The influence of a stochastic interest rate on the n-fold compound option

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Abstract

We reintroduced the idea of an n-fold compound option as a generalization of Geske's (2-fold) compound option in the same framework of constant interest rates. For the valuation of long-term financial agreements (life insurance products) this assumption is not always realistic. So that the stochastic modelling of the interest rates might be a better approach.

According to Miltersen et al., we will use the requirement of simple interest rates over a fixed finite period to be log-normal distributed, instead of the continuously compounded interest rates. With these assumptions, closed-form solutions are determined for the n-fold compound call options written on zero-coupon bonds.

keywords: financial, n-fold compound options, log-normal interest rates.

1 Introduction

In constructing an n-fold compound option model, Thomassen and Van Wouwe [5] used the same economic assumptions as Geske [2] did when introducing his (2-fold) compound option. One of the economic hypothesis is the assumption of a deterministic risk-free interest rate both for the original compound option and the generalized compound option.

If however one looks at possible application fields of the n-fold compound option, it appears that longterm commitments are involved. Indeed, the introduction of for instance new life insurance contracts are longterm processes. So one can start questioning the validity of a constant interest rate. The life insurance practice shows that the currently used approaches to represent interest rates in actuarial models may be broadly categorized as deterministic and stochastic.

The most familiar deterministic approach is still a single interest rate with a slight generalization to a series of interest rates for future years. In recent years one also sees the development of a tremendous profusion of models for valuation of interest rate sensitive financial instruments. Wall street dealers routinely use a multiplicity of models based on widely varying assumptions in different markets. For example, an options desk most likely uses a version of the Black formula to value interest rate caps and floors, implying an approximately log-normal distribution of the interest rates.

A critical issue in selecting an interest rate model is ease of application. It is indeed difficult or impossible to provide efficient valuation algorithms for some models.

For that reason we focused on the literature to find closed form solutions for term structure derivatives with log-normal interest rates. We were inspired by the reasoning developed by Miltersen, Sandmann and Sondermann [3].

They derived a unified model that gives closed form solutions for caps and floors written on interest rates as well as for puts and calls written on zero-coupon bonds. Their crucial assumption is that the simple interest rates over a fixed finite period are log-normal distributed, an assumption that is shown to be consistent with the Heath-Jarrow-Morton model for a specific choice of volatility.

As they pointed out, log-normal distributed interest rate models avoid the problem of negative interest rates but the rates explode with positive probability, implying zero prices for bonds and hence allowing arbitrage opportunities.

The problem of exploding interest rates disappears as shown in Sandmann and Sondermann [4] if instead of assuming the continuously compounded interest rates to be log-normal distributed, one assumes the simple interest rates over a fixed finite period to be log-normal distributed.

The main result of their article is a unified model that provides closed form solutions for interest rate caps and floors as well as for puts and calls written on zero-coupon bonds within the context of a log-normal interest rate model. The closed form valuation is a Black-Scholes alike formula which we could generalize for n-fold compound calls written on zero-coupon bonds in the same framework as by Miltersen et al. [3].

2 A model for simple forward rates over a fixed period

Let $f(t, T, \alpha)$ denote the simple forward rate at time t over a fixed period of length α prevailing at date t for the future time interval $[T, T + \alpha]$. According to Miltersen, Sandmann and Sondermann [3] the simple forward rates will be modelled as log-normal diffusions:

$$df(\cdot, T, \alpha)_t = \mu(t, T, \alpha) \cdot f(t, T, \alpha) dt + \gamma(t, T, \alpha) \cdot f(t, T, \alpha) dW_t \quad (1)$$

Let $P(t, T)$ denote the price at time t of a default-free zero-coupon bond that pays 1 at maturity date T . It then follows that

$$P(t, T + \alpha) = P(t, T) \frac{1}{1 + \alpha f(t, T, \alpha)} \quad (2)$$

If the forward price $F(t, T, \alpha)$ of the contract is defined as the fixed price which the buyer agrees to pay at date T for the bond with maturity $T + \alpha$, such that the value of the forward contract at date t is zero, by no arbitrage assumptions the forward price is given by

$$F(t, T, \alpha) = \frac{P(t, T + \alpha)}{P(t, T)}. \quad (3)$$

3 A closed-form solution for the n-fold compound option

We reintroduce the notion of an n-fold compound option:

Definition 3.1 *By induction:*

A compound call option (of order 2) is a call option on a call option. This can be generalized to a compound call of order n (with exercise date and price given by t_1 and K_1) with an underlying call of order $n - 1$.

Notations:

- V : current market value of the firm,
- t_i : maturity date of investment for the compound call option C_i ,
- K_i : exercise price for the compound call option C_i ,
- C_i : current value of the compound call option on the option C_{i+1} ,
- r : risk-free rate of interest,
- σ^2 : instantaneous variance of the return on the assets of the firm,
- $N_n(a_1, a_2, \dots, a_n; F)$: n-variate cumulative normal distribution function with a_i as upper limits, $-\infty$ as lower limits and F as the correlation matrix.

Theorem 3.1 Suppose for $s = n, n-1, \dots, 2$ the calls C_s of order $n+1-s$ are known and given by:

$$C_s = V N_{n+1-s}(a_s, a_{s+1}, \dots, a_n; F_s^{n+1-s}) - \sum_{m=s}^n K_m e^{-r(t_m-t)} N_{m+1-s}(b_s, b_{s+1}, \dots, b_m; F_s^{m+1-s}), \quad (4)$$

with the notations

$$a_\ell = b_\ell + \sigma \sqrt{t_\ell - t} \quad \ell = 1, 2, \dots, n \quad (5)$$

$$b_\ell = \frac{\ln \frac{V}{\bar{V}_\ell} + (r - \frac{\sigma^2}{2})(t_\ell - t)}{\sigma \sqrt{t_\ell - t}} \quad \ell = 1, 2, \dots, n \quad (6)$$

$$\bar{V}_\ell = \text{solution of the equation} \quad (7)$$

$$C_{\ell+1}(V, t_\ell) = K_\ell \quad \ell = 1, 2, \dots, n-1$$

$$\bar{V}_n = K_n \quad (8)$$

$$\rho_{ij} = \sqrt{\frac{t_i - t}{t_j - t}} \quad i < j \quad (9)$$

$$F_s^\ell = (f_{ij}^\ell)_{i,j=1,2,\dots,\ell} \quad \text{where} \quad \begin{cases} f_{ii} = 1 \\ f_{ij} = \rho_{i+s-1, j+s-1} \quad i < j \end{cases} \quad (10)$$

Then the n -fold compound option can be found:

$$C_1 = V N_n(a_1, a_2, \dots, a_n; F_1^n) - \sum_{m=1}^n K_m e^{-r(t_m-t)} N_m(b_1, b_2, \dots, b_m; F_1^m). \quad (11)$$

For a proof the reader is referred to the article by Thomassen and Van Wouwe [5].

We now make a particular choice for the underlying by taking a forward F instead of the current market value of the firm V . This implies that we will use the forward price $\hat{C}$ for a call option C , defined by

$$\hat{C}(t, F(t, T, \alpha)) \equiv \frac{C}{P(t, T)} \quad (12)$$

With the adapted definition for an n -fold compound option we are in the possibility to value the forward price of an n -fold compound call option. The valuation is proven in the following theorem by induction:

Theorem 3.2 *The following notations are introduced:*

$$\begin{aligned}
\alpha_i &= t_{i+1} - t_i & \forall i = 1, \dots, n \\
s_i &= \int_t^{t_i} \gamma^2(r, t_n, \alpha_n) dr & \forall i = 1, \dots, n \\
F_i &= \text{is solution of } P(t_i, t_{i+1}) \cdot \hat{C}_{i+1}(t_i, F) = K_i & \forall i = 1, \dots, n-1 \\
F_n &= K_n \\
a_i &= b_i + \sqrt{s_i} & \forall i = 1, \dots, n \\
b_i &= \frac{1}{\sqrt{s_i}} \left(\ln \left| \frac{F(1 - F_i)}{F_i(1 - F)} \right| - \frac{s_i}{2} \right) & \forall i = 1, \dots, n
\end{aligned}$$

where F is used as the abbreviated notation for the forward rate $F(t, t_n, \alpha_n)$. Suppose $\forall v = n, n-1, \dots, 3, 2$ the value of the $(n-v+1)$ -fold is known and given by

$$\begin{aligned}
\hat{C}_v &= \prod_{i=v}^{n-1} P(t_i, t_{i+1}) \cdot F \cdot N_{n-v+1}(a_v, a_{v+1}, \dots, a_n; G_n^v) \\
&\quad - \sum_{i=v}^n K_i \cdot F \cdot N_{i-v+1}(a_v, a_{v+1}, \dots, a_i; G_i^v) \cdot \prod_{j=v}^{i-1} P(t_j, t_{j+1}) \\
&\quad - \sum_{i=v}^n K_i \cdot (1 - F) \cdot N_{i-v+1}(b_v, b_{v+1}, \dots, b_i; G_i^v) \cdot \prod_{j=v}^{i-1} P(t_j, t_{j+1}) \quad (13)
\end{aligned}$$

then the value of the n -fold is determined by the following closed-form solution:

$$\begin{aligned}
\hat{C}_1 &= \prod_{i=1}^{n-1} P(t_i, t_{i+1}) \cdot F \cdot N_n(a_1, a_2, \dots, a_n; G_n^1) \\
&\quad - \sum_{i=1}^n K_i \cdot F \cdot N_i(a_1, a_2, \dots, a_i; G_i^1) \cdot \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \\
&\quad - \sum_{i=1}^n K_i \cdot (1 - F) \cdot N_i(b_1, b_2, \dots, b_i; G_i^1) \cdot \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \quad (14)
\end{aligned}$$

where the correlation matrices are defined by the following structure

$$G_y^x = (g_{ab})_{a,b=1,2,\dots,y-x+1} \quad \text{with} \quad \begin{cases} g_{aa} = 1 \\ g_{ab} = g_{ba} = \sqrt{\frac{s_{a+x-1}}{s_{b+x-1}}} \quad \text{if } a < b \end{cases}, \quad (15)$$

(where $\prod$ equals 1 if the upper limit is smaller than the lower limit and where $\sum$ equals 0 if the upper limit is smaller than the lower limit.)

Proof

n = 1

This result was obtained by Miltersen, Sandmann and Sondermann [3].

n > 1

Suppose $\forall v = n, n-1, \dots, 3, 2$ the value of the $(n-v+1)$ -fold $\hat{C}_v$ is given by

$$\begin{aligned} \hat{C}_v &= \prod_{i=v}^{n-1} P(t_i, t_{i+1}) \cdot F \cdot N_{n-v+1}(a_v, a_{v+1}, \dots, a_n; G_n^v) \\ &\quad - \sum_{i=v}^n K_i \cdot F \cdot N_{i-v+1}(a_v, a_{v+1}, \dots, a_i; G_i^v) \cdot \prod_{j=v}^{i-1} P(t_j, t_{j+1}) \\ &\quad - \sum_{i=v}^n K_i \cdot (1-F) \cdot N_{i-v+1}(b_v, b_{v+1}, \dots, b_i; G_i^v) \cdot \prod_{j=v}^{i-1} P(t_j, t_{j+1}) \quad (16) \end{aligned}$$

For the n -fold C_1 we construct the following hedge:

$$C_1 = \psi_t^1 \cdot P(t, t_1) \cdot F(t, t_n, \alpha_n) + \psi_t^2 \cdot P(t, t_1) \quad (17)$$

so that the forward value fulfills

$$\hat{C}_1 = \psi_t^1 \cdot F(t, t_n, \alpha_n) + \psi_t^2 \quad (18)$$

Using Itô calculus and equation (1) the following SDE for the forward can be derived:

$$\begin{aligned} dF(\cdot, t_n, \alpha_n)_t &= \left\{ -\mu(t, t_n, \alpha_n) \cdot \alpha_n \cdot f(t, t_n, \alpha_n) \cdot F^2(t, t_n, \alpha_n) \right. \\ &\quad \left. + \gamma^2(t, t_n, \alpha_n) \cdot \alpha_n^2 \cdot f^2(t, t_n, \alpha_n) \cdot F^3(t, t_n, \alpha_n) \right\} dt \\ &\quad - \gamma(t, t_n, \alpha_n) \cdot \alpha_n \cdot f(t, t_n, \alpha_n) \cdot F^2(t, t_n, \alpha_n) \cdot dW_t \quad (19) \end{aligned}$$

and therefore

$$\begin{aligned} d\hat{C}_1 &= \left\{ \frac{\partial \hat{C}_1}{\partial t} + \frac{1}{2} \gamma^2(t, t_n, \alpha_n) \cdot (1 - F(t, t_n, \alpha_n))^2 \cdot F^2(t, t_n, \alpha_n) \cdot \frac{\partial^2 \hat{C}_1}{\partial F^2} \right\} dt \\ &\quad + \frac{\partial \hat{C}_1}{\partial F} dF \quad (20) \end{aligned}$$

On the other hand, if we suppose $\psi_t^1 = \frac{\partial \hat{C}_1}{\partial F}$, $\psi_t^2 = \hat{C}_1 - \frac{\partial \hat{C}_1}{\partial F} \cdot F$. In combination with the fact of a self-financing portfolio, we obtain

$$d\hat{C}_1 = \psi_t^1 \cdot dF = \frac{\partial \hat{C}_1}{\partial F} \cdot dF \quad (21)$$

From these two equations stands the PDE for the forward price of the call option:

$$\frac{\partial \hat{C}_1}{\partial t} + \frac{1}{2} \gamma^2(t, t_n, \alpha_n) \cdot (1 - F(t, t_n, \alpha_n))^2 \cdot F^2(t, t_n, \alpha_n) \cdot \frac{\partial^2 \hat{C}_1}{\partial F^2} = 0 \quad (22)$$

This equation will be transformed in a number of steps to obtain a diffusion equation. The following substitutions are executed:

$$t \rightarrow s_1 = \int_t^{t_1} \gamma^2(r, t_n, \alpha_n) dr \quad (23)$$

$$F \rightarrow z = \ln \left| \frac{F}{1 - F} \right| \quad (24)$$

$$\text{suppose } F < 1 \text{ so } F = \frac{1}{1 + \exp(-z)}$$

$$g(s_1, z) = \hat{C}_1(t, F) \quad (25)$$

$$a(z) = \frac{1}{\exp(z/2) + \exp(-z/2)} \quad (26)$$

$$b(s_1) = \exp(-s_1/8) \quad (27)$$

$$a(z) \cdot b(s_1) \cdot h(s_1, z) = g(s_1, z) \quad (28)$$

By these substitutions the above equation (22) is rewritten as a diffusion equation:

$$\frac{1}{2} h_{zz} - h_{s_1} = 0 \quad (29)$$

which is solved using a Green's function of the form

$$G(s_1, z - z') = \frac{1}{\sqrt{2\pi s_1}} \exp \left(-\frac{(z - z')^2}{2s_1} \right) \quad (30)$$

Also the boundary conditions have to be adapted to the diffusion equation. Through the different substitutions the boundary conditions are expressed in the following way:

$$\hat{C}_1|_{t_1} = \left. \frac{C_1}{P(t, t_1)} \right|_{t_1} = C_1|_{t_1} \quad (31)$$

$$\hat{C}_1|_{t_1} = \max \{0, C_2(t_1, F) - K_1\} \quad (32)$$

$$= \max \{0, P(t_1, t_2) \cdot \hat{C}_2(t_1, F) - K_1\} \quad (33)$$

$$= \begin{cases} 0 & \text{if } P(t_1, t_2) \cdot \hat{C}_2(t_1, F) < K_1 \\ \prod_{i=1}^{n-1} P(t_i, t_{i+1}) F N_{n-1}(a_2^*, a_3^*, \dots, a_n^*; G_n^{2*}) \\ - \sum_{i=2}^n K_i F N_{i-1}(a_2^*, a_3^*, \dots, a_i^*; G_i^{2*}) \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \\ - \sum_{i=2}^n K_i (1-F) N_{i-1}(b_2^*, b_3^*, \dots, b_i^*; G_i^{2*}) \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \\ - K_1 & \text{if } P(t_1, t_2) \cdot \hat{C}_2(t_1, F) \geq K_1 \end{cases} \quad (34)$$

(where a * means that the variable is evaluated at date $t = t_1$)

$$h(0, z) = \begin{cases} 0 & \text{if } F < F_1 \\ \prod_{i=1}^{n-1} P(t_i, t_{i+1}) \frac{1}{1 + \exp(-z)} N_{n-1}(a_2^{**}, a_3^{**}, \dots, a_n^{**}; G_n^{2*}) \\ - \sum_{i=2}^n K_i \frac{1}{1 + \exp(-z)} N_{i-1}(a_2^{**}, a_3^{**}, \dots, a_i^{**}; G_i^{2*}) \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \\ - \sum_{i=2}^n K_i \frac{\exp(-z)}{1 + \exp(-z)} N_{i-1}(b_2^{**}, b_3^{**}, \dots, b_i^{**}; G_i^{2*}) \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \\ - K_1 & \text{if } F \geq F_1 \end{cases} \quad (35)$$

or

$$= \begin{cases} 0 & \text{if } z < \ln \left| \frac{F_1}{1 - F_1} \right| \\ \prod_{i=1}^{n-1} P(t_i, t_{i+1}) e^{z/2} N_{n-1}(a_2^{**}, a_3^{**}, \dots, a_n^{**}; G_n^{2*}) \\ - \sum_{i=2}^n K_i e^{z/2} N_{i-1}(a_2^{**}, a_3^{**}, \dots, a_i^{**}; G_i^{2*}) \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \\ - \sum_{i=2}^n K_i e^{-z/2} N_{i-1}(b_2^{**}, b_3^{**}, \dots, b_i^{**}; G_i^{2*}) \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \\ - K_1 (e^{z/2} + e^{-z/2}) & \text{if } z \geq \ln \left| \frac{F_1}{1 - F_1} \right| \end{cases} \quad (36)$$

If we introduce the notation $N_0 \equiv 1$, the final form for the boundary conditions becomes

$$h(0, z) = \begin{cases} 0 & \text{if } z < \ln \left| \frac{F_1}{1 - F_1} \right| \\ \prod_{i=1}^{n-1} P(t_i, t_{i+1}) e^{z/2} N_{n-1}(a_2^{**}, a_3^{**}, \dots, a_n^{**}; G_n^{2*}) \\ - \sum_{i=1}^n K_i e^{z/2} N_{i-1}(a_2^{**}, a_3^{**}, \dots, a_i^{**}; G_i^{2*}) \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \\ - \sum_{i=1}^n K_i e^{-z/2} N_{i-1}(b_2^{**}, b_3^{**}, \dots, b_i^{**}; G_i^{2*}) \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \\ \text{if } z \geq \ln \left| \frac{F_1}{1 - F_1} \right| \end{cases} \quad (37)$$

By the use of Green's functions, the solution of the diffusion equation is given by

$$\begin{aligned} & h(s_1, z) \\ &= \int_{-\infty}^{+\infty} h(0, z') \cdot G(s_1, z - z') dz' \\ &= \int_{\ln \left| \frac{F_1}{1 - F_1} \right|}^{+\infty} \prod_{i=1}^{n-1} P(t_i, t_{i+1}) e^{z'/2} N_{n-1}(a_2^{**}, \dots, a_n^{**}; G_n^{2*}) \frac{1}{\sqrt{2\pi s_1}} e^{\frac{-(z-z')^2}{2s_1}} dz' \\ & - \sum_{i=1}^n K_i \int_{\ln \left| \frac{F_1}{1 - F_1} \right|}^{+\infty} e^{z'/2} N_{i-1}(a_2^{**}, \dots, a_i^{**}; G_i^{2*}) \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \frac{1}{\sqrt{2\pi s_1}} e^{\frac{-(z-z')^2}{2s_1}} dz' \\ & - \sum_{i=1}^n K_i \int_{\ln \left| \frac{F_1}{1 - F_1} \right|}^{+\infty} e^{-z'/2} N_{i-1}(b_2^{**}, \dots, b_i^{**}; G_i^{2*}) \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \frac{1}{\sqrt{2\pi s_1}} e^{\frac{-(z-z')^2}{2s_1}} dz' \end{aligned} \quad (38)$$

By the substitution $\rho = \frac{z' - z}{\sqrt{s_1}}$

$$\begin{aligned}
& h(s_1, z) \\
&= \prod_{i=1}^{n-1} P(t_i, t_{i+1}) \\
&\quad \cdot \int_{\left\{ \ln \left| \frac{F_1}{1-F_1} \right| - z \right\} / \sqrt{s_1}}^{+\infty} N_{n-1}(a_2^{***}, \dots, a_n^{***}; G_n^{2*}) \frac{1}{\sqrt{2\pi}} \exp \left[\frac{\sqrt{s_1}\rho + z}{2} - \frac{\rho^2}{2} \right] d\rho \\
&- \sum_{i=1}^n K_i \prod_{j=1} P(t_j, t_{j+1}) \\
&\quad \cdot \int_{\left\{ \ln \left| \frac{F_1}{1-F_1} \right| - z \right\} / \sqrt{s_1}}^{+\infty} N_{i-1}(a_2^{***}, \dots, a_i^{***}; G_i^{2*}) \frac{1}{\sqrt{2\pi}} \exp \left[\frac{\sqrt{s_1}\rho + z}{2} - \frac{\rho^2}{2} \right] d\rho \\
&- \sum_{i=1}^n K_i \prod_{j=1} P(t_j, t_{j+1}) \\
&\quad \cdot \int_{\left\{ \ln \left| \frac{F_1}{1-F_1} \right| - z \right\} / \sqrt{s_1}}^{+\infty} N_{i-1}(b_2^{***}, \dots, b_i^{***}; G_i^{2*}) \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{\sqrt{s_1}\rho + z}{2} - \frac{\rho^2}{2} \right] d\rho
\end{aligned} \tag{39}$$

Substituting $u = \rho - \frac{\sqrt{s_1}}{2}$ in the first two integrals, and $w = \rho + \frac{\sqrt{s_1}}{2}$ in the third, the boundary conditions are translated for the new variables in the following way:

$$\begin{aligned}
s_j^* &= s_j(t_1) \\
&= \int_{t_1}^{t_j} \gamma^2(r, t_n, \alpha_n) dr
\end{aligned} \tag{40}$$

$$\begin{aligned}
b_j^* &= \frac{1}{\sqrt{s_j^*}} \left\{ \ln \left| \frac{F[1-F_j]}{F_j[1-F]} \right| - \frac{s_j^*}{2} \right\} \\
b_j^{**} &= \frac{1}{\sqrt{s_j^*}} \left\{ \ln \left| \frac{1-F_j}{F_j} \right| + z' - \frac{s_j^*}{2} \right\} \\
b_j^{***} &= \frac{1}{\sqrt{s_j^*}} \left\{ \ln \left| \frac{1-F_j}{F_j} \right| + \sqrt{s_1}\rho + z - \frac{s_j^*}{2} \right\} \\
b_j^{****} &= \frac{1}{\sqrt{s_j^*}} \left\{ \sqrt{s_1}w + \ln \left| \frac{F[1-F_j]}{F_j[1-F]} \right| - \frac{s_1 + s_j^*}{2} \right\} \\
a_j^* &= \frac{1}{\sqrt{s_j^*}} \left\{ \ln \left| \frac{F[1-F_j]}{F_j[1-F]} \right| + \frac{s_j^*}{2} \right\}
\end{aligned} \tag{41}$$

$$\begin{aligned}
a_j^{**} &= \frac{1}{\sqrt{s_j^*}} \left\{ \ln \left| \frac{1-F_j}{F_j} \right| + z' + \frac{s_j^*}{2} \right\} \\
a_j^{***} &= \frac{1}{\sqrt{s_j^*}} \left\{ \ln \left| \frac{1-F_j}{F_j} \right| + \sqrt{s_1} \rho + z + \frac{s_j^*}{2} \right\} \\
a_j^{****} &= \frac{1}{\sqrt{s_j^*}} \left\{ \sqrt{s_1} u + \ln \left| \frac{F[1-F_j]}{F_j[1-F]} \right| + \frac{s_1 + s_j^*}{2} \right\}
\end{aligned} \tag{42}$$

Observing that

$$s_1 + s_j^* = s_j,$$

we obtain

$$b_j^{****} = \frac{1}{\sqrt{s_j^*}} \left\{ \sqrt{s_1} w + \ln \left| \frac{F[1-F_j]}{F_j[1-F]} \right| - \frac{s_j}{2} \right\} \tag{43}$$

$$a_j^{****} = \frac{1}{\sqrt{s_j^*}} \left\{ \sqrt{s_1} u + \ln \left| \frac{F[1-F_j]}{F_j[1-F]} \right| + \frac{s_j}{2} \right\} \tag{44}$$

The expression for $h(s_1, z)$ becomes:

$$\begin{aligned}
&h(s_1, z) \\
&= \prod_{i=1}^{n-1} P(t_i, t_{i+1}) \exp \left[\frac{s_1}{8} + \frac{z}{2} \right] \\
&\quad \cdot \int_{\left\{ \ln \left| \frac{F_1}{1-F_1} \right| - z - \frac{s_1}{2} \right\} / \sqrt{s_1}}^{+\infty} N_{n-1}(a_2^{****}, \dots, a_n^{****}; G_n^{2*}) \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{u^2}{2} \right] du \\
&- \sum_{i=1}^n K_i \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \exp \left[\frac{s_1}{8} + \frac{z}{2} \right] \\
&\quad \cdot \int_{\left\{ \ln \left| \frac{F_1}{1-F_1} \right| - z - \frac{s_1}{2} \right\} / \sqrt{s_1}}^{+\infty} N_{i-1}(a_2^{****}, \dots, a_i^{****}; G_i^{2*}) \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{u^2}{2} \right] du \\
&- \sum_{i=1}^n K_i \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \exp \left[\frac{s_1}{8} - \frac{z}{2} \right] \\
&\quad \cdot \int_{\left\{ \ln \left| \frac{F_1}{1-F_1} \right| - z + \frac{s_1}{8} \right\} / \sqrt{s_1}}^{+\infty} N_{i-1}(b_2^{****}, \dots, b_i^{****}; G_i^{2*}) \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{w^2}{2} \right] dw
\end{aligned} \tag{45}$$

with

$$G_j^{2*} = (g_{ab})_{a,b=1,2,\dots,j-1} \quad \text{with} \quad \begin{cases} g_{aa} = 1 \\ g_{ab} = g_{ba} = \sqrt{\frac{s_{a+1}^*}{s_{b+1}^*}} \end{cases} \quad \text{if } a < b$$

Using appendix A the final equation writes:

$$\begin{aligned}
& h(s_1, z) \\
&= \prod_{i=1}^{n-1} P(t_i, t_{i+1}) e^{s_1/8} e^{z/2} \\
&\quad \cdot N_n \left(\frac{1}{\sqrt{s_1}} \left\{ \ln \left| \frac{F[1-F_1]}{F_1[1-F]} \right| + \frac{s_1}{2} \right\}, \frac{1}{\sqrt{s_1+s_2^*}} \left\{ \ln \left| \frac{F[1-F_2]}{F_2[1-F]} \right| + \frac{s_2}{2} \right\}, \right. \\
&\quad \quad \quad \left. \dots, \frac{1}{\sqrt{s_1+s_n^*}} \left\{ \ln \left| \frac{F[1-F_n]}{F_n[1-F]} \right| + \frac{s_n}{2} \right\}; M_n \right) \\
&- \sum_{i=1}^n K_i \prod_{j=1}^{i-1} P(t_j, t_{j+1}) e^{s_1/8} e^{z/2} \\
&\quad \cdot N_i \left(\frac{1}{\sqrt{s_1}} \left\{ \ln \left| \frac{F[1-F_1]}{F_1[1-F]} \right| + \frac{s_1}{2} \right\}, \frac{1}{\sqrt{s_1+s_2^*}} \left\{ \ln \left| \frac{F[1-F_2]}{F_2[1-F]} \right| + \frac{s_2}{2} \right\}, \right. \\
&\quad \quad \quad \left. \dots, \frac{1}{\sqrt{s_1+s_i^*}} \left\{ \ln \left| \frac{F[1-F_i]}{F_i[1-F]} \right| + \frac{s_i}{2} \right\}; M_i \right) \\
&- \sum_{i=1}^n K_i \prod_{j=1}^{i-1} P(t_j, t_{j+1}) e^{s_1/8} e^{-z/2} \\
&\quad \cdot N_i \left(\frac{1}{\sqrt{s_1}} \left\{ \ln \left| \frac{F[1-F_1]}{F_1[1-F]} \right| - \frac{s_1}{2} \right\}, \frac{1}{\sqrt{s_1+s_2^*}} \left\{ \ln \left| \frac{F[1-F_2]}{F_2[1-F]} \right| - \frac{s_2}{2} \right\}, \right. \\
&\quad \quad \quad \left. \dots, \frac{1}{\sqrt{s_1+s_i^*}} \left\{ \ln \left| \frac{F[1-F_i]}{F_i[1-F]} \right| - \frac{s_i}{2} \right\}; M_i \right) \\
&= \prod_{i=1}^{n-1} P(t_i, t_{i+1}) e^{s_1/8} e^{z/2} N_n(a_1, a_2, \dots, a_n; M_n) \\
&- \sum_{i=1}^n K_i \prod_{j=1}^{i-1} P(t_j, t_{j+1}) e^{s_1/8} e^{z/2} N_i(a_1, a_2, \dots, a_i; M_i) \\
&- \sum_{i=1}^n K_i \prod_{j=1}^{i-1} P(t_j, t_{j+1}) e^{s_1/8} e^{-z/2} N_i(b_1, b_2, \dots, b_i; M_i)
\end{aligned} \tag{46}$$

with correlation matrices

$$M_j = (m_{uv})_{u,v=1,2,\dots,j} \quad \text{with} \quad \begin{cases} m_{uu} = 1 \\ m_{uv} = m_{vu} = \sqrt{\frac{s_1 + s_u^*}{s_1 + s_v^*}} = \sqrt{\frac{s_u}{s_v}} \end{cases} \quad \text{if } u < v$$

or in other words: $M_j \equiv G_j^1$.

Returning to the value of the forward n-fold compound call option we conclude:

$$\begin{aligned}\hat{C}_1(t, F) &= \prod_{i=1}^{n-1} P(t_i, t_{i+1}) \cdot F \cdot N_n(a_1, \dots, a_n; G_n^1) \\ &\quad - \sum_{i=1}^n K_i \cdot \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \cdot F \cdot N_i(a_1, \dots, a_i; G_i^1) \\ &\quad - \sum_{i=1}^n K_i \cdot \prod_{j=1}^{i-1} P(t_j, t_{j+1}) \cdot [1 - F] \cdot N_i(b_1, \dots, b_i; G_i^1) \quad (47)\end{aligned}$$

■

A Appendix

To integrate a product of an exponential and an n-variate normal CDF, we need the following theorem:

Theorem A.1

$$\begin{aligned}&\int_{-b_1}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{q^2}{2}\right) N_{i-1}\left(\frac{fq + b_2}{g_2}, \dots, \frac{fq + b_i}{g_i}; R\right) dq \\ &= N_i\left(b_1, \frac{b_2}{\sqrt{f^2 + g_2^2}}, \dots, \frac{b_i}{\sqrt{f^2 + g_i^2}}; H\right) \quad (48)\end{aligned}$$

with correlation matrices

$$R = (r_{ab})_{a,b=1,2,\dots,i-1} \quad (49)$$

$$\text{such that } \begin{cases} r_{aa} = 1 \\ r_{ab} = r_{ba} = \frac{g_{a+1}}{g_{b+1}} \end{cases} \quad \text{for } a < b$$

$$H = (h_{ab})_{a,b=1,2,\dots,i} \quad (50)$$

$$\text{such that } \begin{cases} h_{aa} = 1 \\ h_{ab} = h_{ba} = \sqrt{\frac{f^2 + g_a^2}{f^2 + g_b^2}} \end{cases} \quad \text{for } a < b$$

$$\text{and where for convenience } g_1 \equiv 0 \quad (51)$$

Proof

We use corollary A.1 which was proven by Thomassen and Van Wouwe in an earlier paper [5]:

Corollary A.1 *Let C be a given correlation matrix and suppose $\forall i = 1, \dots, n : m_i \in \mathbb{R}$ are known.*

Let N_{n-1} be the $(n-1)$ -variate normal cumulative distribution function and N_n the n -variate normal cumulative distribution function, the following relation can be proven:

$$N_n(m_1, \dots, m_n; C) = \int_{-m_1}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2} N_{n-1}\left(\frac{m_2 + c_{12}x_1}{\sqrt{1-c_{12}^2}}, \dots, \frac{m_n + c_{1n}x_1}{\sqrt{1-c_{1n}^2}}; B\right) dx_1,$$

with $B = (b_{ij})_{i,j=1,\dots,n-1}$ a symmetric matrix with

$$b_{ij} = \frac{c_{i+1,j+1} - c_{1,i+1} c_{1,j+1}}{\sqrt{(1-c_{1,i+1}^2)(1-c_{1,j+1}^2)}}.$$

Remark

This corollary is used in the following way:

$$N_i\left(b_1, \frac{b_2}{\sqrt{f^2 + g_2^2}}, \dots, \frac{b_i}{\sqrt{f^2 + g_i^2}}; H\right) = \int_{-b_1}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{q^2}{2}\right) N_{i-1}\left(\frac{\frac{b_2}{\sqrt{f^2 + g_2^2}} + qh_{12}}{\sqrt{1-h_{12}^2}}, \dots, \frac{\frac{b_i}{\sqrt{f^2 + g_i^2}} + qh_{1i}}{\sqrt{1-h_{1i}^2}}; C\right) dq$$

$$\text{with } C_{vw} = \frac{h_{v+1,w+1} - h_{1,v+1} \cdot h_{1,w+1}}{\sqrt{(1-h_{1,v+1}^2)(1-h_{1,w+1}^2)}} \quad (52)$$

Now, by replacing the element $h_{i,j}$ by the value $\sqrt{\frac{f^2 + g_i^2}{f^2 + g_j^2}}$, it can easily be seen that

$$\frac{\frac{b_j}{\sqrt{f^2 + g_j^2}} + qh_{1j}}{\sqrt{1-h_{1j}^2}} = \frac{b_j + fq}{g_j} \quad \forall j = 2, 3, \dots, i \quad (53)$$

$$C_{vw} = \frac{g_{v+1}}{g_{w+1}} \quad \forall v < w \quad (54)$$

$$C_{vv} = 1 \quad (55)$$

So it follows that $C = R$.

■

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