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Uniqueness of completion for metrically generated constructs ☆

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Abstract

In this paper, for metrically generated constructs $\mathbf{X}$ in the sense of [E. Colebunders, R. Lowen, Metrically generated theories, Proc. Amer. Math. Soc. 133 (2005) 1547–1556] we study completion as a $\mathcal{U}$-reflector R on the subconstruct $\mathbf{X}_0$ of all T_0-objects, for $\mathcal{U}$ some class of embeddings. Roughly speaking we deal with constructs $\mathbf{X}$ that are generated by the subclass of their metrizable objects and for various types of completion functors R available in that context, we obtain internal descriptions of the largest class $\mathcal{U}$ for which completion is unique. We apply our results to some well known situations. Completion of uniform spaces, of proximity spaces or of non-Archimedian uniform spaces is unique with respect to the class of all epimorphic embeddings, and this class is the largest one. However the largest class of morphisms for which Dieudonné completion of completely regular spaces or of zero dimensional spaces is unique, is strictly smaller than the class of all epimorphic embeddings. The same is true for completion in quantitative theories like uniform approach spaces for which the largest $\mathcal{U}$ coincides with the class of all embeddings that are dense with respect to the metric coreflection. Our results on completion for metrically generated constructs explain these differences.

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1. Introduction

For the subconstruct $\mathbf{X}_0$ consisting of all T_0-objects in some topological construct $\mathbf{X}$, we consider a reflective full subconstruct $\mathbf{R}$ with reflection functor $R:\mathbf{X} \to \mathbf{R}$. For some class $\mathcal{U}$ of morphisms satisfying

- $\mathcal{U}$ is closed under composition,
- $\mathbf{R}$ is $\mathcal{U}$-reflective (in the sense that the reflection morphisms r_X belong to $\mathcal{U}$)

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uniqueness of the reflection with respect to $\mathcal{U}$ means that

- whenever $f: X \to Y$ is in $\mathcal{U}$ with Y in **R**, there exists a unique isomorphism $h: R(X) \to Y$ such that $h \circ r_X = f$.

If $\mathcal{U}$ is a class satisfying these three conditions with respect to the reflective subconstruct **R** then **R** is said to be subfirmly $\mathcal{U}$-reflective.

We denote by $\mathcal{L}(R)$ the class of all morphisms $u: X \to Y$ for which $R(u)$ is an isomorphism. As is well known [1] $\mathcal{L}(R)$ is the largest class $\mathcal{U}$ of morphisms satisfying the three conditions above. **R** is said to be firmly reflective with respect to $\mathcal{L}(R)$.

If we make the further assumption that

- the reflection $R(X)$ of an object X is an extension, i.e. the reflection morphism r_X is an embedding

then we say that the reflection is a completion. Similar to Lemma 1.12 in [1], in the case of a completion the following inclusion can be shown to hold

$$\mathcal{L}(R) \subset \mathrm{EpiEmb}_{\mathbf{X}_0},$$

where $\mathrm{EpiEmb}_{\mathbf{X}_0}$ consists of all epimorphic embeddings in $\mathbf{X}_0$. In some well known examples the class $\mathcal{L}(R)$ is equal to the class $\mathrm{EpiEmb}_{\mathbf{X}_0}$. This is for instance the case for the usual completion functor for metric or for uniform spaces. As we will see this inclusion needs not be an equality in general.

In this paper, for metrically generated constructs **X** in the sense of [7] we study completion as a reflector R on $\mathbf{X}_0$ and for various types of completion functors R available in that context, we obtain internal descriptions for the class $\mathcal{L}(R)$. Roughly speaking we deal with constructs **X** that are generated by the subclass of their metrizable objects. A more precise definition is formulated in the next paragraph. Particular examples we think of are the well known constructs **Unif**, **Prox**, **Creg**, **ZDim**, consisting of all uniform spaces, all proximity spaces, all completely regular spaces or all zero-dimensional spaces respectively, and all of their quantitative counterparts, like for instance **UAp** the construct of uniform approach spaces [13,19].

We obtain our results basically starting from two standard examples, the usual completion functor for uniform spaces for which $\mathcal{L}(R) = \mathrm{EpiEmb}_{\mathbf{X}_0}$ and the usual completion functor for uniform approach spaces for which $\mathcal{L}(R)$ coincides with the class of all embeddings that are dense with respect to the metric coreflection [13]. We prove that these basic constructions carry over to arbitrary metrically generated constructs and we make a comparison of the different completions obtained. We recover many examples, like for instance the reflector on **Creg** to the class of Dieudonné complete spaces [9]. We characterize $\mathcal{L}(R)$ in all these examples and again it follows from our general techniques that the results go through when one restricts to a zero-dimensional or a totally bounded context or when one generalizes to quantified versions of these.

2. Preliminaries

As mentioned in the introduction, the framework we will be working in is that of metrically generated constructs as introduced in [7]. In this section we gather the preliminary material that is needed.

A function $d: X \times X \to [0, \infty]$ is called a quasi-pre-metric if it is zero on the diagonal, we will drop "pre" if d satisfies the triangle inequality and we will drop "quasi" if d is symmetric. Denote by **Met** the construct of quasi-pre-metric spaces and contractions (a map $f: (X, d) \to (X', d')$ is a contraction if $d' \circ f \times f \leqslant d$) and by **Met**$(X)$ the fiber of **Met** structures on X.

A *base category* $\mathcal{C}$ is a full and isomorphism-closed concrete subconstruct of **Met** which is closed for initial morphisms and contains all **Met** indiscrete spaces [7]. In particular we think of the base category consisting of all metric spaces, denoted by $\mathcal{C}^{\Delta,s}$. In this paper we consider only base categories satisfying the following supplementary conditions.

- $\mathcal{C}$ consists of metric spaces, i.e. $\mathcal{C} \subset \mathcal{C}^{\Delta,s}$.
- $\mathcal{C}$ is closed under dense embeddings in $\mathcal{C}^{\Delta,s}$ in the sense that whenever $f: (X, d) \to (Y, d')$ is a $\mathcal{T}_{d'}$-dense embedding in $\mathcal{C}^{\Delta,s}$ with (X, d) belonging to $\mathcal{C}$ then also (Y, d') belongs to $\mathcal{C}$.

The second condition is similar to B6 considered in [6], where it was already observed that the base category consisting of all totally bounded metric spaces or the base category consisting of all ultrametric spaces both satisfy the supplementary conditions listed here. If (X,d) is a $\mathcal{C}$-object, we call d a $\mathcal{C}$-metric.

Given a base category $\mathcal{C}$, a topological construct **X** is called *$\mathcal{C}$-metrically generated* if there exists a concrete functor $K:\mathcal{C}\to\mathbf{X}$ such that K preserves initial morphisms and $K(\mathcal{C})$ is initially dense in **X**.

We now recall that there exists a model category for all $\mathcal{C}$-metrically generated constructs. For any collection $\mathcal{D}$ of quasi-pre-metrics on a set X we put

$$\mathcal{D}\downarrow := \{e\in\mathbf{Met}(X)\mid \exists d\in\mathcal{D}\colon\ e\leqslant d\}.$$

A downset in $\mathbf{Met}(X)$ is a non-empty subset $\mathcal{D}$ such that $\mathcal{D}\downarrow=\mathcal{D}$. We say that a subset $\mathcal{B}$ of $\mathbf{Met}(X)$ is a *basis* for $\mathcal{D}$ if $\mathcal{B}\downarrow=\mathcal{D}$.

$\mathbf{M}^{\mathcal{C}}$ is the construct with objects, pairs $(X,\mathcal{D})$ where X is a set and $\mathcal{D}$ is a downset in $\mathbf{Met}(X)$ which has a basis consisting of $\mathcal{C}$-metrics. $\mathcal{D}$ is called a *$\mathcal{C}$-meter (on X)* and $(X,\mathcal{D})$ a *$\mathcal{C}$-metered space*. For brevity in notation $\mathbf{M}^{\mathcal{C}^{\Delta,s}}$ will shortly be denoted $\mathbf{M}^{\Delta,s}$.

If $(X,\mathcal{D})$ and $(X',\mathcal{D}')$ are $\mathcal{C}$-metered spaces and $f:(X,\mathcal{D})\to(X',\mathcal{D}')$ then we say that f is a *contraction* if d' in $\mathcal{D}'$ implies $d'\circ f\times f\in\mathcal{D}$.

Concretely coreflective subconstructs of $\mathbf{M}^{\mathcal{C}}$ play an important role in this theory and in order to describe them we recall the notion of an expander from [7]. We call ξ an *expander* on $\mathbf{M}^{\mathcal{C}}$ if for any X and any $\mathcal{C}$-meter $\mathcal{D}$, ξ provides us with a $\mathcal{C}$-meter $\xi(\mathcal{D})$ in such a way that the following properties are fulfilled:

- $\mathcal{D}\subset\xi(\mathcal{D})$,
- $\mathcal{D}\subset\mathcal{D}'\Rightarrow\xi(\mathcal{D})\subset\xi(\mathcal{D}')$,
- $\xi(\xi(\mathcal{D}))=\xi(\mathcal{D})$,
- if $f:Y\to X$ and $\mathcal{D}\in\mathbf{M}^{\mathcal{C}}(X)$, then: $\xi(\mathcal{D})\circ f\times f\subset\xi(\mathcal{D}\circ f\times f\downarrow)$.

Given an expander ξ on $\mathbf{M}^{\mathcal{C}}$, we define $\mathbf{M}^{\mathcal{C}}_{\xi}$ as the full coreflective subconstruct of $\mathbf{M}^{\mathcal{C}}$ with objects those metered spaces $(X,\mathcal{D})$ for which $\xi(\mathcal{D})=\mathcal{D}$.

The main result of [7] states that a topological construct is $\mathcal{C}$-metrically generated if and only if it is concretely isomorphic to $\mathbf{M}^{\mathcal{C}}_{\xi}$ for some expander ξ on **M**.

Remark that for any given expander ξ on $\mathbf{M}^{\Delta,s}$, there exists an adapted version $\xi^{\mathcal{C}}$ on $\mathbf{M}^{\mathcal{C}}$ defined by $\xi^{\mathcal{C}}(\mathcal{D})=\{d\in\xi(\mathcal{D})\mid d\ \mathcal{C}\text{-metric}\}\downarrow$. This adapted version can be thought of as a sort of restriction although it is not a restriction in the strict sense since the categories in question are not comparable. Again for simplicity in notation, when writing $\mathbf{M}^{\mathcal{C}}_{\xi}$, ξ is implicitly meant to be the adapted version.

At the end of the paper, in the section devoted to examples, we give the explicit formulation of several expanders on $\mathbf{M}^{\Delta,s}$. When applied to $\mathbf{M}^{\Delta,s}$ these expanders give rise to coreflective subconstructs $\mathbf{M}^{\Delta,s}_{\xi}$ that are isomorphic to **Top**, **UAp** [13], **Unif**, **UG** [14] and $\mathcal{C}^{\Delta,s}$. Here we just give one example, ι, which we will need right away:

$$e\in\iota(\mathcal{D})\quad\text{iff}\quad d\leqslant\sup_{e\in\mathcal{E}}e,\quad\text{for a finite }\mathcal{E}\subset\mathcal{D}.$$

We assume that all the expanders appearing in this paper satisfy two more technical conditions taken from [5].

- $\iota\leqslant\xi$,
- $\xi(\{\mathbf{0}\})=\{\mathbf{0}\}$, where $\mathbf{0}$ is the zero metric.

It should be noted that these conditions are very mild and satisfied in most interesting cases.

For a given expander ξ on $\mathbf{M}^{\mathcal{C}}$, an $\mathbf{M}^{\mathcal{C}}_{\xi}$-object $(X,\mathcal{D})$ is a T_0 object (in the sense that all morphisms from indiscrete objects to $(X,\mathcal{D})$ are constant, [16]) if and only if

$$\forall x,y\in X, x\neq y, \exists d\in\mathcal{D}\colon\ d(x,y)\neq 0.$$

The subcategory of $\mathbf{M}^{\mathcal{C}}_{\xi}$ consisting of all T_0-objects is denoted by $\mathbf{M}^{\mathcal{C}}_{\xi_0}$.

3. Completeness in $\mathbf{M}_{\iota 0}^{\mathcal{C}}$

In this section, given an arbitrary base category $\mathcal{C}$, we will build two different completion theories in the category $\mathbf{M}_{\iota 0}^{\mathcal{C}}$.

The first completeness notion is based on the transition from an arbitrary $\mathbf{M}_{\iota}^{\mathcal{C}}$-object $(X,\mathcal{D})$ to the uniform space $(X,\mathcal{U}\mathcal{D})$. Choosing an arbitrary basis $\mathcal{P}$ for $\mathcal{D}$, consisting of $\mathcal{C}$ metrics, $(X,\mathcal{U}\mathcal{D})$ is generated by $\mathcal{P}$ in the usual way, this means by taking $V_\epsilon^p = \{(x,y) \mid p(x,y) < \epsilon\}$ with $p \in \mathcal{P}$ and $\epsilon > 0$ as subbasic sets.

3.1. Definition. An object $(X,\mathcal{D})$ in $\mathbf{M}_{\iota}^{\mathcal{C}}$ is uniformly complete if the associated space $(X,\mathcal{U}\mathcal{D})$ is complete in the usual sense.

The second completeness notion we consider is based on another transition to uniform spaces. Starting from an arbitrary $\mathbf{M}_{\iota}^{\mathcal{C}}$-object $(X,\mathcal{D})$ we first associate with it the metered space $(X,\{\bigvee\mathcal{D}\}\downarrow)$ and then as before, go over to the associated uniform space $(X,\mathcal{U}\{\bigvee\mathcal{D}\}\downarrow)$.

3.2. Definition. An object $(X,\mathcal{D})$ in $\mathbf{M}_{\iota}^{\mathcal{C}}$ is metrically complete if the associated space $(X,\mathcal{U}\{\bigvee\mathcal{D}\}\downarrow)$ is complete in the usual sense.

Our purpose in the next paragraph is to prove that both completeness notions give rise to reflective subconstructs of $\mathbf{M}_{\iota 0}^{\mathcal{C}}$. In order to avoid repetition of the arguments, we will use a common notation for both constructions. We use $h\mathcal{D}$ for the transformation of $\mathcal{D}$, $\mathcal{U}h\mathcal{D}$ for the associated uniformity and $\mathcal{T}h\mathcal{D}$ for its associated topology. Using this notation, h is the identity in Definition 3.1 and $h\mathcal{D} = \{\bigvee\mathcal{D}\}\downarrow$ in Definition 3.2. We use the terminology "h-complete" to describe the completeness notions defined in 3.1 and 3.2.

3.3. Proposition. *In both cases h satisfies the following properties*:

H1. *h is expansive*: $\mathcal{D} \subset h(\mathcal{D})$;
H2. *h is monotone*: $\mathcal{D} \subset \mathcal{D}'$ *implies* $h\mathcal{D} \subset h\mathcal{D}'$;
H3. *h reflects T_0-objects*: $(X,h\mathcal{D})$ *is* T_0 *implies* $(X,\mathcal{D})$ *is* T_0;
H4. $U_h : \mathbf{M}_{\iota}^{\mathcal{C}} \to \mathbf{Unif} : (X,\mathcal{D}) \to (X,\mathcal{U}h\mathcal{D})$ *is a functor preserving subspaces.*

Proof. The only non-trivial part is H4. For h the identity it suffices to observe that **Unif** is isomorphic to some construct that is coreflectively embedded in $\mathbf{M}_{\iota}^{\Delta,s}$. Moreover $\mathbf{M}_{\iota}^{\mathcal{C}}$ is a full subconstruct of $\mathbf{M}_{\iota}^{\Delta,s}$ which is closed under subspaces.

For $h\mathcal{D} = \{\bigvee\mathcal{D}\}\downarrow$ observe that $\mathbf{M}^{\Delta,s} \to \mathcal{C}^{\Delta,s} : (X,\mathcal{D}) \mapsto (X,h\mathcal{D})$, describes a coreflector. □

Since our metrics are allowed to take the value infinity, we have to agree on the structure we put on $[0,+\infty]$. Equip $[0,+\infty]$ with the metric d_E which on $\mathbb{R}^+$ coincides with the Euclidean metric, which is zero on the diagonal and where $d_E(x,\infty) = d_E(\infty,x) = \infty$, for every $x \in \mathbb{R}^+$. Clearly the associated uniformity $(X,\mathcal{U}_{d_E})$ of $([0,+\infty],d_E)$ is a complete T_0 space.

The next result now follows as in the classical real valued case.

3.4. Proposition. *For every metric p we have that* $p : (X \times X, \mathcal{U}_p \times \mathcal{U}_p) \to ([0,+\infty],\mathcal{U}_{d_E})$ *is uniformly continuous.*

In order to fix notation, starting with $(X,\mathcal{D}) \in \mathbf{M}_{\iota 0}^{\mathcal{C}}$, let $(\widetilde{X},\widetilde{\mathcal{U}h\mathcal{D}})$ be the completion of the T_0 uniform space $(X,\mathcal{U}h\mathcal{D})$. The set $\widetilde{X}$ consists of the minimal $\mathcal{U}h\mathcal{D}$-Cauchy filters.

The map $i : (X,\mathcal{U}h\mathcal{D}) \to (\widetilde{X},\widetilde{\mathcal{U}h\mathcal{D}}) : x \mapsto \mathcal{V}(x)$, which sends a point x to the neighborhood filter of x with respect $(X,\mathcal{T}h\mathcal{D})$, is an epimorphic embedding in $\mathbf{Unif}_0$.

3.5. Proposition. *Let $(X,\mathcal{D}) \in \mathbf{M}^{\mathcal{C}}_{\iota 0}$ and let $\mathcal{P}$ be a basis for $\mathcal{D}$ consisting of $\mathcal{C}$-metrics. For every $p \in \mathcal{P}$ there is a unique uniformly continuous $\mathcal{C}$-metric $\tilde{p}$ such that the diagram*

$$\begin{array}{ccc}
(\widetilde{X},\widetilde{\mathcal{U}h\mathcal{D}}) \times (\widetilde{X},\widetilde{\mathcal{U}h\mathcal{D}}) & & \\
{\scriptstyle i\times i}\uparrow & \searrow^{\tilde{p}} & \\
(X,\mathcal{U}h\mathcal{D}) \times (X,\mathcal{U}h\mathcal{D}) & \xrightarrow[p]{} & ([0,+\infty],\mathcal{U}_{d_E})
\end{array}$$

commutes.

Proof. From H1 we get that the uniformity $\mathcal{U}_p$ on X is coarser than the uniformity $\mathcal{U}h\mathcal{D}$. So clearly p is uniformly continuous and therefore it has a unique uniformly continuous extension $\tilde{p}$. Moreover $\tilde{p}$ is a metric and $i:(X,p) \to (\widetilde{X},\tilde{p})$ is dense. By the assumptions made on the base category and the fact that p is a $\mathcal{C}$-metric it follows that $\tilde{p}$ is a $\mathcal{C}$-metric too. □

Next we put

$$\widetilde{\mathcal{D}} := \{\tilde{p} \mid p \in \mathcal{P}\}\downarrow.$$

Since the extensions are created by continuity and since X is dense in its extension it easily follows that $\widetilde{\mathcal{D}}$ is an ideal. So $(\widetilde{X},\widetilde{\mathcal{D}})$ is an object in $\mathbf{M}^{\mathcal{C}}_{\iota}$.

3.6. Proposition. *For both cases of h we have the following property*

H5. *For every $(X,\mathcal{D}) \in \mathbf{M}^{\mathcal{C}}_{\iota 0}$: $(\widetilde{X},\mathcal{U}h\widetilde{\mathcal{D}}) = (\widetilde{X},\widetilde{\mathcal{U}h\mathcal{D}})$.*

Proof. In the case that h is the identity the statement is exactly the claim that $\widetilde{\mathcal{D}}$ is the gauge of the uniform completion of $(X,\mathcal{U}\mathcal{D})$ [11].

In the case where $h\mathcal{D} = \{\bigvee\mathcal{D}\}\downarrow$ first observe that for $\mathcal{P}$ a basis of $\mathcal{C}$-metrics for $\mathcal{D}$ and $p \in \mathcal{P}$ we have that $\tilde{p} \leqslant \widetilde{\bigvee\mathcal{D}}$. This implies that $\bigvee_{p\in\mathcal{P}}\tilde{p} \leqslant \widetilde{\bigvee\mathcal{D}}$ and so both are uniformly continuous for $\mathcal{U}\,\widetilde{\bigvee\mathcal{D}}\downarrow$. Since they coincide on the dense subset $X \times X$, they are equal. □

3.7. Proposition. *Let $(X,\mathcal{D})$ be in $\mathbf{M}^{\mathcal{C}}_{\iota 0}$. Then $(\widetilde{X},\widetilde{\mathcal{D}})$ is an $\mathbf{M}^{\mathcal{C}}_{\iota 0}$-object which is h-complete.*

Proof. We still have to check that $(\widetilde{X},\widetilde{\mathcal{D}})$ is a T_0-object. The uniform structures $\widetilde{\mathcal{U}h\mathcal{D}}$ and $\mathcal{U}h\widetilde{\mathcal{D}}$ on $\widetilde{X}$ coincide by H5, so $(\widetilde{X},h\widetilde{\mathcal{D}})$ is T_0. Since h reflects the T_0 property it follows that $(\widetilde{X},\widetilde{\mathcal{D}})$ is a T_0-object. Finally, by H5 $(\widetilde{X},\widetilde{\mathcal{D}})$ clearly is h-complete. □

4. Completeness as a reflection

In this paragraph we prove that both completeness notions introduced in the previous section define a reflective subconstruct of $\mathbf{M}^{\mathcal{C}}_{\iota 0}$. As before h stands for either the identity or $h\mathcal{D} = \{\bigvee\mathcal{D}\}\downarrow$. We denote by $h\mathbf{M}^{\mathcal{C}}_{\iota 0}$ the full subconstruct of $\mathbf{M}^{\mathcal{C}}_{\iota 0}$ consisting of all h-complete objects.

Let $(X,\mathcal{D}) \in \mathbf{M}^{\mathcal{C}}_{\iota 0}$. We denote

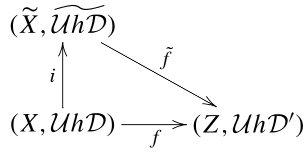
$$r_X : (X,\mathcal{D}) \to (\widetilde{X},\widetilde{\mathcal{D}}) : x \mapsto \mathcal{V}(x),$$

where as before, for x in X, $\mathcal{V}(x)$ denotes the neighborhood filter of x with respect to the underlying topology $\mathcal{T}h\mathcal{D}$ of $\mathcal{U}h\mathcal{D}$. Clearly in view of H5 $r_X : (X,\mathcal{D}) \to (\widetilde{X},\widetilde{\mathcal{D}}) : x \mapsto \mathcal{V}(x)$ is a $\mathcal{T}h\widetilde{\mathcal{D}}$-dense embedding in $\mathbf{M}^{\mathcal{C}}_{\iota 0}$.

4.1. Theorem. *$h\mathbf{M}^{\mathcal{C}}_{\iota 0}$ is reflective in $\mathbf{M}^{\mathcal{C}}_{\iota 0}$.*

Proof. Let $(X,\mathcal{D}) \in \mathbf{M}^{\mathcal{C}}_{\iota 0}$. We will prove that $r_X : (X,\mathcal{D}) \to (\widetilde{X},\widetilde{\mathcal{D}})$ is an $h\mathbf{M}^{\mathcal{C}}_{\iota 0}$-reflection arrow.

Let $f:(X,\mathcal{D}) \to (Z,\mathcal{D}')$ be a contraction in $\mathbf{M}_{\iota_0}^{\mathcal{C}}$, where $(Z,\mathcal{D}')$ is an h-complete object. Then $f:(X,\mathcal{U}h\mathcal{D}) \to (Z,\mathcal{U}h\mathcal{D}')$ is uniformly continuous, so there exists a unique uniformly continuous extension $\tilde{f}:(\widetilde{X},\mathcal{U}h\widetilde{\mathcal{D}}) \to (Z,\mathcal{U}h\mathcal{D}')$ such that the following diagram in **Unif** commutes



For every minimal Cauchy filter $\mathcal{M} \in \widetilde{X}$ the value $\tilde{f}(\mathcal{M})$ is the limit of $f(i^{-1}(\mathcal{V}(\mathcal{M})|_{i(X)}))$ with respect to the topology $\mathcal{T}h\mathcal{D}'$. We now check that $\tilde{f}:(\widetilde{X},\widetilde{\mathcal{D}}) \to (Z,\mathcal{D}')$ is a contraction. Let $d' \in \mathcal{D}'$, then it follows from Theorem 3.4 that

$$d':(Z,\mathcal{T}h\mathcal{D}') \times (Z,\mathcal{T}h\mathcal{D}') \to \big([0,\infty],\mathcal{T}_{d_E}\big)$$

is continuous.

Let $\mathcal{M},\mathcal{N} \in \widetilde{X}$. Then $\tilde{f}(\mathcal{M}) = \lim_{\mathcal{T}h\mathcal{D}'} f(i^{-1}(\mathcal{V}(\mathcal{M})|_{i(X)}))$ and similarly $\tilde{f}(\mathcal{N}) = \lim_{\mathcal{T}h\mathcal{D}'} f(i^{-1}(\mathcal{V}(\mathcal{N})|_{i(X)}))$. Applying the continuity of d' it follows that

$$\begin{aligned} d' \circ \tilde{f} \times \tilde{f}(\mathcal{M},\mathcal{N}) &= d'\big(\lim_{\mathcal{T}h\mathcal{D}'} f\big(i^{-1}\big(\mathcal{V}(\mathcal{M})|_{i(X)}\big)\big), \lim_{\mathcal{T}h\mathcal{D}'} f\big(i^{-1}\big(\mathcal{V}(\mathcal{N})|_{i(X)}\big)\big)\big) \\ &= \lim_{\mathcal{T}_{d_E}} d'\big(f\big(i^{-1}\big(\mathcal{V}(\mathcal{M})|_{i(X)}\big)\big) \times f\big(i^{-1}\big(\mathcal{V}(\mathcal{N})|_{i(X)}\big)\big)\big) \\ &= \lim_{\mathcal{T}_{d_E}} d' \circ f \times f\big(i^{-1}\big(\mathcal{V}(\mathcal{M})|_{i(X)}\big) \times i^{-1}\big(\mathcal{V}(\mathcal{N})|_{i(X)}\big)\big) \\ &= \widetilde{d' \circ f \times f}(\mathcal{M},\mathcal{N}). \end{aligned}$$

Finally observe that the uniqueness of $\tilde{f}$ follows from the uniqueness of the extension in $\mathbf{Unif}_0$. □

5. Description of the class $\mathcal{L}(R)$

As before, given h we consider the reflective subconstruct of $\mathbf{M}_{\iota_0}^{\mathcal{C}}$ consisting of all h-complete objects and we denote the reflector by $R_h:\mathbf{M}_{\iota_0}^{\mathcal{C}} \to h\mathbf{M}_{\iota_0}^{\mathcal{C}}$.

In this paragraph we deal with the main question posed in the introduction, to find an internal characterization of the morphism class $\mathcal{L}(R)$ that is the largest class $\mathcal{U}$ satisfying:

- $\mathcal{U}$ is closed under composition;
- $\mathbf{R} = h\mathbf{M}_{\iota_0}^{\mathcal{C}}$ is $\mathcal{U}$-reflective in the sense that the reflection morphisms belong to $\mathcal{U}$;
- Reflection is unique with respect to $\mathcal{U}$ in the sense that whenever $f:X \to Y$ is in $\mathcal{U}$ with Y in $\mathbf{R} = h\mathbf{M}_{\iota_0}^{\mathcal{C}}$, there exists a unique isomorphism $k:R(X) \to Y$ such that $k \circ r_X = f$.

Using the terminology from [1] it means we are looking for the *firm* class of morphisms $\mathcal{L}(R_h)$ i.e. the *largest subfirm* morphism class with respect to $\mathbf{R} = h\mathbf{M}_{\iota_0}^{\mathcal{C}}$. Remark that, as was noted in the introduction, similar to Lemma 1.12 in [1], it can be shown that $\mathcal{L}(R_h) \subset \mathrm{EpiEmb}_{\mathbf{M}_{\iota_0}^{\mathcal{C}}}$.

In order to internally describe the morphisms in $\mathcal{L}(R_h)$ we introduce the following class of morphisms in $\mathbf{M}_{\iota_0}^{\mathcal{C}}$.

$$\mathcal{W} = \big\{f:(X,\mathcal{D}) \to (X',\mathcal{D}') \mid f \text{ is a } \mathcal{T}h\mathcal{D}'\text{-dense embedding in } \mathbf{M}_{\iota_0}^{\mathcal{C}}\big\}.$$

5.1. Proposition. *$\mathcal{W}$ is a subfirm morphism class with respect to $h\mathbf{M}_{\iota_0}^{\mathcal{C}}$.*

Proof. That $\mathcal{W}$ is closed under composition follows from the fact that by H4 also $\mathbf{M}_{\iota}^{\mathcal{C}} \to \mathbf{Top}:(X,\mathcal{D}) \to (X,\mathcal{T}h\mathcal{D})$ preserves subspaces. That $h\mathbf{M}_{\iota_0}^{\mathcal{C}}$ is $\mathcal{W}$-reflective was shown in 4.1.

To prove that there is uniqueness of completion with respect to $\mathcal{W}$, suppose $f:(X,\mathcal{D})\to(X',\mathcal{D}')$ belongs to $\mathcal{W}$ and $(X',\mathcal{D}')$ belongs to $h\mathbf{M}^{\mathcal{C}}_{\iota 0}$. So f is a $\mathcal{T}h\mathcal{D}'$-dense embedding in $\mathbf{M}^{\mathcal{C}}_{\iota 0}$ and $(X',\mathcal{D}')$ is h-complete. Apply the functor $U_h:\mathbf{M}^{\mathcal{C}}_{\iota}\to\mathbf{Unif}$.

Then $U_h f:(X,\mathcal{U}h\mathcal{D})\to(X',\mathcal{U}h\mathcal{D}')$ is an epimorphic embedding in $\mathbf{Unif}_0$ and $(X',\mathcal{U}h\mathcal{D}')$ is complete. It follows that $\widetilde{U_h f}:(\widetilde{X},\widetilde{\mathcal{U}h\mathcal{D}})\to(X',\mathcal{U}h\mathcal{D}')$ is an isomorphism in $\mathbf{Unif}_0$ and hence it follows that $\tilde{f}:(\widetilde{X},\widetilde{\mathcal{D}})\to(X',\mathcal{D}')$ is a bijective morphism. To show that it is an isomorphism in $\mathbf{M}^{\mathcal{C}}_{\iota 0}$ let $d\in\mathcal{D}$. Since $f:(X,\mathcal{D})\to(X',\mathcal{D}')$ is an embedding there exists a $d'\in\mathcal{D}'$ such that $d\leqslant d'\circ f\times f$. Hence $\tilde{d}\leqslant\widetilde{d'\circ f\times f}$. Moreover $\widetilde{d'\circ f\times f}$ and $d'\circ\tilde{f}\times\tilde{f}$ coincide on a $\mathcal{T}h\widetilde{\mathcal{D}}$-dense subset and so they are equal.

Finally we can conclude that $\tilde{d}\leqslant\widetilde{d'}\circ\tilde{f}\times\tilde{f}$ from which the initiality of $\tilde{f}$ follows. □

The previous result implies that $\mathcal{W}\subset\mathcal{L}(R_h)$. We will now show that in fact both classes of morphisms coincide.

Since we already proved subfirmness of $\mathcal{W}$, by Proposition 1.11 in [1], it suffices to show that $\mathcal{W}$ is *coessential* in the sense that for every two morphisms g and f such that g and $g\circ f$ belong to $\mathcal{W}$ also f belongs to $\mathcal{W}$.

5.2. Theorem. *$\mathcal{W}$ is the firm morphism class associated to $\mathbf{R}=h\mathbf{M}^{\mathcal{C}}_{\iota 0}$, i.e. $\mathcal{W}=\mathcal{L}(R_h)$.*

Proof. Let $f:(X,\mathcal{D})\to(X',\mathcal{D}')$ and $g:(X',\mathcal{D}')\to(X'',\mathcal{D}'')$ be morphisms in $\mathbf{M}^{\mathcal{C}}_{\iota 0}$ such that g and $g\circ f$ belong to $\mathcal{W}$.

Since $g\circ f$ is an embedding it follows that f is an embedding too. By H4 the functor

$$U_h:\mathbf{M}^{\mathcal{C}}_{\iota}\to\mathbf{Unif}:(X,\mathcal{D})\to(X,\mathcal{U}h\mathcal{D})$$

preserves subspaces, so $U_h g:(X',\mathcal{U}h\mathcal{D}')\to(X'',\mathcal{U}h\mathcal{D}'')$ and $U_h(g\circ f):(X,\mathcal{U}h\mathcal{D})\to(X'',\mathcal{U}h\mathcal{D}'')$ are epimorphic embeddings in $\mathbf{Unif}_0$. Since the class of epimorphic embeddings in $\mathbf{Unif}_0$ is coessential, it follows that $U_h f:(X,\mathcal{U}h\mathcal{D})\to(X',\mathcal{U}h\mathcal{D}')$ is an epimorphic embedding in $\mathbf{Unif}_0$. Hence f is $\mathcal{T}h\mathcal{D}'$-dense. □

5.3. Examples. We can now apply the previous result to the two special cases of h we considered before.

(1) When $u\mathbf{M}^{\mathcal{C}}_{\iota 0}$ is the subconstruct consisting of all uniformly complete spaces the reflector $R_u:\mathbf{M}^{\mathcal{C}}_{\iota 0}\to u\mathbf{M}^{\mathcal{C}}_{\iota 0}$ corresponds to the firm class

$$\mathcal{L}(R_u)=\{u:(X,\mathcal{D})\to(X',\mathcal{D}')\mid u\text{ is a }\mathcal{T}\mathcal{D}'\text{-dense embedding in }\mathbf{M}^{\mathcal{C}}_{\iota 0}\}.$$

(2) When $m\mathbf{M}^{\mathcal{C}}_{\iota 0}$ is the subconstruct consisting of all metrically complete spaces, the reflector $R_m:\mathbf{M}^{\mathcal{C}}_{\iota 0}\to m\mathbf{M}^{\mathcal{C}}_{\iota 0}$ corresponds to the firm class

$$\mathcal{L}(R_m)=\{u:(X,\mathcal{D})\to(X',\mathcal{D}')\mid u\text{ is a }\mathcal{T}_{\bigvee\mathcal{D}'}\text{-dense embedding in }\mathbf{M}^{\mathcal{C}}_{\iota 0}\}.$$

Clearly we have $\mathcal{L}(R_m)\subset\mathcal{L}(R_u)\subset\mathrm{EpiEmb}_{\mathbf{M}^{\mathcal{C}}_{\iota 0}}$. Remark that when the base category $\mathcal{C}$ satisfies some mild conditions as formulated in [5], then $\mathcal{L}(R_u)$ coincides with the class of all epimorphic embeddings in $\mathbf{M}^{\mathcal{C}}_{\iota 0}$. This is for instance the case when $\mathcal{C}$ is $\mathcal{C}^{\Delta,s}$ or $\mathcal{C}$ is the base category of all totally bounded metric spaces or of all ultrametric spaces.

The density described in the definition of $\mathcal{L}(R_m)$ is the density in the metric coreflection and in general this morphism class does not coincide with the class of all epimorphic embeddings.

In both cases this density is described by a closure operator on $\mathbf{M}^{\mathcal{C}}_{\iota}$. For recent work in the field of closure operators see e.g. [4].

5.4. Remark. Remark that our main Theorem 5.2 and the preceding propositions and Theorems 3.5, 3.7, 4.1 and 5.1 remain valid for any h satisfying the conditions H1 up to H5.

Another possible generalization of the theory presented so far is obtained by allowing base categories consisting of quasi metrics. The guiding examples for this generalization are the bicompleteness for quasi metric or quasi uniform spaces, [3], or sobriety for approach spaces [10]. Some results in this direction are described in [6].

6. A comparison between uniform and metric completion

From the discussion in 5.3 it follows that $\mathcal{L}(R_m) \subset \mathcal{L}(R_u)$. Since the subconstructs of complete objects associated to these morphism classes consist of the classes of respective injective objects [1], the category of uniformly complete objects is a subcategory of the category of metrically complete objects. In this paragraph we will further investigate the relation between the two completions.

We first investigate uniform completion. We start from the explicit construction of $R_u(X, \mathcal{D})$ for some given metered space in $\mathbf{M}^{\mathcal{C}}_{\iota_0}$.

By definition, for a filter $\mathcal{F}$ we have

- $\mathcal{F}$ is $\mathcal{UD}$-Cauchy $\Leftrightarrow \epsilon > 0,\ \forall d \in \mathcal{D},\ \exists x \in X\colon\ B_d(x, \epsilon) \in \mathcal{F}$

and $(X, \mathcal{D})$ is uniformly complete if every $\mathcal{UD}$-Cauchy filter is $\mathcal{TD}$ convergent. Moreover the uniform completion is given by $R_u(X, \mathcal{D}) = (X^u, \mathcal{D}^u)$ where X^u is the set of minimal $\mathcal{UD}$-Cauchy filters and where

$$\mathcal{D}^u := \{p^u \mid p \in \mathcal{P}\}\downarrow$$

for some $\mathcal{C}$ basis $\mathcal{P}$, with

$$p^u(\mathcal{M}, \mathcal{N}) = \sup_{M \in \mathcal{M}, N \in \mathcal{N}} \inf_{x \in M, y \in N} p(x, y)$$

for $\mathcal{M}$ and $\mathcal{N}$ in X^u. The embedding of X goes via

$$X \to X^u : x \to \mathcal{N}(x),$$

where $\mathcal{N}(x)$ is the neighborhood filter of x in the topology $\mathcal{TD}$.

Next we recall the construction of the ν-completion $R_\nu(X, \mathcal{D})$ from [18], which is based on the completion of the nearness space associated to $(X, \mathcal{D})$.

6.1. Definition. We associate an appropriate nearness space in the sense of [12] to the $\mathbf{M}^{\mathcal{C}}_{\iota_0}$-object $(X, \mathcal{D})$. The structure $\nu_{\mathcal{D}}$ on X consists of all covers refined by some cover of the form

$$\{B_{d_x}(x, \epsilon) \mid x \in X\},$$

with $\epsilon > 0$ and $(d_x)_x \in \mathcal{D}^X$.

In [18] it is shown that this covering structure on X defines a regular T_1 nearness space which we denote $(X, \nu_{\mathcal{D}})$. As is well known for a filter $\mathcal{F}$ we have

- $\mathcal{F}$ $\nu_{\mathcal{D}}$-Cauchy $\Leftrightarrow \forall \epsilon > 0,\ \forall (d_x)_x \in \mathcal{D}^X,\ \exists x \in X\colon\ B_{d_x}(x, \epsilon) \in \mathcal{F}$

or equivalently

- $\mathcal{F}$ is $\nu_{\mathcal{D}}$-Cauchy $\Leftrightarrow \forall \epsilon > 0,\ \exists x \in X,\ \forall d \in \mathcal{D}\colon\ B_d(x, \epsilon) \in \mathcal{F}$.

Remark that this formulation coincides with the notion of a Cauchy filter in the uniform approach space $(X, \delta_{\mathcal{D}})$ with symmetric gauge basis $\mathcal{D}$, [13].

$(X, \mathcal{D})$ is called ν-complete if $(X, \nu_{\mathcal{D}})$ is complete, meaning that every $\nu_{\mathcal{D}}$-Cauchy filter converges in the topological coreflection of the nearness space. Since the nearness space $(X, \nu_{\mathcal{D}})$ is regular it can be completed by using its minimal Cauchy filters as new points [12].

Along the same lines as the proof of 7.1.5 in [13] we now can state

6.2. Proposition. *Every $\nu_{\mathcal{D}}$-Cauchy filter is a $\mathcal{UD}$-Cauchy filter. Moreover every minimal $\nu_{\mathcal{D}}$-Cauchy filter is a minimal $\mathcal{UD}$-Cauchy filter.*

This result in particular implies that the topology $\mathcal{TD}$ on X and the topological coreflection of the nearness space $\nu_{\mathcal{D}}$ coincide. So with the notation of 6.1. $\mathcal{N}(x)$ is the neighborhood filter in this topology. Moreover the result also implies that the ν-completion $R_\nu(X,\mathcal{D})$ can be constructed as a suitable subspace of the completion $R_u(X,\mathcal{D})$.

6.3. Definition. Let X^ν be the subset of X^u consisting of the minimal $\nu_{\mathcal{D}}$-Cauchy filters. Further let

$$\mathcal{D}^\nu := \{p^\nu \mid p \in \mathcal{P}\}\downarrow$$

for some $\mathcal{C}$ basis $\mathcal{P}$, with $p^\nu = p^u|_{X^\nu\times X^\nu}$

6.4. Proposition. *(See [18].) $(X^\nu,\mathcal{D}^\nu)$ is ν-complete and*

$$(X,\mathcal{D}) \to (X^\nu,\mathcal{D}^\nu) : x \to \mathcal{N}(x),$$

is a $\mathcal{T}_{\bigvee\mathcal{D}^\nu}$-dense embedding in $\mathbf{M}^{\mathcal{C}}_{\iota_0}$ and a reflection. $(X^\nu,\mathcal{D}^\nu)$ defines the ν completion $R_\nu(X,\mathcal{D})$ which is firm with respect to

$$\mathcal{L}(R_\nu) = \{f : (X,\mathcal{D}) \to (X',\mathcal{D}') \mid f \text{ is a } \mathcal{T}_{\bigvee\mathcal{D}'}\text{-dense embedding in } \mathbf{M}^{\mathcal{C}}_{\iota_0}\}.$$

Now we turn to metrical completeness. We start from the explicit construction of $R_m(X,\mathcal{D})$ for some given metered space in $\mathbf{M}^{\mathcal{C}}_{\iota_0}$.

By definition, for a filter $\mathcal{F}$ we have

- $\mathcal{F}$ is $\bigvee\mathcal{D}$-Cauchy $\Leftrightarrow \forall\epsilon>0\ \exists x\in X$: $\bigcap_{d\in\mathcal{D}} B_d(x,\epsilon)\in\mathcal{F}$

and the space $(X,\mathcal{D})$ is metrically complete if every $\bigvee\mathcal{D}$-Cauchy filter is convergent in $\mathcal{T}_{\bigvee\mathcal{D}}$ and of course sequences are sufficient in this respect.

Clearly we have

6.5. Proposition. *Every $\bigvee\mathcal{D}$-Cauchy filter is a $\nu_{\mathcal{D}}$-Cauchy filter.*

Remark that the converse is not valid as follows from the following example.

6.6. Example. On the real line $\mathbb{R}$ we consider the usual topology $\mathcal{T}$ and the usual uniformity $\mathcal{U}$. For $\mathcal{D}$ we take

$$\mathcal{D} = \{d \mid d \text{ uniformly continuous metric}\}\downarrow.$$

Then every neighborhood filter $\mathcal{N}(x)$ is a $\nu_{\mathcal{D}}$-Cauchy filter. However $\bigvee\mathcal{D}$ is the discrete metric and hence the only $\bigvee\mathcal{D}$-Cauchy filters are the point filters.

However the complete objects in both structures do coincide. This follows by adapting 6.3.11 in [13] or from the observation that both reflectors R_ν and R_m are firm for the same morphism class $\mathcal{L}(R_\nu)=\mathcal{L}(R_m)$.

6.7. Proposition. *The metric completion can be isomorphically described as $R_m(X,\mathcal{D}) \approx (X^m,\mathcal{D}^m)$ where X^m can be taken as the set of all equivalence classes of Cauchy sequences for $\bigvee\mathcal{D}$. Then we have*

$$\mathcal{D}^m := \{p^m \mid p \in \mathcal{P}\}\downarrow$$

for some $\mathcal{C}$ basis $\mathcal{P}$, with

$$p^m(\langle(x_n)\rangle, \langle(y_n)\rangle) = \lim p(x_n,y_n)$$

for $\langle(x_n)\rangle$ and $\langle(y_n)\rangle$ equivalence classes of Cauchy sequences. The embedding of X goes via

$$X \to X^m : x \to \langle(x)\rangle,$$

where $\langle(x)\rangle$ is the equivalence class in $\mathcal{T}_{\bigvee\mathcal{D}}$ of the constant sequence in x.

The equality $\mathcal{L}(R_\nu) = \mathcal{L}(R_m)$ implies even more. It follows that for a given space $(X, \mathcal{D})$ the completions $R_m(X, \mathcal{D})$ and $R_\nu(X, \mathcal{D})$ are isomorphic. In the next theorem we give an explicit description of the unique isomorphism from $R_m(X, \mathcal{D})$ to $R_\nu(X, \mathcal{D})$ that leaves X pointwise fixed.

6.8. Proposition. *Let $(X, \mathcal{D})$ in $\mathbf{M}^{\mathcal{C}}_{\iota_0}$. The unique isomorphism from $R_m(X, \mathcal{D})$ to $R_\nu(X, \mathcal{D})$ that leaves X pointwise fixed is given by*

$$k : R_m(X, \mathcal{D}) \to R_\nu(X, \mathcal{D}) : \langle (x_n)\rangle \to \mathcal{M}_\nu(x_n)$$

where $\mathcal{M}_\nu(x_n)$ is the minimal $\nu_{\mathcal{D}}$-Cauchy filter contained in the elementary filter generated by the sequence (x_n).

Proof. We use the short notation $\varphi = \bigvee \mathcal{D}$ and we denote by $\mathcal{F}(x_n)$ the elementary filter generated by the sequence (x_n). That k is well defined follows from Proposition 6.5 and that it leaves X pointwise fixed is clear.

First we show injectivity. Suppose that (y_n) and (z_n) are φ-Cauchy sequences which are not φ-equivalent i.e. the sequence $(\varphi(y_n, z_n))$ does not converge to 0.

Without loss of generality we may suppose that there exists an $\epsilon > 0$ such that $\varphi(y_n, z_n) > \epsilon$, for every $n \in \mathbb{N}$. We determine n_0 such that for every $p, q \geqslant n_0$ both $\varphi(y_p, y_q) < \epsilon/3$ and $\varphi(z_p, z_q) < \epsilon/3$. Further we fix a metric $d \in \mathcal{D}$ such that $d(y_{n_0}, z_{n_0}) \geqslant \epsilon$.

Consider the minimal $\nu_{\mathcal{D}}$-Cauchy filter $\mathcal{M}_\nu(y_n)$ contained in $\mathcal{F}(y_n)$. By Proposition 6.2 it follows that $\mathcal{M}_\nu(y_n)$ is a minimal $\mathcal{UD}$-Cauchy filter too. Hence $\bigcup_{s \geqslant n_0} B_d(y_s, \epsilon/3)$ belongs to $\mathcal{M}_\nu(y_n)$.

Let m be arbitrary and fixed. For $r \geqslant m \vee n_0$ and $s \geqslant n_0$ we have

$$d(y_{n_0}, z_{n_0}) \leqslant d(y_{n_0}, y_s) + d(y_s, z_r) + d(z_r, z_{n_0})$$

and since the first and the third term are strictly smaller that $\epsilon/3$, it follows that $z_r \notin B_d(y_s, \epsilon/3)$.

So we can conclude that for arbitrary m, $\{z_r \mid r \geqslant m\}$ is not contained in $\bigcup_{s \geqslant n_0} B_d(y_s, \epsilon/3)$. This implies that $\mathcal{M}_\nu(y_n)$ is not contained in $\mathcal{F}(z_n)$ and therefore $\mathcal{M}_\nu(y_n) \neq \mathcal{M}_\nu(z_n)$.

Secondly we prove that k is surjective. This part of the proof uses some technique developed in the proof of 6.3.11 in [13]. Let $\mathcal{M}$ be a $\nu_{\mathcal{D}}$-minimal Cauchy filter. From the second characterization of a $\nu_{\mathcal{D}}$-Cauchy filter given in Subsection 6.2, we have

$$\forall n > 0,\ \exists x_n \in X,\ \forall d \in \mathcal{D}\colon\ B_d\left(x_n, \frac{1}{n}\right) \in \mathcal{M}.$$

We select a sequence (x_n) satisfying the above condition. From 6.3.11 of [13] we can deduce that the condition ensures that the sequence is a Cauchy sequence for the supremum metric φ and by 6.5 it is also Cauchy for $\nu_{\mathcal{D}}$. We finalize our argument by showing that $\mathcal{F}(x_n)$ is $\nu_{\mathcal{D}}$-equivalent to $\mathcal{M}$. Let $\epsilon > 0$ be arbitrary and choose n_0 such that $\frac{1}{n_0} < \epsilon$ and such that $\varphi(x_p, x_q) < \epsilon$ whenever $p \geqslant n_0$ and $q \geqslant n_0$. Put $x = x_{n_0}$. Now for an arbitrary metric in $\mathcal{D}$ we have $B_d(x, \epsilon) \in \mathcal{M}$ by definition of the sequence. Moreover we also have $\{x_k \mid k \geqslant n_0\} \subset B_d(x, \epsilon)$. So finally $B_d(x, \epsilon) \in \mathcal{M} \cap \mathcal{F}(x_n)$.

From the fact that $\mathcal{M} \cap \mathcal{F}(x_n)$ is a $\nu_{\mathcal{D}}$-Cauchy filter we can conclude that $\mathcal{M} = \mathcal{M}_\nu(x_n)$.

Finally, we check that k is an isomorphism by showing that $p^\nu \circ k \times k = p^m$ for every $\mathcal{C}$-metric p.

Let (x_n) and (y_n) be φ-Cauchy sequences. Let $\mathcal{M}(x_n)$ and $\mathcal{M}(y_n)$ be the minimal $\nu_{\mathcal{D}}$-Cauchy filters contained in their elementary filters. These are equal to the minimal $\mathcal{UD}$-Cauchy filters contained in their elementary filters. So in $(X^u, \mathcal{D}^u)$ the sequences (x_n) and (y_n) converge to the points $\mathcal{M}(x_n)$ and $\mathcal{M}(y_n)$ respectively. By continuity of p^u we have $p(x_n, y_n) \to p^u(\mathcal{M}(x_n), \mathcal{M}(y_n)) = p^\nu(\mathcal{M}(x_n), \mathcal{M}(y_n)) = p^\nu \circ k \times k(\langle (x_n)\rangle, \langle (y_n)\rangle)$. □

7. Uniqueness of completions in subcategories of $\mathbf{M}^{\mathcal{C}}_{\iota_0}$

In this section we will use the results on $\mathbf{M}^{\mathcal{C}}_{\iota_0}$ to find a description of the firm class $\mathcal{L}(S)$ in case $S : \mathbf{X}_0 \to \mathbf{S}$ is a reflector defined on the class of T_0-objects in some metrically generated construct $\mathbf{X}$. Since every such construct can be isomorphically embedded in $\mathbf{M}^{\mathcal{C}}_{\iota}$ (here again we assume that we work with meters that are ideals), the following general result is a starting point. By subconstruct we always mean full and isomorphism closed subconstruct. The proof of the following proposition is straightforward.

7.1. Proposition. *Let* $\mathbf{Z}$ *be a reflective subconstruct of a construct* $\mathbf{Y}$*, with reflector* $R:\mathbf{Y}\to\mathbf{Z}$*. Suppose that* $\mathbf{X}$ *is a full subconstruct of* $\mathbf{Y}$*. The following are equivalent*:

(1) $\mathbf{X}$ *is closed under the reflector in the sense that* $R(\mathbf{X})\subset\mathbf{X}$.
(2) $\mathbf{X}\cap\mathbf{Z}$ *is reflective in* $\mathbf{X}$ *with reflector* S *and*

$$\mathcal{L}(S)=\mathcal{L}(R)\cap \mathrm{Mor}\mathbf{X}.$$

Let $\mathbf{X}$ be a $\mathcal{C}$-metrically generated construct which by the main theorem in [7] is isomorphic to some construct $\mathbf{M}^{\mathcal{C}}_{\xi}$ for a suitable ξ on $\mathbf{M}^{\mathcal{C}}$. We will study completeness on $\mathbf{M}^{\mathcal{C}}_{\xi_0}$ and then draw conclusions for $\mathbf{X}_0$. The results for $\mathbf{M}^{\mathcal{C}}_{\xi_0}$ will be derived with the help of the previous Proposition 7.1. As before h covers two particular cases, it is either the identity as in Definition 3.1 or $h\mathcal{D}=\{\bigvee\mathcal{D}\}\downarrow$ as in Definition 3.2 and again we use the terminology "h-complete" to describe either of the two completeness notions. Given h we consider the reflective subconstruct of $\mathbf{M}^{\mathcal{C}}_{\iota_0}$ consisting of all h-complete objects and we denote the reflector by $R:\mathbf{M}^{\mathcal{C}}_{\iota_0}\to h\mathbf{M}^{\mathcal{C}}_{\iota_0}$. Further let

$$h\mathbf{M}^{\mathcal{C}}_{\xi_0}=\mathbf{M}^{\mathcal{C}}_{\xi_0}\cap h\mathbf{M}^{\mathcal{C}}_{\iota_0}.$$

The following result describes a sufficient condition to obtain the firm class of morphisms associated to $h\mathbf{M}^{\mathcal{C}}_{\xi_0}$.

7.2. Proposition. *With the notations just described, consider the following conditions*:

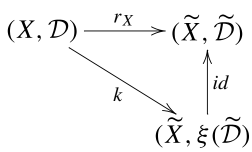
(h1) *For every* $(X,\mathcal{D})$ *in* $\mathbf{M}^{\mathcal{C}}_{\iota_0}$ *we have the equality* $\mathcal{T}h\xi(\mathcal{D})=\mathcal{T}h\mathcal{D}$;
(h2) $\mathbf{M}^{\mathcal{C}}_{\xi_0}$ *is closed under the reflector* $R:\mathbf{M}^{\mathcal{C}}_{\iota_0}\to h\mathbf{M}^{\mathcal{C}}_{\iota_0}$;
(h3) $h\mathbf{M}^{\mathcal{C}}_{\xi_0}$ *is reflective in* $\mathbf{M}^{\mathcal{C}}_{\xi_0}$ *with reflector* $S:\mathbf{M}^{\mathcal{C}}_{\xi_0}\to h\mathbf{M}^{\mathcal{C}}_{\xi_0}$ *with* $S=R|_{\mathbf{M}^{\mathcal{C}}_{\xi_0}}$ *and there is firmness with respect to*

$$\mathcal{L}(S)=\bigl\{f:(X,\mathcal{D})\to(X',\mathcal{D}')\mid f \text{ morphism in } \mathbf{M}^{\mathcal{C}}_{\xi_0},\ \mathcal{T}h\mathcal{D}'\text{-dense, embedding in } \mathbf{M}^{\mathcal{C}}_{\iota_0}\bigr\}.$$

Then (h1) *implies* (h2) *and* (h2) $\Leftrightarrow$ (h3).

Proof. First observe that (h1) implies that $(X,\xi(\mathcal{D}))$ is h-complete whenever $(X,\mathcal{D})$ is h-complete. By H2 and $\mathcal{D}\subset\xi(\mathcal{D})$ it follows that the uniformity $\mathcal{U}h\mathcal{D}$ is coarser than the uniformity $\mathcal{U}h\xi(\mathcal{D})$. Moreover by application of (h1) we get that the underlying topologies of these uniformities coincide.

Assume (h1). In order to prove (h2), let $(X,\mathcal{D})$ belong to $\mathbf{M}^{\mathcal{C}}_{\xi_0}$. Then the diagram



commutes, where r_X is the reflection morphism of $(X,\mathcal{D})$ and k is the image of r_X by the coreflector associated with ξ. So k is an embedding in $\mathbf{M}^{\mathcal{C}}_{\iota_0}$. Application of (h1) to $(\widetilde{X},\widetilde{\mathcal{D}})$ implies that k is $\mathcal{T}h\xi(\widetilde{\mathcal{D}})$-dense. In view of the first part of this proof we have that $(\widetilde{X},\xi(\widetilde{\mathcal{D}}))$ is h-complete.

So we can conclude that $(\widetilde{X},\xi(\widetilde{\mathcal{D}}))$ and $(\widetilde{X},\widetilde{\mathcal{D}})$ are isomorphic and hence $(\widetilde{X},\widetilde{\mathcal{D}})$ belongs to $\mathbf{M}^{\mathcal{C}}_{\xi_0}$.

That (h2) and (h3) are equivalent follows at once from 7.1. □

8. Concrete examples of uniform completeness

In this section we deal with uniform completeness and we adapt the results of the previous section to the case where h is the identity on meters. As in 5.3 we write $u\mathbf{M}^{\mathcal{C}}_{\xi_0}$ for the construct consisting of complete objects in $\mathbf{M}^{\mathcal{C}}_{\xi_0}$. We write (u1), (u2) and (u3) for the adapted versions of (h1), (h2) and (h3) respectively.

Consider the following expanders on $\mathbf{M}^{\Delta,s}$

$$d \in \xi_T(\mathcal{D}) \quad \text{iff} \quad \forall x \in X,\ \forall \varepsilon > 0,\ \exists d_1, \ldots, d_n \in \mathcal{D},\ \exists \delta > 0\colon\ \sup_{i=1}^{n} d_i(x, y) < \delta \Rightarrow d(x, y) < \varepsilon$$

$$d \in \xi_A(\mathcal{D}) \quad \text{iff} \quad \forall x \in X,\ \forall \varepsilon > 0,\ \forall \omega < \infty,\ \exists d_1, \ldots, d_n \in \mathcal{D}\colon\ d(x, y) \wedge \omega \leqslant \sup_{i=1}^{n} d_i(x, y) + \varepsilon$$

$$d \in \xi_U(\mathcal{D}) \quad \text{iff} \quad \forall \varepsilon > 0,\ \exists d_1, \ldots, d_n \in \mathcal{D},\ \exists \delta > 0\colon\ \sup_{i=1}^{n} d_i(x, y) < \delta \Rightarrow d(x, y) < \varepsilon$$

$$d \in \xi_{UG}(\mathcal{D}) \quad \text{iff} \quad \forall \varepsilon > 0,\ \forall \omega < \infty,\ \exists d_1, \ldots, d_n \in \mathcal{D}\colon\ d(x, y) \wedge \omega \leqslant \sup_{i=1}^{n} d_i(x, y) + \varepsilon$$

$$e \in \xi_D(\mathcal{D}) \quad \text{iff} \quad d \leqslant \sup_{e \in \mathcal{D}} e.$$

When applied to $\mathbf{M}^{\Delta,s}$ these expanders give rise to coreflective subconstructs $\mathbf{M}_{\xi}^{\Delta,s}$ that are isomorphic to **Creg** (completely regular spaces), **UAp** (uniform approach spaces [13]), **Unif** (uniform spaces), **UG** (uniform gauge spaces [14], [15]) and $\mathcal{C}^{\Delta,s}$ respectively.

As mentioned in Section 2, for any given expander ξ on $\mathbf{M}^{\Delta,s}$, there exists an adapted version $\xi^{\mathcal{C}}$ on $\mathbf{M}^{\mathcal{C}}$ defined by $\xi^{\mathcal{C}}(\mathcal{D}) = \{d \in \xi(\mathcal{D}) \mid d\ \mathcal{C}\text{-metric}\} \downarrow$.

When applied to $\mathbf{M}^{\mathcal{C}}$, for $\mathcal{C}$ consisting of all totally bounded metric spaces, the coreflective subconstructs $\mathbf{M}_{\xi}^{\mathcal{C}}$ are isomorphic to **Creg**, **UAp**, **Prox** (proximity spaces), **Gap** (Efgap spaces [8]) and to the construct of totally bounded metric spaces respectively.

When applied to $\mathbf{M}^{\mathcal{C}}$, for $\mathcal{C}$ consisting of all ultrametric spaces, the coreflective subconstructs $\mathbf{M}_{\xi}^{\mathcal{C}}$ are isomorphic to **ZDim** (zero-dimensional topological spaces), **ZDAp** (zero-dimensional approach spaces [6]), **NaUnif** (non-Archimedian uniform spaces [17]), **TUG** (transitive uniform gauge spaces [6]) and to the construct of all ultrametric spaces respectively.

The following table summarizes the above.

	All metric spaces	Totally bounded metric spaces	Ultrametric spaces
ξ_T	**Creg**	**Creg**	**ZDim**
ξ_A	**UAp**	**UAp**	**ZDAp**
ξ_U	**Unif**	**Prox**	**NAUnif**
ξ_{UG}	**UG**	**Gap**	**TUG**
ξ_D	all metric spaces	totally bounded metric spaces	ultrametric spaces

8.1. Proposition. *For an arbitrary base category $\mathcal{C}$ all the expanders in the previous list satisfy the equivalent conditions* (u2) *and* (u3).

Proof. By straightforward calculation it follows that condition (u1) is fulfilled by the expander $\xi_T^{\mathcal{C}}$. Then it is easily seen to be fulfilled for the expanders $\xi_A^{\mathcal{C}}$, $\xi_U^{\mathcal{C}}$ and $\xi_{UG}^{\mathcal{C}}$ as well. This implies they all satisfy (u2). For $\xi_D^{\mathcal{C}}$ it is easily checked directly that (u2) is fulfilled. □

8.2. Remark. $\xi_D^{\mathcal{C}}$ need not satisfy condition (u1) as follows from an argument similar to the one in 6.6. Let $\mathcal{C}$ be $\mathcal{C}^{\Delta,s}$. Again use $\mathbb{R}$ with $\mathcal{D}$ having as a basis all uniformly continuous metrics. The topology $\mathcal{T}\mathcal{D}$ is the usual topology whereas $\mathcal{T}\xi_D(\mathcal{D})$ is the discrete topology.

By application of Propositions 9.3, 7.2 and 7.1 we now have

8.3. Proposition. *For any base category $\mathcal{C}$ and for any of the expanders $\xi_T^{\mathcal{C}}$, $\xi_A^{\mathcal{C}}$, $\xi_U^{\mathcal{C}}$, $\xi_{UG}^{\mathcal{C}}$ and $\xi_D^{\mathcal{C}}$, with*

$$u\mathbf{M}_{\xi_0}^{\mathcal{C}} = u\mathbf{M}_{\iota_0}^{\mathcal{C}} \cap \mathbf{M}_{\xi_0}^{\mathcal{C}}$$

we have:

(1) *$u\mathbf{M}_{\xi_0}^{\mathcal{C}}$ is reflective in $\mathbf{M}_{\xi_0}^{\mathcal{C}}$ with reflector $S_u : \mathbf{M}_{\xi_0}^{\mathcal{C}} \to u\mathbf{M}_{\xi_0}^{\mathcal{C}}$*;

(2) $S_u = R_u|_{\mathbf{M}^{\mathcal{C}}_{\xi_0}}$;

(3) *There is firmness with respect to*

$$\mathcal{L}(S_u) = \{ f : (X, \mathcal{D}) \to (X', \mathcal{D}') \mid f \text{ morphism in } \mathbf{M}^{\mathcal{C}}_{\xi_0},\ \mathcal{TD}'\text{-dense, embedding in } \mathbf{M}^{\mathcal{C}}_{\iota_0} \}.$$

Next we investigate how uniform completeness relates to some standard completeness notions.

8.4. Example. $\mathbf{X} = \mathcal{C}$ for an arbitrary base subcategory of $\mathcal{C}^{\Delta,s}$.

Using the isomorphic description $\mathbf{M}^{\mathcal{C}}_{\xi_D}$ it is clear that uniform completeness of a $\mathcal{C}$-metric has exactly the same meaning as ordinary completeness. Moreover the class of morphisms $\mathcal{L}(S_u)$ coincides with the class of all epimorphic embeddings, where embedding refers to $\mathcal{C}$.

8.5. Example. $\mathbf{X} = \mathbf{Unif}$ or $\mathbf{X} = \mathbf{Prox}$ or $\mathbf{X} = \mathbf{NAUnif}$.

The results are based on the isomorphic descriptions of these constructs for suitable base category $\mathcal{C}$ and expander $\xi^{\mathcal{C}}_U$ as formulated above. By definition and by Theorem 8.3 the uniformly complete objects of $\mathbf{X}_0$ are those for which the associated uniformity is complete in the usual sense.

In each case, by Theorem 8.3 for the reflector S_u we have that $\mathcal{L}(S_u) = \mathrm{EpiEmb}_{\mathbf{X}_0}$, the class of all epimorphic embeddings, where embedding now refers to the construct $\mathbf{X}_0$. The fact that these classes of morphisms coincide can be deduced from the fact that the classes of complete objects are the same. Another direct argument that can be used is the classical result 15N. in [11], which in our terminology says that every $f : (X, \mathcal{D}) \to (X', \mathcal{D}')$ which corresponds to a $\mathcal{TD}'$-dense embedding in $\mathbf{X}_0$ is an embedding in $\mathbf{M}^{\mathcal{C}}_{\iota_0}$.

8.6. Example. $\mathbf{X} = \mathbf{AUnif}$ or $\mathbf{X} = \mathbf{Gap}$ or $\mathbf{X} = \mathbf{TAUnif}$.

The results are based on the isomorphic descriptions of these constructs for suitable base category $\mathcal{C}$ and expander $\xi^{\mathcal{C}}_{UG}$ as formulated above. By definition and by Theorem 8.3 the uniformly complete objects of $\mathbf{X}_0$ are those for which the associated uniformity is complete in the usual sense. This notion coincides with the completeness studied in [15] and [8]. In each case by Theorem 8.3 we have $\mathcal{L}(S_u) = \mathrm{EpiEmb}_{\mathbf{X}_0}$, the class of all epimorphic embeddings (where embedding refers to the construct $\mathbf{X}_0$). This fact follows directly from the result proved in [6], that every $f : (X, \mathcal{D}) \to (X', \mathcal{D}')$ which corresponds to a $\mathcal{TD}'$-dense embedding in $\mathbf{X}_0$, is an embedding in $\mathbf{M}^{\mathcal{C}}_{\iota_0}$.

In order to study uniform completeness for **Creg** and **UAp** and their zero-dimensional versions, we first need the following result concerning fine spaces.

8.7. Definition. Let ξ_1 and ξ_2 be two expanders on a model category $\mathbf{M}^{\mathcal{C}}$ such that $\xi_1 \leqslant \xi_2$. The fine $\mathbf{M}^{\mathcal{C}}_{\xi_1}$-space of an $\mathbf{M}^{\mathcal{C}}_{\xi_2}$-object $(X, \mathcal{D})$ is the finest $\mathbf{M}^{\mathcal{C}}_{\xi_1}$-object $(X, \mathcal{D}')$ such that $\xi_2(\mathcal{D}') = \mathcal{D}$.

If we consider the expanders ξ_U and ξ_T on the model category $\mathbf{M}^{\Delta,s}$, the notion of a fine $\mathbf{M}^{\Delta,s}_{\xi_U}$-space of a $\mathbf{M}^{\Delta,s}_{\xi_T}$ space coincides with the notion of fine uniformity of a completely regular topological space. When the adapted versions of the expanders are used on the model category $\mathbf{M}^{\mathcal{C}}$, for $\mathcal{C}$ the construct of all totally bounded metric spaces, then the notion of fine $\mathbf{M}^{\mathcal{C}}_{\xi_U}$ space of a given $\mathbf{M}^{\mathcal{C}}_{\xi_T}$ space coincides with the fine proximity space of a given completely regular topological space. When the adapted versions of the expanders are used on the model category $\mathbf{M}^{\mathcal{C}}$, for $\mathcal{C}$ the construct of all ultrametric metric spaces, then the notion of fine $\mathbf{M}^{\mathcal{C}}_{\xi_U}$ space of a given $\mathbf{M}^{\mathcal{C}}_{\xi_T}$ space coincides with the fine non-Archimedian uniform space of a given zero-dimensional topological space.

Next we prove that the fine space always exists.

8.8. Proposition. *Let ξ_1, ξ_2 be expanders on a model category $\mathbf{M}^{\mathcal{C}}$, such that $\xi_1 \leqslant \xi_2$. The fine $\mathbf{M}^{\mathcal{C}}_{\xi_1}$-space of a $\mathbf{M}^{\mathcal{C}}_{\xi_2}$-space $(X, \mathcal{D})$ is $(X, \mathcal{D})$.*

Proof. Let $(X, \mathcal{D}) \in \mathbf{M}^{\mathcal{C}}_{\xi_2}$. Then the meter $\mathcal{D}$ is saturated for ξ_2 hence it is also ξ_1-saturated. For every $\mathbf{M}^{\mathcal{C}}_{\xi_1}$-space $(X, \mathcal{D}')$ such that $\xi_2(\mathcal{D}') = \mathcal{D}$ we have that $\mathcal{D}' \subset \mathcal{D}$. Hence $(X, \mathcal{D})$ is the finest $\mathbf{M}^{\mathcal{C}}_{\xi_1}$-space such that $\xi_2(\mathcal{D}) = \mathcal{D}$. □

8.9. Example. $\mathbf{X} = \mathbf{CReg}$ or $\mathbf{X} = \mathbf{ZDim}$.

The results are based on the isomorphic descriptions of these constructs for suitable base category $\mathcal{C}$ and expander $\xi_T^{\mathcal{C}}$ as formulated above.

By definition and by Theorem 8.3 a uniformly complete object of $\mathbf{X}_0$ is a completely regular (zero-dimensional) topological T_0-space $(X, \mathcal{T})$ for which the corresponding object $(X, \mathcal{D})$ in $\mathbf{M}^{\mathcal{C}}_{\xi_{T0}}$ generates a complete (non-Archimedian) uniformity $(X, \mathcal{UD})$. Since the meter $\mathcal{D}$ is saturated for $\xi_T^{\mathcal{C}}$ it is also saturated for $\xi_U^{\mathcal{C}}$. It follows from 8.8 that $(X, \mathcal{UD})$ is the fine (fine non-Archimedian) uniformity for $(X, \mathcal{T})$. So the uniformly complete objects are the completely regular (zero-dimensional) topological T_0-spaces with a complete fine (non-Archimedian) uniformity. These objects are complete in the sense of Dieudonné [9].

Note that in this case the class of morphisms $\mathcal{L}(S_u)$ is strictly contained in the class of epimorphic embeddings. In order to see this one can use an example similar to the one in Example 1.8 in [2]. The topological dense embedding of the (zero-dimensional) space $\mathbb{N}$ of natural numbers in its Alexandroff compactification $\mathbb{N}^*$ is not an embedding in $\mathbf{M}^{\mathcal{C}}_{\iota_0}$.

8.10. Example. $\mathbf{X} = \mathbf{UAp}$ or $\mathbf{X} = \mathbf{ZDAp}$.

The situation is analogous to the previous one. Using the isomorphism to $\mathbf{M}^{\mathcal{C}}_{\xi_A}$ for suitable $\mathcal{C}$, we find that the uniformly complete T_0 (zero-dimensional) uniform approach spaces are exactly those with a complete fine (transitive) approach uniformity. We call them again Dieudonné complete. Using essentially the same counterexample as in the topological case, it follows that also in the approach case the class of morphisms $\mathcal{L}(S_u)$ is strictly contained in the class of epimorphic embeddings.

9. Concrete examples of metric completeness

In this section we deal with metric completeness and we adapt the results of Section 6 to the case where h acts as $h\mathcal{D} = \{\bigvee \mathcal{D}\}\downarrow$ on a meter $\mathcal{D}$. As in 5.3 we write $m\mathbf{M}^{\mathcal{C}}_{\xi_0}$ for the construct consisting of metrically complete objects in $\mathbf{M}^{\mathcal{C}}_{\xi_0}$. We write (m1), (m2) and (m3) for the adapted versions of (h1), (h2) and (h3) respectively.

We consider the same collection of expanders as in Section 7 and they give rise to the constructs described there. In order to examine which of the expanders do satisfy (m3), the following result will be useful.

9.1. Proposition. *Let* $\mathbf{X}$ *be* $\mathcal{C}$*-metrically generated, let* ξ *be the expander on* $\mathbf{M}^{\mathcal{C}}$ *such that* $\mathbf{M}^{\mathcal{C}}_{\xi} \approx \mathbf{X}$ *and let* $K_\xi : \mathcal{C} \to \mathbf{M}^{\mathcal{C}}_{\xi} : (X, d) \to \xi(d\downarrow)$. *Then the following are equivalent.*

(1) $\xi \leqslant \xi_D^{\mathcal{C}}$ *on* $\mathbf{M}^{\mathcal{C}}$.
(2) $\xi_D^{\mathcal{C}} \circ \xi = \xi_D^{\mathcal{C}}$ *on* $\mathbf{M}^{\mathcal{C}}$.
(3) $K_\xi : \mathcal{C} \to \mathbf{M}^{\mathcal{C}}_{\xi}$ *is an embedding.*

Proof. (1) $\Rightarrow$ (2): Let $(X, \mathcal{D})$ be a $\mathcal{C}$-metered space. By assumption we have $\mathcal{D} \subset \xi(\mathcal{D}) \subset \xi_D^{\mathcal{C}}(\mathcal{D})$. Applying the expander $\xi_D^{\mathcal{C}}$ we can deduce that $\xi_D^{\mathcal{C}} \circ \xi(\mathcal{D}) = \xi_D^{\mathcal{C}}(\mathcal{D})$.

(2) $\Rightarrow$ (3): Let d, d' be $\mathcal{C}$-metrics on a set X such that $K_\xi(X, d) = K_\xi(X, d')$. This implies that $\xi(d\downarrow) = \xi(d'\downarrow)$. Consequently, by the assumption that $\xi_D^{\mathcal{C}} \circ \xi = \xi_D^{\mathcal{C}}$, it follows that $\xi_D^{\mathcal{C}}(d\downarrow) = \xi_D^{\mathcal{C}}(d'\downarrow)$ and hence we can conclude that $d = d'$. So the functor K_ξ is injective on objects and since it is a concrete functor K_ξ is an embedding.

(3) $\Rightarrow$ (1): Suppose $K_\xi : \mathcal{C} \to \mathbf{M}^{\mathcal{C}}_{\xi}$ is an embedding. Let e and d be $\mathcal{C}$-metrics such that $e \in \xi(d\downarrow)$. Then the identity $(X, \xi(d\downarrow)) \to (X, \xi(e\downarrow))$ is a morphism in $\mathbf{M}^{\mathcal{C}}_{\xi}$. Since K_ξ is full the identity $(X, d) \to (X, e)$ is a morphism too. So $e \in d\downarrow$ from which we can conclude that $\xi(d\downarrow) = d\downarrow$ for every $\mathcal{C}$-metric d. Now suppose $\mathcal{D}$ satisfies $\xi_D^{\mathcal{C}}(\mathcal{D}) = \mathcal{D}$. This implies $\mathcal{D} = d\downarrow$ for some $\mathcal{C}$-metric d, then by the previous result we have $\mathcal{D} = \xi(d\downarrow)$. So we can conclude that $\xi(\mathcal{D}) = \mathcal{D}$. Finally we obtain that $\xi \leqslant \xi_D^{\mathcal{C}}$ on $\mathbf{M}^{\mathcal{C}}$. □

9.2. Proposition. *With the notations of the previous proposition, put* (m0) K_ξ *is an embedding.*

Then (m0) $\Rightarrow$ (m1) $\Rightarrow$ (m2) $\Leftrightarrow$ (m3).

Proof. We use characterization (1) from Proposition 9.1. So for arbitrary $\mathcal{D}$ in $\mathbf{M}^{\mathcal{C}}$ condition (m0) implies that $\xi(\mathcal{D}) \subset \xi_D^{\mathcal{C}}(\mathcal{D})$. Applying h on both sides we get $\{\bigvee \xi(\mathcal{D})\}\downarrow \subset \{\bigvee \xi_D^{\mathcal{C}}(\mathcal{D})\}\downarrow \subset \{\bigvee \mathcal{D}\}\downarrow$. So finally we can conclude $\mathcal{T}\{\bigvee \xi(\mathcal{D})\}\downarrow = \mathcal{T}\{\bigvee \mathcal{D}\}\downarrow$.

The other implications are just special instances of Proposition 7.2. □

9.3. Proposition.

(1) *For an arbitrary base category $\mathcal{C}$ the expanders $\xi_D^{\mathcal{C}}$, $\xi_A^{\mathcal{C}}$ and $\xi_{UG}^{\mathcal{C}}$ satisfy the equivalent conditions* (m2) *and* (m3).
(2) *For any base category that is closed under multiples* (*so for any of our examples of base categories*) *the expanders $\xi_T^{\mathcal{C}}$ and $\xi_U^{\mathcal{C}}$ satisfy* (m2) *and* (m3).

Proof. It is easily seen that for the expanders $\xi_D^{\mathcal{C}}$, $\xi_A^{\mathcal{C}}$ and $\xi_{UG}^{\mathcal{C}}$ the functor K_ξ is an embedding. So these expanders also satisfy the equivalent conditions (m2) and (m3).

Secondly consider the expanders $\xi_T^{\mathcal{C}}$ and $\xi_U^{\mathcal{C}}$. If $(X, \mathcal{D})$ belongs to $\mathbf{M}_{\xi_0}^{\mathcal{C}}$ for one of the expanders ξ_T or ξ_U then $\mathcal{D}$ contains all multiples of metrics in $\mathcal{D}$. It follows that $\bigvee \mathcal{D}$ is discrete and hence complete. So $\mathbf{M}_{\xi_0}^{\mathcal{C}}$ is closed under the reflector $R_m : \mathbf{M}_{\iota_0}^{\mathcal{C}} \to m\mathbf{M}_{\iota_0}^{\mathcal{C}}$. □

9.4. Remark. None of the implications in 9.2 can be reversed.

(1) That (m2) does not imply (m1) follows from the observation that on $\mathcal{C}^{\Delta,s}$ neither ξ_T nor ξ_U satisfy (m1). For $(X, \mathcal{D})$ belonging to $\mathbf{M}_{\iota_0}^{\Delta,s}$ and for the expanders ξ_T or ξ_U, all multiples of metrics in $\mathcal{D}$ belong to $\xi(\mathcal{D})$. This implies that $\mathcal{T}\{\bigvee \xi(\mathcal{D})\}\downarrow$ is discrete, whereas $\mathcal{T}\{\bigvee \mathcal{D}\}\downarrow$ need not be discrete. As a concrete example take for instance $\mathbb{R}$ with $\mathcal{D} = d\downarrow$, where d is the Euclidean metric.

(2) In order to see that (m1) does not imply (m0) we consider the following expander ξ on $\mathbf{M}^{\Delta,s}$. For (X, d) in $\mathcal{C}^{\Delta,s}$ we first put d' the function defined by:

$$d'(x, y) = d(x, y) \quad \text{if } d(x, y) < 1 \quad \text{and} \quad d'(x, y) = \infty \quad \text{if } d(x, y) \geqslant 1$$

and then we put

$$\xi d = \sup\{e \in \mathcal{C}^{\Delta,s} \mid e \leqslant d'\}.$$

Finally for a metered space $(X, \mathcal{D})$ put

$$\xi(\mathcal{D}) = \{\xi d \mid d \in \mathcal{D}\}\downarrow.$$

Then K_ξ is not an embedding, however the equality $\mathcal{T} \vee \xi(\mathcal{D}) = \mathcal{T} \vee \mathcal{D}$ holds.

By application of Propositions 9.3, 7.2 and 7.1 we now immediately have

9.5. Proposition. *For any base category $\mathcal{C}$ and for any of the expanders $\xi_T^{\mathcal{C}}$, $\xi_A^{\mathcal{C}}$, $\xi_U^{\mathcal{C}}$, $\xi_{UG}^{\mathcal{C}}$ and $\xi_D^{\mathcal{C}}$ with*

$$m\mathbf{M}_{\xi_0}^{\mathcal{C}} = m\mathbf{M}_{\iota_0}^{\mathcal{C}} \cap \mathbf{M}_{\xi_0}^{\mathcal{C}}$$

we have:

(1) *$m\mathbf{M}_{\xi_0}^{\mathcal{C}}$ is reflective in $\mathbf{M}_{\xi_0}^{\mathcal{C}}$ with reflector $S_m : \mathbf{M}_{\xi_0}^{\mathcal{C}} \to m\mathbf{M}_{\xi_0}^{\mathcal{C}}$*;
(2) $S_m = R_m|_{\mathbf{M}_{\xi_0}^{\mathcal{C}}}$;
(3) *There is firmness with respect to the class $\mathcal{L}(S_m)$ consisting of all $f : (X, \mathcal{D}) \to (X', \mathcal{D}')$ which are $\mathcal{T}\{\bigvee \mathcal{D}'\}\downarrow$-dense, a morphism in $\mathbf{M}_{\xi_0}^{\mathcal{C}}$, and an embedding in $\mathbf{M}_{\iota_0}^{\mathcal{C}}$.*

It follows from the remarks just made that metrical completeness is not an interesting notion in $\mathbf{M}_{\xi_0}^{\mathcal{C}}$ for the expanders $\xi_T^{\mathcal{C}}$ and $\xi_U^{\mathcal{C}}$ and for base categories that are closed under multiples, since all the objects in the category are metrically complete. However for those expanders for which K_ξ is an embedding we get interesting completeness notions.

9.6. Example. $\mathbf{X} = \mathcal{C}$ for an arbitrary base subcategory of $\mathcal{C}^{\Delta,s}$.

Using the isomorphic description $\mathbf{M}^{\mathcal{C}}_{\xi_D}$ it is clear that metric completeness of a $\mathcal{C}$-metric has again exactly the same meaning as ordinary completeness. So it coincides with uniform completeness we considered before. Hence the class of morphisms $\mathcal{L}(S_m)$ coincides with the class of all epimorphic embeddings, where embedding refers to $\mathcal{C}$.

9.7. Example. $\mathbf{X} = \mathbf{UAp}$ or $\mathbf{ZDAp}$.

Again the results are based on the isomorphic descriptions of these constructs using a suitable base category $\mathcal{C}$ and expander $\xi^{\mathcal{C}}_A$. By definition and by Theorem 9.5 the metrically complete objects of $\mathbf{X}_0$ correspond to those $(X, \mathcal{D})$ for which $\bigvee \mathcal{D}$ is complete in the usual sense. By Theorem 9.5 for the reflector S_m we have that the class $\mathcal{L}(S_m)$ consists of all $f : (X, \mathcal{D}) \to (X', \mathcal{D}')$ which are $\mathcal{T}\{\bigvee \mathcal{D}'\}\downarrow$-*dense*, a morphism in $\mathbf{M}^{\mathcal{C}}_{\xi_{A_0}}$ and an embedding in $\mathbf{M}^{\mathcal{C}}_{\iota_0}$.

In [6] it was shown that for $\mathcal{C} = \mathcal{C}^{\Delta,s}$ or $\mathcal{C}$ the class of all ultrametric spaces, a morphism $f : (X, \mathcal{D}) \to (X', \mathcal{D}')$ which is a $\mathcal{T}\{\bigvee \mathcal{D}'\}\downarrow$-dense embedding in $\mathbf{M}^{\mathcal{C}}_{\xi_{A_0}}$ is an embedding in $\mathbf{M}^{\mathcal{C}}_{\iota_0}$ as well.

So in both examples we can conclude that for metric completeness the class $\mathcal{L}(S_m)$ is given by all $f : (X, \mathcal{D}) \to (X', \mathcal{D}')$ which are $\mathcal{T}\{\bigvee \mathcal{D}'\}\downarrow$-dense embeddings in $\mathbf{M}^{\mathcal{C}}_{\xi_{A_0}}$.

If on the other hand we consider the class of epimorphic embeddings then we know that

$$\mathrm{EpiEmb}_{\mathbf{M}^{\mathcal{C}}_{\xi_{A_0}}} = \big\{ f : (X, \mathcal{D}) \to (X', \mathcal{D}') \mid f \text{ a } \mathcal{T}\mathcal{D}'\text{-dense embedding in } \mathbf{M}^{\mathcal{C}}_{\xi_{A_0}} \big\}.$$

Remark that the morphism class $\mathcal{L}(S_m)$ is strictly contained in $\mathrm{EpiEmb}_{\mathbf{M}^{\mathcal{C}}_{\xi_{A_0}}}$. As an example consider the topological space $\mathbb{N}$ and its Alexandroff compactification $\mathbb{N}^*$, and consider them as uniform approach spaces. Then the canonical injection $j : \mathbb{N} \to \mathbb{N}^*$ is an epimorphic embedding in $\mathbf{M}^{\mathcal{C}}_{\xi_{A_0}}$, but it is not dense for the metric coreflection.

9.8. Example. $\mathbf{X} = \mathbf{AUnif}$ or $\mathbf{X} = \mathbf{Gap}$ or $\mathbf{X} = \mathbf{TAUnif}$.

We use the isomorphic copies $\mathbf{M}^{\mathcal{C}}_{\xi_{UG}}$ for suitable $\mathcal{C}$. The metrically complete objects $(X, \mathcal{D})$ are those objects for which $\bigvee \mathcal{D}$ is complete in the usual sense. Again the class $\mathcal{L}(S_m)$ is given by the $f : (X, \mathcal{D}) \to (X', \mathcal{D}')$ which are $\mathcal{T}\{\bigvee \mathcal{D}'\}\downarrow$-dense morphisms in $\mathbf{M}^{\mathcal{C}}_{\xi_{UG_0}}$ and embeddings in $\mathbf{M}^{\mathcal{C}}_{\iota_0}$ and hence, making use of a property from [6] which we recalled earlier in 8.6, namely that every $f : (X, \mathcal{D}) \to (X', \mathcal{D}')$ which is a $\mathcal{T}\mathcal{D}'$-dense embedding in $\mathbf{M}^{\mathcal{C}}_{\xi_{UG_0}}$ is an embedding in $\mathbf{M}^{\mathcal{C}}_{\iota_0}$, it is the class

$$\Big\{ f : (X, \mathcal{D}) \to (X', \mathcal{D}') \mid f \text{ is a } \mathcal{T}\Big\{\bigvee \mathcal{D}'\Big\}\downarrow\text{-dense embedding in } \mathbf{M}^{\mathcal{C}}_{\xi_{UG_0}} \Big\}.$$

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Further reading

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