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Exploring The λ Copula Construction Method for Archimedean copulas : Discussion of Three λ Types

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We introduce and discuss a new parametric copula builder which is named the “ λ construction method”. The methodology is explained and illustrated using 3 types of λ functions. It shows that the λ method has strong visual advantages for recognizing key dependence characteristics and importing them into the copula model. Furthermore, the λ method facilitates the representation of a copula family as a collection of comparable test spaces as defined in Michiels and De Schepper (2008). As such, the modeling capacity of these families is discussed in a clear way.

Keywords: copula, Kendall’s τ , multiparametric Archimedean copula family, tail dependence

AMS Subject Classification: 62H20, 62P05, 62H12

1. INTRODUCTION

During the last decades, the research in the field of dependencies has gained a lot by the use of copulas. A copula in fact is nothing else than a mathematical expression allowing to split up a general joint distribution into the margins and the mutual dependence between the factors, but due to the fact that is a very flexible tool, it has become almost impossible to imagine dependence modeling without copulas nowadays.

When a practioner is faced with a modeling problem in a certain application, be it in finance, biostatistics, hydrology, geostatistics or another discipline, the basic problem is to obtain the best possible fit for the observed dependence structure. One possibility is to choose among existing copulas for the one which performs best according to a goodness-of-fit test. For the choice of the copula, one could look e.g. at the observed value of the concordance in the data, or at striking characteristics of the observed dependence like symmetries, or one could work with comparable test spaces as we suggested in a previous paper (see Michiels and De Schepper (2008)). Another possibility consists of the construction of a new “ideal” copula, on the basis of a number of parameters that can be estimated from the data.

In this paper we want to contribute to this last domain, by presenting a new method of constructing multiparametric bivariate Archimedean copula families. More specifically, we will show how such new families can be built and we will discuss the advantages of this approach. The central item in our method is the function $\lambda_\theta(t) = \frac{\varphi_\theta(t)}{\varphi_\theta'(t)}$ ($t \in [0, 1]$), where φ is the generator of the Archimedean copula, such that $C(u, v) = \varphi_\theta^{[-1]} \{ \varphi_\theta(u) + \varphi_\theta(v) \}$, with $u, v \in [0, 1]$ and $\varphi_\theta^{[-1]}$ the pseudo-inverse of the generator. See section 2 for a brief summary about these concepts.

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The first important advantage of our construction method can be found in the clear link with Kendall's τ , due to the relation $\tau = 1 + 4 \int_0^1 \lambda_\theta(t) dt$. As a consequence, for any (estimated) value of τ and for any feasible function λ , the possible Archimedean dependence structures can be explored. A second advantage of the construction method, related to the previous one, is the possibility to explicitly import Kendall's τ as the concordance parameter into the copula function. Next, all parameter ranges can be obtained in function of τ , so parameter interpretation becomes straightforward. Finally, it will turn out that our method has also strong visual advantages (see section 4).

The method is explained by a full elaboration of the construction method for three types of λ functions: polynomial functions, rational functions and logarithmic type functions. For the first two λ types, examples with and without intercept are being discussed; for the third type only examples without intercept are presented. For each family, we will derive the parameter ranges and expressions for the tail dependence measures. Furthermore, the power of the visual interpretation of the λ function is shown by studying the effect of a change in the parameters of λ on the simulated (U, V) observations from the affiliated copula. We also compare the modeling power of the three λ types by looking for parallels and differences, e.g. according to the possible lower tail and upper tail combinations. Finally we present an overview of the modeling strengths and weaknesses of the λ types. Because the emphasis in this working paper will be on the methodological aspect of the construction method, both strict and non-strict copula families will be discussed, although the latter have never been used for fitting applications.

The paper is organized as follows. In section 2, a brief summary is given about basic copula theory including the most important copula properties in relation to fitting applications. Next, in section 3, we introduce and discuss the λ method of copula construction. Together with section 4, this is the main part of the paper. In the fourth section, the three λ types are presented, worked out and compared with each other. The visual power of the different λ functions is illustrated by means of a number of simulations. Section 5 concludes.

2. BASIC CONCEPTS

We start with the definition and the most important theorem for copulas. We use the symbol \mathbf{I} to denote the unit interval $[0, 1]$.

Definition 1 *A bivariate copula is a function $C : \mathbf{I}^2 \rightarrow \mathbf{I}$ with the following properties:*

1. *C is 2-increasing, or for all $u_1 \leq u_2, v_1 \leq v_2 \in \mathbf{I}$ it is true that $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$.*
2. *C is grounded, or $C(u, 0) = C(0, v) = 0$ for all $(u, v) \in \mathbf{I}^2$.*
3. *C has uniform $[0, 1]$ margins, or $C(u, 1) = u$ and $C(1, v) = v$ for all $(u, v) \in \mathbf{I}^2$.*

In fact a copula represents the link between the marginal distribution functions and their joint aggregate. This link can be formalized through the following theorem:

Theorem 1 *(Sklar's theorem) Let H be a bivariate joint distribution function with margins F and G . Then there exists a copula C in such a way that $H(x, y) = C(F(x), G(y))$ for all $(x, y) \in \mathbb{R}$.*

If F and G are defined continuously, then C is unique. If not, then C is unique on $\text{im}F \times \text{im}G$. Conversely, if C is a copula and F and G are distribution functions, then H is defined as indicated above.

Sklar's theorem withholds two important facts which are of great value to dependence modeling. The first one includes the observation that copulas facilitate the construction of bivariate distribution functions, in the sense that any combination of margins can be chosen to build their bivariate aggregate.

The second one entails the observation that any bivariate distribution function can be split up into a part only containing information related to the respective variables, the margins, and into a part which captures the dependence structure inherent to the multivariate distribution function, the copula.

A copula usually belongs to one or more copula families. These families are characterized by one or more parameters. When a copula family entails both the Fréchet lower bound $W(u, v) = \max(u + v - 1, 0)$ and upper bound $M(u, v) = \min(u, v)$ and also the independence copula $\Pi(u, v) = uv$ it is called *comprehensive*. Typically a bivariate copula family has one parameter, which determines the degree of dependence the system displays. Additionally, the copula parameter can also be used to display the degree of tail dependence inherent to the system (provided the tail dependence is parameter dependent). But clearly, there is a one-on-one relationship here between the degree of dependence and the degree of tail dependence or any other parameter dependent copula property. Alternatively, bivariate copula families with multiple parameters can be defined. A multiparametric system typically allows more degrees of freedom when modeling dependence, as it, apart from its dependence parameter, also can control e.g. tail dependence properties for a given degree of dependence.

In this paper we will focus on a well-known class of copula families, the Archimedean copula class. This class is characterized by a generator φ , a function which facilitates the construction, parameter estimation and simulation of copulas. For this reason Archimedean copulas form an important copula class and find a wide range of applications. The copula generator φ and its pseudo-inverse are defined in the following way:

Definition 2 A generator φ is a continuous, strictly decreasing convex function defined on \mathbf{I} and image $[0, \infty)$. If $\varphi(0) = \infty$ then the generator is called strict. The pseudo-inverse of φ is the function $\varphi^{[-1]}$ with support $[0, \infty)$ and image \mathbf{I} , given by

$$\varphi^{[-1]} = \begin{cases} \varphi^{-1}(t), & 0 \leq t \leq \varphi(0) \\ 0, & \varphi(0) \leq t < \infty \end{cases}.$$

With this generator the Archimedean copula can be defined as follows:

Definition 3 A bivariate Archimedean copula with generator φ is the function $C : \mathbf{I}^2 \rightarrow \mathbf{I}$ defined as:

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)).$$

If $\varphi^{[-1]}$ has an explicit form, the Archimedean copula can also be written in a closed form. This is generally the case for most known one-parametric Archimedean copula families (see Nelsen (2006) for an overview). An important aspect of the dependence structure is the dependence in the tails. In this respect, the coefficients of tail dependence (see Joe (1997)) are a powerful tool.

Let X and Y be continuous random variables with margins F and G , respectively.

Definition 4 The upper tail dependence parameter λ_U is the limit (provided it exists) of the conditional probability that Y is greater than the t -th percentile of G given that X is greater than the t -th percentile of F as t approaches 1, i.e. $\lambda_U = \lim_{t \rightarrow 1^-} P[Y > G^{[-1]}(t) | X > F^{[-1]}(t)]$. The lower tail dependence parameter λ_L is the limit (provided it exists) of the conditional probability that Y does not exceed the t -th percentile of G given that X does not exceed the t -th percentile of F as t approaches 0, i.e. $\lambda_L = \lim_{t \rightarrow 0^+} P[Y \leq G^{[-1]}(t) | X \leq F^{[-1]}(t)]$

Nelsen (2006) shows that the tail dependence parameters λ_L and λ_U only depend on the derivative diagonal section δ_C of a copula.

Lemma 1 If the diagonal section of a copula is defined as $\delta_C(t) = C(t, t)$, $t \in [0, 1]$, the tail dependence parameters can be obtained as follows

$$\lambda_U = 2 - \lim_{t \rightarrow 1^-} \frac{1 - C(t, t)}{1 - t} = 2 - \delta'_C(1^-)$$

$$\lambda_L = \lim_{t \rightarrow 0^+} \frac{C(t,t)}{t} = \delta'_C(0^+).$$

Indices of regular variation are closely related to the classic tail dependence coefficients. For a sound analysis of these quantities we refer to Charpentier and Segers (2008).

Definition 5 Let C be an Archimedean copula with generator φ . The index of regular variation at zero is defined as $\theta_0 := -\lim_{s \rightarrow 0} \frac{s\varphi'(s)}{\varphi(s)}$ such that $\lim_{s \rightarrow 0} \frac{\varphi(st)}{\varphi(s)} = t^{-\theta_0}$, $t \in (0, \infty)$. The index of regular variation at one is defined as $\theta_1 := -\lim_{s \rightarrow 0} \frac{s\varphi'(1-s)}{\varphi(1-s)}$ such that $\lim_{s \rightarrow 0} \frac{\varphi(1-st)}{\varphi(1-s)} = t^{-\theta_1}$, $t \in (0, \infty)$.

Charpentier and Segers (2008) show that the tail dependence parameters λ_L and λ_U can be written by means of these indices.

Lemma 2 An alternative expression for the coefficients of tail dependence λ_L and λ_U in terms of θ_0 and θ_1 is

$$\begin{aligned}\lambda_U &= 2 - 2^{1/\theta_1} \\ \lambda_L &= 2^{-1/\theta_0}.\end{aligned}$$

A transparent way of representing copulas, is by visualising their *level curves*.

Definition 6 A level curve is defined as the function $C(u, v) = t$ for $t \in [0, 1]$ and $(u, v) \in \mathbf{I}^2$.

By studying the level curves we get more insight about the distribution of the probability mass or C -measure below, on and above the level curves, i.e. the probability of drawing random variables U, V from the subsets of \mathbf{I}^2 below, on and above the level curves. For more information, see Nelsen (2006). Related to this subject we define a second characterisation. It is concerned with the property of having a non-empty zero set $Z(C)$, since it determines the proximity of the zero curve to the countermonotonic curve which has C -measure 1.

Definition 7 The zero set is defined as the set $Z(C) = \{(u, v) \in \mathbf{I} | C_\theta(u, v) = 0\}$.

This zero set $Z(C)$ might contain positive area which naturally influences the shape of the copula. Additional to $Z(C)$ the C -measure of the zero curve $\varphi(u) + \varphi(v) = \varphi(0)$, which will be denoted as C_0 , can be nonnegative for non-strict generators. Hence, it can be used as indicator for its shape.

3. THE λ METHOD

We now turn to a specific aspect of the study of Archimedean copula functions, namely their construction. It must be clear that an Archimedean copula can be obtained in a straightforward way. Indeed, if one defines a feasible generator φ one immediately has a feasible (implicit or explicit) copula. However, the φ function is not *visually informative* and as such it is, in our opinion, not the best choice for constructing a copula. Besides, the functional form of φ does not provide a priori information on C_φ . This can be explained by the following observations:

- Every generator function φ can be rescaled by a constant $c > 0$.
- The tail dependence coefficients rely on C_φ , or on φ and $\varphi^{[-1]}$.
- The C -measure of the zero curve depends on φ and φ' .
- The parameter range of φ , in terms of τ , depends on φ and φ' .

It should be clear that a construction method that makes it possible to deal with these concerns, would create an interesting alternative to the φ method. Therefore, we want to explore such a construction method based on the λ function. It will be shown that this function does entail all the visual information necessary to fully understand the resulting copula function. We first give a brief characterization of the λ function. Next we explain the λ copula construction method.

3.1. The λ function

The λ function was introduced in Genest and Mackay (1986). In general the relationship between the copula parameter and Kendall's τ reads as

$$\tau = 4 \int \int_{I^2} C(u, v) dC(u, v) - 1. \quad (1)$$

Genest and Mackay showed that this expression can be written in terms of the generator φ and its first derivative φ' , or

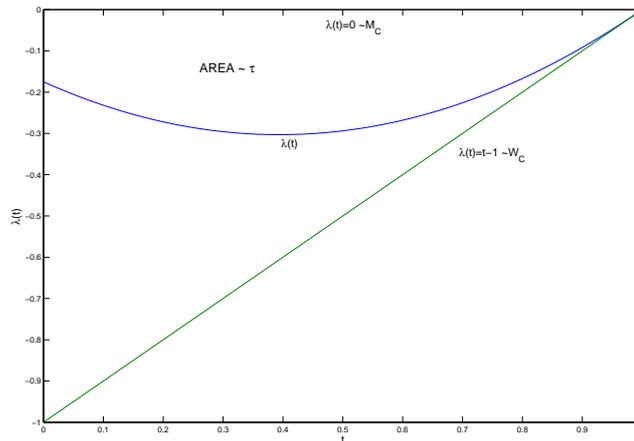
$$\tau = 1 + 4 \int_0^1 \lambda(t) dt \quad (2)$$

where

$$\lambda(t) = \frac{\varphi(t)}{\varphi'(t)}. \quad (3)$$

As such the λ function provides the geometrical interpretation between the copula and Kendall's τ . In Figure 1 a graph of the λ function is shown in the Kendall's τ space which is naturally bounded by the λ versions of the Fréchet bounds $W(u, v) = \max(u + v - 1, 0)$ and $M(u, v) = \min(u, v)$. Another important advantage is that the value of the intercept $\lambda(0)$ is in fact the C-measure of the zero curve of the non-strict copula (see Theorem 4.3.3 in Nelsen (2006)).

Figure 1: Geometrical interpretation between the copula and Kendall's τ .



It must be clear that the λ is more informative than the generator function φ . If we return to the four observations earlier in the text, we can already mention two important facts about the λ function:

- The C-measure of the zero curve is fully determined by λ ,
- There is a straightforward relation between τ and λ .

Also the second observation made earlier, concerning the tail dependence coefficients, can be solved by means of the λ function. Therefore we use a result from Charpentier and Segers (2008) to redefine the

coefficients of upper and lower tail dependence as a function of λ . As such, tail dependence properties can be studied by analysing the λ function. We first state the following results:

Lemma 3 *For any strict Archimedean copula, it is true that*

- $\lambda_L = 0 \Leftrightarrow \lambda'(0^+) = -\infty$
- $\lambda_L = 1 \Leftrightarrow \lambda'(0^+) = 0$
- $\lambda_U = 0 \Leftrightarrow \lambda'(1^-) = 1$
- $\lambda_U = 1 \Leftrightarrow \lambda'(1^-) = 0$.

This result can be proved when looking at the diagonal section (see Lemma 1) for the Fréchet upper bound M and the independence copula $\Pi(u, v) = uv$. Note that for a non-strict copula it is possible that $\lambda'(0^+) < 0$.

From Charpentier and Segers (2008) we recall the result that the coefficient of upper tail dependence λ_U is determined by the index of regular variation θ_1 of φ in 1. In a similar fashion the coefficient of lower tail dependence λ_L is determined by the index of regular variation θ_0 of φ in 0. Consequently the following holds:

Lemma 4 *For any Archimedean copula it is true that*

$$\theta_0 = -\frac{1}{\lambda'(0)}$$

$$\theta_1 = \frac{1}{\lambda'(1)}.$$

Corollary 1 *As a consequence of Lemma 2 and Lemma 4, the coefficients of tail dependence of an Archimedean copula can be rewritten as*

$$\lambda_L = 2^{\lambda'(0^+)}$$

$$\lambda_U = 2 - 2^{\lambda'(1^-)}.$$

As such a third observation can be made:

- With the alternative definition the tail dependence coefficients are completely determined by λ .

A final advantage of the λ function is the fact that it can be used to assess the goodness-of-fit of a (non)parametric copula. This was first proposed in Genest and Rivest (1993), where the relationship between λ and K , with $K(t) = P(C(u, v) \leq t)$ the copula distribution function, is given by $\lambda(t) = t - K(t)$. As such, an empirical λ function can be obtained and directly fitted on using observations coming from K . In a more recent paper by Genest et al. (2007) two goodness-of-fit statistics on K are being discussed and are proven to be powerful. Finally, in a paper by Lambert (2007) the λ function is fitted on *nonparametrically*, using the method of splines in a fitting application.

As conclusion we state that the λ function gives a very good picture of the bivariate copula. Not only does it provide valuable information regarding copula characteristics, it can also be used to assess the goodness-of fit of a copula. Furthermore, simulations of the copula can be done by using the generator based algorithm by Genest and Rivest (1993).

3.2. The λ copula construction method

We now explain how the λ function can be used in order to construct an Archimedean copula (space). As a first step, we note that from (3) we can recover φ by solving the differential equation. This yields

$$\varphi(t) = \varphi(t_0) e^{\int_{t_0}^t \frac{1}{\lambda(z)} dz} \quad (4)$$

for $0 < t_0 < 1$. This function is properly defined, see e.g. Genest and Rivest (1993). Of course not any arbitrarily chosen function λ will lead to a function that represents a generator of an Archimedean copula. In order to define a feasible φ some restrictions need to be imposed on λ (see also Lambert (2007)).

Property 1 *The function φ as in (4) will be a well-defined generator function, provided that λ is a continuously differentiable function with domain $[0, 1]$ for which is true that*

R1 $\lambda(0) \in [-1, 0]$

R2 $\lambda(1) = 0$

R3 $\lambda(t) < 0, t \in (0, 1)$

R4 $\lambda'(t) < 1, t \in (0, 1)$.

Proof.

By construction $\varphi(t) = \varphi(t_0) e^{\int_{t_0}^t \frac{1}{\lambda(z)} dz}$ with $0 < t_0 < 1$ assures $\varphi(t) \geq 0$. The fact that λ is continuous differentiable assures φ to be continuously differentiable. From restriction R3 we derive $\lambda(t) < 0, t \in (0, 1) \iff \frac{\varphi(t)}{\varphi'(t)} < 0, t \in (0, 1) \iff \varphi'(t) < 0, t \in (0, 1) \iff \varphi$ strictly decreasing on $(0, 1)$. From restriction R4 we derive $\left(\frac{\varphi(t)}{\varphi'(t)}\right)' < 1, t \in (0, 1) \iff 1 - \frac{\varphi(t)\varphi''(t)}{(\varphi'(t))^2} < 1, t \in (0, 1) \iff \varphi''(t) > 0, t \in (0, 1) \iff \varphi$ is strictly convex on $(0, 1)$. From restriction R2 we derive $\lambda(0) = 1 \iff \frac{\varphi(1)}{\varphi'(1)} = 0 \iff \varphi(1) = 0$ (since φ' is negative and strictly increasing on $(0, 1)$). Restriction R2-R4 combined guarantee $1 - t \leq \lambda(t) \leq 0$ for $t \in (0, 1)$ and consequently $\varphi(0) \in [-1, 0]$.

As such restriction R1 is superfluous but from a construction point of view it facilitates the choice for the parametric λ form and the derivation of the parameter ranges. Note that the generator can only be strict if $\lambda(0) = 0$, but the fact that $\lambda(0) = 0$ does not guarantee in advance the strictness of the generator.

Any λ function satisfying these requirements will provide a valid copula generator. As such it is arguable that any empirical dependence function C can in fact be represented by an Archimedean copula through (4). If the λ function itself is symbolically integrable, it will be possible to directly import Kendall's τ as the concordance parameter into the system. If the reciprocal form $\frac{1}{\lambda}$ is symbolically integrable, a closed form generator can be obtained, which is practical for means of simulation.

An extra advantage of the λ method is its capability to generate and study *comparable* Archimedean test spaces. A comparable test space is introduced in Michiels and De Schepper (2008) and refers to a collection of copulas that are able to describe a certain degree of dependence and which can be used together in a fitting application. Indeed, by making a restriction on τ it is possible to obtain the vast Archimedean comparable copula space for τ value τ^* by solving the following mathematical problem

find $\lambda_\theta(t)$ for $\theta = f(\tau^*)$

subject to:

$$\lambda_\theta(0) \in [-1, 0]$$

$$\begin{aligned}\lambda_\theta(1) &= 0 \\ \lambda_\theta(t) &\leq 0 \\ \lambda'_\theta(t) &\leq 1 \\ \text{for } t &\in [0, 1], \tau^* \in [-1, 1]\end{aligned}$$

which allows to obtain every possible (Archimedean) dependence structure per degree of dependence. Of course, in order to solve this mathematical program, it is necessary to refine the objective function, by fixing a functional form for λ . This will be subject of section 4, where we suggest, argue and elaborate several possibilities. We will do this by defining multiparametric λ functions. If the parametrization is not too complex, it will be possible to represent the copula as a collection of comparable test spaces. It will also be possible to write all parameters in function of τ . As such, the concordance ordering of the copula family is straightforward, as $C_{1,\theta=f(\tau_1)} \prec C_{1,\theta=f(\tau_2)}, \forall \tau_1 \leq \tau_2$.

Although it seems very promising to depart from the λ method to study the Archimedean copula space and to construct multiparametric families, as far as we know it has not been done in the literature. In the following section we will show how this method can be used to generate general Archimedean copulas. We will investigate 3 types of λ functions and we will explore their associated Archimedean copula spaces.

4. THREE TYPES OF MULTIPARAMETRIC λ FUNCTIONS

We now discuss three types of λ functions: the polynomial λ , denoted as $\lambda_{(POL)}$, the rational λ , denoted as $\lambda_{(RAT)}$ and the log inspired λ , denoted as $\lambda_{(LOG)}$. We will illustrate the procedure by means of examples of low complexity. As already mentioned in the introduction, the focus of this paper lies on the methodology, and therefore also examples containing non-strict copulas will be considered. We discuss parameter interpretations, partially by means of the coefficients of tail dependence. The obtained multiparametric families will be represented as collections of comparable test spaces, which facilitates the parameter interpretation. The practical use of the λ function as a representative of the dependence function will be shown by comparing the characteristics of the λ function and the characteristics of the simulated $(U, V) \sim C_\lambda$ pair. Wherever possible, we generalize properties for n -parametric λ functions.

4.1. Polynomial configuration

Polynomial λ 's are first class candidates for constructing multiparametric Archimedean families, since both their functional form f as their reciprocal form $\frac{1}{f}$ are symbolically integrable. Another advantage of the polynomial class is that it is very wide. We will discuss two types of $\lambda_{(POL)}$: polynomials with intercept, denoted as $\lambda_{(POL,1)}$, and polynomials without intercept, denoted as $\lambda_{(POL,2)}$.

4.1.1. Polynomial λ with intercept

Polynomial λ 's with intercept generally create non-strict Archimedean copulas. We will discuss this λ type by considering a three parameter 2nd degree polynomial:

$$\lambda_{(POL,1)}(t) = \alpha + \beta t + \gamma t^2.$$

Note that for $\alpha = 0$ the resulting copula family will be strict and is known as family (4.12) in Nelsen (2006). For $-1 \leq \alpha < 0$ the copula will be non-strict. When incorporating the restrictions on the λ we get

$$\text{R1: } \lambda(0) = \alpha \Rightarrow \alpha \in [-1, 0]$$

$$\text{R2: } \lambda(1) = \alpha + \beta + \gamma \Rightarrow \gamma = -\alpha - \beta$$

R3: satisfied if

$$\lambda \text{ is convex} \Leftrightarrow \gamma \geq 0$$

$$\lambda \text{ is concave} \Leftrightarrow \gamma \leq 0 \text{ and } \beta \geq -2\gamma$$

R4: satisfied if $\beta \leq 1 - 2\gamma$

Note that from restriction R2 we already lose a parameter, so this basically is a two parameter system. From (2) we deduce $6\alpha + 3\beta + 2\gamma = \frac{3(\tau-1)}{2}$. Combining this relationship with R2 we get

$$\beta(\tau, \alpha) = \frac{3}{2}\tau - \frac{3}{2} - 4\alpha \quad (5)$$

$$\gamma(\tau, \alpha) = -\frac{3}{2}\tau + \frac{3}{2} + 3\alpha. \quad (6)$$

The convexity of λ (R3) results in an additional lower bound for α : $\max(-1, \frac{\tau-1}{2}) \leq \alpha \leq 0$. Restriction R4 leads to an additional upper bound for α : $\max(-1, \frac{\tau-1}{2}) \leq \alpha \leq \min(0, \frac{3\tau-1}{4})$.

When λ is concave R3 leads to a different lower bound for α : $\max(-1, \frac{3(\tau-1)}{4}) \leq \alpha \leq 0$. Summarizing the global α range can be expressed in terms of τ in the following way:

$$\alpha \in [\frac{1}{2}(\tau - 1), \min(\frac{1}{4}(3\tau - 1), 0)]. \quad (7)$$

All parameters are now written as a function of τ . Moreover, β and γ leave the system due to (5) and (6). A comparable test space can be obtained by first fixing a τ value and then obtaining the λ functions which accord to the feasible α range. It can easily be checked that this 2nd degree polynomial can generate comparable test spaces over the complete $\tau \in [-1, 1]$ range. For $\tau \in [\frac{1}{3}, 1]$ the λ function can generate one strict copula per τ value (when $\alpha = 0$). For $\tau < \frac{1}{3}$ the λ function only generates non-strict copulas.

The tail dependence properties can be studied using

$$\lambda_L = 2^{3(\tau-1)}, \text{ (only in case } \alpha = 0) \quad (8)$$

and

$$\lambda_U = 2 - 2^{\frac{3}{2}(1-\tau)+2\alpha}. \quad (9)$$

The λ function leads to the following copula generator:

$$\varphi_{\tau, \alpha}(t) = \left(\frac{1-t}{(1+\mu)t-\alpha} \right)^{\frac{1}{\mu}} \quad (10)$$

$$\text{with } \mu = -\frac{3}{2}\tau + \frac{3}{2} + 2\alpha.$$

As can be seen from its functional form, (10) can be inverted in an explicit way, leading to a closed form copula family.

Next we turn attention to the visual advantage of the λ method when it comes to the construction of copulas. In Figures 2 and 3 visuals are shown from both $\lambda_{(POL,1)}$ and $C_{\lambda_{(POL,1)}}$ for $\tau = 0.5$ and $\tau = -0.5$. The latter is represented by means of simulating observations from $(U, V) \sim C_{\lambda_{(POL,1)}}$. The idea here is to show how well λ behaves as a univariate representative of the affiliated copula. A first observation includes the clear relationship between the $\lambda(0)$ value (Figure 3, 4 (left)) and the size of $Z(C)$ (Figure 3, 4 (right)): the lower the absolute value of the intercept the smaller $Z(C)$ will be. A second observation focuses on the upper tail behaviour of the copula. As can be seen, for example in Figure 3 (a), strong upper tail dependence accords with a λ function which, for $t \rightarrow 1$ is close to the line $\lambda = 0$. In a similar fashion, for example in Figure 4 (a), the case of weak or no upper tail dependence accords with a λ function which, for $t \rightarrow 1$, is close to the line $\lambda = t - 1$. As such, a quick scan of the λ function immediately provides us with key aspects of the associated dependence function.

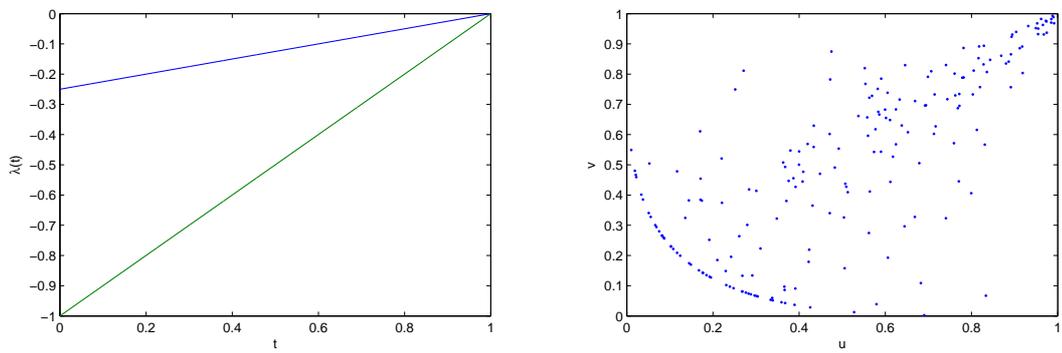
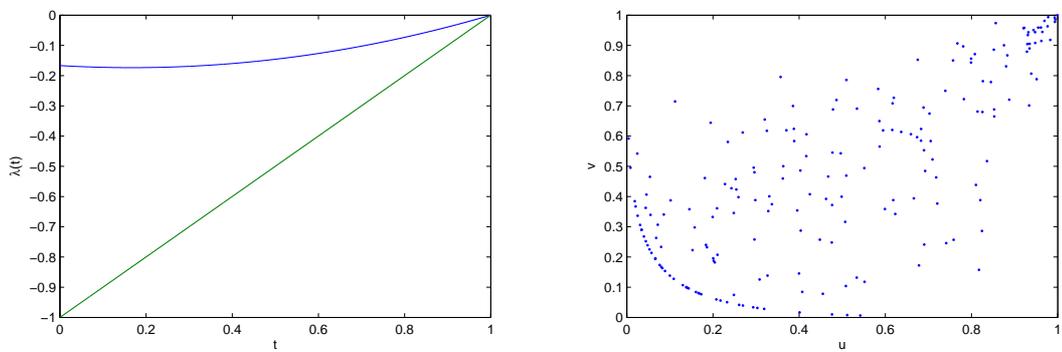
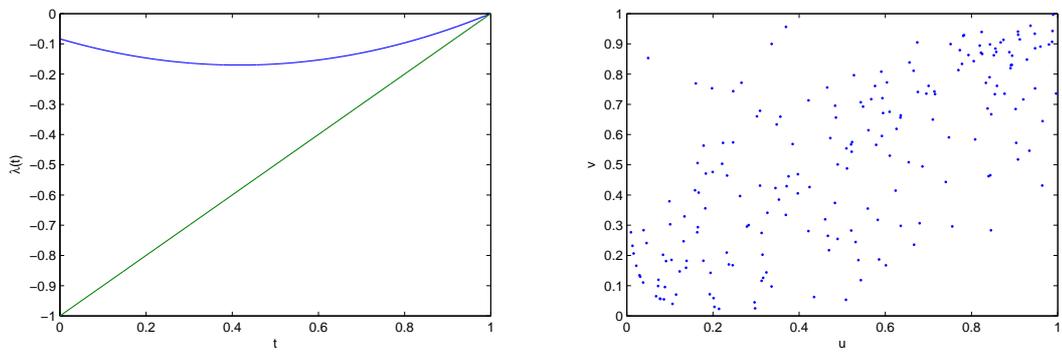
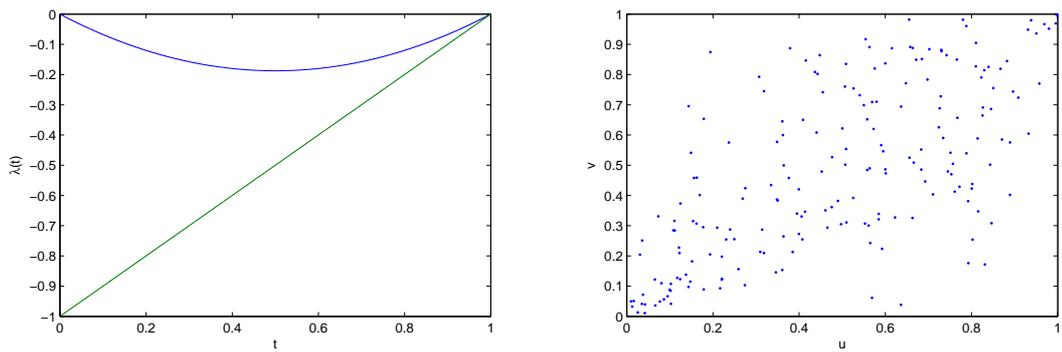
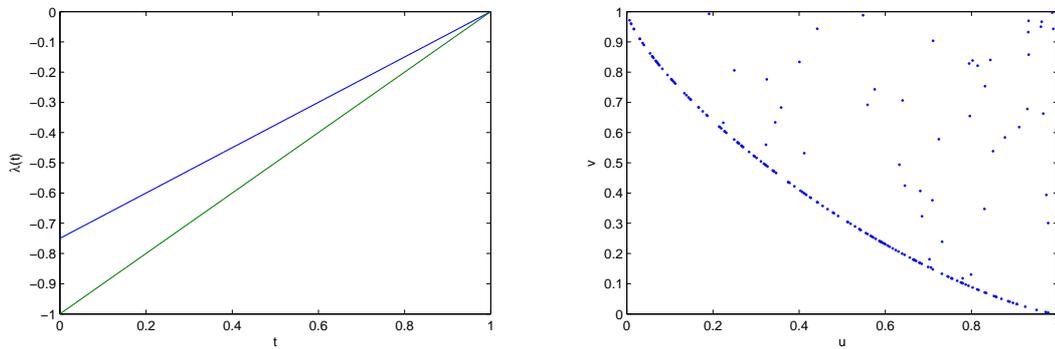
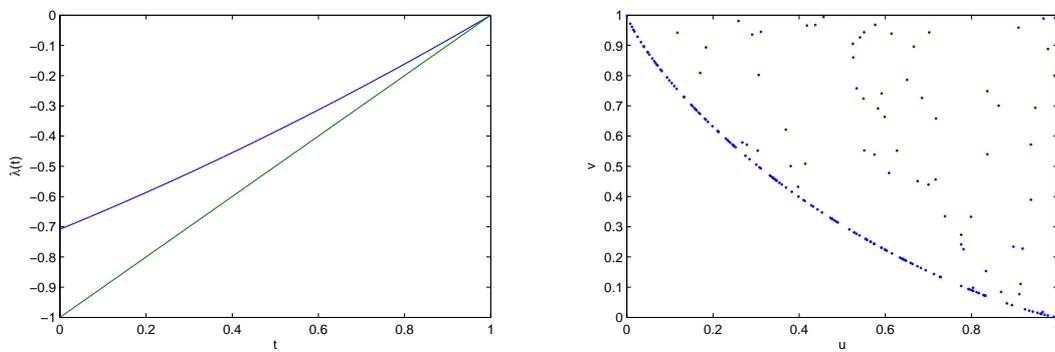
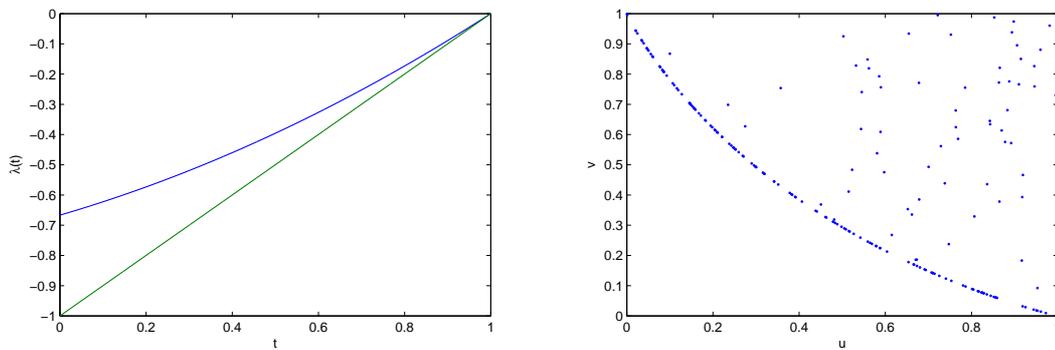
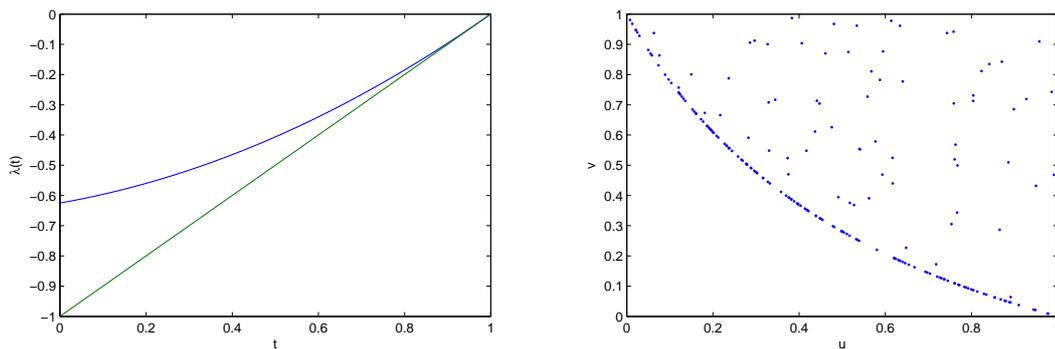
Figure 2: Comparable test space and simulations for $\tau=0.5$ for $\lambda_{(POL,1)}(t) = \alpha + \beta t + \gamma t^2$.(a) $\tau=0.5$, $\alpha=-0.2500$, $C_0=0.2500$, $\lambda_U=0.8513$ (b) $\tau=0.5$, $\alpha=-0.1833$, $C_0=0.1833$, $\lambda_U=0.6957$ (c) $\tau=0.5$, $\alpha=-0.0917$, $C_0=0.0917$, $\lambda_U=0.5189$ (d) $\tau=0.5$, $\alpha=0$, $C_0=0$, $\lambda_L=0.5946$, $\lambda_U=0.3182$

Figure 3: Comparable test space and simulations for $\tau = -0.5$ for $\lambda_{(POL,1)}(t) = \alpha + \beta t + \gamma t^2$.(a) $\tau = -0.5$, $\alpha = -0.7500$, $C_0 = 0.7500$, $\lambda_U = 0.3182$ (b) $\tau = -0.5$, $\alpha = -0.7083$, $C_0 = 0.7083$, $\lambda_U = 0.2182$ (c) $\tau = -0.5$, $\alpha = -0.6667$, $C_0 = 0.6667$, $\lambda_U = 0.1123$ (d) $\tau = -0.5$, $\alpha = -0.6250$, $C_0 = 0.6250$, $\lambda_U = 0$

4.1.2. Polynomial λ without intercept

The polynomial defined in 4.1.1. mainly provides non-strict Archimedean copulas, since for $\alpha \neq 0$ we have positive C_0 . The general τ range is $[-1, 1]$, but strict copulas can only be generated for $\tau \in [\frac{1}{3}, 1]$ and only one strict copula per comparable test space can be produced. However, it is desirable, for example in the field of finance, to work with strict copulas only. Hence, polynomials without intercept then are more appropriate, since they produce only strict Archimedean copulas. Consequently, this type of λ configuration can be used to study the affiliated strict Archimedean test spaces. As example we consider a 3th degree two parameter polynomial:

$$\lambda_{(POL,2)}(t) = t(t-1)(\alpha t + \beta).$$

When incorporating the restrictions on the λ we get

R1: satisfied since $\lambda(0) = 0$

R2: satisfied since $\lambda(1) = 0$

R3: satisfied if

$$\alpha \geq 0 \text{ and } \beta \geq 0$$

$$\alpha \leq 0 \text{ and } \beta \geq -\alpha$$

R4: satisfied if

$$\alpha \geq 0 \text{ and } -1 \leq \beta \leq \min(\alpha, 1 - \alpha)$$

$$\alpha \leq 0 \text{ and } -2\alpha \leq \beta \leq 1 - \alpha$$

$$\alpha \leq 0 \text{ and } -1 \leq \beta \leq \alpha$$

From (2) we derive

$$\beta(\alpha, \tau) = \frac{3(1-\tau) - \alpha}{2}. \quad (11)$$

When combining this relationship with restrictions R3 and R4, we get additional upper and lower bounds for $\alpha \geq 0$:

$$\max(0, 1 - \tau) \leq \alpha \leq \min(3(1 - \tau), 3\tau - 1), \tau \in [\frac{1}{2}, 1]. \quad (12)$$

In a similar fashion restrictions R3 and R4 lead to extra bounds for $\alpha \leq 0$ (note that the third R4 condition cannot be combined with the R3 condition for $\alpha \leq 0$):

$$\tau - 1 \leq \alpha \leq \min(3\tau - 1, 0), \tau \in [0, 1]. \quad (13)$$

Combined this leads to the global α range:

$$\tau - 1 \leq \alpha \leq \min(3\tau - 1, 3(1 - \tau)), \tau \in [0, 1]. \quad (14)$$

Again all parameter ranges are written as a function of τ . Since β leaves the system based on (11) a comparable test space is created by choosing $\tau \in [0, 1]$ and then obtaining the λ functions for the appropriate α space. The 2 parameter function $\lambda_{(POL,2)}$ can only generate strict Archimedean copulas in positive τ test spaces.

The tail dependence measures are given by

$$\lambda_L = 2^{\frac{3(\tau-1)+\alpha}{2}} \quad (15)$$

and

$$\lambda_U = 2 - 2^{\frac{3(1-\tau)+\alpha}{2}}. \quad (16)$$

The λ function leads to the following copula generator:

$$\varphi_{\tau,\alpha}(t) = (1-t)^{\frac{2}{\mu}} t^{-\frac{2}{\mu-2\alpha}} \left(\alpha t + \frac{\mu-2\alpha}{2} \right)^{\frac{4\alpha}{\mu(\mu-2\alpha)}} \quad (17)$$

$$\text{with } \mu = 3 + \alpha - 3\tau.$$

As can be seen from the functional form of (17), it is no longer analytically invertible. Hence, the alternative definition of the coefficients of tail dependence as defined in Section 3 now prove their practical use, as explicit formulas for tail dependence can still be obtained. Furthermore, the strictness property of $\lambda_{(POL,2)}$ can be checked evaluating (17) for $t = 0$.

Again we mention the visual advantage of the λ function. Figure 5 shows visuals of the $\lambda_{(POL,2)}$ and its associated copula as simulations from $(U, V) \sim C_{\lambda_{(POL,2)}}$. Notice how for upper tail dependence the same observation can be made as for $\lambda_{(POL,1)}$. In the case for strong upper tail dependence (Figure 5 (a)) the λ function will be close to the comonotone case ($\lambda = 0$) for $t \rightarrow 0$. Conversely, for weak upper tail dependence (Figure 5 (d)) the opposite is true (closer to the countermonotone case). Analogous observations can be made for lower tail dependence.

Since $\lambda_{(POL,2)}$ provides strict copulas we are interested in its modeling power. The polynomial defined above is a two parameter model. Which combinations of λ_L and λ_U are possible for a specific τ value in this kind of parametrization? In Figure 4 the feasible region is displayed. As can be seen, the number of possible combinations is rather limited per τ space, as λ_L and λ_U cannot move over the $[0, 1]$ range. Also, notice the fact that $\frac{d\lambda_U}{d\lambda_L}$ decreases for increasing τ .

The polynomial as described above can only generate copulas for positive dependence. In order to create strict Archimedean copulas appropriate to describe negative dependence, a higher degree polynomial is necessary to obtain more curvature. However, this can be seen as a disadvantage of polynomial λ functions, as also the necessary amount of parameters will increase and therefore also its complexity. Moreover, it will no longer be possible to obtain explicit bounds for the parameters. An alternative way of obtaining comparable test spaces from higher degree polynomials is by numerically solving the mathematical problem defined in Section 3.

In conclusion, for $\lambda_{(POL,2)}$ we make the following observation: Note that λ_L can never be equal to zero for polynomials, since it would require the first degree variable parameter moving to infinity. Hence, strict copula families generated from a polynomial λ function **always have lower tail dependence**.

Property 2 Any λ function of the form $\lambda(t) = t(t-1)(\beta_0 + \beta_1 t^1 + \dots + \beta_n t^n)$, $t \in [0, 1]$, $n \in [2, \infty)$ will generate copula functions with lower tail dependence $\lambda_L \neq 0$.

4.2. Rational configuration

A second type of λ function which also proves to be workable are rational functions. Like the polynomial class this class of functions is very numerous. As with the polynomial class we first discuss the rational λ with intercept, denoted as $\lambda_{(RAT,1)}$. Next we will discuss the case without intercept, denoted $\lambda_{(RAT,2)}$.

4.2.1. Rational λ with intercept

Rational λ 's with intercept generate non-strict Archimedean copulas. As example we use the 2 parameter first degree rational function:

$$\lambda_{(RAT,1)}(t) = \frac{\alpha(1-t)}{\beta t + 1}.$$

When incorporating the restrictions on the λ we get

$$\text{R1: } \lambda(0) = \alpha \implies \alpha \in [-1, 0]$$

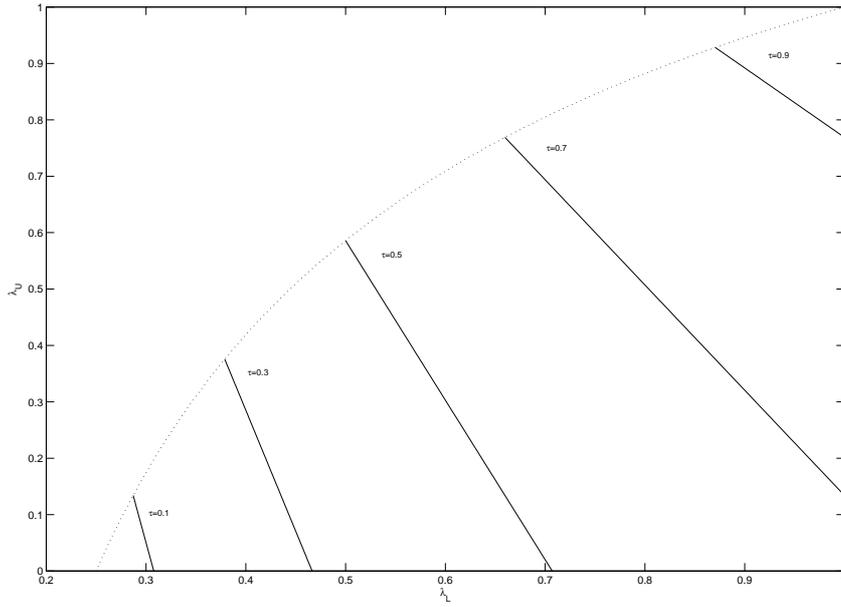


Figure 4: Combinations of λ_U with λ_L in comparable test spaces $\tau \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ for $\lambda_{(POL,2)}(t) = t(t-1)(\alpha t + \beta)$.

$$\text{R2: } \lambda(1) = \frac{0}{\beta+1} = 0 \text{ or } \beta \neq -1$$

$$\text{R3: satisfied if } \beta \in (-1, +\infty), \alpha \leq 0$$

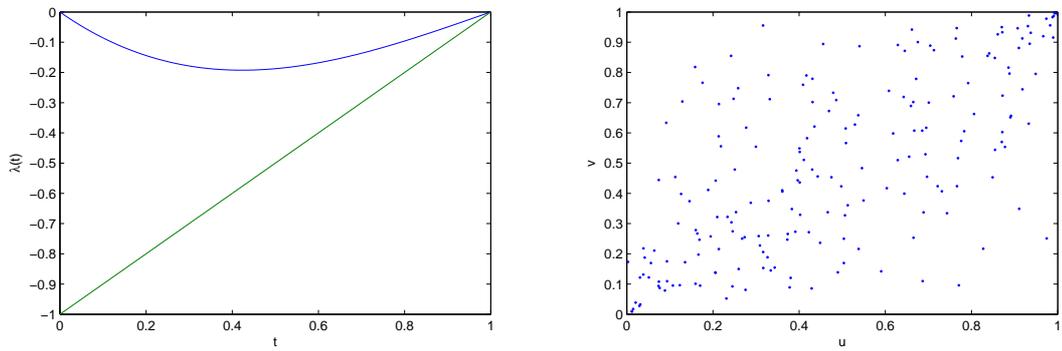
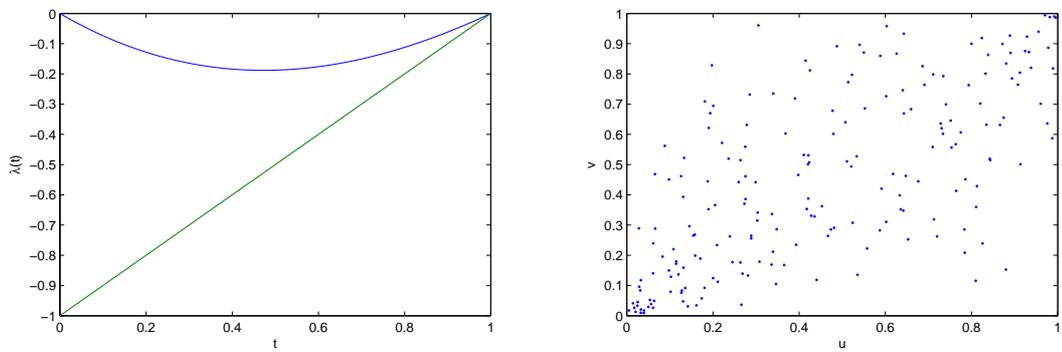
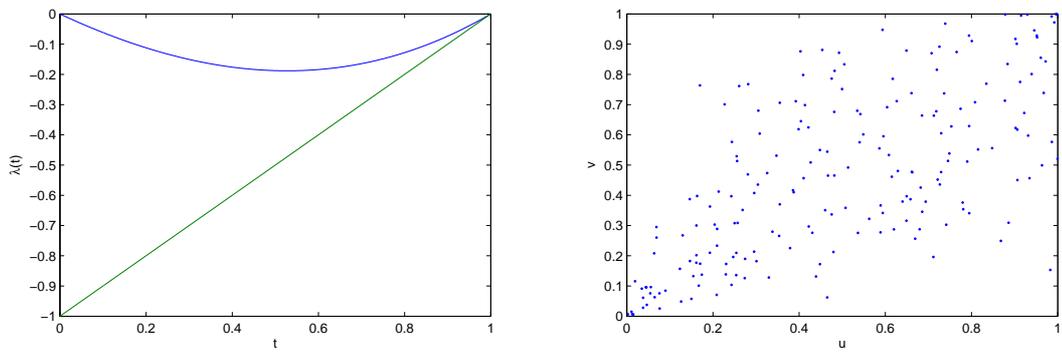
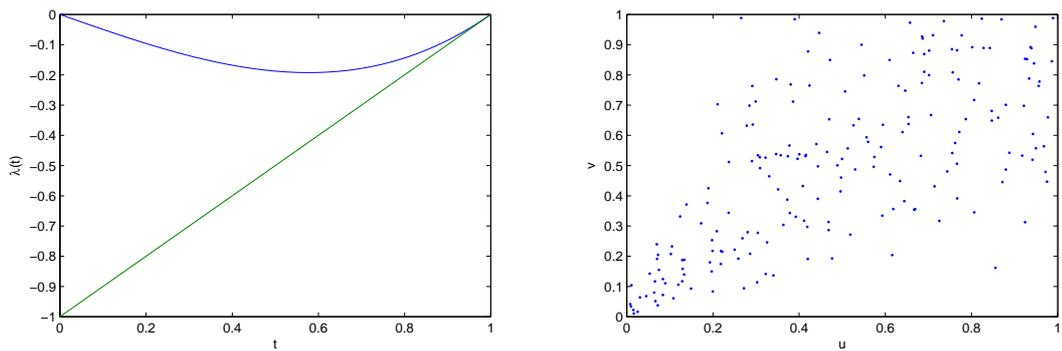
$$\text{R4: satisfied if } -\alpha - 1 \leq \beta \leq -1 - \frac{1}{\alpha}, \alpha \leq 0$$

From (2) we retrieve $\tau = 1 + 4\alpha \left[\ln |1 + \beta|^{\frac{1+\beta}{\beta^2}} - \frac{1}{\beta} \right]$ which can only be made explicit with respect to α :

$$\alpha = \frac{\tau - 1}{4} \left[\ln |1 + \beta|^{\frac{1+\beta}{\beta^2}} - \frac{1}{\beta} \right]^{-1}. \quad (18)$$

A comparable test spaces per τ value is obtained by verifying for $\beta \in (-1, +\infty)$ that expression (18) results in values belonging to $[\max(-\beta - 1, -1, -\frac{1}{1+\beta}), 0]$. As for $\lambda_{(POL,1)}$, non-strict Archimedean copulas can be generated over the complete $\tau \in [-1, 1]$ range. However, $\lambda_{(RAT,1)}$ can only generate non-strict copulas ($\alpha = 0 \iff C = C_M$) per comparable test space, whereas the polynomial $\lambda_{(POL,1)}$ could create strict copulas for $\tau \in [\frac{1}{3}, 1]$. As such only the measure for upper tail dependence is relevant:

$$\lambda_U = 2 - 2^{-\frac{\alpha}{\beta+1}}. \quad (19)$$

Figure 5: Comparable test space and simulations for $\tau=0.5$ for $\lambda_{(POL,2)}(t) = t(t-1)(\alpha t + \beta)$.(a) $\tau=0.5, \alpha=-0.5000, \lambda_L=0.5, \lambda_U=0.5858$ (b) $\tau=0.5, \alpha=-0.1667, \lambda_L=0.5612, \lambda_U=0.4126$ (c) $\tau=0.5, \alpha=0.1667, \lambda_L=0.6300, \lambda_U=0.2182$ (d) $\tau=0.5, \alpha=0.5000, \lambda_L=0.7071, \lambda_U=0$

From restriction R1 it is clear the α value determines C_0 .

The λ function creates the following copula generator, which is again not explicitly invertable:

$$\begin{aligned} \varphi_{\tau,\beta}(t) &= (1-t)^{-\mu_2-\mu_1} e^{\mu_2(1-t)} \\ \text{with } \mu_1 &= \frac{4(\ln|1+\beta| \frac{1+\beta}{\beta^2} - \frac{1}{\beta})}{\tau-1}, \mu_2 = \frac{4\beta(\ln|1+\beta| \frac{1+\beta}{\beta^2} - \frac{1}{\beta})}{\tau-1}. \end{aligned} \quad (20)$$

In Figures 6 and 7 visuals are shown for both $\lambda_{(RAT,1)}$ and its copula simulations. It is not difficult to see how again the λ function captures key copula characteristics. The size of $Z(C)$ clearly diminishes when $|\lambda(0)|$ becomes smaller and the upper tail dependence property can be derived from the behaviour of the right tail of λ , expressed by λ_U .

4.2.2. Rationals without intercept

As with the polynomial configuration with intercept the same remark can be made for $\lambda_{(RAT,1)}$. Although this λ function is able to produce dependence structures over the complete τ range, it provides non-strict copulas (except the comonotonic case) and as such its practical use is almost negligible. Hence we want to study the rational configuration without intercept necessary to create strict Archimedean copulas. We show the flexibility of strict rational λ functions by using a two parameter example of the form

$$\lambda_{(RAT,2)}(t) = \frac{\alpha t(t-1)}{\beta t+1}.$$

When incorporating the restrictions on the λ we get

- R1: satisfied since $\lambda(0) = 0$
- R2: satisfied since $\lambda(1) = \frac{0}{\beta+1} = 0$ or $\beta \neq -1$.
- R3: satisfied if $\beta \in (-1, +\infty)$, $\alpha \geq 0$
- R4: satisfied if $\alpha \in [0, \beta+1]$, $\beta \in (-\infty, +\infty)$

From (2) we retrieve $\tau = 1 + \frac{4\alpha}{\beta^2} \left[-\frac{\beta}{2} - 1 + \ln|1+\beta| \frac{(1+\beta)}{\beta} \right]$ which can only be made explicit with respect to α :

$$\alpha = \frac{\tau-1}{4} \beta^2 \left[-\frac{\beta}{2} - 1 + \ln|1+\beta| \frac{(1+\beta)}{\beta} \right]^{-1}. \quad (21)$$

To obtain comparable test spaces per τ value one needs to verify for $\beta \in (-1, +\infty)$ that expression (21) results in values belonging to $[0, \beta+1]$. It can easily be checked that the τ range for this copula family is $[-1, 1]$, and as such, for the same parametrization, the concordance range of $\lambda_{(RAT,2)}$ is larger than $\lambda_{(POL,2)}$.

The tail dependence properties can be studied immediately, as

$$\lambda_L = 2^{-\alpha} \quad (22)$$

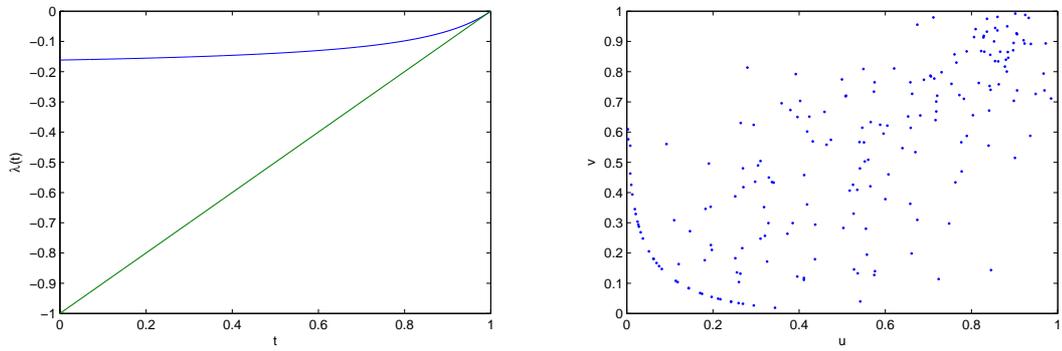
and

$$\lambda_U = 2 - 2^{\frac{\alpha}{\beta+1}}. \quad (23)$$

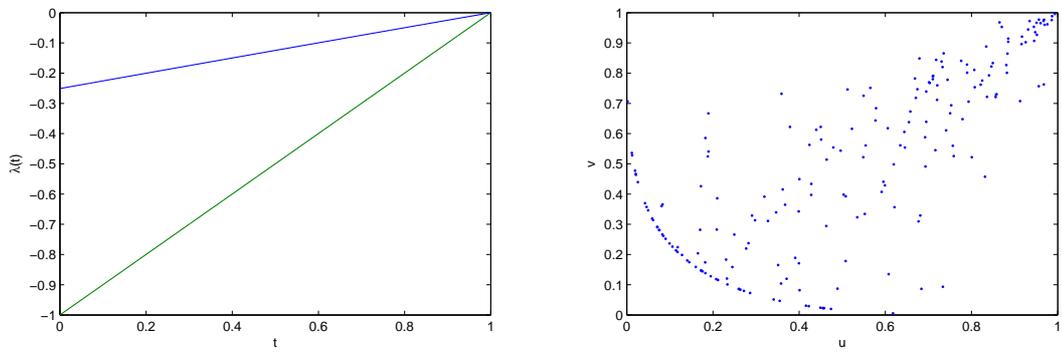
The λ function leads to the following copula generator, which is not analytically invertable:

$$\begin{aligned} \varphi_{\tau,\beta}(t) &= (1-t)^{(\beta+1)\mu} t^{-\mu} \\ \text{with } \mu &= \frac{4}{\beta^2(1-\tau)} \left(\frac{\beta}{2} - 1 + \ln|1+\beta| \frac{1+\beta}{\beta} \right). \end{aligned} \quad (24)$$

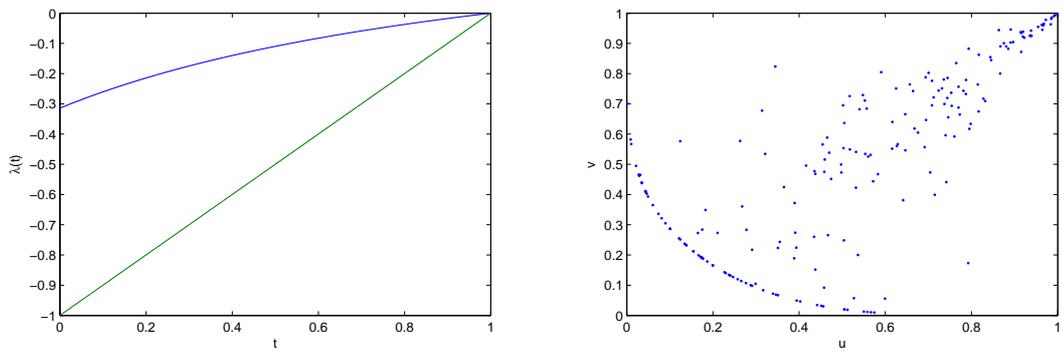
Figure 6: Comparable test space and simulations for $\tau = 0.5$ for $\lambda_{(RAT,1)}(t) = \frac{\alpha(1-t)}{\beta t + 1}$.



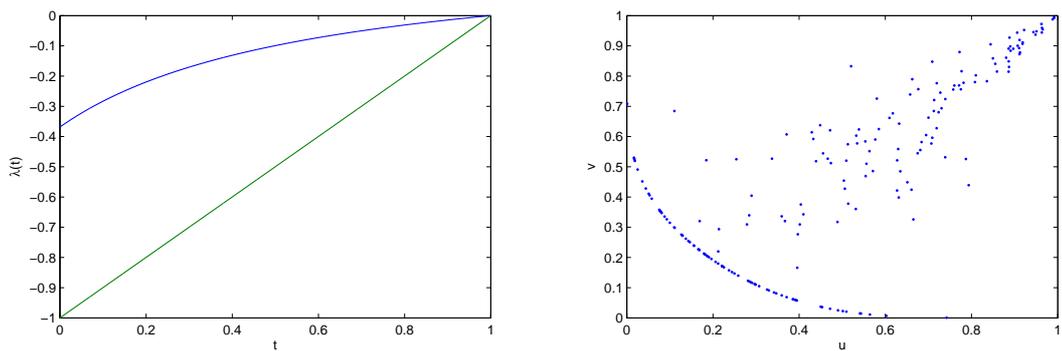
(a) $\tau = 0.5, \alpha = -0.1616, C_0 = 0.1616, \lambda_U = 0$



(b) $\tau = 0.5, \alpha = -0.2510, C_0 = 0.2510, \lambda_U = 0.8124$

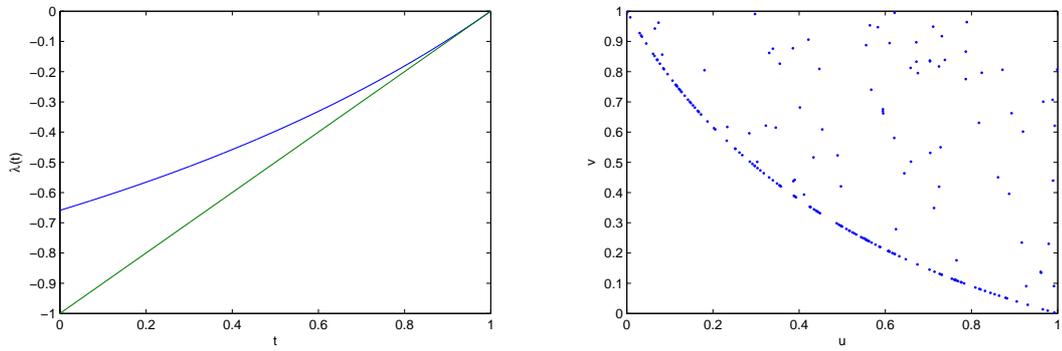


(c) $\tau = 0.5, \alpha = -0.3143, C_0 = 0.3143, \lambda_U = 0.8759$

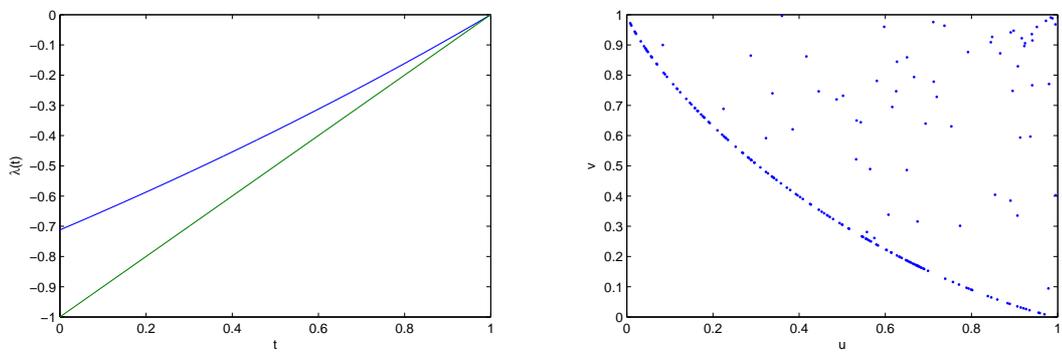


(d) $\tau = 0.5, \alpha = -0.3687, C_0 = 0.3687, \lambda_U = 0.9012$

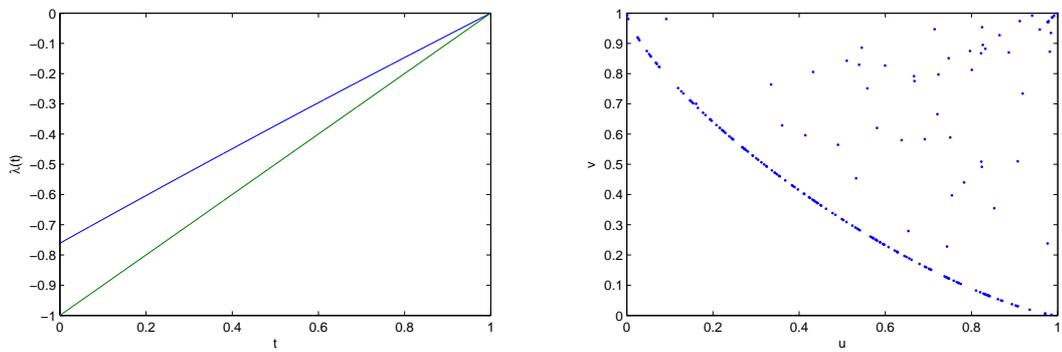
Figure 7: Comparable test space and simulations for $\tau = -0.5$ for $\lambda_{(RAT,1)}(t) = \frac{\alpha(1-t)}{\beta t+1}$.



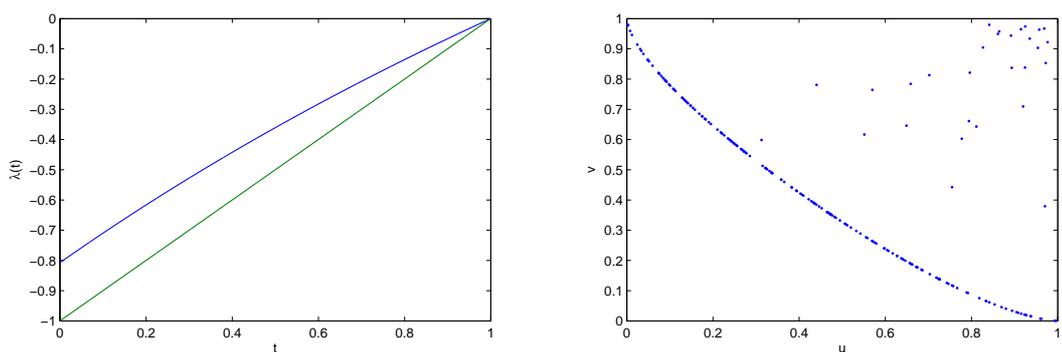
(a) $\tau = -0.5$, $\alpha = -0.6589$, $C_0 = 0.6589$, $\lambda_U = 0$



(b) $\tau = -0.5$, $\alpha = -0.7120$, $C_0 = 0.7120$, $\lambda_U = 0.2154$



(c) $\tau = -0.5$, $\alpha = -0.7612$, $C_0 = 0.7612$, $\lambda_U = 0.3434$



(d) $\tau = -0.5$, $\alpha = -0.8075$, $C_0 = 0.8075$, $\lambda_U = 0.4285$

The strictness property of $\lambda_{(RAT,2)}$ can be checked by evaluating (24) for $t = 0$. We now turn to the modeling aspects of $\lambda_{(RAT,2)}$. In Figures 10 and 11 visuals are shown of both the λ function and simulations from its associated copula for $\tau = 0.5$ and $\tau = -0.5$. As with the former λ functions we clearly see how the tail dependence aspects of the copula are recognizable on the λ function. It also shows that, unlike $\lambda_{(POL,2)}$, $\lambda_{(RAT,2)}$ can generate copulas whose λ_L becomes arbitrary small for $\alpha \rightarrow +\infty$. In Figure 10 the same remarks can be made for a strict Archimedean test space for negative dependence.

Next we investigate the modeling power of $\lambda_{(RAT,2)}$ by obtaining the feasible (λ_L, λ_U) combinations for a given τ value, provided the given parametrization. This is shown in Figure 8. It becomes clear that $\lambda_{(RAT,2)}$ can model tail dependence freely, since $\lambda_{Lmin}, \lambda_{Umin}$ can take the value 0 for $\tau < 1$. Next we investigate the modeling power of $\lambda_{(RAT,2)}$ versus $\lambda_{(POL,2)}$. In Figure 9 the feasible regions for (λ_L, λ_U) combinations are compared for $\lambda_{(POL,2)}$ and $\lambda_{(RAT,2)}$ for $\tau \in [0, 1]$.

The following observations can be made. First of all, the tail dependence range per τ value for $\lambda_{(RAT,2)}$ is more complete than for $\lambda_{(POL,2)}$, as the $\lambda_{(RAT,2)}$ system enables a complete trade-off between λ_L and λ_U . Secondly, for strong positive dependence ($\tau \in \{0.7, 0.9\}$) the $\lambda_{(RAT,2)}$ still holds the possibility to generate copulas with $\lambda_U=0$, while this is not the case for $\lambda_{(POL,2)}$. Finally, notice the difference in trade-off between λ_L and λ_U for $\lambda_{(RAT,2)}$ and $\lambda_{(POL,2)}$. Clearly the $\frac{d\lambda_L}{d\lambda_U} \lambda_{(RAT,2)}$ function has a concave form, while $\frac{d\lambda_L}{d\lambda_U} \lambda_{(POL,2)}$ has a convex form. As can be seen in Figure 9, both λ functions share (λ_L, λ_U) combinations, for $\tau \in [0.5, 1]$.

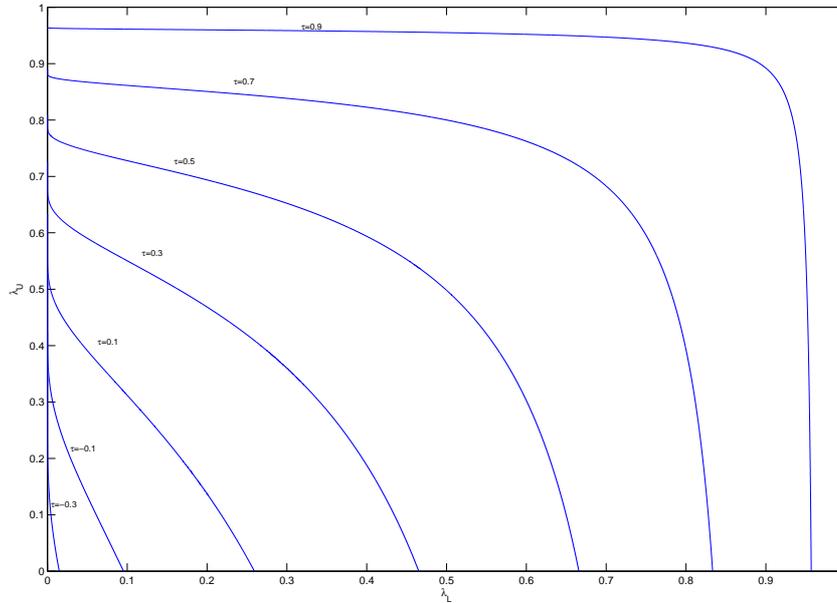


Figure 8: Combinations of λ_U with λ_L in comparable test spaces $\tau \in \{-0.9, -0.7, -0.5, -0.3, -0.1, 0.1, 0.3, 0.5, 0.7, 0.9\}$ for $\lambda_{(RAT,2)}(t) = \frac{\alpha t(t-1)}{\beta t+1}$.

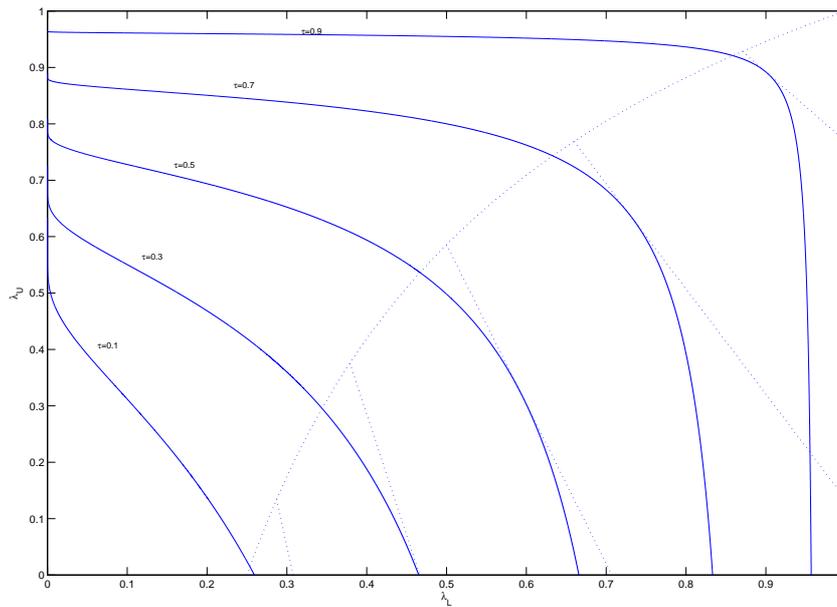


Figure 9: Comparison combinations of λ_U with λ_L in comparable test spaces $\tau \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ for $\lambda_{(POL,2)}(t) = t(t-1)(\alpha t + \beta)$ (dotted line) and $\lambda_{(RAT,2)}(t) = \frac{\alpha t(t-1)}{\beta t+1}$ (solid line).

4.3. Logarithmic configuration

None of the families discussed above actually contains the independence copula $\Pi(u, v) = uv$. Hence a third type of λ function is introduced which contains Π and also proves workable in the sense that both λ and $\frac{1}{\lambda}$ are integrable. In analogy with the preceding two λ types we will discuss two different logarithmic types. The first one provides mainly non-strict copulas and is denoted as $\lambda_{(LOG,1)}$. The second one provides both strict and non-strict copulas but only the strict part will be discussed. This type will be denoted as $\lambda_{(LOG,2)}$.

4.3.1. Logarithmic λ type 1

The first λ type is based on powers of logarithmic functions. As example we discuss a two parameter function of the form

$$\lambda_{(LOG,1)}(t) = \alpha t(\ln(t))^2 + \beta t \ln(t)$$

where $\lambda(0)$ is defined as the limiting value $\lambda(0^+) = 0$. This function provides non-strict copulas for $\alpha \neq 0$. However, since $\lambda(0^+) = 0$ we have $C_0 = 0$ is parameter independent. For $\alpha = 0$ the associated family of copulas is known as the Gumbel-Hougaard family. (The strictness property can be checked by evaluating $\varphi(0)$ in (28).) For $\alpha = 0$ and $\beta = 1$ we have $\lambda_{(LOG,1)} = \lambda_\Pi$.

When incorporating the restrictions on the λ we get

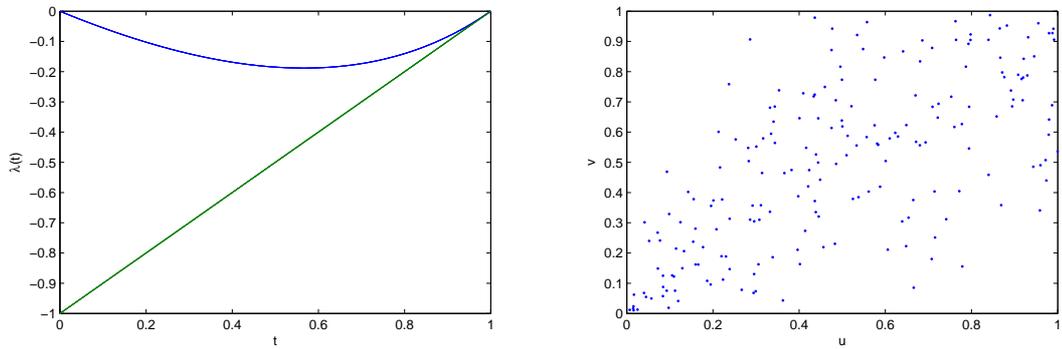
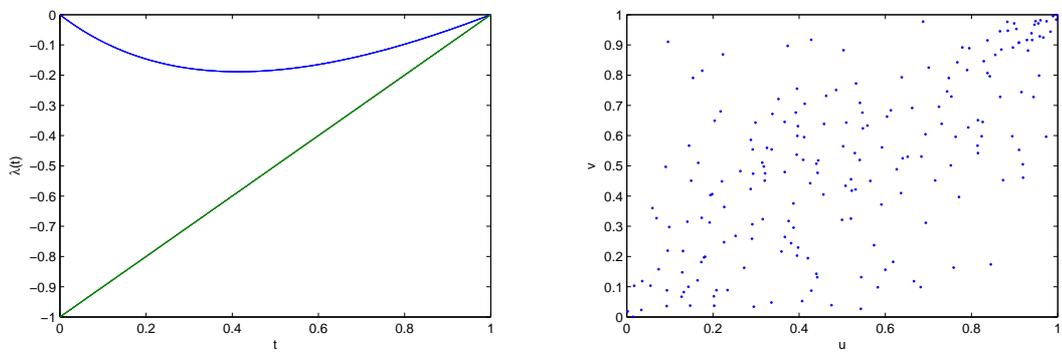
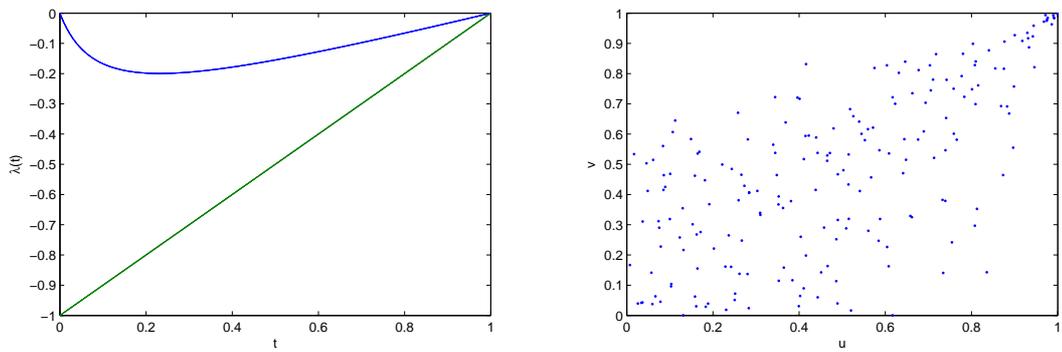
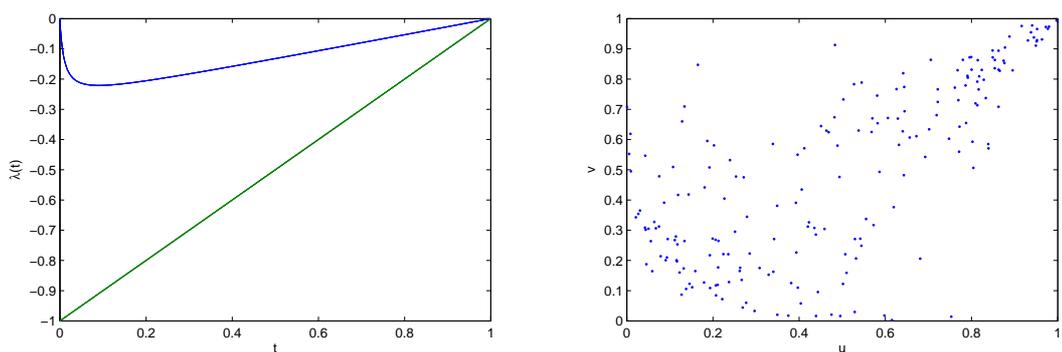
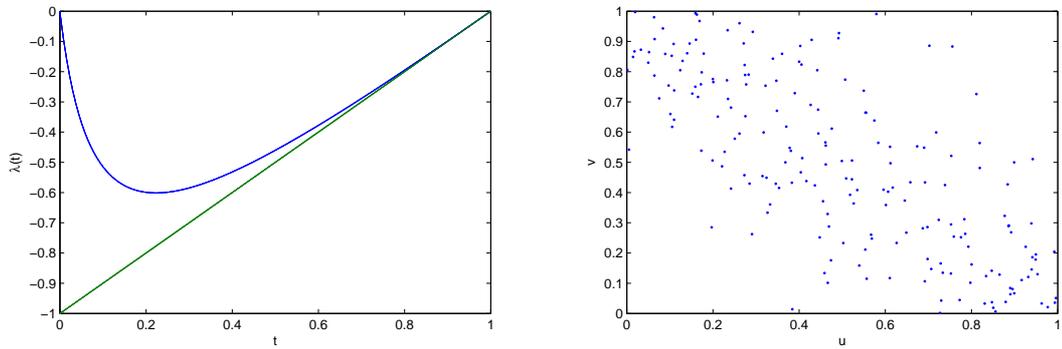
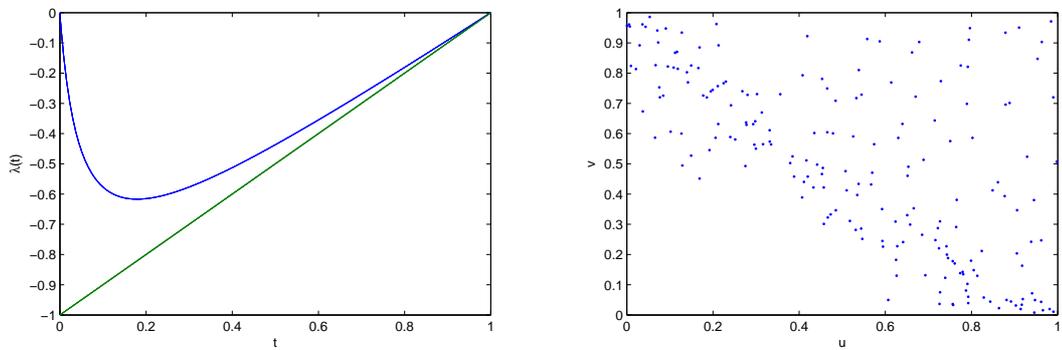
Figure 10: Comparable test space and simulations for $\tau = 0.5$ for $\lambda_{(RAT,2)}(t) = \frac{\alpha t(t-1)}{\beta t+1}$.(a) $\tau = 0.5, \beta = -0.413, \lambda_L = 0.6657, \lambda_U = 1.5305e - 004$ (b) $\tau = 0.5, \beta = 1, \lambda_L = 0.4667, \lambda_U = 0.5362$ (c) $\tau = 0.5, \beta = 10, \lambda_L = 0.0760, \lambda_U = 0.7360$ (d) $\tau = 0.5, \beta = 100, \lambda_L = 7.3929e - 009, \lambda_U = 0.7966$

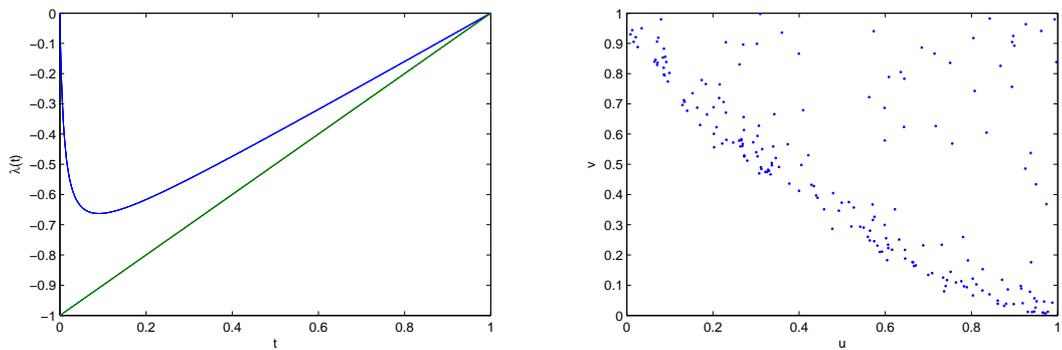
Figure 11: Comparable test space and simulations for $\tau = -0.5$ for $\lambda_{(RAT,2)}(t) = \frac{\alpha t(t-1)}{\beta t + \gamma}$.



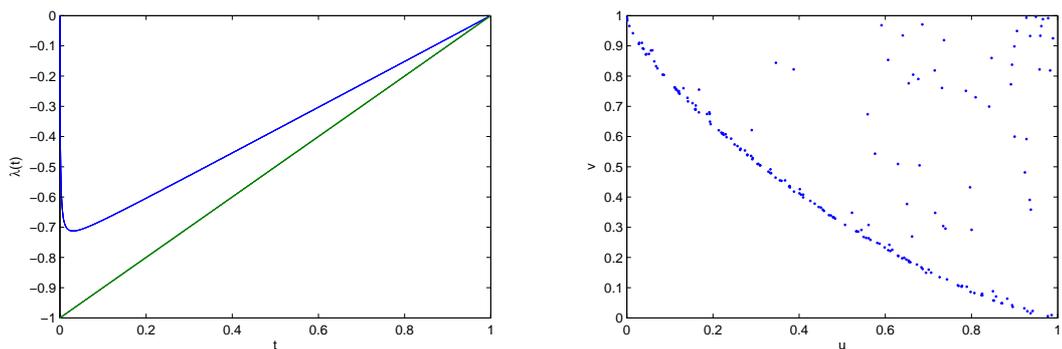
(a) $\tau = -0.5, \beta = 10.9, \lambda_L = 2.6411e - 004, \lambda_U = 8.3008e - 004$



(b) $\tau = 0.5, \beta = 20, \lambda_L = 1.6118e - 006, \lambda_U = 0.1139$



(c) $\tau = 0.5, \beta = 100, \lambda_L = 4.2310e - 025, \lambda_U = 0.2575$



(d) $\tau = 0.5, \beta = 1000, \lambda_L = 5.4890e - 232, \lambda_U = 0.3086$

R1: satisfied since $\lambda(0) = 0$

R2: satisfied since $\lambda(1) = 0$

R3: satisfied if $\alpha \leq 0$ and $\beta \geq 0$

R4: satisfied if

$$\alpha \in \left[\frac{-1-\sqrt{(1-\beta^2)}}{2}, \frac{-1+\sqrt{(1-\beta^2)}}{2} \right], \beta \in [-1, 1]$$

When combining R3 with R4 we get clear bounds for $\alpha \in \left[\frac{-1-\sqrt{(1-\beta^2)}}{2}, 0 \right]$ and $\beta \in [0, 1]$. From (2) we retrieve $\beta = -\tau + 1 + \alpha \in [0, 1]$ or $\tau - 1 \leq \alpha \leq \tau$. Combined this leads to the following bounds on α :

$$\max\left(\frac{1}{5}(\tau - 3 - 2\sqrt{1 + \tau - \tau^2}), \tau - 1\right) \leq \alpha \leq \min\left(\frac{1}{5}(\tau - 3 + 2\sqrt{1 + \tau - \tau^2}), \tau\right), \tau = \left[\frac{1-\sqrt{5}}{2}, 1\right] \quad (25)$$

$$\max\left(\frac{1}{5}(\tau - 3 - 2\sqrt{1 + \tau - \tau^2}), \tau - 1\right) \leq \alpha \leq \min(0, \tau), \tau = \left[-\frac{1}{2}, 1\right]. \quad (26)$$

All parameters can be written as a function of τ . As such, a comparable test space can be obtained by fixing $\tau \in \left[\frac{1-\sqrt{5}}{2}, 1\right]$ and by collecting λ functions according to the appropriate α range. Indeed, λ_L can generate strict copulas for both positive and negative dependence up to $\tau = -0.618$. In order to model copulas with degree of dependence $\tau < -0.618$ a more complex $\lambda_{(LOG,1)}$ is necessary, providing more curvature.

It can easily be checked that this family contains only copulas having no lower tail dependence, since for $\alpha \neq 0$ we have non-strict copulas, and for $\alpha = 0$ we have $\lambda_L = 0$ independent of the parameters. Upper tail dependence can be studied by means of

$$\lambda_U = 2 - 2^\beta. \quad (27)$$

The following generalizations concerning $\lambda_{(LOG,1)}$ can be made:

Property 3 Any λ function of the form $\lambda(t) = \beta_1 t(\ln(t))^1 + \beta_2 t(\ln(t))^2 + \dots + \beta_n t(\ln(t))^n$ will generate copula functions with zero lower tail dependence.

Property 4 Any λ function of the form $\lambda(t) = t(\ln(t)) + \beta_2 t(\ln(t))^2 + \dots + \beta_n t(\ln(t))^n$ will generate copula functions without tail dependence.

Property 5 Any λ function of the form $\lambda(t) = \beta_2 t(\ln(t))^2 + \dots + \beta_n t(\ln(t))^n$ will generate copula functions with $\lambda_U = 1$ and $\lambda_L = 0$.

The $\lambda_{(LOG,1)}$ function leads to the following generator, which is not explicitly invertible:

$$\varphi_{\alpha,\tau}(t) = \left[\left[\frac{\ln t}{\alpha \ln t + 1 + \alpha - \tau} \right]^2 \right]^{\frac{1}{2(1+\alpha-\tau)}}. \quad (28)$$

Next we turn to the modeling capacities of $\lambda_{(LOG,1)}$. Figure 13 and 14 show visuals of $\lambda_{(LOG,1)}$ and simulations from its associated copula in test spaces $\tau = 0.5$ and $\tau = -0.5$. In contrast to $\lambda_{(POL,1)}$ and $\lambda_{(RAT,1)}$, the log type λ has $C_0 = 0$ which is parameter independent. As a consequence notice the more relaxed scatterplot forms with respect to the zero curve of the $\lambda_{(LOG,1)}$ copulas. Indeed, although the copulas associated to $\lambda_{(LOG,1)}$, $\alpha \neq 0$ are non-strict, their form does not imply to actually observe a zero curve, which is the case for $\lambda_{(POL,1)}$ and $\lambda_{(RAT,1)}$. Hence, it can be argued that $\lambda_{(LOG,1)}$ copulas can be used to model real-life bivariate dependence structures.

Due to this fact we compare the modeling capacity of $\lambda_{(LOG,1)}$ with that of $\lambda_{(POL,2)}$ and $\lambda_{(RAT,2)}$ rather than with that of $\lambda_{(POL,1)}$ and $\lambda_{(RAT,1)}$. Clearly, since the former has no lower tail dependence we are interested in the possible (τ, λ_U) combinations for $\tau \in [-0.618, 1]$, and we compare it to the (τ, λ_U) combinations for $\lambda_{(RAT,2)}$ and $\lambda_{(POL,2)}$. In Figure 12 the three feasible regions for λ_U are displayed, given by the boundary functions $\lambda_{U,max}^{\lambda_{(RAT,2)}}$, $\lambda_{U,max}^{\lambda_{(POL,2)}}$, $\lambda_{U,max}^{\lambda_L}$, $\lambda_{U,min}^{\lambda_{(LOG,1)}}$.

A first observation involves the fact that $\lambda_{(RAT,2)}$ is the only λ type that can model λ_U for all $\tau \in [-1, 1]$, given the same parametrization. Next, notice that for $\tau \geq 0$ the function λ_L can model copulas having $\lambda_U = 1$, while the other λ type cannot, except for $\tau = 1$. A third observation focuses on the fact that $\lambda_{(LOG,1)}$ cannot generate copulas with $\lambda_U = 0$ for $\tau > 0$, while the other λ types can. A fourth observation is dedicated to the fact that the λ_U ranges for $\lambda_{(LOG,1)}$ and $\lambda_{(POL,2)}$ are complementary, as the boundary curves $\lambda_{U,max}^{\lambda_{(POL,2)}} = \lambda_{U,min}^{\lambda_{(LOG,1)}}$. Finally, notice that the feasible region of $\lambda_{(POL,2)}$ lies completely in the feasible region of $\lambda_{(RAT,2)}$.

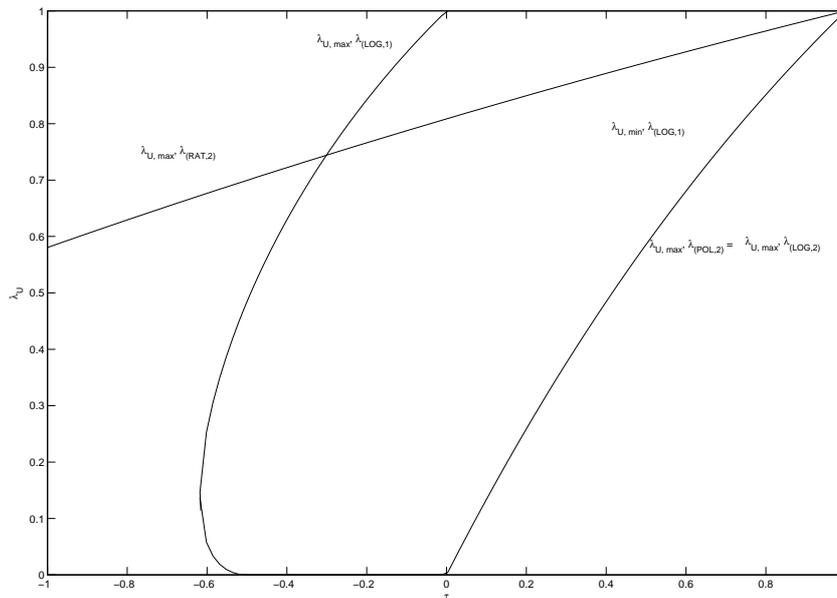


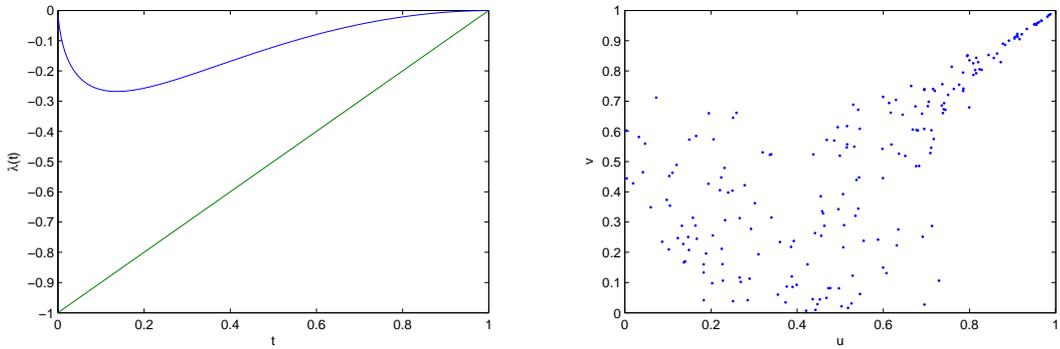
Figure 12: Overview of λ_U ranges for $\lambda_{(LOG,2)}(t) = \beta t^\alpha \ln(t)$, $\lambda_{(LOG,1)}(t) = \alpha t(\ln(t))^2 + \beta t \ln(t)$, $\lambda_{(POL,2)}(t) = t(t-1)(\alpha t + \beta)$ and $\lambda_{(RAT,2)}(t) = \frac{\alpha t(1-t)}{\beta t + 1}$ in the Kendall's tau space

4.3.2. Logarithmic λ type 2

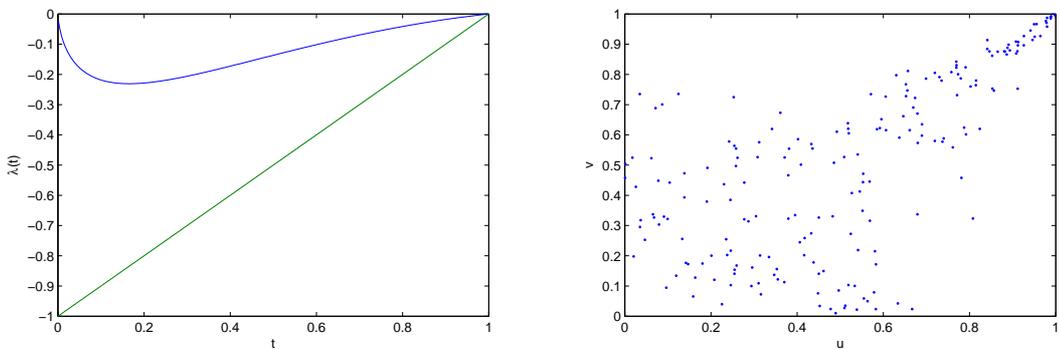
A second type of logarithmic λ involves a parametric extension of λ_{Π} . As an example we discuss a two parameter function of the form

$$\lambda_{(LOG,2)}(t) = \beta t^\alpha \ln(t)$$

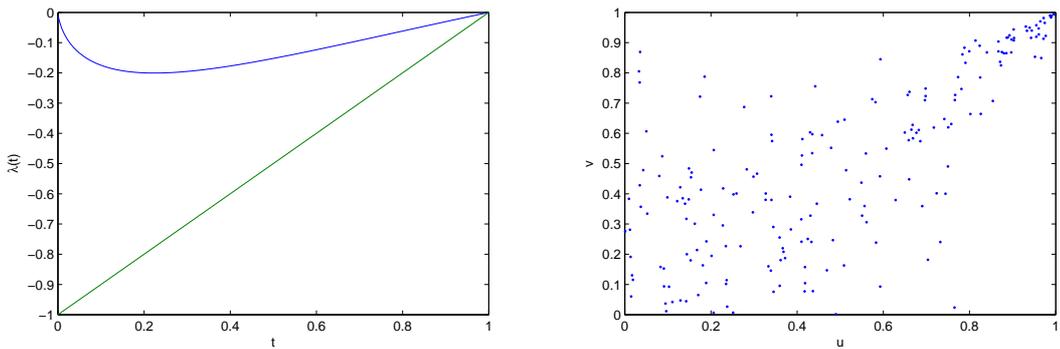
Figure 13: Comparable test space and simulations for $\tau = 0.5$ for $\lambda_{(LOG,1)}(t) = \alpha t(\ln(t))^2 + \beta t \ln(t)$.



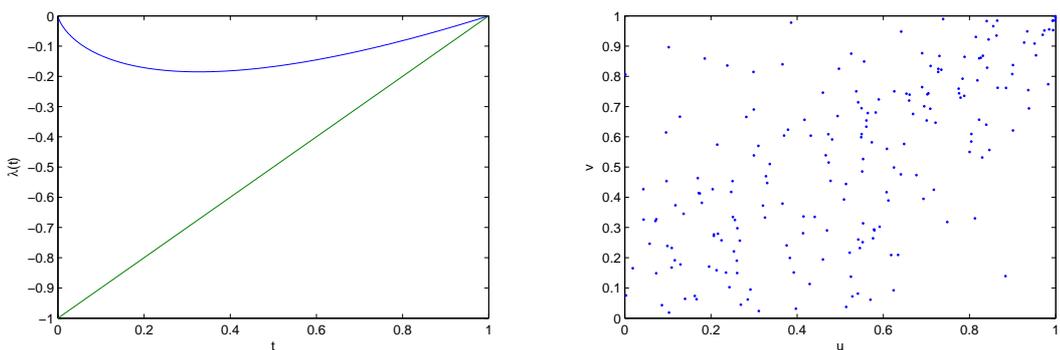
(a) $\tau = 0.5, \alpha = -0.5000, \lambda_U = 1$



(b) $\tau = 0.5, \alpha = -0.3509, \lambda_U = 0.8911$

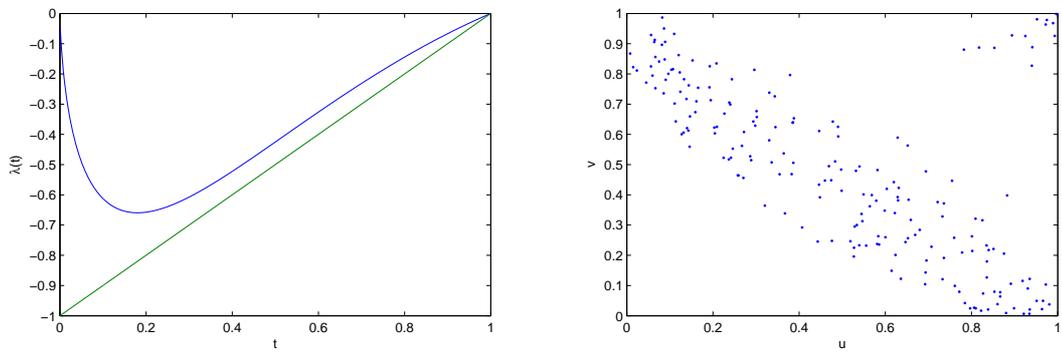


(c) $\tau = 0.5, \alpha = -0.2019, \lambda_U = 0.7705$

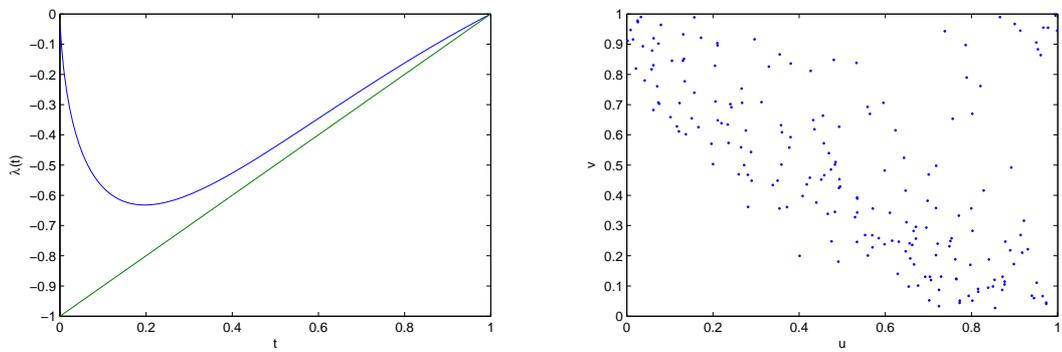


(d) $\tau = 0.5, \alpha = -0.0528, \lambda_U = 0.6366$

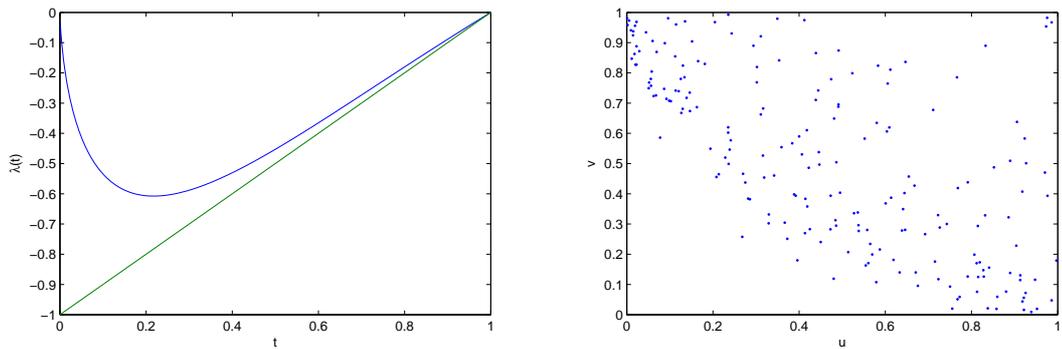
Figure 14: Comparable test space and simulations for $\tau = -0.5$ for $\lambda_{(LOG,1)}(t) = \alpha t(\ln(t))^2 + \beta t \ln(t)$.



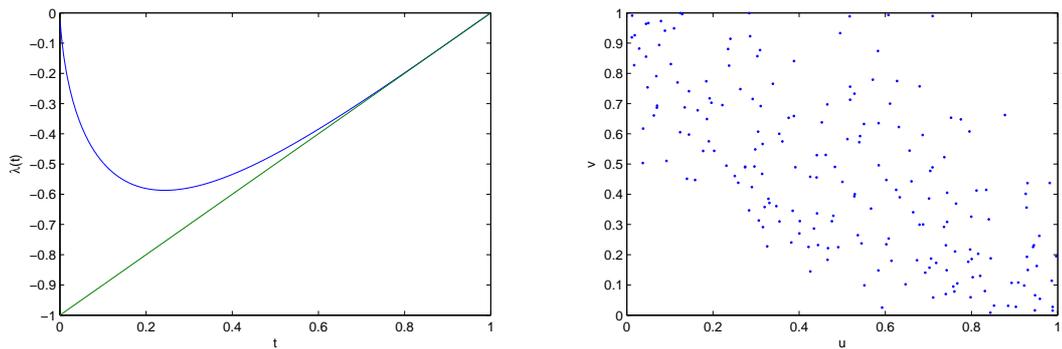
(a) $\tau = -0.5, \alpha = -0.9000, \lambda_U = 0.4843$



(b) $\tau = -0.5, \alpha = -0.7667, \lambda_U = 0.3376$



(c) $\tau = -0.5, \alpha = -0.6333, \lambda_U = 0.1765$



(d) $\tau = -0.5, \alpha = -0.5000, \lambda_U = 0$

where $\lambda(0)$ is defined as the limiting value $\lambda(0^+) = 0$. The function provides strict copulas for $\alpha \geq 1$ and non-strict copulas for $\alpha < 1$. (The strictness property can be checked by evaluating $\varphi(0)$ in (30).) For $\alpha = 1$ and $\beta = 1$ we have $\lambda_{(LOG,2)} = \lambda_{\Pi}$. As mentioned before we only discuss the strict part here, since $\lambda_{(LOG,1)}$ is mainly concerned with non-strict copulas.

When incorporating the restrictions on the λ we get

R1: satisfied since $\lambda(0) = 0, \alpha \neq 0$

R2: satisfied since $\lambda(1) = 0$

R3: satisfied if $\beta \geq 0$ since $t \ln(t) \leq 0, t \in [0, 1]$

R4: satisfied for any feasible (α, β) pair from $\beta t^{\alpha-1} + \alpha \beta \ln(t) t^{\alpha-1} \leq 0, t \in [0, 1]$

In order to obtain feasible parameter ranges for α and β respectively, 2 alternatives exist. The first alternative consists of solving R4 *numerically* and then combining R3 with R4 to obtain lower bounds of α in function of β . However, if one is interested in obtaining only strict copulas from $\lambda_{(LOG,2)}$ a second alternative exist. Following the strictness property we obtain the α lower bound and the appropriate α range is $[1, +\infty)$. An upper bound for β is established to evaluate R4 for $t = 1$ which yields $\lambda_U = 2 - 2^\beta$. As such the feasible β range is $[0, 1]$.

From (2) we retrieve $\tau = 1 + \frac{4\beta}{(\alpha+1)^2}$ and as such a comparable test space can be obtained by fixing τ and obtaining all λ functions for which $\alpha \geq 1$ and $\beta \in [0, 1]$. It can easily be checked that for $\alpha \geq 1$ the appropriate τ range is $[0, 1]$ so only positive dependence can be modeled by the strict part of $\lambda_{(LOG,2)}$.

Next tail dependence properties are discussed. It is not difficult to see that this family contains only copulas having perfect lower tail dependence, since for $\alpha \geq 1$ we have $\lim_{t \rightarrow 0} \lambda_{(LOG,2)}(t)' = 0 \implies \lambda_L = 1$. Upper tail dependence can be studied using

$$\lambda_U = 2 - 2^\beta. \quad (29)$$

The resulting generator is

$$\varphi_{\beta,\tau}(t) = e^{\frac{Ei\left(\ln(t)\left(2 - \sqrt{\frac{4\beta}{1-\tau}}\right)\right)}{\beta}} \quad (30)$$

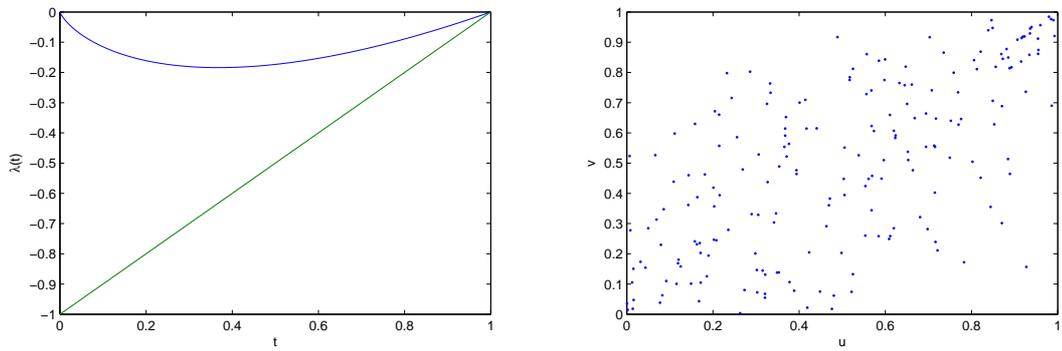
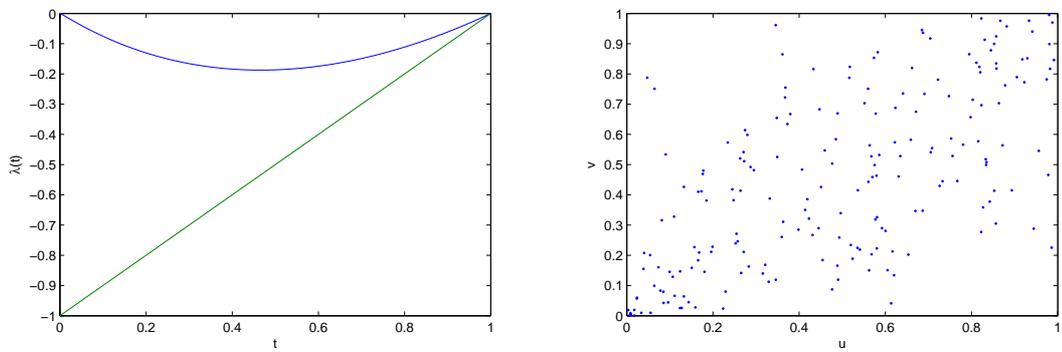
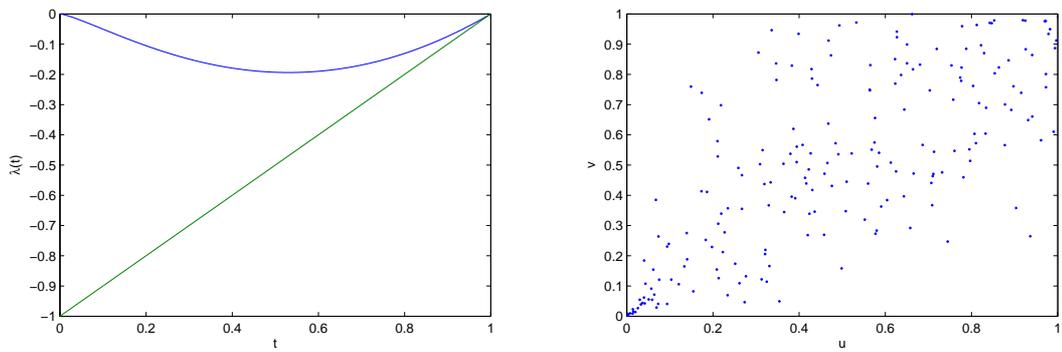
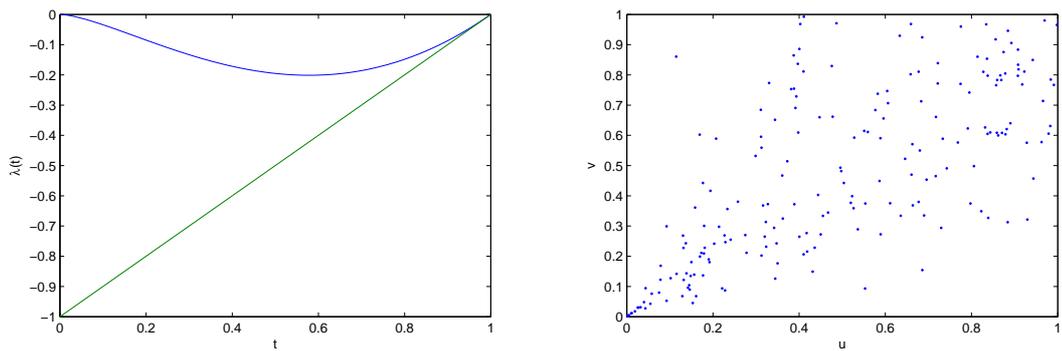
where Ei is defined as the exponential integral $Ei(x) = \int_{-\infty}^x \frac{e^t}{t} dt$. Clearly this generator is not analytically invertible.

Next we turn to the modeling capacities of $\lambda_{(LOG,2)}$. Figure 15 show visuals of $\lambda_{(LOG,2)}$ and simulations from its associated copula in test spaces $\tau = 0.5$. Again it can be seen how well the λ function captures the behaviour of the copula in a one dimensional way.

Next we investigate the modeling power of $\lambda_{(LOG,2)}$. Clearly, since $\lambda_L = 1$ is a constant we compare the possible (τ, λ_U) combinations for $\tau \in [0, 1]$ and compare it with $\lambda_{(POL,2)}$, $\lambda_{(RAT,2)}$ and $\lambda_{(LOG,1)}$. In Figure 12 the four feasible regions for λ_U are displayed, given by the boundary functions $\lambda_{Umax\lambda_{(POL,2)}} = \lambda_{Umax\lambda_{(LOG,2)}}$, $\lambda_{Umax\lambda_{(LOG,1)}}$, $\lambda_{Umin\lambda_{(LOG,1)}}$ and $\lambda_{Umax\lambda_{(RAT,2)}}$. It becomes clear that $\lambda_{(LOG,2)}$ can model the same λ_U range as $\lambda_{(POL,2)}$, making abstract of lower tail dependence. As such, these two families can be seen as complementary in the λ_L field. Finally we want to stress the fact that $\lambda_{(LOG,2)}$ entails the independence copula, which is an absolute advantage when e.g. a modeler would like to simplify a higher dimensional dependence problem by testing for conditional independence.

5. CONCLUSION

In the present contribution attention is drawn to a new and informative copula construction method, namely the λ construction method. It is shown how well the λ function captures essential characteristics of the associated copula and how straightforwardly parameters can be interpreted. The λ method is

Figure 15: Comparable test space and simulations for $\tau = 0.5$ for $\lambda_{(LOG,2)}(t) = \beta t^\alpha \ln(t)$ (a) $\tau = 0.5, \alpha = 1, \lambda_L = 1, \lambda_U = 0.5858$ (b) $\tau = 0.5, \alpha = 1.3094, \lambda_L = 1, \lambda_U = 0.4126$ (c) $\tau = 0.5, \alpha = 1.5820, \lambda_L = 1, \lambda_U = 0.2182$ (d) $\tau = 0.5, \alpha = 1.8284, \lambda_L = 1, \lambda_U = 0$

explored by discussing three functional types: polynomial functions $\lambda_{(POL,1)}$ and $\lambda_{(POL,2)}$, rational functions $\lambda_{(RAT,1)}$ and $\lambda_{(RAT,2)}$ and a log inspired functions $\lambda_{(LOG,1)}$ and $\lambda_{(LOG,2)}$. Examples of low complexity are used to explore the affiliated Archimedean copula spaces and to compare tail dependence properties between the λ types. Table 1 gives an overview of the three λ types and their characteristics.

The functions $\lambda_{(POL,1)}$ and $\lambda_{(RAT,1)}$ are defined as functions with intercept, implying non-strictness and positive C-measure on the zero curve. The function $\lambda_{(LOG,1)}$ also provides non-strict copulas but is defined without intercept. Consequently the dependence structures have a more relaxed form and real life dependence applications are not excluded. The functions $\lambda_{(POL,2)}$, $\lambda_{(RAT,2)}$ and $\lambda_{(LOG,1)}$ are defined to obtain strict generators and can be used for (higher) dependence modeling problems. A comparison of $\lambda_{(POL,2)}$ and $\lambda_{(RAT,2)}$ shows that the region of (λ_L, λ_U) combinations in the τ space for the latter is the larger for the same parametrization (see Figure 11). As such, $\lambda_{(RAT,2)}$ is more flexible than $\lambda_{(POL,2)}$ with respect to these characteristics. Moreover, the incapability of $\lambda_{(POL,2)}$ to model $\lambda_L=0$ has been generalized for n-parametric $\lambda_{(POL,2)}$. An advantage of both $\lambda_{(POL,2)}$ and $\lambda_{(RAT,2)}$ is that their tail dependence coefficients vary with τ . This is not the case for $\lambda_{L,1,2}$ where λ_L is a constant. The advantage the latter have, is that they include the independence copula Π . It turns out that some interesting generalizations for the n-parametric $\lambda_{(LOG,1)}$ type can be made concerning tail dependence properties, namely for $\lambda_{(LOG,1)}$ with no lower tail dependence, $\lambda_{(LOG,1)}$ with no tail dependence and $\lambda_{(LOG,1)}$ with perfect upper tail dependence. The modeling capacities for the four λ types concerning λ_U are compared in the τ space (12). Here it is shown that $\lambda_{(RAT,2)}$ is more flexible with respect to the τ space (it can model situations up to $\tau = -1$). Comparing $\lambda_{(RAT,2)}$ to $\lambda_{(LOG,1)}$ we see that $\lambda_{(LOG,1)}$ can model stronger upper tail dependence in the $\tau \in [0,1]$ region, but this is offset by its incapacity of modeling weak upper tail dependence. Comparing $\lambda_{(POL,2)}$ with $\lambda_{(LOG,1)}$ shows that both regions are complementary. When comparing $\lambda_{(POL,2)}$ with $\lambda_{(RAT,2)}$ we observe that the $\lambda_{(POL,2)}$ region lies completely in the $\lambda_{(RAT,2)}$ region (making abstract of λ_L). Finally, the modeling capacity of λ_U for $\lambda_{(LOG,2)}$ is the same as for $\lambda_{(POL,2)}$, excluding λ_L . As such, they can be seen as complementary in the λ_L field. Finally, we want to stress the extra advantage of the λ function for fitting applications. As such, future research will include the testing of the flexibility of the new multiparametric families in practice.

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λ type	$\lambda_{(POL,1)}$	$\lambda_{(RAT,1)}$	$\lambda_{(LOG,1)}$
function	$\alpha + \beta t + \gamma t^2$	$\frac{\alpha(1-t)}{\beta t+1}$	$\alpha t(\ln(t))^2 + \beta t \ln(t)$
τ	$\frac{2}{3}(6\alpha + 3\beta + 2\gamma) + 1$	$1 + 4\alpha \left \ln 1 + \beta \frac{1+\beta}{\beta^2} - \frac{1}{\beta} \right $	$1 + \alpha - \beta$
τ range	$[-1, 1]$	$[-1, 1]$	$[-0.681, 1]$
strictness	non-strict for $\alpha \neq 0$ strict for $\alpha = 0, \tau \in [\frac{1}{3}, 1]$	non-strict	strict for $\alpha = 0$
C_0	$ \alpha $	$ \alpha $	0
λ_L	$2^{3(\tau-1)}, \tau \in [\frac{1}{3}, 1]$	0	0
λ_U	$2 - 2^{\frac{3}{2}(1-\tau)+2\alpha}$	$2 - 2^{-\frac{\alpha}{\beta+1}}$	$2 - 2^\beta$
λ_{II}	no	no	$\alpha = 0, \beta = 1$
λ type	$\lambda_{(POL,2)}$	$\lambda_{(RAT,2)}$	$\lambda_{(LOG,2)}$
function	$t(t-1)(\alpha t + \beta)$	$\frac{\alpha t(t-1)}{\beta t+1}$	$\beta t^\alpha \ln(t)$
τ	$1 - \frac{2\beta+\alpha}{3}$	$1 + \frac{4\alpha}{\beta^2} \left[-\frac{\beta}{2} - 1 + \ln 1 + \beta \frac{(1+\beta)}{\beta} \right]$	$1 + \frac{4\beta}{(\alpha+1)^2}$
τ range	$[0, 1]$	$[-1, 1]$	$[0, 1], \alpha \geq 0$
strictness	strict	strict	strict for $\alpha \geq 1$
C_0	0	0	0
λ_L	$2^{\frac{3(\tau-1)+\alpha}{2}}$	$2^{-\alpha}$	1
λ_U	$2 - 2^{\frac{3(1-\tau)+\alpha}{2}}$	$2 - 2^{\frac{\alpha}{\beta+1}}$	$2 - 2^\beta$
λ_{II}	no	no	$\alpha = 1, \beta = 1$

Table 1: Overview of the 6 multiparametric Archimedean copula families based on $\lambda_{(POL,1)}$, $\lambda_{(POL,2)}$, $\lambda_{(RAT,1)}$, $\lambda_{(RAT,2)}$, $\lambda_{(LOG,1)}$ and $\lambda_{(LOG,2)}$.

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