



# Time-dependent Lotkaian informetrics incorporating growth of sources and items

L. Egghe\*

Universiteit Hasselt (UHasselt), Campus Diepenbeek, Agoralaan, B-3590 Diepenbeek, Belgium  
 Universiteit Antwerpen (UA), Campus Drie Eiken, Universiteitsplein 1, B-2610 Wilrijk, Belgium

## ARTICLE INFO

### Article history:

Received 13 February 2007  
 Received in revised form 8 January 2008  
 Accepted 11 January 2008

### Keywords:

Lotka  
 Lotkaian informetrics  
 Growth of sources  
 Growth of items  
 Time-dependent  
*h*-index  
 Hirsch  
*g*-index

## ABSTRACT

In a previous article, static Lotkaian theory was extended by introducing a growth function for the items. In this article, a second general growth function – this time for the sources – is introduced. Hence this theory now comprises real growth situations, where items and sources grow, starting from zero, and at possibly different paces. The time-dependent size- and rank-frequency functions are determined and, based on this, we calculate the general, time-dependent, expressions for the *h*- and *g*-index. As in the previous article we can prove that both indices increase concavely with a horizontal asymptote, but the proof is more complicated: we need the result that the generalized geometric average of concavely increasing functions is concavely increasing.

© 2008 Elsevier Ltd. All rights reserved.

## 1. Introduction

Let us have the size-frequency function of Lotka type

$$f(j) = \frac{C}{j^\alpha} \quad (1)$$

with  $j \in [1, +\infty[$  and  $\alpha > 1$ . The theory could be extended to general  $\alpha$  provided  $j$  is limited to a finite closed interval but this complication would make the arguments in this paper more cumbersome while, essentially, nothing new would be added; therefore we limit ourselves to  $j \geq 1$ ,  $\alpha > 1$ . Hence, for every  $j \geq 1$ ,

$$\int_j^\infty f(j') dj' = r \quad (2)$$

denotes the total number of sources with item-density  $j$  or larger. Transforming (1) into a probability density  $\varphi$  leads to

$$\varphi(j) = \frac{\alpha - 1}{j^\alpha} \quad (3)$$

as is readily seen (since  $\alpha > 1$ ).

In [5], we let the items grow in time  $t$ , using a general growth function  $G(t)$  as follows: define the new item-densities at time  $t$  as

$$jG(t) = k. \quad (4)$$

\* Corresponding address: Universiteit Hasselt (UHasselt), Campus Diepenbeek, Agoralaan, B-3590 Diepenbeek, Belgium.  
 E-mail address: [leo.egghe@uhasselt.be](mailto:leo.egghe@uhasselt.be).

Based on this we could prove in [5] that the time-dependent size-frequency density, derived from  $\varphi$  is the function

$$\varphi(k, t) = G(t)^{\alpha-1} \varphi(k) \quad (5)$$

for  $k \geq G(t)$ . Here  $\varphi(k)$  is considered to be  $\varphi(k, \infty)$ , i.e. for  $t = \infty$ .

The rank-frequency function  $\gamma$ , associated with  $f$ , is generally defined as (cf. [4])

$$r = \gamma^{-1}(j) = \int_j^{\infty} f(j') dj' \quad (6)$$

(cf. (2)), where  $\gamma^{-1}$  denotes the inverse of the function  $\gamma$  (note that  $\gamma^{-1}$  strictly decreases so that  $\gamma = (\gamma^{-1})^{-1}$  is defined). Note also that  $r \geq 0$ .

The time-dependent rank-frequency function was proved in [5] to be

$$\gamma(r, t) = G(t) \gamma(r). \quad (7)$$

The Hirsch-index (or  $h$ -index, shortly) is defined on a ranked list of sources (ranked in decreasing order of the number of items they produce or have), cf. [1–3,6–14]. It is the highest rank  $h$  such that all sources on rank 1,  $\dots$ ,  $h$  have  $h$  or more items. In the above time dependent framework it was proved that (see [5]– see also [8]) the time-dependent  $h$ -index is

$$h(t) = G(t)^{\frac{\alpha-1}{\alpha}} h, \quad (8)$$

where  $h$  is the  $h$ -index for  $t = \infty$ .

The  $g$ -index (introduced in [6,7,9]) is a solution for the drawback of the  $h$ -index that it does not increase even when the sources with the most items increase their number of items. We define the  $g$ -index to be the highest rank  $g$  such that the sources on rank 1,  $\dots$ ,  $g$ , together, have  $g^2$  or more items (note that  $h$  satisfies this, by definition, hence  $g \geq h$ ). In the above framework, in [5], we proved that the time-dependent  $g$ -index is

$$g(t) = G(t)^{\frac{\alpha-1}{\alpha}} g, \quad (9)$$

where  $g$  is the  $g$ -index for  $t = \infty$ .

In the next section we will extend this theory by allowing for two general growth functions: one is the function  $G$  as in (4) (from now on called  $G_1$ ) and another function  $G_2$  acting on source ranks. We then determine the general time-dependent Lotkian theory by proving formulae for the time-dependent size- and rank-frequency functions (using both growth functions  $G_1$  and  $G_2$ ).

Formula (5) is extended to

$$\varphi(k, t) = G_2(t) G_1(t)^{\alpha-1} \varphi(k) \quad (10)$$

and formula (7) is extended to

$$\gamma(s, t) = G_1(t) \gamma(r) \quad (11)$$

with

$$r G_2(t) = s \quad (12)$$

(the defining relation for  $G_2$ ).

In the third section we show that the general time-dependent formulae for the  $h$ -index and the  $g$ -index are:

$$h(t) = G_1(t)^{1-\frac{1}{\alpha}} G_2(t)^{\frac{1}{\alpha}} h \quad (13)$$

$$g(t) = G_1(t)^{1-\frac{1}{\alpha}} G_2(t)^{\frac{1}{\alpha}} g, \quad (14)$$

where  $h$  and  $g$  are the  $h$ -index and  $g$ -index for  $t = \infty$ . Hence both  $h(t)$  and  $g(t)$  are the values at  $t = \infty$ , multiplied by a time factor which is a generalized geometric mean of the functions  $G_1(t)$  and  $G_2(t)$ . We prove that such a generalized geometric mean of concavely increasing functions  $G_1$  and  $G_2$  is concavely increasing, hence the same is true for the functions  $h(t)$  and  $g(t)$ , growing to their values  $h$  and  $g$  (formulae proved earlier will be repeated here) for  $t = \infty$ .

## 2. Time-dependent Lotkian theory, incorporating general growth models for items and sources

### 2.1. Basic relations

Basic in this theory is the “static” law of Lotka (1), normalized so that we have a size-frequency distribution (see (3))

$$\varphi(j) = \frac{\alpha - 1}{j^\alpha}, \quad (15)$$

where  $j \geq 1$  and  $\alpha > 1$ . This distribution is considered as the “limiting” situation at time  $t = \infty$ , where  $\varphi(j)$  denotes the density of sources with item density  $j$  (see [4]).

For the growth in time  $t \geq 0$  we assume the following simple relations:

(i) Item growth: Let  $G_1(t)$  be the cumulative growth distribution of the item densities expressed by (cf. (4) with  $G = G_1$ )

$$jG_1(t) = k. \tag{16}$$

Hence, since  $j \geq 1$ , we have that  $k \geq G_1(t)$ .

(ii) Source growth: Let  $G_2(t)$  be the cumulative growth distribution of the rank densities (of the sources) expressed by (new equation)

$$rG_2(t) = s. \tag{17}$$

Hence  $s \geq 0$  since  $r \geq 0$ .

We underline that  $G_1$  and  $G_2$  are arbitrary cumulative growth distributions and, of course  $G_1$  and  $G_2$  can be different. As cumulative growth distributions we can (and will) suppose that  $G_1$  and  $G_2$  are strictly increasing concave functions such that  $G_1(0) = 0$ ,  $G_2(0) = 0$ ,  $\lim_{t \rightarrow \infty} G_1(t) = 1$ ,  $\lim_{t \rightarrow \infty} G_2(t) = 1$ . A classical example of such a function is

$$G_i(t) = 1 - a_i^t, \tag{18}$$

where  $0 < a_i < 1$ ,  $t \geq 0$ , but this is only an example (we do not need (18) for our model further on).

### 2.2. Size-frequency functions, in function of time $t$

Let  $T$  denote the total number of sources at  $t = \infty$ . Since (17) implies

$$\frac{r}{T}G_2(t) = \frac{s}{T} \tag{19}$$

and since

$$\frac{r}{T} = \int_j^\infty \varphi(j') dj' \tag{20}$$

denotes the fraction of sources (at  $t = \infty$ ) with item-density  $j$  or more (cf. (2) divided by  $T$ , hence using (3)), we have, by (19) and (20) that

$$G_2(t) \int_j^\infty \varphi(j') dj' \tag{21}$$

denotes the fraction of sources (with respect to  $t = \infty$ ) with  $k$  or more as item-density, where

$$jG_1(t) = k \tag{22}$$

by (16). Let  $\Phi(k, t)$  denote this fraction of sources. Hence

$$\begin{aligned} \Phi(k, t) &= G_2(t) \int_{\frac{k}{G_1(t)}}^\infty \varphi(j') dj' \\ \Phi(k, t) &= G_2(t) \int_{\frac{k}{G_1(t)}}^\infty \frac{\alpha - 1}{j'^{\alpha}} dj' \\ \Phi(k, t) &= G_2(t) \left( \frac{G_1(t)}{k} \right)^{\alpha-1} \end{aligned} \tag{23}$$

since  $\alpha > 1$ . Since

$$\Phi(k, t) = \int_k^\infty \varphi(k', t) dk' \tag{24}$$

by definition of the time-dependent Lotka function  $\varphi(k, t)$ , we clearly have

$$\varphi(k, t) = -\Phi'(k, t), \tag{25}$$

where  $\Phi'(k, t)$  denotes  $\frac{\partial \Phi(k, t)}{\partial k}$

Consequently, by (23)

$$\varphi(k, t) = G_2(t) (\alpha - 1) \frac{G_1(t)^{\alpha-1}}{k^\alpha} \tag{26}$$

$$\varphi(k, t) = G_2(t) G_1(t)^{\alpha-1} \varphi(k), \tag{27}$$

where  $k \geq G_1(t)$  and by (15). Note that, here, we extend (15) to  $k \geq G_1(t)$  (by (26)).

Note that  $\varphi(k, t)$  is the density of source fractions (with respect to  $t = \infty$ ) (cf. (20) and (21)) with item-density  $k$ . Hence, denoting by  $T$  the total number of sources at  $t = \infty$ ,

$$f(k, t) =: T\varphi(k, t) \tag{28}$$

$$f(k, t) = G_2(t) G_1(t)^{\alpha-1} T\varphi(k) \tag{29}$$

$$f(k, t) = G_2(t) G_1(t)^{\alpha-1} f(k) \tag{30}$$

denotes the actual density of sources (with respect to  $t = \infty$ ) with item density  $k$  at time  $t$  and where  $f(k) = T\varphi(k)$  denotes the same at  $t = \infty$ . Note that also here (15) is extended to  $k \geq G_1(t)$ .

2.3. Total number of sources and items at time  $t$

The above function  $f(k, t)$  already allows one us calculate

$T(t)$  = the total number of sources at time  $t$

$A(t)$  = the total number of items at time  $t$ .

We have, by the definition of  $f(k, t)$  (cf. (29)) and since  $k \geq G_1(t)$ :

$$\begin{aligned} T(t) &= \int_{G_1(t)}^{\infty} f(k, t) dk \\ T(t) &= G_2(t) G_1(t)^{\alpha-1} T \int_{G_1(t)}^{\infty} \varphi(k) dk \\ T(t) &= G_2(t) G_1(t)^{\alpha-1} T G_1(t)^{1-\alpha} \\ T(t) &= G_2(t) T \end{aligned} \tag{31}$$

a logical result. In the same way, for  $\alpha > 2$ , we have (cf. (29))

$$\begin{aligned} A(t) &= \int_{G_1(t)}^{\infty} kf(k, t) dk \\ &= TG_2(t) G_1(t)^{\alpha-1} \int_{G_1(t)}^{\infty} k\varphi(k) dk \\ &= TG_2(t) G_1(t)^{\alpha-1} \frac{\alpha - 1}{\alpha - 2} G_1(t)^{2-\alpha}. \end{aligned} \tag{32}$$

But, as is readily seen, on  $t = \infty$  (cf. [4, Chapter 2]), since  $\alpha > 2$

$$\begin{aligned} \mu = \frac{A}{T} &= \frac{\int_1^{\infty} k\varphi(k) dk}{\int_1^{\infty} \varphi(k) dk} \\ &= \frac{\alpha - 1}{\alpha - 2}. \end{aligned} \tag{33}$$

( $T$  = the total number of sources at  $t = \infty$ ,  $A$  = the total number of items at  $t = \infty$ ).

Hence, (33) in (32) yields

$$A(t) = AG_1(t) G_2(t),$$

also a logical result since items at  $t$  are determined by the items in the already existing sources; the former one is determined by  $G_1$  and the latter one by  $G_2$ .

Since rank-frequency functions are equivalent with size-frequency functions (by the general relation (6), but now  $t$ -dependent) we are now in a position to calculate the time-dependent rank-frequency function  $\gamma(s, t)$ .

2.4. The rank-frequency function, in function of time  $t$

Since, at  $t$ , the rank-densities are denoted by  $s$  via (17) and by (6), interpreted at time  $t$ , we have (the inverse refers to the variable  $k$ )

$$\begin{aligned} s = s(t) &= \gamma^{-1}(k, t) = \int_k^{\infty} f(k', t) dk' \\ &= TG_2(t) G_1(t)^{\alpha-1} \int_k^{\infty} \varphi(k') dk', \end{aligned} \tag{34}$$

by (29). Hence, using (15)

$$s = s(t) = \frac{C}{\alpha - 1} \frac{G_2(t) G_1(t)^{\alpha-1}}{k^{\alpha-1}}$$

denoting  $T(\alpha - 1) = C$  (hence  $f(k) = \frac{C}{k^\alpha}$ ). This yields

$$k = \gamma(s, t) = \frac{C^{\frac{1}{\alpha-1}}}{((\alpha - 1)s)^{\frac{1}{\alpha-1}}} G_2(t)^{\frac{1}{\alpha-1}} G_1(t).$$

Invoking (17) yields

$$k = \gamma(s, t) = \frac{C^{\frac{1}{\alpha-1}}}{((\alpha - 1)r)^{\frac{1}{\alpha-1}}} G_1(t)$$

$$\gamma(s, t) = \gamma(r) G_1(t) \tag{35}$$

with  $r$  and  $s$  related as in (17). Note that (35) follows from the fact that  $f(k) = \frac{C}{k^\alpha}$  and by applying the same calculation as performed here (but for  $t = \infty$ , hence  $G_1(\infty) = G_2(\infty) = 1$ ). In fact the result can also be found in [4, Exercise II.2.2.6 (p. 134)] and its proof is also given in [10, Appendix].

In the next section, we will apply these results in the calculation of the general forms of the time-dependent  $h$ - and  $g$ -indices.

### 3. Formulae for the time-dependent $h$ - and $g$ -indices and properties of these functions of time

#### 3.1. The time-dependent $h$ -index

The  $h$ -index in this model is defined to be  $h(t)$  such that

$$\int_{h(t)}^\infty f(k, t) dk = h(t). \tag{36}$$

Hence by (29) and (15) we have

$$TG_2(t) G_1(t)^{\alpha-1} \int_{h(t)}^\infty \frac{\alpha - 1}{k^\alpha} dk = h(t).$$

So

$$h = h(t) = (TG_2(t))^{\frac{1}{\alpha}} G_1(t)^{\frac{\alpha-1}{\alpha}}. \tag{37}$$

Invoking that, for  $t = \infty$

$$h = h(\infty) = T^{\frac{1}{\alpha}} \tag{38}$$

(see [12,10]). So (37) and (38) yield

$$h(t) = G_1(t)^{1-\frac{1}{\alpha}} G_2(t)^{\frac{1}{\alpha}} h. \tag{39}$$

Note that the time-dependent  $h$ -index is proportional to the generalized geometric mean of the growth functions  $G_1(t)$  and  $G_2(t)$ . In the following we will study properties of this generalized geometric mean.

#### 3.2. The time-dependent $g$ -index

There are two, equivalent, ways to define the  $g$ -index (at time  $t$ ), denoted  $g(t)$ : if

$$\int_k^\infty k' f(k', t) dk' = g(t)^2 \tag{40}$$

then

$$g(t) = \gamma^{-1}(k, t). \tag{41}$$

Equivalently,  $g(t)$  is defined as

$$\int_0^{g(t)} \gamma(r, t) dr = g(t)^2. \tag{42}$$

The latter approach has been followed in [5] (but only using the growth function  $G_1$ —there denoted  $G$ ). Here we will follow the former approach. Formula (40) yields, using (15) and (19) (and the notation  $C = T(\alpha - 1)$ , already used above):

$$\begin{aligned} G_2(t) G_1(t)^{\alpha-1} \int_k^\infty \frac{C}{k'^{\alpha-1}} dk' &= g(t)^2 \\ G_2(t) G_1(t)^{\alpha-1} \frac{C}{\alpha - 2} k^{2-\alpha} &= g(t)^2, \end{aligned} \tag{43}$$

supposing  $\alpha > 2$ . But (34) yields (using (15) again)

$$\begin{aligned}\gamma^{-1}(k, t) &= G_2(t) G_1(t)^{\alpha-1} \int_k^{\infty} \frac{C}{k'^{\alpha}} dk' \\ g(t) = \gamma^{-1}(k, t) &= G_2(t) G_1(t)^{\alpha-1} \frac{C}{\alpha-1} k^{1-\alpha}\end{aligned}\quad (44)$$

by (41). So, solving (44) for  $k$ , yields

$$k = \left( \frac{\alpha-1}{C} \frac{g(t)}{G_2(t)} \right)^{\frac{1}{1-\alpha}} G_1(t) \quad (45)$$

(45) in (43) yields, after some calculation and using that  $T = \frac{C}{\alpha-1}$ :

$$g(t) = G_1(t)^{1-\frac{1}{\alpha}} G_2(t)^{\frac{1}{\alpha}} \left( \frac{\alpha-1}{\alpha-2} \right)^{\frac{\alpha-1}{\alpha}} T^{\frac{1}{\alpha}}. \quad (46)$$

Invoking that, for  $t = \infty$ ,

$$g = g(\infty) = \left( \frac{\alpha-1}{\alpha-2} \right)^{\frac{\alpha-1}{\alpha}} T^{\frac{1}{\alpha}} \quad (47)$$

(see [7]). So (47) in (46) yields

$$g(t) = G_1(t)^{1-\frac{1}{\alpha}} G_2(t)^{\frac{1}{\alpha}} g. \quad (48)$$

Hence  $g(t)$  has the same proportionality factor as  $h(t)$  (see (39)). We will now study this proportionality factor which determines the evolution of  $h(t)$  and  $g(t)$  in time.

### 3.3. Properties of $G_1(t)^{1-\frac{1}{\alpha}} G_2(t)^{\frac{1}{\alpha}}$

Since  $G_1(t)$  and  $G_2(t)$  are cumulative growth distributions we have that

$$\begin{aligned}G_1(0) &= G_2(0) = 0 \\ \lim_{t \rightarrow \infty} G_1(t) &= \lim_{t \rightarrow \infty} G_2(t) = 1\end{aligned}$$

and that  $G_1$  and  $G_2$  are strictly increasing in  $t$  and this increase is concave (example: (18) but we do not need this special case here).

So it is then trivial that  $h(t)$  strictly increases from 0 to  $h$  (its maximal value, being  $h(\infty)$ ) and that  $g(t)$  strictly increases from 0 to  $g$  ( $\geq h$ ).

The lemma in the Appendix shows that the generalized geometric mean of strictly increasing concave functions is strictly increasing and concave so that also  $h(t)$  and  $g(t)$  are strictly increasing and concave. Of course, since the  $g$ -index is larger than or equal to the  $h$ -index we also have, for every  $t$ :

$$g(t) \geq h(t). \quad (49)$$

## 4. Conclusions and open problems

In this paper we studied the aspect of time-dependence in the evolution of a Lotkaian system. Source and item densities are growing in time using two general cumulative growth distributions. Using this general framework yields formulae for the time-dependent size- and rank-frequency functions in the function of the final situation at  $t = \infty$ , where we suppose we have a Lotkaian system. We show that at every time  $t$ , we have a Lotkaian system with the same Lotka exponent, but where the growth is dependent on – in addition to the two growth distributions – this Lotka exponent.

Similar results are proved for the time-dependent  $h$ - and  $g$ -indices. Here we prove that both indices are proportional to their values at  $t = \infty$ , and where the time-dependent proportionality factor is a generalized geometric mean of the two growth distributions. Also here the Lotka exponent is involved. We prove in a lemma in the Appendix that a general geometric mean of strictly increasing concave functions is strictly increasing and concave. Hence the same is true for the time-dependent  $h$ - and  $g$ -indices.

We formulate as an open problem the definition of other growth processes of sources and items, where the growth functions are acting on the source- and item-densities in a nonlinear way and the derivation from this framework of the time-dependent size- and rank-frequency functions and the time-dependent  $h$ - and  $g$ -indices.

## Appendix

Let  $x$  and  $y$  be two positive real numbers. We say that  $z$  is a generalized geometric average of  $x$  and  $y$  if there exists a  $\lambda \in ]0, 1[$  such that

$$z = x^\lambda y^{1-\lambda}. \quad (\text{A.1})$$

The classical geometric mean is given when  $\lambda = \frac{1}{2}$ : then  $z = \sqrt{xy}$ .

In Section 3 we found that

$$h(t) = G_1(t)^{1-\frac{1}{\alpha}} G_2(t)^{\frac{1}{\alpha}} h \quad (\text{A.2})$$

$$g(t) = G_1(t)^{1-\frac{1}{\alpha}} G_2(t)^{\frac{1}{\alpha}} g \quad (\text{A.3})$$

hence  $h(t)$  and  $g(t)$  are equal to their values at  $t = \infty$  multiplied by a generalized geometric mean of the growth functions  $G_1(t)$  and  $G_2(t)$  (note, indeed, that, since  $\alpha > 1$  we have that  $\lambda = 1 - \frac{1}{\alpha} \in ]0, 1[$ ).

We have the following general lemma.

**Lemma.** Let  $G_1, G_2$  be two functions of the variable  $t$  which are strictly increasing and concave. Then their generalized geometric average (for  $\alpha > 1$ )

$$G(t) =: G_1(t)^{1-\frac{1}{\alpha}} G_2(t)^{\frac{1}{\alpha}} \quad (\text{A.4})$$

is strictly increasing and concave.

**Proof.** That  $G$  is strictly increasing in  $t$  is trivial. For the concavity we need to calculate  $G''$ . We have

$$G'(t) = \frac{\alpha-1}{\alpha} \left(\frac{G_1(t)}{G_2(t)}\right)^{-\frac{1}{\alpha}} G_1'(t) + \frac{1}{\alpha} \left(\frac{G_1(t)}{G_2(t)}\right)^{\frac{\alpha-1}{\alpha}} G_2'(t) \quad (\text{A.5})$$

$$\begin{aligned} G''(t) &= \frac{\alpha-1}{\alpha} \left(-\frac{1}{\alpha}\right) \left(\frac{G_1(t)}{G_2(t)}\right)^{-\frac{1}{\alpha}-1} \frac{G_2(t)G_1'(t) - G_1(t)G_2'(t)}{G_2^2(t)} \cdot G_1'(t) + \frac{\alpha-1}{\alpha} \left(\frac{G_1(t)}{G_2(t)}\right)^{-\frac{1}{\alpha}} G_1''(t) \\ &\quad + \frac{1}{\alpha} \frac{\alpha-1}{\alpha} \left(\frac{G_1(t)}{G_2(t)}\right)^{\frac{\alpha-1}{\alpha}-1} \frac{G_2(t)G_1'(t) - G_1(t)G_2'(t)}{G_2^2(t)} \cdot G_2'(t) + \frac{1}{\alpha} \left(\frac{G_1(t)}{G_2(t)}\right)^{\frac{\alpha-1}{\alpha}} G_2''(t). \end{aligned}$$

The second and fourth terms are clearly strictly negative since  $G_1$  and  $G_2$  are concave. The first and third term, together, equal

$$\begin{aligned} &\frac{\alpha-1}{\alpha^2} \left(\frac{G_1(t)}{G_2(t)}\right)^{-\frac{1}{\alpha}} \frac{G_2(t)G_1'(t) - G_1(t)G_2'(t)}{G_2^2(t)} \left[-\frac{G_2(t)}{G_1(t)}G_1'(t) + G_2'(t)\right] \\ &= \frac{\alpha-1}{\alpha^2} \left(\frac{G_1(t)}{G_2(t)}\right)^{-\frac{1}{\alpha}} \frac{1}{G_1(t)G_2^2(t)} \left[-(G_2(t)G_1'(t) - G_1(t)G_2'(t))^2\right] \\ &< 0, \quad \text{since } \alpha > 1. \end{aligned}$$

So  $G''(t) < 0$  for all  $t$  and hence  $G(t)$  is concave.  $\square$

## References

- [1] P. Ball, Index aims for fair ranking of scientists, *Nature* 436 (2005) 900.
- [2] L. Bornmann, H.-D. Daniel, Does the  $h$ -index for ranking of scientists really work? *Scientometrics* 65 (2005) 391–392.
- [3] T. Braun, W. Glänzel, A. Schubert, A Hirsch-type index for journals, *The Scientist* 19 (22) (2005) 8.
- [4] L. Egghe, *Power Laws in the Informetric Production Process: Lotkaian Informetrics*, Elsevier, Oxford, UK, 2005.
- [5] L. Egghe, Item-time-dependent Lotkaian informetrics and applications to the calculation of the time-dependent  $h$ - and  $g$ -index, *Mathematical and Computer Modelling* 45 (7–8) (2007) 864–872.
- [6] L. Egghe, Letter: How to improve the  $h$ -index, *The Scientist* 20 (3) (2006).
- [7] L. Egghe, Theory and practise of the  $g$ -index, *Scientometrics* 69 (1) (2006) 131–152.
- [8] L. Egghe, Dynamic  $h$ -index: The Hirsch index in function of time, *Journal of the American Society for Information Science and Technology* 58 (3) (2007) 452–454.
- [9] L. Egghe, An improvement of the  $h$ -index: The  $g$ -index, *ISSI Newsletter* 2 (1) (2006) 8–9.
- [10] L. Egghe, R. Rousseau, An informetric model for the Hirsch index, *Scientometrics* 69 (1) (2006) 121–129.
- [11] W. Glänzel, On the opportunities and limitations of the  $H$ -index, *Science Focus* 1 (1) (2006) 10–11.
- [12] W. Glänzel, On the  $H$ -index—a mathematical approach to a new measure of publication activity and citation impact, *Scientometrics* 67 (2) (2006) 315–321.
- [13] J.E. Hirsch, An index to quantify an individual's scientific research output, *Proceedings of the National Academy of Sciences of the United States of America* 102 (2005) 16569–16572.
- [14] A.F.J. van Raan, Comparison of the Hirsch-index with standard bibliometric indicators and with peer judgement for 147 chemistry research groups, *Scientometrics* 67 (1) (2006) 491–502.