of the diffusion equation $dX_t = \mu(X_t; \theta)dt + dW_t$, can be written as

$$\begin{aligned} & p(0, x_0; t, x_t) \\ & = \exp\left\{ \int_{x_0}^{x_t} \mu(z; \theta) dz \right\} \cdot I_{[V(x;\theta)]}(0, x_0; t, x_t) \; ; \end{aligned}$$

it satisfies the forward Fokker Planck equation (17) with boundary conditions

$$\lim_{x_t \to +\infty} p(0, x_0; t, x_t) = 0 \quad \text{and} \quad \lim_{x_t \to -\infty} p(0, x_0; t, x_t) = 0.$$

(ii) absorbing boundary in a ($x_0 \geq a$)

The transition probability of the process X, solution of the diffusion equation $dX_t = \mu(X_t; \theta)dt + dW_t$, can be written as

$$\begin{aligned} & p^A(0, x_0; t, x_t) \\ & = \begin{cases} 0 & x_t < a \\ \exp\left\{ \int_{x_0}^{x_t} \mu(z; \theta) dz \right\} \cdot \lim_{\beta \to \infty} I_{[V_\beta^A(x;\theta)]}(0, x_0; t, x_t) & x_t > a \\ \delta(x_t - a) \cdot \left(1 - \int_{a+}^{+\infty} p^A(0, x_0; t, x) dx \right) & x_t = a \end{cases} \end{aligned}$$

with potential $V_\beta^A(x;\theta) = V(x;\theta) + \beta\delta(x-a)$;
it satisfies the forward Fokker Planck equation (17) with boundary conditions

$$\lim_{x_t\to+\infty} p^A(0,x_0;t,x_t) = 0 \quad \text{and} \quad \lim_{x_t\to a+} p^A(0,x_0;t,x_t) = 0.$$

(iii) reflecting boundary in a ($x_0 \geq a$)

The transition probability of the process X, solution of the diffusion equation $dX_t = \mu(X_t;\theta)dt + dW_t + \frac{1}{2}dL_t^{a+}(X)$, where $L_t^{a+}(X)$ is the right-local time of X, can be written as

$$p^R(0,x_0;t,x_t) = \begin{cases} 0 & x_t < a \\ \exp\left\{\int_{x_0}^{x_t} \mu(z;\theta)dz\right\} \cdot \lim_{\beta\to\infty} I_{[V_\beta^R(x;\theta)]}(0,x_0;t,x_t) & x_t \geq a \end{cases}$$

with potential $V_\beta^R(x;\theta) = V(x;\theta) + \frac{\mu(a;\theta)}{2}\delta(x-a+) + \beta\delta'(x-a)$;
it satisfies the forward Fokker Planck equation (17) with boundary conditions

$$\lim_{x_t\to+\infty} p^R(0,x_0;t,x_t) = 0$$

and

$$\lim_{x_t\to a+}\left(\frac{1}{2}\frac{\partial}{\partial x_t}p^R(0,x_0;t,x_t) - \mu(x_t;\theta)p^R(0,x_0;t,x_t)\right) = 0.$$

A proof can be found in appendix A.

Remark 1. The situation of a stochastic differential equation with non-unit volatility,

$$dX_t = \mu(X_t;\theta)dt + \sigma(X_t;\theta)dW_t, \tag{19}$$

can be solved by a transformation of the original process X_t to the process $Y_t = \phi(X_t;\theta)$ with $\phi(x;\theta) = \int^x \frac{dz}{\sigma(z;\theta)}$ applying the Ito lemma or the generalized Tanaka formula (see [18]). The absorbing or reflecting boundary then has to be imposed on $\phi(a;\theta)$ instead of on a. The transition probability of X follows from the transition probability of Y as $p_X(0,x_0;t,x_t) = p_Y\left(0,\phi(x_0;\theta);t,\phi(x_t;\theta)\right)/\sigma(x_t;\theta)$.

In order not to complicate the formulas, for the theoretical part of this paper, we only write out the results in case the volatility is equal to one. However, the numerical illustrations in the last section will discuss a model with non-unit volatility.

Remark 2. The results can be easily adapted for the situation $x_0 \leq a$.

Remark 3. The δ'-function perturbation in the path integral leads to a propagator discontinuous in a. Therefore, the right δ-potential $\delta(x-a+)$ defines the path integral in an unambiguous way. This is closely related to the discontinuity of the symmetric local time of reflecting diffusion mentioned in the introduction.

Remark 4. The result for the reflecting barrier nicely illustrates in the path integral formalism the Girsanov transformation for reflecting diffusion, as proposed by Kinkladze [29].

4 Arrow-Debreu securities

The path integal is a kernel function and permits to solve the Schrödinger equation for any initial condition. Pricing contingent claims on interest rates reduces to solving a backward partial differential equation for appropriate terminal condition determined by the payoff of the claim. Beaglehole and Tenney [5] and Jamshidian [26] solve the general contingent claim pricing equation by means of a kernel function and interpret this particular solution as the prices of fundamental state dependent securities, or Arrow-Debreu securities. A nice generalization of the use of path integrals for the calculation of transition probabilities can be found in the pricing of these basic securities.

4.1 Definition

Assume that the spot rate is defined as a time-homogeneous diffusion process $r = \{r_t : t \geq 0\}$ with $\mu(r;\theta)$ and $\sigma^2(r;\theta)$ the instantaneous mean and variance per unit of time change. An Arrow-Debreu security pays 1 at maturity time only if the spot rate equals a fixed value $\hat{r}$. Any contingent claim can be replicated by purchasing a portfolio of these fundamental claims. The price of an Arrow-Debeu security at time t with maturity T and fixed value $\hat{r}$ can be written as

$$\Gamma(t,r;T,\hat{r}) = E_t^P\left[e^{-\int_t^T r_u du - \frac{1}{2}\int_t^T \lambda^2(r_u,u)du - \int_t^T \lambda(r_u,u)dW_u} \cdot \delta(r_T - \hat{r})\right], \tag{20}$$

where the current spot rate is equal to r, $\lambda(r_t,t)$ is the market risk-premium and P is the historical measure. In the sequel we will assume that $\lambda \equiv 0$, corresponding to the local expectation hypothesis which is compatible with the no-arbitrage requirement in case of no liquidity preference (see e.g. [14]). This assumption has important consequences in practice: the risk-neutral probability measure Q is equal to the historical measure P and estimating the parameter vector θ from real spot rate data is sufficient to specify the term structure.

The price Γ of the Arrow-Debreu security can also be found as the solution of the backward partial differential equation

$$\frac{\partial \Gamma}{\partial t} + \frac{1}{2}\sigma^2(r;\theta)\frac{\partial^2 \Gamma}{\partial r^2} - r \cdot \Gamma + \mu(r;\theta)\frac{\partial \Gamma}{\partial r} = 0 \tag{21}$$

with terminal condition

$$\Gamma(t,r;T,\hat{r}) = \delta(r - \hat{r}) \tag{22}$$

and with boundary conditions $\lim_{r\to\pm\infty}\Gamma(t,r;T,\hat{r}) = 0$, see e.g. [25]. When the spot rate process is restricted to an absorbing or a reflecting barrier, the pricing equation is subject to a specific boundary condition at the location of the barrier.

4.2 From Arrow-Debreu securities to general contingent claims

The price $B(t,r;T)$ of a general interest-rate contingent claim with maturity date T, with continuous payout function $c(r,t)$ and with a final discrete payoff $h(r_T)$, satisfies

the partial differential equation

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2(r;\theta)\frac{\partial^2 B}{\partial r^2} - r \cdot B + \mu(r;\theta)\frac{\partial B}{\partial r} + c(r,t) = 0 \tag{23}$$

with terminal condition

$$B(T, r_T; T) = h(r_T). \tag{24}$$

Following [5], the price can be determined by

$$B(t,r;T) = \int_{-\infty}^{+\infty}\int_t^T \Gamma(t,r;\tau,\hat{r}) \cdot c(\hat{r},\tau)\, d\hat{r}\, d\tau + \int_{-\infty}^{+\infty} \Gamma(t,r;T,\hat{r}) \cdot h(\hat{r})\, d\hat{r}, \tag{25}$$

where $\Gamma(t,r;T,\hat{r})$ is the price of the Arrow-Debreu security.

4.3 Examples

1. For a zero-coupon bond, the continuous payout function and final discrete payoff equal

$$\begin{cases} c(r,t) = 0 \\ h(r_T) = 1; \end{cases}$$

the price for this contingent claim can be determined as

$$B_{ZC}(t,r;T) = \int_{-\infty}^{+\infty} \Gamma(t,r;T,\hat{r})\, d\hat{r}. \tag{26}$$

2. For a European call option on the interest rate, the continuous payout function and final discrete payoff equal

$$\begin{cases} c(r,t) = 0 \\ h(r_T) = \max(r_T - K, 0); \end{cases}$$

the price for this contingent claim can be determined as

$$B_{EC}(t,r;T) = \int_{K}^{+\infty} \Gamma(t,r;T,\hat{r}) \cdot (\hat{r} - K)\, d\hat{r}. \tag{27}$$

3. For a caplet at rate r^*, the continuous payout function and final discrete payoff equal

$$\begin{cases} c(r,t) = \min(r, r^*) \\ h(r_T) = \max(r_T - r^*, 0); \end{cases}$$

the price for this contingent claim can be determined as

$$\begin{aligned} &B_{CL}(t,r;T) \\ &= \textstyle\int_{-\infty}^{+\infty}\int_t^T \Gamma(t,r;\tau,\hat{r}) \cdot \min(\hat{r}, r^*)\, d\hat{r}\, d\tau + \int_{r^*}^{+\infty}(r - r^*)\Gamma(t,r;T,\hat{r})\, d\hat{r}. \end{aligned} \tag{28}$$

4. For a floorlet at rate r^*, the continuous payout function and final discrete payoff equal

$$\begin{cases} c(r,t) = \max(r, r^*) \\ h(r_T) = \max(r^* - r_T, 0); \end{cases}$$

the price for this contingent claim can be determined as

$$\begin{aligned} &B_{FL}(t,r;T) \\ &= \int_{-\infty}^{+\infty} \int_t^T \Gamma(t,r;\tau,\hat{r}) \cdot \max(\hat{r}, r^*)\, d\hat{r}\, d\tau + \int_{-\infty}^{r^*} (r^* - r)\Gamma(t,r;T,\hat{r})\, d\hat{r}. \end{aligned} \tag{29}$$

4. For a swap with fixed rate r^*, the continuous payout function and final discrete payoff equal

$$\begin{cases} c(r,t) = (r - r^*) \\ h(r_T) = 0; \end{cases}$$

the price for this contingent claim[3] can be determined as

$$B_{SW}(t,r;T) = \int_{-\infty}^{+\infty} \int_t^T \Gamma(t,r;\tau,\hat{r}) \cdot (\hat{r} - r^*)\, d\hat{r}\, d\tau. \tag{30}$$

4.4 Arrow-Debreu securities and path integrals

As it was shown for transition probabilities, also the price of an Arrow-Debreu security can be expressed by means of a path integral for any spot rate process r defined as above. Starting from the path integral representation for the price of the Arrow-Debreu security, an expression for the price of general interest-rate contingent claims can then be found as explained in the previous subsection. As before, without loss of generality we can assume a unit volatility (see remark below).

If the function $V : (\mathbb{R}, \Theta) \to \mathbb{R}$ is defined by

$$V(x;\theta) = \frac{1}{2}\mu^2(x;\theta) + \frac{1}{2}\frac{\partial \mu(x;\theta)}{\partial x}, \tag{31}$$

the following results hold :

i) no boundary

The price of the Arrow-Debreu securities (20) for the term structure driven by the spot rate process r, solution of the diffusion equation $dr_t = \mu(r_t;\theta)dt + dW_t$, can be written as

$$\begin{aligned} &\Gamma(t,r;T,\hat{r}) \\ &= \exp\left\{\int_r^{\hat{r}} \mu(z;\theta)dz\right\} \cdot I_{[V(x;\theta)+x]}(t,r;T,\hat{r})\ ; \end{aligned} \tag{32}$$

this price satisfies the backward equation (21) with boundary conditions

$$\lim_{r\to+\infty} \Gamma(t,r;T,\hat{r}) = 0 \quad \text{and} \quad \lim_{r\to-\infty} \Gamma(t,r;T,\hat{r}) = 0.$$

[3]Note that the price of a swap can be negative.

ii) absorbing boundary in a ($r \geq a$)

The price of the Arrow-Debreu securities (20) for the term structure driven by the spot rate process r, solution of the diffusion equation $dr_t = \mu(r_t;\theta)dt + dW_t$, can be written as

$$\Gamma^A(t,r;T,\hat{r}) = \begin{cases} 0 & \hat{r} < a \\ \exp\left\{\int_r^{\hat{r}} \mu(z;\theta)dz\right\} \cdot \lim_{\beta\to+\infty} I_{[V_\beta^A(x;\theta)+x]}(t,r;T,\hat{r}) & \hat{r} > a \\ \delta(\hat{r}-a)\cdot\left(B_{ZC}(t,r;T) - \int_{a+}^{+\infty}\Gamma^A(t,r;T,\hat{r})d\hat{r}\right) & \hat{r} = a \end{cases}$$

with potential $V_\beta^A(x;\theta) = V(x;\theta) + \beta\delta(x-a)$ and with $B_{ZC}(t,r;T)$ the price of the zero coupon bond.
$\Gamma^A(t,r;T,\hat{r})$ satisfies the backward equation (21) with boundary conditions

$$\lim_{r\to+\infty}\Gamma^A(t,r;T,\hat{r}) = 0 \quad \text{and} \quad \lim_{r\to a+}\Gamma^A(t,r;T,\hat{r}) = 0$$

iii) reflecting boundary in a ($r \geq a$)

The price of the Arrow-Debreu securities (20) for the term structure driven by the spot rate process r, solution of the diffusion equation $dr_t = \mu(r_t;\theta)dt + dW_t + \frac{1}{2}dL_t^{a+}(r)$ where $L_t^{a+}(r)$ is the right-local time of r, can be written as

$$\Gamma^R(t,r;T,\hat{r}) = \begin{cases} 0 & \hat{r} < a \\ \exp\left\{\int_r^{\hat{r}} \mu(z;\theta)dz\right\} \cdot \lim_{\beta\to\infty} I_{[V_\beta^R(x;\theta)+x]}(t,r;T,\hat{r}) & \hat{r} \geq a \end{cases}$$

with potential $V_\beta^R(x;\theta) = V(x;\theta) + \frac{\mu(a;\theta)}{2}\delta(x-a+) + \beta\delta'(x-a)$;
this price satisfies the backward equation (21) with boundary conditions

$$\lim_{r\to+\infty}\Gamma^R(t,r;T,\hat{r}) = 0$$

and

$$\lim_{r\to a+}\frac{1}{2}\frac{\partial}{\partial r}\Gamma^R(t,r;T,\hat{r}) = 0.$$

A proof can be found in appendix B.

Remark 1. Again, the situation of a stochastic differential equation with non-unit volatility,

$$dr_t = \mu(r_t;\theta)dt + \sigma(r_t;\theta)dW_t, \tag{33}$$

can be solved by a transformation of the original process r to the process $Y_t = \phi(r_t;\theta)$ with $\phi(r;\theta) = \int^r \frac{dz}{\sigma(z;\theta)}$ applying the Ito lemma or the generalized Tanaka formula

(see [18]). The absorbing or reflecting boundary then has to be imposed on $\phi(a)$ instead of on a. The prices of the Arrow-Debreu securities for r then follow from

$$\begin{aligned}\Gamma(t,r;T,\hat{r}) &= E_t^P\left[e^{-\int_t^T \phi^{-1}(Y_u;\theta)du}\cdot\delta(\phi^{-1}(Y_T;\theta)-\hat{r})\right]\\ &= E_t^P\left[e^{-\int_t^T \phi^{-1}(Y_u;\theta)du}\cdot\frac{\delta(Y_T-\phi(\hat{r};\theta))}{\sigma(\hat{r};\theta)}\right].\end{aligned}$$

Remark 2. The results can be easily adapted for the situation $x_0 \leq a$.

5 Closed form approximation

5.1 Methodology

In the previous sections, we derived path integral representations for the transition probabilities of diffusion processes and for the prices of Arrow-Debreu securities. For the computation of these path integrals, we can rely on results from quantum mechanics (see e.g. [18]). However, in most cases, an exact calculation of the functional integrals is not possible. In this section we will propose a solution appropriate to path integrals with δ and δ'-potentials in the integrand. We derive an analytical upper and lower bound for such path integrals, which can also be combined to give an accurate approximation.

Our methodology is based on a decomposition of the function V appearing in the potential of the path integral $I_{[V(x;\theta)]}(0,x_0;t,x_t)$. This decomposition $V(x;\theta)=V_1(x;\theta)+V_2(x;\theta)$ has to be chosen in such a way that the path integral corresponding to $V_2(x;\theta)$ is perfectly tractable. It is for the more difficult part denoted by $V_1(x;\theta)$, that we will work with an approximation.

It can be shown in a straightforward way, that for any decomposition $V(x;\theta) = V_1(x;\theta)+V_2(x;\theta)$, the path integral $I_{[V(x;\theta)]}(0,x_0;t,x_t)$ can be written as

$$I_{[V(x;\theta)]}(0,x_0;t,x_t) = I_{[V_2(x;\theta)]}(0,x_0;t,x_t)\cdot \mathrm{E}_{(V_2)}\left[e^{-\int_0^t V_1(X(s);\theta)ds}\right]. \tag{34}$$

The expectation here is taken over all paths starting in x_0 at time 0 and arriving in x_t at time t with marginal distribution density

$$f_{(V_2)}(x;s) = \frac{d}{dx}F_{(V_2)}(x;s) = \frac{d}{dx}\mathrm{Prob}[X(s)\leq x \mid X(0)=x_0, X(t)=x_t] \tag{35}$$

given by

$$f_{(V_2)}(x;s) = \frac{I_{[V_2(x;\theta)]}(0,x_0;s,x)I_{[V_2(x;\theta)]}(s,x;t,x_t)}{I_{[V_2(x;\theta)]}(0,x_0;t,x_t)}. \tag{36}$$

Making use of such a decomposition, the path integral is written as an expectation of a functional involving $V_1(x;\theta)$. This means that the problem is reduced to the issue of finding an approximation for such an expectation.

5.2 Upper and lower bound for the expectation

In order to arrive at a closed form approximation for the remaining expectation, we will construct an upper and a lower bound, which eventually can be combined to give an accurate approximation.

1. For the upper bound, we make use of the concept of convex ordering, and the relation with comonotonic risks. The basic idea is to exchange the original stochastic variable for a similar variable, for which it is more likely to reach extreme values. In that case, an expectation of a decreasing exponential functional of this new variable will be larger than for the original one. Some definitions and important results about this powerful tool can be found in appendix C. For more details, we refer to some earlier contributions, e.g. [11, 12].

Relying on the notion of convex ordering, it can be shown that

$$\mathrm{E}_{(V_2)}\left[e^{-\int_0^t V_1(X(s);\theta)ds}\right] \leq \mathrm{E}_U\left[e^{-\int_0^t F^{-1}_{V_1(X(s);\theta)}(U)ds}\right], \tag{37}$$

where U is a $[0,1]$ uniformly distributed variable.

The following notations are used for the distribution functions:

$$F_{X(s)}(x) = F_{(V_2)}(x;s), \qquad \text{(cf supra)} \tag{38}$$

$$F_{V_1(X(s);\theta)}(x) = \mathrm{Prob}[V_1(X(s);\theta) \leq x], \tag{39}$$

and

$$F^{-1}_{V_1(X(s);\theta)}(p) = \inf\{x \in \mathbb{R} : F_{V_1(X(s);\theta)}(x) \geq p\}, \tag{40}$$

with $x \in \mathbb{R}$ and $p \in [0,1]$. A proof can be found in appendix D.

2. For the lower bound, we suggest to make use of the inequality of Jensen (see e.g. [13]), combined with a conditioning on an intermediate time-point.

Relying on the inequality of Jensen, it can be shown that

$$\mathrm{E}_{(V_2)}\left[e^{-\int_0^t V_1(X(s);\theta)ds}\right] \geq \mathrm{E}_{X(\tau)}\left[e^{-\int_0^\tau \mathrm{E}_{(V_2)}[V_1(X(s);\theta)]ds} \cdot e^{-\int_\tau^t \mathrm{E}_{(V_2)}[V_1(X(s);\theta)]ds}\right], \tag{41}$$

where τ is any point between 0 and t.
The expectation $\mathrm{E}_{X(\tau)}[...]$ means an expectation over all possible values of $X(\tau)$.

A proof can be found in appendix E.

Note that a disadvantage of this approach is the fact that the bounds for the transition probabilities and for the prices of Arrow-Debreu securities no longer satisfy the boundary conditions for the partial differential equations. However, due to the high accuracy of the bounds, this is only a minor drawback.

5.3 Application to the two boundary cases

The choice of the decomposition for the absorbing and reflecting boundary cases are obvious, due to the appearance of delta functions and the results of subsections 3.2 and 4.4.

Indeed, for V_1 we choose the difficult part, referring to the drift term in the stochastic differential equation. For V_2, we then work with

i) absorbing boundary in a

$$V_2(x;\theta) = \beta \cdot \delta(x-a) \; ; \tag{42}$$

ii) reflecting boundary in a

$$V_2(x;\theta) = \frac{\mu(a;\theta)}{2} \cdot \delta(x-a+) + \beta \cdot \delta'(x-a) \; . \tag{43}$$

The path integral $I_{[\beta\delta(x-a)]}$ can be interpreted as the propagator of a Brownian particule killed when the time spent in the vicinity of a measured by the local time exceeds an independent random variable T with exponential distribution $P[T \leq t] = 1 - e^{-\beta t}$. As β goes to infinity, $I_{[\beta\delta(x-a)]}$ converges to the transition probability of a Brownian motion instantaneously killed at a. In the case of a reflecting boundary and $\mu(a;\theta) \geq 0$, $I_{[V_2(x;\theta)]}$ converges to the transition probability of an elastic Brownian motion[4] killed with a rate $\mu(a;\theta)/2$, see Ito and McKean [24].

5.4 Remarks

Consider a stochastic process X for which the transition probabilities can be expressed as explained in section 3. We know how to construct a closed form upper and lower bound for the path integral appearing in expressions derived in subsections 3.2 and 4.4. The problem now with these bounds is the fact that we are no longer dealing with density functions, since the total mass is larger or smaller than one.

For the case without boundaries, and for the case with a reflecting boundary, we suggest to solve this problem by making use of a convex combination of the upper and lower bound. Let $p^{upp}(0,x_0;t,x_t)$ and $p^{R,upp}(0,x_0;t,x_t)$ denote the upper bounds for the transition probabilities without boundary and with a reflecting boundary, and analogously let $p^{low}(0,x_0;t,x_t)$ and $p^{R,low}(0,x_0;t,x_t)$ denote the lower bounds.
Then for all $x_t \in \mathbb{R}$ we have

$$p^{low}(0,x_0;t,x_t) \leq p(0,x_0;t,x_t) \leq p^{upp}(0,x_0;t,x_t) \; ,$$

and for all $x_t \geq a$

$$p^{R,low}(0,x_0;t,x_t) \leq p^{R}(0,x_0;t,x_t) \leq p^{R,upp}(0,x_0;t,x_t).$$

[4]The corresponding distribution function $F_{(V_2)}(x;s)$ contains tedious combinations of erf(.) functions; fortunately, accurate algebraic approximations can be found in Abramowitz and Stegun [1].

An approximation for the transition probability, now with mass one, can be found as

$$\tilde{p}(0, x_0; t, x_t) = z \cdot p^{low}(0, x_0; t, x_t) + (1 - z) \cdot p^{upp}(0, x_0; t, x_t), \tag{44}$$

and

$$\tilde{p}^R(0, x_0; t, x_t) = z^R \cdot p^{R,low}(0, x_0; t, x_t) + (1 - z^R) \cdot p^{R,upp}(0, x_0; t, x_t), \tag{45}$$

where z and z^R, depending on the time t and on the starting point x_0, are chosen such that the total mass equals one, or

$$\int_{-\infty}^{+\infty} \tilde{p}(0, x_0; t, x_t)\ dx_t = 1$$

and

$$\int_{a}^{+\infty} \tilde{p}^R(0, x_0; t, x_t)\ dx_t = 1.$$

In the situation of an absorbing boundary, we have to work differently. First, for $x_t > a$ let's denote the upper and lower bound in the same way as before, such that

$$p^{A,low}(0, x_0; t, x_t) \leq p^A(0, x_0; t, x_t) \leq p^{A,upp}(0, x_0; t, x_t) \tag{46}$$

for all $x_t > a$.
In order to get approximations for the real transition probability, a convex combination of both bounds is not necessary, since the total mass can be leveled to one by allocating the remaining part to the probability of absorption. In that case, both $p^{A,low}$ and $p^{A,upp}$ will be density functions, which approximate the exact transition probability.
Note that in that case, from the function $p^{A,upp}$, we can deduce a lower limit for the probability of absorption, and analogously from the function $p^{A,low}$, we can deduce an upper limit. Indeed, it is true that

$$\begin{aligned} 1 - \textstyle\int_{a+}^{+\infty} p^{A,upp}(0, x_0; t, x)dx \\ \leq 1 - \textstyle\int_{a+}^{+\infty} p^{A}(0, x_0; t, x)dx \\ \leq 1 - \textstyle\int_{a+}^{+\infty} p^{A,low}(0, x_0; t, x)dx. \end{aligned} \tag{47}$$

In the case of an absorbing boundary, it is impossible to express directly the price of the Arrow-Debreu security at the barrier by means of a path integral. Indeed, the path integral formulation of $\Gamma^A(r, t; T, a)$ derived in subsection 4.4 involves the price of the zero coupon bond. In general, the price $B_{ZC}(r, t; T)$ is unknown in closed form. In order to avoid simulation, we propose the simple bounds $\Gamma^A(t, r; T, a) = c \cdot p^A(t, r; T, a)$ with

$$E_t^P \left[e^{-\int_t^T r_s ds} | r_T = a+ \right] \leq c \leq e^{-a(T-t)}$$

and

$$E_t^P \left[e^{-\int_t^T r_s ds} | r_T = a+ \right] = \frac{\Gamma^A(t, r; T, a+)}{p^A(t, r; T, a+)}. \tag{48}$$

6 Numerical illustration

In this section, we test the accuracy of the bounds for the evaluation of contingent claims on Japanese interest rates. The data sets consist of the Japanese three-month interbank spot rate monthly sampled for two periods of time: firstly from January 1986 through January 2001 and secondly from January 1994 through January 2001[5]. Figure 1 plots both time series.

We calibrate three one-factor models, namely the Vasicek model, the extended CIR (ECIR) and the extended CEV (ECEV), maximizing the approximate log-likelihood function

$$l(\theta) = \sum_{i=1}^{n-1} \ln\{\tilde{p_\theta}(0, r_{it/n}; t/n, r_{(i+1)t/n})\},$$

where $\tilde{p_\theta}(0, r_{it/n}; t/n, r_{(i+1)t/n})$ is the Euler approximation[6] of the transition density. Table 1 reports the parameter estimates. The asymptotic variance of the estimators is provided by the inverse of the Fisher information matrix $I(\theta)$; the statistical tests are performed using the asymptotic distribution

$$(\hat{\theta} - \theta) \to \sqrt{n} N(0, I(\theta)^{-1}).$$

The sharp fall of the spot rate between 1991 and 1993 explains the high variance of $\hat{\alpha}$ which is not significant for the period $1986 - 2001$.

For our numerical illustrations, we select the ECIR model for the period $1986 - 2001$ and the ECEV model with $\rho = 0$ for the period $1994 - 2001$. For both spot rate models, 0 is an attainable point where an explicit boundary can be imposed.

Figure 2 plots the bounds for the transition probabilities. As it can be observed from the pictures, these bounds turn out to be very accurate. As outlined in remark 5.4, we can obtain intervals for the probability of absorption; these are illustrated in figure 3.

Next to the transition probabilities, we consider the prices of claims contingent on the three-month interbank floating rate. We assume that there is no liquidity preference and we set the risk-premium $\lambda(r_t, t)$ equal to zero. This assumption is tenable as Japan experiences almost a deflation despite the fall of interest rates. An alternative is to calibrate a time-homogeneous risk-premium $\lambda(r_t) = \lambda_1 \cdot (r_t)^{\lambda_2}$ on zero-coupon bonds market data as done in e.g. [7].

[5]Since 2001, the Bank of Japan has activated the so-called Zero Interest Rate Policy (ZIRP) to maintain the short term rate close to zero until deflationary pressure will diminish. More appropriate term structures have been discussed to deal with this particular situation, see e.g. [31].

[6]The Euler approximation is obtained, assuming that the drift $\mu(r)$ and the volatility $\sigma(r)$ are constant and equal to $\mu(r_{it/n})$ and $\sigma(r_{it/n})$ in the time interval $[it/n, (i+1)t/n]$.

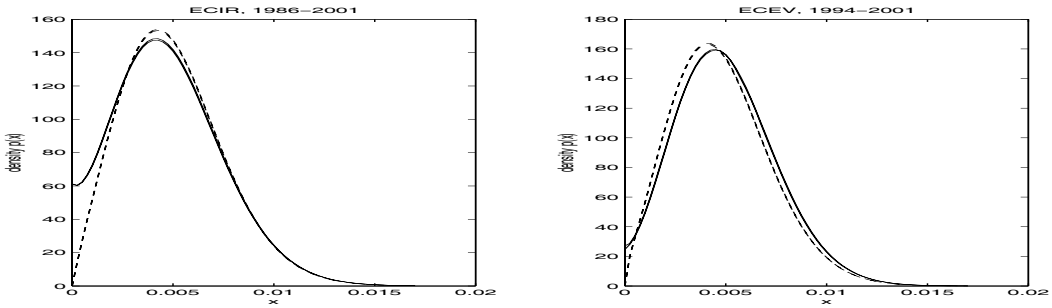

Figure 2: Bounds for the transition probabilities, $r_0 = 0.005$, $t_0 = 0$ and $t_e = 0.5$. The two solid lines refer to the reflecting boundary; the two dashed lines refer to the absorbing boundary.

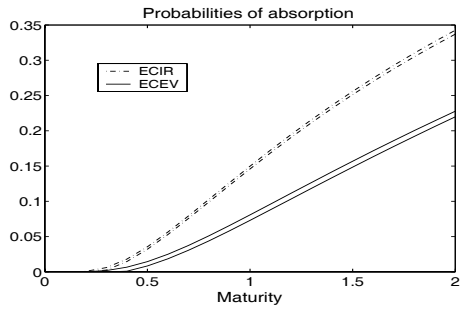

Figure 3: Bounds for the probabilities of absorption as a function of the time to maturity, $r_0 = 0.005$ and $t_0 = 0$. The two dashed lines refer to the $ECIR$ model; the two solid lines refer to the $ECEV$ model.

In figures 4, 5, 6 and 7 we present the bounds for the prices of the zero-coupon bonds as a function of the time to maturity together with the corresponding yield curve $R(t, r; T) = -\ln B_{ZC}(t, r; T)/(T - t)$. As one could expect, the bond prices are very close to 1, which causes the estimation of the yield curve to be rather difficult. Finally, the bounds for the contingent claims proposed in subsection 4.3 with $K = 0.003$ for the European call, $r^* = 0.008$ for the caplet, $r^* = 0.003$ for the floorlet and $r^* = 0.004$ for the swap provide fair intervals for the real prices. Except for the swap, it can be observed that prices are not very sensitive to the choice of the model; the interpretation of the reflecting boundary, however, remains more meaningful.

Table 1: Quasi maximum-likelihood estimates for the Japanese three-month interbank rate, $1986-2001$ and $1994-2001$. This table reports yearly parameter estimates. $\ln L$ is the optimal value of the average log-likelihood function. The standard deviation is computed with the inverse of the Fisher information matrix.

	Japanese interbank rate (1986-2001)				Japanese interbank rate (1986-2001)			
Model	Reflecting spot rate on $[0,+\infty)$	Parameter Estimates	Standard Deviation	p-value	Absorbing spot rate on $[0,+\infty)$	Parameter Estimates	Standard Deviation	p-value
Vasicek	$dr_t = \kappa(\alpha - r_t)dt + \sigma dB_t + \frac{1}{2}dL_t^{0+}(r)$	$\kappa = 0.035208$	0.100344	0.7254	$dr_t = \kappa(\alpha - r_t)dt + \sigma dB_t$	$\kappa = -0.001536$	0.088356	0.9862
	$\ln L = 4.721543$	$\alpha = -0.107436$	0.442179	0.8080	$\ln L = 4.642763$	$\alpha = 2.672470$	152.1394	0.9860
		$\sigma = 0.007880$	$2.71e^{-4}$	0.0000		$\sigma = 0.007371$	$2.16e^{-4}$	0.0000
ECIR	$dr_t = \kappa(\alpha - r_t)dt + \sigma\sqrt{r_t+\rho}dB_t + \frac{1}{2}dL_t^{0+}(r)$	$\kappa = 0.135516$	0.084876	0.1104	$dr_t = \kappa(\alpha - r_t)dt + \sigma\sqrt{r_t+\rho}dB_t$	$\kappa = 0.141480$	0.084948	0.0958
	$\ln L = 4.842074$	$\alpha = 0.001488$	0.014422	0.9178	$\ln L = 4.840035$	$\alpha = 0.002818$	0.012625	0.8234
		$\sigma = 0.040657$	0.002729	0.0000		$\sigma = 0.041398$	0.002691	0.0000
		$\rho = 0.004790$	0.001903	0.0118		$\rho = 0.003845$	0.001587	0.0154
ECEV	$dr_t = \kappa(\alpha - r_t)dt + \sigma(r_t+\rho)^\gamma dB_t + \frac{1}{2}dL_t^{0+}(r)$	$\kappa = 0.136296$	0.085500	0.1109	$dr_t = \kappa(\alpha - r_t)dt + \sigma(r_t+\rho)^\gamma dB_t$	$\kappa = 0.142608$	0.084444	0.0913
	$\ln L = 4.842100$	$\alpha = 0.001910$	0.015755	0.9035	$\ln L = 4.840568$	$\alpha = 0.003944$	0.013721	0.7738
		$\sigma = 0.038336$	0.017652	0.0299		$\sigma = 0.034085$	0.009816	0.0005
		$\rho = 0.003937$	0.006912	0.5689		$\rho = 0.001853$	0.003163	0.5689
		$\gamma = 0.477729$	0.172184	0.0055		$\gamma = 0.431431$	0.096633	0.0000

	Japanese interbank rate (1994-2001)				Japanese interbank rate (1994-2001)			
Model	Reflecting spot rate on $[0,+\infty)$	Parameter Estimates	Standard Deviation	p-value	Absorbing spot rate on $[0,+\infty)$	Parameter Estimates	Standard Deviation	p-value
Vasicek	$dr_t = \kappa(\alpha - r_t)dt + \sigma dB_t + \frac{1}{2}dL_t^{0+}(r)$	$\kappa = 0.382716$	0.268524	0.1541	$dr_t = \kappa(\alpha - r_t)dt + \sigma dB_t$	$\kappa = 0.418848$	0.231276	0.0701
	$\ln L = 5.243775$	$\alpha = 0.001913$	0.007811	0.8065	$\ln L = 5.207560$	$\alpha = 0.003101$	0.005719	0.5877
		$\sigma = 0.004659$	$2.63e^{-4}$	0.0000		$\sigma = 0.004343$	$2.08e^{-4}$	0.0000
ECIR	$dr_t = \kappa(\alpha - r_t)dt + \sigma\sqrt{r_t+\rho}dB_t + \frac{1}{2}dL_t^{0+}(r)$	$\kappa = 0.595260$	0.332580	0.0735	$dr_t = \kappa(\alpha - r_t)dt + \sigma\sqrt{r_t+\rho}dB_t$	$\kappa = 0.623976$	0.329388	0.0582
	$\ln L = 5.272662$	$\alpha = 0.004767$	0.003234	0.1405	$\ln L = 5.269272$	$\alpha = 0.004982$	0.002972	0.0937
		$\sigma = 0.038714$	0.009335	0.0000		$\sigma = 0.041569$	0.007929	0.0000
		$\rho = 0.005265$	0.005361	0.3260		$\rho = 0.003492$	0.003475	0.3149
ECEV	$dr_t = \kappa(\alpha - r_t)dt + \sigma(r_t+\rho)^\gamma dB_t + \frac{1}{2}dL_t^{0+}(r)$	$\kappa = 0.575892$	0.386232	0.1359	$dr_t = \kappa(\alpha - r_t)dt + \sigma(r_t+\rho)^\gamma dB_t$	$\kappa = 0.534816$	0.414252	0.1967
	$\ln L = 5.275091$	$\alpha = 0.004987$	0.003547	0.1597	$\ln L = 5.269558$	$\alpha = 0.003893$	0.004209	0.3550
		$\sigma = 0.017261$	0.015921	0.2783		$\sigma = 0.019835$	0.018099	0.2731
		$\rho = 0.000283$	0.003444	0.9346		$\rho = 3.70e^{-5}$	0.002213	0.9867
		$\gamma = 0.276810$	0.209373	0.1861		$\gamma = 0.303703$	0.199169	0.1273

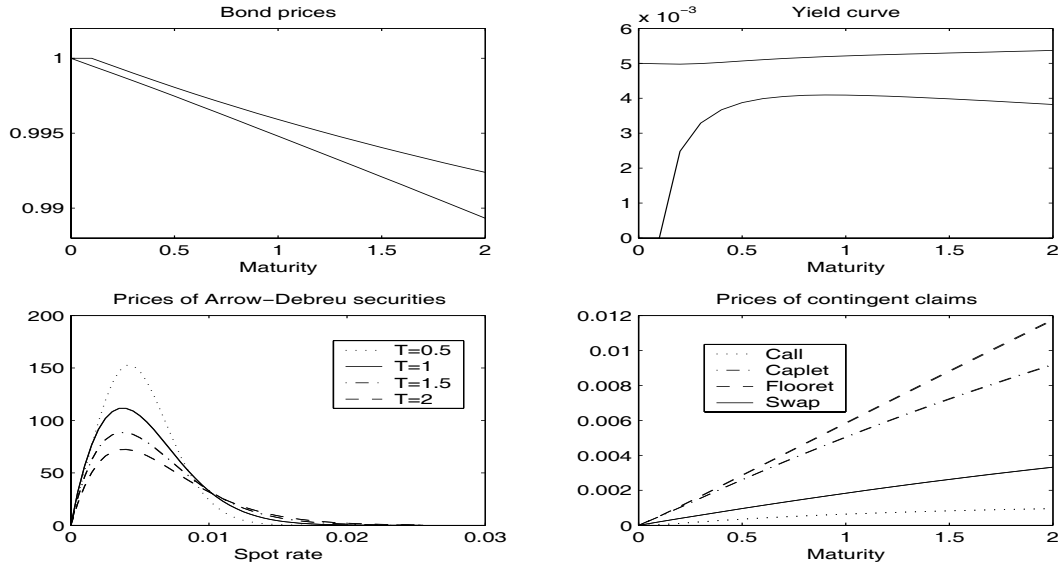

Figure 4: Bounds for the bond prices, the yield curve, the prices of Arrow-Debreu securities and of more general contingent claims under the $ECIR$ model with absorbing boundary at 0, $t = 0$ and r_t=0.005.

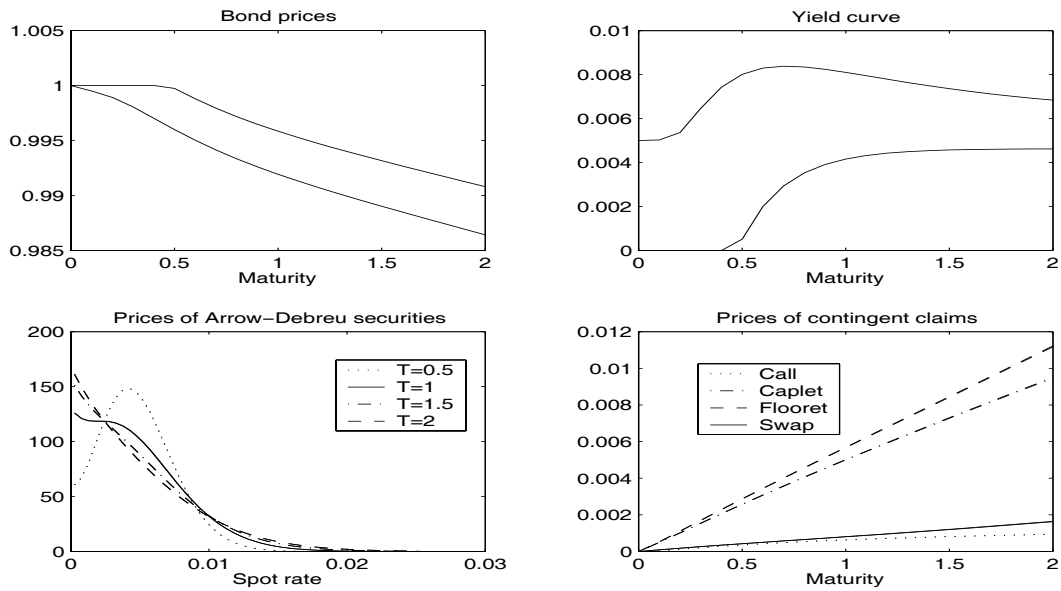

Figure 5: Bounds for the bond prices, the yield curve, the prices of Arrow-Debreu securities and of more general contingent claims under the $ECIR$ model with reflecting boundary at 0, $t = 0$ and r_t=0.005.

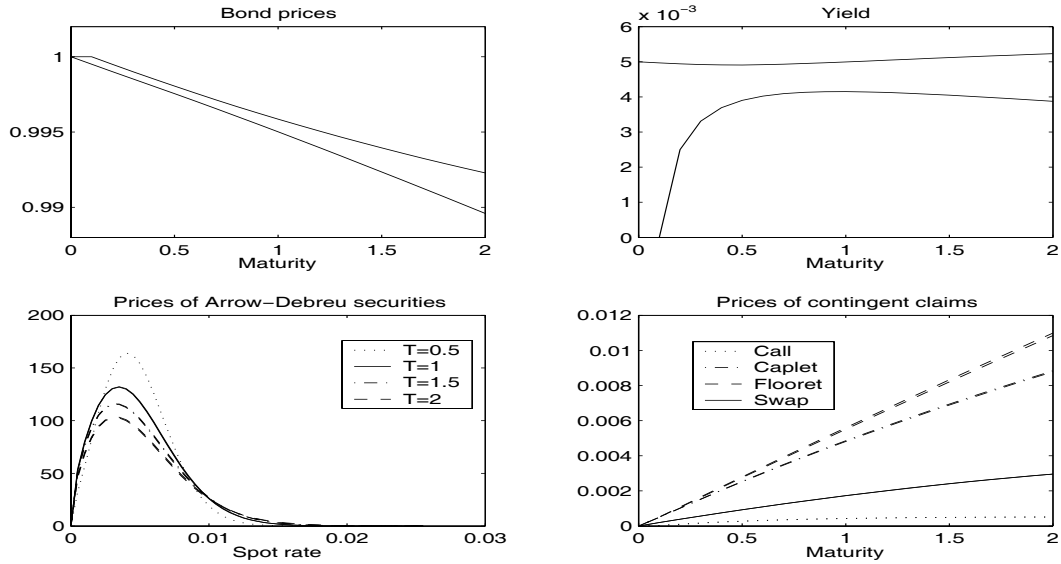

Figure 6: Bounds for the bond prices, the yield curve, the prices of Arrow-Debreu securities and of more general contingent claims under the $ECEV$ model with absorbing boundary at 0, $t = 0$ and r_t=0.005 .

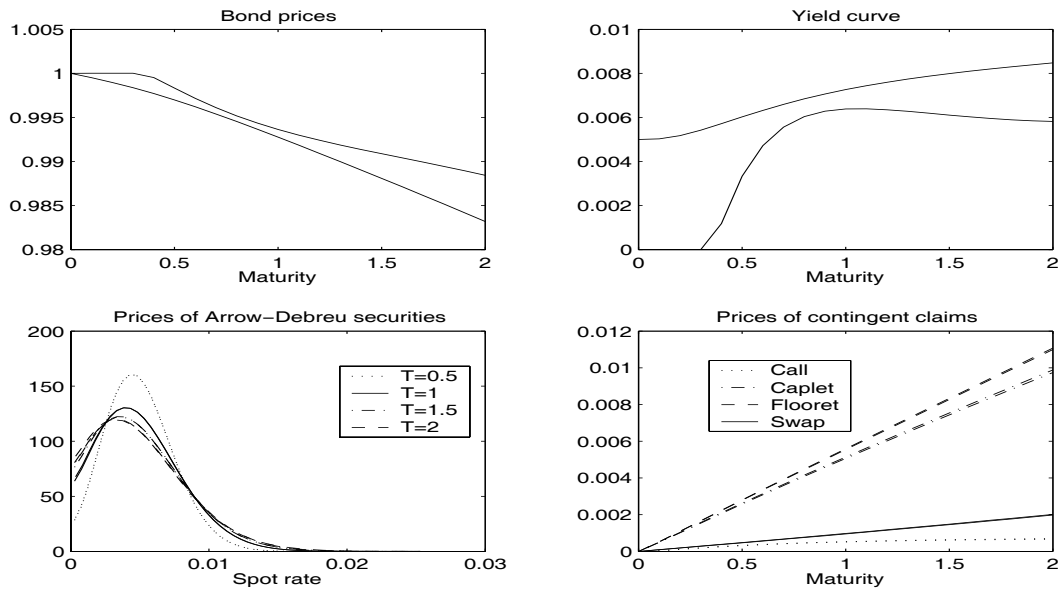

Figure 7: Bounds for the bond prices, the yield curve, the prices of Arrow-Debreu securities and of more general contingent claims under the $ECEV$ model with reflecting boundary at 0, $t = 0$ and r_t=0.005 .

Corresponding author:

Prof.dr. Ann De Schepper
Faculty of Applied Economics
University of Antwerp
Prinsstraat 13
2000 Antwerp
Belgium
tel : +32-3-220.40.77
fax : +32-3-220.48.17
email : ann.deschepper@ua.ac.be

A Proof for section 3

The result for the transition probability in case no boundary is imposed is not new. For a proof, we can refer e.g. to [18].

Assume now that we work with an absorbing boundary in a.
First, we denote by $G_{[V^A_{+\infty}(x;\theta)]}(0, x_0; s, x_t)$ the limit of the Green's function of the path integral $I_{[V^A_\beta(x;\theta)]}(0, x_0; s, x_t)$ when β goes to infinity[7]. From (8) and (10), we know that

$$\begin{aligned}&\tfrac{1}{2}\tfrac{\partial^2}{\partial x_t^2} G_{[V^A_{+\infty}(x;\theta)]}(0, x_0; s, x_t)\\ &\quad = s \cdot G_{[V^A_{+\infty}(x;\theta)]}(0, x_0; s, x_t) + V(x_t;\theta) \cdot G_{[V^A_{+\infty}(x;\theta)]}(0, x_0; s, x_t) - \delta(x_t - x_0)\end{aligned}$$

with conditions

$$\begin{aligned}&\lim_{x_t \to +\infty} G_{[V^A_{+\infty}(x;\theta)]}(0, x_0; s, x_t) = 0\\ &\lim_{x_t \to a+} G_{[V^A_{+\infty}(x;\theta)]}(0, x_0; s, x_t) = 0.\end{aligned}$$

After a multiplication with $\exp\{\int_{x_0}^{x_t} \mu(z;\theta)dz\}$ and an inversion of the Laplace transform, it follows that $p^A(0, x_0; t, x_t)$ is the solution of the forward Fokker-Planck equation. Hence, from probability theory, we know that $p^A(0, x_0; t, x_t)$ is the transition probability of the absorbing process (see e.g. [15]).

Finally, we discuss the case with a reflecting boundary. In analogy with the former case, it can be shown from (8) and (10), that

$$\begin{aligned}&\tfrac{1}{2}\tfrac{\partial^2}{\partial x_t^2} G^N_{[V(x;\theta)]}(0, x_0; s, x_t)\\ &\quad = s \cdot G^N_{[V(x;\theta)]}(0, x_0; s, x_t) + V(x_t;\theta) \cdot G^N_{[V(x;\theta)]}(0, x_0; s, x_t) - \delta(x_t - x_0)\end{aligned}$$

[7] In fact, the function $G_{[V^A_{+\infty}(x;\theta)]}$ is equal to the function $G^D_{[V(x;\theta)]}$ introduced in subsection 2.2.

with conditions ($x_0 > a$)

$$\lim_{x_t \to +\infty} G^N_{[V(x;\theta)]}(0, x_0; s, x_t) = 0$$
$$\lim_{x_t \to a+} \frac{\partial}{\partial x_t} G^N_{[V(x;\theta)]}(0, x_0; s, x_t) = 0.$$

From δ-function perturbation, see equation (10), an expression for $G_{[V^R_{+\infty}(x;\theta)]}(0, x_0; s, x_t)$, denoting the limit of the Green's function of the path integral $I_{[V^R_\beta(x;\theta)]}(0, x_0; s, x_t)$ when β goes to infinity, follows as

$$\begin{aligned}
&G_{[V^R_{+\infty}(x;\theta)]}(0, x_0; s, x_t) \\
&= G^N_{[V(x;\theta)]}(0, x_0; s, x_t) - \frac{G^N_{[V(x;\theta)]}(0, x_0; s, a+)\, G^N_{[V(x;\theta)]}(0, a+; s, x_t)}{G^N_{[V(x;\theta)]}(0, a+; s, a+) + 2/\mu(a;\theta)} \\
&= G^N_{[V(x;\theta)]}(0, x_0; s, x_t) - \frac{\mu(a;\theta)}{2} G_{[V^R_{+\infty}(x;\theta)]}(0, x_0; s, a+)\, G^N_{[V(x;\theta)]}(0, a+; s, x_t).
\end{aligned}$$

As we have $\frac{\partial}{\partial x_t} G^N_{[V(x;\theta)]}(0, x_0; s, a+) = 0$, except when $x_0 = a+$ for which it is true that $\frac{\partial}{\partial x_t} G^N_{[V(x;\theta)]}(0, a+; s, a+) = -2$, it follows that

$$\lim_{x_t \to a+} \left(\frac{\partial}{\partial x_t} G_{[V^R_{+\infty}(x;\theta)]}(0, x_0; s, x_t) - \mu(x_t;\theta) G_{[V^R_{+\infty}(x;\theta)]}(0, x_0; s, x_t) \right) = 0.$$

After a multiplication with $\exp\{\int_{x_0}^{x_t} \mu(z;\theta)dz\}$ and an inversion of the Laplace transform, it follows that $p^R(0, x_0; t, x_t)$ is the solution of the forward Fokker-Planck equation. Hence, from probability theory, we know that $p^R(0, x_0; t, x_t)$ is the transition probability of the reflecting process.

B Proof for section 4

We start by showing that the path integral expression of (32), corresponding to the non-boundary case, satisfies the partial differential equation (21).
This can be done rather easily, when transforming the partial differential equation for Γ into an equation for H, by defining

$$\Gamma(t, r; T, \hat{r}) = e^{\int_r^{\hat{r}} \mu(z;\theta)dz} H(t, r; T, \hat{r}).$$

Indeed, the equation for H becomes

$$\frac{\partial H}{\partial t} + \frac{1}{2}\frac{\partial^2 H}{\partial r^2} - \left\{ \frac{1}{2}\mu^2(r;\theta) + \frac{1}{2}\frac{\partial \mu(r;\theta)}{\partial r} + r \right\} \cdot H = 0\ ;$$

a comparison with equation (5) leads to the desired result.
Note that this result is –of course– consistent with the definition of the price as an expected value of the discount factor, see equation (20), combined with the path integral expression for the transition probability, presented earlier.

Starting from this result, the expressions for the prices of the Arrow-Debreu securities in case a reflecting or absorbing boundary is imposed, are straightforward. Indeed, the potential $V(x;\theta)$ in the path integral has to be completed with the δ and δ'-function, just as explained in the proof for section 3.

C About convex ordering

In this section, we briefly recall the most important definitions and results regarding convex ordering. For proofs and more details, we refer to [11, 12].

C.1 The notion of convex ordering

A variable A is said to be smaller than B in convex ordering,

$$A \leq_{cx} B \ , \tag{49}$$

if for each convex function $u : \mathbb{R} \to \mathbb{R} : x \mapsto u(x)$, the expected values (provided they exist) are ordered as

$$\mathrm{E}[u(A)] \leq \mathrm{E}[u(B)] \ . \tag{50}$$

Since convex functions take on their largest values in the tails, the variable B is more likely to take on extreme values than the variable A, and thus it can be interpreted as more dangerous.

For the application in the present contribution, we attain our goal, if we can construct such a more dangerous or convex larger variable for the integral $\int_0^t V(X(s))ds$. Making use of comonotonic variables, this can perfectly be done.

C.2 Construction of a convex larger variable

If a variable A consists of a sum of random variables $X_1, X_2, ..., X_n$, then replacing the copula of $(X_1, X_2, ..., X_n)$ by the comonotonic copula, yields a sum that is a convex larger variable than this variable A.
Or,

$$\sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} F_{X_i}^{-1}(U) \ , \tag{51}$$

where U is an arbitrary random variable, uniformly distributed on $[0, 1]$.
Note that a vector $\underline{X}^c = (X_1^c, X_2^c, ..., X_n^c)$ is comonotonic, if for all $\underline{x} = (x_1, x_2, ..., x_n)$, we have

$$F_{\underline{X}^c}(\underline{x}) = \min\left(F_{X_1^c}(x_1), F_{X_2^c}(x_2), ..., F_{X_n^c}(x_n)\right) \tag{52}$$

The notation F_{X_i} is used for the distribution function of the variable X_i, and the inverse distribution is defined in the classical way as

$$F_{X_i}^{-1}(p) = \inf\{x \in \mathbb{R} : F_{X_i}(x) \geq p\}. \tag{53}$$

Note that the corresponding terms in both sums are all mutually identically distributed. Both sums differ in the interdependence structure; the convex largest sum is the sum with the most dangerous interdependencies.

This result can be generalized for a sum of functions of variables as follows :

$$\sum_{i=1}^{n} g_i(X_i) \leq_{cx} \sum_{i=1}^{n} F^{-1}_{g_i(X_i)}(U) \ , \tag{54}$$

where U is an arbitrary random variable, uniformly distributed on $[0, 1]$.

Finally, when taking limits, the construction is also valid for integrals :

$$\int_0^t g(X(s))ds \leq_{cx} \int_0^t F^{-1}_{g(X(s))}(U)ds \ , \tag{55}$$

where U is an arbitrary random variable, uniformly distributed on $[0, 1]$.

D Proof for the upper bound of section 5

Following the result of (55), we know that the variable

$$A = \int_0^t V_1(X(s))ds \tag{56}$$

is smaller in convex ordering than

$$B = \int_0^t F^{-1}_{V_1(X(s))}(U)ds. \tag{57}$$

Hence, since the negative exponential function is convex, the result follows immediately from the definition of convex ordering.

E Proof for the lower bound of section 5

We start by writing

$$\mathrm{E}_{(V_2)}\left[e^{-\int_0^t V_1(X(s))ds}\right] = \mathrm{E}_{\Lambda}\left\{\mathrm{E}_{(V_2)}\left[e^{-\int_0^t V_1(X(s))ds}\middle|\Lambda\right]\right\} \tag{58}$$

for an arbitrary variable Λ.
For a choice of $\Lambda \equiv X(\tau)$ with τ between 0 and t, the inequality of Jensen immediately leads to the lower bound of (41).

References

[1] Abramowitz M. & Stegun I.A. (1970). *Handbook of mathematical functions*, Dover, 1046 p.

[2] Ait-Sahalia Y. (1996). "Testing continuous-time models of the spot interest rate", *Review of Financial Studies*, vol.9, p.385-426.

[3] Ait-Sahalia Y. (1999). "Transition probabilities for interest rates and other non-linear diffusions", *Journal of finance*, vol.54(4), p.1361-1395.

[4] Baaquie B.E. (1997). "A path integral approach to option pricing with stochastic volatility: some exact results", *Journal de Physique I*, vol.7(12), p.1733-1753.

[5] Beaglehole D.R. & Tenney M.S. (1991). "General solutions of some interest rate-contingent claim pricing equations", *Journal of fixed income*, vol.2, p.69-83.

[6] Black F. (1995). "Interest rates as options", *Journal of finance*, vol.50, p.1371-1376.

[7] Brennan M.J. & Schwartz E.S. (1979). "A continuous time approach to the pricings of bonds", *Journal of Banking and Finance*, vol.3, p.133-155.

[8] Chiarella C. & El-Hassan N. (1997). "Evaluation of derivative security prices in the Heath-Jarrow-Morton framework with path integrals using fast Fourier transform techniques", *Journal of Financial Engineering*, vol.6(2), p.121-147.

[9] Chiarella C., El-Hassan N. & Kucera A. (1999). "Evaluation of American option prices in a path integral framework using Fourier-Hermite series expansions", *Journal of Economic Dynamics and Control*, vol.23, p.1387-1424.

[10] Cox D.R. & Miller H.D. (1965). *The theory of stochastic processes*, Chapman and Hall, London, 398 p.

[11] Dhaene J., Denuit M., Goovaerts M.J., Kaas R. & Vyncke D. (2002). "The concept of comonotonicity in actuarial science and finance: theory", *Insurance: Mathematics and Economics*, vol.31(1), p.3-34.

[12] Dhaene J., Denuit M., Goovaerts M.J., Kaas R. & Vyncke D. (2002). "The concept of comonotonicity in actuarial science and finance: applications", *Insurance: Mathematics and Economics*, vol.31(2), p.133-162.

[13] Feynman R.P. & Hibbs A.R. (1965). *Quantum mechanics and path integrals*, McGraw-Hill Book Company, 365 p.

[14] Gibson R., Lhabitant F.S., & Talay D. (1998). "Modeling the term structure of interest rates: a review of the literature", Lausanne University – Ecole des HEC, Working Paper 9801, 97 p.

[15] Gikhman I.I. & Skorokhod A.V. (1963). *Introduction to the theory of random processes*, Saunders Mathematics Books, 516 p.

[16] Goldstein R.S. & Keirstead W.P. (1997). "On the term structure of interest rates in the presence of reflecting and absorbing boundaries", Ohio State University – Charles A. Dice Center for Research, Working Paper 97-1, 36 p.

[17] Goldstein R.S. (1997). "Identifying the class of term structure models possessing closed-form solutions for bond and bond-option prices: an expectation approach", Ohio State University – Charles A. Dice Center for Research, Working Paper 97-7, 15 p.

[18] Goovaerts M., De Schepper A. & Decamps M. (2003). "Transition probabilities for diffusion equations by means of path integals", KULeuven Department of Applied Economics, Research Report 0229, 46 p.

[19] Goovaerts M., Babcenco A. & Devreese J. (1973). "A new expansion method in the Feynman path integral formalism: application to the one-dimensional delta-function potential", *Journal of Mathematical Physics*, vol.14(5), p.554-559.

[20] Grosche C. (1990). "Path integrals for potential problems with δ-function perturbation", *Journal of Physics A*, vol.23, p.5205-5234.

[21] Grosche C. (1993). "δ-function perturbations and boundary problems by path integration", *Annalen der Physik*, vol.2(6), p.557-589.

[22] Grosche C. (1995). "δ'-function perturbations and Neumann boundary conditions by path integration", *Journal of Physics A*, vol.28, p.L99-L105.

[23] Harrison J. & Taksar M. (1983). "Instantaneous control of Brownian motion", *Mathematics of Operations Research*, vol.8, p.439-453.

[24] Itô K. & McKean H.P. (1965). *Diffusion processes and their sample paths*, Academic Press, New York, 321 p.

[25] James J. & Webber N. (2000). *Interest rate modelling*, John Wiley & Sons, New York, 654 p.

[26] Jamshidian F. (1991). "Bond and option evaluation in the Gaussian interest rate model", *Research in Finance*, vol.9, p.131-170.

[27] Kac M. (1951). "On some connections between probability theory and differential and integral equations", *Proceedings of the second symposium on Mathematics, Statistics and Probability*, Berkeley, p.189-215.

[28] Karatzas I. & Shreve S.E. (1988). *Brownian motion and stochastic calculus*, Springer-Verlag, New York, 470 p.

[29] Kinkladze G.F. (1982). "A note on the structure of processes the measure of which is absolutely continuous with respect to the Wiener modulus measure", *Stochastics*, vol.8, p.39-44.

[30] Kleinert H. (2002). "Option pricing from path integral for non-Gaussian fluctuations. Natural martingale and application to truncated Levy distributions", *Physica A*, vol.312(1), p.217-242.

[31] Marumo K., Nakayama T., Nishioka S. & Toshida T. (2003). "Extracting Market Expectations on the Duration of the Zero Interest Rate Policy From Japan's Bond Prices", Bank of Japan – Financial Markets Departement, Working Paper 03-E-2, 27 p.

[32] Legall J.F. (1982). "Temps locaux et quations diffrentielles stochastiques", *Lecture Notes in Mathematics*, Springer-Verlag, 986.

[33] Linetsky V. (1998). "The path integral approach to financial modeling and option pricing", *Computational Economics*, vol.11(1-2), p.129-163.

[34] Matacz A. (2002). "Path dependent option pricing: the path integral partial averaging method", *Journal of Computational Finance*, vol.6(2).

[35] Merton R.C. (1973). "An intertemporal capital asset pricing model", *Econometrica*, vol.41, p.867-887.

[36] Montagna G., Moreni N. & Nicrosini O. (2002). "A path integral way to option pricing", *Physica A*, vol.310(3), p.450-466.

[37] Montagna G. & Nicrosini O. (2002). "Efficient option pricing with path integral", *European Physical Journal B*, vol.27(2), p.249-256.

[38] Montagna G., Morelli M., Nicrosini O., Amato P. & Farina M. (2003). "Pricing derivatives by path integral and neural networks", *Physica A*, vol.324(1), p.189-195.

[39] Revuz D. & Yor M. (1991). *Continuous martingales and Brownian motion*, Springer-Verlag, Berlin, 533 p.

[40] Rosa-Clot M. & Taddei S. (2002). "A path integral approach to derivative security pricing", *International Journal of Theoretical and Applied Finance*, vol.5(2), p.123-146.

[41] Stanton R. (1997). "A nonparametric model of term structure dynamics and the market price of interest rate risk", *Journal of finance*, vol.52(5), p.1973-2002.

[42] Vasicek O. (1977). "An equilibrium characterization of the term structure", *Journal of Financial Economics*, vol.5, p.177-188.