The Fueter Theorem by Representation Theory

David Eelbode*, Peter Van Lancker†, Vladimír Souček△

*Department of Mathematics and Computer Science, Antwerp University, Middelheimlaan 1, 2020 Antwerp, Belgium,
†Department of Engineering Sciences, University College Ghent (Member of Ghent University Association), Schoonmeersstraat 52, 9000 Ghent, Belgium
△Mathematical Institute, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, Praha, Czech Republic

Abstract. Odd powers of the Dirac operator are conformally invariant, if the conformal weight of the field is chosen appropriately. In the paper, it is shown how to use the invariance properties of the kernel of powers of the Dirac operator to deduce in a simple way the statement of the (generalized) Fueter theorem.

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INTRODUCTION

The Fueter theorem is a classical fact from Clifford analysis ([1, 2, 6, 11]) going back to the 30's ([5, 14, 18] and many authors have contributed to its various generalizations (a sample of them can be found in papers [5, 7, 12, 13, 16, 8, 9, 10]). The essence of the original theorem is a simple procedure how to construct solutions of the Fueter equation from a given holomorphic function. In later generalizations, further easy objects (e.g., a homogeneous polynomial of a certain type) were added to the input data of the construction. So the basic aim of the Fueter-type theorems was to construct more difficult objects (monogenic functions) from well-known and easy to describe data. Tools for proofs were always analytic.

The aim of the paper is to show that Fueter-type theorems have a purely representation theoretical origin. They are direct consequences of the basic fact saying that the Dirac operator (and its odd powers) are conformally invariant for an appropriate conformal weight. From this point of view, the main idea of the Fueter construction consists of the following simple fact. Any conformal transformation (or its infinitesimal version) maps solutions of (a power of) the Dirac equation again to solutions. So it is possible to try to construct more complicated solutions starting from simpler ones. Maps in the conformal group can be divided into translations, rotations and dilations, and proper conformal transformations. We shall show that the proper conformal transformations (resp. their infinitesimal forms) can be used to generate successively more and more complicated solutions of the powers of the Dirac operator. The construction works in a very general situation but the explicit form of the constructed solutions can be quite complicated. The Fueter theorem describes a special situation, where the set of constructed solutions have a similar form as holomorphic functions in plane.

INFINITESIMAL CONFORMAL TRANSFORMATIONS.

The Dirac equation on $\mathbb{R}^{m+1}$ has a big group G of (first order) symmetries consisting of conformal maps under the condition that the corresponding fields transforms with a (uniquely) given conformal weight. In geometric terms, it means that we are not considering functions but densities of a given weight. The group G contains translations, rotations, dilations and proper conformal transformations. The Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{so}(1,m+2)$. The algebra $\mathfrak{g}$ is $|1|$-graded, i.e., $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, where $\mathfrak{g}_{-1}\simeq\mathfrak{g}_1\simeq\mathbb{R}^{0,m+1}$, and $\mathfrak{g}_0\simeq\mathfrak{so}(0,m+1)\oplus\mathbb{R}$.

The infinitesimal conformal action can be realized by vector fields on $\mathbb{R}^{0,m+1}$. In such a realization, the space $\mathfrak{g}_{-1}$ (infinitesimal translations) has a basis given by derivatives $Y_i=\partial_{x_i}$ with respect to coordinates, and elements of $\mathfrak{g}_0$ (infinitesimal rotations and dilations) are realized by usual angular momenta, resp. by the Euler operator $\mathbb{E}$. A basis of vector fields X_i (infinitesimal versions of proper conformal transformations) giving a realization of elements in $\mathfrak{g}_1$

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has a more complicated form and will be computed below. It can be obtained by a composition of inversions and (infinitesimal) translations. Note that the action of proper conformal transformations depend on a given conformal weight. Coordinates on $\mathbb{R}^{0,m+1}$ will be denoted by $(x_0, x_1, \dots, x_m)$. Let us start first with the definition of (a one parametric) set of inversions.

Definition 1. *If $\alpha \in \mathbb{C}$, we define the inversion operator I_α on $\mathbb{R}_{0,m+1}$-valued functions $f(x)$ by*

$$(I_\alpha f)(x) = \frac{x}{|x|^\alpha} f\left(\frac{x}{|x|^2}\right).$$

It is well known that the action of the inversion operator preserves the space of solutions of the Dirac equation if and only if $\alpha = m+1$. It follows from the known classification of conformally invariant operators (see [15, 17]) that the similar statement is also true for all odd powers of the Dirac operator. The kernel of the operator ∂^{2j+1} is preserved by the inversion I_α if and only if $\alpha = m+1-2j$. Hence the action of inversions can be used to transform solutions of powers of the Dirac operator. The transformation $\mathrm{I}_\alpha \partial_{x_i} \mathrm{I}_\alpha$ is a proper conformal transformation for a (unique) choice of α. Let us now compute an explicit form of such a vector field.

Lemma 2. *The operator $X_0 = I_\alpha \partial_{x_0} I_\alpha$ is acting on $\mathbb{R}_{0,m+1}$-valued maps on $\mathbb{R}^{m+1}$ by*

$$X_0 = -|x|^2 \partial_{x_0} + x_0(2\mathbb{E}_x + \alpha) + x e_0,$$

where $\mathbb{E}_x = \sum_{i=0}^m x_i \partial_{x_i}$ is the Euler operator and $|x|^2 = \sum_{i=0}^m x_i^2$.

Proof:
It is sufficient to compute how X_0 acts on a homogeneous polynomial P_k of a given order k. We get

$$I_\alpha P_k = \frac{x}{|x|^{\alpha+2k}} P_k,\ \partial_{x_0}(I_\alpha P_k) = \frac{e_0}{|x|^{\alpha+2k}} P_k - (\alpha+2k)\frac{x x_0}{|x|^{\alpha+2k+2}} P_k + \frac{x}{|x|^{\alpha+2k}} \partial_{x_0} P_k,$$

$$I_\alpha(\partial_{x_0}(I_\alpha P_k)) = x e_0 P_k + (\alpha+2k) x_0 P_k - |x|^2 \partial_{x_0} P_k.$$

□

It is an important fact that the operator X_0 together with the partial derivative $Y_0 = -\partial_{x_0}$ generate an $\mathfrak{sl}(2,\mathbb{R})$ subalgebra in the conformal Lie algebra $\mathfrak{so}(1, m+2)$. The spaces of monogenic functions, which we shall construct using the Fueter type theorem are modules for this Lie subalgebra $\mathfrak{sl}(2,\mathbb{R})$.

Lemma 3. *The triple $\{X_0, Y_0, H\}$ defined by*

$$X_0 = I_\alpha \partial_{x_0} I_\alpha;\ Y_0 = -\partial_{x_0};\ H = 2\mathbb{E}_x + \alpha - 1$$

satisfies the standard $\mathfrak{sl}(2,\mathbb{R})$ commutation relations $[X_0, Y_0] = H,\ [H, X_0] = 2X_0,\ [H, Y_0] = -2Y_0$.

Proof:
The commutation relations of X_0 and Y_0 with the shifted Euler operator H are obvious and

$$\left[\partial_{x_0}, -|x|^2 \partial_{x_0} + x_0(2\mathbb{E}_x + \alpha) + x e_0\right] = -2x_0 \partial_{x_0} + (2\mathbb{E}_x + \alpha) + 2x_0 \partial_{x_0} - 1\ .$$

□

CONFORMAL INVARIANCE OF ∂^{2j+1}

The Dirac operator and its odd powers are conformally invariant for an appropriate conformal weight (see [15, 17, 3]). In particular, the action of the element $X_0 = \mathrm{I}_\alpha \partial_0 \mathrm{I}_\alpha$ preserves the space of solutions of the power ∂^{2j+1} of the Dirac operator, if $\alpha = m - 2j + 1$. The action of X_0 may be used to create more complicated solutions from a simple one.

Generalizations of the Fueter theorem are usually stated, in fact, for the Cauchy-Riemann operator D instead of the Dirac operator ∂. For its definition, we have to break the $SO(0, m+1)$ invariance down to $SO(0, m)$ by a choice of a direction in $\mathbb{R}^{m+1}$, say, e_0. The Cauchy-Riemann operator D is then defined by $D = -e_0 \partial = \partial_{x_0} - \sum_i e_0 e_i \partial_{x_i}$. It is often formulated in terms of the para-vector variable $x_0 + \sum_{i=0}^m f_i x_i$, $f_i = -e_0 e_i$, then $D = \partial_{x_0} + \sum_i f_i \partial_{x_i}$. Every solution f of the Dirac equation $\partial f = 0$ is, of course, also the solution of the Cauchy-Riemann equation $D f = -e_0 \partial f = 0$, and vice versa. The same statements remains also true in higher dimensions.

THE FUETER THEOREM IN QUATERNIONIC ANALYSIS

The main idea of a Fueter type theorem came from the following disappointing fact in quaternionic analysis. Let $\mathbb{H}$ denote the field of quaternions. It was immediately found that, contrary to the holomorphic case, the quaternionic powers $q^k, q = q_0 + i_1 q_1 + q_2 i_2 + q_3 i_3 \in \mathbb{H}$ are not monogenic, i.e., they do not satisfy the Fueter equation

$$D f = (\partial_0 + i_1 \partial_1 + i_2 \partial_2 + i_3 \partial_3) f = 0.$$

But Fueter noted that $\Delta(q^k)$ are monogenic ([5]). This gives immediately the model form of the Fueter theorem. Let us define the Fueter map Φ from the space of holomorphic functions in the plane to functions of quaternionic variable as follows. If $f(z) = f_0(x_0, x_1) + i f_1(x_0, x_1), z = x_0 + i x_1$ is a holomorphic function in the plane, then the value $\Phi(f)$ is defined by

$$\Phi(f)(q) = f_0(q_0, r) + \omega f_1(q_0, r),\ \omega = \frac{\underline{q}}{|\underline{q}|},\ \underline{q} = i_1 q_1 + i_2 q_2 + i_3 q_3,\ r = |\underline{q}| = \sqrt{q_1^2 + q_2^2 + q_3^2}.$$

Then the Fueter theorem is stating that if $f(z)$ is a holomorhic function in the plane, then $\Delta(\Phi(f))$ is monogenic, i.e. $D(\Delta(\Phi(f))) = 0$. Note that the function $\Phi(f)$ is not necessarily defined in the whole space $\mathbb{R}^{0,m+1}$, it can have singularities due to the factor ω. Indeed, let $f(z)$ be a holomorphic function represented by its Taylor series

$$f(z) = \sum_k a_k z^k,\ a_k = b_k + i c_k$$

is a holomorphic function on the domain of convergence of the series, then the coefficients $a_k = b_k + i c_k$ are mapped by Φ to $b_k + \omega c_k$ and can have singularities.

We can write f as $f(z) = g(z) + i h(z)$, $g(z) = \sum_k b_k z^k$, $h(z) = \sum_k c_k z^k$. The functions g and h in the decomposition are characterized by the property that their values on $\mathbb{R}$ are real. The Fueter map then satisfies $\Phi(f) = \Phi(g) + \omega \Phi(h)$. Hence Φ is only $\mathbb{R}$-linear and images of polynomials are no more polynomials in general. There is an alternative version of the Fueter theorem (which is a special case of the general theorem discussed below).

Theorem 4. *Let us suppose that $f(z) = \sum_k a_k z^k$ is a holomorphic function in the plane, then $\Delta\tilde{\Phi}(f)$ is monogenic, where $\tilde{\Phi}(f) := \sum_k a_k q^k$.*

Hence if $f = g + i h$ is the decomposition above, then $\tilde{\Phi}(f) = \Phi(g) + i\Phi(h)$. Note that $\tilde{\Phi}(f)$ has values in the complex Clifford algebra $\mathbb{C}_{m+1}$ and is $\mathbb{C}$-linear.

There is a close relation between both formulations. The function $\Delta\Phi(f)$ is monogenic for all f holomorphic if and only if the same is true for $\tilde{\Phi}(f)$ and $\omega\tilde{\Phi}(f)$ for all f holomorphic.

A GENERAL FORM OF THE FUETER THEOREM

We shall now state a general form of the Fueter theorem in higher (even) dimensions. Let $m+1$ be even. Let us consider the plane $\mathbb{R}^2$ with coordinates $z = x_0 + i x_1$. Vectors in $\mathbb{R}^{0,m+1}$ are identified with the vector variable $x = e_0 x_0 + \underline{x}, \underline{x} = \sum_{i=1}^m e_i x_i$ and we shall introduce the expression $-e_0 x = x_0 + \sum_{i=1}^m e_i e_0 x_i$.

Theorem 5. *Let $P_\ell(\underline{x})$ be a homogeneous polynomial of order ℓ with values in the Clifford algebra $\mathbb{R}_{0.m+1}$, depending only on the variables $x_1, \ldots, x_m$. Let $f(z) = \sum_k a_k z^k, a_k \in \mathbb{C}$ be a holomorphic function in the plane.*

Then

$$\partial^{m+2\ell}(\tilde{\Phi}(f) P_\ell) = 0,\ \tilde{\Phi}(f)(x) = \sum_k a_k (-e_0 x)^k,$$

hence $\Delta^{\frac{m-1}{2}+\ell}(\tilde{\Phi}(f) P_\ell)$ is monogenic in $\mathbb{R}^{0,m+1}$. The same is true for the function $\omega\tilde{\Phi}(f) P_\ell$ instead of $\tilde{\Phi}(f) P_\ell$.

Note that the word monogenic in the theorem above means that the function is in the kernel of the Dirac operator ∂ as well as in the kernel of the Cauchy-Riemann operator D.

Proof

Let us show first that any power $(-e_0 x)^j$, $j \in \mathbb{N}$ is in the kernel of the operator $\partial^m = \partial \Delta^{\frac{m-1}{2}}$. The kernel of the operator ∂^m is invariant under the action of the inversion I_α for the special value $\alpha = 2$. The constant function 1 is in the kernel, hence the same is true for

$$(X_0)^j(1), j \in \mathbb{N}, X_0 = \mathrm{I}_2 \partial_0 \mathrm{I}_2.$$

Let us denote $\underline{x} = \sum_{k=1}^{m} e_k x_k$. Using the explicit form of the operator X_0, we get

$$
\begin{aligned}
[-|x|^2\partial_0 + x_0(2\mathbb{E}+2) + xe_0] \quad & \left((x_0 - e_0 x)^j\right) = \\
&= -|x|^2\, j\,(x_0 - e_0\underline{x})^{j-1} + 2(j+1)x_0\,(x_0 - e_0\underline{x})^j - (x_0 + e_0\underline{x})\,(x_0 - e_0\underline{x})^j = \\
&= (x_0 - e_0\underline{x})^{j-1}(j+1)\left(x_0^2 - 2e_0x_0\underline{x} - |\underline{x}|^2\right) = (j+1)(x_0 - e_0\underline{x})^{j+1}.
\end{aligned}
$$

Hence we get by induction that

$$(X_0)^j(1) = j!(-e_0 x)^j,\, j \in \mathbb{N}$$

is in the kernel of the operator ∂^m. By $\mathbb{C}$-linearity, the same is true for the function $f(-e_0 x)$, if f is a polynomial.

Let us now discuss the case of the function $\tilde{\Phi}(f)P_\ell$, where $f(z)$ is a polynomial. Let $\alpha = 2 - 2\ell$. We know that the kernel of the operator $\partial^{m+2\ell}$ is invariant with respect to the action of I_α. Hence the operator $X_0 = \mathrm{I}_\alpha \partial_{x_0} \mathrm{I}_\alpha$ is preserving the kernel of the operator $\partial^{m+2\ell}$. Now for any polynomial $f(z)$

$$\mathrm{I}_\alpha[f(-e_0 x)P_\ell] = \frac{x}{|x|^{2-2\ell}} \frac{1}{|x|^{2\ell}} f\left(\frac{-e_0 x}{|x|^2}\right) P_\ell(x) = \mathrm{I}_2\left(f(-e_0 x)\right) P_\ell(x).$$

Similarly, we get that $X_0[f(-e_0 x)P_\ell] = \mathrm{I}_2 \partial_{x_0} \mathrm{I}_2\left(f(-e_0 x)\right) P_\ell(x)$. We know that for $f(z) = 1$, the function P_ℓ is in the kernel of $\partial^{m+2\ell}$. Hence again by induction the same is true for any power $f(z) = z^j$ and by linearity for any polynomial f.

The case of f holomorphic is then treated by taking the limit of holomorphic polynomials.

To prove the second statement of the theorem, it is sufficient to show that ωP_ℓ is also in the kernel of $\partial^{m+2\ell}$ and to apply the same procedure as above. But P_ℓ depends only on coordinates in $\mathbb{R}^m$ and the kernel of the operator $\underline{\partial}^{m+2\ell}$ is invariant with respect to $\mathrm{I}_{1-2\ell}$. Hence $\mathrm{I}_{1-2\ell}(P_\ell) = \frac{\underline{x}}{|\underline{x}|} P_\ell = \omega P_\ell$ is also polymonogenic of order $m + 2l$ in $\mathbb{R}^m - \{0\}$ and thus also in $\mathbb{R}^{m+1} - \mathbb{R}e_0$, since ωP_l does not depend on x_0.

The usual form of the Fueter theorem (using the Fueter map Φ) is then just a simple consequence.

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REFERENCES

1. Brackx, F., Delanghe, R., Sommen, F., *Clifford Analysis*, *Research Notes in Mathematics 76*, Pitman, London (1982).
2. Delanghe, R., Sommen, F., Souček, V., *Clifford analysis and spinor valued functions*, Kluwer Academic Publishers, Dordrecht (1992).
3. Eastwood, M., Ryan, J., Monogenic functions in conformal geometry, SIGMA **3** (2007), 084, 14 pages.
4. Eelbode, D., Monogenic Appell sets as representations of the Heisenberg algebra, to appear in Adv. Appl. Cliff. Alg.
5. Fueter, R., Die funktionentheorie der differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier variablen, Comment. Math. Helv., **7** (1935) 307-330.
6. Guerlebeck, K., Sproessig, W., *Quaternionic and Clifford Calculus for Physicists and Engineers,* Wiley, Chichester, 1998.
7. Kou, K.I., Qian, T., Sommen, F., Generalizations of Fueter's Theorem, Meth. Appl. Anal. **9** No. 2 (2002), pp. 273-290.
8. Peña-Peña, D., Sommen, F., A generalization of Fueter's theorem, Results Math., **49** (3-4) (2006), 301-311.
9. Peña-Peña, D., Cauchy-Kowalevski extensions, Fueter's theorems and boundary values of special systems in Clifford analysis, PhD thesis, Ghent University, 2008, available at http://hdl.handle.net/1854/11636.
10. Peña-Peña, D., Sommen, F. Fueter's theorem: The saga continues, Jour. Math. Anal. Appl., **365** (2010), pp. 29-35.
11. Ryan, J., *Introductory Clifford analysis*, preprint arXiv:math/0303339v1
12. Qian, T., Generalization of Fueter's result to $\mathbb{R}^{m+1}$, Rend. Mat. Acc. Lincei **9** (1997), 111-117.
13. Qian, T., Sommen, F., Deriving harmonic functions in higher dimensional spaces, Z. Anal. Anwend. **22** (2) (2003), 275-288.
14. Sce, M., Osservaziono sulle serie di potenzi nei moduli quadratici, Atti. Acad. Nax. Lincei Rend Sci Fis Mat. Nat, **23** (1957), 220-225.
15. Slovák, J.,: Invariant operators on conformal manifolds, Research Lecture Notes, University of Vienna, 138 pages, 1992.
16. Sommen, F., On a generalization of Fueter's theorem, Z. Anal. Anwend. **19** (4) (2000), 899-902.
17. Souček, V., Higher spin and conformal invariants in Clifford analysis, in Proc. of Symposium "Analytical and numerical methods in Clifford analysis, Seiffen, 1996, 175-186.
18. Sudbery, A., Quaternionic analysis, Math. Proc. of the Cambridge Philosophical Soc., **85** (1979), 199-225.