

Extending irreducible braid representations to the 3-component loop braid group

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(Received September 8, 2015, Accepted February 1, 2016)

Abstract

In a recent paper [3] it is shown that irreducible representations of the three string braid group B_3 of dimension ≤ 5 extend to representations of the three component loop braid group LB_3 . Further, an explicit 6-dimensional irreducible B_3 -representation is given not allowing such an extension.

In this note we give a necessary and sufficient condition, in all dimensions, on the components of irreducible representations of the modular group Γ such that sufficiently general representations extend to $\Gamma *_{C_3} S_3$. As a consequence, the corresponding irreducible B_3 -representations do extend to LB_3 .

1 The strategy

The 3-component loop braid group LB_3 encodes motions of 3 oriented circles in \mathbb{R}^3 . The generator σ_i ($i = 1, 2$) is interpreted as passing the i -th circle under and through the $i + 1$ -th circle ending with the two circles' positions interchanged. The generator s_i ($i = 1, 2$) simply interchanges the circles i and $i + 1$. For physical motivation and graphics we refer to the paper by

Key words and phrases: Loop braid group, quiver representations, TQFT.
AMS (MOS) Subject Classifications: 16G20, 20F36, 81T45, 68Q12.
ISSN 1814-0432, 2016, <http://ijmcs.future-in-tech.net>

John Baez, Derek Wise and Alissa Crans [2]. The defining relations of LB_3 are:

1. $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$
2. $s_1s_2s_1 = s_2s_1s_2$
3. $s_1^2 = s_2^2 = 1$
4. $s_1s_2\sigma_1 = \sigma_2s_1s_2$
5. $\sigma_1\sigma_2s_1 = s_2\sigma_1\sigma_2$

Note that (1) is the defining relation for the 3-string braid group B_3 , (2) and (3) define the symmetric group S_3 , therefore the first three relations describe the free group product $B_3 * S_3$.

Recall that the modular group $\Gamma = C_2 * C_3 = \langle s, t | s^2 = 1 = t^3 \rangle$ is a quotient of B_3 by dividing out the central element $c = (\sigma_1\sigma_2)^3$, so that we can take $t = \bar{\sigma}_1\bar{\sigma}_2$ and $s = \bar{\sigma}_1\bar{\sigma}_2\bar{\sigma}_1$. Hence, any irreducible n -dimensional representation $\phi : B_3 \rightarrow GL_n$ will be isomorphic to one of the form

$$\phi(\sigma_1) = \lambda\psi(\bar{\sigma}_1), \quad \text{and} \quad \phi(\sigma_2) = \lambda\psi(\bar{\sigma}_2)$$

for some $\lambda \in \mathbb{C}^*$ and $\psi : \Gamma \rightarrow GL_n$ an n -dimensional irreducible representation of $\Gamma = \langle s, t \rangle = \langle \bar{\sigma}_1, \bar{\sigma}_2 \rangle$. With $S_3 = \langle s_1, s_2 | s_1s_2s_1 = s_2s_1s_2, s_1^2 = 1 = s_2^2 \rangle$, we consider the amalgamated free product

$$G = \Gamma *_{C_3} S_3$$

in which the generator of C_3 is equal to $t = \bar{\sigma}_1\bar{\sigma}_2$ in Γ and to s_1s_2 in S_3 .

We will impose conditions on ψ such that it extends to a (necessarily irreducible) representations of G . Then, if this is possible, as $\psi(\bar{\sigma}_1\bar{\sigma}_2) = \psi(s_1s_2)$ and as the defining equations (1),(4) and (5) of LB_3 are homogeneous in the σ_i it will follow that

$$\phi(\sigma_i) = \lambda\psi(\bar{\sigma}_i), \quad \text{and} \quad \phi(s_i) = \psi(s_i)$$

is a representation of LB_3 extending the irreducible representation ϕ of B_3 .

2 The result

Bruce Westbury has shown in [7] that the variety $\mathbf{iss}_n \Gamma$ classifying isomorphism classes of n -dimensional semi-simple Γ -representations decomposes as a disjoint union of irreducible components

$$\mathbf{iss}_n \Gamma = \bigsqcup_{\alpha} \mathbf{iss}_{\alpha} \Gamma$$

where $\alpha = (a, b; x, y, z) \in \mathbb{N}^{\oplus 5}$ satisfying $a + b = n = x + y + z$. Moreover, if $xyz \neq 0$ then the component $\mathbf{iss}_{\alpha} \Gamma$ contains a Zariski open and dense subset of irreducible Γ -representations if and only if $\max(x, y, z) \leq \min(a, b)$. In this case, the dimension of $\mathbf{iss}_{\alpha} \Gamma$ is equal to $1 + n^2 - (a^2 + b^2 + x^2 + y^2 + z^2)$. In going from irreducible Γ -representations to irreducible B_3 -representations we multiply by $\lambda \in \mathbb{C}^*$. As a result, it is shown in [7] that there is a μ_6 -action on the components $\mathbf{iss}_{\alpha} \Gamma$ leading to the same component of B_3 -representations. That is, the variety $\mathbf{irr}_n B_3$ classifying isomorphism classes of irreducible n -dimensional B_3 -representations decomposes into irreducible components

$$\mathbf{irr}_n B_3 = \bigcup_{\alpha} \mathbf{irr}_{\alpha} B_3$$

where $\alpha = (a, b; x, y, z)$ satisfies $a + b = n = x + y + z$, $a \geq b \geq x = \max(x, y, z)$.

Theorem 2.1. *A Zariski open and dense subset of irreducible Γ -representations in $\mathbf{iss}_{\alpha} \Gamma$ extends to the group $G = \Gamma *_{C_3} S_3$ if and only if there are natural numbers u, v, w with $w \geq \max(u, v)$ such that*

$$\alpha = (v + w, u + w; u + v, w, w)$$

As a consequence, a Zariski open and dense subset of irreducible B_3 -representations in $\mathbf{irr}_{\alpha} B_3$ extends to the three-component loop braid group LB_3 if there are natural numbers $u \leq v \leq w$ such that $\alpha = (a, b; x, y, z)$ with $x = \max(x, y, z)$ and

$$a = v + w, b = u + w, \{x, y, z\} = \{u + v, w, w\}$$

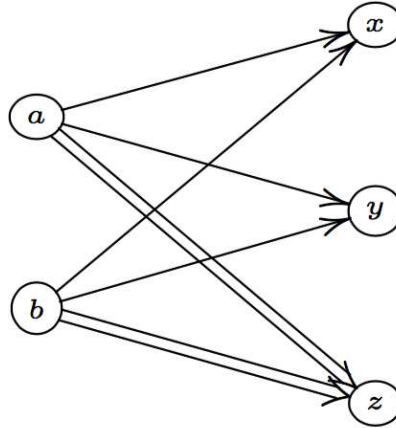
Observe that the first dimension n allowing an admissible 5-tuple not satisfying this condition is $n = 6$ with $\alpha = (3, 3; 3, 2, 1)$.

3 The proof

If V is an n -dimensional $G = \Gamma *_{C_3} S_3 \simeq C_2 * S_3$ -representation, then by restricting it to the subgroups C_2 and S_3 we get decomposition of V into

$$S_+^{\oplus a} \oplus S_-^{\oplus b} = V \downarrow_{C_2} = V = V \downarrow_{S_3} = T^{\oplus x} \oplus S^{\oplus y} \oplus P^{\oplus z}$$

where $\{S_+, S_-\}$ are the 1-dimensional irreducibles of C_2 , T is the trivial S_3 -representation, S the sign representation and P the 2-dimensional irreducible S_3 -representation. Clearly we must have $a + b = n = x + y + 2z$ and once we choose bases in each of these irreducibles we have that V itself determines a representation of the following quiver setting



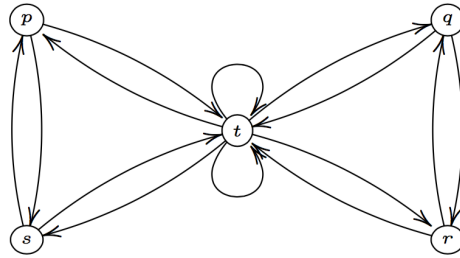
where the arrows give the block-decomposition of the base-change matrix B from the chosen basis of $V \downarrow_{C_2}$ to the chosen basis of $V \downarrow_{S_3}$. Isomorphism classes of irreducible G -representations correspond to isomorphism classes of θ -stable quiver representation of dimension vector $\beta = (a, b; x, y, z)$ for the stability structure $\theta = (-1, -1; 1, 1, 2)$. The minimal dimension vectors of θ -stable representations are

$$\left\{ \begin{array}{l} \alpha_1 = (1, 0; 1, 0, 0) \\ \alpha_2 = (1, 0; 0, 1, 0) \\ \alpha_3 = (0, 1; 1, 0, 0) \\ \alpha_4 = (0, 1; 0, 1, 0) \\ \alpha_5 = (1, 1; 0, 0, 1) \end{array} \right.$$

which give us unique 1-dimensional G -representations S_1, S_2, S_3, S_4 and a 2-parameter family of 2-dimensional irreducible G -representations from which we choose S_5 . By the results of [1], the local structure of the component $\text{iss}_\beta G$ for $\beta = (p + q + t, r + s + t; p + r, q + s, t)$ in a neighborhood of the semi-simple G -representation

$$M = S_1^{\oplus p} \oplus S_2^{\oplus q} \oplus S_3^{\oplus r} \oplus S_4^{\oplus s} \oplus S_5^{\oplus t}$$

is étale equivalent to the local structure of the quiver-quotient variety of the setting below at the zero-representation

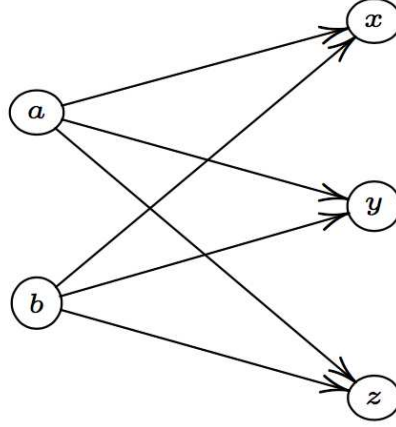


Hence, $\text{iss}_\beta G$ will contain a Zariski open and dense subset of irreducible representations if and only if $\gamma = (p, q, r, s, t)$ is a simple dimension vector for this quiver, which by [5] is equivalent to γ being either $(1, 0, 0, 1, 0)$ or $(0, 1, 1, 0, 0)$ or satisfying the inequalities

$$p \leq s + t, \quad q \leq r + t, \quad r \leq q + t, \quad s \leq p + t$$

Having determined the components containing irreducible G -representations, we have to determine those containing a Zariski open subset which remain irreducible when restricted to Γ .

As $\Gamma = C_2 * C_3$ any Γ -representation V corresponds to a semi-stable quiver representation for the setting



when

$$V \downarrow_{C_2} = S_+^{\oplus a} \oplus S_-^{\oplus b} \quad \text{and} \quad V \downarrow_{C_3} = T_1^{\oplus x} \oplus T_\rho^{\oplus y} \oplus T_{\rho^2}^{\oplus z}$$

with $\{T_1, T_\rho, T_{\rho^2}\}$ the irreducible C_3 -representations. Because $T \downarrow_{C_3} = T_1 = S \downarrow_{C_3}$ and $P \downarrow_{C_3} = T_\rho \oplus T_{\rho^2}$ we have that $M \downarrow_\Gamma$ has dimension vector

$$\alpha = (a, b; x, y, z) = (p + q + t, r + s + t; p + q + r + s, t, t)$$

which satisfies the condition that $\max(x, y, z) \leq \min(a, b)$ if and only if $t \geq r + s$ and $t \geq p + q$. Setting $u = r + s$, $v = p + q$ and $w = t$, the statement of Theorem 1 follows.

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