A study of topological properties in approach theory using monoidal topology

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To Wouter, my point of convergence

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Karen Van Opdenbosch
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Abstract

Monoidal topology is a research area in mathematics providing a common framework for convergence. Two parameters, a Set-monad $\mathbb{T}$ and a quantale $\mathcal{V}$, together with an extension of the monad $\mathbb{T}$ to $\mathcal{V}$-Rel, provide us with a category of lax algebras, denoted $(\mathbb{T}, \mathcal{V})$-Cat. Suitable choices for $\mathbb{T}$ and $\mathcal{V}$ result in a lax algebraic description of ordered spaces, metric spaces, topological spaces and approach spaces.

Our interest goes out to approach spaces and a first lax algebraic description of approach spaces was given by Clementino and Hofmann [CH03] by defining an extension $\overline{\beta}$ of the ultrafilter monad β to numerical relations resulting in (β, P_+)-Cat $\cong$ App.

In this work we look for relational representations of App, i.e. lax algebraic representations only using the quantale 2. We introduce the functional ideal monad $\mathbb{I}$, which is power-enriched, and using the theory of power-enriched monads developed in [HST14] we are able to prove that $(\mathbb{I}, 2)$-Cat $\cong$ App. We also look at the prime functional ideals and their corresponding monad $\mathbb{B}$. We show that $\mathbb{B}$ is a submonad of $\mathbb{I}$ satisfying those properties needed in order to conclude $(\mathbb{I}, 2)$-Cat $\cong$ $(\mathbb{B}, 2)$-Cat.

We also turn our attention to NA-App, the full subcategory of App consisting of non-Archimedean approach spaces. We answer the question of determining which parameters $\mathbb{T}$ and $\mathcal{V}$ should be used in order to capture NA-App as a category of lax algebras. It turns out that the answer lies in switching the quantale P_+ in the presentation of App as lax algebras by Clementino and Hofmann to $\mathsf{P}_\vee$, which results in the isomorphism NA-App $\cong$ $(\beta, \mathsf{P}_\vee)$-Cat.

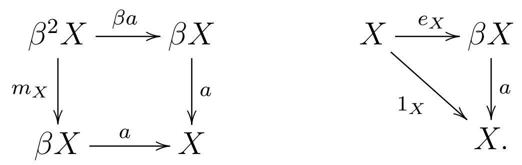
The relational descriptions of App by means of the functional ideal monad and the prime functional ideal monad, and the description of NA-App as a category of lax algebras are the main instruments for an in depth study of new approach invariants. These approach invariants will arise as topological properties in lax algebras depending on the monad $\mathbb{T}$ and the quantale $\mathcal{V}$. We study Hausdorff separation, compactness and regularity to name only a few.

Contents

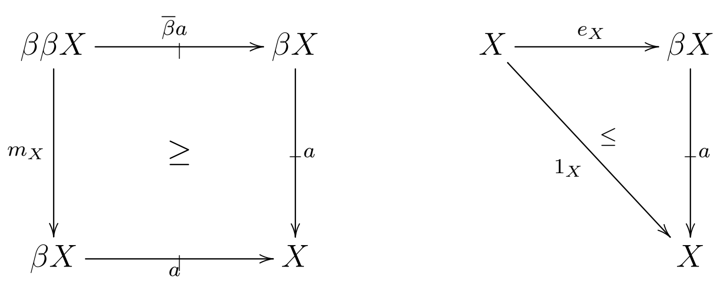
Introduction

Monoidal topology is an active research area in mathematics providing a common framework for "convergence".

A first root of the theory of monoidal topology is Barr's relational representation of topological spaces [Bar70], which forms a generalization of Manes' proof that compact Hausdorff topological spaces are the Eilenberg-Moore algebras of the ultrafilter monad $\mathbb{\beta} = (\beta, m, e)$ [Man69]. In this description, a compact Hausdorff topological space is a set X equipped with a map $a : \beta X \longrightarrow X$ assigning to every ultrafilter on X its unique point of convergence in X, satisfying two axioms that can be represented by the following diagrams



In the work of Barr [Bar70], in order to obtain a characterization of all topological spaces, the map $a : \beta X \longrightarrow X$ was replaced by a relation $a : \beta X \relbar\joinrel\mapstochar\joinrel\rightarrow X$. Now it is no longer assured that every ultrafilter converges (compactness) and that there has to be at most one point of convergence (Hausdorffness). Of course, one knows what βa is when $a : \beta X \longrightarrow X$ a map, but not when $a : \beta X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ a relation, so in order for the next definitions to make sense, the ultrafilter monad $\mathbb{\beta} = (\beta, m, e)$ now has to be extended to Rel, the category of sets and relations, as $\overline{\mathbb{\beta}} = (\overline{\beta}, m, e)$. Considering a lax version of the diagrams above, this gives us



or, in pointwise form, with a and $\overline{\beta}a$ denoted by $\rightarrow$, we get:

$$\text{transitivity: } \mathfrak{X} \rightarrow \mathcal{U} \,\&\, \mathcal{U} \rightarrow z \Rightarrow m_X\mathfrak{X} \rightarrow z$$

and

$$\text{reflexivity: } e_X(x) \rightarrow x,$$

for all $\mathfrak{X} \in \beta\beta X, \mathcal{U} \in \beta X$ and $z, x \in X$.

Barr showed that a set X equipped with a relation $a : \beta X \relbar\!\!\!\mapsto X$ satisfying the two axioms above, transitivity and reflexivity, is a topological space and that every topological space can be represented in such a way. Together with continuous maps described as the convergence preserving maps, this gives a relational presentation of the category Top, the category of topological spaces and continuous maps, which we denote

$$(\mathbb{\beta}, 2)\text{-Cat} \cong \text{Top}.$$

The terminology "transitivity" and "reflexivity" preserves the meaning of transitivity and reflexivity for ordered spaces. The axioms above applied to the identity monad $\mathbb{1}$ instead of the ultrafilter monad $\mathbb{\beta}$ yield a pair (X, a) with $a : X \relbar\!\!\!\mapsto X$ a relation, which can be denoted by $\leq$ satisfying

$$\forall x, y, z \in X : x \leq y \,\&\, y \leq z \Rightarrow x \leq z,$$

the usual transitivity axiom, and

$$\forall x \in X : x \leq x,$$

the usual reflexivity axiom. Hence

$$(\mathbb{1}, 2)\text{-Cat} \cong \text{Ord},$$

where Ord is the category of ordered sets and order preserving maps. Remark that anti-symmetry is not assumed for ordered sets and in literature these structures are often referred to as preordered sets.

A second important step in the development of monoidal topology was the description of metric spaces as (small individual) categories enriched over the extended non-negative real half-line by Lawvere [Law73]. In more familiar terms, this means that if we apply the axioms above to the identity monad $\mathbb{1}$ and change relations to multi-valued relations, one gets a lax algebraic description of quasi-metric spaces. To see this, consider $[0, \infty]$ equipped with the reversed order and structured by $+$ and 0. We denote this by

$$\mathsf{P}_+ = \big([0, \infty], \leq_{\text{op}}, +, 0\big).$$

A quasi-metric space (X, a) is a set X equipped with a map $a : X \times X \longrightarrow [0, \infty]$, or equivalently a $[0, \infty]$-valued relation $a : X \longrightarrow\!\!\!\!\!\!\mapsto X$ satisfying the transitivity axiom

$$\forall x, y, z \in X : a(x, y) + a(y, z) \geq a(x, z),$$

and the reflexivity axiom

$$\forall x \in X : a(x, x) = 0.$$

Hence, the category $q\mathsf{Met}$ of quasi-metric spaces and non-expansive maps is presented by

$$(\mathbb{1}, \mathsf{P}_+)\text{-}\mathsf{Cat} \cong q\mathsf{Met}.$$

Quasi-metric structures do not behave well with respect to the formation of initial structures, in particular products. The product in $q\mathsf{Met}$ of an infinite family of quasi-metric spaces is not compatible with the topological product of the associated underlying topologies. As a remedy to this defect, the common supercategory App (the objects of which are called approach spaces) of Top and $q\mathsf{Met}$ was introduced [Low15]. The basic difference between approach spaces and metric spaces is that in the former, one specifies and axiomatizes point-set distances, where such a point-set distance, unlike the situation for quasi-metric spaces, is not necessarily derivable from the point-point distances.

An approach space is a set X endowed with a function

$$\delta : X \times 2^X \longrightarrow [0, \infty],$$

called a distance satisfying the following properties:

(D1) $\forall x \in X, \forall A \subseteq X : x \in A \Rightarrow \delta(x, A) = 0.$

(D2) $\forall x \in X : \delta(x, \emptyset) = \infty.$

(D3) $\forall x \in X, \forall A, B \subseteq X : \delta(x, A \cup B) = \min\big(\delta(x, A), \delta(x, B)\big).$

(D4) $\forall x \in X, \forall A \subseteq X, \forall \varepsilon \geq 0 : \delta(x, A) \leq \delta\big(x, A^{(\varepsilon)}\big) + \varepsilon,$
with $A^{(\varepsilon)} := \big\{x \in X \mid \delta(x, A) \leq \varepsilon\big\}.$

The value $\delta(x, A)$ is interpreted as the distance from the point x to the set A.

The morphisms in the category App are called contractions and a contraction is a map

$$f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$$

between two approach spaces such that

$$\forall x \in X, \forall A \subseteq X : \delta_Y\big(f(x), f(A)\big) \leq \delta_X(x, A).$$

Approach spaces can be equivalently defined by a set X equipped with a limit operator

$$\lambda : \mathsf{F}X \longrightarrow [0,\infty]^X,$$

on the set $\mathsf{F}X$ of filters on X and satisfying appropriate axioms, where the value $\lambda\mathcal{F}(x)$ is interpreted as the distance that x is away from being a limit point of the filter $\mathcal{F}$. An approach space can also be defined by a tower

$$(\mathfrak{t}_\varepsilon)_{\varepsilon\in\mathbb{R}^+},$$

an ordered family of pretopologies on X indexed by the positive real numbers fulfilling certain coherence conditions, a gauge

$$\mathcal{G} \subseteq q\mathsf{Met}(X),$$

an ideal of quasi-metrics on X satisfying a saturation property, or many other equivalent structures.

The category App of approach spaces and contractions contains both Top and $q\mathsf{Met}$ as fully embedded subcategories, Top is a concretely coreflective and reflective subcategory, and $q\mathsf{Met}$ is a concretely coreflective subcategory of App. Preliminaries on approach spaces will be presented in Section 1.1.

A first lax algebraic characterization of approach spaces was given by Clementino and Hofmann [CH03] by defining an extension $\overline{\beta}$ of the ultrafilter monad β to numerical relations. Using the description of approach spaces in terms of the limit operator, an approach space can be described as (X,a) where X is a set and $a : \beta X \relbar\!\mapsto X$ is a multi-valued relation satisfying the transitivity axiom

$$a\big(m_X\mathfrak{X}, z\big) \leq \overline{\beta}a(\mathfrak{X},\mathcal{U}) + a(\mathcal{U}, z),$$

for all $\mathfrak{X} \in \beta\beta X, \mathcal{U} \in \beta X$ and $z \in X$, and the reflexivity axiom

$$a\big(e_X(x), x\big) = 0,$$

for all $x \in X$. Hence,

$$(\beta, \mathsf{P}_+)\text{-}\mathsf{Cat} \cong \mathsf{App}.$$

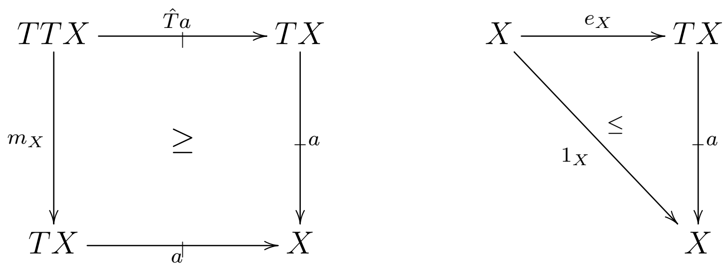
In general, the two parameters of monoidal topology are a Set-monad $\mathbb{T} = (T, m, e)$ and a quantale $\mathcal{V}$ together with a lax extension $\hat{\mathbb{T}}$ of the monad $\mathbb{T}$ to the category $\mathcal{V}$-Rel of sets and $\mathcal{V}$-valued relations. This provides us with the category

$$(\mathbb{T},\mathcal{V})\text{-}\mathsf{Cat}$$

where an object (X,a) is a set X equipped with a $\mathcal{V}$-relation

$$a : TX \relbar\!\mapsto X$$

satisfying transitivity and reflexivity.



So far we have seen monoidal topology provides a common setting to describe ordered spaces, metric spaces, topological spaces and approach spaces. Preliminaries on monoidal topology will be recalled in Chapter 1.

In Chapter 2 we turn our attention to NA-App, the full subcategory of App with objects the non-Archimedean approach spaces. Non-Archimedean approach spaces were introduced by Brock and Kent [BK98] and were also considered by Colebunders, Mynard and Trott in [CMT14] and by Boustique and Richardson [BR17] as certain limit tower spaces.

Non-Archimedean approach spaces are those approach spaces X where the distance δ satisfies the strong triangular inequality

(D4$_\vee$) $\forall x \in X, \forall A \subseteq X, \forall \varepsilon \geq 0 : \delta(x, A) \leq \delta\big(x, A^{(\varepsilon)}\big) \vee \varepsilon.$

They can easily be characterized in terms of the tower. Non-Archimedean approach spaces are those approach spaces with a tower of topologies

$$(\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$$

satisfying the coherence condition: $\forall \varepsilon \in \mathbb{R}^+ : \mathcal{T}_\varepsilon = \bigvee_{\gamma > \varepsilon} \mathcal{T}_\gamma$.

We investigate a characterization of non-Archimedean approach spaces in terms of the gauge. It turns out that non-Archimedean approach spaces are those approach spaces with a gauge basis consisting of quasi-ultrametrics, i.e. quasi-metrics $d : X \times X \longrightarrow [0, \infty]$ satisfying the strong triangular inequality

$$\forall x, y, z \in X : d(x, z) \leq d(x, y) \vee d(y, z).$$

In Section 2.2 we answer the question of determining which parameters $\mathbb{T}$ and $\mathcal{V}$ should be used in order to capture NA-App as a category of lax algebras. Inspired by the known fact that

$$(\mathbb{1}, \mathsf{P}_\vee)\text{-}\mathsf{Cat} \cong q\mathsf{Met}^u,$$

where $q\mathsf{Met}^u$ is the full subcategory of $q\mathsf{Met}$ consisting of all quasi-ultrametric spaces, we succeed in proving that the solution lies in replacing the quantale P_+ by $\mathsf{P}_\vee$ in the representation of App as (β, P_+)-Cat. We find the representation

$$(\beta, \mathsf{P}_\vee)\text{-}\mathsf{Cat} \cong \mathsf{NA}\text{-}\mathsf{App}.$$

In what we discussed so far, App could only be represented as a category of lax algebras by extending the ultrafilter monad to numerical relations. In Chapter 3 we answer the question whether some representation of App is possible in terms of relational algebras. This means we want to focus on lax-algebraic presentations of App using only the quantale $\mathcal{V} = 2$.

Our guiding example is again Top. In [Sea05] Seal described topological spaces as $\mathbb{F}$-monoids for the power-enriched filter monad $\mathbb{F}$. The key to this description is the map

$$X \longrightarrow \mathsf{F}X : x \mapsto \mathcal{V}(x)$$

which sends every point x of a topological space X to its neighborhood filter $\mathcal{V}(x)$. Moreover, since convergence in topological spaces is completely determined by $\big(\mathcal{V}(x)\big)_{x\in X}$, the representation of topological spaces as $\mathbb{F}$-monoids for the power-enriched filter monad $\mathbb{F}$ was shown by Seal to imply the presentation in terms of relational algebras

$$(\mathbb{F}, 2)\text{-}\mathsf{Cat} \cong \mathsf{Top}.$$

In order to tackle the question whether App can be described in terms of relational algebras for some power-enriched monad $\mathbb{T} = (T, m, e)$, we first focus on finding a description of App in terms of $\mathbb{T}$-monoids for a suitable power-enriched monad. The clue to the solution of this problem is the map

$$x \mapsto \mathcal{A}(x),$$

sending every point x of an approach space X to its local approach system, where $\mathcal{A}(x)$ can be derived from the gauge of the approach space by appropriate saturation of the collection

$$\{d(x, \cdot) \mid d \in \mathcal{G}\}.$$

We introduce the monad $\mathbb{I} = (\mathsf{I}, m, e)$ of functional ideals and we prove that it is power-enriched. These investigations lead to the presentation of approach spaces as $\mathbb{I}$-monoids. Moreover as convergence of functional ideals in an approach space X is completely determined by its local approach system $\big(\mathcal{A}(x)\big)_{x\in X}$, we can apply a general theorem from [HST14] about the relation between categories of $\mathbb{T}$-monoids and of relational $\mathbb{T}$-algebras, in order to conclude that

$$(\mathbb{I}, 2)\text{-}\mathsf{Cat} \cong \mathsf{App},$$

describing a representation of App as a category of relational algebras and thus giving a positive answer to the question put forward in this chapter.

We finish this chapter by studying prime functional ideals and their monad $\mathbb{B}$ [LVOV08], [LV08], which is not power-enriched. We show that $\mathbb{B}$ is a submonad of $\mathbb{I}$ satisfying exactly those properties formulated in [HST14] needed to conclude that $(\mathbb{B}, 2)\text{-}\mathsf{Cat} \cong (\mathbb{I}, 2)\text{-}\mathsf{Cat}$, thus recovering the results from [LV08] that

$$(\mathbb{B}, 2)\text{-}\mathsf{Cat} \cong \mathsf{App}.$$

The new descriptions of NA-App as a category of lax algebras obtained in Chapter 2 and of App as a category of relational algebras as developed in Chapter 3 are the main instruments for an in depth study of new approach invariants in the final chapter, Chapter 4. These approach invariants will arise as topological properties of the lax algebras (or relational algebras in case $\mathcal{V} = 2$) involved. Looking at lax algebras (X, a) as spaces and denoting the convergence relation on X by $a : TX \to\!\!\!\!\!\mapsto X$ as before, topological properties for such spaces were introduced based on the convergence notion a and depending on the monad $\mathbb{T}$, the quantale $\mathcal{V}$ and the extension of $\mathbb{T}$ to $\mathcal{V}$-Rel, [HST14]. These notions were applied to the guiding examples $(\beta, 2)\text{-}\mathsf{Cat} \cong \mathsf{Top}$, where they coincide with the usual notions of the respective properties and to $(\beta, \mathsf{P}_+)\text{-}\mathsf{Cat} \cong \mathsf{App}$ where they also coincide with some known approach invariants.

In this introduction we limit ourselves to the discussion of new approach invariants based on Hausdorff separation (at most one point of convergence), compactness (at least one point of convergence) and regularity (reversing the transitivity axiom of a) for lax algebras, but other invariants have also been studied in this thesis.

First we discuss new invariants for non-Archimedean approach spaces. Based on the representation $(\beta, \mathsf{P}_\vee)\text{-}\mathsf{Cat} \cong \mathsf{NA\text{-}App}$, a non-Archimedean approach space X is $(\beta, \mathsf{P}_\vee)$-Hausdorff if and only if for the convergence relation a having finite values $a(\mathcal{U}, x)$ and $a(\mathcal{U}, y)$, with $\mathcal{U}$ an ultrafilter on X, implies $x = y$. On the other hand looking at the tower of X which consists of an indexed family of level topologies $(\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$, we can consider three other notions, "strongly" Hausdorff (all level topologies are Hausdorff), "almost strongly" Hausdorff (all the level topologies at strictly positive levels are Hausdorff) and the property that the topological coreflection $(X, \mathcal{T}_0)$ of X is Hausdorff. It appears that all properties are equivalent except for the last one which is strictly weaker.

We use similar definitions for "strongly" compact, "almost strongly" compact and the topological coreflection $(X, \mathcal{T}_0)$ being compact. It appears that $(\beta, \mathsf{P}_\vee)$-compact is equivalent to almost strongly compact, both being equivalent to the known 0-compactness for approach spaces [Low15]. Strongly compact is equivalent to the topological coreflection $(X, \mathcal{T}_0)$ of X being compact and both are strictly stronger than the former.

We investigate similar links for regularity. The property "strongly" regular was first introduced by Brock and Kent [BK98] and this property was also studied in the context of contractive extensions in [CMT14]. It appears that strongly regular and almost strongly regular are both equivalent to $(\beta, \mathsf{P}_\vee)$-regularity. Regularity for approach spaces as introduced in [Rob92] by Robeys is strictly weaker and implies the topological coreflection $(X, \mathcal{T}_0)$ of X being regular.

Next we turn our attention to relational algebras describing App. We investigate topological properties in App induced by the representation $(\mathbb{I}, 2)\text{-}\mathsf{Cat} \cong \mathsf{App}$ for the power-enriched monad $\mathbb{I}$. Since the improper functional ideal P_b^X consisting of all bounded functions from X to $[0, \infty]$, converges to all points of X, only trivial results can be expected when studying some of the topological properties listed above. Therefore in some cases we will abandon the improper element and restrict to proper functional ideals on X by considering the subfunctor I_p defined by $\mathsf{I}_p X = \mathsf{I} X \setminus \{\mathsf{P}_b^X\}$.

The $(\mathsf{I}_p, 2)$-Hausdorff property explicitly means that when some proper functional ideal converges to both points x and y, then $x = y$. This notion turns out to be equivalent to the approach invariant stating that the topological coreflection of X is Hausdorff.

For a study of compactness, it turns out that abandoning the improper element does not have any effect on the topological property and when considering $(\mathbb{I}, 2)$-compactness in App, this explicitly means that every functional ideal has a point of convergence. In particular there exists $x \in X$ such that $\{0\} \rightarrowtail x$, or equivalently there exists $x \in X$ such that $\mathcal{A}(x) = \{0\}$. This notion will be called supercompactness in App.

The most extensive study in this thesis is the one for regularity. First of all we prove some general results for power-enriched monads. We show that a relational $\mathbb{T}$-algebra (X, a), for $\mathbb{T}$ power-enriched, is $(\mathbb{T}, 2)$-regular if and only if it is indiscrete, even when restricting to proper elements. For our particular monad $\mathbb{I}$ in order to obtain some interesting invariant related to regularity, we restrict ourselves to functional ideals generated by certain selections. In doing so we obtain a property equivalent to regularity for approach spaces, as introduced by Robeys [Rob92], for which we provide a characterization in terms of functional ideals.

Finally we investigate topological properties in App induced by the representation $(\mathbb{B}, 2)\text{-}\mathsf{Cat} \cong \mathsf{App}$. Since $\mathbb{B}$ is not a power-enriched monad, the situation will be different. However, the improper functional ideal P_b^X is a prime functional ideal, hence in some cases we will again abandon the improper element and restrict to proper prime functional ideals on X by considering the subfunctor B_p defined by $\mathsf{B}_p X = \mathsf{B}_p X \setminus \{\mathsf{P}_b^X\}$ in order to get interesting results.

The notion $(\mathsf{B}_p, 2)$-Hausdorff means that when some proper prime functional ideal converges to both points x and y then $x = y$. This notion turns out to be equivalent to the $(\mathsf{I}_p, 2)$-Hausdorff property.

Considering $(\mathbb{B}, 2)$-compactness we get a different property than $(\mathbb{I}, 2)$-compactness. Whereas we called $(\mathbb{I}, 2)$-compactness of X supercompactness in App, $(\mathbb{B}, 2)$-compactness gives different results since $\{0\}$ is not a prime functional ideal. An approach space X is $(\mathbb{B}, 2)$-compact if and only if its topological coreflection is compact.

Due to the fact that there are improper elements, $(\mathbb{B}, 2)$-regularity is uninteresting, since an approach space X is $(\mathbb{B}, 2)$-regular if and only if it is indiscrete. Contrary to the case for the functional ideal monad, restricting to proper prime functional ideals already gives an interesting property. We prove that $(\mathsf{B}_p, 2)$-regularity is equivalent to the approach space being topological and regular. It requires further weakening of the concept to obtain a characterization of the usual regularity property in App [Rob92] in terms of convergence of prime functional ideals.

Chapter 1

Preliminaries

1.1 Approach spaces

Topologists prefer to work in a category like Top (topological spaces and continuous maps), even if this means that they have to abandon an originally metric setup in the category qMet (quasi-metric spaces and non-expansive maps). A reason why we prefer to look at the underlying topology is given by the fact that it is the topology which provides us with the framework in which most of the basic concepts of analysis are defined, such as, for example, convergence, continuity and compactness. The most fundamental problem which arises in this transition is the fact that metric initial structures do not necessarily accord with topological initial structures: the countable product of metrizable spaces is metrizable, but there is no canonical metric for the product topology, and for uncountable products there simply is no metric at all for the product topology. This is the main motivation for looking at a common supercategory of Top and qMet, namely App (the topological construct of approach spaces and contractions), in which we are able to find a suitable solution to cope with this problem. Approach spaces thus are a generalization of both metric spaces and topological spaces.

Approach theory was introduced by Lowen in various papers between 1988 and 1995, resulting in a first book [Low97]. In 2015 a second book on approach theory was published, [Low15] and we refer to this work as a comprehensive source on approach theory.

In this section we give an introduction to approach theory. We list all required results from [Low15] without proofs and we refer to this work for more information.

1.1.1 The objects: equivalent descriptions of approach spaces

Approach spaces can be defined by conceptually very different, but nevertheless equivalent structures. Here we will introduce the structures needed in this work. More information on these structures, and various other structures defining approach spaces, can be found in Sections 1.1 and 1.2 of *Index Analysis* [Low15].

A. Distances

The first structure which we will be considering is that of a distance between points and sets. In a metric space (X, d) a distance between pairs of points is given and a distance between points and sets can be derived using the following formula

$$\delta_d(x, A) := \inf_{a \in A} d(x, a) \quad \forall x \in X, \forall A \subseteq X.$$

Here we start from a concept of distance between points and sets.

The unbounded closed interval $[0, \infty]$ will be denoted by P.

Definition 1.1.1.1. (Distance) A function

$$\delta : X \times 2^X \longrightarrow \mathsf{P}$$

is called a *distance* if it satisfies the following properties.

(D1) $\forall x \in X, \forall A \subseteq X : x \in A \Rightarrow \delta(x, A) = 0.$

(D2) $\forall x \in X : \delta(x, \emptyset) = \infty.$

(D3) $\forall x \in X, \forall A, B \subseteq X : \delta(x, A \cup B) = \min\big(\delta(x, A), \delta(x, B)\big).$

(D4) $\forall x \in X, \forall A \subseteq X, \forall \varepsilon \geq 0 : \delta(x, A) \leq \delta\big(x, A^{(\varepsilon)}\big) + \varepsilon,$
with $A^{(\varepsilon)} := \big\{x \in X \mid \delta(x, A) \leq \varepsilon\big\}.$

The value $\delta(x, A)$ is interpreted as the distance from the point x to the set A.

The following proposition contains some fundamental properties which will be useful in the sequel and can be found as Proposition 1.1.2 in *Index Analysis* [Low15].

Proposition 1.1.1.2. If $\delta : X \times 2^X \longrightarrow \mathsf{P}$ is a distance, then the following properties hold:

1. $\forall x \in X, \forall A, B \subseteq X : A \subseteq B \Rightarrow \delta(x, B) \leq \delta(x, A).$
2. $\forall x \in X, \forall \mathcal{A} \subseteq 2^X, \mathcal{A}$ finite: $\delta\big(x, \bigcup \mathcal{A}\big) = \min_{A \in \mathcal{A}} \delta(x, A).$
3. $\forall x \in X, \forall A, B \subseteq X : \delta(x, A) \leq \delta(x, B) + \sup_{b \in B} \delta(b, A).$

For a given subset $A \subseteq X$, we define

$$\delta_A : X \longrightarrow \mathsf{P} : x \mapsto \delta(x, A)$$

and we call such functions *distance functionals*.

B. Limit operators

The second structure which we will need is the one of a limit operator.

We will use the following notations. $\mathsf{F}X$ stands for the set of all filters on X, and βX stands for the set of all ultrafilters on X. If $\mathcal{F}$ is a given filter on X, then we will denote by $\mathsf{F}(\mathcal{F})$ the collection of all filters on X which are finer than $\mathcal{F}$, and by $\beta(\mathcal{F})$ the collection of all ultrafilters on X which are finer than $\mathcal{F}$.

If $\mathcal{A}$ is a collection of subsets of X, then the stack of $\mathcal{A}$ is defined as

$$\mathrm{stack}\,\mathcal{A} := \{B \subseteq X \mid \exists A \in \mathcal{A} : A \subseteq B\}. \tag{1.1}$$

If $\mathcal{F}$ is a filter on X then the $\sec$ of $\mathcal{F}$ is defined as

$$\sec \mathcal{F} := \bigcup_{\mathcal{U} \in \beta(\mathcal{F})} \mathcal{U} = \{A \subseteq X \mid \forall F \in \mathcal{F} : A \cap F \neq \emptyset\}, \tag{1.2}$$

where $\sec$ stands for secant. If $\mathcal{A}$ is a filterbasis, then $\mathrm{stack}\,\mathcal{A}$ is the filter generated by $\mathcal{A}$. Not to overload notations, we often omit writing stack before a filterbase. In case $\mathcal{A}$ reduces to a single set A, we write $\mathrm{stack}\,A$, or even shorter $\dot{A}$, instead of $\mathrm{stack}\{A\}$, and in case the single set A furthermore reduces to a single point a we write $\dot{a}$.

Consider a filter $\mathcal{F} \in \mathsf{F}\,X$. For a function $f : X \longrightarrow Y$, we define the image of a filter as the filter on Y generated by $\{f(F) \mid F \in \mathcal{F}\}$, i.e.

$$f(\mathcal{F}) := \mathrm{stack}\{f(F) \mid F \in \mathcal{F}\}. \tag{1.3}$$

This definition makes sure that we do not always have to write stack when considering images of filters, making our formulas easier to digest.

We define a diagonal operation for filters. Early references on the *Kowalsky diagonal operation* are Kent [Ken64], Kowalsky [Kow54] and Lowen-Colebunders [LC89].

Given a filter $\mathfrak{F}$ on $\mathsf{F}\,X$, i.e. $\mathfrak{F} \in \mathsf{F}^2\,X$, we define $\Sigma\mathfrak{F}$ as follows

$$\Sigma\mathfrak{F} = \bigcup_{\mathcal{A} \in \mathfrak{F}} \bigcap_{\mathcal{W} \in \mathcal{A}} \mathcal{W} = \{A \subseteq X \mid \tilde{A} \in \mathfrak{F}\}, \tag{1.4}$$

with $\tilde{A} = \{\mathcal{W} \in \mathsf{F}\,X \mid A \in \mathcal{W}\}$. We can even particularize this definition to ultrafilters $\mathfrak{X}$ on βX, i.e. $\mathfrak{X} \in \beta^2 X$. Here $\Sigma\mathfrak{X}$ is defined in exactly the same way:

$$\Sigma\mathfrak{X} = \bigcup_{\mathcal{A} \in \mathfrak{X}} \bigcap_{\mathcal{W} \in \mathcal{A}} \mathcal{W} = \{A \subseteq X \mid \tilde{A} \in \mathfrak{X}\}, \tag{1.5}$$

with $\tilde{A} = \{\mathcal{W} \in \beta X \mid A \in \mathcal{W}\}$. In case all filters involved are ultrafilters, the Kowalsky sum is again an ultrafilter.

Consider now a non-empty set J and a function $\sigma : J \longrightarrow \mathsf{F}\, X$. This function gives us a family of filters $\big(\sigma(j)\big)_{j\in J}$ on X. If we now consider a filter $\mathcal{F}$ on J, then we are able to apply the Kowalsky diagonal operation Σ to $\sigma\mathcal{F}$. This gives us

$$\begin{aligned} \Sigma\sigma(\mathcal{F}) &= \{A \subseteq X \mid \tilde{A} \in \sigma\mathcal{F}\} \\ &= \bigcup_{\mathcal{A}\in\sigma\mathcal{F}} \bigcap_{\mathcal{W}\in\mathcal{A}} \mathcal{W} \\ &= \bigcup_{F\in\mathcal{F}} \bigcap_{\mathcal{W}\in\sigma F} \mathcal{W} \\ &= \bigcup_{F\in\mathcal{F}} \bigcap_{j\in F} \sigma(j). \end{aligned}$$

Before defining the structure of a limit operator, we give two useful purely filter-theoretic result, which we will require multiple times throughout this work. These results can be found in *Index Analysis* [Low15] as Lemma 1.1.4 and Lemma 1.1.5.

Lemma 1.1.1.3. If $\mathcal{F}$ is a filter, and for each ultrafilter $\mathcal{U} \in \beta(\mathcal{F})$ we have selected a set $S(\mathcal{U}) \in \mathcal{U}$, then there exists a finite set $U_S \subseteq \beta(\mathcal{F})$ such that

$$\bigcup_{\mathcal{U}\in U_S} S(\mathcal{U}) \in \mathcal{F}. \tag{1.6}$$

The following formula allows us to interchange liminf and limsup in several instances.

Lemma 1.1.1.4. If $\mathcal{U}$ is an ultrafilter on X and $f : X \longrightarrow \mathsf{P}$ is an arbitrary function, then

$$\sup_{U\in\mathcal{U}} \inf_{y\in U} f(y) = \inf_{U\in\mathcal{U}} \sup_{y\in U} f(y). \tag{1.7}$$

In [Low15] various equivalent definitions of limit operators are given. We introduce the one which is most interesting for our work.

Definition 1.1.1.5. (Limit operator) A function

$$\lambda : \mathsf{F}X \longrightarrow \mathsf{P}^X$$

is called a *limit operator* if it satisfies the following properties.

(L1) $\forall x \in X : \lambda\dot{x}(x) = 0.$

(L2) For a (non-empty) family $(\mathcal{F}_j)_{j\in J}$ of filters on X

$$\lambda\left(\bigcap_{j\in J}\mathcal{F}_j\right) = \sup_{j\in J}\lambda\mathcal{F}_j.$$

(L*) For any set J, for any $\psi : J \longrightarrow X$, for any $\sigma : J \longrightarrow \mathsf{F}X$ and for any $\mathcal{F} \in \mathsf{F}J$

$$\lambda\Sigma\sigma(\mathcal{F}) \leq \lambda\psi(\mathcal{F}) + \inf_{F\in\mathcal{F}}\sup_{j\in F}\lambda\sigma(j)\big(\psi(j)\big).$$

The value $\lambda\mathcal{F}(x)$ is interpreted as the distance that the point x is away from being a limit point of the filter $\mathcal{F}$. The smaller the value $\lambda\mathcal{F}(x)$, the closer x becomes to being a limit point of $\mathcal{F}$. Notice also that it immediately follows from (L2) that

$$\forall \mathcal{F}, \mathcal{G} \in \mathsf{F}X : \mathcal{G} \subseteq \mathcal{F} \Rightarrow \lambda\mathcal{F} \leq \lambda\mathcal{G}.$$

We also mention the following result, which shows that a limit operator can equivalently be defined using ultrafilters instead of filters. For more detail, we refer to Theorem 1.1.11 in *Index Analysis* [Low15].

Theorem 1.1.1.6. *Given a function $\lambda : \beta X \longrightarrow \mathsf{P}^X$ satisfying (L1), the extension to $\mathsf{F}X$ defined by*

$$\overline{\lambda} : \mathsf{F}X \longrightarrow \mathsf{P}^X : \mathcal{F} \mapsto \sup_{\mathcal{U}\in\beta(\mathcal{F})}\lambda\mathcal{U}$$

is a limit operator if and only if it satisfies the following property:

(Lβ^*) For any set J, for any $\psi : J \longrightarrow X$, for any $\sigma : J \longrightarrow \beta X$ and for any $\mathcal{U} \in \beta J$

$$\lambda\Sigma\sigma(\mathcal{U}) \leq \lambda\psi(\mathcal{U}) + \inf_{U\in\mathcal{U}}\sup_{j\in U}\lambda\sigma(j)\big(\psi(j)\big).$$

C. Approach systems

Approach systems can be thought of as a localization of the notion of metric. In each point of the space X we give a collection of P-valued functions, called local distances, each of which measures a distance from the given point to any other point of the space.

A non-empty subset $\mathcal{A}$ of P-valued functions on a given set X is called an ideal in P^X if it is closed under the operation of taking finite suprema and under the operation of taking smaller functions.

Given a collection of functions $\mathcal{A} \subseteq \mathsf{P}^X$ and a function $\varphi \in \mathsf{P}^X$, we will say that φ is *dominated* by $\mathcal{A}$ if

$$\forall\varepsilon > 0, \forall\omega < \infty : \exists\varphi^\omega_\varepsilon \in \mathcal{A} \text{ such that } \varphi \wedge \omega \leq \varphi^\omega_\varepsilon + \varepsilon. \tag{1.8}$$

We will then also say that the family $(\varphi_\varepsilon^\omega)_{\varepsilon>0,\omega<\infty}$ dominates φ.

Further we will say that a collection of functions $\mathcal{A} \subseteq \mathsf{P}^X$ is *saturated*, if any function which is dominated by $\mathcal{A}$ already belongs to $\mathcal{A}$.

Definition 1.1.1.7. (Approach system) A collection of ideals $\big(\mathcal{A}(x)\big)_{x\in X}$ in P^X, indexed by the points of X, is called an *approach system* if for all $x \in X$ the following properties hold:

(A1) $\forall \varphi \in \mathcal{A}(x) : \varphi(x) = 0.$

(A2) $\mathcal{A}(x)$ is saturated.

(A3) $\forall \varphi \in \mathcal{A}(x), \forall \varepsilon > 0, \forall \omega < \infty, \exists (\varphi_z)_{z\in X} \in \prod_{z\in X} \mathcal{A}(z)$ such that

$$\forall z, y \in X : \varphi(y) \wedge \omega \leq \varphi_x(z) + \varphi_z(y) + \varepsilon.$$

For any $x \in X$, a function in $\mathcal{A}(x)$ is called a *local distance* in x. (A3) will be referred to as the *mixed triangular inequality*. The value $\varphi(y)$ of a local distance $\varphi \in \mathcal{A}(x)$ at a point $y \in X$ is interpreted as the "distance from x to y according to φ". The set of local distances in a point can be compared to the set of neighborhoods of a point in a topological space. Each neighborhood determines its own set of points which are considered close by (in the neighborhood of) the given point. In the same way each local distance makes its own measurement of the distance other points in the space lie from the given point.

Often one can determine collections $\mathcal{B}(x), x \in X$, which would be natural candidates to form an approach system, but not all required properties are fulfilled. In particular property (A2) is not often automatically fulfilled. To handle this we introduce a type of basis for approach systems. We recall that a subset $\mathcal{B}$ of P^X is called an ideal basis in P^X if, for any $\varphi, \psi \in \mathcal{B}$, there exists $\mu \in \mathcal{B}$ such that $\varphi \vee \psi \leq \mu$.

Definition 1.1.1.8. A collection of ideal bases $\big(\mathcal{B}(x)\big)_{x\in X}$ in P^X is called an *approach basis* if, for all $x \in X$, the following properties hold:

(B1) $\forall \varphi \in \mathcal{B}(x) : \varphi(x) = 0.$

(B2) $\forall \varphi \in \mathcal{B}(x), \forall \varepsilon > 0, \forall \omega < \infty, \exists (\varphi_z)_{z\in X} \in \prod_{z\in X} \mathcal{B}(z)$ such that

$$\forall z, y \in X : \varphi(y) \wedge \omega \leq \varphi_x(z) + \varphi_z(y) + \varepsilon.$$

Since (B1) corresponds to (A1) and (B2) to (A3), an approach system is also an approach basis, and any result for approach bases will also hold for approach systems. In order to derive the set of all local distances from an approach basis we will also require the following *saturation operation*. Given a subset $\mathcal{B} \subseteq \mathsf{P}^X$ we define

$$\widehat{\mathcal{B}} := \Big\{ \varphi \in \mathsf{P}^X \mid \mathcal{B} \text{ dominates } \varphi \Big\}. \tag{1.9}$$

Definition 1.1.1.9. A collection of ideal bases $\big(\mathcal{B}(x)\big)_{x\in X}$ is called a *basis for an approach system* $\big(\mathcal{A}(x)\big)_{x\in X}$, if for all $x \in X$, $\mathcal{A}(x)$ equals the saturation of $\mathcal{B}(x)$, i.e. $\mathcal{A}(x) = \widehat{\mathcal{B}(x)}$. In this case we also say that $\big(\mathcal{B}(x)\big)_{x\in X}$ generates $\big(\mathcal{A}(x)\big)_{x\in X}$.

Proposition 1.1.1.10. If $\big(\mathcal{B}(x)\big)_{x\in X}$ is an approach basis, then $\big(\widehat{\mathcal{B}(x)}\big)_{x\in X}$ is an approach system with $\big(\mathcal{B}(x)\big)_{x\in X}$ as basis and if $\big(\mathcal{B}(x)\big)_{x\in X}$ is a basis for an approach system $\big(\mathcal{A}(x)\big)_{x\in X}$, then it is an approach basis.

Definition 1.1.1.11. It follows from the saturation condition that the set $\mathcal{A}_b(x)$ of all bounded functions in $\mathcal{A}(x)$ is a particularly interesting basis. It satisfies the saturation condition in a simpler form, which says that for all $\mu \in \mathsf{P}^X$ bounded:

$$\forall \varepsilon > 0, \exists \varphi \in \mathcal{A}_b(x) : \mu \leq \varphi + \varepsilon \Rightarrow \mu \in \mathcal{A}_b(x). \tag{1.10}$$

We refer to this collection as the *bounded approach basis* or *bounded approach system*.

D. Gauges

Given a set X, a map $d : X \times X \longrightarrow \mathsf{P}$ which vanishes on the diagonal and satisfies the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z), \tag{1.11}$$

for all $x, y, z \in X$ will be called a *quasi-metric*. If the map is moreover symmetric then it is called a *metric*. If the underlying topology is Hausdorff, the quasi-metric will be called *separated*. A map $f : (X, d) \longrightarrow (X', d')$ between quasi-metric spaces is called *non-expansive* if

$$\forall x, y \in X : d'\big(f(x), f(y)\big) \leq d(x, y). \tag{1.12}$$

The collection of all quasi-metrics on a set X will be denoted by $q\mathsf{Met}(X)$. The category of all quasi-metric spaces (respectively metric spaces) equipped with non-expansive maps as morphisms is denoted $q\mathsf{Met}$ (respectively Met).

Given a collection $\mathcal{D} \subseteq q\mathsf{Met}(X)$ and a quasi-metric $d \in q\mathsf{Met}(X)$, we will say that d is *locally dominated* by $\mathcal{D}$ if for all $x \in X, \varepsilon > 0$ and $\omega < \infty$ there exists a $d_x^{\varepsilon,\omega} \in \mathcal{D}$ such that

$$d(x, \cdot) \wedge \omega \leq d_x^{\varepsilon,\omega}(x, \cdot) + \varepsilon. \tag{1.13}$$

Further we say that a collection of quasi-metrics $\mathcal{D}$ is *locally saturated*, if any quasi-metric d which is locally dominated by $\mathcal{D}$ already belongs to $\mathcal{D}$.

Definition 1.1.1.12. (Gauge) A subset $\mathcal{G}$ of $q\mathsf{Met}(X)$ is called a *gauge* if it is an ideal in $q\mathsf{Met}(X)$ which fulfills the following property:

(G1) $\mathcal{G}$ is locally saturated.

Similar to the case for approach systems here too, it regularly happens that one has a collection of quasi-metrics which would be a natural candidate to form a gauge but not all conditions are fulfilled. The following type of collection will often be encountered.

Definition 1.1.1.13. A subset $\mathcal{H}$ of $q\mathsf{Met}(X)$ is called *locally directed* if for any $\mathcal{H}_0 \subseteq \mathcal{H}$ finite we have that $\sup_{d\in\mathcal{H}_0} d$ is locally dominated by $\mathcal{H}$.

By definition, a gauge, being an ideal, is locally directed and similar to the situation for approach systems and approach bases, here too, any result shown to hold for locally directed sets will also hold for gauges.

In order to derive the gauge from a locally directed set we will also require a local saturation operation which is perfectly similar to the one for approach systems.

Given a subset $\mathcal{D} \subseteq q\mathsf{Met}(X)$ we define

$$\widehat{\mathcal{D}} := \{d \in q\mathsf{Met}(X) \mid \mathcal{D} \text{ locally dominates } d\}. \tag{1.14}$$

We call $\widehat{\mathcal{D}}$ the *local saturation* of $\mathcal{D}$.

Definition 1.1.1.14. A set $\mathcal{H}$ in $q\mathsf{Met}(X)$ is called a *basis* for the gauge $\mathcal{G}$ if $\widehat{\mathcal{H}} = \mathcal{G}$. In this case we also say that $\mathcal{H}$ generates $\mathcal{G}$ or that $\mathcal{G}$ is generated by $\mathcal{H}$.

Proposition 1.1.1.15. If $\mathcal{H}$ is a locally directed set, then $\widehat{\mathcal{H}}$ is a gauge with $\mathcal{H}$ as basis and if $\mathcal{H}$ is a basis for the gauge $\mathcal{G}$, then it is locally directed.

Definition 1.1.1.16. Here too it is useful to mention that a particularly interesting basis for a gauge $\mathcal{G}$ is given by the set $\mathcal{G}_b$ of all bounded quasi-metrics in $\mathcal{G}$. This set satisfies the saturation condition in a simpler form, namely for any bounded quasi-metric d

$$\forall x \in X, \forall \varepsilon > 0, \exists d_x^\varepsilon \in \mathcal{G}_b : d(x,\cdot) \leq d_x^\varepsilon(x,\cdot) + \varepsilon \Rightarrow d \in \mathcal{G}_b. \tag{1.15}$$

We refer to this collection as the *bounded gauge basis*.

E. Towers

A tower is an ordered family of pre-topologies on X, indexed by the positive real numbers and fulfilling certain coherence conditions. Towers will turn out to be the most interesting characterization of the so called non-Archimedean approach spaces, which will be introduced in Section 2.1.

Definition 1.1.1.17. (Tower) A family of functions

$$\mathfrak{t}_\varepsilon : 2^X \longrightarrow 2^X, \qquad \varepsilon \in \mathbb{R}^+$$

is called a *tower* if it satisfies the following properties:

(T1) $\forall A \in 2^X, \forall \varepsilon \in \mathbb{R}^+ : A \subseteq \mathfrak{t}_\varepsilon(A)$.

(T2) $\forall \varepsilon \in \mathbb{R}^+ : \mathfrak{t}_\varepsilon(\emptyset) = \emptyset$.

(T3) $\forall A, B \in 2^X, \forall \varepsilon \in \mathbb{R}^+ : \mathfrak{t}_\varepsilon(A \cup B) = \mathfrak{t}_\varepsilon(A) \cup \mathfrak{t}_\varepsilon(B)$.

(T4) $\forall A \in 2^X, \forall \varepsilon, \gamma \in \mathbb{R}^+ : \mathfrak{t}_\varepsilon(\mathfrak{t}_\gamma(A)) \subseteq \mathfrak{t}_{\varepsilon+\gamma}(A)$.

(T5) $\forall A \in 2^X, \forall \varepsilon \in \mathbb{R}^+ : \mathfrak{t}_\varepsilon(A) = \bigcap_{\varepsilon<\gamma} \mathfrak{t}_\gamma(A)$.

Note that by (T3) and (T5) we have

$$\forall A \subseteq B \subseteq X, \forall \alpha, \beta \in \mathbb{R}^+ : \alpha \leq \beta \Rightarrow \mathfrak{t}_\alpha(A) \subseteq \mathfrak{t}_\beta(B).$$

F. Functional ideal convergence

In this section we define our last structure, functional ideal convergence. The idea behind this is to embed the numerical information of the theory into ideals of functions in P^X and to use these rather than filters to describe convergence.

This type of structure will be of particular interest in Chapter 3.

Definition 1.1.1.18. An (order theoretic) ideal $\mathfrak{I}$ in P^X is called a *functional ideal* (on X) if it fulfills the following properties:

(I1) Each function $\varphi \in \mathfrak{I}$ is bounded.

(I2) $\mathfrak{I}$ is saturated in the sense that for all $\mu \in \mathsf{P}^X$:

$$\forall \varepsilon > 0, \exists \varphi \in \mathfrak{I} : \mu \leq \varphi + \varepsilon \Rightarrow \mu \in \mathfrak{I}. \tag{1.16}$$

Note that condition (I1) implies that we are actually considering ideals in P_b^X. Given a functional ideal $\mathfrak{I}$ we define its *characteristic value* as

$$c(\mathfrak{I}) := \sup_{\mu\in\mathfrak{I}} \inf_{x\in X} \mu(x) = \sup\{\alpha \mid \alpha \text{ constant}, \alpha \in \mathfrak{I}\}. \tag{1.17}$$

It follows immediately from the definition that there is only one functional ideal which has an infinite characteristic value and this is the functional ideal consisting of all bounded functions, i.e. P_b^X. We denote this functional ideal $\mathfrak{Z}_X$. A functional ideal with a finite characteristic value will be called a *proper* functional ideal and $\mathfrak{Z}_X$ will be called the *improper* functional ideal.

If $\mathcal{B} \subseteq \mathsf{P}_b^X$ is an ideal, then we can saturate it, similarly to the saturation introduced for approach systems:

$$\widehat{\mathcal{B}} := \{\mu \in \mathsf{P}_b^X \mid \forall\varepsilon > 0, \exists\varphi \in \mathcal{B}; \mu \leq \varphi + \varepsilon\}. \tag{1.18}$$

This is a functional ideal and we say that $\mathcal{B}$ is a basis for $\widehat{\mathcal{B}}$.

Definition 1.1.1.19. Given a proper functional ideal $\mathfrak{I}$ on X such that $c(\mathfrak{I}) \leq \alpha < \infty$ we define

$$\mathfrak{f}_\alpha(\mathfrak{I}) := \big\{\{\mu < \beta\} \mid \mu \in \mathfrak{I}, \alpha < \beta\big\},$$

with

$$\{\mu < \beta\} := \{x \in X \mid \mu(x) < \beta\}.$$

This is a filter on X. We will denote $\mathfrak{f}_{c(\mathfrak{I})}(\mathfrak{I})$ simply by $\mathfrak{f}(\mathfrak{I})$. If $\mathcal{F}$ is a filter on X then we define

$$\iota_X(\mathcal{F}) := \big\{\mu \in \mathsf{P}_b^X \mid \forall\alpha \in]0,\infty[: \{\mu < \alpha\} \in \mathcal{F}\big\}.$$

This is a proper functional ideal with characteristic value equal to zero and it is generated by $\{\theta_F^\omega \mid F \in \mathcal{F}, \omega < \infty\}$. For $A \subseteq X$ and $\omega < \infty$, $\theta_A^\omega : X \longrightarrow \mathsf{P}$ is a two-valued map defined by

$$\theta_A^\omega(x) := \begin{cases} 0 & x \in A, \\ \omega & x \notin A. \end{cases}$$

When $\mathcal{F} = \mathcal{P}X$ is the improper filter, $\iota_X(\mathcal{F}) = \mathfrak{Z}_X$. In particular when $A \subseteq X$ and $\mathcal{F} = \dot{A}$, then we get $\iota_X(\dot{A}) = \{\mu \in \mathsf{P}_b^X \mid \mu_{|A} = 0\}$.

If $\mathfrak{I}$ is a functional ideal on X and $\alpha \in \mathsf{P}$, then we define

$$\mathfrak{I} \oplus \alpha := \begin{cases} \{\nu \in \mathsf{P}_b^X \mid \exists\mu \in \mathfrak{I} : \nu \leq \mu + \alpha\} & \alpha \text{ finite}, \\ \mathfrak{Z}_X & \alpha = \infty. \end{cases} \tag{1.19}$$

Obviously $c(\mathfrak{I} \oplus \alpha) = c(\mathfrak{I}) + \alpha$.

If X is a set, we let $\mathsf{I}X$ be the set of all functional ideals on X. The collection of functional ideals on X, $\mathsf{I}X$, is a "conditional" lattice in the following sense. Arbitrary infima always exist and are proper as long as at least one of the functional ideals involved is proper. If $(\mathfrak{I}_i)_{i\in I}$ is a family of functional ideals on X, then the infimum is given by

$$\inf_{i\in I} \mathfrak{I}_i = \bigcap_{i\in I} \mathfrak{I}_i = \big\{ \inf_{i\in I} \mu_i \mid \forall i \in I : \mu_i \in \mathfrak{I}_i \big\}. \tag{1.20}$$

In general the union of an arbitrary family of proper functional ideals on X is no longer a functional ideal on X. The supremum however always exists. If $(\mathfrak{I}_i)_{i\in I}$ is a family of proper functional ideals the supremum of the family is given by

$$\sup_{i\in I} \mathfrak{I}_i = \widehat{\mathcal{A}}, \tag{1.21}$$

where

$$\mathcal{A} = \big\{ \sup_{k\in K} \mu_k \mid K \subseteq I \text{ finite}, \forall k \in K : \mu_k \in \mathfrak{I}_k \big\}.$$

This supremum need not be proper.

For a map $f : X \longrightarrow Y$ if $\mathfrak{I}$ is a functional ideal on X we define and denote its image $\mathsf{I}f(\mathfrak{I})$ as

$$\mathsf{I}f(\mathfrak{I}) := \{\mu \in \mathsf{P}_b^Y \mid \mu \cdot f \in \mathfrak{I}\}. \tag{1.22}$$

It is immediately verified that this is indeed a functional ideal on Y with basis given by

$$\{(f\nu)_\eta \mid \nu \in \mathfrak{I}, \eta < \infty\} \tag{1.23}$$

where

$$(f\nu)_\eta(y) := \begin{cases} \inf_{y=f(x)} \nu(x) & y \in f(X), \\ \eta & y \notin f(X). \end{cases} \tag{1.24}$$

For the following proposition, we refer to Proposition 1.1.46 in *Index Analysis* [Low15].

Proposition 1.1.1.20. The following properties hold with $f : X \longrightarrow Y$ a given map:

1. $\forall \mathfrak{I}_k \in \mathsf{I}X, k \in K$: $\mathsf{I}f(\bigcap_{k\in K} \mathfrak{I}_k) = \bigcap_{k\in K} \mathsf{I}f(\mathfrak{I}_k)$.
2. $\forall \mathfrak{I} \in \mathsf{I}X : c\big(\mathsf{I}f(\mathfrak{I})\big) = c(\mathfrak{I})$.
3. $\forall \mathfrak{I} \in \mathsf{I}X, \forall \alpha \in [c(\mathfrak{I}), \infty[$: $\mathfrak{f}_\alpha\big(\mathsf{I}f(\mathfrak{I})\big) = f\big(\mathfrak{f}_\alpha(\mathfrak{I})\big)$.
4. $\forall \mathcal{F} \in \mathsf{F}X : \mathsf{I}f\big(\iota_X(\mathcal{F})\big) = \iota_Y\big(f(\mathcal{F})\big)$.

We require a diagonal operation for functional ideals which will be crucial in Section 3.1.2. For this we need the definition of the map $l : \mathsf{P}_b^X \longrightarrow \mathsf{P}_b^{\mathsf{I}X} : \mu \mapsto l_\mu$ where $\mathfrak{I} \in \mathsf{I}X$ is mapped to

$$l_\mu(\mathfrak{I}) := \inf\{\alpha \in \mathsf{P} \mid \mu \in \mathfrak{I} \oplus \alpha\}. \tag{1.25}$$

l is well-defined, furthermore, by saturatedness the infimum in the definition is actually a minimum so that for any $\mu \in \mathsf{P}_b^X$ and $\mathfrak{I} \in \mathsf{I}X$ we have $\mu \in \mathfrak{I} \oplus l_\mu(\mathfrak{I})$.

We list some properties of the map l for which we refer to Proposition 1.1.57 in *Index Analysis* [Low15].

Proposition 1.1.1.21. The following properties hold.

1. For any $\mu, \nu \in \mathsf{P}_b^X$ and $\mathfrak{I} \in \mathsf{I}X : l_{\mu\vee\nu}(\mathfrak{I}) = l_\mu(\mathfrak{I}) \vee l_\nu(\mathfrak{I})$.

2. For any $\mathfrak{I} \in \mathsf{I}X$, if θ is constant then $l_\theta(\mathfrak{I}) = \theta \ominus c(\mathfrak{I})$ and in particular $l_\theta \leq \theta$.

3. For any $\mathfrak{I} \in \mathsf{I}X$ and for any $\mu \in \mathsf{P}_b^X$, if θ is constant then

$$l_{\mu+\theta}(\mathfrak{I}) \vee \theta = l_\mu(\mathfrak{I}) + \theta \text{ and } l_{\mu\ominus\theta}(\mathfrak{I}) = l_\mu(\mathfrak{I}) \ominus \theta.$$

4. l_μ is an extension of μ in the sense that $l_\mu\big(\iota_X(\dot{x})\big) = \mu(x)$, for any $x \in X$.

5. For any $\mathfrak{I} \in \mathsf{I}X$ and $\mu \in \mathsf{P}_b^X$: $l_\mu(\mathfrak{I}) = 0$ if and only if $\mu \in \mathfrak{I}$ and in particular $l_\mu(\mathfrak{Z}_X) = 0$.

Given sets J and X, a map $s : J \longrightarrow \mathsf{I}X$ and a functional ideal $\mathfrak{I} \in \mathsf{I}J$ then we define the *diagonal functional ideal* of s with respect to $\mathfrak{I}$ as

$$m_X\big(\mathsf{I}\, s(\mathfrak{I})\big) := \{\mu \in \mathsf{P}_b^X \mid l_\mu \in \mathsf{I}\, s(\mathfrak{I})\}. \tag{1.26}$$

The following useful alternative characterization of $m_X\big(\mathsf{I}\, s(\mathfrak{I})\big)$ can be found in Theorem 1.1.58 in *Index Analysis* [Low15].

Theorem 1.1.1.22. *If X and J are sets, $s : J \longrightarrow \mathsf{I}X$ and $\mathfrak{I} \in \mathsf{I}J$ then*

$$m_X\big(\mathsf{I}\, s(\mathfrak{I})\big) = \bigcup_{\mu\in\mathfrak{I}} \bigcap_{j\in J} s(j) \oplus \mu(j).$$

Definition 1.1.1.23. (Functional ideal convergence) A relation $\rightarrowtail \subseteq \mathsf{I}X \times X$ is called a *functional ideal convergence* if it satisfies the following properties.

(F1) For every $x \in X : \iota_X(\dot{x}) \rightarrowtail x$.

(F2) If $(\mathfrak{I}_i)_{i\in I}$ is a family of functional ideals then $\bigcap_{i\in I}\mathfrak{I}_i \rightarrowtail x$ if and only if $\mathfrak{I}_i \rightarrowtail x$ for every $i \in I$.

(F3) If $s : X \longrightarrow \mathsf{I}X$ is a selection of functional ideals such that $s(z) \rightarrowtail z$ for all $z \in X$ and $\mathfrak{I}$ is a functional ideal such that $\mathfrak{I} \rightarrowtail x$, then $m_X\big(\mathsf{I}s(\mathfrak{I})\big) \rightarrowtail x$.

In Theorem 1.1.67 of [Low15] a useful alternative characterization of functional ideal convergence is established, which entails a weakening of (F2) and a strengthening of (F3).

Theorem 1.1.1.24. *A relation $\rightarrowtail \subseteq \mathsf{I}X \times X$ satisfying (F1) is a functional ideal convergence if and only if it satisfies the properties*

(F2w) For any $\mathfrak{K} \subseteq \mathfrak{I}$ and any $x \in X$: $\mathfrak{K} \rightarrowtail x \Rightarrow \mathfrak{I} \rightarrowtail x$.

(F) For any set A, for any $\psi : A \longrightarrow X$, for any $s : A \longrightarrow \mathsf{I}X$, for any $\mathfrak{I} \in \mathsf{I}A$ and for any $x \in X$

$$\big(\forall a \in A : s(a) \rightarrowtail \psi(a) \text{ and } \mathsf{I}\psi(\mathfrak{I}) \rightarrowtail x\big) \Rightarrow m_X\big(\mathsf{I}s(\mathfrak{I})\big) \rightarrowtail x.$$

G. Transition formulas

In spite of the fact that all concepts which were defined are both conceptually and technically very different from each other, they are all equivalent. One type of structure unambiguously determines a unique structure of each of the other types. A structure derived from another one by such a transition will be referred to as an associated structure. The proofs that all these structures are equivalent, can be found in [Low15]. For easy reference, we will list the transition formulas between the various introduced structures.

1. **Transition formulas from a distance δ**

$$\lambda\mathcal{F}(x) = \sup_{A\in\sec(\mathcal{F})} \delta(x, A). \tag{1.27}$$

$$\mathcal{A}(x) = \big\{\varphi \in \mathsf{P}^X \mid \forall A \subseteq X : \inf_{y\in A} \varphi(y) \leq \delta(x, A)\big\}. \tag{1.28}$$

$$\mathcal{G} = \big\{d \in q\mathsf{Met}(X) \mid \forall A \subseteq X : \inf_{a\in A} d(\cdot, a) \leq \delta_A\big\}. \tag{1.29}$$

$$\mathfrak{t}_\varepsilon(A) = A^{(\varepsilon)} = \{x \in X \mid \delta(x, A) \leq \varepsilon\}. \tag{1.30}$$

2. **Transition formulas from a limit operator λ**

$$\delta(x,A) = \inf_{\mathcal{U}\in\beta(A)} \lambda\mathcal{U}(x). \tag{1.31}$$

$$\mathcal{A}(x) = \big\{\varphi \in \mathsf{P}^X \mid \forall \mathcal{U}\in\beta(X) : \sup_{U\in\mathcal{U}} \inf_{y\in U} \varphi(y) \leq \lambda\mathcal{U}(x)\big\} \tag{1.32}$$

$$\mathcal{G} = \big\{d \in q\mathsf{Met}(X) \mid \forall \mathcal{U}\in\beta(X) : \sup_{U\in\mathcal{U}} \inf_{y\in U} d(\cdot,y) \leq \lambda\mathcal{U}\big\}. \tag{1.33}$$

$$\mathfrak{t}_\varepsilon(A) = \{x\in X \mid \exists \mathcal{F}\in \mathsf{F}(A) : \lambda\mathcal{F}(x)\leq\varepsilon\}. \tag{1.34}$$

$$\mathfrak{I} \rightarrowtail x \Leftrightarrow \forall \alpha \in [c(\mathfrak{I}),\infty[: \lambda\mathfrak{f}_\alpha(\mathfrak{I})(x)\leq\alpha. \tag{1.35}$$

3. **Transition formulas from an approach system $\mathcal{A}$**

$$\delta(x,A) = \sup_{\varphi\in\mathcal{A}(x)} \inf_{y\in A} \varphi(y). \tag{1.36}$$

$$\lambda\mathcal{F}(x) = \sup_{\varphi\in\mathcal{A}(x)} \inf_{F\in\mathcal{F}} \sup_{y\in F} \varphi(y). \tag{1.37}$$

$$\mathcal{G} = \big\{d\in q\mathsf{Met}(X) \mid \forall x\in X : d(x,\cdot)\in\mathcal{A}(x)\big\}. \tag{1.38}$$

$$\mathfrak{I} \rightarrowtail x \Leftrightarrow \mathcal{A}_b(x)\subseteq\mathfrak{I}. \tag{1.39}$$

4. **Transition formulas form a gauge $\mathcal{G}$**

$$\delta(x,A) = \sup_{d\in\mathcal{G}} \inf_{y\in A} d(x,y). \tag{1.40}$$

$$\lambda\mathcal{F}(x) = \sup_{d\in\mathcal{G}} \inf_{F\in\mathcal{F}} \sup_{y\in F} d(x,y). \tag{1.41}$$

$$\mathcal{A}(x) = \big\{\varphi\in\mathsf{P}^X \mid \{d(x,\cdot)\mid d\in\mathcal{G}\} \text{ dominates } \varphi\big\}. \tag{1.42}$$

5. **Transition formulas from a tower $\mathfrak{t}$**

$$\delta(x,A) = \inf\{\varepsilon\in\mathbb{R}^+ \mid x\in\mathfrak{t}_\varepsilon(A)\}. \tag{1.43}$$

$$\lambda\mathcal{F}(x) = \sup_{A\in\sec(\mathcal{F})} \inf\{\varepsilon\in\mathbb{R}^+ \mid x\in\mathfrak{t}_\varepsilon(A)\}. \tag{1.44}$$

$$\mathcal{A}(x) = \big\{\varphi\in\mathsf{P}^X \mid \forall A\subseteq X, \forall\varepsilon>0 : x\in\mathfrak{t}_\varepsilon(A)\Rightarrow \inf_{y\in A}\varphi(y)\leq\varepsilon\big\}. \tag{1.45}$$

$$\mathcal{G} = \big\{d\in q\mathsf{Met}(X) \mid \forall A\subseteq X : \mathfrak{t}_\varepsilon(A)\subseteq\{\inf_{y\in A} d(\cdot,y)\leq\varepsilon\}\big\}. \tag{1.46}$$

6. **Transition formulas from a functional ideal convergence $\rightarrowtail$**

$$\lambda\mathcal{F}(x) = \inf\{\alpha \mid \iota_X(\mathcal{F}) \oplus \alpha \rightarrowtail x\}. \tag{1.47}$$
$$\mathcal{A}_b(x) = \bigcap\{\mathfrak{I} \in \mathsf{I}X \mid \mathfrak{I} \rightarrowtail x\}. \tag{1.48}$$

H. Approach spaces

All introduced structures are equivalent and for proofs we refer to Section 1.2 in *Index Analysis* [Low15]. The following definition makes sense.

Definition 1.1.1.25. A set X endowed with a distance δ, a limit operator λ, an approach system $\big(\mathcal{A}(x)\big)_{x\in X}$, a gauge $\mathcal{G}$, a tower $(\mathfrak{t}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ or a functional ideal convergence $\rightarrowtail$ is called an *approach space*, which is again denoted by X.

1.1.2 The morphisms: contractions

The morphisms which are naturally associated with the structures defined in the foregoing section can most intuitively be defined in terms of distances, but we will immediately give characterizations of the morphisms in terms of each of the introduced structures.

Definition 1.1.2.1. A function $f : X \longrightarrow X'$ between two approach spaces is called a *contraction* if for all $A \subseteq X$ and $x \in X$

$$\delta'\big(f(x), f(A)\big) \leq \delta(x, A).$$

The following theorem gives a characterization of contractions in terms of limit operators, approach systems, gauges, towers and functional ideal convergence. The proof can be found in Theorem 1.3.3 in [Low15].

Theorem 1.1.2.2. *For a function $f : X \longrightarrow X'$ between two approach spaces the following properties are equivalent:*

(i) f *is a contraction.*

(ii) $\forall \mathcal{F} \in \mathsf{F}X : \lambda'\big(f(\mathcal{F})\big) \cdot f \leq \lambda\mathcal{F}$.

(iii) $\forall \mathcal{U} \in \beta X : \lambda'\big(f(\mathcal{U})\big) \cdot f \leq \lambda\mathcal{U}$.

(iv) $\forall x \in X, \forall \varphi' \in \mathcal{A}'\big(f(x)\big) : \varphi' \cdot f \in \mathcal{A}(x)$.

(v) $\forall d' \in \mathcal{G}' : d' \cdot (f \times f) \in \mathcal{G}$.

(vi) $\forall A \subseteq X, \forall \varepsilon \in \mathbb{R}^+ : f\big(t_\varepsilon(A)\big) \subseteq t'_\varepsilon\big(f(A)\big)$.

(vii) $\forall \mathfrak{I} \in \mathsf{I}X, \forall x \in X : \mathfrak{I} \rightarrowtail x \Rightarrow \mathsf{I}f(\mathfrak{I}) \rightarrowtail f(x)$.

1.1.3 The topological construct App

Definition 1.1.3.1. Approach spaces form the objects and contractions form the morphisms of a category which we denote App.

For the fundamental theory of topological constructs, we refer to [AHS06].

App is a topological construct. In the following theorem we list the formulas for the initial approach structures which we will need further on in this work. We refer to Theorem 1.3.12 and Theorem 1.3.18 in *Index Analysis* [Low15].

Theorem 1.1.3.2. *Given approach spaces $(X_j)_{j\in J}$, consider the source*

$$(f_j : X \longrightarrow X_j)_{j\in J}$$

in App.

- *Suppose that, for each $j \in J$, λ_j is the limit operator on X_j. Then the initial limit operator on X is given by*

$$\lambda\mathcal{F} = \sup_{j\in J} \lambda\big(f_j(\mathcal{F})\big) \cdot f_j.$$

- *Suppose that, for each $j \in J$, $\rightarrowtail_i$ is the functional ideal convergence of X_j. Then the initial functional ideal convergence on X is given by*

$$\mathfrak{I} \rightarrowtail x \Leftrightarrow \forall j \in J : \mathsf{I}f_j(\mathfrak{I}) \rightarrowtail_j f_j(x).$$

In any topological construct, on any set there are discrete and indiscrete structures. A structure is called discrete if any function defined on a set with that structure is a morphism and indiscrete if any function to a set with that structure is a morphism.

Given a set X the *discrete approach structure* is determined by any (and all) of the following structures:

1. Distance: $\delta : X \times 2^X \longrightarrow \mathsf{P}$ where, for all $x \in X$ and $A \subseteq X$,

$$\delta(x, A) = \begin{cases} 0 & x \in A, \\ \infty & x \notin A. \end{cases}$$

2. Limit operator: $\lambda : \mathsf{F}X \longrightarrow \mathsf{P}^X$ where, for all $x \in X$ and $\mathcal{F} \in \mathsf{F}X$,

$$\lambda\mathcal{F}(x) = \begin{cases} \theta_x & \mathcal{F} = \dot{x}, \\ \infty & \mathcal{F} \neq \dot{x}. \end{cases}$$

3. Approach system: $\big(\mathcal{A}(x)\big)_{x\in X}$ where, for all $x \in X$,
$$\mathcal{A}(x) = \{\varphi \in \mathsf{P}_b^X \mid \varphi(x) = 0\}.$$
4. Gauge: $\mathcal{G} = q\mathsf{Met}(X)$.
5. Tower: $(\mathfrak{t}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ where, for all $\varepsilon \in \mathbb{R}^+$ and $A \subseteq X$,
$$\mathfrak{t}_\varepsilon(A) = A.$$
6. Functional ideal convergence: If $\mathfrak{I} = \iota(\dot{x}) \oplus \alpha$ for some $x \in X$ and $\alpha \in [0,\infty[$ then $\mathfrak{I} \rightarrowtail x$ and otherwise $\mathfrak{I}$ does not converge to any point. The improper functional ideal $\mathfrak{Z}_X$ converges to all points.

Given a set X the *indiscrete approach structure* is determined by any (and all) of the following structures:

1. Distance: $\delta : X \times 2^X \longrightarrow \mathsf{P}$ where, for all $x \in X$ and $A \subseteq X$,
$$\delta(x, A) = \begin{cases} 0 & A \neq \emptyset, \\ \infty & A = \emptyset. \end{cases}$$
2. Limit operator: $\lambda : \mathsf{F}X \longrightarrow \mathsf{P}^X$ where, for all $\mathcal{F} \in \mathsf{F}X$, $\lambda\mathcal{F} = 0$.
3. Approach system: $\big(\mathcal{A}(x)\big)_{x\in X}$ where, for all $x \in X$, $\mathcal{A}(x) = \{0\}$.
4. Gauge: $\mathcal{G} = \{0\}$.
5. Tower: $(\mathfrak{t}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ where, for all $\varepsilon \in \mathbb{R}^+$ and $A \subseteq X$,
$$\mathfrak{t}_\varepsilon(A) = \begin{cases} X & A \neq \emptyset, \\ \emptyset & A = \emptyset. \end{cases}$$
6. Functional ideal convergence: For all $\mathfrak{I} \in \mathsf{I}X$ and $x \in X$: $\mathfrak{I} \rightarrowtail x$.

1.1.4 Topological and metric approach spaces

Both topological and metric spaces can be viewed as special types of approach spaces. More precisely, both Top and $q\mathsf{Met}$ can be embedded as full and isomorphism-closed subcategories of App. For Top the embedding will turn out to be both concretely reflective and concretely coreflective. For $q\mathsf{Met}$ the embedding will turn out to be concretely coreflective but not reflective.

A. Topological approach spaces

Given a topological space $(X, \mathcal{T})$ we associate with it a natural approach space in the following way:

$$\lambda_{\mathcal{T}} : \mathsf{F}\, X \longrightarrow \mathsf{P}^X,$$

where

$$\lambda_{\mathcal{T}}\mathcal{F}(x) = \begin{cases} 0 & \mathcal{F} \to x \text{ in } (X, \mathcal{T}), \\ \infty & \text{elsewhere} \end{cases}$$

is a limit operator on X.

An approach space of type $(X, \lambda_{\mathcal{T}})$ for some topology $\mathcal{T}$ on X will be called a *topological approach space*. The following proposition gives an internal characterization of these spaces.

Proposition 1.1.4.1. An approach space (X, λ) is topological if and only if for any filter $\mathcal{F} \in \mathsf{F}X$ we have that $\lambda\mathcal{F}(X) \subseteq \{0, \infty\}$.

This shows that topological spaces can be viewed as certain types of approach spaces. That Top is moreover concretely embedded in App is a consequence of the fact that given topological spaces $(X, \mathcal{T})$, $(X', \mathcal{T}')$ and a map between them $f : X \longrightarrow X'$ we have that f is continuous as a map between topological spaces if and only if it is a contraction as a map between the associated approach spaces, as follows from the following observation:

$$f(\mathrm{cl}_{\mathcal{T}} A) \subseteq \mathrm{cl}_{\mathcal{T}'} \big(f(A)\big) \Leftrightarrow \forall \varepsilon \in \mathbb{R}^+ : f\big(A^{(\varepsilon)}\big) \subseteq f(A)^{(\varepsilon')}.$$

Hence the concrete functor from Top to App which takes $(X, \mathcal{T})$ to $(X, \lambda_{\mathcal{T}})$ is a full embedding of Top in App.

Moreover, Top is embedded as a concretely reflective and coreflective subcategory of App. As a corollary of this theorem, we get that Top is closed under the formation of limits and initial structures in App, as well as colimits and final structures. In particular, a product or coproduct in App of a family of topological approach spaces is a topological approach space and, likewise, a subspace or quotient in App of a topological approach space is a topological approach space

The topological coreflection of an approach space will be frequently used in this work. For any approach space (X, δ), its Top-coreflection is determined by the distance δ^{tc} associated with the topological closure operator given by

$$\mathrm{cl}_\delta(A) := \{x \in X \mid \delta(x, A) = 0\}. \tag{1.49}$$

As far as notation is concerned, we will denote the topological coreflection of an approach space X by $\mathrm{C}_{\mathsf{Top}}\, X$.

B. Quasi-metric approach spaces

Given a quasi-metric space (X, d), we associate with it a natural approach space in the following way:

$$\delta_d : X \times 2^X \longrightarrow \mathsf{P}$$

where

$$\delta_d(x, A) = \inf_{a \in A} d(x, a)$$

is a distance on X.

An approach space of type (X, δ_d) for some quasi-metric d on X will be called a quasi-metric approach space. The following proposition gives an internal characterization of these spaces.

Proposition 1.1.4.2. An approach space (X, δ) is quasi-metric if and only if for all $x \in X$ and $A \subseteq X$, we have $\delta(x, A) = \inf_{a \in A} \delta\big(x, \{a\}\big)$.

This shows that quasi-metric spaces can be viewed as certain types of approach spaces. That $q\mathsf{Met}$ is moreover concretely embedded in App is a consequence of the fact that given quasi-metric spaces $(X, d), (X', d')$ and a map $f : X \longrightarrow X'$ between them, we have that f is non-expansive as a map between quasi-metric spaces if and only if it is a contraction as a map between the associated approach spaces. Hence the concrete functor from $q\mathsf{Met}$ to App which takes (X, d) to (X, δ_d) is a full embedding of $q\mathsf{Met}$ in App.

Moreover, $q\mathsf{Met}$ is embedded as a concretely coreflective subcategory of App. For any approach space (X, δ), its $q\mathsf{Met}$-coreflection is determined by the distance δ^{qm} associated with the quasi-metric

$$d : X \times X \longrightarrow \mathsf{P} : (x, y) \mapsto \delta(x, \{y\}). \tag{1.50}$$

As a corollary of this theorem we get that $q\mathsf{Met}$ is closed under the formation of colimits and final structures in App. In particular, a coproduct in App of a family of quasi-metric approach spaces is a quasi-metric approach space and, likewise, a quotient in App of a quasi-metric approach space is a quasi-metric approach space.

If (X, d) is a quasi-metric space, then the Top-coreflection of (X, δ_d) is $(X, \delta_{\mathcal{T}_d})$, where $\mathcal{T}_d$ is the topology generated by d.

A fundamental relationship among the different types of structures which we are considering in approach theory is that of a topology generated by a metric. It is the failure of this relationship to be well behaved with respect to products which is one of the main motivations for considering approach spaces. What the foregoing results tell us is that this relationship is recaptured in App as a canonical functor, namely the Top-coreflector restricted to $q\mathsf{Met}$. In the case of a quasi-metric space the Top-coreflector gives us the underlying topological space. It is therefore natural to extend this interpretation to the whole of App. The situation is clarified in the following diagram.

$$\begin{array}{ccc} q\mathsf{Met} & \xrightarrow{F} & \mathsf{Top} \\ {\scriptstyle E}\downarrow & \nearrow_{\mathrm{C}_{\mathsf{Top}}} & \\ \mathsf{App} & & \end{array}$$

The functor E is the embedding of qMet in App, F is the functor associating with each quasi-metric space its underlying topological space and $\mathrm{C}_{\mathsf{Top}}$ is the Top-coreflector. The diagram commutes and $\mathrm{C}_{\mathsf{Top}}$ thus is an extension of F.

1.2 Monads

Nothing seems to be more benign in algebra than the notion of a monoid, that is, of a set M that comes with an associative binary operation $m : M \times M \longrightarrow M$ and a neutral element, written as a nullary operation $1 : 1 \longrightarrow M$. In particular a monad on Set can be seen as a monoid in the monoidal category of all endofunctors on Set [ML98]. A monad $\mathbb{T} = (T, m, e)$ on Set is given by a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ and two natural transformations, the multiplication and unit of the monad, $m : TT \longrightarrow T$ and $e : 1 \longrightarrow T$, satisfying the multiplication law and right and left unit laws $m \cdot mT = m \cdot Tm$ and $m \cdot eT = 1_T = m \cdot Te$.

Monoids and their actions on monads occur not only everywhere in algebra but also provide the basic ingredients of what is called monoidal topology [HST14].

In this section we introduce the concept of a monad on the category Set and give various important examples.

1.2.1 Monads

Monads can be defined on an arbitrary category X. In this work, however, we restrict ourselves to Set-monads, with Set the category of sets and maps between them. First of all we give the definition of a so called Set-monad.

Definition 1.2.1.1. A *monad* $\mathbb{T} = (T, m, e)$ on the category Set is given by a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ and two natural transformations, the *multiplication* and the *unit* of the monad

$$m : TT \longrightarrow T, \quad e : 1_{\mathsf{Set}} \longrightarrow T,$$

satisfying the multiplication law and the right and left unit laws

$$m \cdot mT = m \cdot Tm, \quad m \cdot eT = 1_T = m \cdot Te; \tag{1.51}$$

equivalently, these equalities mean that the diagrams

$$\begin{array}{ccc} TTT & \xrightarrow{Tm} & TT \\ {\scriptstyle mT}\downarrow & & \downarrow{\scriptstyle m} \\ TT & \xrightarrow{m} & T \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} T & \xrightarrow{eT} & TT & \xleftarrow{Te} & T \\ & {\scriptstyle 1_T}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle 1_T} & \\ & & T & & \end{array}$$

commute. A *morphism of monads* $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ (where $\mathbb{S} = (S, n, d)$ is another Set-monad) is a natural transformation $\alpha : S \longrightarrow T$ that preserves the monad structure:

$$\alpha \cdot n = m \cdot (\alpha * \alpha), \quad \alpha \cdot d = e. \tag{1.52}$$

Here $\alpha * \alpha$ stands for the so called horizontal composition of natural transformations with components defined by

$$(\alpha * \alpha)_X = T\alpha_X \cdot \alpha_{SX} = \alpha_{TX} \cdot S\alpha_X. \tag{1.53}$$

We list the most important examples of Set-monads.

Examples 1.2.1.2. 1. The *identity monad* on Set: $\mathbb{1} = (1_{\mathsf{Set}}, 1, 1)$.

We should note that in most books, the identity monad is denoted $\mathbb{I}$. However, we choose to use the notation $\mathbb{1}$, since in our work $\mathbb{I}$ will be used to denote the so called functional ideal monad, introduced in Chapter 3.

2. The covariant powerset functor $\mathcal{P} : \mathsf{Set} \longrightarrow \mathsf{Set}$, together with the union $m_X : \mathcal{P}\mathcal{P}X \longrightarrow \mathcal{P}X$ and singleton maps $e_X : X \longrightarrow \mathcal{P}X$, defined by

$$m_X(\mathcal{A}) = \bigcup \mathcal{A}, \quad e_X(x) = \{x\},$$

for all $\mathcal{A} \in \mathcal{P}\mathcal{P}X$, $x \in X$, form the *powerset monad* $\mathbb{P} = (\mathcal{P}, m, e)$.

3. The filter functor F on Set, defined by

$$\mathsf{F}X = \{\mathcal{F} \subseteq \mathcal{P}X \mid \mathcal{F} \text{ filter on } X\}$$

and

$$\mathsf{F}f : \mathsf{F}X \longrightarrow \mathsf{F}Y : \mathcal{F} \mapsto f(\mathcal{F}),$$

as defined in (1.3), for all sets X, Y and maps $f : X \longrightarrow Y$, together with multiplication

$$m_X : \mathsf{FF}X \longrightarrow \mathsf{F}X : \mathfrak{X} \mapsto \Sigma\mathfrak{X},$$

given by the Kowalsky diagonal operation (1.4) and unit

$$e_X : X \longrightarrow \mathsf{F}X : x \mapsto \dot{x},$$

where the components are given by the principal filters, form the *filter monad* $\mathbb{F} = (\mathsf{F}, m, e)$.

4. Similar to the filter functor, the ultrafilter functor β on Set, defined by

$$\beta X = \{\mathcal{U} \subseteq \mathcal{P}X \mid \mathcal{U} \text{ ultrafilter on } X\}$$

and

$$\beta f : \beta X \longrightarrow \beta Y : \mathcal{U} \mapsto f(\mathcal{U}),$$

for all sets X, Y and maps $f : X \longrightarrow Y$ together with multiplication

$$m_X : \beta\beta X \longrightarrow \beta X : \mathfrak{X} \mapsto \Sigma\mathfrak{X},$$

given by the Kowalsky diagonal operation (1.5) and unit

$$e_X : X \longrightarrow \beta X : x \mapsto \dot{x},$$

form the *ultrafilter monad* $\mathbb{\beta} = (\beta, m, e)$.

1.3 Lax algebras

In this section we introduce the key category of interest throughout this work, the category $(\mathbb{T}, \mathcal{V})$-$\mathsf{Cat}$ for a quantale $\mathcal{V}$ and a monad $\mathbb{T}$ on Set, laxly extended to the category $\mathcal{V}$-Rel of sets and $\mathcal{V}$-valued relations. The objects of the category $(\mathbb{T}, \mathcal{V})$-$\mathsf{Cat}$ will be called $(\mathbb{T}, \mathcal{V})$-categories, $(\mathbb{T}, \mathcal{V})$-spaces or lax algebras. These categories provide a common setting to describe ordered sets, metric spaces, topological spaces and approach spaces in a natural way.

We will introduce all required concepts and examples in this section without proofs. All results in this section can be found in [HST14], unless explicitly stated otherwise. We refer the reader to [HST14] for more detailed information.

1.3.1 Quantales

A *quantale* $\mathcal{V} = (\mathcal{V}, \otimes, k)$ is a complete lattice $\mathcal{V}$ which carries a monoid structure with neutral element k such that, when the binary operation is denoted as a tensor $\otimes$,

$$a \otimes (-) : \mathcal{V} \longrightarrow \mathcal{V}, \qquad (-) \otimes b : \mathcal{V} \longrightarrow \mathcal{V}$$

are sup-maps, for all $a, b \in \mathcal{V}$; hence the tensor distributes over suprema:

$$a \otimes \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \otimes b_i), \qquad \bigvee_{i \in I} a_i \otimes b = \bigvee_{i \in I} (a_i \otimes b). \tag{1.54}$$

A *lax homomorphism of quantales* $f : \mathcal{V} \longrightarrow \mathcal{W}$ is a monotone map satisfying

$$f(a) \otimes f(b) \leq f(a \otimes b), \qquad l \leq f(k), \tag{1.55}$$

for all $a, b \in \mathcal{V}$ and with l the neutral element of $\mathcal{W}$. Monotonicity of f means equivalently lax preservation of joins, i.e.

$$\bigvee f(A) \leq f(\bigvee A) \tag{1.56}$$

for all $A \subseteq \mathcal{V}$.

A quantale $\mathcal{V}$ is *commutative* if it is commutative as a monoid.

In a quantale $\mathcal{V}$, for every $a \in \mathcal{V}$, the sup-map $a \otimes (-)$ is left adjoint to a map $a \multimap (-) : \mathcal{V} \longrightarrow \mathcal{V}$ which is uniquely determined by

$$a \otimes v \leq b \Leftrightarrow v \leq a \multimap b, \tag{1.57}$$

for all $v, b \in \mathcal{V}$; hence

$$a \multimap b = \bigvee \{v \in \mathcal{V} \mid a \otimes v \leq b\}. \tag{1.58}$$

Likewise, for all $a \in \mathcal{V}$, the sup-map $(-) \otimes a$ is left adjoint to a map $(-) \multimapinv a : \mathcal{V} \longrightarrow \mathcal{V}$. In the case where $\mathcal{V}$ is commutative, $a \multimap (-)$ and $(-) \multimapinv a$ coincide, and either of the two notations may be used.

We list some examples of quantales which we will use frequently throughout this work.

Examples 1.3.1.1. 1. The *two-chain* $2 = \{\text{false} \models \text{true}\} = \{\bot, \top\}$ with $\otimes = \wedge$, $k = \top$. Here $a \multimap b$ is the Boolean truth value of the implication $a \to b$.

2. Allowing for an interval of truth values, we consider the extended real halfline $\mathsf{P} = [0, \infty]$ which is a complete lattice with respect to its natural order $\leq$. We reverse its order, so that $0 = \top$ is the top and $\infty = \bot$ is the bottom element. We consider it a quantale with $\otimes$ given by addition extended via
$$a + \infty = \infty + a = \infty$$
for all $a \in \mathsf{P}$ and $k = 0 = \top$. We denote this quantale
$$\mathsf{P}_{+} = \big([0, \infty]^{\mathrm{op}}, +, 0\big).$$
Here we have
$$b \multimapinv a = b \ominus a := \inf\{v \in \mathsf{P} \mid b \leq a + v\}$$
so that
$$b \ominus a = \begin{cases} b - a & a \leq b < \infty, \\ 0 & b \leq a, \\ \infty & a < b = \infty. \end{cases}$$

When working with $[0, \infty]^{\mathrm{op}}$ and forming infima or suprema, we will denote these by $\inf^{\mathrm{op}}, \bigwedge^{\mathrm{op}}, \sup^{\mathrm{op}}$ or $\bigvee^{\mathrm{op}}$. It means that we will deviate slightly from the conventions made in [HST14], since we will use both the symbols $\inf$ and $\bigwedge$ when forming infima and $\sup$ and $\bigvee$ when forming suprema, referring to the natural order on $[0, \infty]$.

3. Since $[0, \infty]^{\mathrm{op}}$ is a chain, it is a frame, and we may consider it a quantale

$$\mathsf{P}_{\vee} = \big([0, \infty]^{\mathrm{op}}, \vee, 0\big)$$

with its meet operation (which, according to our conventions is the supremum with respect to the natural order of $[0, \infty]$ and will be denoted by $\vee$) and neutral element 0. In [HST14], this quantale is called $\mathsf{P}_{\max}$.

Again, we use the same conventions as introduced for $\mathsf{P}_{_+}$ when forming infima or suprema.

The map $a \vee (-) : \mathsf{P}_{\vee} \longrightarrow \mathsf{P}_{\vee}$ is left adjoint to the map $a \multimapdot (-) : \mathsf{P}_{\vee} \longrightarrow \mathsf{P}_{\vee}$ defined by

$$a \multimapdot b := \inf\{v \in \mathsf{P} \mid b \leq a \vee v\} = \begin{cases} 0 & a \geq b, \\ b & a < b. \end{cases}$$

The quantales 2, $\mathsf{P}_{_+}$ and $\mathsf{P}_{\vee}$ are all commutative.

For the quantale $\mathsf{P}_{\vee}$, we list the following properties of the operation $\multimapdot$. We also add the proofs of these results, since they cannot be found in [HST14].

Proposition 1.3.1.2. The following properties hold.

1. $\forall a, b, c \in \mathsf{P}_{\vee} : (c \multimapdot a) \vee (a \multimapdot b) \geq c \multimapdot b$,

2. $\forall a, b, c \in \mathsf{P}_{\vee} : (a \vee b) \multimapdot c = (a \multimapdot c) \wedge (b \multimapdot c)$.

Proof. To prove the first property, suppose that $(a \multimapdot b) \vee (c \multimapdot a) = \gamma$. Then, for $\varepsilon > 0$, we have that $(a \multimapdot b) < \gamma + \varepsilon$ and $(c \multimapdot a) < \gamma + \varepsilon$. Hence, there exist $v_1, v_2 < \gamma + \varepsilon$ such that $b \leq a \vee v_1$ and $a \leq c \vee v_2$. So we get $b \leq c \vee (v_1 \vee v_2)$, with $v_1 \vee v_2 \leq \gamma + \varepsilon$. This proves that $c \multimapdot b \leq \gamma + \varepsilon$ and by arbitrariness of $\varepsilon > 0$, we get $c \multimapdot b \leq \gamma$.

To prove the second property, note that the right-hand side is c when $a < c$ and $b < c$. In this case $a \vee b < c$ and thus the left-hand side is c as well. In all other cases, the right-hand side is equal to zero. Clearly, $a \geq c$ or $b \geq c$ implies $a \vee b \geq c$, so the left-hand side is zero as well. $\square$

1.3.2 $\mathcal{V}$-relations

A *relation* $r : X \nrightarrow Y$ from a set X to a set Y distinguishes those elements $x \in X$ and $y \in Y$ that are r-related. We write $x\,r\,y$ if x is r-related to Y. We can display r as a two-valued function via

$$r : X \times Y \longrightarrow \{\text{true, false}\} = 2.$$

In order to model situations where quantitative information is available, r can be allowed to take values in any quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$ rather than just $2 = (2, \wedge, \top)$. A *$\mathcal{V}$-relation* $r : X \nrightarrow Y$ from X to Y is therefore presented by a map $r : X \times Y \longrightarrow \mathcal{V}$. As for ordinary relations, a $\mathcal{V}$-relation $r : X \nrightarrow Y$ can be composed with another $\mathcal{V}$-relation $s : Y \nrightarrow Z$ via "matrix multiplication"

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z), \tag{1.59}$$

for all $x \in X, z \in Z$, to yield a $\mathcal{V}$-relation $s \cdot r : X \nrightarrow Z$. This composition is associative and the $\mathcal{V}$-relation $1_X : X \nrightarrow X$ that sends every diagonal element (x, x) to k, and all other elements to the bottom element $\bot$ of $\mathcal{V}$, serves as the identity morphism on X. Thus, sets and $\mathcal{V}$-relations form a category denoted by

$$\mathcal{V}\text{-}\mathsf{Rel}.$$

The set $\mathcal{V}\text{-}\mathsf{Rel}(X, Y)$ of all $\mathcal{V}$-relations from X to Y inherits the pointwise order induced by $\mathcal{V}$. Given $r, r' : X \nrightarrow Y$, then

$$r \leq r' \Leftrightarrow \forall (x, y) \in X \times Y : r(x, y) \leq r'(x, y). \tag{1.60}$$

The canonical isomorphism $X \times Y \cong Y \times X$ induces a bijection between $\mathcal{V}\text{-}\mathsf{Rel}(X, Y)$ and $\mathcal{V}\text{-}\mathsf{Rel}(Y, X)$ so that for every $\mathcal{V}$-relation $r : X \nrightarrow Y$ one has the opposite $\mathcal{V}$-relation $r^\circ : Y \nrightarrow X$ defined by

$$r^\circ(y, x) = r(x, y), \tag{1.61}$$

for all $x \in X, y \in Y$. This operation preserves the order on $\mathcal{V}\text{-}\mathsf{Rel}(X, Y)$:

$$r \leq r' \Rightarrow r^\circ \leq (r')^\circ, \tag{1.62}$$

and one has $1_X^\circ = 1_X$ as well as $r^{\circ\circ} = r$. The equality

$$(s \cdot r)^\circ = r^\circ \cdot s^\circ \tag{1.63}$$

holds when $\mathcal{V}$ is commutative.

Examples 1.3.2.1. 1. $2\text{-Rel} \cong \text{Rel}$, the category of sets and relations between them.

2. For P_+-Rel, the composition of two P_+-relations $r : X \times Y \longrightarrow \mathsf{P}_+$ and $s : Y \times Z \longrightarrow \mathsf{P}_+$ yields

$$s \cdot r(x,z) = \inf\{r(x,y) + s(y,z) \mid y \in Y\} \tag{1.64}$$

for all $x \in X$ and $z \in Z$.

3. For $\mathsf{P}_\vee$-Rel , the composition of two $\mathsf{P}_\vee$-relations $r : X \times Y \longrightarrow \mathsf{P}_\vee$ and $s : Y \times Z \longrightarrow \mathsf{P}_\vee$ yields

$$s \cdot r(x,z) = \inf\{r(x,y) \vee s(y,z) \mid y \in Y\}. \tag{1.65}$$

1.3.3 $\mathcal{V}$-categories and $\mathcal{V}$-functors

Definition 1.3.3.1. A *$\mathcal{V}$-category* (X, a) is a set X together with a transitive and reflexive $\mathcal{V}$-relation $a : X \nrightarrow X$, meaning it is a map $a : X \times X \longrightarrow \mathcal{V}$ satisfying

$$a(x,y) \otimes a(y,z) \leq a(x,z) \quad \text{and} \quad k \leq a(x,x) \tag{1.66}$$

for all $x, y, z \in X$. A *$\mathcal{V}$-functor* $f : (X,a) \longrightarrow (Y,b)$ is a map $f : X \longrightarrow Y$ satisfying

$$a(x,x') \leq b\big(f(x), f(x')\big), \tag{1.67}$$

for all $x, x' \in X$. Since identity maps and composition of $\mathcal{V}$-functors are $\mathcal{V}$-functors, $\mathcal{V}$-categories and $\mathcal{V}$-functors form a category

$$\mathcal{V}\text{-Cat}.$$

Examples 1.3.3.2. 1. For $\mathcal{V} = 2 = \{\text{true, false}\}$, writing $x \leq y$ for $a(x,y) =$ true, the transitivity and reflexivity conditions become

$$(x \leq y \,\&\, y \leq z \Rightarrow x \leq z) \quad \text{and} \quad x \leq x,$$

for all $x, y, z \in X$. Thus, a 2-category $(X, \leq)$ is just an ordered set. A 2-functor $f : (X, \leq) \longrightarrow (Y, \leq)$ is a map $f : X \longrightarrow Y$ with

$$x \leq x' \Rightarrow f(x) \leq f(x'),$$

for all $x, x' \in X$. So we can conclude that

$$2\text{-Cat} \cong \text{Ord},$$

where Ord is the category of ordered sets and order-preserving maps between them.

2. For $\mathcal{V} = \mathsf{P}_+$, a transitive and reflexive P_+-relation is a map $a : X \times X \longrightarrow \mathsf{P}_+$ such that

$$a(x,y) + a(y,z) \geq a(x,z) \quad \text{and} \quad a(x,x) = 0,$$

for all $x, y, z \in X$. Hence a P_+-category (X, a) is a quasi-metric space. A P_+-functor $f : (X,a) \longrightarrow (Y,b)$ is a map $f : X \longrightarrow Y$ such that

$$a(x,x') \geq b\big(f(x), f(x')\big),$$

i.e. it is a non-expansive map. We conclude

$$\mathsf{P}_+\text{-}\mathsf{Cat} \cong q\mathsf{Met}.$$

3. For $\mathcal{V} = \mathsf{P}_\vee$, a transitive and reflexive $\mathsf{P}_\vee$-relation is a map $a : X \times X \longrightarrow \mathsf{P}_\vee$ such that

$$a(x,y) \vee a(y,z) \geq a(x,z) \quad \text{and} \quad a(x,x) = 0,$$

for all $x, y, z \in X$. Hence a $\mathsf{P}_\vee$-category (X, a) is a quasi-ultrametric space. A $\mathsf{P}_\vee$-functor $f : (X,a) \longrightarrow (Y,b)$ is a map $f : X \longrightarrow Y$ such that

$$a(x,x') \geq b\big(f(x), f(x')\big),$$

i.e. it is a non-expansive map. We conclude

$$\mathsf{P}_\vee\text{-}\mathsf{Cat} \cong q\mathsf{Met}^u.$$

1.3.4 Lax extensions of functors and monads

For a given monad $\mathbb{T}$ on Set, we now consider extensions of $\mathbb{T}$ to $\mathcal{V}$-Rel. For this, we first concentrate on the underlying Set-functor T.

Definition 1.3.4.1. For a quantale $\mathcal{V}$ and a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, a *lax extension* $\hat{T} : \mathcal{V}\text{-}\mathsf{Rel} \longrightarrow \mathcal{V}\text{-}\mathsf{Rel}$ of T to $\mathcal{V}$-Rel is given by functions

$$\hat{T}_{X,Y} : \mathcal{V}\text{-}\mathsf{Rel}(X,Y) \longrightarrow \mathcal{V}\text{-}\mathsf{Rel}(TX,TY)$$

for all sets X, Y (with $\hat{T}_{X,Y}$ simply written as $\hat{T}$), such that

1. $r \leq r' \Rightarrow \hat{T}r \leq \hat{T}r'$,
2. $\hat{T}s \cdot \hat{T}r \leq \hat{T}(s \cdot r)$,
3. $Tf \leq \hat{T}f$ and $(Tf)^\circ \leq \hat{T}(f^\circ)$,

for all sets X, Y, Z, $\mathcal{V}$-relations $r, r' : X \longrightarrow\!\!\!\!\!\!\mapsto Y$, $s : Y \longrightarrow\!\!\!\!\!\!\mapsto Z$, and maps $f : X \longrightarrow Y$.

Let us now turn our attention to e and m that we wish to extend from Set to $\mathcal{V}$-Rel together with the functor T.

Definition 1.3.4.2. A triple $\hat{\mathbb{T}} = (\hat{T}, m, e)$ is a *lax extension of the monad* $\mathbb{T} = (T, m, e)$ if $\hat{T}$ is a lax extension of T which makes both $m : \hat{T}\hat{T} \longrightarrow \hat{T}$ and $e : 1_{\mathcal{V}\text{-Rel}} \longrightarrow \hat{T}$ oplax, i.e.

4. $m_Y \cdot \hat{T}\hat{T}r \leq \hat{T}r \cdot m_X$,

5. $e_Y \cdot r \leq \hat{T}r \cdot e_X$,

for all $\mathcal{V}$-relations $r : X \longrightarrow\!\!\!\!\!\!\mapsto Y$.

These inequalities yield the following pointwise expressions:

4*. $\hat{T}\hat{T}r(\mathfrak{X}, \mathfrak{Y}) \leq \hat{T}r\big(m_X(\mathfrak{X}), m_Y(\mathfrak{Y})\big)$,

5*. $r(x, y) \leq \hat{T}r\big(e_X(x), e_Y(y)\big)$,

for all $x \in X, y \in Y, \mathfrak{X} \in TTX, \mathfrak{Y} \in TTY$, and $\mathcal{V}$-relations $r : X \longrightarrow\!\!\!\!\!\!\mapsto Y$.

We say that the extension $\hat{\mathbb{T}} = (\hat{T}, m, e)$ of the monad $\mathbb{T}$ is *flat* if the lax extension $\hat{T}$ of the functor T is flat, i.e. if

$$\hat{T}1_X = T1_X = 1_{TX}. \tag{1.68}$$

1.3.5 $(\mathbb{T}, \mathcal{V})$-categories and $(\mathbb{T}, \mathcal{V})$-functors

Let $\mathcal{V}$ be a quantale and let $\hat{\mathbb{T}} = (\hat{T}, m, e)$ be a lax extension to $\mathcal{V}$-Rel of a monad $\mathbb{T} = (T, m, e)$ on Set.

A $(\mathbb{T}, \mathcal{V})$-relation $a : TX \longrightarrow\!\!\!\!\!\!\mapsto X$ is *transitive* if it satisfies

$$a \cdot \hat{T}a \leq a \cdot m_X. \tag{1.69}$$

In pointwise notation, the transitivity condition becomes

$$\widehat{T}a(\mathfrak{X}, \mathcal{Y}) \otimes a(\mathcal{Y}, z) \leq a\big(m_X(\mathfrak{X}), z\big), \tag{1.70}$$

for all $\mathfrak{X} \in TTX, \mathcal{Y} \in TX$ and $z \in X$. A $(\mathbb{T}, \mathcal{V})$-relation $a : TX \longrightarrow\!\!\!\!\!\!\mapsto X$ is *reflexive* if it satisfies

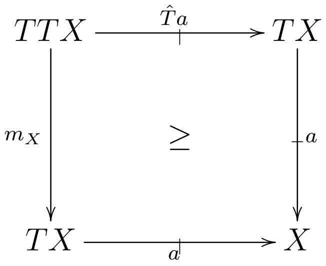
$$1_X \leq a \cdot e_X. \tag{1.71}$$

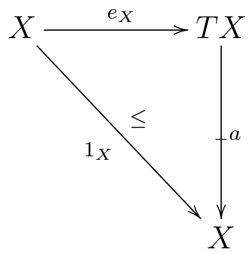
In pointwise notation, the reflexivity condition becomes

$$k \leq a\big(e_X(x), x\big), \tag{1.72}$$

for all $x \in X$.

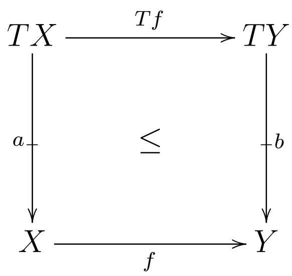
Definition 1.3.5.1. A $(\mathbb{T}, \mathcal{V})$-*category*, also referred to as a $(\mathbb{T}, \mathcal{V})$-*algebra* or a $(\mathbb{T}, \mathcal{V})$-*space*, is a pair (X, a) consisting of a set X and a transitive and reflexive $(\mathbb{T}, \mathcal{V})$-relation $a : TX \nrightarrow X$; i.e. it is a set X with a relation $a : TX \nrightarrow X$ satisfying (1.69) and (1.71).





A map $f : X \longrightarrow Y$ between $(\mathbb{T}, \mathcal{V})$-categories (X, a) and (Y, b) is a $(\mathbb{T}, \mathcal{V})$-*functor* if it satisfies

$$f \cdot a \leq b \cdot Tf. \tag{1.73}$$



We can transcribe this condition equivalently as $a \leq f^{\circ} \cdot b \cdot Tf$, which, in pointwise notation, reads as

$$a(\mathcal{X}, x) \leq b\big(Tf(\mathcal{X}), f(x)\big), \tag{1.74}$$

for all $\mathcal{X} \in TX$ and $x \in X$.

The identity map $1_X : (X, a) \longrightarrow (X, a)$ is a $(\mathbb{T}, \mathcal{V})$-functor, and so is the composite of $(\mathbb{T}, \mathcal{V})$-functors. Hence, $(\mathbb{T}, \mathcal{V})$-categories and $(\mathbb{T}, \mathcal{V})$-functors form a category, denoted by

$$(\mathbb{T}, \mathcal{V})\text{-}\mathsf{Cat}.$$

When $\mathbb{T} = \mathbb{1}$ is identically extended to $\mathcal{V}$-Rel, we get

$$(\mathbb{1}, \mathcal{V})\text{-}\mathsf{Cat} = \mathcal{V}\text{-}\mathsf{Cat}.$$

In order to compose compose $(\mathbb{T}, \mathcal{V})$-relations $r : TX \nrightarrow Y$ and $s : TY \nrightarrow Z$, we introduce the Kleisli convolution.

Definition 1.3.5.2. Given a lax extension $\hat{\mathbb{T}} = (\hat{T}, m, e)$ of a monad $\mathbb{T} = (T, m, e)$, the *Kleisli convolution* $s \circ r : TX \to\!\!\!\!\!\!\mapsto Z$ of $(\mathbb{T}, \mathcal{V})$-relations $r : TX \to\!\!\!\!\!\!\mapsto Y$ and $s : TY \to\!\!\!\!\!\!\mapsto Z$ is the $(\mathbb{T}, \mathcal{V})$-relation defined by

$$s \circ r := s \cdot \hat{T}r \cdot m_X^\circ$$

The $(\mathbb{T}, \mathcal{V})$-relation $e_X^\circ : TX \to\!\!\!\!\!\!\mapsto X$ is a lax identity for this composition: one has

$$e_Y^\circ \circ r = e_Y^\circ \cdot \hat{T}r \cdot m_X^\circ \geq r \cdot e_{TX}^\circ \cdot m_X^\circ = r,$$

with equality holding if $e^\circ = (e_X^\circ)_X : \hat{T} \longrightarrow 1$ is a natural transformation, and

$$r \circ e_X^\circ = r \cdot \hat{T}(e_X^\circ) \cdot m_X^\circ \geq r \cdot (Te_X)^\circ \cdot m_X^\circ = r,$$

with equality holding when $\hat{T}$ is flat.

We determine the $(\mathbb{T}, \mathcal{V})$-relations for which the opposite inequalities hold. These will be called unitary $(\mathbb{T}, \mathcal{V})$-relations.

Definition 1.3.5.3. A $(\mathbb{T}, \mathcal{V})$-relation $r : TX \to\!\!\!\!\!\!\mapsto Y$ is *right unitary* if

$$r \circ e_X^\circ \leq r,$$

and it is *left unitary* if

$$e_Y^\circ \circ r \leq r.$$

In terms of the relational composition, these conditions amount to the following inequalities

$$r \cdot \hat{T}1_X \leq r \qquad \text{and} \qquad e_Y^\circ \cdot \hat{T}r \cdot m_X^\circ \leq r,$$

respectively, and hence equalities.

For a $(\mathbb{T}, \mathcal{V})$-category (X, a), the transitive and reflexive relation $a : TX \to\!\!\!\!\!\!\mapsto X$ is unitary, i.e. it is both right unitary

$$a \cdot \hat{T}1_X \leq a, \tag{1.75}$$

and left unitary

$$e_X^\circ \cdot \hat{T}a \cdot m_X^\circ \leq a \tag{1.76}$$

The Kleisli convolution is not necessarily associative. Associativity of this operation turns out to depend on the lax extension of the monad.

Definition 1.3.5.4. A lax extension $\hat{\mathbb{T}}$ to $\mathcal{V}$-Rel of a monad $\mathbb{T} = (T, m, e)$ is associative whenever the Kleisli convolution of unitary $(\mathbb{T}, \mathcal{V})$-relations is associative, i.e.

$$t \circ (s \circ r) = (t \circ s) \circ r,$$

for all unitary $(\mathbb{T}, \mathcal{V})$-relations $r : TX \to\!\!\!\!\!\!\mapsto Y$, $s : TY \to\!\!\!\!\!\!\mapsto Z$ and $t : TZ \to\!\!\!\!\!\!\mapsto W$.

The following turns out to be an easy characterization of associative lax extensions.

Proposition 1.3.5.5. Let $\hat{\mathbb{T}}$ be a lax extension to $\mathcal{V}$-Rel of a monad $\mathbb{T} = (T, m, e)$ on Set. The following are equivalent:

(i) $\hat{\mathbb{T}}$ is an *associative* lax extension of the monad $\mathbb{T}$,

(ii) $\hat{T} : \mathcal{V}\text{-Rel} \longrightarrow \mathcal{V}\text{-Rel}$ preserves composition and $m^\circ : \hat{T} \longrightarrow \hat{T}\hat{T}$ is natural.

1.3.6 A lax-algebraic characterization of Top

Inspired by the work of Manes [Man69] on the role of the ultrafilter monad $\beta = (\beta, m, e)$ on Set in the characterization of compact Hausdorff spaces, Barr [Bar70] observed that for describing convergence in an arbitrary topological space one needs convergence relations instead of maps, thus obtaining a lax-algebraic characterization of topological spaces as relational β-algebras, by means of two convergence axioms, transitivity

$$a\big(\overline{\beta}a(\mathscr{U})\big) \leq a\big(m_X(\mathscr{U})\big)$$

and reflexivity

$$x \leq a\big(e_X(x)\big)$$

for all $\mathscr{U} \in \beta\beta X$ and $x \in X$ with $a : \beta X \nrightarrow X$ the convergence relation. These conditions make sense after extending the ultrafilter monad to the category Rel. With convergence preserving morphisms, the category $(\beta, 2)$-Cat of relational β-algebras, is isomorphic to Top.

The Barr extension of the filter and the ultrafilter monad

In [HST14] the *Barr extension* of a functor T is introduced and it is shown under which conditions this results in a lax extension of a monad $\mathbb{T} = (T, m, e)$ on Set to Rel. We will not give the technical details, but simply give the needed constructions for the filter monad $\mathbb{F} = (\mathsf{F}, m, e)$ and the ultrafilter monad $\beta = (\beta, m, e)$. In these cases the Barr extension yields a lax extension of the monads $\mathbb{F}$ and β to Rel.

For the filter functor $\mathsf{F} : \text{Set} \longrightarrow \text{Set}$, the Barr extension $\overline{F}$ is obtained as follows. For filters $\mathcal{A} \in \mathsf{F}X, \mathcal{B} \in \mathsf{F}Y$, and a relation $r : X \nrightarrow Y$ we define

$$\mathcal{A}\,(\overline{\mathsf{F}}r)\,\mathcal{B} \Leftrightarrow \exists \mathcal{C} \in \mathsf{F}R\,(\pi_1(\mathcal{C}) = \mathcal{A}\,\&\,\pi_2(\mathcal{C}) = \mathcal{B}), \qquad (1.77)$$

where R denotes the representation of the relation $r : X \to\!\!\!\!\!\!+ Y$ as a subset of $X \times Y$. If such a filter $\mathcal{C}$ exists, then, for all $A \in \mathcal{A}$, one has: $C := \pi_1^{-1}(A) \in \mathcal{C}$, and the set

$$r(A) := \{y \in Y \mid \exists y \in A\,(x\,r\,y)\} \tag{1.78}$$

must be in $\mathcal{B}$, as it contains $\pi_2(C)$ and $\pi_2(C) \in \pi_2(\mathcal{C}) = \mathcal{B}$. Similarly, one observes that $r^\circ(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. Conversely, if $r(A) \in \mathcal{B}$ and $r^\circ(B) \in \mathcal{A}$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the sets $C_{A,B} = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$ form a filter base for $\mathcal{C} \in \mathsf{F}R$ such that $\pi_1(\mathcal{C}) = \mathcal{A}$ and $\pi_2(\mathcal{C}) = \mathcal{B}$.

Therefore, the Barr extension of the filter functor is given by

$$\mathcal{A}\,(\overline{\mathsf{F}}r)\,\mathcal{B} \Leftrightarrow r(\mathcal{A}) \subseteq \mathcal{B}\,\&\,r^\circ(\mathcal{B}) \subseteq \mathcal{A}, \tag{1.79}$$

for all $\mathcal{A} \in \mathsf{F}X$ and $\mathcal{B} \in \mathsf{F}Y$, and relations $r : X \to\!\!\!\!\!\!+ Y$, where $r(\mathcal{A})$ is the filter generated by the filter base $\{r(A) \mid A \in \mathcal{A}\}$.

If both $\mathcal{A}$ and $\mathcal{B}$ are ultrafilters, then

$$r(\mathcal{A}) \subseteq \mathcal{B} \Leftrightarrow r^\circ(\mathcal{B}) \subseteq \mathcal{A}. \tag{1.80}$$

The Barr extension of the ultrafilter functor β is therefore described by

$$\mathcal{A}\,(\overline{\beta}r)\,\mathcal{B} \Leftrightarrow r(\mathcal{A}) \subseteq \mathcal{B} \Leftrightarrow r^\circ(\mathcal{B}) \subseteq \mathcal{A}, \tag{1.81}$$

for all $\mathcal{A} \in \beta X, \mathcal{B} \in \beta Y$ and relations $r : X \to\!\!\!\!\!\!+ Y$, or equivalently

$$\mathcal{A}\,(\overline{\beta}r)\,\mathcal{B} \Leftrightarrow \forall A \in \mathcal{A}, B \in \mathcal{B}\,\exists x \in A, y \in B : x\,r\,y. \tag{1.82}$$

The Barr extensions $\overline{F}$ and $\overline{\beta}$ both result in lax extensions $\overline{\mathbb{F}} = (\overline{F}, m, e)$ and $\overline{\mathbb{\beta}} = (\overline{\beta}, m, e)$ of the corresponding monads to Rel.

Topological spaces as relational algebras

Any topological space $(X, \mathcal{T})$ can be equivalently described as a pair (X, a), with $a : \beta X \to\!\!\!\!\!\!+ X$ a relation representing convergence which, when we denote both a and $\overline{\beta}a$ by $\to$, satisfies

$$\mathfrak{X} \to \mathcal{U}\,\&\,\mathcal{U} \to z \Rightarrow \Sigma\mathfrak{X} \to z \quad \text{and} \quad \dot{x} \to x,$$

for all $z, x \in X, \mathcal{U} \in \beta X$ and $\mathfrak{X} \in \beta\beta X$. In this context, the continuous maps $f : (X, a) \longrightarrow (Y, b)$ are exactly convergence preserving maps, i.e. the maps $f : X \longrightarrow Y$ such that

$$\mathcal{U} \to x \Rightarrow f(\mathcal{U}) \to f(x),$$

for all $x \in X$ and $\mathcal{U} \in \beta X$.

This results in the following theorem.

Theorem 1.3.6.1. *The category* $(\mathbb{\beta}, 2)$-Cat *of relational algebras for the Barr extension* $\overline{\mathbb{\beta}} = (\overline{\beta}, m, e)$ *to* Rel *of the ultrafilter monad* $\mathbb{\beta} = (\beta, m, e)$ *on* Set *is isomorphic to* Top*:*

$$(\mathbb{\beta}, 2)\text{-Cat} \cong \text{Top}.$$

1.3.7 A lax-algebraic characterization of App

A first lax-algebraic description of approach spaces was established by Clementino and Hofmann in [CH03]. The construction involves the ultrafilter monad $\mathbb{\beta} = (\beta, m, e)$, the quantale P_+ and an extension of $\mathbb{\beta}$ to P_+-Rel. It was shown that, using the appropriate extension, we get

$$(\mathbb{\beta}, \mathsf{P}_+)\text{-Cat} \cong \text{App}.$$

This will be our guiding example in Section 2.2 to find a lax-algebraic description of the so called non-Archimedean approach spaces. Therefore, we will give the construction of the isomorphism. The proof of Clementino and Hofmann makes a detour via distances. We choose to follow the alternative proof by Lowen [Low15], which gives a direct link between lax algebraic structures for the ultrafilter monad and limit operators describing approach spaces.

The lax extension of the ultrafilter monad to P_+-Rel

We start by describing the lax extension of the ultrafilter monad $\mathbb{\beta} = (\beta, m, e)$ to P_+-Rel.

For a P_+-relation $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and $\alpha \in \mathsf{P}$, we define the relation $r_\alpha : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ by

$$x \, r_\alpha \, y \Leftrightarrow r(x, y) \leq \alpha. \tag{1.83}$$

For $A \subseteq X$ we put $r_\alpha(A) = \{y \in Y \mid \exists x \in A : x \, r_\alpha \, y\}$, and for $\mathcal{A} \subseteq \mathcal{P}(X)$ we let $r_\alpha(\mathcal{A}) = \big\{r_\alpha(A) \mid A \in \mathcal{A}\big\}$. Then, for $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ a P_+-relation, we let

$$\overline{\beta} r(\mathcal{U}, \mathcal{W}) := \inf\{\alpha \in \mathsf{P} \mid \mathcal{U} \, \overline{\beta}(r_\alpha) \, \mathcal{W}\}, \tag{1.84}$$

for $\mathcal{U} \in \beta X$ and $\mathcal{W} \in \beta Y$, where $\overline{\beta}$ in this formula stands for the Barr-extension as defined in (1.81).

$\overline{\mathbb{\beta}} = (\overline{\beta}, m, e)$ is a lax extension of $\mathbb{\beta} = (\beta, m, e)$ to P_+-Rel.

The following formula, which can be found both in [Low15] and [HST14] will be useful for further calculations.

Lemma 1.3.7.1. Consider a P_+-relation $a : \beta X \relbar\joinrel\mapstochar\joinrel\rightarrow X$. Then, for all $\mathfrak{X} \in \beta\beta X$ and $\mathcal{U} \in \beta X$, we have

$$\overline{\beta} a(\mathfrak{X}, \mathcal{U}) = \sup_{\mathcal{A} \in \mathfrak{X}} \sup_{U \in \mathcal{U}} \inf_{\mathcal{W} \in \mathcal{A}} \inf_{x \in U} a(\mathcal{W}, x). \tag{1.85}$$

Approach spaces as lax algebras

The following theorem gives a lax algebraic characterization of approach spaces via the ultrafilter monad $\mathbb{\beta} = (\beta, m, e)$, laxly extended to P_+-Rel.

Theorem 1.3.7.2. *The category* $(\mathbb{\beta}, \mathsf{P}_+)$-$\mathsf{Cat}$ *of lax algebras for the* P_+-Rel-*extension of the ultrafilter monad is isomorphic to* App*:*

$$(\mathbb{\beta}, \mathsf{P}_+)\text{-}\mathsf{Cat} \cong \mathsf{App}.$$

The proof from [Low15] is based on the fact that given a $(\mathbb{\beta}, \mathsf{P}_+)$-algebra (X, a), the reflexive and transitive P_+-relation $a : \beta X \to\!\!\!\!\!\mapsto X$ can be identified as a limit operator on X satisfying all axioms stated in Theorem 1.1.1.6. Given a limit operator λ on X satisfying those axioms, this map can be seen as a relation satisfying the reflexivity and transitivity condition. Once objects in both categories are identified, it is an easy consequence of the characterization of contractions via ultrafilters in Theorem 1.1.2.2 to see that morphisms in both categories coincide as well.

1.4 Kleisli monoids

1.4.1 Power-enriched monads

For a morphism $\tau : \mathbb{P} \longrightarrow \mathbb{T}$ of monads on Set, the underlying set TX is equipped with the separated order given by

$$\mathcal{X} \leq \mathcal{Y} \Leftrightarrow m_X \cdot \tau_{TX}\big(\{\mathcal{X}, \mathcal{Y}\}\big) = \mathcal{Y}, \tag{1.86}$$

for all $\mathcal{X}, \mathcal{Y} \in TX$. The hom-sets $\mathsf{Set}(X, TY)$ become separated ordered sets via the induced point-wise order:

$$f \leq g \Leftrightarrow \forall x \in X : f(x) \leq g(x),$$

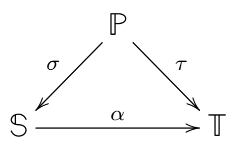
for all $f, g : X \longrightarrow TY$.

Definition 1.4.1.1. A *power-enriched* monad is a pair $(\mathbb{T}, \tau)$ composed of a monad $\mathbb{T}$ on Set and a monad morphism $\tau : \mathbb{P} \longrightarrow \mathbb{T}$ such that

$$f \leq g \Rightarrow m_Y \cdot Tf \leq m_Y \cdot Tg, \tag{1.87}$$

for all $f, g : X \longrightarrow TY$.

A morphism $\alpha : (\mathbb{S}, \sigma) \longrightarrow (\mathbb{T}, \tau)$ of power-enriched monads is a monad morphism $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ such that $\tau = \alpha \cdot \sigma$:



For maps $f : X \longrightarrow TY$ and $g : Y \longrightarrow TZ$, we introduce the following notations which will simplify some formulas in the sequel.

$$f^{\mathbb{T}} := m_Y \cdot Tf, \tag{1.88}$$

and

$$g \circ f := g^{\mathbb{T}} \cdot f = m_Z \cdot Tg \cdot f. \tag{1.89}$$

For a power-enriched monad, the order (1.86) makes TX a complete lattice and turns $m_X : TTX \longrightarrow TX$ and $Tf : TX \longrightarrow TY$ into sup-maps, for all sets X, Y and maps $f : X \longrightarrow Y$. We recall the following formula that can be deduced from IV.1.5.1 in [HST14].

$$m_X \cdot \tau_{TX} = \bigvee_{TX}, \tag{1.90}$$

where the supremum on the right-hand side is taken in TX with respect to the order defined in (1.86).

Examples 1.4.1.2.

1. The filter monad $\mathbb{F}$ is power-enriched via the principal filter natural transformation $\tau : \mathcal{P} \longrightarrow \mathsf{F}$ which yields a monad morphism $\tau : \mathbb{P} \longrightarrow \mathbb{F}$. The order on $\mathsf{F}X$ defined by (1.86) is the refinement order

$$\mathcal{F} \leq \mathcal{G} \Leftrightarrow \mathcal{F} \supseteq \mathcal{G}, \tag{1.91}$$

   and suprema in $\mathsf{F}X$ are given by intersections.

2. The ultrafilter monad β is not power-enriched: for the set $X = \emptyset$, one observes that $\beta X = \emptyset$ cannot be a complete lattice.

1.4.2 $\mathbb{T}$-monoids

We introduce the category $\mathbb{T}$-Mon of $\mathbb{T}$-monoids and $\mathbb{T}$-morphisms for a power-enriched monad $\mathbb{T}$.

Definition 1.4.2.1. Let $(\mathbb{T}, \tau)$ be a power-enriched monad on Set. The category $\mathbb{T}$-Mon of $\mathbb{T}$-*monoids* (or *Kleisli monoids*) has as objects pairs (X, ν), where X is a set, and its structure $\nu : X \longrightarrow TX$ is transitive and reflexive:

$$\nu \circ \nu \leq \nu, \qquad e_X \leq \nu \tag{1.92}$$

(where $\circ$ is the composition as defined in (1.89)). A $\mathbb{T}$-*morphism* $f : (X, \nu) \longrightarrow (Y, \mu)$ is a map $f : X \longrightarrow Y$ satisfying:

$$Tf \cdot \nu \leq \mu \cdot f. \tag{1.93}$$

The order on the hom-sets in the formulas above depends on τ.

In the presence of the reflexivity condition, transitivity can be expressed as an equality $\nu \circ \nu = \nu$, since

$$\nu = \nu \circ e_X \leq \nu \circ \nu \leq \nu. \tag{1.94}$$

We also have

$$\nu^{\mathbb{T}} \cdot \nu^{\mathbb{T}} = (\nu \circ \nu)^{\mathbb{T}} = \nu^{\mathbb{T}}. \tag{1.95}$$

1.4.3 Kleisli extension

A power-enriched monad can be extended to Rel by means of the Kleisli extension.

For a relation $r : X \relbar\joinrel\mapsto Y$, we put $r^\flat : Y \longrightarrow \mathcal{P}X$ the map defined by

$$x \in r^\flat(y) \Leftrightarrow x\, r\, y, \tag{1.96}$$

and $r^\tau : TY \longrightarrow TX$ the map defined by

$$r^\tau := m_X \cdot T(\tau_X \cdot r^\flat). \tag{1.97}$$

Definition 1.4.3.1. Given a power-enriched monad $(\mathbb{T}, \tau)$, the *Kleisli extension* $\check{T}$ of T to Rel with respect to τ is described by the functions

$$\check{T} = \check{T}_{X,Y} : \mathsf{Rel}(X, Y) \longrightarrow \mathsf{Rel}(TX, TY),$$

with

$$\mathcal{X}\, (\check{T}r)\, \mathcal{Y} \Leftrightarrow \mathcal{X} \leq r^\tau(\mathcal{Y}) \tag{1.98}$$

for all relations $r : X \relbar\joinrel\mapsto Y$, and $\mathcal{X} \in TX, \mathcal{Y} \in TY$, or equivalently

$$(\check{T}r)^\flat = \downarrow_{TX} \cdot r^\tau : TY \longrightarrow PTX. \tag{1.99}$$

The Kleisli-extension $\check{T}$ to Rel of a power-enriched monad $(\mathbb{T}, \tau)$ is not only a lax-extension of T, but yields a lax extension of the monad $\mathbb{T}$ to Rel.

Proposition 1.4.3.2. Given a power-enriched monad $(\mathbb{T}, \tau)$, the Kleisli extension $\check{T}$ of T to Rel yields a lax extension $\check{\mathbb{T}} = (\check{T}, m, e)$ of $\mathbb{T} = (T, m, e)$ to Rel.

Since the Kleisli extension provides the monad $\mathbb{T}$ with a lax extension, there is a natural order on TX associated with $\check{T}$:

$$\mathcal{X} \leq \mathcal{Y} \Leftrightarrow \mathcal{X}\, (\check{T}1_X)\, \mathcal{Y}. \tag{1.100}$$

There is also the order (1.86) induced by the monad morphism τ. Both orders on TX are equivalent.

The following theorem gives a crucial connection between the category $\mathbb{T}$-Mon, for a power-enriched monad $(\mathbb{T}, \tau)$ and its corresponding category of relational algebras for the Kleisli extension.

Theorem 1.4.3.3. *Given a power-enriched monad* $(\mathbb{T}, \tau)$ *equipped with its Kleisli extension* $\check{T}$*, there is an isomorphism*

$$(\mathbb{T}, 2)\text{-}\mathsf{Cat} \cong \mathbb{T}\text{-}\mathsf{Mon}.$$

1.4.4 Topological spaces as $\mathbb{F}$-monoids and relational $\mathbb{F}$-algebras

In 1.3.6 we recalled that topological spaces can be characterized in terms of convergence of ultrafilters by means of two axioms, transitivity and reflexivity.

The natural question, whether similar results can be obtained in terms of filter convergence, was answered positively by Seal in [Sea05].

We know that the filter monad $\mathbb{F} = (\mathsf{F}, m, e)$ is power-enriched. Hence, we know by Theorem 1.4.3.3 that

$$(\mathbb{F}, 2)\text{-}\mathsf{Cat} \cong \mathbb{F}\text{-}\mathsf{Mon}.$$

So instead of giving an explicit description of the isomorphism between Top and $(\mathbb{F}, 2)$-Cat, one can look at Top and $\mathbb{F}$-Mon instead.

The solution starts from the description of Top in terms of neighborhood filters, namely by axioms for maps $\nu : X \longrightarrow \mathsf{F}\,X$, into the set of all filters on X, as observed by Gähler in [Gäh92].

Given a topological space $(X, \mathcal{T})$, we define a map

$$\nu : X \longrightarrow \mathsf{F}\,X : x \mapsto \nu(x),$$

sending a point of the space to its neighborhood filter defined by

$$A \in \nu(x) \Leftrightarrow \exists U \in \mathcal{T} : x \in U \subseteq A.$$

Then clearly

$$\nu(x) \subseteq e_X(x) = \dot{x} = \{A \subseteq X \mid x \in A\},$$

for all $x \in X$, and therefore by (1.91)

$$e_X \leq \nu.$$

For $A \subseteq X$ we define

$$A^{\mathbb{F}} = \{\mathcal{F} \in \mathsf{F}\,X \mid A \in \mathcal{F}\}.$$

By the fact that an open set is a neighborhood of each of its points we get for all $x \in X$ and $A \subseteq X$

$$\begin{aligned}
A \in \nu(x) &\Leftrightarrow \exists B \in \nu(x)\, \forall y \in B : A \in \nu(y)\\
&\Leftrightarrow \exists B \in \nu(x)\, \forall y \in B : \nu(y) \in A^{\mathbb{F}}\\
&\Leftrightarrow \exists B \in \nu(x) : B \subseteq \nu^{-1}(A^{\mathbb{F}})\\
&\Leftrightarrow \nu^{-1}(A^{\mathbb{F}}) \in \nu(x)\\
&\Leftrightarrow A^{\mathbb{F}} \in \mathsf{F}\,\nu \cdot \nu(x)\\
&\Leftrightarrow A \in m_X \cdot \mathsf{F}\,\nu \cdot \nu(x) = \nu \circ \nu(x).
\end{aligned}$$

Again by (1.91) we get

$$\nu \circ \nu \leq \nu.$$

Conversely, if (X, ν) is an object in $\mathbb{F}$-Mon, we can define a topology $\mathcal{T}$ on X by

$$U \in \mathcal{T} \Leftrightarrow \forall x \in X : \big(x \in U \Rightarrow U \in \nu(x)\big).$$

Now consider two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ with $\nu : X \longrightarrow \mathsf{F}X$ and $\mu : Y \longrightarrow \mathsf{F}Y$ the corresponding neighborhood filter maps. Now consider a map $f : X \longrightarrow Y$. Suppose f is continuous. Then

$$B \in \mu\big((f(x)\big) \Rightarrow f^{-1}(B) \in \nu(x)$$

and by

$$f^{-1}(B) \in \nu(x) \Leftrightarrow B \in \mathsf{F}f \cdot \nu(x),$$

we get $\mu \cdot f(x) \subseteq \mathsf{F}f \cdot \nu(x)$ for all $x \in X$, which by (1.91) means

$$\mathsf{F}f \cdot \nu \leq \mu \cdot f.$$

Conversely, if $f : X \longrightarrow Y$ satisfies $\mathsf{F}f \cdot \nu \leq \mu \cdot f$, we get that $U \in \mathcal{S}$ implies $f^{-1}(U) \in \mathcal{T}$.

These constructions give us two functors

$$\mathsf{Top} \to \mathbb{F}\text{-}\mathsf{Mon} \quad \text{and} \quad \mathbb{F}\text{-}\mathsf{Mon} \to \mathsf{Top}$$

whose composites are the identities on Top and $\mathbb{F}$-Mon.

Theorem 1.4.4.1. *The category* Top *and the category* $\mathbb{F}$-Mon *are isomorphic.*

$$\mathsf{Top} \cong \mathbb{F}\text{-}\mathsf{Mon}$$

Applying Theorem 1.4.3.3, we get

$$\mathsf{Top} \cong (\mathbb{F}, 2)\text{-}\mathsf{Cat}.$$

This gives a description of topological spaces in terms of convergence of filters by means of two axioms, transitivity and reflexivity. A topological space is a pair (X, a) with $a : \mathsf{F}X \relbar\joinrel\mapsto X$ a relation representing convergence which, when we denote both a and $\check{\mathsf{F}}a$ by $\to$, satisfies

$$\mathfrak{F} \to \mathcal{F} \,\&\, \mathcal{F} \to z \Rightarrow \Sigma\mathfrak{F} \to z \quad \text{and} \quad \dot{x} \to x,$$

for all $\mathfrak{F} \in \mathsf{F}\mathsf{F}X, \mathcal{F} \in \mathsf{F}X$ and $x, z \in X$, where $\check{\mathsf{F}}$ is the Kleisli extension of F to Rel.

Chapter 2

$\mathsf{NA\text{-}App}$ as a category of lax algebras

In this chapter we investigate the full subcategory $\mathsf{NA\text{-}App}$ of App with objects the non-Archimedean approach spaces. Non-Archimedean approach spaces were first introduced by Brock and Kent [BK98] and were also considered by Colebunders, Mynard and Trott in [CMT14] and by Boustique and Richardson as certain limit tower spaces [BR17].

In the first section of this chapter we introduce various equivalent ways to define a non-Archimedean approach space as given by Brock and Kent [BK98]. To the known structures defining these spaces, we add a characterization in terms of the gauge. We show that non-Archimedean approach spaces are those approach spaces whose gauge has a basis consisting of quasi-ultrametrics.

In the second section we investigate whether we can find parameters, i.e. a monad $\mathbb{T}$ and a quantale $\mathcal{V}$, such that the category of non-Archimedean approach spaces $\mathsf{NA\text{-}App}$ can be represented as $(\mathbb{T},\mathcal{V})\text{-}\mathsf{Cat}$. Our solution is inspired by the known fact that the category of quasi-ultrametric spaces $q\mathsf{Met}^u$ is isomorphic to $(\mathbb{1},\mathsf{P}_\vee)\text{-}\mathsf{Cat}$ [HST14]. We adapt the result from Clementino and Hofmann [CH03], $\mathsf{App}\cong(\beta,\mathsf{P}_+)\text{-}\mathsf{Cat}$ by replacing the quantale P_+ to $\mathsf{P}_\vee$ to find a lax algebraic characterization of $\mathsf{NA\text{-}App}$.

We will locate the category $\mathsf{NA\text{-}App}\cong(\beta,\mathsf{P}_\vee)\text{-}\mathsf{Cat}$ relative to more familiar categories $q\mathsf{Met}^u\cong\mathsf{P}_\vee\text{-}\mathsf{Cat}$, $\mathsf{Top}\cong(\beta,2)\text{-}\mathsf{Cat}$ and $\mathsf{App}\cong(\beta,\mathsf{P}_+)\text{-}\mathsf{Cat}$ via reflective and/or coreflective embeddings.

Furthermore we describe the construction of coproducts and quotients in the category $\mathsf{NA\text{-}App}$.

We end this chapter by giving three initially dense objects in the category $\mathsf{NA\text{-}App}$, which shows that $\mathsf{NA\text{-}App}$ is finitely generated.

2.1 Equivalent descriptions of non-Archimedean approach spaces

In this section we define equivalent characterizations of non-Archimedean approach spaces, in terms of distances, limit operators, towers and gauges.

2.1.1 Non-Archimedean limit operators

Non-Archimedean limit operators were first introduced by Brock and Kent in [BK98] by strengthening the triangular inequality in the definition of a limit operator. This corresponds to strengthening axiom (L*) in the definition of a limit operator which we use, see Definition 1.1.1.5 and it is also equivalent to strengthening axiom (Lβ^*) from Theorem 1.1.1.6, which is of special interest throughout this work.

Definition 2.1.1.1. A limit operator $\lambda : \beta X \longrightarrow \mathsf{P}^X$ on a set X satisfying the strong triangular inequality

(L$\beta^*_\vee$) For any set J, for any $\psi : J \longrightarrow X$, for any $\sigma : J \longrightarrow \beta X$ and for any $\mathcal{U} \in \beta J$

$$\lambda \Sigma \sigma(\mathcal{U}) \leq \lambda \psi(\mathcal{U}) \vee \sup_{U \in \mathcal{U}} \inf_{j \in U} \lambda \sigma(j)\big(\psi(j)\big),$$

is called a *non-Archimedean limit operator*.

Definition 2.1.1.2. A set X equipped with a non-Archimedean limit operator $\lambda : \beta X \longrightarrow \mathsf{P}^X$ is called a *non-Archimedean approach spaces*. NA-App is the full subcategory of App consisting of all non-Archimedean approach spaces.

2.1.2 Non-Archimedean distances

First we associate with a limit operator λ of an approach space X, its unambiguously defined distance $\delta : X \times 2^X \longrightarrow \mathsf{P}$. When starting with a non-Archimedean limit operator, we get a non-Archimedean distance, i.e. a distance which satisfies a stronger triangular inequality. Non-Archimedean distance operators were also considered by Brock and Kent [BK98].

Theorem 2.1.2.1. *If $\lambda : \beta X \longrightarrow \mathsf{P}^X$ is the limit operator of a non-Archimedean approach space X, then the associated distance, given by* (1.31) *satisfies the strong triangular inequality (D4$_\vee$)*

$$\delta(x, A) \leq \delta\big(x, A^{(\varepsilon)}\big) \vee \varepsilon,$$

for all $x \in X, A \subseteq X$ *and* $\varepsilon \in \mathsf{P}$*, with* $A^{(\varepsilon)} := \{x \in X \mid \delta(x, A) \leq \varepsilon\}$.

If $\delta : X \times 2^X \longrightarrow \mathsf{P}$ *is a distance of an approach space* X *satisfying the strong triangular inequality (D4$_\vee$), then the associated limit operator, given by* (1.27) *is a non-Archimedean limit operator.*

Proof. In order to prove that δ satisfies (D4$_\vee$), take $A \subseteq X$, $\varepsilon \in \mathbb{R}^+$ and $\mathcal{W} \in \beta\big(A^{(\varepsilon)}\big)$ arbitrary. As in 1.2.2 in [Low15] for all $y \in A^{(\varepsilon)}$, there exists $\sigma(y) \in \beta A$ such that $\lambda\sigma(y)(y) \leq \varepsilon$. For $y \notin A^{(\varepsilon)}$, let $\sigma(y) = \dot{y}$ and set $\varepsilon' := \sup_{W \in \mathcal{W}} \inf_{y \in W} \lambda\sigma(y)(y)$. Since $A \in \bigcap_{y \in A^{(\varepsilon)}} \sigma(y)$ we obtain $\Sigma\sigma(\mathcal{W}) \in \beta A$. Hence

$$\delta(x, A) \leq \lambda\Sigma\sigma(\mathcal{W})(x) \leq \lambda\mathcal{W}(x) \vee \varepsilon' \leq \lambda\mathcal{W}(x) \vee \varepsilon,$$

where the second inequality follows from (L$\beta^*_\vee$) with $J = X$ and $\psi = \mathrm{id}$.

Now suppose that δ is a distance satisfying the strong triangular inequality (D4$_\vee$). We show that the associated limit operator λ satisfies (L$\beta^*_\vee$). Take a set J, a map $\psi : J \longrightarrow X$, a selection $\sigma : J \longrightarrow \beta X$ and an ultrafilter $\mathcal{U} \in \beta J$ and put $\varepsilon := \sup_{U \in \mathcal{U}} \inf_{j \in U} \lambda\sigma(j)\big(\psi(j)\big)$. Take $D \in \Sigma\sigma(\mathcal{U})$ arbitrary. Then, as in 1.2.1 in [Low15], there exists $U \in \mathcal{U}$ such that for all $y \in U$ we get that $D \in \sigma(y)$. Hence, for $y \in U$, we have $\delta\big(\psi(y), D\big) \leq \inf_{U \in \mathcal{U}} \sup_{j \in U} \lambda\sigma(j)\big(\psi(j)\big) = \varepsilon$ and thus $\psi(y) \in D^{(\varepsilon)}$. This shows $D^{(\varepsilon)} \in \psi(\mathcal{U})$. From (D4$_\vee$) we get

$$\delta(x, D) \leq \delta\big(x, D^{(\varepsilon)}\big) \vee \varepsilon \leq \lambda\psi(\mathcal{U})(x) \vee \varepsilon$$

and by arbitrariness of $D \in \Sigma\sigma(\mathcal{U})$,

$$\lambda\Sigma\sigma(\mathcal{U})(x) = \sup_{D \in \Sigma\sigma(\mathcal{U})} \delta(x, D) \leq \lambda\psi(\mathcal{U})(x) \vee \varepsilon.$$

□

Distances satisfying the strong triangular inequality (D4$_\vee$) will be called *non-Archimedean distances*.

The following inequality is a strengthening of the third property in Proposition 1.1.1.2. It will be useful later on and has a straightforward proof.

Proposition 2.1.2.2. If $\delta : X \times 2^X \longrightarrow \mathsf{P}$ is a non-Archimedean distance, then the following inequality holds

$$\delta(x, A) \leq \delta(x, B) \vee \sup_{b \in B} \delta(b, A),$$

for all $x \in X$ and $A, B \subseteq X$.

We give an example of a non-Archimedean approach space on $\mathsf{P}_\vee$ which will play an important role later on in Section 2.5.

Example 2.1.2.3. Define $\delta_{\mathsf{P}_\vee} : \mathsf{P}_\vee \times 2^{\mathsf{P}_\vee} \longrightarrow \mathsf{P}_\vee$ by

$$\begin{aligned}\delta_{\mathsf{P}_\vee}(x, A) &:= \begin{cases} \sup A {-\!\bullet} x & A \neq \emptyset, \\ \infty & A = \emptyset. \end{cases} \\ &= \begin{cases} 0 & A \neq \emptyset \text{ and } x \leq \sup A, \\ x & A \neq \emptyset \text{ and } x > \sup A, \\ \infty & A = \emptyset. \end{cases}\end{aligned}$$

Then $\delta_{\mathsf{P}_\vee}$ is a non-Archimedean distance on $\mathsf{P}_\vee$. The associated non-Archimedean limit operator is defined as follows. Given an ultrafilter $\mathcal{U}$ on $\mathsf{P}_\vee$ and an element $x \in X$, then

$$\begin{aligned}\lambda_{\mathsf{P}_\vee}\mathcal{U}(x) &= \sup_{U\in\mathcal{U}}(\sup U {-\!\bullet} x) \\ &= \begin{cases} 0 & \forall U \in \mathcal{U} : x \leq \sup U, \\ x & \exists U \in \mathcal{U} : x > \sup U. \end{cases}\end{aligned}$$

2.1.3 Non-Archimedean towers

Next, with a distance of an approach space X, we associate its unambiguously defined tower $(\mathsf{t}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ of closure operators. When starting from a non-Archimedean distance, we get the so called non-Archimedean towers. These structures also appear in [BK98] under the name 'strongly approachable limit towers'. Since the notations and the context in this paper differ from our work, for completeness we add the proof which clarifies the explicit transition between non-Archimedean distances and non-Archimedean towers.

Theorem 2.1.3.1. *If $\delta : X \times 2^X \longrightarrow \mathsf{P}$ is the distance of a non-Archimedean approach space X, then all levels of the associated tower $(\mathsf{t}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$, given by* (1.30)*, are topological closure operators.*

If $(\mathsf{t}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ is the tower of an approach space X, where all levels are topological closure operators, then the associated distance, given by (1.43)*, is a non-Archimedean distance on X.*

Proof. Consider a non-Archimedean distance. We prove that all levels of the corresponding tower are topological closure operators, meaning $\mathsf{t}_\varepsilon\big(\mathsf{t}_\varepsilon(A)\big) = \mathsf{t}_\varepsilon(A)$, for all $\varepsilon \in \mathbb{R}^+$ and $A \subseteq X$. One inclusion is clear, the other follows from (D4$_\vee$).

$$\begin{aligned} x \in \mathsf{t}_\varepsilon\big(\mathsf{t}_\varepsilon(A)\big) &\Leftrightarrow \delta\big(x, A^{(\varepsilon)}\big) \leq \varepsilon \\ &\Rightarrow \delta(x, A) \leq \delta\big(x, A^{(\varepsilon)}\big) \vee \varepsilon \leq \varepsilon \\ &\Rightarrow x \in \mathsf{t}_\varepsilon(A). \end{aligned}$$

To prove the converse, suppose $(\mathsf{t}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ is the tower of an approach space X where all levels are topological closure operators. We show that the associated distance satisfies (D4$_\vee$). For $x \in X$ and α with

$$\delta\big(x, A^{(\varepsilon)}\big) = \inf\{\beta \in \mathbb{R}^+ \mid x \in \mathsf{t}_\beta\big(A^{(\varepsilon)}\big)\} < \alpha,$$

we have

$$x \in \mathsf{t}_\alpha\big(A^{(\varepsilon)}\big) = \mathsf{t}_\alpha\big(\mathsf{t}_\varepsilon(A)\big) \subseteq \mathsf{t}_{\alpha\vee\varepsilon}\big(\mathsf{t}_{\alpha\vee\varepsilon}(A)\big) = \mathsf{t}_{\alpha\vee\varepsilon}(A)$$

and therefore $\delta(x, A) \leq \alpha \vee \varepsilon$. □

Towers of approach spaces that satisfy the stronger condition of the previous theorem are called *non-Archimedean towers*. At each level the closure operator t_ε defines a topology $\mathcal{T}_\varepsilon$, so we will denote the structure also by $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$, when working with open sets at each level, or $(\mathcal{C}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$, when using closed sets.

We now get the following characterization of non-Archimedean approach spaces in terms of towers.

Corollary 2.1.3.2. A collection $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ of topologies on a set X defines a tower for some non-Archimedean approach space if and only if it satisfies the *coherence condition*

$$\mathcal{T}_\varepsilon = \bigvee_{\gamma>\varepsilon} \mathcal{T}_\gamma, \tag{2.1}$$

where the supremum is taken in Top.

We include more examples of non-Archimedean approach spaces that will be useful in Section 4.2. The construction of the non-Archimedean approach spaces in the following examples is based on Corollary 2.1.3.2.

Example 2.1.3.3. Let X be a set and $\mathcal{S}$ a given topology on X. Let $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ be defined by

$$\mathcal{T}_\varepsilon = \begin{cases} \mathcal{P}(X) & \text{whenever } 0 \leq \varepsilon < 1, \\ \mathcal{S} & \text{whenever } 1 \leq \varepsilon < 2, \\ \{X, \emptyset\} & \text{whenever } 2 \leq \varepsilon. \end{cases}$$

Clearly the coherence condition is satisfied and so $\big(X, (\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\big)$ defines a non-Archimedean approach space which we will denote by $X_\mathcal{S}$.

Example 2.1.3.4. Let $X =]0, \infty[$, endowed with a topology $\mathcal{T}$ with neighborhoodfilters $\big(\mathcal{V}(x)\big)_{x\in X}$ and assume $\mathcal{T}$ is finer than the right order topology. We define $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ with $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{T}_\varepsilon$ at level $\varepsilon > 0$ having neighborhoodfilters

$$\mathcal{V}_\varepsilon(x) = \begin{cases} \{X\} & \text{whenever } x \leq \varepsilon, \\ \mathcal{V}(x) & \text{whenever } \varepsilon < x, \end{cases}$$

at $x \in X$.

For a fixed level ε and $\varepsilon < x$ the set $\big\{G \in \mathcal{T} \mid x \in G, G \subseteq]\varepsilon, +\infty[\big\}$ is an open base of $\mathcal{V}_\varepsilon(x)$. So clearly $(\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$ is a descending chain of topologies. To check the other inclusion of the coherence condition, let $0 \leq \varepsilon$ and $x \in]0, \infty[$. Either $x \leq \varepsilon$ and then $\mathcal{V}_\varepsilon(x) = \{X\} \subseteq \mathcal{V}_\gamma(x)$ for every γ. Or $\varepsilon < x$, then choose γ with $\varepsilon < \gamma < x$. We have $\mathcal{V}_\varepsilon(x) = \mathcal{V}_\gamma(x) = \mathcal{V}(x)$. So $\big(X, (\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}\big)$ defines a non-Archimedean approach space.

2.1.4 Non-Archimedean gauges

Finally, we look at gauges. We introduce a new and very elegant characterization of non-Archimedean approach spaces in terms of the gauge.

Theorem 2.1.4.1. *Consider a non-Archimedean approach space X with tower of topologies $(\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$. Then the associated gauge, given by* (1.46)*, has a basis consisting of quasi-ultrametrics.*

Proof. For $n \in \mathbb{N}_0$ a strictly positive natural number we consider $\big\{\frac{k}{n} \mid k \in \{0, \cdots, n^2\}\big\}$ on $[0, n]$. At level $\frac{k}{n}$ we choose a finite $\mathcal{T}_{\frac{k}{n}}$-open cover $\mathcal{C}_{\frac{k}{n}}$ in such a way that for $k > 0$ the finite $\mathcal{T}_{\frac{k-1}{n}}$-open cover $\mathcal{C}_{\frac{k-1}{n}}$ is a refinement of $\mathcal{C}_{\frac{k}{n}}$. The following notations are frequently used in the setting of quasi-uniform spaces. For $x \in X$ and $k \in \{0, \cdots, n^2\}$

$$A^x_{\mathcal{C}_{\frac{k}{n}}} = \bigcap \{C \mid C \in \mathcal{C}_{\frac{k}{n}}, x \in C\}$$

and

$$U_{\mathcal{C}_{\frac{k}{n}}} = \bigcup_{x \in X} \{x\} \times A^x_{\mathcal{C}_{\frac{k}{n}}}.$$

We employ a standard technique as used with developments in [Low15] to construct a function depending on n and on the choice of $\mathcal{C}_{\frac{1}{n}}, \cdots, \mathcal{C}_{\frac{n^2}{n}}$ by letting

$$p_n = \inf_{k=1}^{n^2} \left(\frac{k-1}{n} + \theta_{U_{\mathcal{C}_{\frac{k}{n}}}} \right) \wedge n, \tag{2.2}$$

where for $Z \subseteq X$, we use the notation

$$\theta_Z : X \to \mathsf{P} : x \mapsto \begin{cases} 0 & x \in Z \\ \infty & x \notin Z. \end{cases}$$

Clearly for $k \in \{1, \cdots, n^2\}$ we have $U_{\mathcal{C}_{\frac{k-1}{n}}} \subseteq U_{\mathcal{C}_{\frac{k}{n}}}$ and every $U_{\mathcal{C}_{\frac{k}{n}}}$ is a preorder on X. This implies that p_n is a quasi-ultrametric on X. That p_n is zero on the diagonal is clear. We check the nontrivial case of the strong triangular inequality. If $p(x,y) = \frac{i-1}{n}$ and $p(y,z) = \frac{j-1}{n}$ then $(x,y) \in U_{\mathcal{C}_{\frac{i}{n}}}$ and $(y,z) \in U_{\mathcal{C}_{\frac{j}{n}}}$. Hence $(x,y) \in U_{\mathcal{C}_{\frac{i\vee j}{n}}}$ and $(y,z) \in U_{\mathcal{C}_{\frac{i\vee j}{n}}}$. This implies $(x,z) \in U_{\mathcal{C}_{\frac{i\vee j}{n}}}$ from which we conclude $p(x,z) \leq \frac{(i\vee j)-1}{n} \leq \frac{i-1}{n} \vee \frac{j-1}{n}$.

Next we show that each p_n belongs to the gauge $\mathcal{G}$. Fix $\varepsilon \geq 0$ and $\alpha > \varepsilon$. Either $\varepsilon \geq n$ and then the open ball $B_p(x,\alpha) = X \in \mathcal{T}_\varepsilon$, or $\varepsilon \in [\frac{k-1}{n}, \frac{k}{n}[$ for some $k \in \{1, \cdots, n^2\}$. Then for $y \in A^x_{\mathcal{C}_{\frac{k}{n}}}$ we have $(x,y) \in U_{\mathcal{C}_{\frac{k}{n}}}$ which implies $p_n(x,y) \leq \frac{k-1}{n} \leq \varepsilon < \alpha$. So $A^x_{\mathcal{C}_{\frac{k}{n}}} \subseteq B_p(x,\alpha)$ and again $B_p(x,\alpha) \in \mathcal{T}_\varepsilon$.

Let $\mathcal{H}$ be the collection of all quasi-ultrametrics p_n, for arbitrary choices of n and $\mathcal{C}_{\frac{1}{n}}, \cdots, \mathcal{C}_{\frac{n^2}{n}}$. We prove that $\mathcal{H}$ is a basis for the gauge $\mathcal{G}$. Let $d \in \mathcal{G}$, $x \in X$ and $n \in \mathbb{N}_0$. Consider $\left\{\frac{k}{n} \mid k \in \{0, \cdots, n^2\}\right\}$ on $[0,n]$ and at level $\frac{k}{n}$ choose the cover

$$\mathcal{C}_{\frac{k}{n}} = \left\{B_d(x, \frac{k}{n} + \frac{1}{2n}), X\right\}.$$

Since $d \in \mathcal{G}$ the inclusion $\mathcal{T}^d_{\frac{k}{n}} \subseteq \mathcal{T}_{\frac{k}{n}}$ holds, so the cover $\mathcal{C}_{\frac{k}{n}}$ is $\mathcal{T}_{\frac{k}{n}}$-open for every k. Moreover, $\mathcal{C}_{\frac{k-1}{n}}$ refines $\mathcal{C}_{\frac{k}{n}}$ at each level. Let p_n be the associated quasi-ultrametric as in (2.2). We show that

$$d(x,\cdot) \wedge n \leq p_n(x,\cdot) + \frac{2}{n}. \tag{2.3}$$

Let $y \in X$. Either $p_n(x,y) + \frac{2}{n} \geq n$ and then we are done, or $p_n(x,y) + \frac{2}{n} = \alpha \in [\frac{k}{n}, \frac{k+1}{n}[$ for some k. In this case $p_n(x,y) = \alpha - \frac{2}{n} \in [\frac{k-2}{n}, \frac{k-1}{n}[$ which implies $(x,y) \in U_{\mathcal{C}_{\frac{k-1}{n}}}$. So we have $y \in B_d(x, \frac{k-1}{n} + \frac{1}{2n})$, by which $d(x,y) < \frac{k-1}{n} + \frac{1}{2n} < \frac{k}{n} \leq \alpha$. Since (2.3) holds for every $n \in \mathbb{N}_0$ it now follows that also (1.13) is fulfilled for every $\varepsilon > 0$ and $\omega < \infty$, so we can conclude that $\widehat{\mathcal{H}} = \mathcal{G}$. □

Theorem 2.1.4.2. *Consider an approach space X with gauge $\mathcal{G}$, having a basis $\mathcal{H}$ consisting of quasi-ultrametrics. Then the associated distance is a non-Archimedean distance.*

Proof. The distance δ associated with the gauge $\mathcal{G}$ can be derived directly from the basis $\mathcal{H}$ by

$$\delta(x,A) = \sup_{d\in\mathcal{H}} \inf_{a\in A} d(x,a).$$

We only have to show that this distance satisfies (D4$_\vee$). Take $x \in X, A \subseteq X$ and $\varepsilon \in \mathsf{P}$ arbitrary. Then, for any $b \in A^{(\varepsilon)}$, $d \in \mathcal{H}$ and $\theta > 0$, there exists $a_d \in A$ such that $d(b, a_d) < \varepsilon + \theta$. Consequently,

$$d(x,a_d) \leq d(x,b) \vee d(b,a_d) \leq d(x,b) \vee (\varepsilon + \theta),$$

which proves that

$$\inf_{a\in A} d(x,a) \leq \inf_{b\in A^{(\varepsilon)}} d(x,b) \vee (\varepsilon + \theta).$$

Since this holds for all $d \in \mathcal{H}$, it follows that $\delta(x, A) \leq \delta\big(x, A^{(\varepsilon)}\big) \vee \varepsilon$.

□

A gauge with a basis consisting of quasi-ultrametrics will be called a *non-Archimedean gauge*.

2.2 A lax-algebraic characterization of NA-App

In this section we answer the question whether parameters $\mathbb{T}$ and $\mathcal{V}$ can be found such that NA-App can be described as a category of lax algebras $(\mathbb{T}, \mathcal{V})$-Cat. As an inspiration, we look at the known result $q\mathsf{Met}^u \cong \mathsf{P}_\vee$-Cat.

2.2.1 Change-of-base functors

Before generalizing the constructions of P_+-Cat and (β, P_+)-Cat established in Examples 1.3.3.2 and Section 1.3.7 to the quantale $\mathsf{P}_\vee$, as introduced in Examples 1.3.1.1, we give some information concerning change-of-base functors. More details can be found in [HST14].

Given a monad $\mathbb{T} = (T, m, e)$, we can consider lax extensions $\hat{\mathbb{T}}$ and $\check{\mathbb{T}}$ of $\mathbb{T}$ to $\mathcal{V}$-Rel and $\mathcal{W}$-Rel, respectively. Let $\varphi : \mathcal{V} \longrightarrow \mathcal{W}$ be a lax homomorphism of quantales. Then φ induces a lax functor

$$\varphi : \mathcal{V}\text{-}\mathsf{Rel} \longrightarrow \mathcal{W}\text{-}\mathsf{Rel}$$

which leaves objects unchanged and sends $r : X \times Y \longrightarrow \mathcal{V}$ to $\varphi r : X \times Y \longrightarrow \mathcal{W}$. For a Set-map f, we have

$$f \leq \varphi f \qquad \text{and} \qquad f^\circ \leq \varphi(f^\circ)$$

where f and f° are considered as $\mathcal{W}$-relations when appearing on the left of the inequality sign, and as $\mathcal{V}$-relations when appearing on the right. Furthermore, we assume that φ is compatible with the respective lax extensions $\hat{\mathbb{T}}$ and $\check{\mathbb{T}}$ of $\mathbb{T}$ to $\mathcal{V}$-Rel and $\mathcal{W}$-Rel, that is $\check{T}(\varphi r) \leq \varphi(\hat{T} r)$, for all $\mathcal{V}$-relations r.

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathsf{Rel} & \xrightarrow{\ \hat{T}\ } & \mathcal{V}\text{-}\mathsf{Rel} \\ {\scriptstyle \varphi}\downarrow & \leq & \downarrow{\scriptstyle \varphi} \\ \mathcal{W}\text{-}\mathsf{Rel} & \xrightarrow[\ \check{T}\ ]{} & \mathcal{W}\text{-}\mathsf{Rel} \end{array}$$

Under these conditions, φ induces a functor

$$B_\varphi : (\mathbb{T}, \mathcal{V})\text{-}\mathsf{Cat} \longrightarrow (\mathbb{T}, \mathcal{W})\text{-}\mathsf{Cat},$$

called a change-of-base functor associated with φ, sending (X, a) to $(X, \varphi a)$ and leaving maps unchanged. Indeed $(X, \varphi a)$ is a $(\mathbb{T}, \mathcal{W})$-category, since

$$e_X^\circ \leq \varphi(e_X^\circ) \leq \varphi a,$$

and

$$\begin{aligned} \varphi a \cdot \check{T}(\varphi a) \cdot m_X^\circ &\leq \varphi a \cdot \varphi(\hat{T} a) \cdot \varphi(m_X^\circ) \\ &\leq \varphi(a \cdot \hat{T} a \cdot m_X^\circ) \\ &\leq \varphi(a). \end{aligned}$$

Moreover, given a $(\mathbb{T}, \mathcal{V})$-functor $f : (X, a) \longrightarrow (Y, b)$, $f : (X, \varphi a) \longrightarrow (Y, \varphi b)$ is a $(\mathbb{T}, \mathcal{W})$-functor, since

$$f \cdot \varphi a \leq \varphi f \cdot \varphi a \leq \varphi(f \cdot a) \leq \varphi(b \cdot Tf) = \varphi b \cdot Tf.$$

2.2.2 $\mathsf{P}_\vee$-Cat

The map $\varphi : \mathsf{P}_\vee \longrightarrow \mathsf{P}_+$ with $\varphi(v) = v$, for all $v \in \mathsf{P}_\vee$ is a lax homomorphism of quantales. This induces a lax-functor $\varphi : \mathsf{P}_\vee\text{-}\mathsf{Rel} \longrightarrow \mathsf{P}_+\text{-}\mathsf{Rel}$, which leaves objects unaltered and sends $r : X \times Y \longrightarrow \mathsf{P}_\vee$ to $\varphi \cdot r : X \times Y \longrightarrow \mathsf{P}_+$. Obviously, φ is compatible with the identical lax extension of the identity monad $\mathbb{1}$ to $\mathsf{P}_\vee$-Rel and P_+-Rel. Hence, this lax-functor induces a change-of-base functor $B_\varphi : \mathsf{P}_\vee\text{-}\mathsf{Cat} \longrightarrow \mathsf{P}_+\text{-}\mathsf{Cat}$. Moreover, this change-of-base functor is an embedding. In Examples 1.3.3.2, we recalled that $\mathsf{P}_+\text{-}\mathsf{Cat} \cong q\mathsf{Met}$, the category of extended quasi-metric spaces and non-expansive maps, and $\mathsf{P}_\vee\text{-}\mathsf{Cat} \cong q\mathsf{Met}^u$, the category of extended quasi-ultrametric spaces and non-expansive maps.

2.2.3 The extension of the ultrafilter monad to $\mathsf{P}_\vee$-Rel

We will define the extension of the ultrafilter monad $\beta = (\beta, m, e)$ to $\mathsf{P}_\vee$-Rel analogously to the extension of β to P_+-Rel, constructed in 1.3.7.

The result in this section follow from more general results by Clementino and Tholen [CT03]. For completeness we add these results to the thesis, clarifying the constructions needed further on in this work.

For a $\mathsf{P}_\vee$-relation $r : X \nrightarrow Y$ and $\alpha \in \mathsf{P}$, we can define the relation $r_\alpha : X \nrightarrow Y$ by

$$x \, r_\alpha \, y \Leftrightarrow r(x, y) \leq \alpha. \tag{2.4}$$

For $A \subseteq X$ we put $r_\alpha(A) = \{y \in Y \mid \exists x \in A : x\, r_\alpha\, y\}$, and for $\mathcal{A} \subseteq \mathcal{P}(X)$ we let $r_\alpha(\mathcal{A}) = \big\{r_\alpha(A) \mid A \in \mathcal{A}\big\}$. Then, for $r : X \nrightarrow Y$ a $\mathsf{P}_\vee$-relation, we let

$$\overline{\beta}r(\mathcal{U}, \mathcal{W}) := \inf\{\alpha \in \mathsf{P} \mid \mathcal{U}\, \overline{\beta}(r_\alpha)\, \mathcal{W}\}, \tag{2.5}$$

for $\mathcal{U} \in \beta X$ and $\mathcal{W} \in \beta Y$, which can be equivalently expressed by

$$\overline{\beta}r(\mathcal{U}, \mathcal{W}) = \sup_{A \in \mathcal{U}, B \in \mathcal{W}} \inf_{x \in A, y \in B} r(x, y). \tag{2.6}$$

The following result is a special instance of a more general result from [CT03]. We only provide a proof of this theorem to keep the text self-contained.

Theorem 2.2.3.1. *[CT03] The extension of the ultrafilter monad* $\mathbb{\beta}$ *to* $\mathsf{P}_\vee$-Rel *is a flat and associative lax extension, which we again denote by* $\overline{\mathbb{\beta}} = (\overline{\beta}, m, e)$.

Proof. First of all, we show that $\overline{\mathbb{\beta}}$ is indeed a lax extension of $\mathbb{\beta}$ to $\mathsf{P}_\vee$-Rel. The only difference with the P_+-situation, is that $\mathsf{P}_\vee$-relations follow a different composition rule. Since the extension of $\mathbb{\beta}$ is defined analogously to the P_+-case, the only axiom of a lax extension of monads needed to be checked is 2. from Definition 1.3.4.1, since this is the only axiom where composition of relations is involved.

Hence, take two arbitrary $\mathsf{P}_\vee$-relations $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$. Then we show that

$$\overline{\beta}s \cdot \overline{\beta}r \leq_{\text{op}} \overline{\beta}(s \cdot r).$$

Take $\mathcal{U} \in \beta X$ and $\mathcal{W} \in \beta Z$ arbitrary. Let

$$\alpha_1 := \overline{\beta}(s \cdot r)(\mathcal{U}, \mathcal{W})$$

and

$$\alpha_2 := \overline{\beta}s \cdot \overline{\beta}r(\mathcal{U}, \mathcal{W}) = \inf\big\{\overline{\beta}r(\mathcal{U}, \mathcal{V}) \vee \overline{\beta}s(\mathcal{V}, \mathcal{W}) \mid \mathcal{V} \in \beta Y\big\}.$$

We have to prove that $\alpha_1 \leq \alpha_2$. Take $\varepsilon > 0$ arbitrary. There exists $\mathcal{V} \in \beta Y$ such that $\overline{\beta}r(\mathcal{U}, \mathcal{V}) \vee \overline{\beta}(\mathcal{V}, \mathcal{W}) < \alpha_2 + \varepsilon$. Let

$$\gamma_1 := \overline{\beta}r(\mathcal{U}, \mathcal{V}) = \inf\{\alpha \in \mathsf{P} \mid r_\alpha(\mathcal{U}) \subseteq \mathcal{V}\}$$

and

$$\gamma_2 := \overline{\beta}s(\mathcal{V}, \mathcal{W}) = \inf\{\alpha \in \mathsf{P} \mid s_\alpha(\mathcal{V}) \subseteq \mathcal{W}\}.$$

This implies that $r_{\gamma_1+\varepsilon}(\mathcal{U}) \subseteq \mathcal{V}$ and $s_{\gamma_2+\varepsilon}(\mathcal{V}) \subseteq \mathcal{W}$. For $U \in \mathcal{U}$ arbitrary, we have

$$s_{\gamma_2+\varepsilon}\big(r_{\gamma_1+\varepsilon}(U)\big) \in \mathcal{W}.$$

Now we claim that

$$s_{\gamma_2+\varepsilon}\big(r_{\gamma_1+\varepsilon}(U)\big) \subseteq (s \cdot r)_{(\gamma_1 \vee \gamma_2)+\varepsilon}(U).$$

Indeed, for $z \in s_{\gamma_2+\varepsilon}\big(r_{\gamma_1+\varepsilon}(U)\big)$, we get

$$\begin{aligned} z \in s_{\gamma_2+\varepsilon}\big(r_{\gamma_1+\varepsilon}(U)\big) \;&\Leftrightarrow\; \exists y \in r_{\gamma_1+\varepsilon}(U) : y\, s_{\gamma_2+\varepsilon}\, z \\ &\Rightarrow\; \exists y \in Y, \exists x \in U : x\, r_{\gamma_1+\varepsilon}\, y \,\&\, y\, s_{\gamma_2+\varepsilon}\, z \\ &\Leftrightarrow\; \exists y \in Y\, \exists x \in U : r(x,y) \leq \gamma_1 + \varepsilon \,\&\, s(y,z) \leq \gamma_2 + \varepsilon \\ &\Rightarrow\; \exists x \in U : (s \cdot r)(x,z) = \inf\big\{r(x,y) \vee s(y,z) \mid y \in Y\big\} \\ &\qquad\qquad\qquad\qquad\qquad \leq (\gamma_1 \vee \gamma_2) + \varepsilon \\ &\Rightarrow\; \exists x \in U : x\,(s \cdot r)_{(\gamma_1 \vee \gamma_2)+\varepsilon}\, z \\ &\Leftrightarrow\; z \in (s \cdot r)_{(\gamma_1 \vee \gamma_2)+\varepsilon}(U). \end{aligned}$$

This gives us $(s \cdot r)_{(\gamma_1 \vee \gamma_2)+\varepsilon}(U) \in \mathcal{W}$. By arbitrariness of $U \in \mathcal{U}$, we get

$$\inf\big\{\alpha \in \mathsf{P} \mid (s \cdot r)_\alpha(\mathcal{U}) \subseteq \mathcal{W}\big\} \leq (\gamma_1 \vee \gamma_2) + \varepsilon.$$

Hence, we conclude

$$\alpha_1 \leq (\gamma_1 \vee \gamma_2) + \varepsilon < (\alpha_2 + \varepsilon) + \varepsilon,$$

and by arbitrariness of ε, we get

$$\alpha_1 \leq \alpha_2.$$

Next we show that this is an associative lax extension. We do this using the characterization in Proposition 1.3.5.5.

First we show that $\overline{\beta} : \mathsf{P}_\vee\text{-}\mathsf{Rel} \longrightarrow \mathsf{P}_\vee\text{-}\mathsf{Rel}$ preserves composition. To this end, take two arbitrary $\mathsf{P}_\vee$-relations $r : X \relbar\joinrel\mapsto Y$ and $s : Y \relbar\joinrel\mapsto Z$. We already showed that $\overline{\beta}(s \cdot r) \leq \overline{\beta}s \cdot \overline{\beta}r$.

To prove the other inequality, take $\mathcal{U} \in \beta X, \mathcal{W} \in \beta Z$ and $u \in \mathsf{P}$ such that

$$\overline{\beta}(s \cdot r)(\mathcal{U}, \mathcal{W}) = \sup_{A \in \mathcal{U}, C \in \mathcal{W}} \inf_{x \in A, z \in C} s \cdot r(x,z) < u.$$

Hence, for every $A \in \mathcal{U}$ and $C \in \mathcal{W}$ there exist $x \in A$, $z \in C$ and $y \in Y$ such that

$$r(x,y) \vee s(y,z) < u.$$

Define

$$B_{A,C} := \big\{y \in Y \mid \exists x \in A, \exists z \in C : r(x,y) \vee s(y,z) < u\big\} \neq \emptyset.$$

Since $B_{A \cap A', C \cap C'} \subseteq B_{A,C} \cap B_{A',C'}$, the set

$$\{B_{A,C} \mid A \in \mathcal{U}, C \in \mathcal{W}\}$$

is a filterbase on Y. Let $\mathcal{V}$ be an ultrafilter on Y containing it.

Then

$$\overline{\beta}r(\mathcal{U},\mathcal{V}) = \sup_{A\in\mathcal{U},B\in\mathcal{V}} \inf_{x\in A,y\in B} r(x,y) \leq u,$$

because for $A \in \mathcal{U}$ and $B \in \mathcal{V}$ one has $B \cap B_{A,C} \neq \emptyset$. Similarly

$$\overline{\beta}s(\mathcal{V},\mathcal{W}) \leq u$$

and thus

$$\overline{\beta}s \cdot \overline{\beta}r(\mathcal{U},\mathcal{W}) = \inf\left\{\overline{\beta}r(\mathcal{U},\mathcal{V}) \vee \overline{\beta}(\mathcal{V},\mathcal{W}) \mid \mathcal{V} \in \beta Y\right\} \leq u.$$

Next we show that m° is natural. Since $\overline{\beta}$ commutes with the involution on $\mathsf{P}_\vee$-Rel, for $m^\circ : \overline{\beta} \longrightarrow \overline{\beta}\overline{\beta}$ to be natural, we only need to show the equality $m_Y \cdot \overline{\beta}\,\overline{\beta}r = \overline{\beta}r \cdot m_X$. One inequality already follows from the fact that $\overline{\beta}$ is a lax extension of β. To show that $m_Y \cdot \overline{\beta}\,\overline{\beta}r \leq \overline{\beta}r \cdot m_X$, let $\mathfrak{X} \in \beta\beta X$ and $\mathcal{V} \in \beta Y$ such that

$$\overline{\beta}r(m_X\mathfrak{X},\mathcal{V}) < u'' < u' < u.$$

Since the Barr extension of β to Rel is associative, there is some $\mathfrak{Y} \in \beta\beta Y$ with $\mathfrak{X}\,(\overline{\beta}\,\overline{\beta}r_{u''})\,\mathfrak{Y}$, so that one has $\mathfrak{X}\overline{\beta}(\overline{\beta}r)_{u'}\mathfrak{Y}$ and $\mathfrak{X}(\overline{\beta}\,\overline{\beta}r)_u\mathfrak{Y}$, i.e. $\overline{\beta}\,\overline{\beta}r(\mathfrak{X},\mathfrak{Y}) \leq u$.

To prove that $\overline{\beta}$ is a flat extension, it suffices to note that the lax extension of β to P_+-Rel is flat. Hence, since both extensions are defined analogously, we immediately get

$$\overline{\beta}(1_X) = \beta 1_X = 1_{\beta X}.$$

□

2.2.4 $(\beta, \mathsf{P}_\vee)$-Cat

Now $(\beta, \mathsf{P}_\vee)$-Cat is the category of lax algebras (X, a) for the $\mathsf{P}_\vee$-Rel-extension of the ultrafilter monad. Given a $(\beta, \mathsf{P}_\vee)$-category (X, a), then $a : \beta X \longrightarrow\!\!\!\!\!\!\!\mapsto X$ is a $\mathsf{P}_\vee$-relation which is reflexive, meaning

$$a(\dot{x}, x) = 0, \tag{2.7}$$

for all $x \in X$, and transitive, meaning

$$a\big(m_X(\mathfrak{X}), x\big) \leq \overline{\beta}a(\mathfrak{X},\mathcal{U}) \vee a(\mathcal{U}, x), \tag{2.8}$$

for all $\mathfrak{X} \in \beta\beta X, \mathcal{U} \in \beta X$ and $x \in X$. A $(\beta, \mathsf{P}_\vee)$-functor $f : (X, a) \longrightarrow (Y, b)$ is a map satisfying

$$b\big(f(\mathcal{U}), f(x)\big) \leq a(\mathcal{U}, x),$$

for all $\mathcal{U} \in \beta X$ and $x \in X$.

It is clear that the lax-homomorphism of quantales $\varphi : \mathsf{P}_\vee \longrightarrow \mathsf{P}_+$, as defined in 2.2.2, is compatible with the lax-extension of the ultrafilter monad β to $\mathsf{P}_\vee$-Rel and P_+-Rel. Hence, this induces another change-of-base functor

$$C_\varphi : (\beta, \mathsf{P}_\vee)\text{-}\mathsf{Cat} \longrightarrow (\beta, \mathsf{P}_+)\text{-}\mathsf{Cat}.$$

Again, this change-of-base functor is an embedding. In 1.3.7, we already explained that the category (β, P_+)-Cat is isomorphic to App.

Theorem 2.2.4.1. *The category* $(\beta, \mathsf{P}_\vee)$-Cat *of lax algebras for the* $\mathsf{P}_\vee$-Rel-*extension of the ultrafilter monad is isomorphic to* NA-App.

Proof. This proof is similar to the proof of Theorem 2.2.4.1, for which we refer to Theorem 12.7.2 in [Low15].

Reflexivity clearly is equivalent to (L1). So all that remains to be shown is that transitivity is equivalent to (L$\beta^*_\vee$).

Let λ be a non-Archimedean limit operator on X and let $\mathfrak{X} \in \beta\beta X$ and $\mathcal{U} \in \beta X$. Put

$$\varepsilon := \overline{\beta}a(\mathfrak{X}, \mathcal{U}) = \sup_{\mathcal{A} \in \mathfrak{X}} \sup_{U \in \mathcal{U}} \inf_{\mathcal{W} \in \mathcal{A}} \inf_{x \in U} a(\mathcal{W}, x).$$

Let $\rho > 0$ and put

$$J := \big\{(\mathcal{G}, y) \in \beta X \times X \mid \lambda\mathcal{G}(y) \leq \varepsilon + \rho\big\},$$

and consider the projections

$$\begin{array}{ccc} J & \xrightarrow{\psi := \mathrm{pr}_2} & X \\ {\scriptstyle \sigma := \mathrm{pr}_1}\big\downarrow & & \\ \beta X & & \end{array}$$

Note that by definition of ε and ρ, the filter $\mathfrak{X} \times \mathcal{U}$ has a trace on J and consequently we can choose an ultrafilter $\mathcal{R} \in \beta J$ finer than $\mathfrak{X} \times \mathcal{U}$. It then follows that

$$\mathfrak{X} = \mathrm{pr}_1(\mathcal{R}) = \sigma(\mathcal{R}) \text{ and } \mathcal{U} = \mathrm{pr}_2(\mathcal{R}) = \psi(\mathcal{R}),$$

and because (L$\beta^*_\vee$) holds we obtain, for any $x \in X$

$$\lambda\Sigma\sigma(\mathcal{R})(x) \leq \lambda\psi(\mathcal{R})(x) \vee \sup_{R \in \mathcal{R}} \inf_{z \in R} \lambda\sigma(z)\big(\psi(z)\big)$$

and thus

$$\begin{aligned} \lambda m_X(\mathfrak{X})(x) &\leq \lambda\mathcal{U}(x) \vee \sup_{R \in \mathcal{R}, R \subseteq J} \inf_{(\mathcal{G}, y) \in R} \lambda\mathcal{G}(y) \\ &\leq \lambda\mathcal{U}(x) + \varepsilon + \rho. \end{aligned}$$

Consequently, by arbitrariness of ρ and the definition of ε it follows that λ satisfies the transitivity axiom.

Conversely let $a : \beta X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ satisfy the transitivity axiom and let J be a set, $\psi : J \longrightarrow X, \sigma : J \longrightarrow \beta X$ and $\mathcal{F} \in \beta J$. Put

$$\mathfrak{X} := \sigma(\mathcal{F}) \text{ and } \mathcal{U} := \psi(\mathcal{F}).$$

Then it follows that, for any $x \in X$

$$\begin{aligned} a\Big(m_X\big(\sigma(\mathcal{F})\big), x\Big) &\leq a\big(\psi(\mathcal{F}), x\big) \vee \sup_{\mathcal{A}\in\sigma(\mathcal{F})} \sup_{U\in\psi(\mathcal{F})} \inf_{\mathcal{V}\in\mathcal{A}} \inf_{y\in U} a(\mathcal{V}, y) \\ &\leq a\big(\psi(\mathcal{F}), x\big) \vee \sup_{F\in\mathcal{F}} \inf_{\mathcal{V}\in\sigma(F)} \inf_{y\in\psi(F)} a(\mathcal{V}, y) \\ &\leq a\big(\psi(\mathcal{F}), x\big) \vee \sup_{F\in\mathcal{F}} \inf_{z\in F} a\big(\sigma(z), \psi(z)\big) \\ &= a\big(\psi(\mathcal{F}), x)\big) \vee \inf_{F\in\mathcal{F}} \sup_{z\in F} a\big(\sigma(z), \psi(z)\big), \end{aligned}$$

where we used Lemma 1.1.1.4 to interchange inf and sup, since $\mathcal{F} \in \beta J$. This shows that a satisfies (L$\beta^*_\vee$).

That via identification of lax algebraic structures on the one hand with non-Archimedean limit operators on the other hand, the morphisms in both categories coincide is an immediate consequence of the characterization of contractions via ultrafilters and the definitions of the morphisms in $(\beta, \mathsf{P}_\vee)$-Cat. □

2.3 NA-App related to more familiar categories

2.3.1 The embedding $q\mathsf{Met}^u \hookrightarrow$ NA-App

Restricting the coreflector App⟶qMet from Section 1.1.4(B) to NA-App, a non-Archimedean approach space X with limit operator λ (distance δ) is sent to its underlying quasi-ultrametric space (X, d_λ) ((X, d_δ)) given by

$$d_\lambda(x, y) = \lambda(\dot{y})(x) = \delta(x, \{y\}) = d_\delta(x, y) \tag{2.9}$$

for x, y in X. Moreover restricting the embedding qMet $\hookrightarrow$ App from Section 1.1.4(B) to $q\mathsf{Met}^u$, a quasi-ultrametric space (X, d) is mapped to a non-Archimedean approach space X with limit operator defined by

$$\lambda_d(\mathcal{U})(x) = \sup_{U\in\mathcal{U}} \inf_{u\in U} d(x, u), \tag{2.10}$$

for all $\mathcal{U} \in \beta X$ and $x \in X$ and distance

$$\delta_d(x, A) = \inf_{a\in A} d(x, a) \tag{2.11}$$

for all $A \subseteq X$ and $x \in X$. We can conclude that $q\mathsf{Met}^u$ is concretely coreflectively embedded in NA-App and the coreflector is the restriction of the well known coreflector App $\longrightarrow q$Met.

Considering the lax extension $\overline{\beta}$ to $\mathsf{P}_\vee$-Rel, the lax extension of the identity monad $\mathbb{1}$ to $\mathsf{P}_\vee$-Rel and the associated morphism $(1, \overline{1}) \longrightarrow (\beta, \overline{\beta})$ of lax extensions, the induced algebraic functor

$$(\beta, \mathsf{P}_\vee)\text{-Cat} \longrightarrow \mathsf{P}_\vee\text{-Cat}, \tag{2.12}$$

sends a $(\beta, \mathsf{P}_\vee)$-algebra (X, a) to its underlying $\mathsf{P}_\vee$-algebra $(X, a \cdot e_X)$, as introduced in section III.3.4 of [HST14]. Using the isomorphisms described in 2.2.2 and in Theorem 2.2.4.1, this functor sends a $(\beta, \mathsf{P}_\vee)$-algebra (X, a), corresponding to a non-Archimedean approach space X with limit operator λ, to its underlying quasi-ultrametric space (X, d^-_λ). This functor has a left adjoint $\mathsf{P}_\vee$-Cat $\hookrightarrow (\beta, \mathsf{P}_\vee)$-Cat, which associates with a $\mathsf{P}_\vee$-algebra (X, d), the $(\beta, \mathsf{P}_\vee)$-algebra corresponding to (X, λ_{d^-}).

2.3.2 The embedding Top $\hookrightarrow$ NA-App

We consider the lax homomorphism $\iota : 2 \longrightarrow \mathsf{P}_\vee$, sending $\top$ to 0 and $\bot$ to ∞, which is compatible with the lax extensions of the ultrafilter monad to Rel and $\mathsf{P}_\vee$-Rel. Analogous to the situation for P_+ [HST14] the change-of-base functor associated with the lax homomorphism ι constitutes an embedding $(\beta, 2)$-Cat $\hookrightarrow (\beta, \mathsf{P}_\vee)$-Cat. Using the isomorphisms described in 2.2.2, Theorem 2.2.4.1, and the well known isomorphism $(\beta, 2)$-Cat $\cong$ Top (see Section 1.3.6) this gives an embedding of Top in NA-App.
In terms of the limit operator or the distance the embedding Top $\hookrightarrow$ NA-App associates the limit operator $\lambda_\mathcal{T}$ (distance $\delta_\mathcal{T}$) with a topological space $(X, \mathcal{T})$ by $\lambda_\mathcal{T}\mathcal{U}(x) = 0$ if $\mathcal{U}$ converges to x in $(X, \mathcal{T})$ ($\delta_\mathcal{T}(x, A) = 0$ if $x \in \overline{A}$) and with values ∞ in all other cases, for $\mathcal{U} \in \beta X, A \subseteq X, x \in X$. Later on we will also make use of the embedding of Top described in terms of the tower. All levels of the approach tower $(X, (\mathsf{t}_\varepsilon)_{\varepsilon \geq 0})$ associated with a topological space $(X, \mathcal{T})$ coincide, so we have $\mathcal{T}_\varepsilon = \mathcal{T}$ for all $\varepsilon \geq 0$. These formulations of the embedding Top $\hookrightarrow$ NA-App are the codomain restrictions of the embedding of Top in App as described in Section 1.1.4(A).

The map ι has a right adjoint $p : \mathsf{P}_\vee \longrightarrow 2$, where $p(0) = \top$ and $p(v) = \bot$ otherwise, that is again a lax homomorphism of quantales. The map p is also compatible with the lax extensions of the ultrafilter monad β to Rel and $\mathsf{P}_\vee$-Rel and provides the embedding with a right adjoint $(\beta, \mathsf{P}_\vee)$-Cat $\longrightarrow (\beta, 2)$-Cat. This functor can also be obtained by restricting the coreflector $\mathrm{C}_{\mathsf{Top}}$: App $\longrightarrow$ Top, as described in Section 1.1.4(A) to NA-App and we will continue to use the notation $\mathrm{C}_{\mathsf{Top}}$. This coreflector sends a non-Archimedean approach space X to a topological space $\mathrm{C}_{\mathsf{Top}}(X)$ in which an ultrafilter $\mathcal{U}$ converges to a point x precisely when $\lambda\mathcal{U}(x) = 0$ or in which a point x is in the closure of a set A precisely when $\delta(x, A) = 0$. In terms of the tower $(\mathcal{T}_\varepsilon)_{\varepsilon \geq 0}$ the topological space $\mathrm{C}_{\mathsf{Top}}(X)$ is precisely $(X, \mathcal{T}_0)$.

Now define the map $o : \mathsf{P}_\vee \longrightarrow 2$ by $o(v) = \top$ if and only if $v < \infty$. Analogous to the situation of P_+ in [HST14] the map o is a lax homomorphism of quantales, however it is not compatible with the lax extensions of the ultrafilter monad. Nevertheless, given a $(\beta, \mathsf{P}_\vee)$-algebra (X, a), one can still consider the pair (X, oa), where $oa : \beta X \rightharpoonup X$ is defined by $\mathcal{U}\, oa\, x$ precisely when $a(\mathcal{U}, x) < \infty$. This structure satisfies the reflexivity but not the transitivity condition. In other words, (X, oa) is a pseudotopological space. Now we can apply the left adjoint of the full reflective embedding $\mathsf{Top} \hookrightarrow \mathsf{PsTop}$ to (X, oa) to obtain a topological space and thereby a left adjoint $(\beta, \mathsf{P}_\vee)$-$\mathsf{Cat} \longrightarrow (\beta, 2)$-$\mathsf{Cat}$ to the embedding $(\beta, 2)$-$\mathsf{Cat} \hookrightarrow (\beta, \mathsf{P}_\vee)$-$\mathsf{Cat}$. This functor can also be obtained by restricting the reflector $\mathsf{App} \longrightarrow \mathsf{Top}$, as described in Section 1.1.4(A) to NA-App, where the Top-reflection of a non-Archimedean approach space (X, δ) is determined by the non-Archimedean distance associated with the topological reflection of the pretopological closure operator cl, defined by $\mathrm{cl}(A) := \{x \in A \mid \delta(x, A) < \infty\}$.

2.3.3 The embedding NA-App $\hookrightarrow$ App

Based on the characterization of non-Archimedean approach spaces in terms of non-Archimedean gauges, we can also say something more about the embedding of NA-App in App corresponding to the change of base functor

$$C_\varphi : (\beta, \mathsf{P}_\vee)\text{-}\mathsf{Cat} \longrightarrow (\beta, \mathsf{P}_+)\text{-}\mathsf{Cat}.$$

Theorem 2.3.3.1. NA-App *is a concretely reflective subcategory of* App. *If X is an approach space with gauge $\mathcal{G}$, then its* NA-App*-reflection $1_X : X \longrightarrow X^u$ is given by the approach space X^u having $\mathcal{G} \cap q\mathsf{Met}^u(X)$ as basis for its gauge.*

Proof. Since $\mathcal{G} \cap q\mathsf{Met}^u(X)$ is stable under finite suprema, it is locally directed and therefore $\widehat{\mathcal{G}} \cap q\mathsf{Met}^u(X)$ defines a gauge. Let X^u be the associated approach space. Suppose $f : X \longrightarrow Y$ is a contraction with $Y \in$ NA-App with a gauge basis $\mathcal{H}$ consisting of quasi-ultrametrics. Then $d \in \mathcal{H}$ clearly implies $d \cdot f \times f \in \mathcal{G} \cap q\mathsf{Met}^u(X)$. By the characterization of a contraction in terms of a gauge basis, we have that $f : X^u \longrightarrow Y$ is contractive. □

The results in this section can be summarized in the following diagram.

$$\begin{array}{ccccc}
\mathsf{Top} \cong (\beta, 2)\text{-}\mathsf{Cat} & \hookrightarrow & \mathsf{NA\text{-}App} \cong (\beta, \mathsf{P}_\vee)\text{-}\mathsf{Cat} & \hookrightarrow & \mathsf{App} \cong (\beta, \mathsf{P}_+)\text{-}\mathsf{Cat} \\
\uparrow & & \uparrow & & \uparrow \\
\mathsf{Ord} \cong 2\text{-}\mathsf{Cat} & \hookrightarrow & q\mathsf{Met}^u \cong \mathsf{P}_\vee\text{-}\mathsf{Cat} & \hookrightarrow & q\mathsf{Met} \cong \mathsf{P}_+\text{-}\mathsf{Cat}
\end{array}$$

2.4 Coproducts and quotients in NA-App

In Theorem 2.3.3.1, we showed that NA-App is embedded as a concretely reflective subconstruct of App. This shows that NA-App is closed under formation of limits and initial structures in App. In particular, a product in App of a family of non-Archimedean approach spaces is a non-Archimedean approach space and, likewise, a subspace in App of a non-Archimedean approach space, is a non-Archimedean approach space.

In this section, we will give a construction for coproducts and quotients of non-Archimedean approach spaces. In order to create coproducts and quotients in NA-App, we recall the constructions of coproducts and quotients in $q\mathsf{Met}^u$ [Lem84].

2.4.1 Coproducts of non-Archimedean approach spaces

We recall the construction of coproducts in $q\mathsf{Met}^u$[Lem84].

Proposition 2.4.1.1. Let $\big((X_i, d_i)\big)_{i\in I}$ be a family of quasi-ultrametric spaces. The coproduct in $q\mathsf{Met}^u$, $\coprod_{i\in I} X_i = \bigcup_{i\in I} \big(X_i \times \{i\}\big)$, is structured by the following quasi-ultrametric:

$$\coprod_{i\in I} d_i\big((x,j),(y,k)\big) := \begin{cases} d_j(x,y) & j = k, \\ \infty & j \neq k. \end{cases}$$

Given this result, we can construct coproducts of non-Archimedean approach spaces.

Theorem 2.4.1.2. *Let* $\big((X_i, \mathcal{G}_i)\big)_{i\in I}$ *be a family of non-Archimedean approach spaces with* $\mathcal{H}_i$ *a gauge basis for* $\mathcal{G}_i$*, for all* $i \in I$*, consisting of quasi-ultrametrics. The coproduct in* NA-App*,* $\coprod_{i\in I} X_i = \bigcup_{i\in I} \big(X_i \times \{i\}\big)$*, is structured by the following non-Archimedean gauge basis:*

$$\coprod_{i\in I} \mathcal{H}_i := \Big\{ \coprod_{i\in I} d_i \mid \forall i \in I : d_i \in \mathcal{H}_i \Big\} \downarrow,$$

where $\coprod_{i\in I} d_i$ *is defined as in Proposition 2.4.1.1.*

Proof. It is clear that $\coprod_{i\in I}\mathcal{H}_i$ is a basis for a non-Archimedean gauge on $\coprod_{i\in I}X_i$.

Next we show that the canonical injections $\mathrm{in}_k : (X_k,\mathcal{H}_k)\longrightarrow(\coprod_{i\in I}X_i,\coprod_{i\in I}\mathcal{H}_i)$ are contractions, for all $k\in I$. Take a member $\coprod_{i\in I}d_i$ in the gauge basis $\coprod\mathcal{H}_i$. For $x,y\in X_k$ arbitrary, we have

$$\coprod_{i\in I}d_i\big(\mathrm{in}_k(x),\mathrm{in}_k(y)\big)=\coprod_{i\in I}d_i\big((x,k),(y,k)\big)=d_k(x,y),$$

hence $\coprod_{i\in I}d_i\cdot\mathrm{in}_k\times\mathrm{in}_k\in\mathcal{H}_k$.

Finally, we show that the sink

$$\big(\mathrm{in}_k : (X_k,d_k)\longrightarrow(\coprod_{i\in I}X_i,\coprod_{i\in I}\mathcal{H}_i)\big)_{k\in I}$$

is final. Therefore, consider a non-Archimedean approach space Y with gauge basis $\mathcal{H}_Y$ consisting of quasi-ultrametrics and a map

$$f:(\coprod_{i\in I}X_i,\coprod_{i\in I}\mathcal{H}_i)\longrightarrow(Y,\mathcal{H}_Y).$$

If f is a contraction, then it is clear that all maps $f\cdot\mathrm{in}_k$ are contractions, for all $k\in I$. To prove the converse, take $e\in\mathcal{H}_Y$ arbitrary. Take $(x,j),(y,k)\in\coprod_{i\in I}X_i$ arbitrary. If $j=k$, then

$$e\big(f(x,j),f(y,k)\big)=e\big(f(\mathrm{in}_j(x)),f(\mathrm{in}_j(y))\big),$$

and since $f\cdot\mathrm{in}_j$ is a contraction, there exists $\coprod_{i\in I}d_i\in\coprod_{i\in I}\mathcal{H}_i$ such that $e\cdot(f\cdot\mathrm{in}_j\times f\cdot\mathrm{in}_j)\leq\coprod_{i\in I}d_i$. If $j\neq k$, then $\coprod_{i\in I}d_i\big((x,j),(y,k)\big)=\infty$, for all $\coprod_{i\in I}d_i\in\coprod_{i\in I}\mathcal{H}_i$. Hence, we can conclude that for all $e\in\mathcal{H}_Y$, there exists $\coprod_{i\in I}d_i\in\mathcal{H}_i$ such that $e\cdot f\times f\leq\coprod_{i\in I}d_i$ and thus

$$e\in\Big\{\coprod_{i\in I}d_i\mid\forall i\in I: d_i\in\mathcal{H}_i\Big\}\downarrow.$$

□

It follows immediately from the foregoing construction and proof that NA-App is closed under coproducts in App.

2.4.2 Quotients of non-Archimedean approach spaces

We recall the construction of quotients in $q\mathsf{Met}^u$ [Lem84].

Proposition 2.4.2.1. Let (X,d) be a quasi-ultrametric space. A surjective map $f : X \longrightarrow Y$ is a quotient in $q\mathsf{Met}^u$ if Y is endowed with the largest quasi-ultrametric u_{g_d} such that $u_{g_d} \leq g_d$, where

$$g_d : Y \times Y \longrightarrow \mathsf{P} : (y,y') \mapsto \inf_{x \in f^{-1}(y), x' \in f^{-1}(y')} d(x,x').$$

Theorem 2.4.2.2. *Let $(X, \mathcal{G}_X)$ be a non-Archimedean approach space with gauge basis $\mathcal{H}_X$ consisting of quasi-ultrametrics. A surjective map $f : X \longrightarrow Y$ is a quotient in* NA-App *if Y is endowed with the gauge basis*

$$\mathcal{H}_Y = \{u_{g_d} \mid d \in \mathcal{H}_X\} \downarrow,$$

where $u_{g_d}$is defined as in Proposition 2.4.2.1.

Proof. First, we argue that $f : (X, \mathcal{H}_X) \longrightarrow (Y, \mathcal{H}_Y)$ is a contraction. Take $d \in \mathcal{H}_X$ arbitrary. For $x, x' \in X$ arbitrary, we have

$$u_{g_d}\big(f(x), f(x')\big) \leq g_d\big(f(x), f(x')\big) \leq d(x,x'),$$

hence $u_{g_d} \cdot f \times f \in \mathcal{H}_X$, for every $d \in \mathcal{H}_X$.

To prove that $f : (X, \mathcal{H}_X) \longrightarrow (Y, \mathcal{H}_Y)$ is final, consider a non-Archimedean approach space $(Z, \mathcal{H}_Z)$ and a map $h : (Y, \mathcal{H}_Y) \longrightarrow (Z, \mathcal{H}_Z)$. It is clear that if h is a contraction, $h \cdot f$ is a contraction as well. To prove the converse, suppose $h \cdot f$ is a contraction and take $e \in \mathcal{H}_Z$ arbitrary. Then $e \cdot \big((h \cdot f) \times (h \cdot f)\big) \in \mathcal{G}_X$ and thus for all $x \in X, \varepsilon > 0$ and $\omega < \infty$, there exists $d_x^{\varepsilon,\omega} \in \mathcal{H}_X$ such that

$$e\Big(h\big(f(x)\big), h\big(f(\cdot)\big)\Big) \wedge \omega \leq d_x^{\varepsilon,\omega}(x, \cdot) + \varepsilon.$$

Now take $y \in Y, \varepsilon > 0$ and $\omega < \infty$ arbitrary. Then, by surjectivity of f,

$$e\big(h(y), h(\cdot)\big) \wedge \omega \leq d_x^{\varepsilon,\omega}(x, \cdot) + \varepsilon,$$

for every $x \in f^{-1}(y)$. Set $d^{\varepsilon,\omega} = \inf_{x \in f^{-1}(y)} d_x^{\varepsilon,\omega} \in \mathcal{H}_X$. Then

$$e\big(h(y), h(\cdot)\big) \wedge \omega \leq d^{\varepsilon,\omega}(x, \cdot) + \varepsilon.$$

Since this holds for any $x \in f^{-1}(y)$, we get

$$e\big(h(y), h(\cdot)\big) \wedge \omega \leq \inf_{x \in f^{-1}(y)} d^{\varepsilon,\omega}(x, \cdot) + \varepsilon.$$

Hence

$$e\big(h(y), h(\cdot)\big) \wedge \omega \leq g_{d^{\varepsilon,\omega}}(y, \cdot) + \varepsilon,$$

and by definition of $u_{g_{d^{\varepsilon,\omega}}}$ we find

$$e\big(h(y), h(\cdot)\big) \wedge \omega \leq u_{g_{d^{\varepsilon,\omega}}}(y, \cdot) + \varepsilon.$$

Since $u_{g_{d^{\varepsilon,\omega}}} \in \mathcal{H}_Y$, this shows $e \cdot (h \times h) \in \mathcal{G}_Y$. □

2.5 Initially dense objects in NA-App

At this point, we can introduce two new examples of non-Archimedean approach spaces.

Example 2.5.1.3. Consider the quasi-ultrametrics $d_{\mathsf{P}_\vee}$ and $d^-_{\mathsf{P}_\vee}$ on $\mathsf{P}_\vee$ defined by

$$d_{\mathsf{P}_\vee} : \mathsf{P}_\vee \times \mathsf{P}_\vee \longrightarrow \mathsf{P}_\vee : (x, y) \mapsto x \multimap y,$$

and

$$d^-_{\mathsf{P}_\vee} : \mathsf{P}_\vee \times \mathsf{P}_\vee \longrightarrow \mathsf{P}_\vee : (x, y) \mapsto y \multimap x.$$

In this section we show that each of the non-Archimedean approach spaces $(\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})$ of Example 2.1.2.3 and $(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ and $(\mathsf{P}_\vee, d^-_{\mathsf{P}_\vee})$ as introduced above, are initially dense objects in NA-App.

Given a source $\big(f_i : X \longrightarrow (X_i, \lambda_i)\big)_{i \in I}$, the formulas for the initial approach structure on X can be found in Theorem 1.1.3.2.

Proposition 2.5.1.4. For any non-Archimedean approach space X with distance δ and for $A \subseteq X$, the distance functional

$$\delta_A : (X, \delta) \longrightarrow (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee}) : x \mapsto \delta(x, A)$$

is a contraction.

Proof. Let $x \in X$ and $B \subseteq X$. By application of Proposition 2.1.2.2 for $B \neq \emptyset$ and $A \neq \emptyset$, we have

$$\begin{aligned} \delta_{\mathsf{P}_\vee}\big(\delta_A(x), \delta_A(B)\big) &= \big(\sup_{b \in B} \delta(b, A)\big) \multimap \delta(x, A) \\ &\leq \big(\sup_{b \in B} \delta(b, A)\big) \multimap \Big(\delta(x, B) \vee \sup_{b \in B} \delta(b, A)\Big) \\ &\leq \delta(x, B). \end{aligned}$$

□

Theorem 2.5.1.5. $(\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})$ *is initially dense in* NA-App. *More precisely, for any non-Archimedean approach space X, the source*

$$\big(\delta_A : X \longrightarrow (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})\big)_{A \in 2^X}$$

is initial.

Proof. If λ_{in} stands for the initial limit operator on X, then we already know by Proposition 2.5.1.4 that $\lambda_{in} \leq \lambda$. Conversely, take $\mathcal{U} \in \beta X$ and $x \in X$. Then

$$\begin{aligned}
\lambda_{in}\mathcal{U}(x) &= \sup_{A \in 2^X} \lambda_{\mathsf{P}_\vee}\big(\delta_A(\mathcal{U})\big)\big(\delta_A(x)\big) \\
&= \sup_{A \in 2^X} \sup_{U \in \mathcal{U}} \delta_{\mathsf{P}_\vee}\big(\delta_A(x), \delta_A(U)\big) \\
&\geq \sup_{U \in \mathcal{U}} \delta_{\mathsf{P}_\vee}\big(\delta_U(x), \delta_U(U)\big) \\
&= \sup_{U \in \mathcal{U}} \delta_{\mathsf{P}_\vee}\big(\delta_U(x), \{0\}\big) \\
&= \sup_{U \in \mathcal{U}} \delta(x, U) \\
&= \lambda\mathcal{U}(x).
\end{aligned}$$

□

Theorem 2.5.1.6. *Both* $(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ *and* $(\mathsf{P}_\vee, d^-_{\mathsf{P}_\vee})$ *are initially dense objects in* NA-App.

Proof. Since we already know that $(\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})$ is initially dense in NA-App, it suffices to show that we can obtain it via an initial lift of sources of either of the two objects above.

First of all, consider the following source

$$\big(g_\alpha : (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee}) \longrightarrow (\mathsf{P}_\vee, \delta_{d_{\mathsf{P}_\vee}})\big)_{\alpha \in \mathbb{R}^+},$$

where g_α is defined as follows:

$$g_\alpha : (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee}) \longrightarrow (\mathsf{P}_\vee, \delta_{d_{\mathsf{P}_\vee}}) : x \mapsto \begin{cases} 0 & x > \alpha, \\ \alpha & x \leq \alpha. \end{cases}$$

To show that g_α is a contraction, for any $\alpha \in \mathbb{R}^+$, take $x \in \mathsf{P}_\vee$ and $B \subseteq \mathsf{P}_\vee$ arbitrary. For $B \neq \emptyset$, we consider two cases. First suppose $x \leq \alpha$, then

$$\delta_{d_{\mathsf{P}_\vee}}\big(g_\alpha(x), g_\alpha(B)\big) = \inf_{b \in B} g_\alpha(x) {-\!\bullet}\, g_\alpha(b) = \inf_{b \in B} \alpha {-\!\bullet}\, g_\alpha(b) = 0.$$

In case $x > \alpha$, we have

$$\delta_{d_{\mathsf{P}_\vee}}\big(g_\alpha(x), g_\alpha(B)\big) = \inf_{b \in B} g_\alpha(x) {-\!\bullet}\, g_\alpha(b) = \inf_{b \in B} 0 {-\!\bullet}\, g_\alpha(b).$$

If there exists $b \in B$, such that $b > \alpha$, then $\delta_{d_{\mathsf{P}_\vee}}\big(g_\alpha(x), g_\alpha(B)\big) = 0 \leq \delta_{\mathsf{P}_\vee}(x, B)$. If for all $b \in B$ we have that $b \leq \alpha$, then $\sup B \leq \alpha < x$, and thus

$$\delta_{d_{\mathsf{P}_\vee}}\big(g_\alpha(x), g_\alpha(B)\big) = \alpha < x = \delta_{\mathsf{P}_\vee}(x, B).$$

It remains to show that the source $(g_\alpha)_{\alpha\in\mathbb{R}^+}$ is initial. Let λ_{in} stand for the initial limit operator on $\mathsf{P}_\vee$, then we know that $\lambda_{in} \leq \lambda_{\mathsf{P}_\vee}$. To prove the other inequality, take $\mathcal{U} \in \beta\mathsf{P}_\vee$ and $x \in \mathsf{P}_\vee$ arbitrary. If $\lambda_{\mathsf{P}_\vee}\mathcal{U}(x) = 0$, then the inequality is clear. In case $\lambda_{\mathsf{P}_\vee}\mathcal{U}(x) = x$, there exists $A \in \mathcal{U}$ such that $x > \sup A$. Take α arbitrary in the interval $[\sup A, x[$. Then

$$\begin{aligned}
\lambda_{d_{\mathsf{P}_\vee}} g_\alpha(\mathcal{U})\big(g_\alpha(x)\big) &= \sup_{U\in\mathcal{U}} \inf_{y\in U} d_{\mathsf{P}_\vee}\big(g_\alpha(x), g_\alpha(y)\big) \\
&\geq \inf_{y\in A} d_{\mathsf{P}_\vee}\big(g_\alpha(x), g_\alpha(y)\big) \\
&= 0 \multimap \alpha \\
&= \alpha.
\end{aligned}$$

Hence

$$\lambda_{in}\mathcal{U}(x) = \sup_{\alpha\in\mathbb{R}^+} \lambda_{d_{\mathsf{P}_\vee}} g_\alpha(\mathcal{U})\big(g_\alpha(x)\big) \geq \sup_{\alpha\in[\sup A, x[} \alpha = x = \lambda_{\mathsf{P}_\vee}\mathcal{U}(x).$$

So we can conclude that the source is initial.

In a similar way, we can prove that the source

$$\big(f_\alpha : (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee}) \longrightarrow (\mathsf{P}_\vee, \delta_{d^-_{\mathsf{P}_\vee}})\big)\alpha_{\in\mathbb{R}^+},$$

with

$$f_\alpha : (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee}) \longrightarrow (\mathsf{P}_\vee, \delta_{d^-_{\mathsf{P}_\vee}} : x \mapsto \begin{cases} \alpha & x > \alpha, \\ 0 & x \leq \alpha, \end{cases}$$

is initial as well. □

Chapter 3

Approach spaces as relational algebras

In Section 1.3.7 we recalled how to describe approach spaces as lax algebras for the ultrafilter monad, as developed by Clementino and Hofmann [CH03]. Here we focus on presentations of App only using the quantale 2, i.e. presentations of App as relational $\mathbb{T}$-algebras for a suitable monad $\mathbb{T}$.

By Theorem 1.4.3.3, we know that in order to find $\mathsf{App} \cong (\mathbb{T}, 2)\text{-}\mathsf{Cat}$ for some monad $\mathbb{T}$, it suffices to describe approach spaces as Kleisli monoids for a power-enriched monad $\mathbb{T}$,

$$\mathsf{App} \cong \mathbb{T}\text{-}\mathsf{Mon}.$$

In Section 3.1 we introduce the monad $\mathbb{I} = (\mathsf{I}, m, e)$ on Set which we call the functional ideal monad. We prove that $\mathbb{I}$ is power-enriched which leads us to the category $\mathbb{I}$-Mon of all $\mathbb{I}$-monoids with structure preserving maps. We show that this category is isomorphic to App. Through the concrete isomorphism an $\mathbb{I}$-monoid (X, ν) corresponds to an approach space $\big(X, (\mathcal{A}_b(x))_{x \in X}\big)$ described in terms of its bounded local approach system.

We extend I to Rel using the Kleisli extension $\check{\mathsf{I}}$ and since $\mathbb{I}$ is a power-enriched monad, by Theorem 1.4.3.3 we conclude that $\mathbb{I}$-Mon and $(\mathbb{I}, 2)$-Cat are isomorphic. We obtain the result that App can be isomorphically described in terms of convergence of functional ideals using two axioms, transitivity and reflexivity. We compare these axioms to the ones on convergence of functional ideals put forward in [Low15].

In Section 3.2 we focus on prime functional ideals and introduce the prime functional ideal monad $\mathbb{B}$. The case of the prime functional ideal monad is completely different, since it is not power-enriched. We show that it is a submonad of the functional ideal monad $\mathbb{I}$ and that it is both sup-dense and interpolating in $\mathbb{I}$. From general results in [HST14] we get that $(\mathbb{I}, 2)$-Cat and $(\mathbb{B}, 2)$-Cat are isomorphic. We recover the result from [LV08] that App is isomorphic to $(\mathbb{B}, 2)$-Cat. To the known axioms describing App in terms of functional ideal convergence, we add some equivalent new characterizations in terms of prime functional ideal convergence.

3.1 Functional ideal convergence

3.1.1 Functional ideals

In this section we add some new results to the theory of functional ideals, as introduced in Section 1.1.1.

The following proposition provides us with an easy characterization of certain types of proper functional ideals.

Proposition 3.1.1.1. For a proper functional ideal $\mathfrak{I}$ on X the following are equivalent:

(i) There exists a filter $\mathcal{G}$ on X such that $\mathfrak{I} = \iota_X(\mathcal{G}) \oplus c(\mathfrak{I})$.

(ii) For all α, γ with $c(\mathfrak{I}) \leq \alpha < \infty$, $c(\mathfrak{I}) \leq \gamma < \infty : \mathfrak{f}_\alpha(\mathfrak{I}) = \mathfrak{f}_\gamma(\mathfrak{I})$.

Proof. The proof of (i) $\Rightarrow$ (ii) is straightforward.

In order to prove (ii) $\Rightarrow$ (i) let $\mathcal{G} = \mathfrak{f}_\alpha(\mathfrak{I})$ for all $c(\mathfrak{I}) \leq \alpha < \infty$. We show that $\mathfrak{I} = \iota_X(\mathcal{G}) \oplus c(\mathfrak{I})$. For $\mu \in \mathfrak{I}$ put $\omega = \sup \mu$. For ε arbitrary and β such that $c(\mathfrak{I}) < \beta < c(\mathfrak{I}) + \varepsilon$ we have $\mu \leq \theta^\omega_{\{\mu<\beta\}} + c(\mathfrak{I}) + \varepsilon$.

For the other inclusion let $G \in \mathcal{G}$ and $\omega < \infty$. Choose $\omega + c(\mathfrak{I}) < \alpha$ and then $\alpha < \beta$ and $\mu \in \mathfrak{I}$ such that $\{\mu < \beta\} \subseteq G$. Then we have $\theta^\omega_G + c(\mathfrak{I}) \leq \mu \vee c(\mathfrak{I})$ and hence $\theta^\omega_G + c(\mathfrak{I}) \in \mathfrak{I}$. □

For $\Phi \in \mathsf{I}\mathsf{I} X$, a functional ideal on $\mathsf{I} X$, the diagonal operation in formula (1.26), using $J = X$ and $s = \mathrm{id}$, becomes

$$m_X(\Phi) = \{\mu \in \mathsf{P}^X_b \mid l_\mu \in \Phi\}.$$

Let $\mathfrak{I}$ be a functional ideal on X, we define the functional ideal $m^*_X(\mathfrak{I})$ on $\mathsf{I} X$ generated by

$$\{l_\mu \mid \mu \in \mathfrak{I}\}.$$

Proposition 3.1.1.2. m_X and m^*_X fulfill the following properties:

1. $m_X : \mathsf{I}\mathsf{I} X \longrightarrow \mathsf{I} X$ and $m^*_X : \mathsf{I} X \longrightarrow \mathsf{I}\mathsf{I} X$ are both well defined and monotone.

2. $m_X \cdot m^*_X = 1_{\mathsf{I} X}$ and $m^*_X \cdot m_X \leq 1_{\mathsf{I} X}$.

3. m_X is right adjoint and therefore preserves intersections in the sense that for any family $(\Phi_i)_{i\in I}$ of functional ideals on $\mathsf{I}X$ we have
$$m_X\Big(\bigcap_{i\in I}\Phi_i\Big) = \bigcap_{i\in I} m_X(\Phi_i).$$

Proof. We check the inequality in 2. $m_X(m_X^*(\mathfrak{I})) \subseteq \mathfrak{I}$ for a functional ideal $\mathfrak{I} \in \mathsf{I}X$, the rest follows easily from the definitions. Take $\nu \in \mathsf{P}_b^X$ then $l_\nu \leq l_\mu + \varepsilon$ for $\mu \in \mathfrak{I}$ implies that
$$l_{((\nu-\varepsilon)\vee 0)} \leq (l_\nu - \varepsilon) \vee 0 \leq l_\mu.$$
This yields $\nu - \varepsilon \leq \mu$ and $\nu \leq \mu + \varepsilon$ for arbitrary $\varepsilon > 0$ and thus $\nu \in \mathfrak{I}$.

□

There are various ways to express $m_X(\Phi)$, as we already mentioned in Section 1.1.1. We add some new formulas, which were published in [CVO16].

Proposition 3.1.1.3. For $\Phi \in \mathsf{I}\mathsf{I}X$, we have
$$m_X(\Phi) = \bigcup_{\varphi\in\Phi}\bigcap_{\mathfrak{I}\in\mathsf{I}X} \mathfrak{I} \oplus \varphi(\mathfrak{I}) \tag{3.1}$$
and the characteristic value of $m_X(\Phi)$ is given by
$$c\big(m_X(\Phi)\big) = \sup_{\varphi\in\Phi}\inf_{\mathfrak{I}\in\mathsf{I}X} c(\mathfrak{I}) + \varphi(\mathfrak{I}). \tag{3.2}$$
In both expressions the union or the supremum respectively, can be restricted to a basis for Φ.

Proof. To prove the first equality, we start with some μ bounded for which there exists $\varphi \in \Phi$ such that $\mu \in \mathfrak{I} \oplus \varphi(\mathfrak{I})$ for every $\mathfrak{I} \in \mathsf{I}X$. It follows that $l_\mu(\mathfrak{I}) \leq \varphi(\mathfrak{I})$ for every $\mathfrak{I} \in \mathsf{I}X$ and therefore $l_\mu \in \Phi$. For the other direction, suppose $\mu \in m_X(\Phi)$. Then we have $\mu \in \bigcap_{\mathfrak{I}\in\mathsf{I}X} \mathfrak{I} \oplus l_\mu(\mathfrak{I})$.

Next we calculate the characteristic value
$$c\big(m_X(\Phi)\big) = \sup_{\varphi\in\Phi}\sup_{\mu\in\mathfrak{K}_\varphi}\inf_{x\in X}\mu(x)$$
with $\mathfrak{K}_\varphi = \bigcap_{\mathfrak{I}\in\mathsf{I}X} \mathfrak{I} \oplus \varphi(\mathfrak{I})$. So we have
$$\begin{aligned} c\big(m_X(\Phi)\big) &= \sup_{\varphi\in\Phi}\ \sup_{\psi\in\prod_{\mathfrak{I}\in\mathsf{I}X}\mathfrak{I}}\ \inf_{x\in X}\inf_{\mathfrak{I}\in\mathsf{I}X}\big(\psi(x) + \varphi(\mathfrak{I})\big) \\ &= \sup_{\varphi\in\Phi}\inf_{\mathfrak{I}\in\mathsf{I}X}\sup_{\psi\in\mathfrak{I}}\inf_{x\in X}\big(\psi(x) + \varphi(\mathfrak{I})\big) \end{aligned}$$

and using the fact that $c(\Phi) = \sup_{\varphi\in\Phi} \inf_{\mathfrak{I}\in \mathsf{I}X} \varphi(\mathfrak{I})$ we get the result.

In order to prove that the first expression can be restricted to a basis $\mathfrak{B}$ for Φ, for $\mu \in \bigcup_{\varphi\in\Phi}\bigcap_{\mathfrak{I}\in\mathsf{I}X} \mathfrak{I}\oplus\varphi(\mathfrak{I})$ choose $\varphi\in\Phi$ and for $\mathfrak{I}\in\mathsf{I}X$ some $\nu_{\mathfrak{I}}\in\mathfrak{I}$ such that $\mu = \inf_{\mathfrak{I}\in\mathsf{I}X}\nu_{\mathfrak{I}}+\varphi(\mathfrak{I})$. For $\varepsilon>0$ there exists $\psi_\varepsilon\in\mathfrak{B}$ with $\varphi\leq\psi_\varepsilon+\varepsilon$. So we have $\mu\leq\inf_{\mathfrak{I}\in\mathsf{I}X}\nu_{\mathfrak{I}}+\psi_\varepsilon(\mathfrak{I})+\varepsilon$ from which the result follows. The proof for the second expression is analogous. □

Proposition 3.1.1.4. For every $\Phi\in\mathsf{I}^2 X$, we have the inclusion

$$m_X(\Phi)\subseteq\bigvee_{\mathcal{A}\in\mathfrak{f}(\Phi)}\bigcap_{\mathfrak{I}\in\mathcal{A}}\mathfrak{I}\oplus c(\Phi).$$

Whenever $\Phi=\iota_{\mathsf{I}X}(\mathcal{F})\oplus c(\Phi)$ for some filter $\mathcal{F}$ on $\mathsf{I}X$ and $c(\Phi)<\infty$, the formulas simplify to

$$m_X(\Phi)=\bigvee_{\mathcal{A}\in\mathcal{F}}\bigcap_{\mathfrak{I}\in\mathcal{A}}\mathfrak{I}\oplus c(\Phi), \tag{3.3}$$

and

$$c\big(m_X(\Phi)\big)=\sup_{\mathcal{A}\in\mathcal{F}}\inf_{\mathfrak{I}\in\mathcal{A}}c(\mathfrak{I})+c(\Phi). \tag{3.4}$$

Proof. In order to prove the inclusion, first note that when Φ is improper, then by definition so is the filter $\mathfrak{f}(\Phi)$ and consequently also the right-hand side. Next let $\mu\in m_X(\Phi)$ and $\varepsilon>0$. Then

$$\mathcal{A}_\varepsilon=\{l_\mu<c(\Phi)+\varepsilon\}\in\mathfrak{f}(\Phi).$$

For $\mathfrak{I}\in\mathcal{A}_\varepsilon$ we have $l_\mu(\mathfrak{I})<c(\Phi)+\varepsilon$ and hence $\mu\in\mathfrak{I}\oplus\big(c(\Phi)+\varepsilon\big)$. Put $\nu_\varepsilon=(\mu-\varepsilon)\vee 0$ then we have $\mu\leq\nu_\varepsilon+\varepsilon$ and $\nu_\varepsilon\in\bigcap_{\mathfrak{I}\in\mathcal{A}_\varepsilon}\mathfrak{I}\oplus c(\Phi)$.

Under the extra assumptions made on Φ let $\mu\in\bigvee_{\mathcal{A}\in\mathcal{F}}\bigcap_{\mathfrak{I}\in\mathcal{A}}\mathfrak{I}\oplus c(\Phi)$ and $\varepsilon>0$. Choose ν_ε and $\mathcal{A}_\varepsilon\in\mathcal{F}$ such that $\nu_\varepsilon\in\mathfrak{I}\oplus c(\Phi)$, for all $\mathfrak{I}\in\mathcal{A}_\varepsilon$ and such that $\mu\leq\nu_\varepsilon+\varepsilon$. Then we have $\mathcal{A}_\varepsilon\subseteq\{l_{\nu_\varepsilon}\leq c(\Phi)\}$ and hence $\{l_{\nu_\varepsilon}\leq c(\Phi)\}\in\mathcal{F}$. It follows that $l_\mu\leq l_{\nu_\varepsilon}+\varepsilon$ with $l_{\nu_\varepsilon}\in\Phi$.

Next we calculate the characteristic value:

$$\begin{aligned}
c\big(m_X(\Phi)\big) &= \sup_{\mathcal{A}\in\mathcal{F},\omega<\infty}\inf_{\mathfrak{I}\in\mathsf{I}X}\big(c(\mathfrak{I})+\theta^\omega_{\mathcal{A}}(\mathfrak{I})+c(\Phi)\big)\\
&= \sup_{\mathcal{A}\in\mathcal{F},\omega<\infty}\Big(\inf_{\mathfrak{I}\in\mathcal{A}}\big(c(\mathfrak{I})+c(\Phi)\big)\wedge\inf_{\mathfrak{I}\notin\mathcal{A}}\big(c(\mathfrak{I})+\omega+c(\Phi)\big)\Big)\\
&= \sup_{\mathcal{A}\in\mathcal{F}}\inf_{\mathfrak{I}\in\mathcal{A}}\big(c(\mathfrak{I})+c(\Phi)\big)\wedge\infty\\
&= \sup_{\mathcal{A}\in\mathcal{F}}\inf_{\mathfrak{I}\in\mathcal{A}}\big(c(\mathfrak{I})+c(\Phi)\big).
\end{aligned}$$

□

Remark that from the previous formula it immediately follows that $m_X(\Phi)$ can be improper, even if Φ is proper. For instance, take $X = \mathbb{N}$ and $\mathfrak{I} = \iota_{\mathsf{I}\,\mathbb{N}}\mathcal{F}$ for $\mathcal{F}$ a filter on $\mathsf{I}\,\mathbb{N}$ with basis $\{\mathcal{A}_m \mid m \in \mathbb{N}\}$ and

$$\mathcal{A}_m = \{\iota_X(\dot{n}) \oplus n \mid n \geq m\}.$$

Then $c\big(m_X(\Phi)\big) = \sup_{m\in\mathbb{N}} \inf_{n\geq m} n = \infty$.

In some cases the ideal Φ is generated by a selection $s : A \longrightarrow \mathsf{I}\,X$ and some $\mathfrak{I} \in \mathsf{I}\,A$ such that $\Phi = \mathsf{I}\,s(\mathfrak{I})$. In this case the formulas in Proposition 3.1.1.3 can be further simplified and coincide with the expressions considered in Section 1.1.1.

Proposition 3.1.1.5. In case Φ is the image of a selection on A, $s : A \longrightarrow \mathsf{I}\,X$ and $\Phi = \mathsf{I}\,s(\mathfrak{I})$ for some functional ideal $\mathfrak{I}$ on A, the formulas have the following form

$$m_X(\Phi) = \bigvee_{\nu\in\mathfrak{I}} \bigcap_{a\in A} s(a) \oplus \nu(a), \tag{3.5}$$

$$c\big(m_X(\Phi)\big) = \sup_{\nu\in\mathfrak{I}} \inf_{a\in A} c\big(s(a)\big) + \nu(a). \tag{3.6}$$

If moreover $\mathfrak{I}$ is of a special type $\mathfrak{I} = \iota_A(\mathcal{G}) \oplus c(\mathfrak{I})$, for $\mathcal{G} \in \mathsf{F}\,X$, then

$$m_X(\Phi) = \bigvee_{G\in\mathcal{G}} \bigcap_{a\in G} s(a) \oplus c(\mathfrak{I}), \tag{3.7}$$

$$c\big(m_X(\Phi)\big) = \sup_{G\in\mathcal{G}} \inf_{a\in G} c\big(s(a)\big) + c(\mathfrak{I}). \tag{3.8}$$

Proof. The proofs for the various expressions are analogous. We give the explicit calculations for the characteristic value in equation (3.6). We apply Proposition 3.1.1.3 to the basis $\{(s\nu)_\eta \mid \nu \in \mathfrak{I}, \eta < \infty\}$ of $\mathsf{I}\,s(\mathfrak{I})$.

$$\begin{aligned}
c\Big(m_X\big(\mathsf{I}\,s(\mathfrak{I})\big)\Big) &= \sup_{\nu\in\mathfrak{I},\eta<\infty} \inf_{\mathfrak{K}\in\mathsf{I}\,X} c(\mathfrak{K}) + (s\nu)_\eta(\mathfrak{K}) \\
&= \sup_{\nu\in\mathfrak{I},\eta<\infty} \big[\inf_{\mathfrak{K}\in s(A)} c(\mathfrak{K}) + (s\nu)_\eta(\mathfrak{K}) \wedge \inf_{\mathfrak{K}\notin s(A)} c(\mathfrak{K}) + \eta\big] \\
&= \sup_{\nu\in\mathfrak{I}} \inf_{\mathfrak{K}\in s(A)} \big(c(\mathfrak{K}) + \inf_{\mathfrak{K}=s(a')} \nu(a')\big) \wedge \sup_{\eta<\infty} \big(\inf_{\mathfrak{K}\notin s(A)} c(\mathfrak{K}) + \eta\big) \\
&= \sup_{\nu\in\mathfrak{I}} \inf_{a\in A} \Big(c\big(s(a)\big) + \inf_{s(a')=s(a)} \nu(a')\Big) \\
&= \sup_{\nu\in\mathfrak{I}} \inf_{a\in A} c\big(s(a)\big) + \nu(a).
\end{aligned}$$

□

Next we suppose that $\sigma : A \longrightarrow \mathsf{F}X$ is a selection of proper filters and $\mathcal{G}$ is a filter on A. Assume that the ideals $s(a)$ in a selection $s : A \longrightarrow \mathsf{I}X$ are defined as $s(a) = \iota_X\big(\sigma(a)\big) \oplus c\big(s(a)\big)$. Then we have the following result.

Proposition 3.1.1.6. Let $\Phi = \mathsf{I}s(\mathfrak{I})$ be generated by $\mathfrak{I} \in \mathsf{I}A$ and by a selection $s : A \longrightarrow \mathsf{I}X$ with $s(a) = \iota_X\big(\sigma(a)\big) \oplus c\big(s(a)\big)$, whenever $a \in A$ for a selection of proper filters $\sigma : A \longrightarrow \mathsf{F}X$.

Suppose $c\big(m_X(\Phi)\big) = c(\mathfrak{I}) < \infty$, then for α with $c(\mathfrak{I}) \leq \alpha < \infty$ and $\inf_{F\in\mathfrak{f}_\alpha(\mathfrak{I})} \sup_{a\in F} c\big(s(a)\big) = 0$ we have

$$\mathfrak{f}_\alpha\big(m_X(\Phi)\big) \subseteq \Sigma\sigma\big(\mathfrak{f}_\alpha(\mathfrak{I})\big). \tag{3.9}$$

Let $\Phi = \mathsf{I}s(\mathfrak{I})$ be generated by $\mathfrak{I} \in \mathsf{I}A$ and by a selection $s : A \longrightarrow \mathsf{I}X$ with $s(a) = \iota_X\big(\sigma(a)\big) \oplus c\big(s(a)\big)$, whenever $a \in A$ for a selection of proper filters $\sigma : A \longrightarrow \mathsf{F}X$. Suppose $\mathfrak{I} = \iota_A(\mathcal{G}) \oplus c(\mathfrak{I})$ for some filter $\mathcal{G}$ on A and $c\big(m_X(\Phi)\big) < \infty$, then for α with $c\big(m_X(\Phi)\big) \leq \alpha < \infty$ we have the other inclusion

$$\Sigma\sigma(\mathcal{G}) \subseteq \mathfrak{f}_\alpha\big(m_X(\Phi)\big). \tag{3.10}$$

Proof. Under the first assumptions, let $\mu \in m_X(\Phi)$ and $\alpha < \beta$. By (3.5) choose $\varphi \in \mathfrak{I}$ such that $\mu \in \iota_X\big(\sigma(a)\big) \oplus \Big(c\big(s(a)\big) + \varphi(a)\Big)$, whenever $a \in A$. For $a \in A$ choose $S_a \in \sigma(a)$ and $\omega < \infty$ such that $\mu \leq \theta^\omega_{S_a} + c\big(s(a)\big) + \varphi(a)$.

Choose $\varepsilon > 0$ with $\alpha + 2\varepsilon < \beta$ and $F \in \mathfrak{f}_\alpha(\mathfrak{I})$ with $\sup_{a\in F} c\big(s(a)\big) < \varepsilon$. Then there exist $\eta > \alpha$ and $\psi \in \mathfrak{I}$ with $\{\psi < \eta\} \subseteq F$.

Let $\zeta = \varphi \vee \psi \in \mathfrak{I}$ and choose $\alpha < \rho < \eta$, $\rho < \alpha + \varepsilon$ then $\{\zeta < \rho\} \in \mathfrak{f}_\alpha(\mathfrak{I})$. We now claim that for $a \in A$ with $\zeta(a) < \rho$ we have $\{\mu < \beta\} \in \sigma(a)$.

Indeed for such a we have that $\psi(a) < \eta$ implies $a \in F$ and thus $c\big(s(a)\big) < \varepsilon$. Since also $\varphi(a) < \rho$, it follows that $S_a \subseteq \{\mu < \beta\}$.

Under the second assumptions first note that since $m_X(\Phi)$ is proper, also Φ, the filter $\mathcal{G}$ and the functional ideal $\mathfrak{I}$ are proper.

Let $Z \in \Sigma\sigma(\mathcal{G})$, $G \in \mathcal{G}$ with $Z \in \bigcap_{a\in G} \sigma(a)$ and let $\alpha < \gamma < \omega$. By (3.7) for θ^ω_Z we have

$$\theta^\omega_Z \in \bigcap_{a\in G} \iota_X\big(\sigma(a)\big) \Rightarrow \theta^\omega_Z + c(\mathfrak{I}) \in m_X(\Phi) \Rightarrow \{\theta^\omega_Z + c(\mathfrak{I}) < \gamma\} \in \mathfrak{f}_\alpha\big(m_X(\Phi)\big).$$

It follows that $Z \in \mathfrak{f}_\alpha\big(m_X(\Phi)\big)$. □

From the previous result, we obtain the following easy form.

Proposition 3.1.1.7. Let $\Phi = \mathsf{I}s(\mathfrak{I})$ be generated by $\mathfrak{I} \in \mathsf{I}A$ and by a selection $s : A \longrightarrow \mathsf{I}X$ with $s(a) = \iota_X\big(\sigma(a)\big) \oplus c\big(s(a)\big)$, whenever $a \in A$ for a selection of proper filters $\sigma : A \longrightarrow \mathsf{F}X$. Suppose $\mathfrak{J} = \iota_X(\mathcal{W}) \oplus c(\mathfrak{J})$ for some ultrafilter $\mathcal{W}$ on A, then we have

$$m_X(\Phi) = \iota_X\big(\Sigma\sigma(\mathcal{W})\big) \oplus \Big(c(\mathfrak{J}) + \sup_{W \in \mathcal{W}} \inf_{w \in W} c\big(s(w)\big)\Big). \tag{3.11}$$

Proof. By (3.8) for the characteristic value we have

$$c\big(m_X(\Phi)\big) = \sup_{W \in \mathcal{W}} \inf_{w \in W} c\big(s(w)\big) + c(\mathfrak{J}).$$

If $m_X(\Phi)$ is improper then (3.11) clearly holds.

In case $m_X(\Phi)$ is proper, for every α with $c\big(m_X(\Phi)\big) \leq \alpha < \infty$, by application of (3.10) we get $\Sigma\sigma(\mathcal{W}) \subseteq \mathfrak{f}_\alpha\big(m_X(\Phi)\big)$ and since the left-hand side is an ultrafilter, both filters coincide. Applying Proposition 3.1.1.1 we are done.

□

3.1.2 The functional ideal monad

The *functional ideal monad* $\mathbb{I} = (\mathsf{I}, m, e)$ on Set with

$$\mathsf{I} : \mathsf{Set} \longrightarrow \mathsf{Set} : \begin{cases} X \mapsto \mathsf{I}X, \\ f \mapsto \mathsf{I}f, \end{cases}$$

where $\mathsf{I}X$ and $\mathsf{I}f$ are defined as in Section 1.1.1, has multiplication and unit defined by the components

$$m_X : \mathsf{I}\mathsf{I}X \longrightarrow \mathsf{I}X : \Phi \mapsto m_X\Phi = \{\mu \in \mathsf{P}_b^X \mid l_\mu \in \Phi\}$$

and

$$e_X : X \longrightarrow \mathsf{I}X : x \mapsto \iota_X(\dot{x}).$$

Considering the identity functor $1 : \mathsf{Set} \longrightarrow \mathsf{Set}$ together with the functors $\mathsf{I} : \mathsf{Set} \longrightarrow \mathsf{Set}$ and $I^2 : \mathsf{Set} \longrightarrow \mathsf{Set}$ it is easily verified (see also [CLR11]) that e and m are indeed natural transformations

$$e : 1 \longrightarrow \mathsf{I} \text{ and } m : \mathsf{I}^2 \longrightarrow \mathsf{I}$$

and that the following diagrams commute.

Lemma 3.1.2.1. For any $\mu \in \mathsf{P}_b^X$, we have $l_\mu \cdot m_X = l_{l_\mu}$.

Proof. Indeed, for the improper functional ideal $\mathsf{P}_b^{\mathsf{I}X}$ the result follows easily and if $\Phi \in \mathsf{I}^2 X$ is proper, then $l_\mu \in \Phi \oplus \alpha$ if and only if there exists ζ such that $\mu \leq \zeta + \alpha$ and $l_\zeta \in \Phi$ and hence the result follows. □

Proposition 3.1.2.2. $\mathbb{I} = (\mathsf{I}, m, e)$ is a monad on Set.

Proof. That

$$\begin{array}{ccc} \mathsf{I}^3 X & \xrightarrow{m_{\mathsf{I}X}} & \mathsf{I}^2 X \\ {\scriptstyle \mathsf{I} m_X}\downarrow & & \downarrow{\scriptstyle m_X} \\ \mathsf{I}^2 X & \xrightarrow{m_X} & \mathsf{I} X \end{array}$$

commutes follows from the fact that all maps preserve being improper and that for any proper $\Gamma \in \mathsf{I}^3 X$, by Lemma 3.1.2.1

$$\begin{aligned} \mu \in m_X \cdot m_{\mathsf{I}X}(\Gamma) &\Leftrightarrow l_\mu \in m_{\mathsf{I}X}(\Gamma) \\ &\Leftrightarrow l_{l_\mu} \in \Gamma \\ &\Leftrightarrow l_\mu \cdot m_X \in \Gamma \\ &\Leftrightarrow l_\mu \in \mathsf{I} m_X(\Gamma) \\ &\Leftrightarrow \mu \in m_X \cdot \mathsf{I} m_X(\Gamma). \end{aligned}$$

That

$$\begin{array}{ccccc} \mathsf{I}X & \xrightarrow{\mathsf{I}e_X} & \mathsf{I}^2 X & \xleftarrow{e_{\mathsf{I}X}} & \mathsf{I}X \\ & {\scriptstyle 1_{\mathsf{I}X}}\searrow & \downarrow{\scriptstyle m_X} & \swarrow{\scriptstyle 1_{\mathsf{I}X}} & \\ & & \mathsf{I}X & & \end{array}$$

commutes, for the left triangle follows from the fact that for any $\mu \in \mathsf{P}_b^X$ we have that

$$\begin{aligned} \mu \in m_X \cdot \mathsf{I} e_X(\mathfrak{J}) &\Leftrightarrow l_\mu \in \mathsf{I} e_X(\mathfrak{J}) \\ &\Leftrightarrow l_\mu \cdot e_X \in \mathfrak{J} \\ &\Leftrightarrow \mu \in \mathfrak{J}, \end{aligned}$$

and for the right triangle it follows from

$$\begin{aligned} \mu \in m_X \cdot e_{\mathsf{I}X}(\mathfrak{J}) &\Leftrightarrow l_\mu \in e_{\mathsf{I}X}(\mathfrak{J}) \\ &\Leftrightarrow l_\mu(\mathfrak{J}) = 0 \\ &\Leftrightarrow \mu \in \mathfrak{J}. \end{aligned}$$

□

3.1.3 The functional power monad

We give an isomorphic description of the monad $\mathbb{I}$ as a submonad of the supermonad introduced in [CLR11] which was called the functional power monad. In order to avoid confusion in notation with the powerset monad $\mathbb{P}$ introduced earlier we change the name of this functional power monad (which was called $\mathbb{P}$ in [CLR11]) to $\mathbb{FP}$. Let $\mathsf{FP}\,X$ stand for the set of all functions

$$\mathfrak{f} : \mathsf{P}_b^X \longrightarrow \mathsf{P}$$

that satisfy the following conditions:

(FP1) $\mathfrak{f}$ is order-preserving.

(FP2) For any $\mu \in \mathsf{P}_b^X$ and α constant: $\mathfrak{f}(\mu \ominus \alpha) = \mathfrak{f}(\mu) \ominus \alpha$.

(FP3) $\mathfrak{f}(0) = 0$.

From (FP2) and (FP3) it follows that for any α constant also $\mathfrak{f}(\alpha) \leq \alpha$. Applying (FP1) it follows that with $\mu \leq \alpha$ also $\mathfrak{f}(\mu) \leq \alpha$, whenever $\mathfrak{f} \in \mathsf{FP}\,X$. We use the following notations. For $x \in X$ we let

$$\mathrm{ev}_x : \mathsf{P}_b^X \longrightarrow \mathsf{P} : \nu \mapsto \nu(x)$$

and for $\nu \in \mathsf{P}_b^X$ we let

$$\mathrm{ev}_\nu : \mathsf{FP}\,X \longrightarrow \mathsf{P} : \mathfrak{f} \mapsto \mathfrak{f}(\nu).$$

Note that in view of the previous observations, both functions are bounded. We define the triple (FP, μ, η) as follows:

$$\mathsf{FP} : \mathsf{Set} \longrightarrow \mathsf{Set} : \begin{cases} X \mapsto \mathsf{FP}\,X, \\ (f : X \longrightarrow Y) \mapsto \big(\mathsf{FP}\,f : \mathsf{FP}\,X \longrightarrow \mathsf{FP}\,Y : \mathfrak{f} \mapsto \mathfrak{f}(- \cdot f)\big). \end{cases}$$

For any set X, let

$$\eta_X : X \longrightarrow \mathsf{FP}\,X : x \mapsto \mathrm{ev}_x$$

and

$$\mu_X : \mathsf{FP}^2\,X \longrightarrow \mathsf{FP}\,X : \mathfrak{F} \mapsto \big(\mathsf{P}_b^X \longrightarrow \mathsf{P} : \nu \mapsto \mathfrak{F}(\mathrm{ev}_\nu)\big).$$

The triple (FP, μ, η) is a monad, called the *functional power monad* on Set and denoted by $\mathbb{FP}$.

Definition 3.1.3.1. For any set X consider the following extra condition on the functions $\mathfrak{f} \in \mathsf{FP}\,X$

(FPI) $\mathfrak{f}$ preserves binary suprema.

The subset of $\mathsf{FP}\,X$ of all maps $\mathfrak{f} : \mathsf{P}_b^X \longrightarrow \mathsf{P}$ satisfying (FPI) will be denoted by $\mathsf{FP}_I\,X$.

Let $a_X : \mathsf{FP}_I\,X \longrightarrow \mathsf{FP}\,X$ be the canonical subset injection.

Observe that for any $f : X \longrightarrow Y$ we have that $\mathsf{FP}_I(f) : \mathsf{FP}_I\,X \longrightarrow \mathsf{FP}_I\,Y$ is well defined as being the restriction of $\mathsf{FP}(f)$ to $\mathsf{FP}_I\,X$. Moreover for any set $x \in X$ we have $\mathrm{ev}_x \in \mathsf{FP}_I\,X$ for every $x \in X$. With ev_ν^I the restriction of ev_ν to FP_I we have the following result.

Proposition 3.1.3.2. $\mathbb{FP}_I = (\mathsf{FP}_I, \mu^I, \eta^I)$ with

$$\eta_X^I : X \longrightarrow \mathsf{FP}_I\,X : x \mapsto \mathrm{ev}_x$$

and

$$\mu_X^I : \mathsf{FP}_I^2\,X \longrightarrow \mathsf{FP}_I\,X : \mathfrak{F} \mapsto \big(\mathsf{P}_b^X \longrightarrow \mathsf{P} : \nu \mapsto \mathfrak{F}(\mathrm{ev}_\nu^I)\big)$$

is a submonad of $\mathbb{FP}$.

Proposition 3.1.3.3. For any set X there is a one-to-one correspondence between functional ideals on X and functions $\mathfrak{f} \in \mathsf{FP}_I\,X$. This correspondence is given by

$$\mathsf{I}\,X \longrightarrow \mathsf{FP}_I\,X : \mathfrak{I} \mapsto l_{(-)}(\mathfrak{I}) \quad \text{and} \quad \mathsf{FP}_I\,X \longrightarrow \mathsf{I}\,X : \mathfrak{f} \mapsto Z(\mathfrak{f})$$

with $Z(\mathfrak{f})$ the zeroset determined by $\mathfrak{f}$. Moreover

$$l_{(-)}\big(Z(\mathfrak{f})\big) = \mathfrak{f} \quad \text{and} \quad Z\big(l_{(-)}(\mathfrak{I})\big) = \mathfrak{I}, \tag{3.12}$$

for every $\mathfrak{f} \in \mathsf{FP}_I\,X$ and $\mathfrak{I} \in \mathsf{I}\,X$.

Proof. First we need to show that for any $\mathfrak{f} \in \mathsf{FP}_I\,X$ which satisfies the given properties $Z(\mathfrak{f})$ is a functional ideal on X. The ideal properties follow from the fact that $\mathfrak{f}$ is order- and binary sup-preserving. To prove the saturation property, let $\nu \in \mathsf{P}_b^X$ be such that for all $\varepsilon > 0$ there exists a $\nu_\varepsilon \in Z(\mathfrak{f})$ with $\nu \leq \nu_\varepsilon + \varepsilon$. By (FP2), we get

$$0 = \mathfrak{f}(\nu_\varepsilon) = \mathfrak{f}(\nu_\varepsilon + \varepsilon) \ominus \varepsilon$$

so that $\mathfrak{f}(\nu_\varepsilon + \varepsilon) \leq \varepsilon$ and hence by (FP1) and the arbitrariness of ε it follows that $\mathfrak{f}(\nu) = 0$.

Conversely, that for any $\mathfrak{I} \in \mathsf{I}\,X$, $l_{(-)}(\mathfrak{I})$ satisfies (FP1), (FP2), (FP3) and (FPI) was shown in Proposition 1.1.1.21.

Finally, given $\mathfrak{f} \in \mathsf{FP}_I\, X$ and making use of the appropriate properties, we have

$$\begin{aligned} l_\nu\big(Z(\mathfrak{f})\big) &= \inf\{\alpha \in \mathsf{P} \mid \nu \in Z(\mathfrak{f}) \oplus \alpha\} \\ &= \inf\{\alpha \in \mathsf{P} \mid \exists \mu \in \mathsf{P}_b^X \text{ such that } \mathfrak{f}(\mu) = 0 : \nu \leq \mu + \alpha\} \\ &= \inf\{\alpha \in \mathsf{P} \mid \mathfrak{f}(\nu) \leq \alpha\} \\ &= \mathfrak{f}(\nu). \end{aligned}$$

Given $\mathfrak{I} \in \mathsf{I}\, X$ it follows from the saturation property of functional ideals that

$$Z\big(l_{(-)}(\mathfrak{I})\big) = \{\nu \in \mathsf{P}_b^X \mid l_\nu(\mathfrak{I}) = 0\} = \{\nu \in \mathsf{P}_b^X \mid \nu \in \mathfrak{I}\} = \mathfrak{I}.$$

□

Theorem 3.1.3.4. *The monads* $\mathbb{I} = (\mathsf{I}, m, e)$ *and* $\mathbb{FP}_I = (\mathsf{FP}_I, \mu^I, \eta^I)$ *are isomorphic.*

Proof. Let $\sigma : \mathsf{FP}_\mathsf{I} \longrightarrow \mathsf{I}$ have components

$$\sigma_X : \mathsf{FP}_\mathsf{I}\, X \longrightarrow \mathsf{I}\, X : \mathfrak{f} \mapsto Z(\mathfrak{f}).$$

Then σ is a natural transformation. Indeed, for $f : X \longrightarrow Y$ the diagram

$$\begin{array}{ccc} \mathsf{FP}_\mathsf{I}\, X & \xrightarrow{\mathsf{FP}_\mathsf{I} f} & \mathsf{FP}_\mathsf{I}\, Y \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ \mathsf{I}\, X & \xrightarrow{\mathsf{I} f} & \mathsf{I}\, Y \end{array}$$

commutes. To see this, take $\mathfrak{f} \in \mathsf{FP}_\mathsf{I}\, X$ arbitrary, then

$$\begin{aligned} \sigma_Y\big(\mathsf{FP}_\mathsf{I} f(\mathfrak{f})\big) &= Z\big(\mathsf{FP}_\mathsf{I} f(\mathfrak{f})\big) \\ &= Z\big(\mathfrak{f}(- \cdot f)\big) \\ &= \mathsf{I} f\big(Z(\mathfrak{f})\big) \\ &= \mathsf{I} f\big(\sigma_X(\mathfrak{f})\big), \end{aligned}$$

where the third equality follows from the fact that

$$\begin{aligned} \nu \in Z\big(\mathfrak{f}(- \cdot f)\big) &\Leftrightarrow \mathfrak{f}(\nu \cdot f) = 0 \\ &\Leftrightarrow \nu \cdot f \in Z(\mathfrak{f}) \\ &\Leftrightarrow \nu \in \mathsf{I} f\big(Z(\mathfrak{f})\big), \end{aligned}$$

for $\nu \in \mathsf{P}_b^Y$ arbitrary.

$\sigma : \mathbb{I} \longrightarrow \mathbb{F}\mathbb{P}_{\mathsf{I}}$ is a morphism of monads. Therefore we check that

$$\sigma \cdot \mu^I = m \cdot (\sigma * \sigma) \tag{3.13}$$

and

$$\sigma \cdot \eta^I = e, \tag{3.14}$$

where the formula for $\sigma * \sigma$ can be found in (1.53).

We start with (3.13). The following diagram commutes

$$\begin{array}{ccc} \mathsf{FP}_{\mathsf{I}}^2\, X & \xrightarrow{\mu_X^I} & \mathsf{FP}_{\mathsf{I}}\, X \\ {\scriptstyle (\sigma*\sigma)_X}\downarrow & & \downarrow{\scriptstyle \sigma_X} \\ \mathsf{I}^2\, X & \xrightarrow{m_X} & \mathsf{I}\, X \end{array}$$

To see this, first note that for $\nu \in \mathsf{P}_b^X$ and $\mathfrak{f} \in \mathsf{FP}_I\, X$ we have

$$\begin{aligned} l_\nu \cdot \sigma_X(\mathfrak{f}) &= l_\nu\big(Z(\mathfrak{f})\big) \\ &= \inf\{\alpha \in \mathsf{P} \mid \nu \in Z(\mathfrak{f}) \oplus \alpha\} \\ &= \mathfrak{f}(\nu) \\ &= \mathrm{ev}_\nu^{\mathsf{I}}(\mathfrak{f}). \end{aligned}$$

Therefore, for $\mathfrak{F} \in \mathsf{FP}_I^2\, X$ arbitrary, we have

$$\begin{aligned} m_X\big((\sigma * \sigma)_X(\mathfrak{F})\big) &= \{\nu \in \mathsf{P}_b^X \mid l_\nu \in (\sigma * \sigma)_X(\mathfrak{F})\} \\ &= \{\nu \in \mathsf{P}_b^X \mid l_\nu \in \sigma_{\mathsf{I}\, X}(\mathsf{FP}_{\mathsf{I}}\, \sigma_X(\mathfrak{F}))\} \\ &= \{\nu \in \mathsf{P}_b^X \mid l_\nu \in Z(\mathsf{FP}_{\mathsf{I}}\, \sigma_X(\mathfrak{F}))\} \\ &= \{\nu \in \mathsf{P}_b^X \mid \mathsf{FP}_I\, \sigma_X(\mathfrak{F})(l_\nu) = 0\} \\ &= \{\nu \in \mathsf{P}_b^X \mid \mathfrak{F}(l_\nu \cdot \sigma_X) = 0\} \\ &= \{\nu \in \mathsf{P}_b^X \mid \mathfrak{F}(\mathrm{ev}_\nu^I) = 0\} \\ &= \{\nu \in \mathsf{P}_b^X \mid \mu_X^I(\mathfrak{F})(\nu) = 0\} \\ &= Z\big(\mu_X^I(\mathfrak{F})\big) \\ &= \sigma_X\big(\mu_X^I(\mathfrak{F})\big). \end{aligned}$$

That (3.14) holds, follows from the fact that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X^I} & \mathsf{FP}_{\mathsf{I}}\, X \\ & {\scriptstyle e_X}\searrow & \downarrow{\scriptstyle \sigma_X} \\ & & \mathsf{I}\, X \end{array}$$

commutes as well. Take $x \in X$ arbitrary, then

$$\begin{aligned}\sigma_X(\eta_X^I(x)) &= Z(\mathrm{ev}_x)\\ &= \{\nu \in \mathsf{P}_b^X \mid \mathrm{ev}_x(\nu) = 0\}\\ &= \{\nu \in \mathsf{P}_b^X \mid \nu(x) = 0\}\\ &= e_X(x).\end{aligned}$$

That σ is a natural isomorphism follows from Proposition 3.1.3.3 which shows that all components are isomorphisms. □

3.1.4 $\mathbb{I}$ is power-enriched

Proposition 3.1.4.1. We consider $\tau : \mathbb{P} \longrightarrow \mathbb{I}$ defined by its components

$$\tau_X : \mathcal{P}X \longrightarrow \mathsf{I}X : A \mapsto \iota_X(\dot{A}) = \{\mu \in \mathsf{P}_b^X \mid \mu_{|A} = 0\}.$$

1. τ is a monad morphism.

2. The order on $\mathsf{I}X$ associated with τ is the opposite of the inclusion order of functional ideals.

3. $(\mathbb{I}, \tau)$ is power-enriched.

Proof. 1. That τ is a natural transformation is straightforward. Moreover for $x \in X$ we clearly have $\tau_X(\{x\}) = \iota_X(\dot{x}) = e_X(x)$. Next consider $\mathcal{A} \subseteq \mathcal{P}X$. We have

$$(\tau * \tau)_X(\mathcal{A}) = \Big\{\varphi \in \mathsf{P}_b^{\mathsf{I}X} \mid \forall A \in \mathcal{A} : \varphi\big(\iota_X(\dot{A})\big) = 0\Big\}.$$

It follows that

$$\begin{aligned}\tau_X(\bigcup \mathcal{A}) &= \{\mu \in \mathsf{P}_b^{\mathsf{I}X} \mid \mu_{|\bigcup\mathcal{A}} = 0\}\\ &= \{\mu \in \mathsf{P}_b^{\mathsf{I}X} \mid \forall A \in \mathcal{A} : \mu_{|A} = 0\}\\ &= \{\mu \in \mathsf{P}_b^{\mathsf{I}X} \mid \forall A \in \mathcal{A} : l_\mu\big(\iota_X(\dot{A})\big) = 0\}\\ &= m_X \cdot (\tau * \tau)_X(\mathcal{A}).\end{aligned}$$

So we can conclude that τ preserves the monad structure.

2. For $\mathfrak{I}, \mathfrak{K} \in \mathsf{I}X$, we have

$$\mathfrak{I} \leq \mathfrak{K} \Leftrightarrow m_X \cdot \tau_{\mathsf{I}X}\big(\{\mathfrak{I}, \mathfrak{K}\}\big) = \mathfrak{K}.$$

For $\mathcal{A} \subseteq \mathsf{I}X$ we have $m_X \cdot \tau_{\mathsf{I}X}(\mathcal{A}) = \{\mu \in \mathsf{P}_b^X \mid l_\mu \in \iota_{\mathsf{I}X}(\dot{\mathcal{A}})\} = \bigcap_{\mathfrak{I}\in\mathcal{A}} \mathfrak{I}$. So in particular, for $\mathcal{A} = \{\mathfrak{I}, \mathfrak{K}\}$ we obtain for the order $\leq$ associated with τ

$$\mathfrak{I} \leq \mathfrak{K} \Leftrightarrow \mathfrak{I} \cap \mathfrak{K} = \mathfrak{K} \Leftrightarrow \mathfrak{K} \subseteq \mathfrak{I}.$$

So it coincides with the reversed inclusion order on functional ideals.

3. Let $f \leq g$ for $f, g : X \longrightarrow \mathsf{I}Y$ and let $\mathfrak{I} \in \mathsf{I}X$. For $\mu \in m_Y \cdot \mathsf{I}g(\mathfrak{I})$ and $x \in X$ we have $g(x) \subseteq f(x)$ and so

$$\begin{aligned} l_\mu \cdot f(x) &= \inf\{\alpha \mid \exists \nu \in f(x) : \mu \leq \nu + \alpha\} \\ &\leq \inf\{\alpha \mid \exists \nu \in g(x) : \mu \leq \nu + \alpha\} \\ &= l_\mu \cdot g(x). \end{aligned}$$

It follows that $\mu \in m_Y \cdot \mathsf{I}f(\mathfrak{I})$. Hence we have $m_Y \cdot \mathsf{I}g(\mathfrak{I}) \subseteq m_Y \cdot \mathsf{I}f(\mathfrak{I})$. □

3.1.5 Approach spaces as 𝕀-monoids

In this section we prove that the categories 𝕀-Mon and App are isomorphic. The constructed isomorphism links 𝕀-monoids to approach spaces in terms of their bounded local approach systems, see Definition 1.1.1.11.

First of all we prove that an 𝕀-monoid gives rise to a bounded local approach system. We will need the following preliminary result.

Proposition 3.1.5.1. For any $\nu : X \longrightarrow \mathsf{I}X$, for $\mu \in \mathsf{P}_b^X$ and $x \in X$ the following are equivalent:

(i) $\exists \beta \in \nu(x), \forall z \in X : \mu \in \nu(z) \oplus \beta(z)$.

(ii) $l_\mu \cdot \nu \in \nu(x)$.

Proof. (i) ⇒ (ii): Suppose (i). For $z \in X$ determine $\eta \in \nu(z)$ such that $\mu \leq \eta + \beta(z)$. This implies

$$l_\mu \cdot \nu(z) = \inf\{\alpha \in \mathsf{P} \mid \exists \eta \in \nu(z) : \mu \leq \eta + \alpha\} \leq \beta(z)$$

for any $z \in X$. Since $\beta \in \nu(x)$ we get $l_\mu \cdot \nu \in \nu(x)$.

(ii) ⇒ (i): Suppose $l_\mu \cdot \nu \in \nu(x)$. So there exists $\beta \in \nu(x)$ with $l_\mu \cdot \nu \leq \beta$. For $z \in X$ we have $l_\mu \cdot \nu(z) \leq \beta(z)$ which implies $l_\mu \cdot \nu(z) \leq \alpha$ for every α with $\beta(z) \leq \alpha$. By definition of l_μ it follows that

$$\beta(z) \leq \alpha \Rightarrow \mu \in \nu(z) \oplus \alpha,$$

so we can conclude $\mu \in \nu(z) \oplus \beta(z)$. □

Theorem 3.1.5.2. *If* (X, ν) *with* $\nu : X \longrightarrow \mathsf{I}X$ *is an* $\mathbb{I}$*-monoid, then* $\big(\nu(x)\big)_{x\in X}$ *is a bounded local approach system on* X.

Proof. By definition $\nu(x)$ is a functional ideal and by reflexivity $\nu(x) \subseteq \{\mu \in \mathsf{P}_b^X \mid \mu(x) = 0\}$ for every $x \in X$. By transitivity we have $\nu \cdot \nu \leq \nu$, hence for all $x \in X : \nu(x) \subseteq m_X \cdot \mathsf{I}\nu \cdot \nu(x)$ and thus

$$\begin{aligned} \mu \in \nu(x) &\Rightarrow \mu \in m_X \cdot \mathsf{I}\nu \cdot \nu(x) \\ &\Rightarrow l_\mu \cdot \nu \in \nu(x). \end{aligned}$$

By Proposition 3.1.5.1 this means that there exists $\beta \in \nu(x)$ such that for all $z \in X$

$$\mu \in \nu(z) \oplus \beta(z).$$

For $z \in X$ choose $\varphi_z \in \nu(z)$ such that $\mu \leq \varphi_z + \beta(z)$ and put $\psi_z = \varphi_z$ for $z \neq x$ and $\psi_x = \varphi_x \vee \beta$. For z and y arbitrary we now have

$$\mu(y) \leq \psi_z(y) + \psi_x(z).$$

□

In order to prove the reverse implication we first need a preliminary result from [Low15] which we formulate without proof.

Proposition 3.1.5.3. If an approach space is given by its bounded local approach system $\big(\mathcal{A}_b(x)\big)_{x\in X}$ then the operator $\mathfrak{u} : \mathsf{P}_b^X \longrightarrow \mathsf{P}_b^X$ defined by

$$\mathfrak{u}(\mu)(x) = \inf_{\varphi\in\mathcal{A}_b(x)} \sup_{y\in X} \big(\mu(y) - \varphi(y)\big)$$

and called the *upper hull operator*, is expansive and idempotent, preserves finite suprema and satisfies $\mathfrak{u}(0) = 0$ and $\mathfrak{u}(\mu + \alpha) = \mathfrak{u}(\mu) + \alpha$, for every $\alpha < \infty$.

Moreover the bounded approach system can be recovered from $\mathfrak{u}$ by means of the following equivalence:

$$\varphi \in \mathcal{A}_b(x) \Leftrightarrow \mathfrak{u}(\varphi)(x) = 0$$

for every $x \in X$.

Proposition 3.1.5.4. Let $\big(\mathcal{A}_b(x)\big)_{x\in X}$ be a bounded local approach system on X. For $x \in X$ and $\varphi \in \mathcal{A}_b(x)$ there exists $\beta \in \mathcal{A}_b(x)$ such that for all $z \in X$

$$\varphi \in \mathcal{A}_b(z) \oplus \beta(z).$$

Proof. Given $\varphi \in \mathcal{A}_b(x)$ let $\beta = \mathfrak{u}(\varphi)$ be the upper hull of φ in the sense of Proposition 3.1.5.3. In view of Theorem 1.2.40 in [Low15] it follows that $\beta \in \mathcal{A}_b(x)$.

Next since $\mathfrak{u}(\beta) = \beta$, for $z \in X$ arbitrary, we have

$$\beta(z) = \inf_{\varphi \in \mathcal{A}_b(z)} \sup_{y \in X} \big(\beta(y) - \varphi(y)\big).$$

This means that for $\varepsilon > 0$ there exists $\varphi^{\varepsilon} \in \mathcal{A}_b(z)$ such that

$$\big(\beta - \beta(z)\big) \vee 0 \leq \varphi^{\varepsilon} + \varepsilon.$$

So we have $\big(\beta - \beta(z)\big) \vee 0 \in \mathcal{A}_b(z)$ and using the fact that $\varphi \leq \beta \leq \big(\beta - \beta(z)\big) \vee 0 + \beta(z)$ we are done. □

Theorem 3.1.5.5. *Let $\big(\mathcal{A}_b(x)\big)_{x \in X}$ be a bounded approach system on X, then*

$$\mathcal{A}_b : X \longrightarrow \mathsf{I}X : x \mapsto \mathcal{A}_b(x)$$

is an $\mathbb{I}$-monoid.

Proof. Reflexivity follows easily from the fact that $\mathcal{A}_b(x) \subseteq \iota_X(\dot{x})$ for every $x \in X$. In order to prove transitivity let $\mu \in \mathcal{A}_b(x)$. Apply Proposition 3.1.5.4 to find $\beta \in \mathcal{A}_b(x)$ such that $\mu \in \mathcal{A}_b(z) \oplus \beta(z)$ for all $z \in X$ and then Proposition 3.1.5.1 to conclude $l_\mu \cdot \mathcal{A}_b \in \mathcal{A}_b(x)$. It follows that

$$\mu \in m_X \cdot \mathsf{I}\mathcal{A}_b \cdot \mathcal{A}_b(x).$$

□

Theorem 3.1.5.6. *The categories* $\mathbb{I}$-Mon *and* App *are isomorphic.*

Proof. By Theorem 3.1.5.2 and Theorem 3.1.5.5, the objects of $\mathbb{I}$-Mon and App correspond.

To see that the morphisms correspond as well, let $f : X \longrightarrow Y$ be a map and suppose X and Y are endowed with $\mathbb{I}$-monoids $\nu_X : X \longrightarrow \mathsf{I}X$ and $\nu_Y : Y \longrightarrow \mathsf{I}Y$ respectively. Based on Theorem 3.1.5.2 and Theorem 3.1.5.5 these structures are in bijective correspondence with bounded approach systems $\big(\nu_X(x)\big)_{x \in X}$ and $\big(\nu_Y(y)\big)_{y \in Y}$. Clearly we have

$$\mathsf{I}f \cdot \nu_X \leq \nu_Y \cdot f \Leftrightarrow \forall x \in X, \forall \varphi \in \nu_Y\big(f(x)\big) : \varphi \cdot f \in \nu_X(x).$$

□

3.1.6 Approach spaces as relational algebras

Let $\check{\mathbb{I}}$ be the Kleisli extension of $\mathbb{I}$ to Rel, as introduced in (1.98), which is a lax extension of the monad $\mathbb{I}$. The monad $\mathbb{I}$ and its Kleisli extension to Rel give the category of relational $\mathbb{I}$-algebras, $(\mathbb{I}, 2)$-Cat, with objects pairs (X, a) with X a set and $a : \mathsf{I}X \rightarrow\!\!\!\!\!+\, X$ a relation that satisfies transitivity

$$\Phi\,(\check{\mathsf{I}}a)\,\mathfrak{K}\ \&\ \mathfrak{K}\,a\,x \Rightarrow m_X(\Phi)\,a\,x, \tag{3.15}$$

for any $\Phi \in \mathsf{I}\mathsf{I}X, \mathfrak{K} \in \mathsf{I}X$ and $x \in X$, and reflexivity

$$\iota_X(\dot{x})\,a\,x, \tag{3.16}$$

for any $x \in X$, and morphisms $f : (X, a_X) \longrightarrow (Y, a_Y)$ satisfying

$$f \cdot a_X \leq a_Y \cdot \mathsf{I}f. \tag{3.17}$$

By Theorem 1.4.3.3, we get $\mathbb{I}$-Mon $\cong (\mathbb{I}, 2)$-Cat and the isomorphism between the categories $\mathbb{I}$-Mon and $(\mathbb{I}, 2)$-Cat is built on the following equivalence. For an $\mathbb{I}$-monoid (X, ν) the corresponding relational $\mathbb{I}$-algebra (X, a) is given by

$$\mathfrak{I}\,a\,x \Leftrightarrow \mathfrak{I} \leq \nu(x).$$

For a relational $\mathbb{I}$-algebra (X, a) the corresponding $\mathbb{I}$-monoid (X, ν) is given by

$$\nu(x) = \bigcap_{\mathfrak{I}\,a\,x} \mathfrak{I}.$$

Theorem 3.1.6.1. *The categories* App *and* $(\mathbb{I}, 2)$-Cat *are isomorphic,*

$$\mathsf{App} \cong (\mathbb{I}, 2)\text{-}\mathsf{Cat}.$$

Proof. This follow from Theorem 3.1.5.6 were we showed

$$\mathsf{App} \cong \mathbb{I}\text{-}\mathsf{Mon}$$

and from Theorem 1.4.3.3 which implies

$$(\mathbb{I}, 2)\text{-}\mathsf{Cat} \cong \mathbb{I}\text{-}\mathsf{Mon}.$$

□

Since in App convergence of functional ideals is defined by

$$\mathfrak{I} \rightarrowtail x \Leftrightarrow \mathcal{A}_b(x) \subseteq \mathfrak{I},$$

and by Theorems 3.1.5.5 and 3.1.5.2, we get that functional ideal convergence coincides with the reflexive and transitive relation $a : \mathsf{I}X \rightarrow\!\!\!\!\!+\, X$ as above. In what follows we will be writing

$$\mathfrak{I} \rightarrowtail x \text{ instead of } \mathfrak{I}\,a\,x.$$

Using this notation, it is clear that (3.17) coincides with the definition of a contraction in terms of functional ideal convergence.

3.1.7 Approach spaces via functional ideal convergence

On one hand functional ideal convergence is characterized by the two axioms (3.15) and (3.16) above. In Section 1.1.1, we have seen that functional ideal convergence is described using the axioms (F1), (F2) and (F3), or equivalently, (F1), (F2w) and (F).

In order to explain the direct connection between the two equivalent axiom systems, for a relation $r : X \relbar\joinrel\mapsto Y$, we give an explicit description of $\check{\mathrm{I}}r$ appearing in (3.15). Given $r : X \relbar\joinrel\mapsto Y$, for $\nu \in \mathsf{P}^Y$ and $\mu \in \mathsf{P}^X$ we define

$$r^{\wedge}\nu(x) := \inf_{(x,y)\in r} \nu(y)$$

where as usual the value is ∞ if no such y exists, and

$$r^{\vee}\mu(y) := \sup_{(x,y)\in r} \mu(x)$$

where as usual the value is 0 if no such x exists.

The following verifications are easy.

Proposition 3.1.7.1. For any $r : X \relbar\joinrel\mapsto Y$, for $\nu \in \mathsf{P}^Y$ and $\mu \in \mathsf{P}^X$ we have

1. $r^{\wedge} : \mathsf{P}^Y \longrightarrow \mathsf{P}^X$ and $r^{\vee} : \mathsf{P}^X \longrightarrow \mathsf{P}^Y$ are well defined and monotone.
2. $r^{\wedge} \cdot r^{\vee}\mu \geq \mu$ and $r^{\vee} \cdot r^{\wedge}\nu \leq \nu$.
3. $r^{\wedge}$ is right adjoint and therefore preserves infima.
4. $r^{\vee}$ is left adjoint and therefore preserves suprema.

For a relation $r : X \relbar\joinrel\mapsto Y$ and $\mathfrak{B} \subseteq \mathsf{P}_b^Y$ we define

$$r^{\wedge}\mathfrak{B} := \{\mu \in \mathsf{P}_b^X \mid \exists \nu \in \mathfrak{B} : \mu \leq r^{\wedge}\nu\}.$$

Proposition 3.1.7.2. If $\mathfrak{K}$ is a functional ideal on Y, then the collection $r^{\wedge}\mathfrak{K}$ is a functional ideal on X.

Proof. That $r^{\wedge}\mathfrak{K}$ is an ideal follows immediately from Proposition 3.1.7.1. To see that it is saturated let μ be bounded on X and suppose that for every $\varepsilon > 0$ there is some μ_ε bounded and $\nu_\varepsilon \in \mathfrak{K}$ with $\mu \leq \mu_\varepsilon + \varepsilon \leq r^{\wedge}\nu_\varepsilon + \varepsilon = r^{\wedge}(\nu_\varepsilon + \varepsilon)$. For $\nu = \inf_{\varepsilon>0}(\nu_\varepsilon + \varepsilon)$ and again applying Proposition 3.1.7.1, we have

$$\mu \leq \inf_{\varepsilon>0} r^{\wedge}(\nu_\varepsilon + \varepsilon) = r^{\wedge}(\inf_{\varepsilon>0} \nu_\varepsilon + \varepsilon) = r^{\wedge}(\nu)$$

and hence we can conclude that $\mu \in r^{\wedge}\mathfrak{K}$. □

Proposition 3.1.7.3. If $f : X \longrightarrow Y$ is a map then

1. For any $\nu \in \mathsf{P}_b^Y : f^\wedge\nu = \nu \cdot f$ and $f^\wedge\nu \in \mathsf{P}_b^X$.
2. $\mathsf{I} f : \mathsf{I} X \longrightarrow \mathsf{I} Y$ is right adjoint since for any $\mathfrak{J} \in \mathsf{I} X$ and $\mathfrak{K} \in \mathsf{I} Y$, $f^\wedge\mathfrak{K} \subseteq \mathfrak{J}$ if and only if $\mathfrak{K} \subseteq \mathsf{I} f(\mathfrak{J})$.

Proof. 1. is immediate. To prove 2. let $\nu \in \mathfrak{K} \subseteq \mathsf{I} f(\mathfrak{J})$ then obviously we have $\nu \cdot f \in \mathfrak{J}$ and thus $f^\wedge\mathfrak{K} \subseteq \mathfrak{J}$. Conversely if $\nu \in \mathfrak{K}$ and $f^\wedge\mathfrak{K} \subseteq \mathfrak{J}$ then this implies $\nu \cdot f \in \mathfrak{J}$ and thus $\nu \in \mathsf{I} f(\mathfrak{J})$. □

Proposition 3.1.7.4. Let $r : X \longrightarrow\!\!\!\!\!\!\!\mapsto Y$ be a relation, $\mathfrak{J}$ a functional ideal on X and $\mathfrak{B}$ an ideal basis in P_b^Y. If $r^\wedge\mathfrak{B} \subseteq \mathfrak{J}$ then $r^\wedge\widehat{\mathfrak{B}} \subseteq \mathfrak{J}$.

Proof. Let $r^\wedge\mathfrak{B} \subseteq \mathfrak{J}$. Take $\nu \in \widehat{\mathfrak{B}}, \omega < \infty$ and $\varepsilon > 0$. There is $\nu' \in \mathfrak{B}$ such that $\nu \leq \nu' + \varepsilon$. This yields

$$\begin{aligned} r^\wedge\nu \wedge \omega &\leq r^\wedge(\nu' + \varepsilon) \wedge \omega \\ &= (r^\wedge\nu' + \varepsilon) \wedge \omega \\ &\leq r^\wedge\nu' \wedge \omega + \varepsilon. \end{aligned}$$

Because $r^\wedge\nu' \wedge \omega \in \mathfrak{J}$ and ε was arbitrary we obtain $r^\wedge\nu \wedge \omega \in \mathfrak{J}$. □

Proposition 3.1.7.5.

$$(\mathfrak{J}, \mathfrak{K}) \in \check{\mathsf{I}} r \text{ if and only if } r^\wedge\mathfrak{K} \subseteq \mathfrak{J}.$$

Proof. Let $r : X \longrightarrow\!\!\!\!\!\!\!\mapsto Y$ and for $y \in Y$ let

$$\mathfrak{J}_y = \{\eta \in \mathsf{P}_b^X \mid \eta(x) = 0 \text{ whenever } (x, y) \in r\}$$

with $\mathfrak{J}_y = \mathsf{P}_b^X$ if no such x exists. Further let $j_r : Y \longrightarrow \mathsf{I} X$ be the map that sends y to $\mathfrak{J}_y$. For the Kleisli extension and functional ideals $\mathfrak{J}$ and $\mathfrak{K}$ we have

$$\mathfrak{J} \, \check{\mathsf{I}} r \, \mathfrak{K} \Leftrightarrow m_X(\mathsf{I} j_r \mathfrak{K}) \subseteq \mathfrak{J}.$$

In order to see that this condition is equivalent to $r^\wedge\mathfrak{K} \subseteq \mathfrak{J}$, observe that

$$m_X(\mathsf{I} j_r \mathfrak{K}) \subseteq \mathfrak{J} \Leftrightarrow (\forall \mu \in \mathsf{P}_b^X : l_\mu \cdot j_r \in \mathfrak{K} \Rightarrow \mu \in \mathfrak{J})$$

where

$$\begin{aligned} l_\mu \cdot j_r(y) &= \inf\left\{\alpha \mid \exists \eta \in \mathsf{P}_b^X, \eta(x) = 0 \text{ whenever } (x, y) \in r, \mu \leq \eta + \alpha\right\} \\ &= \sup_{(x,y) \in r} \mu(x) \\ &= r^\vee\mu(y). \end{aligned}$$

By the right adjointness of $r^\wedge$ we have $\mu \in r^\wedge\mathfrak{K} \Leftrightarrow r^\vee\mu \in \mathfrak{K}$ which implies

$$m_X(\mathsf{I}\, j_r\mathfrak{K}) \subseteq \mathfrak{I} \Leftrightarrow r^\wedge\mathfrak{K} \subseteq \mathfrak{I}.$$

□

Note that $\check{\mathsf{I}}$ is not a flat extension in the sense of (1.68), meaning that for a map $f : X \longrightarrow Y$, the relation $\check{\mathsf{I}}f$ may differ from $\mathsf{I}\,f$. For ideals $\mathfrak{I}$ and $\mathfrak{K}$ we have $(\mathfrak{I},\mathfrak{K}) \in \check{\mathsf{I}}f$ if and only if $\mathfrak{K} \subseteq \mathsf{I}\,f(\mathfrak{I})$. In particular if $\mathfrak{I} = \iota_X(\mathcal{F}) \oplus \alpha$ for some filter $\mathcal{F}$ on X then all functional ideals $\mathfrak{K}$ of type $\iota_Y(f(\mathcal{F})) \oplus \beta$, for some $\beta \leq \alpha$ satisfy $(\mathfrak{I},\mathfrak{K}) \in \check{\mathsf{I}}f$.

Now we are ready to show the direct correspondence between the axioms for functional ideal convergence $(X, \rightarrowtail)$ either as an $(\mathbb{I}, 2)$-Cat space, or as an object of App as described in Definition 1.1.1.23 and Theorem 1.1.1.24.

Theorem 3.1.7.6. *For a functional ideal convergence* $(X, \rightarrowtail)$ *the following conditions are equivalent:*

(i) $(X, \rightarrowtail)$ *belongs to* $(\mathbb{I}, 2)$-Cat, *meaning it satisfies* (3.16) *and* (3.15).

(ii) $(X, \rightarrowtail)$ *satisfies (F1), (F2w) and (F) from Theorem 1.1.1.24.*

(iii) $(X, \rightarrowtail)$ *satisfies (F1), (F2w), (F2) and (F).*

Proof. (i) ⇒ (ii) (3.16) implies (F1). Next we prove (F2w). Suppose $\mathfrak{I}$ and $\mathfrak{K}$ are given functional ideals on X with $\mathfrak{K} \subseteq \mathfrak{I}$. We show that there exists a functional ideal Φ on $\mathsf{I}\,X$ with $a^\wedge\mathfrak{K} \subseteq \Phi$ and $m_X(\Phi) = \mathfrak{I}$. Put $\Phi = \hat{\mathfrak{B}}$ with

$$\mathfrak{B} = \{\varphi \vee l_\mu \mid \varphi \in \mathsf{P}_b^{\mathsf{I}X}, \varphi \in a^\wedge\mathfrak{K}, \mu \in \mathfrak{I}\}.$$

Remark that the collection $\mathfrak{B}$ is an ideal basis on $\mathsf{I}\,X$ and that $m_X^*(\mathfrak{I}) \subseteq \Phi$, which already implies $\mathfrak{I} \subseteq m_X(\Phi)$. In order to show the other inclusion let $\nu \in m_X(\Phi)$ and $\varepsilon > 0$. There exist $\varphi_\varepsilon \in a^\wedge\mathfrak{K}$ bounded and $\mu_\varepsilon \in \mathfrak{I}$ such that

$$l_\nu \leq (\varphi_\varepsilon \vee l_{\mu_\varepsilon}) + \varepsilon.$$

Choose $\eta_\varepsilon \in \mathfrak{K}$ such that $\varphi_\varepsilon \leq a^\wedge\eta_\varepsilon$. Then for $x \in X$ we have

$$\begin{aligned} \nu(x) &= l_\nu\big(\iota_X(\dot{x})\big) \\ &\leq \Big(a^\wedge\eta_\varepsilon\big(\iota_X(\dot{x})\big) \vee \mu_\varepsilon(x)\Big) + \varepsilon \\ &\leq \inf_{\iota_X(\dot{x})\rightarrowtail z} \big(\eta_\varepsilon(z) \vee \mu_\varepsilon(x)\big) + \varepsilon \\ &\leq \big(\eta_\varepsilon(x) \vee \mu_\varepsilon(x)\big) + \varepsilon. \end{aligned}$$

Then we have $\Phi \check{\mathsf{I}} a\, \mathfrak{K}$ and $\mathfrak{K} \rightarrowtail x$ and (3.15) implies $m_X\Phi = \mathfrak{I} \rightarrowtail x$.

In order to prove condition (F), let $\psi : A \to X$ and $s : A \longrightarrow \mathsf{I}X$ a selection satisfying $s(z) \rightarrowtail \psi(z)$ for every $z \in A$ and let $\mathfrak{I}$ be a functional ideal on A such that $\mathsf{I}\,\psi(\mathfrak{I}) \rightarrowtail x$. Put $\mathfrak{K} = \mathsf{I}\,\psi(\mathfrak{I})$ and $\Phi = \mathsf{I}\,s(\mathfrak{I})$. We claim that $a^\wedge\mathfrak{K} \subseteq \Phi$. Indeed for φ bounded and $\mu \in \mathfrak{K}$ with $\varphi \leq a^\wedge\mu$ and $z \in A$ we have

$$\varphi \cdot s(z) \leq a^\wedge\mu\big(s(z)\big) = \inf_{s(z)\rightarrowtail y} \mu(y) \leq \mu\big(\psi(z)\big).$$

Since $\mu \cdot \psi \in \mathfrak{I}$ we have $\varphi \in \Phi$. It follows that $m_X(\Phi) \rightarrowtail x$.

(ii) $\Rightarrow$ (iii) This part of the proof can be found in [Low15].

(iii) $\Rightarrow$ (i) By definition we have $\iota_X(\dot{x}) \rightarrowtail x$ and so (3.16) holds. In order to prove (3.15) let $\mathfrak{K} \in \mathsf{I}\,X$ with $\mathfrak{K} \rightarrowtail x$ and $a^\wedge\mathfrak{K} \subseteq \Phi$ on $\mathsf{I}\,X$. Consider the selection $s : X \longrightarrow \mathsf{I}\,X$ defined by $s(z) = \bigcap_{\mathfrak{I}\rightarrowtail z} \mathfrak{I}$. We prove that

$$m_X\big(\mathsf{I}\,s(\mathfrak{K})\big) \subseteq m_X(a^\wedge\mathfrak{K}).$$

Applying (3.5), let $\nu \in m_X\big(\mathsf{I}\,s(\mathfrak{K})\big) = \bigcup_{\varphi\in\mathfrak{K}} \bigcap_{z\in X} s(z) \oplus \varphi(z)$. Choose $\varphi \in \mathfrak{K}$ and $(\varphi_z)_{z\in X}$ with $\varphi_z \in s(z)$ for every $z \in X$ such that

$$\nu = \inf_{z\in X} \big(\varphi_z + \varphi(z)\big).$$

Let $\mathfrak{I} \in \mathsf{I}\,X$. For every $z \in X$ with $\mathfrak{I} \rightarrowtail z$ we have $s(z) \subseteq \mathfrak{I}$ and therefore $l_\nu(\mathfrak{I}) \leq \varphi(z)$. Let $\omega = \sup l_\nu$, then it follows that $l_\nu(\mathfrak{I}) \leq \inf_{(\mathfrak{I},z)\in a} \varphi(z) \wedge \omega = (a^\wedge\varphi \wedge \omega)(\mathfrak{I})$. So we have shown that

$$l_\nu \leq a^\wedge\varphi \wedge \omega,$$

from which we conclude that $l_\nu \in a^\wedge\mathfrak{K}$. It follows that

$$m_X\big(\mathsf{I}\,s(\mathfrak{K})\big) \subseteq m_X(\Phi)$$

and applying (F2w) we have $m_X(\Phi) \rightarrowtail x$. □

3.2 Prime functional ideal convergence

In this section we consider prime functional ideals and their monad $\mathbb{B}$, which we introduce as a submonad of $\mathbb{I}$ through a monad morphism $\gamma : \mathbb{B} \longrightarrow \mathbb{I}$. We show that $\mathbb{B}$ is both sup-dense and interpolating in $\mathbb{I}$ in the sense of [HST14] and as a corollary we obtain an isomorphism between $(\mathbb{I}, 2)$-Cat and $(\mathbb{B}, 2)$-Cat. Since we already know that $(\mathbb{I}, 2)$-Cat is isomorphic to App, we obtain the results from [LV08] that the category $(\mathbb{B}, 2)$-Cat, of relational $\mathbb{B}$-algebras and structure preserving morphisms is isomorphic to App. We give an easy characterization of convergence of prime functional ideals in App.

3.2.1 Prime functional ideals and their monad

Prime functional ideals have been studied in [LVOV08] and [LV08].

Definition 3.2.1.1. A functional ideal $\mathfrak{J}$ on X is called *prime* if for all bounded functions μ, ν

$$\mu \wedge \nu \in \mathfrak{J} \Rightarrow \mu \in \mathfrak{J} \text{ or } \nu \in \mathfrak{J}.$$

The set of all prime functional ideals on X will be denoted as $\mathsf{B}\,X$. Note that $\mathsf{P}_b^X \in \mathsf{B}\,X$ which is called improper as before.

We have the following relation between prime functional ideals on a set X and ultrafilters on X. For a proof, we refer to Theorem 1.1.51 in [Low15].

Theorem 3.2.1.2. *A proper functional ideal $\mathfrak{J}$ on X is prime if and only if there exists an ultrafilter $\mathcal{U}$ on X such that*

$$\mathfrak{J} = \iota_X(\mathcal{U}) \oplus c(\mathfrak{J}). \tag{3.18}$$

Proposition 3.2.1.3. If $\mathfrak{J}$ is a prime functional ideal and $\mathfrak{H}$ is a finer functional ideal, then $\mathfrak{H}$ is prime too and moreover there exists $\alpha \geq 0$ such that $\mathfrak{H} = \mathfrak{J} \oplus \alpha$.

The collection of all prime functional ideals finer than $\mathfrak{J}$ is denoted by $\mathsf{B}(\mathfrak{J})$. If $\mathfrak{J}$ is a proper functional ideal then the set of all finer prime functional ideals has minimal elements. The collection of minimal prime functional ideals finer than $\mathfrak{J}$ is denoted by $\mathsf{B}_m(\mathfrak{J})$ and we have

$$\mathfrak{J} = \bigcap \{\mathfrak{H} \mid \mathfrak{H} \in \mathsf{B}_m(\mathfrak{J})\}. \tag{3.19}$$

The following proposition will be useful and can be found in [Low15] as Proposition 1.1.56.

Proposition 3.2.1.4. If $\mathfrak{J}$ is a proper functional ideal, and for each minimal prime functional ideal $\mathfrak{K} \in \mathsf{B}_m(\mathfrak{J})$ we have a function $\rho(\mathfrak{K}) \in \mathfrak{K}$ then for any $\alpha \in [c(\mathfrak{J}), \infty[$ there exists a finite set $\mathfrak{B}_\alpha \subseteq \mathsf{B}_m(\mathfrak{J})$ such that

$$\inf_{\mathfrak{K} \in \mathfrak{B}_\alpha} \rho(\mathfrak{K}) \in \mathfrak{J} \vee \iota_X\big(\mathfrak{f}_\alpha(\mathfrak{J})\big).$$

Proposition 3.2.1.5. If $\mathfrak{J}$ is a proper functional ideal, then there exists $\mathfrak{H} \in \mathsf{B}_m(\mathfrak{J})$ with $c(\mathfrak{H}) = c(\mathfrak{J})$.

Proof. Suppose for every $\mathfrak{H} \in \mathsf{B}_m(\mathfrak{J})$ we have $c(\mathfrak{H}) > c(\mathfrak{J})$. By application of Proposition 3.2.1.4 we can choose a finite number of these ideals $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ with

$$\beta = \min_{i=1,\cdots,n} c(\mathfrak{H}) \in \mathfrak{J} \vee \iota_X\big(\mathfrak{f}(\mathfrak{J})\big).$$

It means that for $\varepsilon > 0$ with $c(\mathfrak{I}) + 2\varepsilon < \beta$ there exist $\omega < \infty$ and $\nu \in \mathfrak{I}$ and $c(\mathfrak{I}) < \gamma < c(\mathfrak{I}) + \varepsilon$ and $\mu \in \mathfrak{I}$ with

$$\beta \leq (\mu + \theta^{\omega}_{\{\nu<\gamma\}}) + \varepsilon.$$

Since $\mu \vee \nu \in \mathfrak{I}$ there exists $x \in X$ with $\mu \vee \nu(x) < \gamma$. Since $\nu(x) < \gamma$ we would have

$$\beta \leq \big(\mu(x) \vee 0\big) + \varepsilon < \gamma + \varepsilon < \beta.$$

$\square$

For a map $f : X \longrightarrow Y$, if $\mathfrak{H}$ is a prime functional ideal on X, then clearly also $\{\mu \in \mathsf{P}_b^Y \mid \mu \cdot f \in \mathfrak{H}\}$ is prime, so the restriction of $\mathsf{I}\, f$ to $\mathsf{B}\, X$ becomes a map $\mathsf{B}\, f : \mathsf{B}\, X \longrightarrow \mathsf{B}\, Y$ sending $\mathfrak{H}$ to $\mathsf{B}\, f(\mathfrak{H}) = \{\mu \in \mathsf{P}_b^Y \mid \mu \cdot f \in \mathfrak{H}\}$. With these definitions

$$\mathsf{B} : \mathsf{Set} \longrightarrow \mathsf{Set}$$

sending X to $\mathsf{B}\, X$ and f to $\mathsf{B}\, f$ defines a functor on Set.

Next we consider the diagonal operation for prime functional ideals. For given X let $\gamma_X : \mathsf{B}\, X \longrightarrow \mathsf{I}\, X$ be the canonical injection. For $\Theta \in \mathsf{B}\,\mathsf{B}\, X$ we consider

$$\Theta_{\mathsf{I}\, X} = \{\varphi \in \mathsf{P}_b^{\mathsf{I}\, X} \mid \varphi \cdot \gamma_X \in \Theta\} \in \mathsf{I}\,\mathsf{I}\, X.$$

Further let

$$l^{\mathsf{B}} : \mathsf{P}_b^X \longrightarrow \mathsf{P}_b^{\mathsf{B}\, X} : \mu \mapsto l^{\mathsf{B}}_\mu = l_\mu \cdot \gamma_X$$

so

$$l^{\mathsf{B}}_\mu(\mathfrak{M}) = \inf\{\alpha \in \mathsf{P} \mid \mu \in \mathfrak{M} \oplus \gamma\}$$

whenever $\mathfrak{M} \in \mathsf{B}\, X$. Clearly l^{B} is well-defined. So we can put

$$n_X(\Theta) = \{\mu \in \mathsf{P}_b^X \mid l^{\mathsf{B}}_\mu \in \Theta\}.$$

Then $n_X(\Theta)$ is a prime functional ideal and we have $\gamma_X\big(n_X(\Theta)\big) = m_X(\Theta_{\mathsf{I}\, X})$.

The *prime functional ideal monad* $\mathbb{B} = (\mathsf{B}, n, e)$ on Set has multiplication and unit with components

$$n_X : \mathsf{B}\,\mathsf{B}\, X \longrightarrow \mathsf{B}\, X : \Theta \mapsto n_X(\Theta)$$

and

$$e_X : X \longrightarrow \mathsf{B}\, X : x \mapsto \iota_X(\dot{x})$$

and $\mathbb{B}$ is a submonad of $\mathbb{I}$ via the monad morphism $\gamma : \mathbb{B} \longrightarrow \mathbb{I}$ with components γ_X as defined above. The monad $\mathbb{B}$ is not power-enriched, as in general $\mathsf{B}\, X$ is not a complete lattice.

3.2.2 Relational $\mathbb{B}$-algebras

The extension to Rel of $\mathbb{B} = (\mathsf{B}, n, e)$ is defined as the *initial extension* induced by the Kleisli extension $\check{\mathbb{I}}$. So for any relation $r : X \rightarrow\!\!\!\!\!\!+ \, Y$, we have

$$\mathfrak{M}\, \check{\mathsf{B}}r\, \mathfrak{N} \Leftrightarrow \gamma_X(\mathfrak{M})\, \check{\mathsf{I}}r\, \gamma_Y(\mathfrak{N}) \Leftrightarrow r^{\wedge}\gamma_Y(\mathfrak{N}) \subseteq \gamma_X(\mathfrak{M})$$

for prime functional ideals $\mathfrak{M} \in \mathsf{B}\,X$ and $\mathfrak{N} \in \mathsf{B}\,Y$. It coincides with the extension that was considered in [LVOV08] and defines a lax extension of the monad $\mathbb{B} = (\mathsf{B}, n, e)$ to Rel.

The category $(\mathbb{B}, 2)$-Cat of relational $\mathbb{B}$-algebras, associated with this lax extension $\check{\mathbb{B}} = (\check{\mathsf{B}}, n, e)$ has as objects pairs (X, a) with X a set and $a : \mathsf{B}\,X \rightarrow\!\!\!\!\!\!+ \, X$ a relation satisfying transitivity

$$\Theta\, \check{\mathsf{B}}a\, \mathfrak{M} \,\&\, \mathfrak{M}\, a\, x \Rightarrow n_X(\Theta)\, a\, x \tag{3.20}$$

for any $\Theta \in \mathsf{B}\,\mathsf{B}\,X, \mathfrak{M} \in \mathsf{B}\,X$ and $x \in X$ and reflexivity

$$\iota_X(\dot{x})\, a\, x \tag{3.21}$$

for any $x \in X$. Morphisms $f : (X, a_X) \longrightarrow (Y, a_Y)$ satisfy

$$f \cdot a_X \leq a_Y \cdot \mathsf{B}\, f. \tag{3.22}$$

We recall some terminology from [HST14]. Consider a monad morphism $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$, where $\mathbb{T} = (T, m, e)$ is a power-enriched monad with structure $\tau : \mathbb{P} \longrightarrow \mathbb{T}$ and equipped with its Kleisli extension $\check{T}$, and $\mathbb{S} = (S, n, d)$ is a monad equipped with its initial extension $\hat{S}$ induced by α. The monad morphism is sup-dense if one has

$$\forall \mathcal{X} \in TX \exists \mathcal{A} \subseteq SX : \mathcal{X} = \bigvee \alpha_X(\mathcal{A}). \tag{3.23}$$

When $\mathbb{S}$ is a sub-monad of $\mathbb{T}$ and the embedding is sup-dense, we simply say that $\mathbb{S}$ is *sup-dense* in $\mathbb{T}$.

Proposition 3.2.2.1. The monad $\mathbb{B}$ is sup-dense in $\mathbb{I}$.

Proof. As we recalled earlier every functional ideal $\mathfrak{K}$ can be written as the intersection

$$\mathfrak{K} = \bigcap \{\mathfrak{H} \mid \mathfrak{H} \in \mathsf{B}(\mathfrak{K})\}$$

of all prime functional ideals finer than $\mathfrak{K}$. □

Next we show that the monad morphism $\gamma : \mathbb{B} \longrightarrow \mathbb{I}$ is *interpolating* in the sense of [HST14]. We recall the notations used in Proposition 3.1.7.5. For a relation $r : \mathsf{B}\, X \longrightarrow\!\!\!\!\!\!\mapsto X$ we have a map

$$j_r : X \longrightarrow \mathsf{I}\,\mathsf{B}\, X : y \mapsto \Phi_y$$

where

$$\Phi_y = \{\varphi \in \mathsf{P}_b^{\mathsf{B}\,X} \mid \varphi(\mathfrak{H}) = 0, \forall \mathfrak{H} \text{ with } \mathfrak{H}\, r\, y\} = r^{\wedge}\iota_X(\dot{y}).$$

Proposition 3.2.2.2. The monad morphism $\gamma : \mathbb{B} \longrightarrow \mathbb{I}$ is *interpolating*, that is for every relation $r : \mathsf{B}\, X \longrightarrow\!\!\!\!\!\!\mapsto X$ and $y \in X$ we have:

$$\mathfrak{H} \in \mathsf{B}\, X, m_X\big(r^{\wedge}\iota_X(\dot{y})\big) \subseteq \mathfrak{H} \Rightarrow \exists \Theta \in \mathsf{B}\,\mathsf{B}\, X, r^{\wedge}\iota_X(\dot{y}) \subseteq \Theta \text{ and } n_X(\Theta) \subseteq \mathfrak{H}.$$

Proof. Let $\mathfrak{H} \in \mathsf{B}\, X$ with $m_X\big(r^{\wedge}\iota_X(\dot{y})\big) \subseteq \mathfrak{H}$.

We claim that with $C_y = \{\mathfrak{K} \in \mathsf{B}\, X \mid \mathfrak{K}\, r\, y\}$, there exists an ultrafilter $\mathcal{W}$ on $\mathsf{B}\, X$, with $C_y \in \mathcal{W}$ and such that

$$\bigcup_{W \in \mathcal{W}} \bigcap_{\mathfrak{I} \in \mathsf{B}\, X, \mathfrak{I} \in W} \mathfrak{I} \subseteq \mathfrak{H}.$$

Suppose on the contrary that for every ultrafilter $\mathcal{W}$ containing C_y there is a set $W \in \mathcal{W}$ with $\bigcap_{\mathfrak{I} \in \mathsf{B}\, X, \mathfrak{I} \in X} \mathfrak{I} \nsubseteq \mathfrak{H}$. Choose a finite subcollection $(\mathcal{W}_i)_{i \in J}$, of these ultrafilters and corresponding sets $W_i \in \mathcal{W}_i$ such that $C_y \subseteq \bigcup_{i \in J} W_i$ and further for every $i \in J$ some $\mu_i \in \bigcap_{\mathfrak{I} \in W_i} \mathfrak{I}$ with $\mu_i \notin \mathfrak{H}$. Put $\mu = \bigwedge_{i \in J} \mu_i$ then we have $\mu \notin \mathfrak{H}$ since this is a prime functional ideal. On the other hand we have

$$\mu \in \bigcap_{\bigcup W_i} \mathfrak{I} \subseteq \bigcap_{C_y} \mathfrak{I} \subseteq \{\mu \in \mathsf{P}_b^X \mid l_{\mu_{|C_y}} = 0\} = m_X\big(r^{\wedge}\iota_X(\dot{y})\big) \subseteq \mathfrak{H}.$$

A contradiction follows and so we are done with the proof of our claim.

Now let $\Theta = \iota_{\mathsf{B}\, X}(\mathcal{W})$. For every $\varphi \in \mathsf{P}_b^{\mathsf{B}\, X}$ with $\varphi_{|C_y} = 0$ we have $\varphi \leq \theta^{\alpha}_{C_y}$ for $\alpha = \sup \varphi$. So we have $r^{\wedge}\iota_X(\dot{y}) \subseteq \iota_{\mathsf{B}\, X}(\mathcal{W})$. Moreover by (3.3)

$$n_X(\Theta) = \bigvee_{\mathcal{A} \in \mathcal{W}} \bigcap_{\mathfrak{I} \in \mathcal{A}} \mathfrak{I} \subseteq \mathfrak{H}.$$

$\square$

Using both sup-density and the interpolating property, the results in [HST14] imply that $(\mathbb{B}, 2)$-Cat and $(\mathbb{I}, 2)$-Cat are isomorphic and combining with the results in Section 3.1, it follows that $(\mathbb{B}, 2)$-Cat and App are isomorphic too, which was already established in [LV08].

Theorem 3.2.2.3. $(\mathbb{B}, 2)$-Cat *and* $(\mathbb{I}, 2)$-Cat *are isomorphic categories and both provide an isomorphic description of* App.

3.2.3 Convergence for prime functional ideals

In what follows we will be writing

$$\mathfrak{K} \rightsquigarrow x \text{ instead of } \mathfrak{K}\, a\, x$$

for the convergence of prime functional ideals satisfying (3.20) and (3.21) and one obtains a restriction of functional ideal convergence to prime functional ideals describing convergence in App.

As in (3.22) morphisms $f : (X, \rightsquigarrow_X) \longrightarrow (Y, \rightsquigarrow_Y)$ can be equivalently described as preserving convergence in the sense that

$$\mathfrak{K} \rightsquigarrow_X x \Rightarrow \mathsf{B}\, f(\mathfrak{K}) \rightsquigarrow_Y f(x),$$

for all $\mathfrak{K} \in \mathsf{B}\, X$ and $x \in X$.

Next we prove that the objects in App can be described by means of an equivalent set of axioms on convergence of prime functional ideals. Consider the following conditions:

Definition 3.2.3.1. (B1) For every $x \in X : \iota_X(\dot{x}) \rightsquigarrow x$.

(B1*) For every $x \in X$ and $\alpha < \infty$: $\iota_X(\dot{x}) \oplus \alpha \rightsquigarrow x$.

(B2) For any two prime functional ideals with $\mathfrak{M} \subseteq \mathfrak{N}$ and $x \in X$

$$\mathfrak{M} \rightsquigarrow x \Rightarrow \mathfrak{N} \rightsquigarrow x.$$

(B) For any set A, for any $\psi : A \longrightarrow X$, for any $s : A \longrightarrow \mathsf{B}\, X$, for any $\mathfrak{M} \in \mathsf{B}(A)$ and for any $x \in X$

$$s(a) \rightsquigarrow \psi(a) \text{ whenever } a \in A \,\&\, \mathsf{B}\, \psi(\mathfrak{M}) \rightsquigarrow x \Rightarrow n_X\big(\mathsf{B}\, s(\mathfrak{M})\big) \rightsquigarrow x.$$

(B*) For any set A, for any $\psi : A \longrightarrow X$, for any $s : A \longrightarrow \mathsf{B}\, X$, for any $\mathfrak{M} \in \mathsf{B}(A)$ and for any $x \in X$ and $\alpha < \infty$

$$s(a) \rightsquigarrow \psi(a) \text{ whenever } a \in A \,\&\, \mathsf{B}\, \psi(\mathfrak{M}) \rightsquigarrow x \Rightarrow n_X\big(\mathsf{B}\, s(\mathfrak{M})\big) \oplus \alpha \rightsquigarrow x.$$

Proposition 3.2.3.2. For a relation $\rightsquigarrow \subseteq \mathsf{B}\, X \times X$ the following are equivalent:

(B1) + (B*) ⇔ (B1*) + (B) ⇔ (B1) + (B) + (B2) ⇔ (B1*) + (B*).

Proof. We prove the non-trivial implications. Suppose $\rightsquigarrow \subseteq \mathsf{B}\, X \times X$ satisfies (B1) + (B*). In order to prove (B1*) let $x \in X$ and $\alpha < \infty$ and consider $A = X, \psi : X \longrightarrow X : z \mapsto z$, $\mathfrak{M} = \iota_X(\dot{x})$ and the selection $s(z) = \iota_X(\dot{z})$ for $z \in X$. Then by (3.5) $n_X\big(\mathsf{B}\, s(\mathfrak{M})\big) \oplus \alpha = \iota_X(\dot{x}) \oplus \alpha$ and hence the conclusion follows.

Next we show that (B1*) + (B) implies (B2). Let $\mathfrak{K}, \mathfrak{I} \in \mathsf{B}\,X$ so that $\mathfrak{K} \subseteq \mathfrak{I}$. Then from Proposition 3.2.1.3 we know there exists $\alpha \geq 0$ such that $\mathfrak{I} = \mathfrak{K} \oplus \alpha$. Put $\psi : X \longrightarrow X : z \mapsto z$, $s : X \longrightarrow \mathsf{B}\,X : z \mapsto \iota_X(\dot{z}) \oplus \alpha$, then it follows that

$$\begin{aligned} n_X \,\mathsf{B}\, s(\mathfrak{K}) &= \{\mu \mid l_\mu \cdot s \in \mathfrak{K}\} \\ &= \{\mu \mid (\mu - \alpha) \vee 0 \in \mathfrak{K}\} \\ &= \mathfrak{K} \oplus \alpha, \end{aligned}$$

and hence, since $s(z) \rightsquigarrow z$ for each $z \in X$, we obtain $\mathfrak{K} \rightsquigarrow x$ implies $\mathfrak{K} \oplus \alpha \rightsquigarrow x$. □

Proposition 3.2.3.3. For a relation $a : \mathsf{B}\,X \longrightarrow\!\!\!\!\!\!\!\mapsto X$ satisfying (B1) and a functional ideal $\mathfrak{K}$ on X we have $c(\mathfrak{K}) = c(a^\wedge \mathfrak{K})$ and in particular $a^\wedge \mathfrak{K}$ is improper if and only if $\mathfrak{K}$ is improper.

Proof. We prove that $\mathfrak{K}$ and $a^\wedge \mathfrak{K}$ contain the same constants. Let $\alpha \in \mathfrak{K}$, then clearly $\alpha \leq a^\wedge \alpha$ and therefore $\alpha \in a^\wedge \mathfrak{K}$. Conversely, let $\alpha \in a^\wedge \mathfrak{K}$ and choose some $\nu \in \mathfrak{K}$ such that $\alpha \leq a^\wedge \nu$. For $z \in X$ arbitrary we have

$$\alpha \leq a^\wedge \nu\big(\iota_X(\dot{z})\big) = \inf_{\iota_X(\dot{z}) \rightsquigarrow y} \nu(y) \leq \nu(z).$$

We can conclude that $\alpha \leq \nu$, so we have $\alpha \in \mathfrak{K}$. □

Theorem 3.2.3.4. *For $(X, \rightsquigarrow)$ a convergence of prime functional ideals, the following properties are equivalent:*

(i) $(X, \rightsquigarrow)$ *satisfies* (3.20) *and* (3.21).

(ii) $(X, \rightsquigarrow)$ *satisfies (B1) and (B*).*

(iii) $(X, \rightsquigarrow)$ *satisfies (B1), (B2) and (B).*

Proof. (i) ⇒ (ii) Clearly (3.21) implies condition (B1). Next we have to check condition (B*). Let A be a set and $\psi : A \to X$, $s : A \to \mathsf{B}\,X$, $\mathfrak{L} \in \mathsf{B}(A)$ and $x \in X$. Suppose $s(z) \rightsquigarrow \psi(z)$, $\forall z \in A$ and $\mathsf{B}\,\psi(\mathfrak{L}) \rightsquigarrow x$. Then put $\mathsf{B}\,s(\mathfrak{L}) \oplus \alpha = \Theta \in \mathsf{B}\,\mathsf{B}\,X$ for $\alpha < \infty$.

For $\nu \in a^\wedge \,\mathsf{B}\,\psi(\mathfrak{L})$, let μ be bounded on A such that $\mu \cdot \psi \in \mathfrak{L}$ and $\nu \leq a^\wedge \mu$. Then it is easily seen that $\nu \cdot s \leq \mu \cdot \psi$ on A. So we may conclude that $\nu \cdot s \in \mathfrak{L}$ and $\nu \in \mathsf{B}\,s(\mathfrak{L}) \oplus \alpha$. It follows that $a^\wedge \,\mathsf{B}\,\psi(\mathfrak{L}) \subseteq \Theta$ and therefore $n_X(\Theta) \rightsquigarrow x$. So we can conclude that $n_X(\mathsf{B}\,s(\mathfrak{L})) \oplus \alpha \rightsquigarrow x$.

(ii) ⇒ (iii) follows from Proposition 3.2.3.2.

(iii) ⇒ (i) (3.21) follows at once from (B1). Let $x \in X$, $\Theta \in \mathsf{B}\,\mathsf{B}\,X$ and $\mathfrak{K} \in \mathsf{B}\,X$ with $\mathfrak{K} \rightsquigarrow x$ and $a^\wedge \mathfrak{K} \subseteq \Theta$. We may assume that $n_X(\Theta)$ is proper, so Θ

is proper, and in view of Proposition 3.2.3.3 $\mathfrak{K}$ is proper too. By Theorem 3.2.1.2 there exist $\mathcal{U}, \mathscr{U}, \gamma$ and δ such that $\mathfrak{K} = \iota_X(\mathcal{U}) \oplus \gamma$ and $\Theta = \iota_{\mathsf{B}\,X}(\mathscr{U}) \oplus \delta$, where both γ and δ are finite.
As a first step in the proof we assume that $\delta = \gamma$. Let

$$A = \{(z, \mathfrak{I}) \mid \mathfrak{I} \rightsquigarrow z, \mathfrak{I} \in \mathsf{B}\,X, z \in X\}.$$

We show that $\mathcal{U} \times \mathscr{U}$ has a trace on A. For $U \in \mathcal{U}$ and $\mathcal{A} \in \mathscr{U}$ and $\gamma < \omega < \infty$ the function

$$\varphi = (a^\wedge(\theta_U^\omega + \gamma) \wedge \omega) \vee (\theta_{\mathcal{A}}^\omega + \gamma)$$

belongs to Θ. We evaluate φ in an arbitrary $\mathfrak{L} \in \mathsf{B}\,X$.
Either $\mathfrak{L} \notin \mathcal{A}$ and then the second term equals $\omega + \gamma \geq \omega$, or $\mathfrak{L} \in \mathcal{A}$. In case $\mathfrak{L}$ diverges the first term equals ω. In case $\mathfrak{L}$ does converge, but never to a point of U, the first term equals $\omega + \gamma \geq \omega$. Since φ cannot be larger than ω in every $\mathfrak{L}$, we can conclude that $\exists \mathfrak{L}_0 \in \mathcal{A}$ that converges to some $z_0 \in U$. Then $(z_0, \mathfrak{L}_0) \in A \cap (U \times \mathcal{A})$.

Let $\mathcal{W}$ be an ultrafilter on A finer than the trace of $\mathcal{U} \times \mathscr{U}$ on A, $\psi : A \to X$ the restriction of the first projection, and $s : A \to \mathsf{B}\,X$ the restriction of the second projection. Then we have $\psi(\mathcal{W}) = \mathcal{U}$ and $s(\mathcal{W}) = \mathscr{U}$ and $s(z) \rightsquigarrow \psi(z), \forall z \in A$ by construction. With $\mathfrak{I} = \iota_A(\mathcal{W}) \oplus \gamma$ we now have $\Theta = \mathsf{B}\,s(\mathfrak{I})$ and $\mathfrak{K} = \mathsf{B}\,\psi(\mathfrak{I})$. By (iii) we can conclude that $n_X(\Theta) \rightsquigarrow x$.

As a second step we now assume that $\delta \neq \gamma$. In view of Proposition 3.2.3.3 we have $\gamma \leq \delta$. Let $\mathfrak{K}' = \iota_X(\mathcal{U}) \oplus \delta$ we show that $a^\wedge \mathfrak{K}' \subseteq \Theta$.

To see this let $U \in \mathcal{U}, \omega > \delta, \eta > \omega + \gamma, \eta > \omega + \delta$ and let $0 < \varepsilon < \gamma$. Choose $\mathcal{A} \in \mathscr{U}$ and $\omega' \geq \omega$ such that for $\varphi_\gamma = a^\wedge(\theta_U^\omega + \gamma) \wedge \eta$ one has

$$\varphi_\gamma \leq \theta_{\mathcal{A}}^{\omega'} + \delta + \varepsilon.$$

We claim that also $\varphi_\delta = a^\wedge(\theta_U^\omega + \delta) \wedge \eta$ satisfies

$$\varphi_\delta \leq \theta_{\mathcal{A}}^{\omega'} + \delta + \varepsilon.$$

To evaluate both sides of the inequality in a prime functional ideal $\mathfrak{L}$, observe that the right-hand side takes values $\delta + \varepsilon$ (when $\mathfrak{L} \in \mathcal{A}$) and $\omega' + \delta + \varepsilon$ (when $\mathfrak{L} \notin \mathcal{A}$). So the inequalities in case where $\mathfrak{L}$ diverges or converges to at least one point of U are trivially fulfilled. In case $\mathfrak{L}$ converges but never to a point of U, $\mathfrak{L} \in \mathcal{A}$ would imply $\varphi_\gamma(\mathfrak{L}) = \omega + \gamma \leq \delta + \varepsilon$ which is impossible. So we may assume $\mathfrak{L} \notin \mathcal{A}$ and then $\varphi_\delta(\mathfrak{L}) \leq \delta + \varepsilon$.

By (B2) we have $\mathfrak{K}' \rightsquigarrow x$ and by application of the first step we can conclude that $n_X(\Theta) \rightsquigarrow x$.

□

Chapter 4

Topological properties in App and NA-App

In this chapter we look at $(\mathbb{T}, \mathcal{V})$-categories as spaces. When considering a $(\mathbb{T}, \mathcal{V})$-category (X, a) and denoting $a : TX \to\!\!\!\!\!\mapsto X$ by $\to \subseteq TX \times X$, it gives us a notion of convergence on X. Inspired by the topological properties in Top and the fact that Top $\cong (\beta, 2)$-Cat, topological properties were introduced in arbitrary $(\mathbb{T}, \mathcal{V})$-categories [HST14].

We start this chapter by recalling the definitions of topological properties in $(\mathbb{T}, \mathcal{V})$-Cat, as can be found in Chapter V of [HST14]. We also recall the known results for these properties in Top $\cong (\beta, 2)$-Cat and App $\cong (\beta, \mathsf{P}_+)$-Cat.

In Section 4.2, we apply these definitions to the lax algebraic representation of the category of non-Archimedean approach spaces, NA-App $\cong (\beta, \mathsf{P}_\vee)$-Cat, as introduced in Chapter 2.

In Section 4.3, we apply this theory to relational algebras and in particular to the relational representations of App as $(\mathbb{I}, 2)$-Cat and $(\mathbb{B}, 2)$-Cat as developed in Chapter 3.

4.1 Basic concepts

4.1.1 Hausdorff separation and lower separation axioms

Definition 4.1.1.1. A $(\mathbb{T}, \mathcal{V})$-space (X, a) is $(\mathbb{T}, \mathcal{V})$-*Hausdorff* if

$$a \cdot a^\circ \leq 1_X. \tag{4.1}$$

The following characterization of $(\mathbb{T}, \mathcal{V})$-Hausdorff can be found as Proposition V.1.1.2 in [HST14].

Proposition 4.1.1.2. A $(\mathbb{T},\mathcal{V})$-space (X,a) is $(\mathbb{T},\mathcal{V})$-Hausdorff if and only if, for all $x,y\in X$ and $\mathcal{Z}\in TX$,

$$\bot< a(\mathcal{Z},x)\otimes a(\mathcal{Z},y)\Rightarrow x=y \quad\text{and}\quad a(\mathcal{Z},x)\otimes a(\mathcal{Z},y)\leq k. \tag{4.2}$$

Examples 4.1.1.3.

1. In $\mathsf{Top}\cong(\beta,2)\text{-}\mathsf{Cat}$ the notion of $(\beta,2)$-Hausdorff coincides with the usual Hausdorff separation property. For a relational β-algebra $(X,\to)$ the condition becomes

$$\forall x,y\in X,\forall\mathcal{Z}\in\beta X:(\mathcal{Z}\to x\,\&\,\mathcal{Z}\to y\Rightarrow x=y). \tag{4.3}$$

We conclude

$$(\beta,2)\text{-}\mathsf{Cat}_{(\beta,2)-\mathsf{Haus}}\cong\mathsf{Haus}.$$

2. For $\mathsf{App}\cong(\beta,\mathsf{P}_+)\text{-}\mathsf{Cat}$ we have seen in Theorem 1.3.7.2 that there is a one-to-one correspondence between limit operators on X and reflexive and transitive (β,P_+)-relations $a:\beta X\to\!\!\!\!\!\mapsto X$. An approach space (X,λ) is (β,P_+)-Hausdorff if and only if

$$\forall x,y\in X,\forall\mathcal{Z}\in\beta X:\lambda\mathcal{Z}(x)<\infty\,\&\,\lambda\mathcal{Z}(y)<\infty\Rightarrow x=y. \tag{4.4}$$

This notion is clearly stronger than the classical notion of Hausdorff separation in approach theory [LS03], where an approach space is called Hausdorff if its topological coreflection $\mathrm{C}_{\mathsf{Top}}X$ is Hausdorff, i.e.

$$\forall x,y\in X,\forall\mathcal{Z}\in\beta X:\lambda\mathcal{Z}(x)=0\,\&\,\lambda\mathcal{Z}(y)=0\Rightarrow x=y. \tag{4.5}$$

For a $(\mathbb{T},\mathcal{V})$-space, we now consider an array of low separation and symmetry conditions, with the terminology borrowed from the role model $\mathsf{Top}\cong(\beta,2)\text{-}\mathsf{Cat}$.

Definition 4.1.1.4. Let (X,a) be a $(\mathbb{T},\mathcal{V})$-space.

1. (X,a) is $(\mathbb{T},\mathcal{V})$-*T_0* if

$$(a\cdot e_X)\wedge(a\cdot e_X)^\circ\leq 1_X. \tag{4.6}$$

2. (X,a) is $(\mathbb{T},\mathcal{V})$-*T_1* if

$$a\cdot e_X\leq 1_X. \tag{4.7}$$

3. (X,a) is $(\mathbb{T},\mathcal{V})$-*R_0* if

$$(a\cdot e_X)^\circ\leq a\cdot e_X. \tag{4.8}$$

4. (X,a) is $(\mathbb{T},\mathcal{V})$-*R_1* if

$$a\cdot a^\circ\leq a\cdot e_X. \tag{4.9}$$

We recall Proposition V.2.2.1 for [HST14].

Proposition 4.1.1.5. The following implications hold for a $(\mathbb{T},\mathcal{V})$-space (X,a):

$$\begin{array}{ccc} (\mathbb{T},\mathcal{V})\text{-Hausdorff} & \Longleftrightarrow & (\mathbb{T},\mathcal{V})\text{-T}_1 \;\&\; (\mathbb{T},\mathcal{V})\text{-R}_1 \\ \Downarrow & & \Downarrow \qquad\qquad \Downarrow \\ (\mathbb{T},\mathcal{V})\text{-T}_1 & \Longleftrightarrow & (\mathbb{T},\mathcal{V})\text{-T}_0 \;\&\; (\mathbb{T},\mathcal{V})\text{-R}_0 \end{array}$$

Examples 4.1.1.6. 1. In Top $\cong$ $(\beta,2)$-Cat all conditions coincide with the usual lower separation and symmetry axioms in Top.

We recall that a topological space X is R_0 if

$$\forall x,y\in X : x\in \mathrm{cl}\{y\} \Leftrightarrow y\in \mathrm{cl}\{x\}.$$

A topological space X is R_1 if

$$\forall x,y\in X : \mathcal{V}(x)\vee\mathcal{V}(y) \text{ is a proper filter on X} \Rightarrow \mathcal{V}(x)=\mathcal{V}(y).$$

2. In App $\cong$ (β,P_+)-Cat one has the following characterizations:

$$(X,\lambda) \text{ is } (\beta,\mathsf{P}_+)\text{-T}_0 \Leftrightarrow \forall x,y\in X : \lambda\dot{x}(y)<\infty \,\&\, \lambda\dot{y}(x)<\infty \Rightarrow x=y, \tag{4.10}$$

$$(X,\lambda) \text{ is } (\beta,\mathsf{P}_+)\text{-T}_1 \Leftrightarrow \forall x,y\in X : \lambda\dot{x}(y)<\infty \Rightarrow x=y; \tag{4.11}$$

$$(X,\lambda) \text{ is } (\beta,\mathsf{P}_+)\text{-R}_0 \Leftrightarrow \forall x,y\in X : \lambda\dot{x}(y)=\lambda\dot{y}(x); \tag{4.12}$$

$$(X,\lambda) \text{ is } (\beta,\mathsf{P}_+)\text{-R}_1 \Leftrightarrow \forall x,y\in X, \forall\mathcal{Z}\in\beta X : \lambda\dot{x}(y)\le\lambda\mathcal{Z}(x)+\lambda\mathcal{Z}(y). \tag{4.13}$$

The following definitions of separation properties in App were introduced by Lowen and Sioen in [LS03].

$$(X,\lambda) \text{ is } \mathrm{T}_0 \Leftrightarrow \forall x,y\in X : \lambda\dot{x}(y)=0 \,\&\, \lambda\dot{y}(x)=0 \Rightarrow x=y; \tag{4.14}$$

$$(X,\lambda) \text{ is } \mathrm{T}_1 \Leftrightarrow \forall x,y\in X : \lambda\dot{x}(y)=0 \Rightarrow x=y. \tag{4.15}$$

For now we omit the notions R_0 and R_1 (the last one called R in [LS03]) in App, but we will get back to these in 4.3.3 in more detail.

It is clear that (β, P_+)-T_0 ((β, P_+)-T_1 respectively) implies T_0 in App (T_1 in App respectively) and that X is T_0 in App (T_1 respectively) if and only if the topological coreflection $\mathrm{C}_{\mathsf{Top}} X$ is T_0 (T_1 respectively) in Top.

The following counterexample shows that the converse does not hold. Consider $X = (\{x, y\}, d)$ where d is the metric structure $d(x, y) = d(y, x) = \alpha$, for $0 < \alpha < \infty$, and $d(x, x) = d(y, y) = 0$. Then X is T_1 (and hence T_0) since $\lambda\dot{a}(b) = 0$ if and only if $d(b, a) = 0$, for $a, b \in X$. However, X is not (β, P_+)-T_0 (and hence not (β, P_+)-T_1), since $\lambda\dot{x}(y) = d(y, x) = \alpha < \infty$ and $\lambda\dot{y}(x) = \alpha < \infty$, but $x \neq y$.

4.1.2 Compactness

Definition 4.1.2.1. A $(\mathbb{T}, \mathcal{V})$-space (X, a) is $(\mathbb{T}, \mathcal{V})$-*compact* if

$$1_{TX} \leq a^\circ \cdot a. \tag{4.16}$$

The following characterization can be found as Proposition V.1.1.2 in [HST14].

Proposition 4.1.2.2. A $(\mathbb{T}, \mathcal{V})$-space (X, a) is $(\mathbb{T}, \mathcal{V})$-compact if and only if, for all $\mathcal{X} \in TX$,

$$k \leq \bigvee_{z \in X} a(\mathcal{X}, z) \otimes a(\mathcal{X}, z). \tag{4.17}$$

Examples 4.1.2.3.

1. In Top $\cong (\beta, 2)$-Cat the notion of $(\beta, 2)$-compactness coincides with the usual compactness property. For a relational β-algebra $(X, \to)$ the condition becomes

$$\forall \mathcal{X} \in \beta X \exists x \in X : \mathcal{X} \to x. \tag{4.18}$$

We conclude

$$(\beta, 2)\text{-}\mathsf{Cat}_{(\beta,2)-\mathsf{Comp}} \cong \mathsf{Comp}.$$

2. For App $\cong (\beta, \mathsf{P}_+)$-Cat, an approach space (X, λ) is (β, P_+)-compact if and only if

$$\inf_{x \in X} \lambda\mathcal{X}(x) = 0, \tag{4.19}$$

for every $\mathcal{X} \in \beta X$. This property is called *0-compact* in [Low15].

4.1.3 Regularity

Regularity conditions for topological and approach spaces have been obtained under some known lax-algebraic description involving a Set monad $\mathbb{T}$ and a quantale $\mathcal{V}$ using a general lax-algebraic notion of regularity that goes back to Möbus

[M¨81]. Situations where the lax-algebraic notion of regularity coincides with the usual notion of regularity for topological and approach spaces, can be found in Chapter V of [HST14].

We assume that $\mathcal{V}$ is commutative.

In order to formulate regularity for a $(\mathbb{T},\mathcal{V})$-space (X,a), we use the $\mathcal{V}$-relation $\hat{a}: TX \rightarrow\!\!\!\!\!\mapsto TX$, where

$$\hat{a} := \widehat{T}a \cdot m_X^\circ. \tag{4.20}$$

While the inequality $a \cdot \hat{a} \leq a$ is exactly the transitivity condition for a, the condition $a \cdot \hat{a}^\circ \leq a$ encodes regularity for (X,a).

Definition 4.1.3.1. A $(\mathbb{T},\mathcal{V})$-space (X,a) is $(\mathbb{T},\mathcal{V})$-*regular* if

$$a \cdot \hat{a}^\circ \leq a, \tag{4.21}$$

that is $a \cdot m_X \cdot (\widehat{T}a)^\circ \leq a$, or, in pointwise form,

$$\widehat{T}a(\mathfrak{X},\mathcal{X}) \otimes a\big(m_X(\mathfrak{X}),z\big) \leq a(\mathcal{X},z), \tag{4.22}$$

for all $\mathfrak{X} \in TTX, \mathcal{X} \in TX$ and $z \in X$.

Examples 4.1.3.2.

1. If $\mathbb{T} = \mathbb{1}$ is the identity monad identically extended to $\mathcal{V}$-Rel, then $a \cdot \hat{a}^\circ = a \cdot a^\circ$, so that $a \cdot a^\circ \leq a$ if and only if $a = a^\circ$. Hence, for $\mathcal{V}$-spaces regularity means symmetry.

2. A topological space considered as a $(\beta, 2)$-space (X,a) is $(\beta, 2)$-regular if and only if it is regular in the usual sense, meaning that for any $x \in X$ and any $A \subseteq X$ closed with $x \notin A$, there exist open sets $U, V \subseteq X$ with $x \in U, A \subseteq V$ and $U \cap V = \emptyset$.

3. An approach space (X,λ), considered as a (β, P_+)-space, is (β, P_+)-regular if and only if

$$\lambda\mathcal{U}(z) \leq \lambda\Sigma\mathfrak{X}(z) + \sup_{\mathcal{A}\in\mathfrak{X}, B\in\mathcal{U}} \inf_{\mathcal{W}\in\mathcal{A}, b\in B} \lambda\mathcal{W}(b), \tag{4.23}$$

for all $\mathfrak{X} \in \beta^2 X, \mathcal{U} \in \beta X$ and $z \in X$. This condition coincides with the classical notion of regularity in App, as introduced by Robeys [Rob92].

We recall that for an approach space (X,λ) the following are equivalent and for a proof we refer to [Rob92].

(i) For $\mathcal{F} \in \mathsf{F}X$, for $\gamma \in [0,\infty]$, and for $x \in X$ we have

$$\lambda\mathcal{F}^{(\gamma)}(x) \leq \lambda\mathcal{F}(x) + \gamma, \tag{4.24}$$

(ii) For $\mathcal{W},\mathcal{U} \in \beta X$, and for $\gamma \in [0,\infty]$ we have

$$\mathcal{W}^{(\gamma)} \subseteq \mathcal{U} \Rightarrow \lambda\mathcal{U}(x) \leq \lambda\mathcal{W}(x) + \gamma. \tag{4.25}$$

An approach space is called *regular* if it satisfies one and hence both conditions. Brock and Kent [BK98] have shown that regularity of an approach space is equivalent to the following conditions in terms of selections, one is a version based on filters, the other uses only ultrafilters. These conditions were also used in [CMT14] were the role of regularity was investigated in the context of contractive extensions.

(iii) If A is a set, $\psi : A \longrightarrow X, \sigma : A \longrightarrow \mathsf{F}_p X$, and $\mathcal{G} \in \mathsf{F}_p A$, then

$$\lambda\psi(\mathcal{G}) \leq \lambda\Sigma\sigma(\mathcal{G}) + \sup_{y\in A} \lambda\sigma(y)\big(\psi(y)\big). \tag{4.26}$$

(iv) If A is a set, $\psi : A \longrightarrow X, \sigma : A \longrightarrow \beta X$, and $\mathcal{U} \in \beta A$, then

$$\lambda\psi(\mathcal{U}) \leq \lambda\Sigma\sigma(\mathcal{U}) + \sup_{y\in A} \lambda\sigma(y)\big(\psi(y)\big). \tag{4.27}$$

In Top we know that regularity implies R_1. In Exercise 2.D in Chapter V.2 in [HST14] it is shown that every $(\mathbb{T},\mathcal{V})$-regular $(\mathbb{T},\mathcal{V})$-space is $(\mathbb{T},\mathcal{V})$-R_{0+}. Such a $(\mathbb{T},\mathcal{V})$-space (X,a) is $(\mathbb{T},\mathcal{V})$-R_{0+} if

$$\hat{a} \cdot e_X \leq a^\circ, \tag{4.28}$$

this condition being stronger than $(\mathbb{T},\mathcal{V})$-R_0.

The following proposition shows that for (β,P_+)-regularity and (β,P_+)-R_1 the same implication holds as in the classical case in the category Top.

Proposition 4.1.3.3. An approach space X is (β,P_+)-R_1 if it is (β,P_+)-regular.

Proof. Take $\mathcal{U} \in \beta X$ and $x,y \in X$ arbitrary. We can assume that $\lambda\mathcal{U}(x) < \infty$ and $\lambda\mathcal{U}(y) < \infty$. Suppose $\lambda\mathcal{U}(x) < \alpha$ and $\lambda\mathcal{U}(y) < \beta$. If $\lambda\mathcal{U}(x) < \alpha$, then for all $A \in \mathcal{U}$ we have $\delta(x,A) < \alpha$, hence $x \in A^{(\alpha)}$ and thus $A^{(\alpha)} \in \dot{x}$. This shows that $\mathcal{U}^{(\alpha)} \subseteq \dot{x}$.

Then

$$\begin{aligned} \lambda\dot{x}(y) &\leq \lambda\mathcal{U}^{(\alpha)}(y) \\ &\leq \lambda\mathcal{U}(y) + \alpha \\ &< \alpha + \beta, \end{aligned}$$

which shows that X is (β,P_+)-R_1. □

4.1.4 Normality

Throughout this subsection we assume that $\mathcal{V}$ is commutative and that $\widehat{\mathbb{T}}$ is associative.

Definition 4.1.4.1. A $(\mathbb{T},\mathcal{V})$-space (X,a) is $(\mathbb{T},\mathcal{V})$-*normal* if

$$\hat{a}\cdot\hat{a}^\circ \leq \hat{a}^\circ\cdot\hat{a}, \tag{4.29}$$

or, in pointwise form, if

$$\hat{a}(\mathcal{Z},\mathcal{X})\otimes\hat{a}(\mathcal{Z},\mathcal{Y}) \leq \bigvee_{\mathcal{W}\in TX} \big(\hat{a}(\mathcal{X},\mathcal{W})\otimes\hat{a}(\mathcal{Y},\mathcal{W})\big), \tag{4.30}$$

for all $\mathcal{X},\mathcal{Y},\mathcal{Z}\in TX$.

Examples 4.1.4.2.

1. A topological space considered as a $(\beta,2)$-space (X,a) is $(\beta,2)$-normal if and only if it is normal in the usual sense, meaning that for all closed subsets $A,B\subseteq X$ with $A\cap B=\emptyset$, there are open sets U and V such that $A\subseteq U$, $B\subseteq V$ and $U\cap V=\emptyset$.

2. In $\mathsf{App}\cong(\beta,\mathsf{P}_+)$-$\mathsf{Cat}$, an approach space (X,a) is (β,P_+)-normal if and only if for any $\mathcal{X},\mathcal{Y},\mathcal{Z}\in\beta X$,

$$\hat{a}(\mathcal{Z},\mathcal{X})+\hat{a}(\mathcal{Z},\mathcal{Y}) \geq \inf_{\mathcal{W}\in\beta X}\hat{a}(\mathcal{X},\mathcal{W})+\hat{a}(\mathcal{Y},\mathcal{W}), \tag{4.31}$$

where

$$\hat{a}(\mathcal{X},\mathcal{Y}) = \inf\{u\in\mathsf{P}\mid \forall A\in\mathcal{X} : A^{(u)}\in\mathcal{Y}\}. \tag{4.32}$$

In order to list some results, we first give the following lemma, which can be found as Lemma V.2.5.1 in [HST14].

Lemma 4.1.4.3. For subsets A,B of an approach space X and any real $u>0$, the following are equivalent:

(i) $\forall C\subseteq X : A\cap C^{(u)}\neq\emptyset$ or $B\cap(X\setminus C)^{(u)}\neq\emptyset$;

(ii) $\exists\mathcal{X},\mathcal{Y},\mathcal{Z}\in\beta X$ such that $\forall C\in\mathcal{Z} : A\cap C^{(u)}\in\mathcal{X}$ and $B\cap C^{(u)}\in\mathcal{Y}$.

The following theorem, Theorem V.2.5.2 from [HST14], shows that (β,P_+)-normality is a very strong condition.

Theorem 4.1.4.4. *For an approach space (X,a), each of the following statements implies the next:*

(a) (X,a) *is* (β,P_+)*-normal,*

(b) for all ultrafilters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ on X and any real $w > 0$,

$$\hat{a}(\mathcal{Z}, \mathcal{X}) < w \,\&\, \hat{a}(\mathcal{Z}, \mathcal{Y}) < w \Rightarrow \exists \mathcal{W} \in \beta X : \begin{cases} \hat{a}(\mathcal{X}, \mathcal{W}) < 2w \\ \hat{a}(\mathcal{Y}, \mathcal{W}) < 2w, \end{cases}$$

(c) for all $A, B \subseteq X$ and any real $v > 0$,

$$A^{(v)} \cap B^{(v)} = \emptyset \Rightarrow \exists u > 0 \exists C \subseteq X : A^{(u)} \cap C^{(u)} = \emptyset = B^{(u)} \cap (X \setminus C)^{(u)},$$

(d) for all ultrafilters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ on X,

$$\hat{a}(\mathcal{Z}, \mathcal{X}) = 0 = \hat{a}(\mathcal{Z}, \mathcal{Y}) \Rightarrow \exists \mathcal{W} \in \beta X : \hat{a}(\mathcal{X}, \mathcal{W}) = 0 = \hat{a}(\mathcal{Y}, \mathcal{W}).$$

None of the three implications is reversible and for counterexamples we refer to Remark V.2.5.3 in [HST14]. It was shown by Van Olmen that condition (c) is equivalent to approach frame normality of the lower regular function frame on X [VO05]. For more results on normality in App, we refer to [CSVDH18a] and [SVDH16].

4.1.5 Extremal disconnectedness

Throughout this subsection we assume that $\mathcal{V}$ is commutative and $\widehat{\mathbb{T}}$ is associative.

Reversing the inequality in (4.29) has an interesting topological meaning. It leads us to consider extremally disconnected objects.

Definition 4.1.5.1. A $(\mathbb{T}, \mathcal{V})$-space (X, a) is $(\mathbb{T}, \mathcal{V})$-*extremally disconnected* if

$$\hat{a}^{\circ} \cdot \hat{a} \leq \hat{a} \cdot \hat{a}^{\circ}, \tag{4.33}$$

i.e.

$$\hat{a}(\mathcal{X}, \mathcal{Z}) \otimes \hat{a}(\mathcal{Y}, \mathcal{Z}) \leq \bigvee_{\mathcal{W} \in TX} \big(\hat{a}(\mathcal{W}, \mathcal{X}) \otimes \hat{a}(\mathcal{W}, \mathcal{Y})\big) \tag{4.34}$$

for all $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in TX$.

Remark that a $\mathcal{V}$-space (X, a) is $(\mathbb{1}, \mathcal{V})$-normal if and only if (X, a°) is $(\mathbb{1}, \mathcal{V})$-extremally disconnected.

Examples 4.1.5.2. 1. Recall that a topological space (X, a) is extremally disconnected if and only if the closure of every open set in X is open. For a topological space X presented as a $(\beta, 2)$-space (X, a) the following conditions are equivalent:

(i) X is extremally disconnected,

(ii) for all open subsets U, W of X, if $U \cap W = \emptyset$ then $\overline{U} \cap \overline{W} = \emptyset$,

(iii) $\hat{a}^\circ \cdot \hat{a} \leq \hat{a} \cdot \hat{a}^\circ$.

This shows that for a $(\beta, 2)$-space (X, a) $(\beta, 2)$-extremal disconnectedness coincides with the classical notion of extremally disconnectedness in topological spaces.

2. In $\mathsf{App} \cong (\beta, \mathsf{P}_+)$-$\mathsf{Cat}$, an approach space (X, a) is (β, P_+)-extremally disconnected if and only if for any $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \beta X$,

$$\hat{a}(\mathcal{X}, \mathcal{Z}) + \hat{a}(\mathcal{Y}, \mathcal{Z}) \geq \inf_{\mathcal{W} \in \beta X} \big(\hat{a}(\mathcal{W}, \mathcal{X}) + \hat{a}(\mathcal{W}, \mathcal{Y})\big). \tag{4.35}$$

4.2 Topological properties in NA-App

In Theorem 2.2.4.1 we showed that NA-$\mathsf{App} \cong (\beta, \mathsf{P}_\vee)$-$\mathsf{Cat}$, hence in this section we explore topological properties in NA-App following the relational calculus introduced in Section 4.1 for $(\mathbb{T}, \mathcal{V})$-properties. We introduce low separation properties, Hausdorff separation, compactness, regularity, normality and extremal disconnectedness as an application to $(\beta, \mathsf{P}_\vee)$-Cat of the corresponding $(\mathbb{T}, \mathcal{V})$-properties for arbitrary $(\mathbb{T}, \mathcal{V})$-$\mathsf{Cat}$. For each of the properties p we characterize the property $(\beta, \mathsf{P}_\vee)$-p in the context NA-App. On the other hand we make use of the well-known meaning of these properties in the setting of Top. For a non-Archimedean approach space X with tower of topologies $(\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$ (see Corollary 2.1.3.2) we compare the property $(\beta, \mathsf{P}_\vee)$-p to the properties

- X has p at level 0: meaning $(X, \mathcal{T}_0)$ has p in Top.
- X strongly has p: meaning $(X, \mathcal{T}_\varepsilon)$ has p in Top for every $\varepsilon \in \mathbb{R}^+$.
- X almost strongly has p: meaning $(X, \mathcal{T}_\varepsilon)$ has p in Top for every $\varepsilon \in \mathbb{R}_0^+$.

4.2.1 Hausdorff separation and lower separation axioms

As explained in Examples 1.3.1.1, the quantales P_+ and $\mathsf{P}_\vee$ have the same lattice structure, but a different tensor. This results in the fact that relations in P_+-Rel and $\mathsf{P}_\vee$-Rel have a different compositional structure, see Examples 1.3.2.1.

If we study separation axioms, as introduced in Section 4.1.1, in $(\beta, \mathsf{P}_\vee)$-$\mathsf{Cat} \cong$ NA-App, the axioms that do not involve composition of relations, will not generate new properties and will simply coincide with the properties (β, P_+)-p in (β, P_+)-$\mathsf{Cat} \cong \mathsf{App}$.

When we look at the definitions of $(\mathbb{T}, \mathcal{V})$-$\mathrm{T}_1$ and $(\mathbb{T}, \mathcal{V})$-$\mathrm{T}_0$, Definition 4.1.1.4, it is immediately clear that these properties do not depend on the compositional structure in $\mathcal{V}$-Rel. Hence, for non-Archimedean approach spaces the property $(\beta, \mathsf{P}_\vee)$-T_1 ($(\beta, \mathsf{P}_\vee)$-T_0 respectively) will simply coincide with the property (β, P_+)-T_1 ((β, P_+)-T_0) as studied in Examples 4.1.1.6.

At first glance the situation looks different for Hausdorff separation. If we look at the definition of $(\mathbb{T}, \mathcal{V})$-Hausdorff, Definition 4.1.1.1, we see a composition of relations and the formula in Proposition 4.1.1.2 clearly shows the tensor of $\mathcal{V}$. However, when we look at the pointwise form of (β, P_+)-Hausdorff (4.4), we see that the tensor disappears, since

$$\lambda\mathcal{Z}(x) + \lambda\mathcal{Z}(y) < \infty \Leftrightarrow \lambda\mathcal{Z}(x) < \infty \,\&\, \lambda\mathcal{Z}(y) < \infty,$$

for all $\mathcal{Z} \in \beta X$ and $x, y \in X$. The same happens for $(\beta, \mathsf{P}_\vee)$-Hausdorff, since we also have

$$\lambda\mathcal{Z}(x) \vee \lambda\mathcal{Z}(y) < \infty \Leftrightarrow \lambda\mathcal{Z}(x) < \infty \,\&\, \lambda\mathcal{Z}(y) < \infty,$$

for all $\mathcal{Z} \in \beta X$ and $x, y \in X$. Hence, the properties $(\beta, \mathsf{P}_\vee)$-Hausdorff and (β, P_+)-Hausdorff coincide in NA-App.

Theorem 4.2.1.1. *For a non-Archimedean approach space X the following properties are equivalent*

(i) X is strongly Hausdorff (strongly T_1, strongly T_0 respectively),

(ii) X is almost strongly Hausdorff (almost strongly T_1, almost strongly T_0 respectively),

(iii) X is $(\beta, \mathsf{P}_\vee)$-Hausdorff ($(\beta, \mathsf{P}_\vee)$-T_1, $(\beta, \mathsf{P}_\vee)$-T_0 respectively),

and imply that X is Hausdorff (T_1, T_0 respectively) at level 0.

Proof. That (i) and (ii) are equivalent is straightforward. The equivalence of (i) and (iii) is based on $\lambda\mathcal{U}(x) < \infty \Leftrightarrow \exists\varepsilon \in \mathbb{R}^+, \lambda\mathcal{U}(x) \leq \varepsilon \Leftrightarrow \exists\varepsilon \in \mathbb{R}^+, \mathcal{U} \to x$ in $(X, \mathcal{T}_\varepsilon)$, for $\mathcal{U} \in \beta X$ and $x \in X$. The proofs of the other cases follow analogously. □

That $(\beta, \mathsf{P}_\vee)$-Hausdorff ($(\beta, \mathsf{P}_\vee)$-T_1, $(\beta, \mathsf{P}_\vee)$-T_0) is not equivalent to the Hausdorff (T_1, T_0) property at level 0 follows from example $X_\mathcal{S}$ in Example 2.1.3.3 with $\mathcal{S} = \{X, \emptyset\}$.

In the definition of $(\mathbb{T}, \mathcal{V})$-$R_0$, Definition 4.1.1.4, there is no composition of relations present. Similar to the T_1 and T_0 properties we get that for non-Archimedean approach spaces $(\beta, \mathsf{P}_\vee)$-R_0 and (β, P_+)-R_0 (4.12) coincide.

R_1 is the only separation axiom that depends on the tensor.

Definition 4.2.1.2. A non-Archimedean approach space X is $(\beta, \mathsf{P}_\vee)$-*R*$_1$ if

$$\lambda\dot{x}(y) \leq \lambda\mathcal{Z}(x) \vee \lambda\mathcal{Z}(y), \tag{4.36}$$

for all $x, y \in X$ and $\mathcal{Z} \in \beta X$.

Theorem 4.2.1.3. *For a non-Archimedean approach space X the following properties are equivalent*

(i) X is strongly R_1 (strongly R_0 respectively),

(ii) X is almost strongly R_1 (almost strongly R_0 respectively),

(iii) X is $(\beta, \mathsf{P}_\vee)$-R_1 ($(\beta, \mathsf{P}_\vee)$-R_0 respectively),

and imply that X is R_1 (R_0 respectively) at level 0.

Proof. That (i) and (ii) are equivalent is straightforward. The equivalence of (i) and (iii) is based on the fact that $\lambda\mathcal{U}(x) \leq \varepsilon$ if and only if $\mathcal{U} \to x$ in $(X, \mathcal{T}_\varepsilon)$, for any $\varepsilon \in \mathbb{R}^+$. □

That $(\beta, \mathsf{P}_\vee)$-R$_1$ ($(\beta, \mathsf{P}_\vee)$-R$_0$ respectively) is not equivalent to R$_1$ at level 0 (R$_0$ at level 0 respectively) follows from Example 2.1.3.4 where $\mathcal{T}$ is the right order topology.

It is clear that $(\beta, \mathsf{P}_\vee)$-R$_1$ implies (β, P_+)-R$_1$. The following counterexample shows that the converse does not hold.

Example 4.2.1.4. Let X be infinite and $\mathcal{S}$ the cofinite topology on X. Consider the non-Archimedean approach space $X_\mathcal{S}$ as in Example 2.1.3.3. Later on, in Example 4.2.3.4, we will establish that this space is (β, P_+)-regular. In Proposition 4.1.3.3 it is shown that this implies that $X_\mathcal{S}$ is (β, P_+)-R$_1$.

The topological space $(X, \mathcal{S})$ is not Hausdorff, but it is T$_1$. By Proposition 4.1.1.5,$(X, \mathcal{S})$ is therefore not R$_1$. Hence $X_\mathcal{S}$ is not strongly R$_1$, and by Theorem 4.2.1.3 it is not $(\beta, \mathsf{P}_\vee)$-R$_1$.

4.2.2 Compactness

The next property that we will consider is $(\beta, \mathsf{P}_\vee)$-compactness. Looking at Proposition 4.1.2.2 and the characterization of (β, P_+)-compactness (4.19), it is again clear that this property does not depend on the tensor. Hence a non-Archimedean approach space is $(\beta, \mathsf{P}_\vee)$-compact if and only if it is (β, P_+)-compact.

The proofs of the following results are straightforward.

Theorem 4.2.2.1. *For a non-Archimedean approach space X the following equivalences hold:*

1. $(\beta, \mathsf{P}_\vee)$-*compact* $\Leftrightarrow$ *almost strongly compact,*

2. *compact at level* $0 \Leftrightarrow$ *strongly compact.*

That $(\beta, \mathsf{P}_\vee)$-compactness does not imply a compact topological coreflection is well known in the setting of App (see Example 4.3.16 in [Low15]). That this is not the case either in the setting of NA-App follows from the following example.

Example 4.2.2.2. Consider the example in Example 2.1.3.4 on $]0,\infty[$ with $\mathcal{T}$ the right order topology. Clearly $\mathcal{T}$ is not compact, whereas the topologies $\mathcal{T}_\varepsilon$ at strictly positive levels are all compact.

Proposition 4.2.2.3. Let X be a non-Archimedean approach space. If X is $(\beta, \mathsf{P}_\vee)$-compact and $(\beta, \mathsf{P}_\vee)$-Hausdorff, then it is a compact Hausdorff topological space.

Proof. Suppose that the non-Archimedean approach space X is both $(\beta, \mathsf{P}_\vee)$-compact and $(\beta, \mathsf{P}_\vee)$-Hausdorff. Then $(X, \mathcal{T}_\varepsilon)$ is a compact Hausdorff topological space, for every $\varepsilon > 0$. By the coherence condition of the non-Archimedean tower, we have $\mathcal{T}_\gamma \subseteq \mathcal{T}_\varepsilon$, for $\varepsilon \leq \gamma$ and therefore $\mathcal{T}_\gamma = \mathcal{T}_\varepsilon$, for every $\gamma, \varepsilon > 0$. Moreover, since $\mathcal{T}_0 = \bigvee_{\gamma > 0} \mathcal{T}_\gamma$, all levels of the non-Archimedean tower are equal. This implies that X is topological. $\square$

4.2.3 Regularity

Next we investigate the notion of regularity. We recall the definition of the $(\beta, \mathsf{P}_\vee)$-regularity property in $(\beta, \mathsf{P}_\vee)$-Cat by giving the pointwise interpretation through the isomorphism in Theorem 2.2.4.1 in terms of the limit operator.

Definition 4.2.3.1. A non-Archimedean approach space X is $(\beta, \mathsf{P}_\vee)$-regular if

$$\lambda\mathcal{U}(x) \leq \lambda\Sigma\mathfrak{X}(x) \vee \sup_{\mathcal{A}\in\mathfrak{X}, B\in\mathcal{U}} \inf_{\mathcal{W}\in\mathcal{A}, b\in B} \lambda\mathcal{W}(b), \tag{4.37}$$

for all $\mathfrak{X} \in \beta\beta X, \mathcal{U} \in \beta X$ and $x \in X$.

The following result gives a characterization in terms of the level topologies.

Theorem 4.2.3.2. *For a non-Archimedean approach space* X *the following are equivalent:*

(i) X *is strongly regular.*

(ii) X *is almost strongly regular.*

(iii) For all $\mathcal{U}, \mathcal{W} \in \beta X$ *and for all* $\gamma \geq 0$*:* $\mathcal{W}^{(\gamma)} \subseteq \mathcal{U} \Rightarrow \lambda\mathcal{U} \leq \lambda\mathcal{W} \vee \gamma$.

(iv) X *is* $(\beta, \mathsf{P}_\vee)$*-regular.*

Proof. To see that (i) and (ii) are equivalent, it suffices to observe that $\bigcup_{\varepsilon>0} \mathcal{C}_\varepsilon$ is a closed basis for the topology $\mathcal{T}_0$, where $\mathcal{C}_\varepsilon$ are the closed sets in $(X, \mathcal{T}_\varepsilon)$.

To prove that (i) implies (iii), take $\mathcal{W}, \mathcal{U} \in \beta X$ and suppose $\mathcal{W}^{(\gamma)} \subseteq \mathcal{U}$ for $\gamma \geq 0$. Suppose $\lambda\mathcal{W}(x) = \varepsilon < \infty$. Then $\lambda\mathcal{W}(x) \leq \varepsilon \vee \gamma$ and thus $\mathcal{W}$ converges to x in $(X, \mathcal{T}_{\varepsilon\vee\gamma})$. By regularity of $(X, \mathcal{T}_{\varepsilon\vee\gamma})$ also $\mathcal{W}^{(\varepsilon\vee\gamma)}$ converges to x in $(X, \mathcal{T}_{\varepsilon\vee\gamma})$. Hence

$$\lambda\mathcal{U}(x) \leq \lambda\mathcal{W}^{(\gamma)}(x) \leq \lambda\mathcal{W}^{(\varepsilon\vee\gamma)}(x) \leq \varepsilon \vee \gamma.$$

Next we prove that (iii) implies (iv) Take $\mathfrak{X} \in \beta\beta X, \mathcal{U} \in \beta X$ and $x \in X$. Put $\gamma = \lambda\Sigma\mathfrak{X}(x)$ and $\varepsilon = \sup_{\mathcal{A}\in\mathfrak{X}, B\in\mathcal{U}} \inf_{\mathcal{W}\in\mathcal{A}, b\in B} \lambda\mathcal{W}(b)$. It is sufficient to assume that both γ and ε are finite. Let $0 < \rho < \infty$ be arbitrary and consider

$$S := \big\{(\mathcal{G}, y) \mid \lambda\mathcal{G}(y) \leq \varepsilon + \rho\big\} \subseteq \beta X \times X.$$

By definition of ε, the filter base $\mathfrak{X} \times \mathcal{U}$ has a trace on S, so we can choose $\mathcal{R} \in \beta S$ refining this trace. For $A \in \Sigma\mathfrak{X}$ and $U \in \mathcal{U}$ there exist $R_1 \in \mathcal{R}$ and $R_2 \in \mathcal{R}$ such that $A \in \bigcap_{z\in R_1} \pi_1 z$ and $U = \pi_2 R_2$, with π_1 and π_2 the projections restricted to S. For $z \in R_1 \cap R_2$ we have $\lambda\pi_1 z(\pi_2 z) \leq \varepsilon + \rho$ and $\pi_2 z \in A^{(\varepsilon+\rho)} \cap U$. Finally we can conclude that $\Sigma\mathfrak{X}^{(\varepsilon+\rho)} \subseteq \mathcal{U}$, which implies

$$\lambda\mathcal{U}(x) \leq \lambda\Sigma\mathfrak{X}(x) \vee (\varepsilon + \rho) = \gamma \vee (\varepsilon + \rho).$$

By arbitrariness of ρ our conclusion follows.

To prove that (iv) implies (i), we use a technique similar to the one used in the proof of Theorem 9 in [BK98]. Let $\mathcal{W}$ be an ultrafilter converging to x in $(X, \mathcal{T}_\gamma)$ for $\gamma \geq 0$ and let $\mathcal{U} \in \beta X$, such that $\mathcal{W}^{(\gamma)} \subseteq \mathcal{U}$. We may assume that γ is finite. Let $0 < \rho < \infty$ be arbitrary. Consider

$$S = \big\{(\mathcal{G}, y) \mid \lambda\mathcal{G}(y) \leq \gamma + \rho\big\} \subseteq \beta X \times X$$

and the filter base $\{S_W \mid W \in \mathcal{W}\}$ on S, where $S_W = \big\{(\mathcal{G}, y) \in S \mid W \in \mathcal{G}\big\}$ whenever $W \in \mathcal{W}$. Let $\mathcal{S}_\mathcal{W}$ be the filter generated. Using the restrictions π_1 and π_2 of the projections to S, we observe the following facts:

a) $\pi_2\mathcal{S}_\mathcal{W} \subseteq \mathcal{W}^{(\gamma)}$: This follows from the fact that $y \in W^{(\gamma)}$ implies the existence of $\mathcal{G} \in \beta X$ with $W \in \mathcal{G}$ and $\lambda\mathcal{G}(y) \leq \gamma + \rho$.

b) There exists $\mathcal{R} \in \beta S$ satisfying $\mathcal{S}_\mathcal{W} \subseteq \mathcal{R}$ and $\pi_2\mathcal{R} = \mathcal{U}$: Suppose the contrary, i.e. for every $\mathcal{R} \in \beta S$ with $\mathcal{S}_\mathcal{W} \subseteq \mathcal{R}$ there exists $U_\mathcal{R} \in \mathcal{U}$ and $R \in \mathcal{R}$ such that $U_\mathcal{R} \cap \pi_2 R = \emptyset$. We can select a finite number of these sets with $U_{\mathcal{R}_i} \cap \pi_2 R_i = \emptyset$ and such that $\bigcup_i R_i \in \mathcal{S}_\mathcal{W}$. In view of a) there exists an index j such that $\pi_2 R_j \in \mathcal{U}$, which is a contradiction.

c) With $\mathfrak{X} = \pi_1\mathcal{R}$ we have $\Sigma\mathfrak{X} = \mathcal{W}$ since $W \in \mathcal{W}$ implies $W \in \bigcap_{\mathcal{G}\in\pi_1 S_W} \mathcal{G}$.

Combining these results, we now have:

$$\begin{aligned}\lambda\mathcal{U}(x) &\leq \lambda\Sigma\mathfrak{X}(x) \vee \sup_{\mathcal{A}\in\mathfrak{X}, B\in\mathcal{U}} \inf_{\mathcal{V}\in\mathcal{A}, b\in B} \lambda\mathcal{V}(b)\\ &= \lambda\mathcal{W}(x) \vee \sup_{R\in\mathcal{R}, R'\in\mathcal{R}} \inf_{\mathcal{V}\in\pi_1 R, b\in\pi_2 R'} \lambda\mathcal{V}(b)\\ &\leq \gamma \vee \sup_{R\in\mathcal{R}} \inf_{\mathcal{V}\in\pi_1 R, b\in\pi_2 R} \lambda\mathcal{V}(b)\\ &\leq \gamma \vee \sup_{R\in\mathcal{R}} \inf_{z\in R} \lambda\pi_1(z)\big(\pi_2(z)\big)\\ &\leq \gamma \vee (\gamma+\rho).\end{aligned}$$

By arbitrariness of $\rho > 0$, we can conclude that $\mathcal{U}$ converges to x in $(X, \mathcal{T}_\gamma)$. $\square$

Clearly we can express condition (iii) in terms of arbitrary filters as $\lambda\mathcal{F}^{(\gamma)} \leq \lambda\mathcal{F} \vee \gamma$, for every $\mathcal{F} \in \mathsf{F}X$ and $x \in X$. In this form the property was considered in [BK98]. In more generality this condition was also considered in [CMT14] in the context of contractive extensions.

Clearly for non-Archimedean approach spaces we have

$$(\beta, \mathsf{P}_\vee)\text{-regular} \Rightarrow (\beta, \mathsf{P}_+)\text{-regular} \Rightarrow \text{regular at level } 0.$$

The following examples based on the construction in Example 2.1.3.3 show that none of the implications is reversible.

Example 4.2.3.3. Let $X = \{0, 1\}$ and $\mathcal{S}$ the Sierpinski topology on X with $\{1\}$ open. The approach space $X_\mathcal{S}$ is not (β, P_+)-regular. This can be seen by taking $\mathcal{F} = \dot{1}$ and $\gamma = 1$. For this choice we have $\mathcal{F}^{(\gamma)} = \{X\}$ and $\lambda\{X\}(1) = 2 \nleq \lambda\dot{1}(1) + 1$. However $(X, \mathcal{T}_0)$ is discrete and hence regular.

Example 4.2.3.4. Let X be infinite and $\mathcal{S}$ the cofinite topology on X. The approach space $X_\mathcal{S}$ is not $(\beta, \mathsf{P}_\vee)$-regular. However it is (β, P_+)-regular. To see this let $1 \leq \gamma < 2$. A filter $\mathcal{F}$ on X either contains a finite set and then $\mathcal{F}^{(\gamma)} = \mathcal{F}$ and $\lambda\mathcal{F}^{(\gamma)} \leq \lambda\mathcal{F} + \gamma$, or $\mathcal{F}$ does not contain a finite set. In that case we have $\mathcal{F}^{(\gamma)} = \{X\}$ and $\lambda\mathcal{F}^{(\gamma)} = 2 \leq \lambda\mathcal{F} + \gamma$. For $\gamma < 1$ or $2 \leq \gamma$ the condition $\lambda\mathcal{F}^{(\gamma)} \leq \lambda\mathcal{F} + \gamma$ is clearly fulfilled.

The following proposition shows that in $(\beta, \mathsf{P}_\vee)$-Cat we also find that $(\beta, \mathsf{P}_\vee)$-regularity implies $(\beta, \mathsf{P}_\vee)$-R$_1$.

Proposition 4.2.3.5. An non-Archimedean approach space X is $(\beta, \mathsf{P}_\vee)$-R$_1$ if it is $(\beta, \mathsf{P}_\vee)$-regular.

Proof. This is just an easy adaptation of the proof from Proposition 4.1.3.3. $\square$

4.2.4 Normality

In this section we investigate normality. We recall the definition of $(\beta, \mathsf{P}_\vee)$-p for p the normality property in $(\beta, \mathsf{P}_\vee)$-Cat.

Definition 4.2.4.1. A non-Archimedean approach space X is $(\beta, \mathsf{P}_\vee)$-*normal* if

$$\hat{a}(\mathcal{U}, \mathcal{A}) \vee \hat{a}(\mathcal{U}, \mathcal{B}) \geq \inf \left\{ \hat{a}(\mathcal{A}, \mathcal{W}) \vee \hat{a}(\mathcal{B}, \mathcal{W}) \mid \mathcal{W} \in \beta X \right\}, \tag{4.38}$$

for all ultrafilters $\mathcal{A}, \mathcal{B}, \mathcal{U}$ on X, with $\hat{a}(\mathcal{U}, \mathcal{A}) = \inf\{u \in \mathsf{P} \mid \mathcal{U}^{(u)} \subseteq \mathcal{A}\}$.

Turning the $\vee$ in the formula into $+$ we have (β, P_+)-normality as introduced in the second example of Examples 4.1.4.2. In case X is a topological approach space, both notions coincide with

$$\overline{\mathcal{U}} \subseteq \mathcal{A} \,\&\, \overline{\mathcal{U}} \subseteq \mathcal{B} \Rightarrow \exists \mathcal{W} \in \beta X : \overline{\mathcal{A}} \subseteq \mathcal{W} \,\&\, \overline{\mathcal{B}} \subseteq \mathcal{W}, \tag{4.39}$$

for all ultrafilters $\mathcal{U}, \mathcal{A}, \mathcal{B}$ on X. As explained in the first example of Examples 4.1.4.2 this condition coincides with the usual notion of normality on the topological (approach) space.

The following lemma will enable us to formulate a result on normality for ultrametric approach spaces.

Lemma 4.2.4.2. For a non-Archimedean approach space X, the following equality holds

$$\hat{a}(\mathcal{U}, \mathcal{A}) = \sup_{U \in \mathcal{U}, A \in \mathcal{A}} \delta(A, U), \tag{4.40}$$

for all $\mathcal{U}, \mathcal{A} \in \beta X$, where $\delta(A, U) = \inf_{z \in A} \delta(z, U)$.

Proof. To prove one inequality, suppose $\hat{a}(\mathcal{U}, \mathcal{A}) < \alpha$. Then choose $\rho, \gamma \in [0, \infty]$ arbitrary such that $\hat{a}(\mathcal{U}, \mathcal{A}) < \rho < \gamma < \alpha$. Then for all $U \in \mathcal{U}, U^{(\rho)} \in \mathcal{A}$. This implies that for all $U \in \mathcal{U}$ and $A \in \mathcal{A}$ $U^{(\rho)} \cap A \neq \emptyset$ and thus for all $U \in \mathcal{U}$ and $A \in \mathcal{A}$, there exists $z \in A$ such that $\delta(z, U) \leq \rho$. Therefore, for all $U \in \mathcal{U}$ and $A \in \mathcal{A}$ $\inf_{z \in A} \delta(z, U) < \gamma$ and thus $\sup_{U \in \mathcal{U}, A \in \mathcal{A}} \delta(A, U) < \alpha$.

To prove the other inequality, suppose $\sup_{U \in \mathcal{U}, A \in \mathcal{A}} \delta(A, U) < \alpha$. Then choose $\gamma \in [0, \infty]$ arbitrary such that $\sup_{U \in \mathcal{U}, A \in \mathcal{A}} \delta(A, U) < \gamma < \alpha$. Then, for $U \in \mathcal{U}$ and $A \in \mathcal{A}$ arbitrary, we have $\delta(A, U) < \gamma$. Hence, there exists $z \in A$ such that $\delta(z, U) < \gamma$ and thus $z \in U^{(\gamma)}$. By arbitrariness of A, this implies $U^{(\gamma)} \in \mathcal{A}$ and by arbitrariness of U, we get $\mathcal{U}^{(\gamma)} \subseteq \mathcal{A}$. Hence $\hat{a}(\mathcal{U}, \mathcal{A}) \leq \gamma < \alpha$. □

By application of this lemma, we get the following result.

Proposition 4.2.4.3. An ultrametric non-Archimedean approach space is $(\beta, \mathsf{P}_\vee)$-normal.

Proof. By symmetry of the ultrametric and the characterization of δ in terms of this ultrametric, we get symmetry of $\hat{a}$. Hence, the ultrafilter $\mathcal{W}$ on the right-hand side in the formula for $(\mathbb{\beta}, \mathsf{P}_\vee)$-normality, can be taken equal to $\mathcal{U}$. □

Without symmetry, we will encounter examples of quasi-ultrametric approach spaces that are $(\mathbb{\beta}, \mathsf{P}_\vee)$-normal and others that are not.

Next we give some useful characterizations of $(\mathbb{\beta}, \mathsf{P}_\vee)$-normality.

Proposition 4.2.4.4. For a non-Archimedean approach space X, the following properties are equivalent:

(i) X is $(\mathbb{\beta}, \mathsf{P}_\vee)$-normal.

(ii) $\hat{a}(\mathcal{U}, \mathcal{A}) < v \,\&\, \hat{a}(\mathcal{U}, \mathcal{B}) < v \Rightarrow \exists \mathcal{W} \in \beta X : \hat{a}(\mathcal{A}, \mathcal{W}) < v \,\&\, \hat{a}(\mathcal{B}, \mathcal{W}) < v$, for all $\mathcal{U}, \mathcal{A}, \mathcal{B} \in \beta X$.

(iii) $A^{(v)} \cap B^{(v)} = \emptyset \Rightarrow \forall u < v, \exists C \subseteq X : A^{(u)} \cap C^{(u)} = \emptyset \,\&\, (X \setminus C)^{(u)} \cap B^{(u)} = \emptyset$, for all $A, B \subseteq X$ and $v > 0$.

Proof. That (i) and (ii) are equivalent is straightforward.

To show that (ii) implies (iii), let $A^{(v)} \cap B^{(v)} = \emptyset$, with $A, B \subseteq X$ for some $v > 0$ and let $u < v$ arbitrary. By way of contradiction, suppose that for all $C \subseteq X$, $A^{(u)} \cap C^{(u)} \neq \emptyset$ or $(X \setminus C)^{(u)} \cap B^{(u)} \neq \emptyset$. By Lemma 4.1.4.3, there exist ultrafilters $\mathcal{U}, \mathcal{A}, \mathcal{B}$ on X satisfying

$$\forall U \in \mathcal{U} : A^{(u)} \cap U^{(u)} \in \mathcal{A} \,\&\, B^{(u)} \cap U^{(u)} \in \mathcal{B}.$$

It follows that $\hat{a}(\mathcal{U}, \mathcal{A}) \leq u < v$ and $\hat{a}(\mathcal{U}, \mathcal{B}) \leq u < v$. By (ii) there exists $\mathcal{W} \in \beta X$ with $\hat{a}(\mathcal{A}, \mathcal{W}) < v \,\&\, \hat{a}(\mathcal{B}, \mathcal{W}) < v$. Since $\mathcal{A}^{(v)} \subseteq \mathcal{W}, A^{(u)} \in \mathcal{A}$ and $A^{(u)(v)} \subseteq A^{(v)(v)} = A^{(v)}$ we have $A^{(v)} \in \mathcal{W}$. In the same way we have $B^{(v)} \in \mathcal{W}$ which contradicts $A^{(v)} \cap B^{(v)} = \emptyset$.

Next we show that (iii) implies (ii) Let $\mathcal{U}, \mathcal{A}, \mathcal{B}$ be ultrafilters on X and $v > 0$ with $\hat{a}(\mathcal{U}, \mathcal{A}) < v \,\&\, \hat{a}(\mathcal{U}, \mathcal{B}) < v$. Choose ε, δ satisfying $\hat{a}(\mathcal{U}, \mathcal{A}) < \varepsilon < \delta < v \,\&\, \hat{a}(\mathcal{U}, \mathcal{B}) < \varepsilon < \delta < v$. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We claim that for all $C \subseteq X$:

$$A^{(\varepsilon)} \cap C^{(\varepsilon)} \neq \emptyset \text{ or } B^{(\varepsilon)} \cap (X \setminus C)^{(\varepsilon)} \neq \emptyset.$$

Indeed, in view of $\mathcal{U}^{(\varepsilon)} \subseteq \mathcal{A}$ and $\mathcal{U}^{(\varepsilon)} \subseteq \mathcal{B}$, the assertion that there exists $C \subseteq X$ with $A^{(\varepsilon)} \cap C^{(\varepsilon)} = \emptyset \,\&\, B^{(\varepsilon)} \cap (X \setminus C)^{(\varepsilon)}$ would imply $C \notin \mathcal{U}$ and $X \setminus C \notin \mathcal{U}$, which is impossible. By (iii) we have $A^{(\delta)} \cap B^{(\delta)} \neq \emptyset$. So there exists an ultrafilter $\mathcal{W}$ on X refining

$$\left\{A^{(\delta)} \cap B^{(\delta)} \mid A \in \mathcal{A}, B \in \mathcal{B}\right\}.$$

Clearly $\mathcal{W}$ satisfies $\mathcal{A}^{(\delta)} \subseteq \mathcal{W}$ and $\mathcal{B}^{(\delta)} \subseteq \mathcal{W}$. So we can conclude that $\hat{a}(\mathcal{A}, \mathcal{W}) < v \,\&\, \hat{a}(\mathcal{B}, \mathcal{W}) < v$. □

Remark that for a non-Archimedean approach space X, condition (ii) in Proposition 4.2.4.4 implies condition (b) in Theorem 4.1.4.4 and therefore also (c), the Van Olmen normality condition.

Theorem 4.2.4.5. *For a non-Archimedean approach X with tower $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$, consider the following properties:*

1. *X is strongly normal.*
2. *X is almost strongly normal.*
3. *X is $(\beta, \mathsf{P}_\vee)$-normal.*
4. *X is normal at level 0.*

The following implications hold: 1. ⇒ 4. and 1. ⇒ 2. ⇒ 3.

Proof. 1. ⇒ 4. and 1. ⇒ 2. are straightforward. We show 2. ⇒ 3. Suppose $(X, \mathcal{T}_u)$ is normal, for every $u > 0$ and let $\hat{a}(\mathcal{U}, \mathcal{A}) < v \,\&\, \hat{a}(\mathcal{U}, \mathcal{B}) < v$ for some $v > 0$. Take u such that $\hat{a}(\mathcal{U}, \mathcal{A}) < u < v \,\&\, \hat{a}(\mathcal{U}, \mathcal{B}) < u < v$. Then for the topological space $(X, \mathcal{T}_u)$ we have $\mathcal{U}^{(u)} \subseteq \mathcal{A}$ and $\mathcal{U}^{(u)} \subseteq \mathcal{B}$. By normality of $(X, \mathcal{T}_u)$ there exists $\mathcal{W} \in \beta X$ satisfying $\mathcal{A}^{(u)} \subseteq \mathcal{W}$ and $\mathcal{B}^{(u)} \subseteq \mathcal{W}$. It follows that $\hat{a}(\mathcal{A}, \mathcal{W}) \leq u < v$ and $\hat{a}(\mathcal{B}, \mathcal{W}) \leq u < v$. □

There are no other valid implications between the properties considered in the previous theorem. This is shown by the following examples.

Example 4.2.4.6. On $X =]0, \infty[$, we consider an approach space X as in Example 2.1.3.4. We make a particular choice for the topology

$$\mathcal{T} = \big\{B^c \mid B \subseteq]0, \infty[, B \text{ bounded}\big\} \cup \{\emptyset\},$$

which is finer than the right order topology, and consider the approach space $\big(X, (\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\big)$. The topology $\mathcal{T}_0$ is not normal, since there are no non-empty distinct and disjoint open subsets, although disjoint non-empty closed subsets do exist. So the non-Archimedean approach space is not strongly normal either.

Let $\varepsilon > 0$ and consider the topological space $(X, \mathcal{T}_\varepsilon)$. It is a normal topological space, since for $x \leq \varepsilon$ we have $x \in A^{(\varepsilon)}$ for every non-empty subset A. So 2. and hence 3. from Theorem 4.2.4.5 are satisfied.

Example 4.2.4.7. Let $X =]0, \infty[$ and let $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ be the tower defined in Example 4.2.4.6 starting from $\mathcal{T} = \big\{B^c \mid B \subseteq]0, \infty[, B \text{ bounded}\big\} \cup \{\emptyset\}$. We define another tower $(\mathcal{S}_\gamma)_{\gamma\in\mathbb{R}^+}$ on X as follows

$$\mathcal{S}_\gamma := \begin{cases} \mathcal{P}(X) & \text{whenever } 0 \leq \gamma < 1, \\ \mathcal{T}_{\gamma-1} & \text{whenever } 1 \leq \gamma. \end{cases}$$

Clearly the tower $(\mathcal{S}_\gamma)_{\gamma\in\mathbb{R}^+}$ defines a non-Archimedean approach space on X. For the topology at level 0 we have $\mathcal{T}_0 = \mathcal{P}(X)$ is normal, but the topological space $(X, \mathcal{S}_1) = (X, \mathcal{T})$ is not normal. So 1. and 2. from Theorem 4.2.4.5 do not hold. However the approach space is $(\beta, \mathsf{P}_\vee)$-normal. Let A and B be non-empty subsets with $A^{(v)} \cap B^{(v)} = \emptyset$ for some $v > 0$ and let $u < v$ be arbitrary. Clearly $v \leq 1$. In that case the level topology for u is discrete. It follows that $C = B$ satisfies condition 3. in Theorem 4.2.4.4.

Example 4.2.4.8. Consider example (2) in remark V.2.5.3 in [HST14]. $X = \big(\{x, y, z, w\}, d\big)$ where d is the quasi-ultrametric structure $d(x, z) = d(y, z) = d(w, z) = 1, d(w, x) = d(w, y) = 3, d(x', x') = 0$ for any $x' \in X$ and $d(x', y') = \infty$ elsewhere. The topology $\mathcal{T}_0$ is discrete and hence normal. However $X = \big(\{x, y, z, w\}, d\big)$ is not $(\beta, \mathsf{P}_\vee)$-normal. Let $\mathcal{U} = \dot{z}, \mathcal{A} = \dot{x}$ and $\mathcal{B} = \dot{y}$. Then, using Lemma 4.2.4.2, we get $\hat{a}(\mathcal{U}, \mathcal{A}) = d(x, z) = 1$ and $\hat{a}(\mathcal{U}, \mathcal{B}) = d(y, z) = 1$. For $\mathcal{W} = \dot{w}$, we have $\hat{a}(\mathcal{A}, \mathcal{W}) = d(w, x) = 3$ and $\hat{a}(\mathcal{B}, \mathcal{W}) = d(w, y) = 3$ and for all other choices of $\mathcal{W}$ we obtain values ∞. So the space X does not satisfy 1. or 2. from Theorem 4.2.4.5 either.

Next we prove that the notions $(\beta, \mathsf{P}_\vee)$-normal and (β, P_+)-normal are unrelated. Again this can be shown by looking at finite non-Archimedean approach spaces that are therefore structured by some quasi-ultrametric.

Example 4.2.4.9. Let $X = \{x, y, z\}$ endowed with the quasi-ultrametric d defined by $d(y, x) = 2, d(x, z) = 1, d(y, z) = 1$ and $d(x', x') = 0$ for all $x' \in X$ and $d(x', y') = \infty$ elsewhere. Clearly the only inequality that has to be checked is $d(x, z) + d(y, z) = 2 \geq d(y, x)$ which is no longer valid when $+$ is changed into $\vee$. The space is (β, P_+)-normal but not $(\beta, \mathsf{P}_\vee)$-normal.

Example 4.2.4.10. Let $X = \{x, y, z, w\}$ be endowed with the quasi-ultrametric d defined by $d(x, z) = 1, d(y, z) = 2, d(w, z) = 2, d(w, y) = 2, d(w, x) = 2$ and $d(x', x') = 0$ for all $x' \in X$ and $d(x', y') = \infty$ elsewhere. The space is not (β, P_+)-normal since $d(y, z) + d(x, z) = 3 < 4 = d(w, y) + d(w, x)$. However X is strongly normal, and therefore $(\beta, \mathsf{P}_\vee)$-normal. To see this observe that the level topologies for $0 \leq \varepsilon < 1$ are discrete and hence normal. For levels $1 \leq \varepsilon < 2$ all points are isolated except x for which the smallest neighborhood is $\{x, z\}$. This topology is normal since all closed sets are open. For levels $2 \leq \varepsilon$ we have smallest neighborhoods $V_w = X, V_y = \{y, z\}, V_z = \{z\}$ and $V_x = \{x, z\}$. These levels are normal too since there are no disjoint non-empty closed sets.

4.2.5 Extremal disconnectedness

We proceed by investigating extremal disconnectedness. We recall the definition of $(\beta, \mathsf{P}_\vee)$-p for p extremal disconnectedness in $(\beta, \mathsf{P}_\vee)$-Cat.

Definition 4.2.5.1. A non-Archimedean approach space X is $(\beta, \mathsf{P}_\vee)$-*extremally disconnected* if

$$\hat{a}(\mathcal{A},\mathcal{U}) \vee \hat{a}(\mathcal{B},\mathcal{U}) \geq \inf\left\{\hat{a}(\mathcal{W},\mathcal{A}) \vee \hat{a}(\mathcal{W},\mathcal{B}) \mid \mathcal{W} \in \beta X\right\},$$

for all ultrafilters $\mathcal{U}, \mathcal{A}, \mathcal{B}$ on X.

Turning the $\vee$ in the formula into $+$ we get (β, P_+)-extremal disconnectedness. In case X is a topological (approach) space both notions coincide with

$$\overline{\mathcal{A}} \subseteq \mathcal{U} \& \overline{\mathcal{B}} \subseteq \mathcal{U} \Rightarrow \exists \mathcal{W} \in \beta X : \overline{\mathcal{W}} \subseteq \mathcal{A} \,\&\, \overline{\mathcal{W}} \subseteq \mathcal{B},$$

for all ultrafilters $\mathcal{U}, \mathcal{A}, \mathcal{B}$ on X. As is shown in Example 4.1.5.2 this condition coincides with the usual notion of extremal disconnectedness on the topological (approach) spaces.

Using Lemma 4.2.4.2, we get the following result for ultrametric (approach) spaces, which is similar to Proposition 4.2.4.3.

Proposition 4.2.5.2. An ultrametric non-Archimedean approach space is $(\beta, \mathsf{P}_\vee)$-extremal disconnected.

Proof. The proof follows a similar argument as the proof of Proposition 4.2.4.3. □

Without symmetry we will encounter examples of quasi-ultrametric approach spaces that are $(\beta, \mathsf{P}_\vee)$-extremally disconnected and others that are not.

Proposition 4.2.5.3. For a non-Archimedean approach space X with tower $(\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$, the following properties are equivalent:

(i) X is $(\beta, \mathsf{P}_\vee)$-extremally disconnected.

(ii) $\hat{a}(\mathcal{A},\mathcal{U}) < v \& \hat{a}(\mathcal{B},\mathcal{U}) < v \Rightarrow \exists \mathcal{W} \in \beta X : \hat{a}(\mathcal{W},\mathcal{A}) < v \,\&\, \hat{a}(\mathcal{W},\mathcal{B}) < v$, for all $\mathcal{U}, \mathcal{A}, \mathcal{B}$ ultrafilters on X and $v > 0$.

(iii) $A \cap B = \emptyset$ with $A, B \in \mathcal{T}_v \Rightarrow \forall u < v : A^{(u)} \cap B^{(u)} = \emptyset$, for all $A, B \subseteq X$ and $v > 0$.

Proof. The equivalence of (i) and (ii) is straightforward.

First we prove that (ii) implies (iii). If (iii) does not hold, then there exists $v > 0$ and $A, B \in \mathcal{T}_v$ with $A \cap B = \emptyset$ and $u < v$ such that $A^{(u)} \cap B^{(u)} \neq \emptyset$. Take an ultrafilter $\mathcal{U}$ on X containing both $A^{(u)}$ and $B^{(u)}$. Let $\mathcal{U}_{\mathcal{T}_u}$ be the ultrafilter on X generated by the $\mathcal{T}_u$-open sets in $\mathcal{U}$. Then clearly $\mathcal{U}_{\mathcal{T}_u} \vee \{A\}$ is a proper filter. We claim that there exists an $\mathcal{A} \in \beta X$ such that $\mathcal{U}_{\mathcal{T}_u} \vee \{A\} \subseteq \mathcal{A}$ with $\mathcal{A}^{(u)} \subseteq \mathcal{U}$. Suppose on the contrary that for every such $\mathcal{A}_i \in \beta X$ with $\mathcal{U}_{\mathcal{T}_u} \vee \{A\} \subseteq \mathcal{A}_i$ there exists $A'_i \in \mathcal{A}_i$ with $A_i^{'(u)} \notin \mathcal{U}$. A finite subcollection of ultrafilters $\{\mathcal{A}_j \mid j \in J\}$ exists, for which the corresponding sets $A'_j \in \mathcal{A}_j$ satisfy $\bigcup_{j\in J} A'_j \in \mathcal{U}_{\mathcal{T}_u} \vee \{A\}$. Let $U \in \mathcal{U}_{\mathcal{T}_u}$ such that $U \cap A^{(u)} \subseteq \bigcup_{j\in J} A'_j$. From $A^{(u)} \in \mathcal{U}$ we deduce that $(U \cap A)^{(u)}$ and hence also $\bigcup_{j\in J} A'_j$ belongs to $\mathcal{U}$, which is a contradiction. In the same way there exists $\mathcal{B} \in \beta X$ with $\mathcal{U}_{\mathcal{T}_u} \vee \{B\} \subseteq \mathcal{B}$ and $\mathcal{B}^{(u)} \subseteq \mathcal{U}$. By (ii) there exists $\mathcal{W} \in \beta X$ with $\mathcal{W}^{(u)} \subseteq \mathcal{A}$ and $\mathcal{W}^{(u)} \subseteq \mathcal{B}$. For every $W \in \mathcal{W}$ we have $W^{(u)} \cap A \in \mathcal{A}$ and $W^{(u)} \cap B \in \mathcal{B}$. By Lemma 4.1.4.3 we get $A \cap C^{(u)} \neq \emptyset$ or $B \cap (X \setminus C)^{(u)} \neq \emptyset$, for all $C \subseteq X$, which is impossible. Hence we have (iii).

Next we prove (iii) implies (ii). Let $\mathcal{A},\mathcal{B},\mathcal{U} \in \beta X$ with $\hat{a}(\mathcal{A},\mathcal{U}) < \gamma$ and $\hat{a}(\mathcal{B},\mathcal{U}) < \gamma$ and choose $u, v > 0$ such that $\hat{a}(\mathcal{A},\mathcal{U}) < u < v < \gamma$ and $\hat{a}(\mathcal{B},\mathcal{U}) < u < v < \gamma$. For $A \in \mathcal{A} \cap \mathcal{T}_v$ and $B \in \mathcal{B} \cap \mathcal{T}_v$ we have $A^{(u)} \cap B^{(u)} \neq \emptyset$. By (iii) we have $A \cap B \neq \emptyset$. So

$$\{A \cap B \mid A \in \mathcal{A} \cap \mathcal{T}_v, B \cap \mathcal{T}_v\}$$

is a filter base and a finer ultrafilter exists. Applying the second equivalence used in the proof of (ii) ⇔ (iii) in V.2.4.4. in [HST14] to the topology $\mathcal{T}_v$ we obtain $\mathcal{W} \in \beta X$ satisfying $\mathcal{W}^{(v)} \subseteq \mathcal{A}$ and $\mathcal{W}^{(v)} \subseteq \mathcal{B}$. So we can conclude that $\hat{a}(\mathcal{W},\mathcal{A}) \leq v < \gamma$ and $\hat{a}(\mathcal{W},\mathcal{B}) \leq v \leq \gamma$. □

Theorem 4.2.5.4. *For a non-Archimedean approach space with tower* $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$*, consider the following properties:*

1. *X is strongly extremally disconnected.*

2. *X is almost strongly extremally disconnected.*

3. *X is* $(\beta, \mathsf{P}_\vee)$*-extremally disconnected.*

4. *X is extremally disconnected at level* 0.

The following implications hold: 1. ⇒ 2. ⇒ 3. and 1. ⇒ 4.

Proof. 1. ⇒ 2. and 1. ⇒ 4. are straightforward.

We show that 2. ⇒ 3. using characterization (iii) in Theorem 4.2.5.3. Suppose $A \cap B = \emptyset$ with $A, B \in \mathcal{T}_v$ and let $u < v$. Since $(X, \mathcal{T}_v)$ is extremally disconnected we have $A^{(v)} \cap B^{(v)} = \emptyset$ and since $\mathcal{T}_v \subseteq \mathcal{T}_u$ the conclusion follows. □

There are no other valid implications between the properties considered in the previous theorem. This is shown by the next examples.

Example 4.2.5.5. We use the ultrametric space $X = \big(\mathsf{P}, d_M\big)$ that played a crucial role in the development of the Banaschewski compactification in [CS17]. On P let d_M be defined by

$$d_M(x,y) := \begin{cases} x \vee y & x \neq y, \\ 0 & x = y. \end{cases}$$

X clearly is an ultrametric space. When it is considered as an approach space it has a tower $(\mathcal{R}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ of zero-dimensional topologies. Since d_M is symmetric, the space $\big(\mathsf{P}, d_M\big)$ is $(\beta, \mathsf{P}_\vee)$-extremally disconnected. The topology $\mathcal{T}_0$ is discrete in all points $x \neq 0$ and has the usual neighborhood filter in 0 for the Euclidean topology. The set $A = \{\frac{1}{n} \mid n > 0\}$ is open in $(X, \mathcal{T}_0)$ but its closure is $A \cup \{0\}$ which clearly is not open. So $\mathcal{T}_0$ is not extremally disconnected and hence X is not strongly extremally disconnected.

Let $\varepsilon > 0$ and consider the level topology $\big(\mathsf{P}, \mathcal{R}_\varepsilon\big)$. Consider the open set $A = \{\frac{1}{n} + \varepsilon \mid n > 0\}$, then its closure $A^{(\varepsilon)} = A \cup [0, \varepsilon]$ which is not open at the level ε. So although 3. is fulfilled, none of the other conditions in Theorem 4.2.5.4 hold.

Example 4.2.5.6. Let $X = [0, \infty[$, endowed with the topology $\mathcal{T}$ with neighborhood filters $\big(\mathcal{V}(x)\big)_{x\in X}$ where $\mathcal{V}(0)$ is the usual neighborhood filter in the Euclidean topology, and $\mathcal{V}(x) = \dot{x}$ whenever $x \neq 0$, so at level 0 we use the same topology as in the previous Example 4.2.5.5. We define $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ with $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{T}_\varepsilon$ at level $0 < \varepsilon$ having a neighborhood filter

$$\mathcal{V}_\varepsilon(x) = \begin{cases} \text{stack}\{[0, x]\} & \text{whenever } 0 < x \leq \varepsilon, \\ \mathcal{V}(x) & \text{whenever } \varepsilon < x \text{ or } x = 0 \end{cases}$$

at $x \in X$. The topology at level 0 is $([0, \infty[, \mathcal{T})$. As we know from Example 4.2.5.5, it is not extremally disconnected.

Next we consider level $\varepsilon > 0$ and $([0, \infty[, \mathcal{T}_\varepsilon)$. Let $A \in \mathcal{T}_\varepsilon$ then either there exists $x \leq \varepsilon$ with $x \in A$. In this case $A^{(\varepsilon)} = [0, \varepsilon] \cup \big(A \cap]\varepsilon, \infty[\big)$ which is open. Or there is no $x \leq \varepsilon$ with $x \in A$. In this case we have $A^{(\varepsilon)} = A$ open. X is extremally disconnected at every strictly positive level, so 2. and 3. from Theorem 4.2.5.4 are satisfied whereas 1. and 4. are not.

Example 4.2.5.7. Let (X, d) be as in Example 4.2.4.8 and now consider (X, d^-). Then we have a space with $\mathcal{T}_0$ extremally disconnected which is not $(\beta, \mathsf{P}_\vee)$-extremally disconnected, so it satisfies 4., but none of the other conditions in Theorem 4.2.5.4.

To see that the notions $(\beta, \mathsf{P}_\vee)$-extremally disconnected and (β, P_+)-extremally disconnected are unrelated, we can consider examples (X, d^-) for each of the spaces described in Example 4.2.4.9 and Example 4.2.4.10.

4.2.6 Topological properties of the initially dense objects

We come back to the examples introduced in Example 2.1.2.3 and Example 2.5.1.3 and investigate their topological properties.

Example 4.2.6.1. First we consider $X = (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})$ as described in Example 2.1.2.3. Clearly the limit operator satisfies $\lambda\mathcal{U}(0) = 0$, for any $\mathcal{U} \in \beta\mathsf{P}_\vee$. Thus the space is compact at level 0.

$\mathsf{P}_\vee$ is T_0 at level 0 since $\lambda\dot{x}(y) = 0 = \lambda\dot{y}(x)$ implies $x = y$. However, the space is not $(\beta, \mathsf{P}_\vee)$-T_0. To see this, take $x, y \in [0, \infty[$ such that $x < y$. Then $\lambda\dot{x}(y) = y < \infty$ and $\lambda\dot{y}(x) = 0 < \infty$, but $x \neq y$.

Since $\delta_{\mathsf{P}_\vee}(0, A) = 0$ for every non-empty subset A, we have $0 \in \mathrm{cl}\,A$ in the topology $\mathcal{T}_0$ for every non-empty subset A. So clearly $\mathcal{T}_0$ is neither T_1 nor regular. Therefore $(\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})$ is not $(\beta, \mathsf{P}_\vee)$-T_1 nor $(\beta, \mathsf{P}_\vee)$-regular.

We know that T_0 and R_0 at level 0 is equivalent to T_1 at level 0. Since X is T_0 at level 0 but not T_1 at level 0, this implies that X is not R_0 at level 0 and neither R_1 at level 0.

$(\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})$ is strongly normal and strongly extremally disconnected. This follows from the fact that at level 0 (and hence all other levels) all non-empty closed sets contain 0. We conclude that all levels are normal. On the other hand there are no non-empty disjoint open sets at level 0 since $A \cap A' = \emptyset$ and $\infty \in \mathsf{P}_\vee \setminus A$ implies $\mathsf{P}_\vee \subseteq \mathsf{P}_\vee \setminus A$. So all levels are extremally disconnected.

Example 4.2.6.2. Next we consider the examples $(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ and $(\mathsf{P}_\vee, d^-_{\mathsf{P}_\vee})$ as described in Example 2.5.1.3. For $X = (\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ and for any $\mathcal{U} \in \beta\mathsf{P}_\vee$, we have that $\lambda_{d_{\mathsf{P}_\vee}}\mathcal{U}(\infty) = \inf_{U\in\mathcal{U}} \sup_{y\in U} d_{\mathsf{P}_\vee}(\infty, y) = 0$, so we get that $(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ is compact at level 0.

For $x, y \in \mathsf{P}_\vee$ arbitrary, we have that $\lambda_{d_{\mathsf{P}_\vee}}\dot{x}(y) = 0$ if and only if $x \leq y$. This proves that $(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ is T_0 at level 0. However, $(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ is not $(\beta, \mathsf{P}_\vee)$-T_0. To see this take $x, y \in [0, \infty[$ such that $x < y$. Then $\lambda_{d_{\mathsf{P}_\vee}}\dot{x}(y) = 0 < \infty$ and $\lambda_{d_{\mathsf{P}_\vee}}\dot{y}(x) = y < \infty$, but $x \neq y$.

Since $d_{\mathsf{P}_\vee}(\infty, z) = 0$, for all $z \in \mathsf{P}_\vee$, we have $\infty \in \mathrm{cl}(A)$ in the topology $\mathcal{T}_0$ for every non-empty subset A. So clearly $\mathcal{T}_0$ is not T_1. Therefore $(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ is not $(\beta, \mathsf{P}_\vee)$-T_1.

We know that T_0 and R_0 at level 0 is equivalent to T_1 at level 0. Since X is T_0 at level 0 but not T_1 at level 0, this implies that X is not R_0 at level 0 and neither R_1 at level 0.

$(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ is not $(\beta, \mathsf{P}_\vee)$-regular since the quasi-ultrametric is not symmetric. Moreover as $\infty \in \mathrm{cl}(A)$ for every subset A, the topology $\mathcal{T}_0$ is not regular either.

$(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ is strongly normal and strongly extremally disconnected. To see this, take $A \subseteq \mathsf{P}_\vee$ non-empty. We notice that at level 0 (and hence at all other levels) $\infty \in \mathrm{cl}(A)$. So all non-empty closed sets contain ∞. We conclude that all levels are normal. On the other hand all non-empty open sets contain 0. So all levels are extremally disconnected.

The results for $(\mathsf{P}_\vee, d^-_{\mathsf{P}_\vee})$ follow analogously.

4.2.7 Compact Hausdorff non-Archimedean approach spaces

In Proposition 4.2.2.3 we proved that a non-Archimedean approach space X that is at the same time $(\beta, \mathsf{P}_\vee)$-compact and $(\beta, \mathsf{P}_\vee)$-Hausdorff is topological. In this section we consider the axioms compact at level 0 in combination with Hausdorff at level 0, in the sense that the topology $\mathcal{T}_0$ is both compact and Hausdorff. In [Low15] these conditions are called compact and Hausdorff and in sections 4.2.7 and 4.2.8 we will use this simplified terminology. We consider $\mathsf{NA\text{-}App}_2$, the full subcategory of $\mathsf{NA\text{-}App}$ consisting of all Hausdorff non-Archimedean approach spaces and $\mathsf{NA\text{-}App}_{c2}$ the full subcategory of $\mathsf{NA\text{-}App}_2$ consisting of all compact Hausdorff non-Archimedean approach spaces. For terminology used in this section, we refer to [AHS06].

$\mathsf{NA\text{-}App}_2$ is epireflective in $\mathsf{NA\text{-}App}$ and is closed under the construction of finer structures, hence it is a quotient reflective subcategory of $\mathsf{NA\text{-}App}$. So $\mathsf{NA\text{-}App}_2$ is monotopological, i.e. for a family $(X_i)_{i\in I}$ of Hausdorff non-Archimedean approach spaces and any point-separating source $(f_i : X \longrightarrow X_i)_{i\in I}$ there exists a unique initial lift on X. Our next aim in this section is to determine the epimorphisms and extremal monomorphisms in this category.

Theorem 4.2.7.1. *A contraction $f : X \longrightarrow Y$ in $\mathsf{NA\text{-}App}_2$ is an epimorphism in $\mathsf{NA\text{-}App}_2$ if and only if $f(X)$ is $\mathrm{C}_{\mathsf{Top}}\, Y$-dense, i.e. $\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}\, Y}\, f(X) = Y$ in the topological coreflection $\mathrm{C}_{\mathsf{Top}}\, Y$ of Y.*

Proof. One implication is straightforward. If $f : X \longrightarrow Y$ is a contraction in $\mathsf{NA\text{-}App}_2$ and $f(X)$ is $\mathrm{C}_{\mathsf{Top}}\, Y$-dense, then $f : \mathrm{C}_{\mathsf{Top}}\, X \longrightarrow \mathrm{C}_{\mathsf{Top}}\, Y$ is an epimorphism in Haus. This implies f is an epimorphism in $\mathsf{NA\text{-}App}_2$, since for any two contractions $u, v : Y \longrightarrow Z$ in $\mathsf{NA\text{-}App}_2$ with $u \cdot f = v \cdot f$, also $\mathrm{C}_{\mathsf{Top}}\, u \cdot \mathrm{C}_{\mathsf{Top}}\, f = \mathrm{C}_{\mathsf{Top}}\, v \cdot \mathrm{C}_{\mathsf{Top}}\, f$ in Haus.

In order to prove the converse, suppose $f : X \longrightarrow Y$ is a contraction in $\mathsf{NA\text{-}App}_2$ with $f(X)$ not $\mathrm{C}_{\mathsf{Top}}\, Y$-dense. Set $M = \mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}\, Y}\big(f(X)\big)$. We use a technique based on the amalgamation, for which we refer to [DGT88]. Assume Y has a gauge $\mathcal{H}$ consisting of quasi-ultrametrics. The gauge basis of the amalgamation $Y \coprod_M Y$ is given by

$$\mathcal{H}_{Y \coprod_M Y} = \{u_{g_d} \mid d \in \mathcal{H}\} \downarrow,$$

where

$$u_{g_d}(\overline{x_i}, \overline{y_j}) = \begin{cases} \inf_{m\in M} d(x,m) \vee d(m,y) & i \neq j, x, y \notin M, \\ d(x,y) & \text{elsewhere.} \end{cases}$$

Moreover since M is $\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}} Y}$-closed in Y, the space $Y \coprod_M Y$ is Hausdorff. Both statements follow analogously to the construction and results in [CCG07]. Now consider the diagram

$$X \xrightarrow{f} Y \underset{j_2}{\overset{j_1}{\rightrightarrows}} Y \coprod Y \xrightarrow{\varphi} Y \coprod_M Y$$

Let $u = \varphi \cdot j_1$ and $v = \varphi \cdot j_2$. Then $u \cdot f = v \cdot f$, but $u \neq v$. Hence f is not an epimorphism in $\mathsf{NA\text{-}App}_2$. □

We proceed by determining the extremal monomorphisms in $\mathsf{NA\text{-}App}_2$. Since $\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}$ is idempotent and weakly hereditary, $\mathsf{NA\text{-}App}_2$ is an $(\mathcal{E}^{\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}}, \mathcal{M}^{\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}})$-category, where $\mathcal{E}^{\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}} = \mathsf{Epi}(\mathsf{NA\text{-}App}_2)$. The following theorem determines the extremal monomorphisms in $\mathsf{NA\text{-}App}_2$.

Theorem 4.2.7.2. *The following classes of morphisms in* $\mathsf{NA\text{-}App}_2$ *coincide:*

1. *The class of all regular monomorphisms;*
2. *The class of all extremal monomorphisms;*
3. *The class of all* $\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}$*-closed embeddings, i.e. embeddings* $f : X \longrightarrow Y$ *where* $\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}} Y} f(X) = f(X)$ *in the topological coreflection* $\mathrm{C}_{\mathsf{Top}} Y$ *of* Y.

Proof. To prove that extremal monomorphisms are $\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}$-closed embeddings, take $\mathcal{E}^{\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}}$ the class of all $\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}$-dense contractions and $\mathcal{M}^{\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}}$ the class of all $\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}$-closed embeddings. By the theorem in section 2.4 of [DT95], we have that $\mathsf{NA\text{-}App}_2$ is an $(\mathcal{E}^{\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}}, \mathcal{M}^{\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}})$-category. Since by Theorem 4.2.7.1, we know that $\mathcal{E}^{\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}} =$
$\mathsf{Epi}(\mathsf{NA\text{-}App}_2)$, it follows that $\mathsf{ExtrMono}(\mathsf{NA\text{-}App}_2) \subseteq \mathcal{M}^{\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}}$ [AHS06].

In order to prove that every $\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}}(\cdot)}$-closed embedding is a regular monomorphism, take $f : X \longrightarrow Y$ a $\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}} Y}$-closed embedding in $\mathsf{NA\text{-}App}_2$. Consider the construction of the amalgamation of Y with respect to $f(X)$:

$$X \xrightarrow{f} Y \underset{j_2}{\overset{j_1}{\rightrightarrows}} Y \coprod Y \xrightarrow{\varphi} Y \coprod_{f(X)} Y$$

Since $f(X)$ is $\mathrm{cl}_{\mathrm{C}_{\mathsf{Top}} Y}$-closed, the amalgamation is Hausdorff. f is the equalizer of the pair $(\varphi \cdot j_1, \varphi \cdot j_2)$. Hence f is a regular monomorphism. □

We recall the following definitions from section 8.1 in [DT95]. Consider a functor $F : \mathsf{X} \longrightarrow \mathsf{Y}$. For an object $Y \in \mathsf{Y}$, let $F^{-1}(Y) := \{X \in \mathsf{X} \mid FX = Y\}$ be the class of objects of the *fiber* of F at Y, and let $\widetilde{F^{-1}}(Y) := \{X \in \mathsf{X} \mid FX \cong Y\}$ be its *replete closure* in Y. F is called *transportable* if for every $B \in \widetilde{F^{-1}}(Y)$, there is $A \in F^{-1}Y$ with $A \cong B$.

Proposition 4.2.7.3. The restriction of the coreflector $\mathrm{C}_{\mathsf{Top}} : \mathsf{NA\text{-}App}_2 \longrightarrow \mathsf{Haus}$ is transportable.

Proof. Take $(Y, \mathcal{T})$ an arbitrary object in Haus. Take $\big(X, (\mathcal{S}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\big) \in \widetilde{\mathrm{C}_{\mathsf{Top}}}^{-1}(Y, \mathcal{T})$ arbitrary. There exists an isomorphism $f : (X, \mathcal{S}_0) \longrightarrow (Y, \mathcal{T})$ in Haus. For any $\varepsilon > 0$, define the topology $\mathcal{T}_\varepsilon$ on Y by transportation of the topology on X by putting $\mathcal{T}_\varepsilon = \big\{f(A) \mid A \in \mathcal{S}_\varepsilon\big\}$ and let $\mathcal{T}_0 = \mathcal{T}$. Then $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ is a non-Archimedean tower on Y in the fiber of $(Y, \mathcal{T})$ and $f : \big(X, (\mathcal{S}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\big) \longrightarrow \big(Y, (\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\big)$ is an isomorphism in $\mathsf{NA\text{-}App}_2$. □

Now we can conclude the following.

Theorem 4.2.7.4. $\mathsf{NA\text{-}App}_2$ *is cowellpowered.*

Proof. By Theorem 4.2.7.1 we have $\mathrm{C}_{\mathsf{Top}}\big(\mathsf{Epi}(\mathsf{NA\text{-}App}_2)\big) \subseteq \mathsf{Epi}(\mathsf{Haus})$. Since $\mathrm{C}_{\mathsf{Top}}$ is fiber small and transportable, the theorem from section 8.1 in [DT95] implies cowellpoweredness of $\mathsf{NA\text{-}App}_2$. □

Theorem 4.2.7.5. $\mathsf{NA\text{-}App}_{c2}$ *is an epireflective subcategory of* $\mathsf{NA\text{-}App}_2$.

Proof. By Theorem 2.3.3.1, NA-App is closed in App under the formation of products and subspaces. Since the class of compact Hausdorff approach spaces is well known to be closed under the formation of products and closed subspaces in App [Low15], we can conclude that $\mathsf{NA\text{-}App}_{c2}$ is closed under formation of products and closed subspaces in $\mathsf{NA\text{-}App}_2$. Since by Theorem 4.2.7.1, Theorem 4.2.7.2 and Theorem 4.2.7.4 $\mathsf{NA\text{-}App}_2$ is a cowellpowered (Epi, Extremal mono)-category, $\mathsf{NA\text{-}App}_{c2}$ is epireflective in $\mathsf{NA\text{-}App}_2$. □

4.2.8 Non-Archimedean Hausdorff compactifications

Due to the foregoing Theorem 4.2.7.5, there is a categorical construction of an epireflector $E : \mathsf{NA\text{-}App}_2 \longrightarrow \mathsf{NA\text{-}App}_{c2}$ with epireflection morphisms $e_X : X \longrightarrow KX$, for every $X \in \mathsf{NA\text{-}App}_2$. The question remains whether the epireflection morphisms e_X are embeddings. The following proposition shows that in general, this is not the case.

Proposition 4.2.8.1. A Hausdorff non-Archimedean approach space X that can be embedded in a compact Hausdorff non-Archimedean approach space Y has a topological coreflection $\mathrm{C}_{\mathsf{Top}} X$ that is a Tychonoff space.

Proof. If $f : X \longrightarrow Y$ is an embedding with $Y \in \mathsf{NA\text{-}App}_{c2}$, then $\mathrm{C}_{\mathsf{Top}} f : \mathrm{C}_{\mathsf{Top}} X \longrightarrow \mathrm{C}_{\mathsf{Top}} Y$ is an embedding in Top with $\mathrm{C}_{\mathsf{Top}} Y$ a compact Hausdorff topological space. So $\mathrm{C}_{\mathsf{Top}} X$ is a Tychonoff space. □

In Theorem 4.2.8.3 below we will formulate sufficient conditions on a Hausdorff non-Archimedean approach space X, to ensure that there exists an embedding into a compact Hausdorff non-Archimedean approach space. Our construction is based on Shanin's compactification of topological spaces.

Given $\mathfrak{S}$, a collection of closed sets of a topological space X satisfying the conditions (i) $\emptyset, X \in \mathfrak{S}$ and (ii) $G_1, G_2 \in \mathfrak{S} \Rightarrow G_1 \cup G_2 \in \mathfrak{S}$, a $\mathfrak{S}$-*family* is a non-empty collection of sets belonging to $\mathfrak{S}$ satisfying the finite intersection property (f.i.p.). A $\mathfrak{S}$-family $\mathcal{F}$ is called *vanishing* if $\bigcap_{F\in\mathcal{F}} F = \emptyset$. A *maximal* $\mathfrak{S}$-family is a $\mathfrak{S}$-family which is not contained in any $\mathfrak{S}$-family as a proper subcollection. If $\mathfrak{S}$ moreover is a closed basis of X, then a compact topological space $\sigma(X, \mathfrak{S}) = (S, \mathcal{S})$ in which X is densely embedded is constructed in [Nag68] on the set $S = X \cup X'$ with X' the set of all maximal vanishing $\mathfrak{S}$-families. S is endowed with the topology $\mathcal{S}$ with $\{S(G) \mid G \in \mathfrak{S}\}$ as a closed basis, where $S(G) = G \cup \{p \in X' \mid G \in p\}$.

Given a non-Archimedean approach space X we will construct an extension of X by first considering a special closed base $\mathfrak{S}$ for $\mathrm{C}_{\mathsf{Top}} X$ and constructing $\sigma(\mathrm{C}_{\mathsf{Top}} X, \mathfrak{S})$.

Theorem 4.2.8.2. *Any non-Archimedean approach space X, given by its tower of topologies $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ and tower of closed sets $(\mathcal{C}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$, can be densely embedded in a compact non-Archimedean approach space $\Sigma(X, \mathfrak{S})$, constructed from the closed basis $\mathfrak{S} = \bigcup_{\varepsilon>0} \mathcal{C}_\varepsilon$ of $\mathrm{C}_{\mathsf{Top}} X$ and such that the topological coreflection $\mathrm{C}_{\mathsf{Top}} \Sigma(X, \mathfrak{S})$ is the Shanin compactification $\sigma(\mathrm{C}_{\mathsf{Top}} X, \mathfrak{S})$ of the topological coreflection $\mathrm{C}_{\mathsf{Top}} X$.*

Proof. Let X be a non-Archimedean approach space given by its tower of topologies $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ and tower of closed sets $(\mathcal{C}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$, and take $\mathfrak{S} = \bigcup_{\varepsilon>0} \mathcal{C}_\varepsilon$. By the coherence condition of the non-Archimedean tower, $\mathfrak{S}$ is a closed basis for the topological space $\mathrm{C}_{\mathsf{Top}} X = (X, \mathcal{T}_0)$ and moreover $\mathfrak{S}$ clearly satisfies the assumptions (i) and (ii) made above. Let $\sigma(\mathrm{C}_{\mathsf{Top}} X, \mathfrak{S}) = (S, \mathcal{S})$ be Shanin's compactification.

For every $\varepsilon \in \mathbb{R}^+$ the collection $\{S(G) \mid G \in \mathcal{C}_\varepsilon\}$ is a basis for the closed sets $\mathcal{D}_\varepsilon$ of a topology $\mathcal{R}_\varepsilon$ on S. However, for $(\mathcal{R}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ the coherence condition from Corollary 2.1.3.2 need not be satisfied. In order to force the coherence condition, we define the following tower of topologies on S; for $\alpha \in \mathbb{R}^+$, let

$$\mathcal{S}_\alpha = \bigvee_{\beta>\alpha} \mathcal{R}_\beta,$$

with the supremum taken in Top. The set S endowed with the tower $(\mathcal{S}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ defines a non-Archimedean approach space which we denote $\Sigma(X,\mathfrak{S})$.

Next we investigate its topological coreflection $\mathrm{C}_{\mathsf{Top}}\,\Sigma(X,\mathfrak{S}) = (S,\mathcal{S}_0)$. By definition, the collection $\bigcup_{\beta>0}\mathcal{D}_\beta$ is a closed basis for $(S,\mathcal{S}_0)$ and therefore the collection

$$\{S(G) \mid G\in\mathcal{C}_\varepsilon,\ \text{for some } \varepsilon>0\} = \{S(G)\mid G\in\mathfrak{S}\}$$

is a basis for its closed sets too. It follows that $(S,\mathcal{S}_0) = \sigma(\mathrm{C}_{\mathsf{Top}}\,X,\mathfrak{S})$, so $\Sigma(X,\mathfrak{S})$ is a compact non-Archimedean approach space.

Let $j : X \longrightarrow S$ be the canonical injection. By construction, for $\varepsilon\in\mathbb{R}^+$ the map $j : (X,\mathcal{T}_\varepsilon)\longrightarrow(S,\mathcal{R}_\varepsilon)$ is an embedding in Top. Since $\mathcal{T}_\varepsilon = \bigvee_{\gamma>\varepsilon}\mathcal{T}_\gamma$, the source $\big(j : (X,\mathcal{T}_\varepsilon)\longrightarrow(S,\mathcal{R}_\gamma)\big)_{\gamma>\varepsilon}$ is initial in Top, which then implies that $j : (X,\mathcal{T}_\varepsilon)\longrightarrow(S,\mathcal{S}_\varepsilon)$ is initial too. So finally we have that

$$j : \big(X,(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\big)\longrightarrow\big(S,(\mathcal{S}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\big)$$

is an embedding in NA-App. That the embedding j is dense follows already from the result at level 0. □

In general, this compactification is not Hausdorff. To ensure the Hausdorff property, we need stronger conditions on X.

Theorem 4.2.8.3. *Let X be a non-Archimedean approach space given by its tower of closed sets $(\mathcal{C}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$. The compactification described in Theorem 4.2.8.2 is Hausdorff if and only if the following conditions are fulfilled:*

1. *X is Hausdorff,*
2. $\forall\varepsilon>0 : \forall G\in\mathcal{C}_\varepsilon, \forall x\notin G : \exists 0<\gamma\le\varepsilon, \exists H\in\mathcal{C}_\gamma$ *such that* $x\in H$ *and* $H\cap G=\emptyset$,
3. $\forall\varepsilon>0 : \forall F,G\in\mathcal{C}_\varepsilon, F\cap G=\emptyset : \exists 0<\gamma\le\varepsilon, \exists H,K\in\mathcal{C}_\gamma$ *such that* $F\cap H=\emptyset$, $G\cap K=\emptyset$, $H\cup K = X$.

Proof. Remark that the collection $\mathfrak{S} = \bigcup_{\varepsilon>0}\mathcal{C}_\varepsilon$ used in the construction of Theorem 4.2.8.2 is closed under finite intersections. Using this fact, condition 2. is equivalent to condition C) and 3. is equivalent to condition D) in [Nag68]. So Theorem IV.3 C) and D) in [Nag68] can be applied. □

Corollary 4.2.8.4. For any non-Archimedean approach space X that is Hausdorff, $(\beta,\mathsf{P}_\vee)$-regular and $(\beta,\mathsf{P}_\vee)$-normal, the compactification described in Theorem 4.2.8.2 is Hausdorff.

Proof. Condition 1. in Theorem 4.2.8.3 is fulfilled. By Theorem 4.2.3.2 all strictly positive level topologies are regular, so in 2. of Theorem 4.2.8.3 we can take $\gamma = \varepsilon$. In order to see that also 3. is fulfilled let $\varepsilon > 0$ and $F, G \in \mathcal{C}_\varepsilon$. Let $0 < \gamma < \varepsilon$ be arbitrary. Since X is $(\beta, \mathsf{P}_\vee)$-normal, by Theorem 4.2.4.4 there exists $C \subseteq X$ with $F \cap C^{(\gamma)} = \emptyset$ and $(X \setminus C)^{(\gamma)} \cap G = \emptyset$. Then $H = C^{(\gamma)}$ and $K = (X \setminus C)^{(\gamma)}$ satisfy the conditions needed in 3. □

Remark that in the previous corollary the condition $(\beta, \mathsf{P}_\vee)$-normal can be weakened to Van Olmen's normality, condition (c) in Theorem 4.1.4.4. Note that if X is a topological (non-Archimedean approach) space, $\mathcal{T}_\varepsilon = \mathcal{T}_0$ and $\mathcal{C}_\varepsilon = \mathcal{C}_0$, for every $\varepsilon > 0$. In this case condition 3. is equivalent to X being a normal topological space.

We get the following corollary.

Corollary 4.2.8.5. Let X be a non-Archimedean approach space such that the conditions from Theorem 4.2.8.3 are fulfilled. Then the topological coreflection $\mathrm{C}_{\mathsf{Top}}\, X$ is Tychonoff.

Proof. The conditions from Theorem 4.2.8.3 imply that X can be embedded in a compact non-Archimedean approach space $\Sigma(X, \mathfrak{S})$, with $\mathrm{C}_{\mathsf{Top}}\, \Sigma(X, \mathfrak{S})$ the Shanin compactification of $\mathrm{C}_{\mathsf{Top}}\, X$, a compact Hausdorff topological space. □

As the Shanin compactification need not be a reflection, we end this section by noting that in general the dense embedding $j : X \longrightarrow \Sigma(X, \mathfrak{S})$ constructed above, is not a reflection.

4.3 Topological properties in relational algebras

In this section we investigate topological properties p in the context of relational algebras in $(\mathbb{T}, 2)$-Cat.

We give special attention to power-enriched monads $\mathbb{T}$ together with their Kleisli extension to Rel and prove some general results. We will focus on the examples $(\mathbb{F}, 2)$-Cat $\cong$ Top and $(\mathbb{I}, 2)$-Cat $\cong$ App. Some results for $(\mathbb{F}, 2)$-Cat $\cong$ Top are known and those can be found in [HST14], but we also add some new results concerning $(\mathbb{F}, 2)$-regularity.

The prime functional ideal monad $\mathbb{B}$ is not power-enriched, but it is a submonad of $\mathbb{I}$ with interesting properties. Therefore we also explore the topological properties in $(\mathbb{B}, 2)$-Cat $\cong$ App.

4.3.1 Improper elements

Let $\mathfrak{p}$ be the least element of TX which we call improper, all other elements are called proper.

From [HST14] we know that for a relational $\mathbb{T}$-algebra (X, a) we have $\mathfrak{p}\, a\, x$, for all $x \in X$. Hence, in order to get interesting results, we will often need to restrict to proper elements.

We put

$$T_pX = TX \setminus \{\mathfrak{p}\}$$

the collection of all proper elements of TX and $j_X : T_pX \longrightarrow TX$ the canonical injection. For $f : X \longrightarrow Y$ the map $Tf : TX \longrightarrow TY$ is a sup-map, so we have $Tf(\mathcal{X})$ proper implies $\mathcal{X}$ proper. In order to have equivalence, we make the following assumption:

$$T_pX = (Tf)^{\circ}(T_pY) \tag{4.41}$$

for all X, Y and $f : X \longrightarrow Y$, that is

$$\mathcal{X} \text{ proper } \Leftrightarrow Tf(\mathcal{X}) \text{ proper}$$

for all $f : X \longrightarrow Y$, $\mathcal{X} \in TX$. Consequently, under this assumption

$$\begin{array}{ccc} T_pX & \xrightarrow{T_pf} & T_pY \\ {\scriptstyle j_X}\downarrow & & \downarrow{\scriptstyle j_Y} \\ TX & \xrightarrow{Tf} & TY \end{array}$$

is a pullback diagram for all $f : X \longrightarrow Y$, and T_pf the restriction of Tf. In particular, for such a monad $\mathbb{T}$, we may consider T_p as a subfunctor of T. However, in general one does not obtain a submonad of $\mathbb{T}$ as will be the case in some of our examples.

4.3.2 Hausdorff separation

Recall that an object (X, a) in $(\mathbb{T}, 2)$-Cat is $(\mathbb{T}, 2)$-*Hausdorff* if

$$\mathcal{X}\, a\, x \,\&\, \mathcal{X}\, a\, y \Rightarrow x = y, \tag{4.42}$$

for all $\mathcal{X} \in TX$ and $x, y \in X$.

The following result from [HST14] gives a characterization of $(\mathbb{T}, 2)$-Hausdorff for power-enriched monads $(\mathbb{T}, \tau)$ together with their Kleisli extension.

Proposition 4.3.2.1. Given a monad $\mathbb{T}$ power-enriched by $\tau : \mathbb{P} \longrightarrow \mathbb{T}$ together with its Kleisli extension $\check{\mathbb{T}}$ to Rel. Let $(X, \to)$ be an object in $(\mathbb{T}, 2)$-Cat.

$(X, \to)$ is $(\mathbb{T}, 2)$-Hausdorff if and only if $|X| \leq 1$.

To get a more interesting notion of Hausdorff separation in relational $\mathbb{T}$-algebras, we will restrict to proper elements. By $(T_p, 2)$-Hausdorff we mean the following property.

Definition 4.3.2.2. An object (X, a) in $(\mathbb{T}, 2)$-Cat for a power-enriched monad $\mathbb{T}$ together with its Kleisli extension is $(T_p, 2)$-*Hausdorff* if

$$\mathcal{X}\, a\, x \,\&\, \mathcal{X}\, a\, y \Rightarrow x = y, \tag{4.43}$$

for all $\mathcal{X} \in T_p X$ and $x, y \in X$.

Hausdorff separation in $(\mathbb{F}, 2)$-Cat

The filter monad $\mathbb{F}$ is a power-enriched monad, see Example 1.4.1.2, and Top is known to be isomorphically described as $(\mathbb{F}, 2)$-Cat, see Section 1.4.4.

The properties $(\mathbb{F}, 2)$-Hausdorff and $(\mathsf{F}_p, 2)$-Hausdorff are known and were studied in [HST14].

Since $\mathbb{F}$ is power-enriched, a topological space, seen as an $(\mathbb{F}, 2)$-algebra, is $(\mathbb{F}, 2)$-Hausdorff if and only if it has at most one point. When removing the least element $\mathfrak{p}$, which is the improper filter $\mathcal{P}X$, the filter monad is replaced by a submonad $\mathbb{F}_p = (\mathsf{F}_p, m, e)$ where $\mathsf{F}_p X = \mathsf{F}X \setminus \{\mathcal{P}X\}$ is the set of all proper filters on X. In this case the Hausdorff property restricted to F_p assumes its usual meaning: A topological space, seen as a relational $\mathbb{F}$-algebra, is $(F_p, 2)$-Hausdorff if and only if it is Hausdorff in the classical sense.

Hausdorff separation in $(\mathbb{I}, 2)$-Cat and $(\mathbb{B}, 2)$-Cat

We proceed by studying Hausdorff separation in $(\mathbb{I}, 2)$-Cat, for the power-enriched monad $\mathbb{I}$.

By Proposition 4.3.2.1, we immediately know that $(\mathbb{I}, 2)$-Hausdorff gives uninteresting results. Therefore we immediately restrict to proper elements.

Recall from Proposition 3.1.4.1 that the order induced by $\tau : \mathbb{P} \longrightarrow \mathbb{I}$, or equivalently the order induced by the Kleisli extension, coincides with the reversed inclusion order on functional ideals

$$\mathfrak{K} \leq \mathfrak{I} \Leftrightarrow \mathfrak{I} \subseteq \mathfrak{K},$$

for all $\mathfrak{I}, \mathfrak{K} \in \mathsf{I}X$.

The functional ideal $\mathsf{P}_b^X = \mathfrak{Z}_X$ is the least element (largest for the inclusion order) and $\tau_X(X) = \{0\}$ the largest one. We let

$$\mathsf{I}_p X = \mathsf{I}X \setminus \{\mathsf{P}_b^X\}$$

the collection of proper functional ideals. Condition (4.41) is fulfilled and I_p is a subfunctor of I. Remark that it does not generate a submonad. This can for instance be seen by taking $X = \{x\}$ a one point set. In this case we have $\mathsf{I}X = \big\{[0,\alpha] \mid \alpha < \infty\big\} \cup \{[0,\infty[\}$. Let $s : \mathsf{P} \longrightarrow \mathsf{I}X$ be the selection defined by $s(a) = [0,a]$ whenever a is finite, and $s(\infty) = [0,\infty[$. Let $\mathcal{G}$ be the filter on P generated by $\{[\alpha,\infty[| \ \alpha < \infty\}$, and $\mathfrak{I} = \iota_{\mathsf{P}}(\mathcal{G})$. Then for $\Phi = \mathsf{I}s(\mathfrak{I})$ we have $c(m_X(\Phi)) = \infty$. It follows that the multiplication can not be restricted to a map $\mathsf{I}_p\mathsf{I}_pX \longrightarrow \mathsf{I}_pX$.

Definition 4.3.2.3. Given an approach space X with functional ideal convergence $\rightarrowtail$, X is $(\mathsf{I}_p, 2)$-*Hausdorff* if

$$\left.\begin{array}{l}\mathfrak{I} \rightarrowtail x\\ \mathfrak{I} \rightarrowtail y\end{array}\right\} \Rightarrow x = y, \tag{4.44}$$

for all $\mathfrak{I} \in \mathsf{I}_p X$ and $x, y \in X$.

Later on in Proposition 4.3.2.6 we show that this definition of Hausdorff separation in $(\mathbb{I}, 2)$-$\mathsf{Cat} \cong \mathsf{App}$ coincides with the property (β, P_+)-Hausdorff.

$\mathbb{B}$ is not power-enriched, nevertheless $\mathsf{B}\,X$ has an improper functional ideal $\mathfrak{Z}_X$. This improper element will imply trivial results once again.

Proposition 4.3.2.4. An approach space X is $(\mathbb{B}, 2)$-Hausdorff if and only if $|X| \leq 1$.

Proof. Consider an approach space X that is $(\mathbb{B}, 2)$-Hausdorff. For $x \in X$ arbitrary, we get $e_X(x) \rightsquigarrow x$ and $e_X(x) \subseteq \mathfrak{Z}_X$, hence $\mathfrak{Z}_X \rightsquigarrow x$. By $(\mathbb{B}, 2)$-Hausdorff, $|X| \leq 1$. The converse is clear. □

In order to get interesting results, we will again abandon improper elements. For a set X, let $\mathsf{B}_p\,X = \mathsf{B}\,X \setminus \{\mathsf{P}_b^X\}$. Then B_p can be considered as a subfunctor of B, however it is again not a submonad.

Definition 4.3.2.5. Given an approach space X with prime functional ideal convergence $\rightsquigarrow$. X is $(\mathsf{B}_p, 2)$-*Hausdorff* if

$$\left.\begin{array}{l}\mathfrak{M} \rightsquigarrow x\\ \mathfrak{M} \rightsquigarrow y\end{array}\right\} \Rightarrow x = y, \tag{4.45}$$

for all $\mathfrak{M} \in \mathsf{B}_p\,X$ and $x, y \in X$.

The following proposition shows that $(\mathsf{I}_p, 2)$-Hausdorff and $(\mathsf{B}_p, 2)$-Hausdorff coincide and are equivalent to the known property (β, P_+)-Hausdorff from Example 4.1.1.3.

Proposition 4.3.2.6. For an approach space X the following are equivalent:

(i) X is $(\mathsf{I}_p, 2)$-Hausdorff,

(ii) X is $(\mathsf{B}_p, 2)$-Hausdorff,

(iii) X is (β, P_+)-Hausdorff.

Proof. Since $\mathsf{B}_p X \subseteq \mathsf{I}_p X$ and $\mathfrak{M} \rightsquigarrow x$ implies $\mathfrak{M} \rightarrowtail x$, for all $\mathfrak{M} \in \mathsf{B}_p X$, it is clear that (i) implies (ii).

To prove (ii) implies (i), suppose X is $(\mathsf{B}_p, 2)$-Hausdorff and take $\mathfrak{I} \in \mathsf{I}_p X$ and $x, y \in X$ arbitrary. Suppose $\mathfrak{I} \rightarrowtail x$ and $\mathfrak{I} \rightarrowtail y$. Since $\mathfrak{I}$ is proper, there exists $\mathfrak{M} \in \mathsf{B}_m(\mathfrak{I})$ which is proper as well. Since $\mathfrak{I} \subseteq \mathfrak{M}$, we get $\mathfrak{M} \rightarrowtail x$ and $\mathfrak{M} \rightarrowtail y$ and thus, since $\mathfrak{M}$ is prime, $\mathfrak{M} \rightsquigarrow x$ and $\mathfrak{M} \rightsquigarrow y$. By $(\mathsf{B}_p, 2)$-Hausdorff separation we get $x = y$.

To show that (ii) implies (iii), take $\mathcal{U} \in \beta X$ and $x, y \in X$ arbitrary such that $\lambda\mathcal{U}(x) < \infty$ and $\lambda\mathcal{U}(y) < \infty$. Take $\alpha < \infty$ such that $\lambda\mathcal{U}(x) \leq \alpha$ and $\lambda\mathcal{U}(y) \leq \alpha$. Then $\iota_X(\mathcal{U}) \oplus \alpha \rightsquigarrow x$ and $\iota_X(\mathcal{U}) \oplus \alpha \rightsquigarrow y$, with $\iota_X(\mathcal{U}) \oplus \alpha \in \mathsf{B}_p X$. This implies $x = y$.

To prove (iii) implies (ii), take $\mathfrak{M} \in \mathsf{B}_p X$ and $x, y \in X$ such that $\mathfrak{M} \rightsquigarrow x$ and $\mathfrak{M} \rightsquigarrow y$. Theorem 3.2.1.2 gives us the existence of an ultrafilter $\mathcal{U} \in \beta X$ such that $\mathfrak{M} = \iota_X(\mathcal{U}) \oplus c(\mathfrak{M})$. Then $\lambda\mathcal{U}(x) \leq c(\mathfrak{M})$ and $\lambda\mathcal{U}(y) \leq c(\mathfrak{M})$. Since $\mathfrak{M} \in \mathsf{B}_p X$, $c(\mathfrak{M}) < \infty$ and thus $x = y$. □

4.3.3 Lower separation axioms

First of all, we recall Definition 4.1.1.4 for $\mathcal{V} = 2$. Let (X, a) be an object in $(\mathbb{T}, 2)$-Cat. Then, (X, a) is

1. $(\mathbb{T}, 2)$-T_0 if
$$e_X(x)\, a\, y \,\&\, e_X(y)\, a\, x \Rightarrow x = y, \tag{4.46}$$
for all $x, y \in X$.

2. $(\mathbb{T}, 2)$-T_1 if
$$e_X(x)\, a\, y \Rightarrow x = y, \tag{4.47}$$
for all $x, y \in X$.

3. $(\mathbb{T}, 2)$-R_0 if
$$e_X(x)\, a\, y \Leftrightarrow e_X(y)\, a\, x \tag{4.48}$$
for all $x, y \in X$.

4. $(\mathbb{T}, 2)$-R_1 if
$$\mathcal{X}\, a\, x \,\&\, \mathcal{X}\, a\, y \Rightarrow e_X(x)\, a\, y, \tag{4.49}$$
for all $x, y \in X$ and $\mathcal{X} \in TX$.

Lower separation axioms in $(\mathbb{F},2)$-Cat

Adapting these definitions of $(\mathbb{T},2)$-p, for p a lower separation property, to the situation of $(\mathbb{F},2)$-Cat, gives rise to the following topological properties in the category Top.

Let $(X,\to)$ be a topological space.

1. X is $(\mathbb{F},2)$-T_0 if
$$\dot{x}\to y\,\&\,\dot{y}\to x\Rightarrow x=y, \tag{4.50}$$
for all $x,y\in X$.
2. X is $(\mathbb{F},2)$-T_1 if
$$\dot{x}\to y\Rightarrow x=y, \tag{4.51}$$
for all $x,y\in X$.
3. X is $(\mathbb{F},2)$-R_0 if
$$\dot{x}\to y\Leftrightarrow \dot{y}\to x \tag{4.52}$$
for all $x,y\in X$.
4. X is $(\mathbb{F},2)$-R_1 if
$$\mathcal{F}\to x\,\&\,\mathcal{F}\to y\Rightarrow \dot{x}\to y, \tag{4.53}$$
for all $x,y\in X$ and $\mathcal{F}\in FX$.

The notions $(\mathbb{F},2)$-T_0, $(\mathbb{F},2)$-T_1 and $(\mathbb{F},2)$-R_0 coincide with the classical separation properties T_0, T_1 and R_0 in Top.

For $(\mathbb{F},2)$-R_1 the situation is different.

Proposition 4.3.3.1. A topological space X is $(\mathbb{F},2)$-R_1 if and only if it is indiscrete.

Proof. One implication is clear. To see the other one, note that the improper filter converges to any point of X. Hence $(\mathbb{F},2)$-R_1 implies $\dot{x}\to y$, for all $x,y\in X$. This coincides with X being indiscrete. □

In order to avoid these uninteresting results, we will restrict to proper elements.

Definition 4.3.3.2. Let $(X,\to)$ be a topological space. X is $(\mathsf{F}_p,2)$-R_1 if
$$\mathcal{F}\to x\,\&\,\mathcal{F}\to y\Rightarrow \dot{x}\to y, \tag{4.54}$$
for all $x,y\in X$ and $\mathcal{F}\in F_pX$.

We give the following alternative characterization of $(\mathsf{F}_p,2)$-R_1.

Proposition 4.3.3.3. For a topological space $(X, \to)$, the following are equivalent:

(i) X is $(\mathsf{F}_p, 2)$-R$_1$,

(ii) If $\mathcal{V}(x) \vee \mathcal{V}(y)$ is a proper filter, then $\mathcal{V}(x) = \mathcal{V}(y)$, for all $x, y \in X$.

Proof. To prove (i) ⇒ (ii), suppose $\mathcal{V}(x) \vee \mathcal{V}(y)$ is proper. Then $\mathcal{V}(x) \vee \mathcal{V}(y) \to x$ and $\mathcal{V}(x) \vee \mathcal{V}(y) \to y$, and thus $\dot{x} \to y$ and $\dot{y} \to x$. Hence $\mathcal{V}(y) \subseteq \mathcal{V}(x)$ and $\mathcal{V}(x) \subseteq \mathcal{V}(y)$ and thus $\mathcal{V}(x) = \mathcal{V}(y)$.

To prove (ii) ⇒ (i), let $\mathcal{F}$ be a proper filter on X such that $\mathcal{F} \to x$ and $\mathcal{F} \to y$. Then $\mathcal{V}(x) \subseteq \mathcal{F}$ and $\mathcal{V}(y) \subseteq \mathcal{F}$. This implies that $\mathcal{V}(x) \vee \mathcal{V}(y)$ is proper and therefore $\mathcal{V}(x) = \mathcal{V}(y)$. This then implies $\dot{x} \to y$. □

If we restrict to proper elements, $(\mathsf{F}_p, 2)$-R$_1$ coincides with the classical R$_1$-property in Top.

Lower separation axioms in $(\mathbb{I}, 2)$-Cat and $(\mathbb{B}, 2)$-Cat

Adapting the definitions of $(\mathbb{T}, 2)$-p, for p a lower separation property to $(\mathbb{I}, 2)$-Cat and $(\mathbb{B}, 2)$-Cat gives the following topological properties in App. We immediately translate the formulas in terms of the limit operator, whenever convenient.

Since $\mathbb{B}$ is a submonad of $\mathbb{I}$, it follows immediately from (4.46), (4.47) and (4.48) that the properties T$_0$, T$_1$ and R$_0$ for $\mathbb{I}$ and $\mathbb{B}$ will coincide. Only the R$_1$ property for both monads will be different.

Let X be an approach space with functional ideal convergence $\rightarrowtail$, prime functional ideal convergence $\rightsquigarrow$ and limit operator λ.

1. X is $(\mathbb{I}, 2)$-T_0 (and thus $(\mathbb{B}, 2)$-T_0) if

$$\iota_X(\dot{x}) \rightarrowtail y \,\&\, \iota_X(\dot{y}) \rightarrowtail x \Rightarrow x = y, \tag{4.55}$$

for all $x, y \in X$, or, equivalently, if

$$\lambda\dot{x}(y) = 0 = \lambda\dot{y}(x) \Rightarrow x = y, \tag{4.56}$$

for all $x, y \in X$.

2. X is $(\mathbb{I}, 2)$-T_1 (and thus $(\mathbb{B}, 2)$-T_1) if

$$\iota_X(\dot{x}) \rightarrowtail y \Rightarrow x = y, \tag{4.57}$$

for all $x, y \in X$, or, equivalently, if

$$\lambda\dot{x}(y) = 0 \Rightarrow x = y, \tag{4.58}$$

for all $x, y \in X$.

3. X is $(\mathbb{I}, 2)$-*R*$_0$ (and thus $(\mathbb{B}, 2)$-*R*$_0$) if

$$\iota_X(\dot{x}) \rightarrowtail y \Leftrightarrow \iota_X(\dot{y}) \rightarrowtail x, \tag{4.59}$$

   for all $x, y \in X$, or, equivalently, if

$$\lambda\dot{x}(y) = 0 \Leftrightarrow \lambda\dot{y}(x) = 0, \tag{4.60}$$

   for all $x, y \in X$.

4. X is $(\mathbb{I}, 2)$-*R*$_1$ if

$$\mathfrak{I} \rightarrowtail x \,\&\, \mathfrak{I} \rightarrowtail y \Rightarrow \iota_X(\dot{x}) \rightarrowtail y, \tag{4.61}$$

   for all $\mathfrak{I} \in \mathsf{I}\, X$ and $x, y \in X$.

5. X is $(\mathbb{B}, 2)$-*R*$_1$ if

$$\mathfrak{H} \rightsquigarrow x \,\&\, \mathfrak{H} \rightsquigarrow y \Rightarrow \iota_X(\dot{x}) \rightsquigarrow y, \tag{4.62}$$

   for all $\mathfrak{H} \in \mathsf{B}\, X$ and $x, y \in X$.

Using the characterizations of these properties in terms of the limit operator, we immediately get the following equivalences.

Proposition 4.3.3.4. Given an approach space X, the following are equivalent:

(i) X is $(\mathbb{I}, 2)$-T$_0$ ($(\mathbb{I}, 2)$-T$_1$, $(\mathbb{I}, 2)$-R$_0$ respectively),

(ii) The topological coreflection $\mathrm{C}_{\mathsf{Top}}\, X$ is T$_0$ (T$_1$, R$_0$ respectively) in Top.

We will compare $(\mathbb{I}, 2)$-R$_0$ (4.59) in App to the notion of R$_0$ in App introduced by Lowen and Sioen [LS03], which we omitted in Example 4.1.1.6.

Definition 4.3.3.5. An approach space X satisfies *R*$_0$ *in the sense of Lowen-Sioen* [LS03] if

$$\forall x \in X : \mathcal{A}(x) = \bigcap_{y:\delta(y,\{x\})=0} \mathcal{A}(y).$$

Proposition 4.3.3.6. For an approach space X the following are equivalent

(i) The topological coreflection $\mathrm{C}_{\mathsf{Top}}\, X$ is R$_0$ in Top, i.e.

$$\forall x, y \in X : \lambda\dot{x}(y) = 0 \Leftrightarrow \lambda\dot{y}(x) = 0,$$

   in terms of the limit operator, or

$$\forall x, y \in X : \delta\big(y, \{x\}\big) = 0 \Leftrightarrow \delta\big(x, \{y\}\big) = 0,$$

   in terms of the distance.

(ii) X is R_0 in the sense of Lowen-Sioen.

Proof. First we prove (i) $\Rightarrow$ (ii). Since $\delta\big(x, \{x\}\big) = 0$, one inclusion is clear. To show the other inclusion, take $\varphi \in \mathcal{A}(x)$ and let $y \in X$ be such that $\delta\big(y, \{x\}\big) = 0$. By (i) we know that also $\delta\big(x, \{y\}\big) = 0$. By the transition formula (1.28), we have

$$\forall A \subseteq X : \varphi \in \mathcal{A}(x) \Leftrightarrow \inf_{z \in A} \varphi(z) \leq \delta(x, A).$$

By Proposition 1.1.1.2, we have $\delta(x, A) \leq \delta\big(x, \{y\}\big) + \delta(y, A) = \delta(y, A)$, hence

$$\forall A \subseteq X : \inf_{z \in A} \varphi(z) \leq \delta(y, A).$$

This implies $\varphi \in \mathcal{A}(y)$. Since this holds for all $y \in X$ such that $\delta\big(y, \{x\}\big) = 0$, we get

$$\varphi \in \bigcap_{y : \delta\left(y, \{x\}\right)} \mathcal{A}(y).$$

To prove (ii) $\Rightarrow$ (i), suppose $\delta\big(y, \{x\}\big) = 0$. Then clearly, by (ii), $\mathcal{A}(x) \subseteq \mathcal{A}(y)$ and thus by the transition formula (1.36)

$$\begin{aligned} \delta\big(x, \{y\}\big) &= \sup_{\varphi \in \mathcal{A}(x)} \varphi(y) \\ &\leq \sup_{\varphi \in \mathcal{A}(y)} \varphi(y) \\ &= 0. \end{aligned}$$

□

Combining Proposition 4.3.3.4 and Proposition 4.3.3.6, we get the following equivalences.

Corollary 4.3.3.7. For an approach space X the following are equivalent

(i) X is $(\mathbb{I}, 2)$-R_0,

(ii) X is $(\mathbb{B}, 2)$-R_0,

(iii) X is R_0 in the sense of Lowen-Sioen,

(iv) The topological coreflection $\mathrm{C}_{\mathsf{Top}} X$ is R_0 in Top.

It is immediately clear that (β, P_+)-R_0, as introduced in Example 4.1.1.6, implies (iv) (and hence all of the equivalent properties) from Corollary 4.3.3.7. The following example shows that the converse does not hold.

Example 4.3.3.8. Consider $X = \big(\{x,y\},d\big)$ where d is the quasi-metric structure $d(x,y)=\alpha$, $d(y,x)=\beta$, $d(x,x)=d(y,y)=0$ with $\alpha\neq\beta$ and $\alpha,\beta>0$. Then X is R_0 in the sense of Corollary 4.3.3.7, since

$$\begin{aligned}\lambda\dot{a}(b)=0 &\Leftrightarrow d(b,a)=0\\ &\Leftrightarrow a=b\\ &\Leftrightarrow \lambda\dot{b}(a)=0,\end{aligned}$$

for $a,b\in X$. However, X is not (β, P_+)-R_1, since

$$\lambda\dot{x}(y)=d(y,x)=\beta\neq\alpha=d(x,y)=\lambda\dot{y}(x).$$

Since $\mathsf{I}\,X$ and $\mathsf{B}\,X$ both have an improper element $\mathfrak{Z}_X$ that converges to all points of X, the properties $(\mathbb{I},2)$-R_1 and $(\mathbb{B},2)$-R_1 are uninteresting. We get that an approach space X is $(\mathbb{I},2)$-R_1 or $(\mathbb{B},2)$-R_1 if and only if its topological coreflection $C_{\mathsf{Top}}\,X$ is an indiscrete topological space. Therefore we restrict to proper elements.

Definition 4.3.3.9. Let X be an approach space with functional ideal convergence $\rightarrowtail$ and prime functional ideal convergence $\rightsquigarrow$.

1. X is $(\mathsf{I}_p,2)$-R_1 if

$$\mathfrak{I}\rightarrowtail x \,\&\, \mathfrak{I}\rightarrowtail y \Rightarrow \iota_X(\dot{x})\rightarrowtail y, \tag{4.63}$$

   for all $\mathfrak{I}\in\mathsf{I}_p\,X$ and $x,y\in X$.

2. X is $(\mathsf{B}_p,2)$-R_1 if

$$\mathfrak{H}\rightsquigarrow x \,\&\, \mathfrak{H}\rightarrowtail y \Rightarrow \iota_X(\dot{x})\rightsquigarrow y, \tag{4.64}$$

   for all $\mathfrak{H}\in\mathsf{B}_p\,X$ and $x,y\in X$.

The following proposition shows that both properties coincide and gives some alternative characterizations.

Proposition 4.3.3.10. For an approach space X the following are equivalent

(i) X is $(\mathsf{I}_p,2)$-R_1,

(ii) X is $(\mathsf{B}_p,2)$-R_1,

(iii)

$$\lambda\mathcal{U}(x)<\infty \,\&\, \lambda\mathcal{U}(y)<\infty \Rightarrow \lambda\dot{x}(y)=0, \tag{4.65}$$

for all $\mathcal{U}\in\beta X$ and $x,y\in X$,

(iv)
$$\lambda\mathcal{U}(x) < \infty \,\&\, \lambda\mathcal{U}(y) < \infty \Rightarrow \mathcal{A}(x) = \mathcal{A}(y), \tag{4.66}$$
for all $\mathcal{U} \in \beta X$ and $x, y \in X$.

Proof. Since $\mathsf{B}_p X \subset \mathsf{I}_p X$, the implication (i) $\Rightarrow$ (ii) is straightforward. To show (ii) $\Rightarrow$ (i), suppose $\mathfrak{I} \rightarrowtail x$ and $\mathfrak{I} \rightarrowtail y$ for $\mathfrak{I} \in \mathsf{I}_p X$. Then there exists $\mathfrak{H} \in \mathsf{B}_m(\mathfrak{I})$ proper and thus $\mathfrak{H} \rightsquigarrow x$ and $\mathfrak{H} \rightsquigarrow y$ and therefore $\iota_X(\dot{x}) \rightsquigarrow y$ and thus $\iota_X(\dot{x}) \rightarrowtail y$.

(ii) $\Leftrightarrow$ (iii) follows immediately from the transition between proper prime functional ideals and ultrafilters, Theorem 3.2.1.2.

To show (iii) $\Rightarrow$ (iv) suppose $\lambda\mathcal{U}(x) < \infty$ and $\lambda\mathcal{U}(y) < \infty$ for $\mathcal{U} \in \beta X$. By (iii) we have $\lambda\dot{x}(y) = 0 = \lambda\dot{y}(x)$. Using (1.41), we get
$$\sup_{d \in \mathcal{G}} d(y, x) = 0 = \sup_{d \in \mathcal{G}} d(x, y),$$
hence for all $d \in \mathcal{G}$ we have $d(y, x) = 0 = d(x, y)$. Now take $d \in \mathcal{G}$ and $z \in X$ arbitrary. Then
$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) = d(y, z) \\ d(y, z) &\leq d(y, x) + d(x, z) = d(x, z). \end{aligned}$$
Hence, for all $d \in \mathcal{G}$ we have
$$d(x, \cdot) = d(y, \cdot).$$
Using (1.42), this gives us
$$\begin{aligned} \mathcal{A}(x) &= \langle d(x, \cdot) \mid d \in \mathcal{G} \rangle \\ &= \langle d(y, \cdot) \mid d \in \mathcal{G} \rangle \\ &= \mathcal{A}(y). \end{aligned}$$
Finally, to show (iv) $\Rightarrow$ (iii) suppose $\lambda\mathcal{U}(x) < \infty$ and $\lambda\mathcal{U}(y) < \infty$, for $\mathcal{U} \in \beta X$ and $x, y \in X$. By (iv) $\mathcal{A}(x) = \mathcal{A}(y)$ and thus, by (1.37)
$$\begin{aligned} \lambda\dot{x}(y) &= \sup_{\varphi \in \mathcal{A}(y)} \varphi(x) \\ &= \sup_{\varphi \in \mathcal{A}(x)} \varphi(x) \\ &= 0. \end{aligned}$$
□

From (iii) it follows immediately that $(\mathsf{B}_p, 2)$-R$_1$ implies $(\mathbb{\beta}, \mathsf{P}_+)$-R$_1$ from Example 4.1.1.6. The following example shows that the converse does not hold.

Example 4.3.3.11. Consider $X = (\{x, y, z\}, d)$, where d is the metric defined by $d(a, a') = \beta$, $0 < \beta < \infty$, for $a, a' \in X$ with $a \neq a'$ and $d(a, a) = 0$ for $a \in X$.

For $\mathcal{U} \in \beta X$ and $a \in X$, we have $\lambda\mathcal{U}(a) = 0$ if $\mathcal{U} = \dot{a}$ and otherwise $\lambda\mathcal{U}(a) = \beta$. Hence $\lambda\dot{a}(b) \leq \lambda\mathcal{U}(a) + \lambda\mathcal{U}(b)$ for all $\mathcal{U} \in \beta X$ and $a, b \in X$ and therefore X is $(\mathbb{\beta}, \mathsf{P}_+)$-$\mathrm{R}_1$.

However, X is not $(\mathsf{B}_p, 2)$-R_1, since $\lambda\dot{z}(x) = \beta = \lambda\dot{z}(y)$, but $\lambda\dot{x}(y) \neq 0$.

An approach space that is $(\mathsf{B}_p, 2)$-R_1 has a topological coreflection which is R_1 in Top. The following example shows that again the converse is not true.

Example 4.3.3.12. Consider $X = (\{x, y, z\}, d)$, where d is a quasi-metric defined by $d(x, y) = d(y, x) = d(z, y) = d(z, x) = \infty$ and $d(x, z) = d(y, z) = \alpha$, with $0 < \alpha < \infty$.

Then $\mathrm{C}_{\mathsf{Top}}\, X$ is R_1 in Top, since X is finite and hence $\lambda\mathcal{U}(a) = 0 = \lambda\mathcal{U}(b)$ if and only if $\mathcal{U} = \dot{a} = \dot{b}$ and this implies $a = b$.

X is not $(\mathbb{\beta}, \mathsf{P}_+)$-$\mathrm{R}_1$ and therefore also not $(\mathsf{B}_p, 2)$-R_1, since

$$\lambda\dot{x}(y) = d(y, x) = \infty$$

but

$$\lambda\dot{z}(x) + \lambda\dot{z}(y) = d(x, z) + d(y, z) = 2\alpha.$$

Finally we finish this section by comparing $(\mathsf{B}_p, 2)$-R_1 to the property R introduced by Lowen and Sioen in [LS03].

For an approach space X with approach system $\big(\mathcal{A}(x)\big)_{x \in X}$, we define the relation $\sim_{\mathrm{R}}$ on X by

$$\begin{aligned} & x \sim_{\mathrm{R}} y \qquad (4.67)\\ \Leftrightarrow\quad & \exists x_1 := x, \cdots, x_n := y \in X : \forall i \in \{1, \cdots, n-1\} : \\ & c\big(\mathcal{A}(x_i) \vee \mathcal{A}(x_{i+1})\big) = 0. \end{aligned}$$

Definition 4.3.3.13. An approach space X satisfies the condition R if it fulfills the following condition

$$\forall x \in X : \mathcal{A}(x) = \bigcap_{y \sim_{\mathrm{R}} x} \mathcal{A}(y). \tag{4.68}$$

Proposition 4.3.3.14. For an approach space X, the following are equivalent

(i) The topological coreflection $\mathrm{C}_{\mathsf{Top}}\, X$ is R_1 in Top,

(ii)
$$c\big(\mathcal{A}(x) \vee \mathcal{A}(y)\big) = 0 \Rightarrow \mathcal{A}(x) = \mathcal{A}(y), \tag{4.69}$$
for all $x, y \in X$,

(iii) X satisfies R.

Proof. To prove (i) ⇒ (ii) suppose $c\big(\mathcal{A}(x) \vee \mathcal{A}(y)\big) = 0$. Then

$$\begin{aligned} c\big(\mathcal{A}(x) \vee \mathcal{A}(y)\big) = 0 \quad &\Leftrightarrow \quad \sup_{\varphi \in \mathcal{A}(x)} \sup_{\psi \in \mathcal{A}(y)} \inf_{z \in X} \varphi(z) \vee \psi(z) = 0 \\ &\Leftrightarrow \quad \forall \varepsilon > 0 \forall \varphi \in \mathcal{A}(x) \forall \psi \in \mathcal{A}(y) \exists z \in X : \\ &\qquad \varphi(z) \vee \psi(z) < \varepsilon \\ &\Leftrightarrow \quad \forall \varepsilon > 0 \forall \varphi \in \mathcal{A}(x) \forall \psi \in \mathcal{A}(y) : \\ &\qquad \{\varphi < \varepsilon\} \cap \{\psi < \varepsilon\} \neq \emptyset. \end{aligned}$$

Hence $\mathcal{V}_{\mathrm{C}_{\mathsf{Top}} X}(x) \vee \mathcal{V}_{\mathrm{C}_{\mathsf{Top}} X}(y)$ is a proper filter on X. Since $\mathrm{C}_{\mathsf{Top}} X$ is R_1, by Proposition 4.3.3.3, this implies $\mathcal{V}_{\mathrm{C}_{\mathsf{Top}} X}(x) = \mathcal{V}_{\mathrm{C}_{\mathsf{Top}} X}(y)$ and thus $\mathcal{A}(x) = \mathcal{A}(y)$.

To prove (ii) implies (i), let $\mathcal{U} \in \beta X$ such that $\lambda\mathcal{U}(x) = 0 = \lambda\mathcal{U}(y)$. Then $\mathcal{V}_{\mathrm{C}_{\mathsf{Top}} X}(x) \subseteq \mathcal{U}$ and $\mathcal{V}_{\mathrm{C}_{\mathsf{Top}} X}(y) \subseteq \mathcal{U}$. This implies that for all $\varphi \in \mathcal{A}(x)$ and $\psi \in \mathcal{A}(y)$ and for all $\varepsilon > 0$ $\{\varphi < \varepsilon\} \cap \{\psi < \varepsilon\} \neq \emptyset$ and thus $\inf_{z \in X}(\varphi \vee \psi)(z) \leq \varepsilon$. Hence $c\big(\mathcal{A}(x) \vee \mathcal{A}(y)\big) = 0$, therefore, by (ii), $\mathcal{A}(x) = \mathcal{A}(y)$ and thus $\lambda\dot{x}(y) = \sup_{\varphi \in \mathcal{A}(y)} \varphi(x) = \sup_{\varphi \in \mathcal{A}(x)} \varphi(x) = 0$.

That (ii) implies (iii) is straightforward.

To prove (iii) implies (ii) suppose $c\big(\mathcal{A}(x) \vee \mathcal{A}(y)\big) = 0$. Then $x \sim_{\mathrm{R}} y$ and thus $\mathcal{A}(y) = \bigcap_{z \sim_R y} \mathcal{A}(z) \subseteq \mathcal{A}(x)$. Similarly $\mathcal{A}(x) \subseteq \mathcal{A}(y)$ and thus $\mathcal{A}(x) = \mathcal{A}(y)$. □

4.3.4 Compactness

Recall that an object (X, a) in $(\mathbb{T}, 2)$-Cat is $(\mathbb{T}, 2)$-*compact* if

$$\forall \mathcal{X} \in TX \; \exists x \in X \text{ such that } \mathcal{X} \, a \, x. \tag{4.70}$$

The following result from [HST14] gives a characterization of $(\mathbb{T}, 2)$-compactness for power-enriched monads $(\mathbb{T}, \tau)$ and their Kleisli extension.

In the case where $\mathbb{T} = (T, m, e)$ is power-enriched via $\tau : \mathbb{P} \longrightarrow \mathbb{T}$ the induced order on an object $(X, \rightarrow)$ in $(\mathbb{T}, 2)$-Cat is defined by

$$x \leq y \Leftrightarrow \tau\big(\{x\}\big) \rightarrow y \Leftrightarrow e_X(x) \rightarrow y. \tag{4.71}$$

Proposition 4.3.4.1. Given a monad $\mathbb{T}$ power-enriched by $\tau : \mathbb{P} \longrightarrow \mathbb{T}$ together with its Kleisli extension $\check{\mathbb{T}}$ to Rel. Let $(X, \rightarrow)$ be an object in $(\mathbb{T}, 2)$-Cat.

If $(X, \rightarrow)$ is $(\mathbb{T}, 2)$-compact, X has a largest element for the induced order (4.71), with the converse statement holding when $\tau(X)$ is the largest element in TX.

Remark 4.3.4.2. Consider $(X, a) \in (\mathbb{T}, 2)\text{-}\mathsf{Cat}$. We know that for the improper element $\mathfrak{p}$ of TX we have

$$\mathfrak{p}\, a\, x, \quad \forall x \in X.$$

Therefore restricting to proper elements will not generate a new compactness notion.

Compactness in $(\mathbb{F}, 2)\text{-}\mathsf{Cat}$

The property of $(\mathbb{F}, 2)$-compactness was studied in [HST14], but for completeness we recall the result.

Consider a topological space as an $(\mathbb{F}, 2)$-algebra (X, a). Suppose that X is $(\mathbb{F}, 2)$-compact. When considering the filter $\{X\}$, one sees that this implies that there must be a point whose only neighborhood is X. Since every other filter converges to this point as well, this characterizes $(\mathbb{F}, 2)$-compactness. Spaces that satisfy this property are called *supercompact* spaces.

By Remark 4.3.4.2 we know that restricting to proper elements will give the same results.

Compactness in $(\mathbb{I}, 2)\text{-}\mathsf{Cat}$ and $(\mathbb{B}, 2)\text{-}\mathsf{Cat}$

Definition 4.3.4.3. Given an approach space X with functional ideal convergence $\rightarrowtail$, X is $(\mathbb{I}, 2)$-*compact* if

$$\forall \mathfrak{I} \in \mathsf{I}\, X\ \exists x \in X \text{ such that } \mathfrak{I} \rightarrowtail x. \tag{4.72}$$

When we consider the functional $\{0\}$, we see that there must be a point $x \in X$ such that $\{0\} \rightarrowtail x$, or equivalently $\mathcal{A}(x) = \{0\}$. Since every other functional ideal converges to this point as well, this characterizes $(\mathbb{I}, 2)$-compactness. Approach spaces that satisfy this property will be called *supercompact* approach spaces.

Now we turn our attention to the prime functional ideal monad $\mathbb{B}$. Here the situation is different, since the functional ideal $\{0\}$ is not prime.

Definition 4.3.4.4. Given an approach space X with prime functional ideal convergence $\rightsquigarrow$, X is $(\mathbb{B}, 2)$-*compact* if

$$\forall \mathfrak{M} \in \mathsf{B}\, X\ \exists x \in X \text{ such that } \mathfrak{M} \rightsquigarrow x. \tag{4.73}$$

The following proposition characterizes $(\mathbb{B}, 2)$-compactness.

Proposition 4.3.4.5. An approach space X is $(\mathbb{B}, 2)$-compact if and only if its topological coreflection $\mathrm{C}_{\mathsf{Top}}\, X$ is compact as a topological space.

Proof. To prove that $(\mathbb{B}, 2)$-compactness gives rise to a compact topological coreflection, take $\mathcal{U} \in \beta X$ arbitrary. Then $\iota_X(\mathcal{U}) \in \mathsf{B}\,X$ and there exists $x \in X$ such that $\iota_X(\mathcal{U}) \rightsquigarrow x$. Then $\lambda\mathcal{U}(x) = 0$.

To prove the converse, take $\mathfrak{M} \in \mathsf{B}\,X$ arbitrary. Since $\mathfrak{Z}_X$ converges to any $x \in X$, we may assume $\mathfrak{M}$ proper. By proposition 3.2.1.2, there exists $\mathcal{U} \in \beta X$ such that $\mathfrak{M} = \iota_X(\mathcal{U}) \oplus c(\mathfrak{M})$. For this $\mathcal{U} \in \beta X$, there exists $x \in X$ such that $\lambda\mathcal{U}(x) = 0$. Hence $\iota_X(\mathcal{U}) \rightsquigarrow x$ and since $\iota_X(\mathcal{U}) \subseteq \iota_X(\mathcal{U}) \oplus c(\mathfrak{M}) = \mathfrak{M}$, it follows that $\mathfrak{M} \rightsquigarrow x$. □

As a consequence of this proposition, we immediately get that $(\mathbb{B}, 2)$-compactness implies (β, P_+)-compactness. The converse is not true and for a counterexample, we refer to Example 4.2.2.2.

$(\mathbb{I}, 2)$-compactness clearly implies $(\mathbb{B}, 2)$-compactness,since $\mathsf{B}\,X \subseteq \mathsf{I}\,X$ for every approach space X. Again the converse does not hold.

Example 4.3.4.6. Let $X = (\{x, y, z\}, d)$ where d is the metric defined by $d(x, y) = 2$, $d(x, z) = d(y, z) = 1$ and $d(x, x) = d(y, y) = d(z, z) = 0$. Then the topological coreflection $\mathrm{C}_{\mathsf{Top}}\,X$ is compact, since X is finite and therefore X is $(\mathsf{B}, 2)$-compact. This space is not $(\mathbb{I}, 2)$-compact since the bases for the approach systems on X are given by

$$\begin{aligned} \mathcal{B}(x) &= \{d(x, \cdot)\}, \\ \mathcal{B}(y) &= \{d(y, \cdot)\}, \\ \mathcal{B}(z) &= \{d(z, \cdot)\}. \end{aligned}$$

Hence, there is no $a \in X$ such that $\mathcal{A}(a) = \{0\}$.

By Remark 4.3.4.2 we know that restricting to proper elements will give the same results.

4.3.5 Regularity

In this section we consider the topological property of $(\mathbb{T}, 2)$-regularity.

First of all we present some general results on regularity for relational $\mathbb{T}$-algebras, where $\mathbb{T}$ is some power-enriched monad. We prove that for a power-enriched monad $\mathbb{T}$, even when restricting to proper elements, $(\mathbb{T}, 2)$-regularity of a relational $\mathbb{T}$-algebra (X, a) is too strong since in most cases it implies (X, a) to be indiscrete.

Then we get back to Top. The classical notion of regularity in Top is known to be equivalent to the following expression in terms of convergence of proper filters and the Kowalsky sum on selected filters. For every set A, $\mathcal{G}$ a filter on A,

$\psi : A \longrightarrow X$ and σ a selection of filters with $\sigma(z) \longrightarrow \psi(z)$ whenever $z \in A$, the filters $\mathfrak{F} = \sigma(\mathcal{G})$ and $\mathcal{F} = \psi(\mathcal{G})$ satisfy

$$\Sigma\mathfrak{F} \longrightarrow x \Rightarrow \mathcal{F} \longrightarrow x, \tag{4.74}$$

for all $x \in X$. The equivalence of (4.74) with the usual regularity of a topological space was established by Cook and Fisher in [CF67]. When considering topological spaces as relational $\mathbb{F}$-algebras, $(\mathbb{F}, 2)$-regularity gives trivial results, even when restricting to proper filters. Therefore we investigate a weaker notion by restricting to filters generated by selections. In this way we end up with a formulation equivalent to (4.74).

Finally we investigate regularity for approach spaces presented as relational $\mathbb{I}$-algebras. Again by general results on power-enriched monads, even when restricting to proper ideals, the notion of $(\mathbb{I}, 2)$-regularity is too strong and we investigate some weaker concept, by restricting to ideals generated by certain selections. In doing so we obtain a characterization of regular approach spaces in terms of functional ideals.

Restricting the prime functional ideal monad $\mathbb{B}$ to proper elements already gives more interesting results. We prove that $(\mathsf{B}_p, 2)$-regularity is equivalent to the approach space being topological and regular. However it requires further weakening of the concept to obtain a characterization of the usual regularity in App in terms of convergence of prime functional ideals.

Regularity in $(\mathbb{T}, 2)$-Cat for a power-enriched monad $\mathbb{T}$

We start by recalling the definition of $(\mathbb{T}, 2)$-regularity for a relational $\mathbb{T}$-algebra.

Definition 4.3.5.1. An object (X, a) in $(\mathbb{T}, 2)$-Cat is $(\mathbb{T}, 2)$-*regular* if

$$\mathfrak{X}\,(\hat{T}a)\,\mathcal{X}\,\&\,m_X(\mathfrak{X})\,a\,x \Rightarrow \mathcal{X}\,a\,x,$$

for all $\mathfrak{X} \in TTX, \mathcal{X} \in TX$ and $x \in X$.

Next we apply $(\mathbb{T}, 2)$-regularity to the situation of a power-enriched monad $(\mathbb{T}, \tau)$.

Proposition 4.3.5.2. Let $(\mathbb{T}, \tau)$ be a power-enriched monad together with its Kleisli extension $\check{\mathbb{T}}$ to Rel, then a relational $\mathbb{T}$-algebra (X, a) is $(\mathbb{T}, 2)$-regular if and only if it is indiscrete.

Proof. Let (X, a) be a $(\mathbb{T}, 2)$-regular relational $\mathbb{T}$-algebra with $X \neq \emptyset$. Let $\mathfrak{X}$ be the least element of TTX and $\mathcal{X} \in TX$ and $x \in X$ be arbitrary. Then clearly we have $\mathfrak{X} \leq a^{\tau}(\mathcal{X})$. Since m_X is sup-preserving, $m_X(\mathfrak{X})$ is the least element in TX and therefore, by right unitarity of a (1.75), we have $m_X(\mathfrak{X})\,a\,x$. Regularity implies that $\mathcal{X}\,a\,x$. By arbitrariness of $\mathcal{X} \in TX$ and $x \in X$ we can conclude that (X, a) is indiscrete. $\square$

In Top it is well-known that regularity implies R_1. By Proposition 4.3.5.2 we find that in general, for a power-enriched monad $\mathbb{T}$ together with its Kleisli extension $\check{\mathbb{T}}$, $(\mathbb{T},2)$-regularity also implies $(\mathbb{T},2)$-R_1.

Proposition 4.3.5.3. Let $(\mathbb{T},\tau)$ be a power-enriched monad together with its Kleisli extension $\check{\mathbb{T}}$ to Rel, then a relational $\mathbb{T}$-algebra (X,a) is $(\mathbb{T},2)$-R_1 if it is $(\mathbb{T},2)$-regular.

In order to avoid uninteresting results, we will restrict to proper elements and by $(T_p,2)$-regularity, we mean the following property.

Definition 4.3.5.4. An object (X,a) in $(\mathbb{T},2)$-Cat is $(T_p,2)$-regular if

$$\mathfrak{X}\,(\hat{T}a)\,\mathcal{X}\,\&\,m_X(\mathfrak{X})\,a\,x \Rightarrow \mathcal{X}\,a\,x,$$

for all $\mathfrak{X} \in Tj_X(TT_pX)$ with $m_X(\mathfrak{X}) \in T_pX$, $\mathcal{X} \in T_pX$ and $x \in X$.

Despite the restriction, $(T_p,2)$-regularity is still too strong as will be shown in Proposition 4.3.5.5. In this result we use the following notation. For $a : TX \nrightarrow X$ we let $a_p : T_pX \nrightarrow X$ be defined as

$$a_p = a \cdot j_X.$$

Proposition 4.3.5.5. Let $(\mathbb{T},\tau)$ be a power-enriched monad, Kleisli extended to Rel, satisfying (4.41) and let (X,a) be a relational $\mathbb{T}$-algebra satisfying

$$\text{For all } x \in X \text{ the smallest element } \mathfrak{p} \in TX \text{ satisfies } \mathfrak{p} \neq e_X(x). \tag{4.75}$$

Then (X,a) is $(T_p,2)$-regular if and only if it is indiscrete.

Proof. Let (X,a) be a relational $\mathbb{T}$-algebra satisfying (4.75) and suppose it is $(T_p,2)$-regular. Let $X \neq \emptyset, x \in X$ arbitrary and $\mathfrak{q}$ the largest element in TX.

Consider $\mathfrak{X} = Tj_X(\mathfrak{Y})$ for $\mathfrak{Y} = a_p^\tau \cdot e_X(x)$. The extension operator linking the monad $\mathbb{T}$ to its associated Kleisli triple, turns the map $g = \tau_{T_pX} \cdot a_p^\flat : X \longrightarrow TT_pX$ into $g_p^{\mathbb{T}} = m_{T_pX} \cdot Tg_p : TX \longrightarrow TT_pX$ and satisfies $g_p^{\mathbb{T}} \cdot e_X = g_p$. It follows that

$$a_p^\tau \cdot e_X(x) = m_{T_pX} \cdot Tg_p \cdot e_X(x) = \tau_{T_pX} \cdot a_p^\flat(x) \in TT_pX.$$

Similar calculations show that $a^\tau \cdot e_X(x) = \tau_{TX} \cdot a^\flat(x)$.

By naturality of τ, since τ_{TX} is monotone, since $e_X(x) \leq \mathfrak{q}$ and since both m_{TX} and $T(\tau_{TX} \cdot a^\flat)$ are sup-maps, we have

$$\begin{aligned}
\mathfrak{X} &= Tj_X \cdot \tau_{T_pX} \cdot a_p^\flat(x) \\
&= \tau_{TX} \cdot \mathcal{P}j_X \cdot a_p^\flat(x) \\
&\leq \tau_{TX} \cdot a^\flat(x) \\
&= a^\tau \cdot e_X(x) \\
&\leq a^\tau(\mathfrak{q}).
\end{aligned}$$

So we have both

$$\mathfrak{X}\,(\check{T}a)\,e_X(x) \text{ and } \mathfrak{X}\,(\check{T}a)\,\mathfrak{q}. \tag{4.76}$$

Applying the fact that a is left unitary (1.76), to the first expression, we have $m_X(\mathfrak{X})\,a\,x$. Moreover, by applying naturality of τ and $m_X \cdot \tau_{TX} = \bigvee_{TX}$ we get

$$\begin{aligned} m_X(\mathfrak{X}) &= m_X \cdot \tau_{TX} \cdot \mathcal{P}j_X \cdot a_p^\flat(x) \\ &= \bigvee \mathcal{P}j_X \cdot a_p^\tau(x) \\ &= \bigvee \{j(\mathcal{X}) \mid \mathcal{X} \neq \mathfrak{p}, \mathcal{X}\,a\,x\} \\ &\geq e_X(x) \\ &> \mathfrak{p} \end{aligned}$$

and therefore $m_X(\mathfrak{X}) \in T_pX$. Applying $(T_p, 2)$-regularity to the second expression in (4.76) we conclude that $\mathfrak{q}\,a\,x$. Applying the fact that a is right unitary (1.75), we can conclude that (X, a) is indiscrete. □

By Proposition 4.3.5.5 we also find the classical relation between $(T_p, 2)$-regularity and $(T_p, 2)$-R$_1$.

Proposition 4.3.5.6. Let $(\mathbb{T}, \tau)$ be a power-enriched monad, Kleisli extended to Rel, satisfying (4.41) and let (X, a) be a relational $\mathbb{T}$-algebra satisfying (4.75) Then (X, a) is $(T_p, 2)$-R$_1$ if it is $(T_p, 2)$-regular.

Regularity in $(\mathbb{F}, 2)$-Cat

As we already know from the section on regularity for power-enriched monads, $(\mathbb{F}, 2)$-regularity for a relational $\mathbb{F}$-algebra

$$\mathscr{F}\,(\check{F}a)\,\mathcal{F}\,\&\,\Sigma\mathscr{F} \to x \Rightarrow \mathcal{F} \to x,$$

for all $\mathscr{F} \in \mathsf{FF}X, \mathcal{F} \in \mathsf{F}X$ and $x \in X$ is equivalent to the corresponding topological space being indiscrete.

We remove the least element $\mathfrak{p} = \mathcal{P}X$. Condition (4.75) is clearly fulfilled. By Proposition 4.3.5.5, $(F_p, 2)$-regularity for a relational $\mathbb{F}$-algebra

$$\mathscr{F}\,(\check{F}a)\,\mathcal{F}\,\&\,\Sigma\mathscr{F} \to x \Rightarrow \mathcal{F} \to x, \tag{4.77}$$

for all $\mathscr{F} \in \mathsf{F}j_X(\mathsf{FF}_pX)$ with $\Sigma\mathscr{F} \in \mathsf{F}_pX$, $\mathcal{F} \in \mathsf{F}_pX$ and $x \in X$, is again equivalent to the topological space being indiscrete.

Next we restrict the expression (4.77) to pairs of filters generated by selections.

Let A be a set, $\mathcal{G} \in \mathsf{F}_pA$ and $\psi : A \longrightarrow X$ and $\sigma : A \longrightarrow \mathsf{F}_pX$, selections with $\sigma(z) \longrightarrow \psi(z)$, for all $z \in A$. Then the filters $\mathscr{F} = \mathsf{F}j_X\big(\sigma(\mathcal{G})\big)$ and $\mathcal{F} = \psi(\mathcal{G})$

satisfy $\mathscr{F}\,(\check{F}a)\,\mathcal{F}$, $\mathscr{F}\in \mathsf{FF}_pX$, $\Sigma\mathscr{F}\in \mathsf{F}_pX$ and $\mathcal{F}\in \mathsf{F}_pX$. So in the contexts of selections, the property becomes:

For every set A, $\mathcal{G}\in \mathsf{F}_pA$ and $\psi : A \longrightarrow X$ and $\sigma : A \longrightarrow \mathsf{F}_pX$ selections with $\sigma(z)\longrightarrow\psi(z)$ whenever $z\in A$, the filters $\mathscr{F} = \mathsf{F}j_X\big(\sigma(\mathcal{G})\big)$ and $\mathcal{F}=\psi(\mathcal{G})$ satisfy

$$\Sigma\mathscr{F}\longrightarrow x \Rightarrow \mathcal{F}\longrightarrow x, \tag{4.78}$$

for all $x\in X$. This property is clearly equivalent to (4.74) and was shown to be equivalent to the usual regularity of a topological space by Cook and Fisher in [CF67]. It was considered by Kent and Richardson in [KR96] under the name *DF*.

By Proposition 4.3.5.3 and Proposition 4.3.5.6, we get that $(\mathbb{F},2)$-regularity implies $(\mathbb{F},2)$-R_1 and that $(\mathsf{F}_p,2)$-regularity implies $(\mathsf{F}_p,2)$-R_1.

Regularity in $(\mathbb{I},2)$-Cat

The condition (4.75) is clearly fulfilled for a relational $\mathbb{I}$-algebra (X,a) so from previous results on regularity for power-enriched monads we know the following are equivalent:

(i) (X,a) is $(\mathbb{I},2)$-regular,

(ii) (X,a) is $(\mathsf{I}_p,2)$-regular,

(iii) The associated approach space is indiscrete.

In order to get a characterization of regularity in approach theory in terms of functional ideals, we weaken the condition even further to functional ideals generated by selections.

Theorem 4.3.5.7. *The following are equivalent:*

(i) $m_X(\Phi)\rightarrowtail x\,\&\,s(z)\oplus\delta\rightarrowtail\psi(z)$*, whenever* $z\in A$*, and* $c(\mathfrak{J})=c\big(m_X(\Phi)\big)$ *and* $\inf_{F\in\mathfrak{f}_\alpha(\mathfrak{J})}\inf_{z\in F}c\big(s(z)\big)=0$ *whenever* $c(\mathfrak{J})\le\alpha<\infty$*, imply*

$$\mathfrak{K}\oplus\delta\rightarrowtail x$$

for all $x\in X,\delta\ge 0,\Phi=\mathsf{I}j_X\big(\mathsf{I}s(\mathfrak{J})\big)$ *and* $\mathfrak{K}=\mathsf{I}\psi(\mathfrak{J})$ *generated by* $\mathfrak{J}\in\mathsf{I}_pA$ *and selections* $s:A\longrightarrow\mathsf{I}_pX$ *and* $\psi:A\longrightarrow X$.

(ii) X *is a regular approach space: If* A *is a set,* $\psi:A\longrightarrow X,\sigma:A\longrightarrow\mathsf{F}_pX$*, and* $\mathcal{G}\in\mathsf{F}_pA$*, then*

$$\lambda\psi(\mathcal{G})\le\lambda\Sigma\sigma(\mathcal{G})+\sup_{z\in A}\lambda\sigma(z)\big(\psi(z)\big).$$

(iii) If A is a set, $\psi : A \longrightarrow X$, $\sigma : A \longrightarrow \mathsf{F}_p X$, and $\mathcal{G} \in \mathsf{F}_p A$, then

$$\lambda\psi\mathcal{G} \leq \lambda\Sigma\sigma(\mathcal{G}) + \inf_{G\in\mathcal{G}} \sup_{z\in G} \lambda\sigma(z)\big(\psi(z)\big).$$

Proof. (i) $\Rightarrow$ (ii): Let A be a set, $\psi : A \longrightarrow X, \sigma : A \longrightarrow \mathsf{F}_p\, X, \mathcal{G} \in \mathsf{F}_p\, A$ and $x \in X$, then we consider $s : A \longrightarrow \mathsf{I}_p\, X$ defined by $s(z) = \iota_X\big(\sigma(z)\big)$ for $z \in A$. Let $\lambda\Sigma\sigma(\mathcal{G})(x) = \varepsilon < \infty$ and $\sup_{z\in A} \lambda\sigma(z)\big(\psi(z)\big) = \delta < \infty$ and consider $\mathfrak{I} = \iota_A(\mathcal{G}) \oplus \varepsilon$. The formulas (3.1) and (3.2) for $\Phi = \mathsf{I}\, j_X\big(\mathsf{I}\, s(\mathfrak{I})\big)$ give the simpler forms

$$m_X(\Phi) = \bigvee_{G\in\mathcal{G}} \bigcap_{z\in G} \iota_X\big(\sigma(z)\big) \oplus \varepsilon$$

and $c\big(m_X(\Phi)\big) = \varepsilon$. Moreover, by (3.10) in Proposition 3.1.1.6 for every α with $\varepsilon \leq \alpha < \infty$ we have

$$\Sigma\sigma(\mathcal{G}) \subseteq \mathfrak{f}_\alpha\big(m_X(\Phi)\big).$$

Applying the transition formula from a limit operator to functional ideal convergence (1.35) we get $m_X(\Phi) \rightarrowtail x$ and $s(z) \oplus \delta \rightarrowtail \psi(z)$, whenever $z \in A$. Moreover since $c\big(s(z)\big) = 0$ for every $z \in A$ also

$$\inf_{F\in\mathfrak{f}_\alpha(\mathfrak{I})} \sup_{z\in F} c\big(s(z)\big) = 0,$$

for all α with $c(\mathfrak{I}) \leq \alpha < \infty$. By (i) we now have that $\mathsf{I}\,\psi(\mathfrak{I}) \oplus \delta \rightarrowtail x$ and by the transition formula from functional ideal convergence to limit operator (1.47), we obtain

$$\lambda\psi(\mathcal{G})(x) \leq \lambda\Sigma\sigma(\mathcal{G})(x) + \sup_{z\in A} \lambda\sigma(z)\big(\psi(z)\big).$$

(ii) $\Rightarrow$ (iii): Let A be a set, $\psi : A \longrightarrow X, \sigma : A \longrightarrow \mathsf{F}_p\, X$ and $\mathcal{G} \in \mathsf{F}_p\, A$ with $\inf_{G\in\mathcal{G}} \sup_{z\in G} \lambda\sigma(z)\big(\psi(z)\big) = \delta$. For $\varepsilon > 0$ choose $G_0 \in \mathcal{G}$ such that $\sup_{z\in G_0} \lambda\sigma(z)\big(\psi(z)\big) < \delta + \varepsilon$ and let $\sigma' : A \longrightarrow \mathsf{F}_p\, X$ be defined as

$$\sigma'(z) := \begin{cases} \sigma(z) & \text{for all } z \in G_0, \\ \dot{\psi}(z) & \text{for all } z \notin G_0. \end{cases}$$

Now apply regularity with as input $\mathcal{G}$ and the selections ψ and σ' on A. Then for $\mathcal{F} = \psi(\mathcal{G})$ and $\mathscr{F} = \sigma'(\mathcal{G})$ we have

$$\begin{aligned} \lambda\mathcal{F}(x) &\leq \lambda\Sigma\mathscr{F}(x) + \sup_{z\in A} \lambda\sigma'(z)\big(\psi(z)\big) \\ &= \lambda\Sigma\mathscr{F}(x) + \sup_{z\in G_0} \lambda\sigma(z)\big(\psi(z)\big) \\ &< \lambda\Sigma\mathscr{F}(x) + \delta + \varepsilon. \end{aligned}$$

By arbitrariness of ε we are done.

(iii) ⇒ (i): Let $x \in X, \delta \geq 0, \Phi = \mathsf{I}\, j_X\big(\mathsf{I}\, s(\mathfrak{I})\big)$ and $\mathfrak{K} = \mathsf{I}\,\psi(\mathfrak{I})$ generated by selections $s : A \longrightarrow \mathsf{I}_p X$ and $\psi : A \longrightarrow X$ and $\mathfrak{I} \in \mathsf{I}_p A$, and assume

$$m_X(\Phi) \rightarrowtail x \,\&\, s(z) \oplus \delta \rightarrowtail \psi(z), \text{ for all } z \in A$$

and

$$c(\mathfrak{I}) = c\big(m_X(\Phi)\big) \,\&\, \inf_{F \in \mathfrak{f}_\alpha(\mathfrak{I})} \sup_{z \in F} c\big(s(z)\big) = 0, \text{ for all } \alpha \text{ with } c(\mathfrak{I}) \leq \alpha < \infty.$$

Apply the result from Proposition 3.2.1.5. For every $z \in A$ choose $r(z) \in \mathsf{B}_m\big(s(z)\big)$ with $c\big(r(z)\big) = c\big(s(z)\big)$. Clearly by Proposition 3.1.1.3 we have $m_X(\mathsf{I}\, r(\mathfrak{I})) \rightarrowtail x$ and $c\Big(m_X\big(\mathsf{I}\, r(\mathfrak{I})\big)\Big) = c\Big(m_X\big(\mathsf{I}\, s(\mathfrak{I})\big)\Big)$. Moreover $\inf_{F \in \mathfrak{f}_\alpha(\mathfrak{I})} \sup_{z \in F} c\big(r(z)\big) = 0$, for all α with $c(\mathfrak{I}) \leq \alpha < \infty$. For $z \in A$ we write $r(z) = \iota_X\big(\sigma(z)\big) \oplus c\big(s(z)\big)$. Since $r(z) \oplus \delta \rightarrowtail \psi(z)$ we have $\lambda\sigma(z)\big(\psi(z)\big) \leq c\big(r(z)\big) + \delta$ for every $z \in A$. Let $\gamma \geq c(\mathfrak{I}) + \delta$ and $\alpha \geq c(\mathfrak{I})$ with $\gamma = \alpha + \delta$. With $\mathcal{G} = \mathfrak{f}_\alpha(\mathfrak{I})$ now apply (iii) to $A, \psi, \sigma, x, \mathcal{G}$. Using equation (3.9) in Proposition 3.1.1.6 we get

$$\begin{aligned}
\lambda\psi(\mathcal{G})(x) &\leq \lambda\Sigma\sigma(\mathcal{G})(x) + \inf_{G \in \mathcal{G}} \sup_{z \in G} \lambda\sigma(z)\big(\psi(z)\big) \\
&\leq \lambda\mathfrak{f}_\alpha\Big(m_X\big(\mathsf{I}\, r(\mathfrak{I})\big)\Big)(x) + \inf_{G \in \mathcal{G}} \sup_{z \in G} c\big(r(z)\big) + \delta \\
&\leq \lambda\mathfrak{f}_\alpha\Big(m_X\big(\mathsf{I}\, r(\mathfrak{I})\big)\Big)(x) + \delta \\
&\leq \alpha + \delta = \gamma.
\end{aligned}$$

Since $\mathfrak{f}_\gamma\big(\mathsf{I}\,\psi(\mathfrak{I}) \oplus \delta\big) = \mathfrak{f}_\alpha\big(\mathsf{I}\,\psi(\mathfrak{I})\big) = \psi(\mathcal{G})$, it follows that $\mathsf{I}\,\psi(\mathfrak{I}) \oplus \delta \rightarrowtail x$. □

By Proposition 4.3.5.3 and Proposition 4.3.5.6, we get that $(\mathbb{I}, 2)$-regularity implies $(\mathbb{I}, 2)$-R_1 and that $(\mathsf{I}_p, 2)$-regularity implies $(\mathsf{I}_p, 2)$-R_1.

Removing the condition $c(\mathfrak{I}) = c\big(m_X(\Phi)\big)$ we obtain a characterization of regular topological spaces.

Theorem 4.3.5.8. *The following are equivalent:*

(i) $m_X(\Phi) \rightarrowtail x \,\&\, s(z) \rightarrowtail \psi(z)$, *whenever* $z \in A \,\&\, c\big(m_X(\Phi)\big) < \infty \,\&$ $\inf_{F \in \mathfrak{f}(\mathfrak{I})} \sup_{z \in F} c\big(s(z)\big) < \infty$ *imply*

$$\mathfrak{K} \rightarrowtail x$$

for all $x \in X$, $\Phi = \mathsf{I} j_X\big(\mathsf{I} s(\mathfrak{I})\big)$ *and* $\mathfrak{K} = \mathsf{I}\psi(\mathfrak{I})$ *generated by* $\mathfrak{I} \in \mathsf{I}_p A$ *and selections* $s : A \longrightarrow \mathsf{I}_p X$ *and* $\psi : A \longrightarrow X$.

(ii) X is a regular topological space.

Proof. (i) $\Rightarrow$ (ii): In order to prove that X is topological let $\mathcal{F} \in \mathsf{F}_p X$, $x \in X$ and suppose $\lambda\mathcal{F}(x) < \infty$. Consider $A = X$, $\psi = 1_X$ and $s : X \longrightarrow \mathsf{I}_p X$ defined by $s(z) = \iota_X\big(\sigma(z)\big) \oplus \lambda\mathcal{F}(x)$ with $\sigma(z) = \dot{z}$ for $z \in X$, $\mathfrak{I} = \iota_X(\mathcal{F})$ and $\Phi = \mathsf{I}\, j_X\big(\mathsf{I}\, s(\mathfrak{I})\big)$.

Clearly $s(z) \rightarrowtail \psi(z)$, for all $z \in A$ and by (3.2) in Proposition 3.1.1.3 we have $c\big(m_X(\Phi)\big) = \lambda\mathcal{F}(x) < \infty$. By (3.10) in Proposition 3.1.1.6, $\mathcal{F} = \Sigma\sigma(\mathcal{F}) \subseteq \mathfrak{f}_\alpha(m_X(\Phi))$ for every $\alpha \geq \lambda\mathcal{F}(x)$. Therefore $\lambda\mathfrak{f}_\alpha(m_X(\Phi))(x) \leq \lambda\mathcal{F}(x)$ for every $\alpha \geq \lambda\mathcal{F}(x)$ and $m_X(\Phi) \rightarrowtail x$. Moreover since $c(s(z)) = \lambda\mathcal{F}(x)$ for every $z \in A$, also $\inf_{F\in\mathfrak{f}(\mathfrak{I})} \sup_{z\in F} c(s(z)) < \infty$. It follows that $\iota_X(\mathcal{F}) \rightarrowtail x$ and so $\lambda\mathcal{F}(x) = 0$. We may conclude that the space is topological.

Next we prove regularity via (4.78). Let A a set, $\mathcal{G} \in \mathsf{F}_p A$ and $\psi : A \to X$ and $\sigma : A \to \mathsf{F}_p X$, with $\sigma(z) \to \psi(z)$ whenever $z \in A$, $\mathscr{F} = \mathsf{F}\, j_X(\sigma(\mathcal{G}))$ and $\mathcal{F} = \psi(\mathcal{G})$ with $\Sigma\mathscr{F} \to x$ for $x \in X$.

For $\mathfrak{I} = \iota_A(\mathcal{G})$ and $s(z) = \iota_X(\sigma(z))$, for all $z \in A$ and $\Phi = \mathsf{I}\, j_X(\mathsf{I}\, s(\mathfrak{I}))$, $\mathfrak{K} = \mathsf{I}\,\psi(\mathfrak{I})$ we clearly have $s(z) \rightarrowtail \psi(z)$, $c(s(z)) = 0$, whenever $z \in A$. Further $c(m_X(\Phi)) = 0$ and $\mathfrak{f}_\alpha(m_X(\Phi)) = \Sigma\sigma(\mathcal{G})$ for all $\alpha \geq 0$ by (3.9) and (3.10) in Proposition 3.1.1.6. So $m_X(\Phi) = \iota_X(\Sigma\sigma(\mathcal{G})) \rightarrowtail x$. From (i) we have $\mathfrak{K} = \iota_X(\mathcal{F}) \rightarrowtail x$ and therefore $\mathcal{F} \to x$.

(ii) $\Rightarrow$ (i): Suppose X is a regular topological space and let $x \in X$, $\Phi = \mathsf{I}\, j_X(\mathsf{I}\, s(\mathfrak{I}))$ and $\mathfrak{K} = \mathsf{I}\,\psi(\mathfrak{I})$ generated by selections $s : A \to \mathsf{I}_p X$ and $\psi : A \to X$ and $\mathfrak{I} \in \mathsf{I}_p A$, with $m_X(\Phi) \rightarrowtail x$, $s(z) \rightarrowtail \psi(z)$, whenever $z \in A$ and $c(m_X(\Phi)) < \infty$, $\inf_{F\in\mathfrak{f}(\mathfrak{I})} \sup_{z\in F} c(s(z)) < \infty$.

As in the proof of Theorem 4.3.5.7 applying Proposition 3.2.1.5, for every $z \in A$ we can choose $r(z) \in \mathsf{B}_m(s(z))$ with $c(r(z)) = c(s(z))$. Then by Proposition 3.1.1.3 we have $m_X(\mathsf{I}\, j_X(\mathsf{I}\, r(\mathfrak{I}))) \rightarrowtail x$, $c(m_X(\mathsf{I}\, j_X(\mathsf{I}\, r(\mathfrak{I})))) = c(m_X(\mathsf{I}\, j_X(\mathsf{I}\, s(\mathfrak{I}))))$ and $\inf_{F\in\mathfrak{f}(\mathfrak{I})} \sup_{z\in F} c(r(z)) < \infty$.

For $z \in A$ we write $r(z) = \iota_X(\sigma(z)) \oplus c(s(z))$. Since $r(z) \rightarrowtail \psi(z)$ we have $\lambda\sigma(z)(\psi(z)) \leq c(r(z))$ and since X is topological $\lambda\sigma(z)(\psi(z)) = 0$ for every $z \in A$.

For $c(\mathfrak{I}) \leq \alpha < \infty$ arbitrary, we claim that there exists γ with

$$c(m_X(\mathsf{I}\, j_X(\mathsf{I}\, r(\mathfrak{I})))) \leq \gamma < \infty$$

such that

$$\mathfrak{f}_\gamma(m_X(\mathsf{I}\, j_X(\mathsf{I}\, r(\mathfrak{I})))) \subseteq \Sigma\, \mathsf{F}\, j_X(\sigma(\mathfrak{f}_\alpha(\mathfrak{I}))).$$

The proof goes along the same lines as for (3.9) in Proposition 3.1.1.6. We put

$$\gamma \geq \max\{c(m_X(\mathsf{I}\, j_X(\mathsf{I}\, r(\mathfrak{I})))), 2 \inf_{F\in\mathfrak{f}(\mathfrak{I})} \sup_{z\in F} c(r(z)), 2\alpha\}.$$

Let $\mu \in m_X(\mathsf{I}\, j_X(\mathsf{I}\, r(\mathfrak{I})))$ and $\gamma < \beta$. By (3.1) choose $\varphi \in \mathfrak{I}$ such that $\mu \in \iota_X(\sigma(z)) \oplus (c(r(z)) + \varphi(z))$, whenever $z \in A$. For $z \in A$ choose $S_z \in \sigma(z)$ and $\omega < \infty$ such that $\mu \leq \theta^{\omega}_{S_z} + c(r(z)) + \varphi(z)$.

Choose $F \in \mathfrak{f}_\alpha(\mathfrak{I})$ with $\sup_{z\in F} c(r(z)) < \frac{\beta}{2}$ and then $\psi \in \mathfrak{I}$ and $\eta > \alpha$ with $\{\psi < \eta\} \subseteq F$. Let $\zeta = \varphi \vee \psi \in \mathfrak{I}$ and choose $\rho < \frac{\beta}{2}$ with $\alpha < \rho < \eta$ then $\{\zeta < \rho\} \in \mathfrak{f}_\alpha(\mathfrak{I})$. We now claim that for $z \in A$ with $\zeta(z) < \rho$ we have $\{\mu < \beta\} \in \sigma(z)$. Indeed for such z we have

$$\psi(z) < \eta \Rightarrow z \in F \Rightarrow c(r(z)) < \frac{\beta}{2}.$$

Since also $\varphi(z) < \rho$, it follows that $S_z \subseteq \{\mu < \beta\}$.

Since $\mathfrak{f}_\gamma(m_X(\mathsf{I}\, j_X(\mathsf{I}\, r(\mathfrak{I})))) \to x$ also $\Sigma\, \mathsf{F}\, j_X(\sigma(\mathfrak{f}_\alpha(\mathfrak{I}))) \to x$ and applying regularity to $\mathcal{F} = \psi(\mathfrak{f}_\alpha(\mathfrak{I}))$ and $\mathscr{F} = \mathsf{F}\, j_X(\sigma(\mathfrak{f}_\alpha(\mathfrak{I})))$ also $\psi(\mathfrak{f}_\alpha(\mathfrak{I})) \to x$. By arbitrariness of α we can conclude that $\mathsf{I}\, \psi(\mathfrak{I}) \rightarrowtail x$. □

Regularity in $(\mathbb{B}, 2)$-Cat

The explicit expression for $(\mathbb{B}, 2)$-regularity of a relational $\mathbb{B}$-algebra (X, a) becomes

$$\Theta\, (\check{\mathsf{B}}a)\, \mathfrak{K} \,\&\, n_X(\Theta) \rightsquigarrow x \Rightarrow \mathfrak{K} \rightsquigarrow x$$

for all $\Theta \in \mathsf{BB}X$, $\mathfrak{K} \in \mathsf{B}X$ and $x \in X$.

For a relational $\mathbb{B}$-algebra (X, a), due to the fact that Θ is allowed to be improper, we have the following equivalence

(i) (X, a) is $(\mathbb{B}, 2)$-regular,

(ii) The associated approach space is indiscrete.

Applying Proposition 4.3.5.3, we again get that $(\mathbb{B}, 2)$-regularity implies $(\mathbb{B}, 2)$-R_1.

Again it is clear that we have to exclude the improper element P^X_b.

$(\mathsf{B}_p, 2)$-regularity becomes

$$\Theta\, (\check{\mathsf{B}}a)\, \mathfrak{K} \,\&\, n_X(\Theta) \rightsquigarrow x \Rightarrow \mathfrak{K} \rightsquigarrow x, \tag{4.79}$$

for all $\Theta \in \mathsf{B}\, j_X(\mathsf{B}\, \mathsf{B}_p\, X)$ with $n_X(\Theta) \in \mathsf{B}_p\, X$, $\mathfrak{K} \in \mathsf{B}_p\, X$ and $x \in X$.

One of the equivalent formulations in the next proposition will make use of selections $s : A \longrightarrow \mathsf{B}_p\, X$ and $\psi : A \longrightarrow X$ and $\mathfrak{I} \in \mathsf{B}_p\, A$ and will deal with ideals $\Theta = \mathsf{B}\, j_X(\mathsf{B}\, s(\mathfrak{I}))$ and $\mathfrak{K} = \mathsf{B}\, \psi(\mathfrak{I})$ generated by these selections.

Theorem 4.3.5.9. *For a relational $\mathbb{B}$-algebra (X, a), the following are equivalent:*

(i) (X, a) is $(\mathsf{B}_p, 2)$-regular.

(ii) $\Theta\,(\check{\mathsf{B}}a)\,\mathfrak{K}\,\&\,n_X(\Theta) \rightsquigarrow x\,\&\,c(\mathfrak{K}) = c(\Theta) \Rightarrow \mathfrak{K} \rightsquigarrow x$*, for all* $\Theta \in \mathsf{B}\,j_X(\mathsf{B}\,\mathsf{B}_p\,X)$ *with* $n_X(\Theta) \in \mathsf{B}_p\,X$*,* $\mathfrak{K} \in \mathsf{B}_p\,X$ *and* $x \in X$.

(iii) $n_X\Theta \rightsquigarrow x\,\&\,s(z) \rightsquigarrow \psi(z)$*, whenever* $z \in A \Rightarrow \mathfrak{K} \rightsquigarrow x$ *for all* $x \in X, \Theta = \mathsf{B}\,j_X\,\mathsf{B}\,s(\mathfrak{I})$ *with* $n_X(\Theta) \in \mathsf{B}_p\,X$ *and* $\mathfrak{K} = \mathsf{B}\,\psi(\mathfrak{I})$*, generated by* $\mathfrak{I} \in \mathsf{B}_p\,A$ *and selections* $s : A \longrightarrow \mathsf{B}_p\,X$ *and* $\psi : A \longrightarrow X$.

(iv) X *is a regular topological space.*

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): Suppose selections $s : A \to \mathsf{B}_p\,X$ and $\psi : A \to X$ and $\mathfrak{I} \in \mathsf{B}_p\,A$ are given, $\Theta = \mathsf{B}\,j_X\,\mathsf{B}\,s(\mathfrak{I})$ and $\mathfrak{K} = \mathsf{B}\,\psi(\mathfrak{I})$ with $n_X(\Theta) \in \mathsf{B}_p\,X$ and $n_X(\Theta) \rightsquigarrow x$. Clearly $c(\mathfrak{K}) = c(\Theta)$. Moreover the condition $s(z) \rightsquigarrow \psi(z)$, whenever $z \in A$ implies $a^\wedge\mathfrak{K} \subseteq \Theta$. This follows from the fact that from $\mu \in \mathsf{P}_b^{\mathsf{B}\,X}$, $\mu \leq a^\wedge\nu$ with $\nu \cdot \psi \in \mathfrak{I}$ we have $\mu \cdot s \leq \nu \cdot \psi$ which implies $\mu \cdot s \in \mathfrak{I}$. By (ii) the conclusion follows.

(iii) $\Rightarrow$ (iv) First we prove that the corresponding approach space X is topological. Let $\mathcal{W} \in \beta X$, $x \in X$ and assume $\lambda\mathcal{W}(x) < \infty$. Consider the selections on $A = X$, $\psi = \mathrm{id}$ and $s(z) = \iota_X(\dot{z}) \oplus \lambda\mathcal{W}(x)$ and $\mathfrak{I} = \iota_X(\mathcal{W})$. Then in view of (3.11) in Proposition 3.1.1.7, for $\Theta = \mathsf{B}\,j_X\,\mathsf{B}\,s(\iota_X(\mathcal{W}))$ we have $n_X(\Theta) = \iota_X(\mathcal{W}) \oplus \lambda\mathcal{W}(x)$. So clearly all conditions in (iii) are fulfilled in order to conclude that $\mathfrak{K} = \iota_X(\mathcal{W})$ satisfies $\mathfrak{K} \rightsquigarrow x$. It follows that $\lambda\mathcal{W}(x) = 0$. So we can conclude that the limit operator is two-valued on ultrafilters and that X is topological.

Next we prove that the regularity condition in (4.27) is fulfilled. Consider selections $\psi : A \to X$ and $\sigma : A \to \beta X$, and $\mathcal{W} \in \beta A$ and let $x \in X$, $\mathcal{U} = \psi(\mathcal{W})$ and $\mathscr{U} = \sigma(\mathcal{W})$. Suppose the right-hand side of (4.27) is finite. Let $\mathfrak{I} = \iota_A(\mathcal{W})$ and for $z \in A$ put

$$s(z) = \iota_A(\sigma(z)) \oplus (\lambda\sigma(z)(\psi(z)) \vee \lambda\Sigma\mathscr{U}(x)).$$

Clearly $s(z) \rightsquigarrow \psi(z)$, for all $z \in A$. Moreover $\Theta = \mathsf{B}\,j_X\,\mathsf{B}\,s(\mathfrak{I})$ satisfies

$$n_X(\Theta) = \iota_X(\Sigma\mathscr{U}) \oplus \sup_{W \in \mathcal{W}} \inf_{z \in W} (\lambda\sigma(z)(\psi(z)) \vee \lambda\Sigma\mathscr{U}(x))$$

and hence all conditions in (iii) hold. So for $\mathfrak{K} = \mathsf{B}\,\psi(\mathfrak{I}) = \iota_X(\psi(\mathcal{W}))$ we have $\mathfrak{K} \rightsquigarrow x$. Since $\lambda\psi(\mathcal{W}) = 0$ the inequality in (4.27) holds.

(iv) $\Rightarrow$ (iii): Let $x \in X$, $\Theta = \mathsf{B}\,j_X\,\mathsf{B}\,s(\mathfrak{I})$ and $\mathfrak{K} = \mathsf{B}\,\psi(\mathfrak{I})$ generated by selections $s : A \to \mathsf{B}_p\,X$ and $\psi : A \to X$ and $\mathfrak{I} \in \mathsf{B}_p\,A$ with $n_X(\Theta) \in \mathsf{B}_p\,X$ and assume $n_X(\Theta) \rightsquigarrow x$ and $s(z) \rightsquigarrow \psi(z)$, for all $z \in A$. For $z \in A$ we denote $s(z) = \iota_X(\sigma(z)) \oplus \alpha_z$ and $\mathfrak{I} = \iota_A(\mathcal{W}) \oplus c(\mathfrak{I})$. With $\mathscr{U} = \sigma(\mathcal{W})$ we have

$$n_X(\Theta) = \iota_X(\Sigma\mathscr{U}) \oplus (c(\mathfrak{I}) + \sup_{W \in \mathcal{W}} \inf_{z \in W} \alpha_z).$$

It follows that both $\lambda\Sigma\mathscr{U}(x)$ and $\sup_{W\in\mathcal{W}}\inf_{z\in W}\lambda\sigma(z)(\psi(z))$ are finite and hence zero. Since $\mathcal{W}$ is an ultrafilter, by application of Lemma 1.1.1.4, we can interchange $\sup$ and $\inf$ and we get

$$\inf_{W\in\mathcal{W}}\sup_{z\in W}\lambda\sigma(z)(\psi(z)) = 0.$$

We use a technique as in the proof of Theorem 4.3.5.7 and choose $W_0 \in \mathcal{W}$ such that $\sup_{z\in W_0}\lambda\sigma(z)(\psi(z)) = 0$. Let $\sigma' : A \to \beta X$ be defined as

$$\sigma'(z) := \begin{cases} \sigma(z) & \text{for all } z \in W_0 \\ \psi(z) & \text{for all } z \notin W_0. \end{cases}$$

Observe that $\sigma(\mathcal{W}) = \sigma'(\mathcal{W}) = \mathscr{U}$. Now apply regularity with as input $\mathcal{W}$ on A and the selections ψ and σ'. Then for $\mathcal{U} = \psi(\mathcal{W})$ we have

$$\begin{aligned} \lambda\mathcal{U}(x) &\leq \lambda\Sigma\mathscr{U}(x) + \sup_{z\in A}\lambda\sigma'(z)(\psi(z)) \\ &= \lambda\Sigma\mathscr{U}(x) + \sup_{z\in W_0}\lambda\sigma(z)(\psi(z)) \\ &= 0 \end{aligned}$$

Since $\mathfrak{K} = \iota_X(\mathcal{U}) \oplus c(\mathfrak{I})$ we can conclude that $\mathfrak{K} \rightsquigarrow x$.

(iii) $\Rightarrow$ (ii) Let $x \in X$, $\Theta = \mathsf{B}\, j_X(\Psi)$ for $\Psi \in \mathsf{B}\,\mathsf{B}_p X$ and $\mathfrak{K} \in \mathsf{B}_p X$ with $n_X(\Theta) \in \mathsf{B}_p X$, $n_X(\Theta) \rightsquigarrow x$, $a^\wedge\mathfrak{K} \subseteq \Theta$ and $c(\mathfrak{K}) = c(\Theta)$. We denote $\Theta = \iota_{\mathsf{B}\,X}(\mathscr{U}) \oplus \gamma$ and $\mathfrak{K} = \iota_X(\mathcal{U}) \oplus \gamma$. Let

$$A = \{(y, \mathfrak{I}) \mid \mathfrak{I} \rightsquigarrow y, \mathfrak{I} \in \mathsf{B}_p X, y \in X\}.$$

We show that $\mathcal{U} \times \mathscr{U}$ has a trace on A. For $U \in \mathcal{U}$ and $\mathcal{A} \in \mathscr{U}$ and $\gamma < \omega < \infty$ the function

$$\varphi = (a^\wedge(\theta_U^\omega + \gamma) \wedge \omega) \vee (\theta_{\mathcal{A}}^\omega + \gamma)$$

belongs to Θ and so $\varphi \cdot j_X \in \Psi$. We evaluate φ in an arbitrary $\mathfrak{L} \in \mathsf{B}_p X$.

Either $\mathfrak{L} \notin \mathcal{A}$ and then the second term equals $\omega + \gamma \geq \omega$, or $\mathfrak{L} \in \mathcal{A}$. In case $\mathfrak{L}$ diverges the first term equals ω. In case $\mathfrak{L}$ does converge, but never to a point of U, the first term equals $\omega + \gamma \geq \omega$. Since φ cannot be larger than ω in every $\mathfrak{L}$, we can conclude that there is $\mathfrak{L}_0 \in \mathcal{A}$ that converges to some $y_0 \in U$. Then $(y_0, \mathfrak{L}_0) \in A \cap (U \times \mathcal{A})$.

Let $\mathcal{W}$ be an ultrafilter on A finer than the trace of $\mathcal{U}\times\mathscr{U}$ on A, $\psi : A \longrightarrow X$ the restriction of the first projection and $s : A \longrightarrow \mathsf{B}_p X$ the restriction of the second projection. Then we have $\psi(\mathcal{W}) = \mathcal{U}$ and $s(\mathcal{W}) = \mathscr{U}$ and $s(z) \rightsquigarrow \psi(z)$ for all $z \in A$, by construction. With $\mathfrak{I} = \iota_A(\mathcal{W}) \oplus \gamma$ we now have $\Theta = \mathsf{B}\, j_X\, \mathsf{B}\, s(\mathfrak{I})$ and $\mathfrak{K} = \mathsf{B}\,\psi(\mathfrak{I})$. By (iii) we can conclude that $\mathfrak{K} \rightsquigarrow x$.

(ii) $\Rightarrow$ (i): Let $x \in X$, $\Theta = \mathsf{B}\, j_X(\Psi)$ for $\Psi \in \mathsf{B}\,\mathsf{B}_p\, X$ and $\mathfrak{K} \in \mathsf{B}_p\, X$ with $n_X(\Theta) \in \mathsf{B}_p\, X$, $n_X(\Theta) \rightsquigarrow x$ and $a^\wedge \mathfrak{K} \subseteq \Theta$. We denote $\mathfrak{K} = \iota_X(\mathcal{U}) \oplus \gamma$ and $\Theta = \iota_{\mathsf{B}\, X}(\mathscr{U}) \oplus \delta$ where both γ and δ are finite. For $\mathfrak{K}' = \iota_X(\mathcal{U}) \oplus \delta$ we show that $a^\wedge \mathfrak{K}' \subseteq \Theta$.

To see this let $U \in \mathcal{U}, \omega > \delta, \eta > \omega + \gamma, \eta > \omega + \delta$ and let $0 < \varepsilon < \gamma$. Choose $\mathcal{A} \in \mathscr{U}$ and $\omega' \geq \omega$ such that for $\varphi_\gamma = a^\wedge(\theta_U^\omega + \gamma) \wedge \eta$ one has

$$\varphi_\gamma \leq \theta_{\mathcal{A}}^{\omega'} + \delta + \varepsilon.$$

We claim that also $\varphi_\delta = a^\wedge(\theta_U^\omega + \delta) \wedge \eta$ satisfies

$$\varphi_\delta \leq \theta_{\mathcal{A}}^{\omega'} + \delta + \varepsilon.$$

To evaluate both sides in a prime ideal $\mathfrak{L}$, observe that the right-hand side takes values $\delta + \varepsilon$ (when $\mathfrak{L} \in \mathcal{A}$) and $\omega' + \delta + \varepsilon$ (when $\mathfrak{L} \notin \mathcal{A}$). So the inequalities in case where $\mathfrak{L}$ diverges or converges to at least one point of U are trivially fulfilled. In case $\mathfrak{L}$ converges but never to a point of U, $\mathfrak{L} \in \mathcal{A}$ would imply $\varphi_\gamma(\mathfrak{L}) = \omega + \gamma \leq \delta + \varepsilon$ which is impossible. So we may assume $\mathfrak{L} \notin \mathcal{A}$ and then $\varphi_\delta(\mathfrak{L}) \leq \delta + \varepsilon$.

Application of (ii) to Θ and $\mathfrak{K}'$ implies $\mathfrak{K}' \rightsquigarrow x$ which means that $\lambda\mathcal{U}(x) \leq \delta$. Since we already proved that (ii) implies the approach space X is topological, we have $\lambda\mathcal{U}(x) = 0$ and hence also $\mathfrak{K} \rightsquigarrow x$. □

This result immediately gives that $(\mathsf{B}_p, 2)$-regularity implies $(\mathsf{B}_p, 2)$-R_1.

Proposition 4.3.5.10. An approach space X is $(\mathsf{B}_p, 2)$-R_1 if it is $(\mathsf{B}_p, 2)$-regular.

Proof. We will use the characterization of $(\mathsf{B}_p, 2)$-R_1 from (4.65). Let $\mathcal{U} \in \beta X$ and suppose $\lambda\mathcal{U}(x) < \infty$ and $\lambda\mathcal{U}(y) < \infty$. Since X is $(\mathsf{B}_p, 2)$-regular, it is topological and thus $\lambda\mathcal{U}(x) = 0 = \lambda\mathcal{U}(y)$. Since regular topological spaces are R_1, we get $\lambda\dot{x}(y) = 0$. □

In order to characterize regularity for approach spaces (not just the topological ones) in terms of the prime functional ideal monad, we need conditions based on the following construction. For a function $\varphi : \mathsf{B}\, X \longrightarrow \mathsf{P}$ and $\delta \geq 0$ we let $\varphi^\delta : \mathsf{B}\, X \longrightarrow \mathsf{P}$ defined by

$$\varphi^\delta(\mathfrak{L}) = \varphi(\mathfrak{L} \oplus \delta).$$

For a functional ideal Φ on $\mathsf{B}\, X$ the collection

$$\{\varphi^\delta | \varphi \in \Phi\}$$

is an ideal basis on $\mathsf{B}\, X$. Let Φ^δ be the functional ideal generated by this basis.

Theorem 4.3.5.11. *The following properties are equivalent:*

(i) $n_X(\Theta) \rightsquigarrow x$ & $(a^\wedge \mathfrak{K})^\delta \subseteq \Theta$ & $c(\mathfrak{K}) = c(\Theta) = c(n_X(\Theta))$, *imply*

$$\mathfrak{K} \oplus \delta \rightsquigarrow x$$

for all $x \in X$, $\delta \geq 0$, $\Theta \in \mathsf{B}\,\mathsf{B}\,X$ *and* $\mathfrak{K} \in \mathsf{B}\,X$.

(ii) $n_X(\Theta) \rightsquigarrow x$ & $s(z) \oplus \delta \rightsquigarrow \psi(z)$, $\forall z \in A$ & $c(\mathfrak{I}) = c(n_X(\Theta))$, *imply*

$$\mathfrak{K} \oplus \delta \rightsquigarrow x$$

for all $x \in X$, $\delta \geq 0$, $\Theta = \mathsf{B}\,s(\mathfrak{I})$ *and* $\mathfrak{K} = \mathsf{B}\,\psi(\mathfrak{I})$ *generated by* $\mathfrak{I} \in \mathsf{B}\,A$ *and selections* $s : A \to \mathsf{B}\,X$ *and* $\psi : A \to X$.

(iii) X is a regular approach space.

Proof. (i) $\Rightarrow$ (ii): Let $x \in X$, $\delta \geq 0$, $\Theta = \mathsf{B}\,s(\mathfrak{I})$ and $\mathfrak{K} = \mathsf{B}\,\psi(\mathfrak{I})$ generated by $\mathfrak{I} \in \mathsf{B}\,A$ and selections $s : A \to \mathsf{B}\,X$ and $\psi : A \to X$ with $n_X(\Theta) \rightsquigarrow x$, $s(z) \oplus \delta \rightsquigarrow \psi(z)$, for all $z \in A$ and $c(\mathfrak{I}) = c(n_X(\Theta))$. It suffices to show that $(a^\wedge \mathfrak{K})^\delta \subseteq \Theta$. Let φ be bounded on $\mathsf{B}\,X$ with $\sup \varphi = \omega$ and $\varphi \leq a^\wedge \mu$ for some bounded function μ with $\mu \cdot \psi \in \mathfrak{I}$. Put $\eta : \mathsf{B}\,X \to \mathsf{P}$

$$\eta(\mathfrak{L}) = \begin{cases} \inf_{\{y \mid s(y) = \mathfrak{L}\}} \mu \cdot \psi(y) & \text{whenever } \mathfrak{L} \in s(A) \\ \omega & \text{whenever } \mathfrak{L} \notin s(A). \end{cases}$$

For $z \in A$ we clearly have $\eta \cdot s(z) \leq \mu \cdot \psi(z)$ so $\eta \cdot s \in \mathfrak{I}$ and $\eta \in \Theta$.

We claim that $\varphi^\delta \leq \eta$. For $\mathfrak{L} \in \mathsf{B}\,X$ either $\mathfrak{L} \notin s(A)$ and then $\varphi^\delta(\mathfrak{L}) \leq \omega = \eta(\mathfrak{L})$. Or $\mathfrak{L} \in s(A)$. For $z \in A$ with $s(z) = \mathfrak{L}$ we have $\mathfrak{L} \oplus \delta \rightsquigarrow \psi(z)$. Hence

$$\varphi^\delta(\mathfrak{L}) = \varphi(\mathfrak{L} \oplus \delta) \leq \inf_{\{y \mid \mathfrak{L} \oplus \delta \rightsquigarrow y\}} \mu(y) \leq \inf_{\{z \mid s(z) = \mathfrak{L}\}} \mu(\psi(z)).$$

(ii) $\Rightarrow$ (iii): In order to prove (4.27), let A a set, $\psi : A \longrightarrow X, \sigma : A \longrightarrow \beta X$, and $\mathcal{U} \in \beta A$ and suppose $\lambda \Sigma \sigma(\mathcal{U}) = \varepsilon < \infty$ and $\sup_{z \in A} \lambda \sigma(z)(\psi(z)) = \delta < \infty$. For $z \in A$ consider $s(z) = \iota_X(\sigma(z))$ and $\mathfrak{I} = \iota_A(\mathcal{U}) \oplus \varepsilon$. Clearly $s(z) \rightsquigarrow \psi(z)$, for all $z \in A$ and since $\Theta = \mathsf{B}\,s(\mathfrak{I})$ satisfies

$$n_X(\Theta) = \iota_X(\Sigma \sigma(\mathcal{U})) \oplus \varepsilon,$$

also $n_X(\Theta) \rightsquigarrow x$ and $c(\mathfrak{I}) = c(n_X(\Theta))$ hold. Then (ii) implies that $\mathfrak{K} = \mathsf{B}\,\psi(\mathfrak{I})$ satisfies $\mathfrak{K} \oplus \delta \rightsquigarrow x$. Since $\mathfrak{K} \oplus \delta = \iota_X(\psi(\mathcal{U})) \oplus (\varepsilon + \delta)$ the conclusion $\lambda \psi(\mathcal{U}) \leq \varepsilon + \delta$ follows.

(iii) $\Rightarrow$ (ii): Suppose X is a regular approach space. Let $x \in X$, $\delta \geq 0$, $\Theta = \mathsf{B}\,s(\mathfrak{I})$ and $\mathfrak{K} = \mathsf{B}\,\psi(\mathfrak{I})$ generated by $\mathfrak{I} \in \mathsf{B}\,A$ and selections $s : A \to \mathsf{B}\,X$

and $\psi : A \to X$ satisfying $n_X(\Theta) \rightsquigarrow x$, $s(z) \oplus \delta \rightsquigarrow \psi(z)$, for all $z \in A$ and $c(\mathfrak{I}) = c(n_X(\Theta))$. For $z \in A$ suppose $s(z) = \iota_X(\sigma(z)) \oplus \alpha_z$ with $\sigma(z) \in \beta X$ and $\mathfrak{I} = \iota_A(\mathcal{W}) \oplus \gamma$. By

$$\gamma = c(n_X(\mathsf{B}\, s(\mathfrak{I}))) = \gamma + \sup_{W \in \mathcal{W}} \inf_{z \in W} \alpha_z$$

we have $\sup_{W \in \mathcal{W}} \inf_{z \in W} \alpha_z = 0$ and $n_X(\mathsf{B}\, s(\mathfrak{I})) = \iota_X(\Sigma\sigma(\mathcal{W})) \oplus \gamma \rightsquigarrow x$. This implies $\lambda\Sigma\sigma(\mathcal{W})(x) \leq \gamma$. Since $\lambda\sigma(z)(\psi(z)) \leq \alpha_z + \delta$, for all $z \in A$ and since $\mathcal{W}$ is an ultrafilter, by application of Lemma 1.1.1.4, it follows that

$$\sup_{W \in \mathcal{W}} \inf_{z \in W} \lambda\sigma(z)(\psi(z)) = \inf_{W \in \mathcal{W}} \sup_{z \in W} \lambda\sigma(z)(\psi(z)) \leq \delta.$$

For $\varepsilon > 0$ as in the proof of Theorem 4.3.5.7, choose $W_0 \in \mathcal{W}$ such that $\sup_{z \in W_0} \lambda\sigma(z)(\psi(z)) < \delta + \varepsilon$ and let $\sigma' : A \to \beta X$ be defined as

$$\sigma'(z) = \begin{cases} \sigma(z) \text{ for all } z \in W_0 \\ \psi(z) \text{ for all } z \notin W_0. \end{cases}$$

Now apply regularity (4.27) with as input $\mathcal{W}$ on A and the selections ψ and σ'. Then for $\mathcal{U} = \psi(\mathcal{W})$ we have

$$\begin{aligned} \lambda\mathcal{U}(x) &\leq \lambda\Sigma(\mathscr{U})(x) + \sup_{z \in A} \lambda\sigma'(z)(\psi(z)) \\ &= \lambda\Sigma\mathscr{U}(x) + \sup_{z \in W_0} \lambda\sigma(z)(\psi(z)) \\ &< \gamma + \delta + \varepsilon \end{aligned}$$

By arbitrariness of ε we can conclude that $\mathfrak{K} = \iota_X(\mathcal{U}) \oplus \gamma \rightsquigarrow x$.

(ii) $\Rightarrow$ (i): Let $x \in X$, $\delta \geq 0$, $\Theta \in \mathsf{B}\,\mathsf{B}\, X$ and $\mathfrak{K} \in \mathsf{B}\, X$ with $n_X(\Theta) \rightsquigarrow x$, $(a^\wedge\mathfrak{K})^\delta \subseteq \Theta$ and $c(\mathfrak{K}) = c(\Theta) = c(n_X(\Theta))$. Assume $\Theta = \iota_{\mathsf{B}\, X}(\mathscr{U}) \oplus \gamma$ and $\mathfrak{K} = \iota_X(\mathcal{U}) \oplus \gamma$. We may assume that $\mathfrak{K}$ is proper, so $\gamma < \infty$. Let

$$A = \{(y, \mathfrak{L}) \mid \mathfrak{L} \oplus \delta \rightsquigarrow y, \mathfrak{L} \in \mathsf{B}\, X, y \in X\}.$$

We show that $\mathcal{U} \times \mathscr{U}$ has a trace on A. Let $U_0 \in \mathcal{U}, \mathcal{A}_0 \in \mathscr{U}, \gamma < \omega < \infty$. Consider

$$\varphi = (a^\wedge(\theta^\omega_{U_0} + \gamma) \wedge \omega)^\delta \vee (\theta^\omega_{\mathcal{A}_0} + \gamma).$$

Since $\varphi \in \Theta$ and $c(\Theta) = \gamma$ one should have $\inf_{\mathfrak{L} \in \mathsf{B}\, X} \varphi(\mathfrak{L}) \leq \gamma$. We have

$$\begin{aligned} \inf_{\mathfrak{L} \in \mathsf{B}\, X} \varphi(\mathfrak{L}) &= (\inf_{\mathfrak{L} \notin \mathcal{A}_0} \varphi(\mathfrak{L})) \\ &\wedge (\inf_{\mathfrak{L} \in \mathcal{A}_0, \mathfrak{L} \oplus \delta \text{ divergent}} \varphi(\mathfrak{L})) \\ &\wedge (\inf_{\mathfrak{L} \in \mathcal{A}_0, \mathfrak{L} \oplus \delta \text{ convergent, } \nexists y \in U_0, \mathfrak{L} \oplus \delta \rightsquigarrow y} \varphi(\mathfrak{L})) \\ &\wedge (\inf_{\mathfrak{L} \in \mathcal{A}_0, \exists y \in U_0, \mathfrak{L} \oplus \delta \rightsquigarrow y} \varphi(\mathfrak{L})). \end{aligned}$$

Since the first three terms are all larger than or equal to ω, the last infimum cannot be infinite. So we can conclude that there exists $\mathfrak{L} \in \mathcal{A}_0$, $\mathfrak{L} \oplus \delta \rightsquigarrow y$ for some $y \in U_0$ and for such $\mathfrak{L}$ and y we have $(y, \mathfrak{L}) \in A \cap U_0 \times \mathcal{A}_0$.

Let $\mathcal{W}$ be an ultrafilter on A that refines the trace of $\mathcal{U} \times \mathscr{U}$ on A. Further let $\psi : A \longrightarrow X$ and $s : A \longrightarrow \mathsf{B}\, X$ be the restrictions of the first and second projections. Then we have $\psi(\mathcal{W}) = \mathcal{U}$ and $s(\mathcal{W}) = \mathscr{U}$. For $\mathfrak{I} = \iota_A(\mathcal{W}) \oplus \gamma$ we have $\mathsf{B}\, \psi(\mathfrak{I}) = \mathfrak{K}$ and $\mathsf{B}\, s(\mathfrak{I}) = \Theta$. Applying (ii) we obtain $\mathfrak{K} \oplus \delta \rightsquigarrow x$.

$\square$

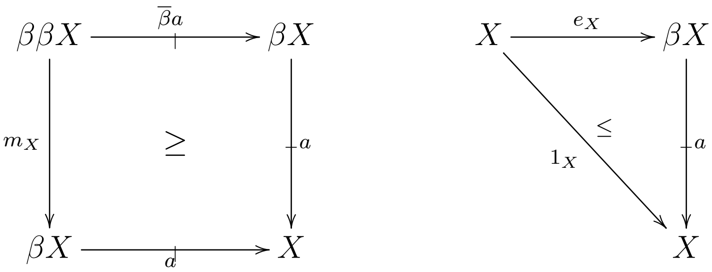
Nederlandse samenvatting

Monoidale topologie is een actief onderzoeksgebied binnen de wiskunde dat een gemeenschappelijk kader verschaft voor "convergentie".

Een eerste bouwsteen voor de ontwikkeling van monoidale topologie is de representatie van topologische ruimten door Barr [Bar70], die een veralgemening vormt van het bewijs van Manes dat compacte Hausdorff topologische ruimten precies de Eilenberg-Moore algebra's zijn voor de ultrafilter monad $\pmb{\beta} = (\beta, m, e)$ [Man69]. In deze beschrijving is een compacte Hausdorff ruimte een verzameling X uitgerust met een afbeelding $a : \beta X \longrightarrow X$ die aan elke ultrafilter op X zijn unieke convergentiepunt in X toekent en voldoet aan twee axioma's die men kan voorstellen door middel van volgende diagrammen

$$\begin{array}{ccc} \beta^2 X & \xrightarrow{\beta a} & \beta X \\ {\scriptstyle m_X}\downarrow & & \downarrow{\scriptstyle a} \\ \beta X & \xrightarrow{a} & X \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{e_X} & \beta X \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle a} \\ & & X. \end{array}$$

Om een beschrijving te krijgen van alle topologische ruimten werd in het werk van Barr [Bar70] de afbeelding $a : \beta X \longrightarrow X$ vervangen door een relatie $a : \beta X \relbar\joinrel\mapstochar\joinrel\rightarrow X$. Daardoor is het niet langer gegarandeerd dat elke ultrafilter convergeert (compactheid) en dat er hoogstens één convergentiepunt is (Hausdorff-separatie). Natuurlijk weet men wat βa betekent wanneer $a : \beta X \longrightarrow X$ een afbeelding is, maar niet wanneer $a : \beta X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ een relatie betreft, dus om ervoor te zorgen dat de volgende definities zinvol zijn, moet men de ultrafilter monad $\pmb{\beta} = (\beta, m, e)$ eerst op gepaste wijze uitbreiden naar Rel, de categorie bestaande uit verzamelingen en relaties. We beschouwen dan ook een lax versie van de bovenstaande diagrammen:



of, in puntsgewijze vorm, waarbij we a en $\overline{\beta}a$ beiden noteren als $\rightarrow$, krijgen we volgende axioma's:

$$\text{transitiviteit: } \mathfrak{X} \rightarrow \mathcal{U} \,\&\, \mathcal{U} \rightarrow z \Rightarrow m_X\mathfrak{X} \rightarrow z$$

en

$$\text{reflexiviteit: } e_X(x) \rightarrow x,$$

voor alle $\mathfrak{X} \in \beta\beta X, \mathcal{U} \in \beta X$ en $z, x \in X$.

Barr toonde aan dat een verzameling X uitgerust met een relatie $a : \beta X \rightarrow\!\!\!\!\!\!+\;\, X$ die voldoet aan beide axioma's hierboven, transitiviteit en reflexiviteit, een topologische ruimte is en dat elke topologische ruimte op zulke manier kan beschreven worden. Samen met de continue afbeeldingen beschreven als de afbeeldingen die de convergentie bewaren, geeft dit een relationele beschrijving van de categorie Top, de categorie van topologische ruimten en continue afbeeldingen, dewelke we noteren als

$$(\mathbb{B}, 2)\text{-}\mathsf{Cat} \cong \mathsf{Top}.$$

De terminologie "transitiviteit" en "reflexiviteit" behoudt de betekenis van transitiviteit en reflexiviteit voor geordende verzamelingen. Als we bovenstaande axioma's toepassen op de identiteitsmonad $\mathbb{1}$ in plaats van op de ultrafilter monad $\mathbb{B}$, krijgen we een paar (X, a) met $a : X \rightarrow\!\!\!\!\!\!+\;\, X$ een relatie dewelke we met $\leq$ kunnen noteren en die voldoet aan

$$\forall x, y, z \in X : x \leq y \,\&\, y \leq z \Rightarrow x \leq z,$$

de klassieke transitiviteit en

$$\forall x \in X : x \leq x,$$

de klassieke reflexiviteit. Vandaar geldt

$$(\mathbb{1}, 2)\text{-}\mathsf{Cat} \cong \mathsf{Ord},$$

waar Ord de categorie is van geordende verzamelingen en ordebewarende afbeeldingen. Merk op dat we anti-symmetrie niet opleggen als axioma bij geordende verzamelingen.

Een tweede belangrijke bouwsteen in de ontwikkeling van monoidale topologie was de beschrijving door Lawvere van metrische ruimten als (kleine individuele) categorieën verrijkt over $[0,\infty]$ [Law73]. In meer begrijpbare taal betekent dit dat als we de bovenstaande axioma's toepassen op de identiteitsmonad $\mathbb{1}$ en de relaties vervangen door $[0,\infty]$-waardige relaties, we een lax algebraïsche beschrijving krijgen van quasi-metrische ruimten. Om dit te begrijpen kunnen we $[0,\infty]$ beschouwen samen met de omgekeerde orde, de bewerking $+$ en neutraal element 0. We noteren dit als

$$\mathsf{P}_+ = \big([0,\infty], \leq_{\text{op}}, +, 0\big).$$

Een quasi-metrische ruimte (X,a) is een verzameling X uitgerust met een afbeelding $a : X \times X \longrightarrow [0,\infty]$, of nog, een $[0,\infty]$-waardige relatie $a : X \to\!\!\!\!\!+ X$ die transitief

$$\forall x,y,z \in X : a(x,y) + a(y,z) \geq a(x,z),$$

en reflexief

$$\forall x \in X : a(x,x) = 0$$

is. Aldus kan men de categorie $q\mathsf{Met}$ van quasi-metrische ruimten en niet-expansieve afbeeldingen beschrijven als

$$(\mathbb{1}, \mathsf{P}_+)\text{-}\mathsf{Cat} \cong q\mathsf{Met}.$$

Quasi-metrische structuren gedragen zich niet goed ten opzichte van het vormen van initiale structuren, producten in het bijzonder. Het product in $q\mathsf{Met}$ van een oneindige familie van quasi-metrische ruimten is niet compatibel met het topologische product van de geassocieerde onderliggende topologieën. Als een oplossing voor dit probleem werd de gemeenschappelijke bovencategorie App (waarvan de objecten approach ruimten genoemd worden) van Top en $q\mathsf{Met}$ ingevoerd [Low15]. Het voornaamste verschil tussen approach ruimten en metrische ruimten is het feit dat in approach ruimten men de afstanden tussen punten en verzamelingen beschrijft en hiervoor axioma's opstelt. Zulke afstanden kunnen, in tegenstelling tot wat het geval is in quasi-metrische ruimten, niet afgeleid worden uit de afstand tussen twee punten.

Een approach ruimte is een verzameling X uitgerust met een functie

$$\delta : X \times 2^X \longrightarrow [0,\infty],$$

die men de distance noemt en voldoet aan volgende axioma's:

(D1) $\forall x \in X, \forall A \subseteq X : x \in A \Rightarrow \delta(x,A) = 0.$

(D2) $\forall x \in X : \delta(x,\emptyset) = \infty.$

(D3) $\forall x \in X, \forall A, B \subseteq X : \delta(x, A \cup B) = \min\big(\delta(x, A), \delta(x, B)\big)$.

(D4) $\forall x \in X, \forall A \subseteq X, \forall \varepsilon \geq 0 : \delta(x, A) \leq \delta\big(x, A^{(\varepsilon)}\big) + \varepsilon$,
waarbij $A^{(\varepsilon)} := \big\{x \in X \mid \delta(x, A) \leq \varepsilon\big\}$.

De waarde $\delta(x, A)$ wordt dan geïnterpreteerd als de afstand van het punt x tot de verzameling A.

De morfismen in de categorie App noemt men contracties en een contractie is een afbeelding

$$f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$$

tussen twee approach ruimten die voldoet aan

$$\forall x \in X, \forall A \subseteq X : \delta_Y\big(f(x), f(A)\big) \leq \delta_X(x, A).$$

Approach ruimten kunnen op equivalente mannier beschreven worden als een verzameling X uitgerust met een limietoperator

$$\lambda : \mathsf{F}X \longrightarrow [0, \infty]^X,$$

gedefinieerd op de verzameling van alle filters op X, dewelke voldoet aan gepaste axioma's. De waarde $\lambda\mathcal{F}(x)$ wordt dan geïnterpreteerd als de maat waarin x een convergentiepunt is van de filter $\mathcal{F}$. Een approach ruimte kan ook beschreven worden door middel van een tower

$$(\mathsf{t}_\varepsilon)_{\varepsilon \in \mathbb{R}^+},$$

een geordende familie van pre-topologieën op X geïndexeerd door de positieve reële getallen die voldoen aan zekere coherentie condities, of een gauge

$$\mathcal{G} \subseteq q\mathsf{Met}(X),$$

een ideaal van quasi-metrieken op X dat voldoet aan een saturatie-eigenschap, of nog andere equivalente structuren.

De categorie App bestaande uit approach ruimten en contracties bevat zowel Top als qMet als vol ingebedde deelcategorieën. Top is een concreet coreflectieve en reflectieve deelcategorie en qMet is een concreet coreflectieve deelcategorie van App. Alle voorkennis over approach ruimten die nodig is zal herhaald worden in Sectie 1.1.

Een eerste lax algebraïsche karakterisatie van approach ruimten werd geconstrueerd door Clementino en Hofmann [CH03] door een extensie $\overline{\beta}$ van de ultrafilter monad β naar numerieke relaties te bouwen. Gebruik makend van de beschrijving van approach ruimten met behulp van de limietoperator kan een approach ruimte gezien worden als een paar (X, a) waar X een verzameling is en $a : \beta X \nrightarrow X$ een $[0, \infty]$-waardige relatie die voldoet aan het transitiviteitsaxioma

$$a\big(m_X \mathfrak{X}, z\big) \leq \overline{\beta} a(\mathfrak{X}, \mathcal{U}) + a(\mathcal{U}, z),$$

voor alle $\mathfrak{X} \in \beta\beta X, \mathcal{U} \in \beta X$ en $z \in X$, en het reflexiviteitsaxioma

$$a\big(e_X(x), x\big) = 0,$$

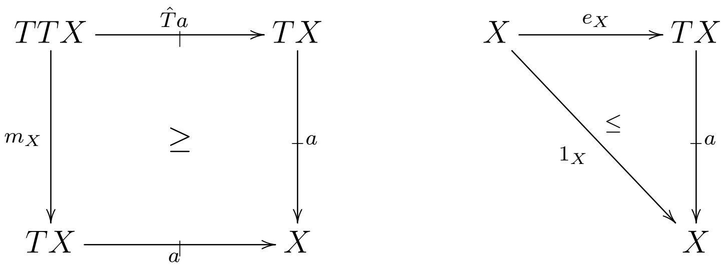
voor alle $x \in X$. Aldus geldt,

$$(\beta, \mathsf{P}_+)\text{-}\mathsf{Cat} \cong \mathsf{App}.$$

In het algemeen maakt monoidale topologie gebruik van twee parameters; een Set-monad $\mathbb{T} = (T, m, e)$ en een quantale $\mathcal{V}$ samen met een lax extensie $\hat{\mathbb{T}}$ van de monad $\mathbb{T}$ tot de categorie $\mathcal{V}$-Rel bestaande uit verzamelingen en $\mathcal{V}$-waardige relaties. Dit levert ons de categorie $(\mathbb{T}, \mathcal{V})$-Cat op met objecten (X, a) waar X een verzamling is uitgerust met een $\mathcal{V}$-relatie

$$a : TX \nrightarrow X$$

die transitief en reflexief is.



Tot hiertoe hebben we gezien dat monoidale topologie een gemeeschappelijk kader vormt om geordende ruimten, metrische ruimten, topologische ruimten en approach ruimten te beschrijven. Alle voorkennis over monoidale topologie die nodig is om deze thesis te begrijpen zal herhaald worden in Hoofdstuk 1.

In Hoofdstuk 2 richten we onze aandacht op NA-App, de volle deelcategorie van App met objecten niet-Archimedische approach ruimten. Niet-Archimedische approach ruimten werden geïntroduceerd door Brock en Kent [BK98] en werden ook beschouwd door Colebunders, Mynard en Trott in [CMT14] en door Boustique en Richardson [BR17] als bepaalde “limit tower spaces”.

Niet-Archimedische approach ruimten zijn die approach ruimten X waar de distance δ voldoet aan de sterke driehoeksongelijkheid

(D4$_\vee$) $\forall x \in X, \forall A \subseteq X, \forall \varepsilon \geq 0 : \delta(x, A) \leq \delta\big(x, A^{(\varepsilon)}\big) \vee \varepsilon.$

Ze kunnen ook makkelijk gekarakteriseerd worden aan de hand van de tower. Niet-Archimedische approach ruimten zijn deze approach ruimten waar de tower bestaat uit topologieën

$$(\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$$

die voldoen aan de coherentie conditie: $\forall \varepsilon \in \mathbb{R}^+ : \mathcal{T}_\varepsilon = \bigvee_{\gamma > \varepsilon} \mathcal{T}_\gamma$.

We onderzoeken een karakterisatie van niet-Archimedische approach ruimten in functie van de gauge. Het blijkt dat niet-Archimedische approach ruimten precies die approach ruimten zijn die een basis voor de gauge hebben bestaande uit quasi-ultrametrieken, i.e. quasi-metrieken $d : X \times X \longrightarrow [0, \infty]$ die voldoen aan de sterke driehoeksongelijkheid

$$\forall x, y, z \in X : d(x, z) \leq d(x, y) \vee d(y, z).$$

In Sectie 2.2 beantwoorden we de vraag welke parameters $\mathbb{T}$ en $\mathcal{V}$ we kunnen gebruiken om NA-App voor te stellen als een categorie van lax algebra's. Hiervoor laten we ons inspireren door het gekende resultaat

$$(\mathbb{1}, \mathsf{P}_\vee)\text{-}\mathsf{Cat} \cong q\mathsf{Met}^u,$$

waar $q\mathsf{Met}^u$ de volle deelcategorie van $q\mathsf{Met}$ is bestaande uit alle quasi-ultrametrische ruimten. We slagen er in om te bewijzen dat de oplossing er uit bestaat het quantale P_+ in de representatie van App als (β, P_+)-Cat door $\mathsf{P}_\vee$ te vervangen. We vinden de representatie

$$(\beta, \mathsf{P}_\vee)\text{-}\mathsf{Cat} \cong \mathsf{NA}\text{-}\mathsf{App}.$$

In wat vooraf ging kon App enkel voorgesteld worden als een categorie van lax algebra's door de ultrafilter monad uit te breiden naar numerieke relaties. In Hoofdstuk 3 beantwoorden we de vraag of een representatie van App mogelijk is in termen van relationele algebras. Dit wil zeggen dat we ons zullen toespitsen op lax algebraïsche representaties van App waar we enkel gebruik maken van het quantale $\mathcal{V} = 2$.

Ons voornaamste voorbeeld is nog maar eens Top. In [Sea05] toonde Seal aan dat topologische ruimten kunnen beschreven worden als $\mathbb{F}$-monoïden voor de filter monad $\mathbb{F}$ die power-enriched is. De sleutel tot dit bewijs is de afbeelding

$$X \longrightarrow \mathsf{F}X : x \mapsto \mathcal{V}(x)$$

die elk punt x van de topologische ruimte X naar zijn omgevingenfilter $\mathcal{V}(x)$ stuurt. Aangezien convergentie in topologische ruimten volledig bepaald wordt door $\big(\mathcal{V}(x)\big)_{x\in X}$, impliceert de representatie van topologische ruimten als $\mathbb{F}$-monoïden voor de filter monad $\mathbb{F}$ die power-enriched is de presentatie in termen van relationele algebra's

$$(\mathbb{F}, 2)\text{-}\mathsf{Cat} \cong \mathsf{Top}.$$

Om de vraag te beantwoorden of we App kunnen beschrijven in termen van relationele algebra's voor een zekere monad $\mathbb{T} = (T, m, e)$ die power-enriched is, zullen we eerst focussen op de beschrijving van App in termen van $\mathbb{T}$-monoïden voor een gepaste monad die power-enriched is. De leiddraad voor de oplossing is de afbeelding

$$x \mapsto \mathcal{A}(x),$$

die elk punt x van een approach ruimte X naar het bijhorende lokale approach systeem stuurt, waar $\mathcal{A}(x)$ kan afgeleid worden uit de gauge $\mathcal{G}$ door de collectie

$$\{d(x, \cdot) \mid d \in \mathcal{G}\}$$

op gepaste wijze te satureren.

We voeren de monad $\mathbb{I} = (\mathsf{I}, m, e)$ van functionele idealen in en bewijzen dat deze power-enriched is. Deze bevindingen leiden uiteindelijk tot de presentatie van approach ruimten als $\mathbb{I}$-monoïden. Meer nog, aangezien convergentie van functionele idealen in een approach ruimte X volledig bepaald wordt door de lokale approach systemen $\big(\mathcal{A}(x)\big)_{x\in X}$, kunnen we een algemene stelling uit [HST14] toepassen over de relatie tussen categorieën van $\mathbb{T}$-monoïden en relationele $\mathbb{T}$-algebra's om te kunnen concluderen dat

$$(\mathbb{I}, 2)\text{-}\mathsf{Cat} \cong \mathsf{App},$$

wat een presentatie van App geeft als een categorie van relationele algebra's en de vraag die we in dit hoofdstuk voorop stelden positief beantwoordt.

We eindigen dit hoofdstuk met een studie van de priem functionele idealen en de bijbehorende monad $\mathbb{B}$ [LVOV08], [LV08], dewelke niet power-enriched is. We tonen aan dat $\mathbb{B}$ een submonad is van $\mathbb{I}$ die precies die eigenschappen heeft die in [HST14] geformuleerd worden om te kunnen concluderen dat $(\mathbb{B}, 2)\text{-}\mathsf{Cat} \cong (\mathbb{I}, 2)\text{-}\mathsf{Cat}$, wat ons het resultaat uit [LV08] geeft dat

$$(\mathbb{B}, 2)\text{-}\mathsf{Cat} \cong \mathsf{App}.$$

De nieuwe beschrijvingen van NA-App als een categorie van lax algebra's, bekomen in Hoofdstuk 2, en van App als een categorie van relationele algebra's, ontwikkeld in Hoofdstuk 3 zijn de voornaamste werktuigen voor een diepgaande studie van nieuwe approach invarianten in het laatste hoofdstuk, Hoofdstuk 4. Deze approach invarianten zullen onstaan als topologische eigenschappen van de betrokken lax algebra's (of relationele algebra's in het geval $\mathcal{V} = 2$). We kunnen de lax algebra's (X, a) beschouwen als ruimten en noteren de convergentierelatie op X als $a : TX \nrightarrow X$, zoals voordien. Topologische eigenschappen voor zulke ruimten werden geïntroduceerd gebaseerd op de notie van convergentie, a, en afhankelijk van de monad $\mathbb{T}$, het quantale $\mathcal{V}$ en de extensie van $\mathbb{T}$ naar $\mathcal{V}$-Rel, [HST14]. Deze noties werden vervolgens toegepast op de voorbeelden $(\beta, 2)$-Cat $\cong$ Top, waar ze samenvallen met de gekende respectievelijke eigenschappen, en op (β, P_+)-Cat $\cong$ App waar ze ook samen vallen met enkele gekende approach invarianten.

In deze samenvatting beperken we ons tot het bespreken van nieuwe approach invarianten gebaseerd op Hausdorff separatie (ten hoogste één convergentiepunt), compactheid (ten minste één convergentiepunt) en regulariteit (het omdraaien van het transitiviteitsaxioma van a), maar ook andere invarianten komen aan bod in deze thesis.

Allereerst bespreken we nieuwe invarianten voor niet-Archimedische approach ruimten. Gebaseerd op de representatie $(\beta, \mathsf{P}_\vee)$-Cat $\cong$ NA-App is een niet-Archimedische approach ruimte X $(\beta, \mathsf{P}_\vee)$-Hausdorff als en slechts als eindige waarden van de convergentierelatie a voor $a(\mathcal{U}, x)$ en $a(\mathcal{U}, y)$ met $\mathcal{U}$ een ultrafilter op X impliceert dat $x = y$. Anderzijds weten we dat de tower bestaat uit een geïndexeerde familie van topologieën $(\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$, dus kunnen we ook drie andere noties beschouwen: "strongly" Hausdorff (alle niveaus van de tower zijn Hausdorff), "almost strongly" Hausdorff (alle strikt positieve niveaus van de tower zijn Hausdorff) en de eigenschap dat de topologische coreflectie $(X, \mathcal{T}_0)$ van X Hausdorff is. Alle eigenschappen blijken equivalent te zijn, behalve de laatste dewelke strikt zwakker is.

We gebruiken gelijkaardige definities voor "strongly" compact, "almost strongly" compact en de topologische coreflectie $(X, \mathcal{T}_0)$ is compact. $(\beta, \mathsf{P}_\vee)$-compact is equivalent met almost strongly compact, en beiden zijn equivalent met het gekende begrip 0-compact voor approach ruimten [Low15]. Strongly compact is equivalent met het compact zijn van de topologische coreflectie $(X, \mathcal{T}_0)$ van X en beiden zijn strikt sterker dan de vorige eigenschap.

We onderzoeken ook gelijkaardige links voor regulariteit. De eigenschap "strongly" regulier werd geïntroduceerd door Brock en Kent in [BK98] en werd ook bestudeerd in de context van contractieve extensies in [CMT14]. We concluderen dat strongly regulier en almost strongly regulier beiden equivalent zijn met $(\beta, \mathsf{P}_\vee)$-regulariteit. Regulariteit voor approach ruimten, zoals ingevoerd in [Rob92] door Robeys is strikt zwakker en impliceert dat de topologische coreflectie $(X, \mathcal{T}_0)$ van X regulier is.

Vervolgens richten we onze aandacht op de relationele algebra's die App beschrijven. We onderzoeken topologische eigenschappen in App geïnduceerd door de representatie $(\mathbb{I}, 2)$-Cat $\cong$ App voor de monad $\mathbb{I}$ die power-enriched is. Aangezien het improper functionele ideaal P_b^X bestaande uit alle begrensde functies van X naar $[0, \infty]$, naar alle punten van X convergeert, zullen we in sommige gevallen van de topologische eigenschappen die hierboven vernoemd werden enkel triviale resultaten kunnen verwachten. Daarom zullen we in sommige gevallen het improper element moeten uitsluiten en zullen we ons beperken tot functionele idealen op X die proper zijn door de deelfunctor I_p gedefinieerd door $\mathsf{I}_p X = \mathsf{I} X \setminus \{\mathsf{P}_b^X\}$ te beschouwen.

De eigenschap $(\mathsf{I}_p, 2)$-Hausdorff betekent dat wanneer een proper functioneel ideaal naar zowel x als y convergeert, dan $x = y$. Deze notie blijkt equivalent te zijn met de approach invariant die stelt dat de topologische coreflectie van X Hausdorff is.

Voor een studie van compactheid zal het uitsluiten van impropere elementen geen invloed hebben op de topologische eigenschap. Wanneer we $(\mathbb{I}, 2)$-compactheid in App beschouwen, wil dit zeggen dat elk functioneel ideaal een convergentiepunt moet hebben. In het bijzonder bestaat er een $x \in X$ zodat $\{0\} \rightarrowtail x$, of nog $\mathcal{A}(x) = \{0\}$. Dit begrip zullen we supercompactheid in App noemen.

De meest uitgebreide studie in deze thesis is die voor regulariteit. Eerst tonen we enkele algemene resultaten voor monads die power-enriched zijn. We tonen dat een relationele $\mathbb{T}$-algebra (X, a), voor $\mathbb{T}$ een monad die power-enriched is, $(\mathbb{T}, 2)$-regulier is als en slechts als de ruimte indiscreet is, zelfs als we ons beperken tot elementen die proper zijn. Om voor onze monad $\mathbb{I}$ een interessante invariant te krijgen die gerelateerd is met regulariteit, zullen we ons beperken tot functionele idealen die gegenereerd worden door zekere selecties. Hierdoor zullen we een eigenschap bekomen die equivalent is met regulariteit in approach ruimten, zoals geïntroduceerd door Robeys [Rob92], waarvoor we een karakterisatie geven in termen van functionele idealen.

Tot slot zullen we topologische eigenschappen bestuderen in App die geïnduceerd worden door de representatie $(\mathbb{B}, 2)$-Cat $\cong$ App. Aangezien $\mathbb{B}$ niet power-enriched is, zal de situatie hier verschillend zijn. Het impropere functionele ideaal P_b^X is echter priem, waardoor we ook hier in sommige gevallen het impropere element zullen moeten uitsluiten. We zullen ons dan moeten beperken tot propere functionele idealen die priem zijn door de deelfunctor B_p te beschouwen waarbij $\mathsf{B}_p X = \mathsf{B}_p X \setminus \{\mathsf{P}_b^X\}$ om interessante resultaten te bekomen.

De eigenschap $(\mathsf{B}_p, 2)$-Hausdorff betekent dat wanneer een proper priem functioneel ideaal naar zowel x als y convergeert volgt dat $x = y$. Deze eigenschap zal equivalent blijken met de eigenschap $(\mathsf{I}_p, 2)$-Hausdorff.

Wanneer we $(\mathbb{B}, 2)$-compactheid beschouwen krijgen we meteen een andere eigenschap dan $(\mathbb{I}, 2)$-compactheid. Terwijl $(\mathbb{I}, 2)$-compactheid van X door ons supercompactheid in App genoemd werd, geeft $(\mathbb{B}, 2)$-compactheid andere resultaten aangezien $\{0\}$ geen priemideaal is. Een approach ruimte X is $(\mathbb{B}, 2)$-compact als en slechts als de topologische coreflectie compact is.

Wegens de aanwezigheid van impropere elementen is $(\mathbb{B}, 2)$-regulariteit een oninteressante eigenschap, aangezien een approach ruimte X $(\mathbb{B}, 2)$-regulier is enkel en alleen indien ze indiscreet is. In tegenstelling tot wat het geval is voor de monad $\mathbb{I}$, zal de restrictie tot propere elementen hier wel een interessante eigenschap opleveren. We tonen aan dat $(\mathsf{B}_p, 2)$-regulariteit equivalent is met het topologisch en regulier zijn van de approach ruimte. Om een karakterisatie van de gebruikelijke regulariteitseigenschap in App [Rob92] te krijgen in termen van convergentie voor priem functionele idealen, zullen we het concept nog verder moeten afzwakken.

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