



# Curves and quivers

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## Abstract

In this paper we associate to a  $\mathbb{k}$ -curve  $A$  (formerly known as a quasi-free algebra [J. Cuntz, D. Quillen, Algebra extensions and nonsingularity, J. Amer. Math. Soc. 8 (1995) 251–289] or formally smooth algebra [M. Kontsevich, A. Rosenberg, Noncommutative smooth spaces, math.AG/9812158, 1998]) the *one-quiver*  $Q_1(A)$  and dimension vector  $\alpha_1(A)$ . This pair contains enough information to reconstruct for all  $n \in \mathbb{N}$  the  $GL_n$ -étale local structure of the representation scheme  $\text{rep}_n A$ . In an appendix we indicate how one might extend this to curves over non-algebraically closed fields. Further, we classify all finitely generated groups  $G$  such that the group algebra  $kG$  is a  $k$ -curve. If  $\text{char}(k) = 0$  these are exactly the virtually free groups. We determine the one-quiver setting in this case and indicate how it can be used to study the finite-dimensional representations of virtually free groups. As this approach also applies to *fundamental algebras of graphs of separable  $k$ -algebras*, we state the results in this more general setting.

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## 1. Curves

In this paper,  $k$  is a commutative field with algebraic closure  $\mathbb{k}$ . Algebras will be associative  $k$ -algebras with unit and (usually) finitely generated over  $k$ . For a  $k$ -algebra  $A$  let

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$A'$  be the  $k$ -vectorspace  $A/k.1$  and define (following [6, Section 1]) the graded algebra of *non-commutative differential forms*

$$\Omega A = \bigoplus_{i=0}^{\infty} \Omega^i A \quad \text{with } \Omega^i A = A \otimes A'^{\otimes i}$$

with multiplication defined by the maps  $\Omega^n A \otimes \Omega^{k-1} A \rightarrow \Omega^{n+k-1} A$ , where

$$(a_0, \dots, a_n) \cdot (a_{n+1}, \dots, a_{n+k}) = \sum_{i=0}^n (-1)^{n-i} (a_0, \dots, a_i a_{i+1}, \dots, a_{n+k}).$$

As  $\Omega^0 A = A$  this multiplication defines an  $A$ -bimodule structure on each  $\Omega^n A$  and one proves [6, Proposition 2.3] that  $\Omega A \cong T_A(\Omega^1 A)$  the tensor algebra of the  $A$ -bimodule  $\Omega^1 A$ . We remark that the standard assumption of [6] is that  $k = \mathbb{C}$  the field of complex numbers. However, with minor modifications most results remain valid over an arbitrary basefield and we will refer to statements in [6] whenever the argument can be repeated verbatim.

**Definition 1.** A finitely generated  $k$ -algebra  $A$  is said to be a  *$k$ -curve* (or *quasi-free* [6] or *formally smooth* [11]) if either of the following two equivalent conditions is satisfied:

- the universal bimodule  $\Omega^1(A)$  of derivations is a projective  $A$ -bimodule;
- $A$  satisfies the lifting property modulo nilpotent ideals in  $k$ -alg, i.e. the category of  $k$ -algebras. That is, for every  $k$ -algebra morphism  $A \xrightarrow{\phi} B/I$  where  $I \triangleleft B$  is a nilpotent ideal, there exists an algebra morphism  $A \xrightarrow{\psi} B$  inducing  $\phi$ .

Whereas the lifting property extends Grothendieck’s characterization of commutative regular algebras (see, for example, [10]) to the non-commutative setting, such algebras are known to be *hereditary* by [6, Proposition 5.1] and hence they behave *quite like curves*.

Recall that a finite-dimensional  $k$ -algebra  $S$  is said to be *separable* if and only if  $S$  is the direct sum of simple algebras each of which has a center which is a separable field extension of  $k$ . For example, the group algebra  $kG$  of a finite group  $G$  is separable if and only if the order of  $G$  is a unit in  $k$ . Separable  $k$ -algebras are known to be  $k$ -curves by [6, Section 4] but should be thought of as corresponding to *points*. In fact, they are characterized by either of the following two equivalent conditions:

- $A$  is a projective  $A$ -bimodule;
- $A$  satisfies the *conjugate* lifting property modulo nilpotent ideals in  $k$ -alg.

That is, if  $I \triangleleft B$  is a nilpotent ideal and if  $\bar{\phi}, \bar{\psi} : S \rightrightarrows B/I$  are two  $k$ -algebra morphisms which are conjugate by a unit  $\bar{b} \in B/I$  then there exist algebra lifts  $\phi, \psi : S \rightrightarrows B$  and a unit  $b \in B$  (mapping to  $\bar{b}$ ) conjugating  $\phi$  and  $\psi$ , see [6, Proposition 6.1.2].

Genuine examples of  $k$ -curves are: the free algebra  $k\langle x_1, \dots, x_m \rangle$ , the path algebra  $kQ$  of a finite quiver  $Q$  and the coordinate ring  $k[C]$  of a smooth affine commutative curve  $C$ .

From these more complicated examples are constructed by universal constructions such as taking algebra free products  $A * A'$  or universal localizations  $A_\Sigma$ . In the next section we will introduce a new class of  $k$ -curve examples.

For an  $k$ -algebra  $A$  recall that the *representation scheme*  $\text{rep}_n A$  is the affine  $k$ -scheme representing the functor

$$k\text{-commalg} \rightarrow \text{sets} \quad \text{defined by} \quad C \mapsto \text{Hom}_{k\text{-alg}}(A, M_n(C)),$$

where  $k\text{-commalg}$  is the category of commutative  $k$ -algebras. A major motivation for studying  $k$ -curves comes from the result mentioned in [11,12] and proved in [15, (2.2)].

**Proposition 1.** *If  $A$  is a  $k$ -curve, then all representation schemes  $\text{rep}_n A$  are smooth affine varieties (possibly having several connected components).*

## 2. Curves from graphs

In this section we will imitate the Bass–Serre theory of the fundamental group of a graph of groups, see [7,24], to construct a large class of examples of  $k$ -curves.

**Definition 2.** Let  $G = (V, E)$  be a finite graph with vertex-set  $V$  and edges  $E$ . A  $G$ -graph of  $k$ -curves  $\mathcal{Q}_G$  is the assignment of:

- an  $k$ -curve  $A_v$  to every vertex  $v \in V$ ;
- a separable  $k$ -algebra  $S_e$  to every edge  $e \in E$ ;
- inclusions of  $k$ -algebras

$$S_e \xrightarrow{i_{e,v}} A_v \quad \text{and} \quad S_e \xrightarrow{i_{e,w}} A_w \quad \text{for every edge } \textcircled{v} \text{---}^e \text{---} \textcircled{w}.$$

If, moreover, all vertex-algebras are separable algebras  $S_v$  we call this data a  $G$ -graph of separable algebras and denote it by  $\mathcal{S}_G$ .

In order to construct the *fundamental algebra*  $\pi_1(\mathcal{Q}_G)$  of a  $G$ -graph of curves  $\mathcal{Q}_G$  we need to have  $k$ -algebra equivalents for the notions of *amalgamated group products* [24, Section 1.2] and of the *HNN construction* [24, Section 1.4]. If  $S$  is a separable  $k$ -algebra and if  $A$  and  $A'$  are  $S$ -algebras, then the *coproduct*  $A *_S A'$  is the algebra representing the functor

$$\text{Hom}_{S\text{-alg}}(A, -) \times \text{Hom}_{S\text{-alg}}(A', -)$$

in the category  $S\text{-alg}$  of  $S$ -algebras, see, for example, [22, Chapter 2] for its construction and properties. As for the HNN-construction, let  $\alpha, \beta : S \hookrightarrow A$  be two  $k$ -algebra embeddings of  $S$  in  $A$ , consider the algebra

$$A *_S^{\alpha, \beta} = \frac{A * k[t, t^{-1}]}{(\beta(s) - t^{-1}\alpha(s)t : \forall s \in S)}.$$

**Lemma 1.** *Let  $S$  be a separable  $k$ -algebra,  $A$  and  $A'$   $k$ -curves and  $k$ -embeddings  $\alpha, \beta : S \hookrightarrow A$  and  $S \hookrightarrow A'$ . Then, the  $k$ -algebras*

$$A *_S A' \quad \text{and} \quad A *_S^{\alpha, \beta}$$

are again  $k$ -curves.

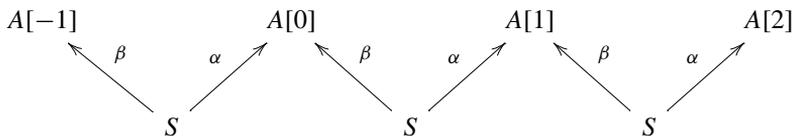
**Proof.** A morphism  $A *_S A' \xrightarrow{g} B/I$  is fully determined by morphisms  $A \xrightarrow{f} B/I$  and  $A' \xrightarrow{f'} B/I$  such that  $f|_S = f'|_S$ . As  $A$  and  $A'$  are quasi-free one has  $k$ -algebra lifts  $\tilde{f} : A \rightarrow B$  and  $\tilde{f}' : A' \rightarrow B$  whence two morphisms on  $S$  which have to be conjugated by an  $b \in B^*$  such that  $\tilde{b} = 1_{B/I}$ , that is  $f'(s) = b^{-1}f(s)b$  for all  $s \in S$ . But then, we have a lift  $A *_S A' \rightarrow B$  determined by the morphisms  $b^{-1}fb$  and  $f'$ .

A morphism  $A *_S^{\alpha, \beta} \xrightarrow{g} B/I$  determines (and is determined by) a morphism  $A \xrightarrow{f} B/I$  and a unit  $\tilde{b} = g(t)$  such that  $f \circ \alpha$  and  $f \circ \beta : S \rightarrow B/I$  are conjugated via  $\tilde{b}$ . Because  $A$  is quasi-free we have a lift  $\tilde{f} : A \rightarrow B$  and algebra maps  $\tilde{f} \circ \alpha$  and  $\tilde{f} \circ \beta : S \rightarrow B$  which reduce to  $\tilde{b}$  conjugate morphisms. But then there is a unit  $b \in B^*$  conjugating  $\tilde{f} \circ \alpha$  to  $\tilde{f} \circ \beta$  and mapping  $t$  to  $b$  produces the required lift  $A *_S^{\alpha, \beta} \rightarrow B$ .  $\square$

However, as often with universal constructions, we have to take care not to end up with the trivial algebra! Because  $S$  is semi-simple and  $A$  and  $A'$  are faithful  $S$ -algebras it follows from [22, Chapter 2] that there are inclusions  $A \hookrightarrow A *_S A'$  and  $A' \hookrightarrow A *_S A'$ . To prove that  $A \hookrightarrow A *_S^{\alpha, \beta}$  we give another description of the HNN-construction mimicking [24, Section 1.4]. For any  $n \in \mathbb{Z}$  take  $A[n] \simeq A$  and construct the amalgamated products

$$A_0 = A, \quad A_1 = A[-1] *_S A_0 *_S A[1], \quad \dots, \quad A_k = A[-k] *_S A_{k-1} *_S A[k]$$

with respect to the following embeddings:



As  $S$  is semi-simple we have by [22, Chapter 2] embeddings  $A_0 \subset A_1 \subset A_2 \subset \dots$  and hence  $A$  embeds in the limit  $\tilde{A} = \varinjlim A_n$ . The shift-identity

$$\dots \rightarrow A[k-1] \xrightarrow{\text{id}} A[k] \xrightarrow{\text{id}} A[k+1] \rightarrow \dots$$

induces an automorphism  $\phi$  on  $\tilde{A}$  and as the two algebras below have the same universal property they are isomorphic:

$$A *_S^{\alpha, \beta} \simeq \tilde{A}[t, t^{-1}, \phi] \quad \text{whence} \quad A \hookrightarrow A *_S^{\alpha, \beta}.$$

Here,  $\tilde{A}[t, t^{-1}, \phi]$  is the skew Laurent algebra with  $ta = \phi(a)t$  for all  $a \in \tilde{A}$ , see also [8] for more details on the HNN construction for rings.

**Definition 3.** Let  $\mathcal{Q}_G$  be a graph of  $k$ -curves and let  $T$  be a maximal subtree of  $G$ . We construct the  $k$ -algebra  $A_T$  by induction on the number  $t$  of edges in  $T$ . If  $t = 0$  so  $V = \{v\}$  then  $A_T = A_v$ . If  $t > 0$ , consider a leaf vertex  $v$  with connecting edge  $\textcircled{v} \xrightarrow{e} \textcircled{w}$  in  $T$ . Construct a new tree  $T'$  on  $t - 1$  edges by dropping the vertex  $v$  and edge  $e$  and construct a new graph of  $k$ -curves  $\mathcal{Q}'_{T'}$  by

$$A'_w = A_v *_S A_w, \quad A'_u = A_u, \quad \text{for } v \neq u \in V, \quad A'_f = A_f, \quad \text{for } e \neq f \in E$$

then  $A_T \simeq A_{T'}$ . Observe that there are embeddings  $A_u \xrightarrow{i_u} A_T$  for every  $u \in V$ .

Let  $G - T = \{e_1, \dots, e_r\}$  and take  $A_0 \simeq A_T$ . For every edge  $\textcircled{v} \xrightarrow{e_i} \textcircled{w}$  in  $G - T$  there are two embeddings

$$\alpha_i : S_e \xrightarrow{i_{e_i,v}} S_v \xrightarrow{i_v} A_{i-1} \quad \text{and} \quad \beta_i : S_e \xrightarrow{i_{e_i,w}} S_w \xrightarrow{i_w} A_{i-1}$$

and we define

$$A_i \simeq A_{i-1} *_S^{\alpha_i, \beta_i}.$$

The algebra  $A_r$  is then called the *fundamental algebra of the graph of  $k$ -curves*  $\mathcal{Q}_G$  and is denoted by  $\pi_1(\mathcal{Q}_G)$ .

**Theorem 1.** *If  $\mathcal{Q}_G$  is a graph of  $k$ -curves, the fundamental algebra  $\pi_1(\mathcal{Q}_G)$  is again a  $k$ -curve.*

**Proof.** Immediate from the construction and Lemma 1.  $\square$

Presumably, similar to the case of groups [24], this construction is independent of the choice of maximal subtree. At the moment it may be better to use the notation  $\pi_1(\mathcal{Q}_G)$  and talk about the fundamental algebra for a fixed maximal subtree  $T$ . In the remainder of this paper, we will always fix such a maximal subtree.

### 3. Curve group algebras

The classification of  $k$ -curves is way out of reach at the moment so it is important to have partial classifications. In [6, Section 6] the finite-dimensional  $k$ -curves were shown to be the hereditary finite-dimensional  $k$ -algebras (and hence Morita equivalent to path algebras  $kQ$  of a finite quiver  $Q$  without oriented cycles). In this section we will classify the group algebras  $kH$  for  $H$  a finitely generated group which are  $k$ -curves. The desired answer is that these are precisely the  $kH$  with  $H$  a *virtually free group* (that is,  $H$  has a free subgroup of finite index) but we have to take the characteristic of  $k$  into account (observe that finite groups are virtually free).

If  $\mathcal{G}_G$  is a graph of *finite groups* as in [24] such that all orders are invertible in  $k$ , then we can associate to it a graph of separable  $k$ -algebras  $\mathcal{S}_G$  by taking

$$S_v = kG_v \quad \forall v \in V \quad \text{and} \quad S_e = kG_e \quad \forall e \in E$$

with embeddings determined by the group-embeddings. If  $\pi_1(\mathcal{G}_G)$  is the *fundamental group* of  $\mathcal{G}_G$  as in [24, Section 5.1] then the point of the construction in the previous section is that

$$k\pi_1(\mathcal{G}_G) \simeq \pi_1(\mathcal{S}_G)$$

and hence these group algebras are  $k$ -curves. The connection with virtually free groups is provided by a result of Karrass, Pietrowski and Solitar, see, for example, [26, Theorem 3.5]. The following statements are equivalent for a finitely-generated group  $H$ :

- $H = \pi_1(\mathcal{G}_G)$  for a graph of finite groups;
- $H$  is a virtually free group.

For example, all congruence subgroups in the modular group  $SL_2(\mathbb{Z})$  are virtually free. On the other hand, the third braid group  $B_3 = \langle s, t \mid s^2 = t^3 \rangle$  is not virtually free. Note that very little is known about simple representations of congruence subgroups. For some low-dimensional classifications of  $SL_2(\mathbb{Z})$ -representations see [25].

**Theorem 2.** *The following statements are equivalent for a finitely generated group  $H$ :*

- (1) *the group algebra  $kH$  is an  $k$ -curve;*
- (2)  *$H$  is a virtually free group and is isomorphic to  $\pi_1(\mathcal{G}_G)$  for some graph of finite groups such that all orders of the vertex groups  $G_v$  are finite and invertible in  $k$ .*

**Proof.** If  $kH$  is a quasi-free  $k$ -algebra, it has to be hereditary by [6, Proposition 5.1] and hence, in particular, its augmentation ideal  $\omega_H$  must be a projective left  $kH$ -module. By a result of Dunwoody, see [7, Theorem IV.2.12] this is equivalent to  $H$  being the fundamental group of a graph of finite groups  $\mathcal{G}_G$  such that all vertex-group orders are invertible in  $k$ , whence (2) follows. The converse implication follows from the discussion preceding the statement and the last section.  $\square$

If  $\text{char}(k) = 0$  it follows from this and Proposition 1 that all representation schemes  $\text{rep}_n kH$  are smooth affine varieties whenever  $H$  is a finitely generated virtually free group.

Shortly after the present paper was posted on the arXiv [17], A. Ardizzoni posted a paper [3] extending the foregoing result to a larger class of Hopf-curves. If  $H$  is an Hopf algebra with antipode  $S$  and counit  $\epsilon$ , then  $\lambda \in H^*$  is called an *ad-invariant integral* if the following conditions are satisfied:

- (1)  $h_{(1)}\lambda(h_{(2)}) = 1_H\lambda(h)$  for all  $h \in H$ ;
- (2)  $\lambda(h_{(1)}xS(h_{(2)})) = \epsilon(h)\lambda(x)$  for all  $h, x \in H$ ;
- (3)  $\lambda(1_H) = 1_{\mathbb{k}}$ .

Remark that a groupalgebra  $\mathbb{k}G$  has an ad-invariant integral taking  $\lambda(g) = \delta_{e,g}$ . The result below summarizes some of [3].

**Theorem 3** (Ardizzoni). *Let  $H$  be a Hopf algebra with antipode  $S$ , counit  $\epsilon$  and ad-invariant integral. The following statements are equivalent:*

- (1)  $H$  is a  $\mathbb{k}$ -curve;
- (2) the augmentation ideal  $\text{Ker}(\epsilon)$  is a projective left  $H$ -module.

#### 4. The component semigroup

From now on we will mostly work over the algebraically closed field  $\mathbb{k}$ . Results which hold over an arbitrary base field  $k$  will be stated as such. If  $A$  is an  $\mathbb{k}$ -curve we know from Proposition 1 that all representation schemes are smooth affine varieties.

**Definition 4.** For an  $\mathbb{k}$ -curve  $A$  the smooth variety  $\text{rep}_n A$  decomposes into connected (equivalently, irreducible) components

$$\text{rep}_n A = \bigsqcup_{|\alpha|=n} \text{rep}_\alpha A,$$

where  $\alpha$  is a label. We call  $\alpha$  a *dimension vector* of *total dimension*  $|\alpha| = n$ .

An  $\mathbb{k}$ -point of  $\text{rep}_n A$  is an  $n$ -dimensional left  $A$ -module and the direct sum of modules defines the *sum maps*

$$\text{rep}_n A \times \text{rep}_m A \longrightarrow \text{rep}_{n+m} A.$$

If we decompose these varieties into their connected components and use the fact that the image of two connected varieties is again connected, we can define a semigroup.

**Definition 5.** The *component semigroup*  $\text{comp}(A)$  is the set of all dimension vectors  $\alpha$  equipped with the addition  $\alpha + \beta = \gamma$  where  $\gamma$  determines the unique component  $\text{rep}_\gamma A$  of  $\text{rep}_{n+m} A$  containing the image of  $\text{rep}_\alpha A \times \text{rep}_\beta A$  under the sum map

$$\bigsqcup_{|\alpha|=n} \text{rep}_\alpha A \times \bigsqcup_{|\beta|=m} \text{rep}_\beta A \longrightarrow \bigsqcup_{|\gamma|=n+m} \text{rep}_\gamma A.$$

$\text{comp}(A)$  is a commutative semigroup with an augmentation map  $\text{comp}(A) \rightarrow \mathbb{N}$  sending a dimension vector  $\alpha$  to its total dimension  $|\alpha|$ .

Here are some examples:

- for  $A = M_{n_1}(\mathbb{k}) \oplus \dots \oplus M_{n_k}(\mathbb{k})$  semi-simple,  $\text{comp}(A) = (\mathbb{N}n_1, \dots, \mathbb{N}n_k) \subset \mathbb{N}^k$ ;

- for  $A = \mathbb{k}Q$  a path algebra we have  $comp(A) = \mathbb{N}^k$  where  $k$  is the number of vertices of the quiver  $Q$ ;
- for a direct sum  $A = A_1 \oplus A_2$  we have  $comp(A) = comp(A_1) \oplus comp(A_2)$ ;
- for a free algebra product  $A = A_1 * A_2$  we have that  $comp(A_1)$  is the fibered product (using the augmentation)  $comp(A_1) \times_{\mathbb{N}} comp(A_2)$ , see [19, Proposition 1].

In [19, Question 2] K. Morrison asked whether  $comp(A)$  is always a free abelian semigroup (as in the examples above). However, even for  $A$  an  $\mathbb{k}$ -curve, reality is more complex as one can remove components by the process of universal localization (see, for example, [22] for definition and properties of universal localization).

**Proposition 2.** *For every subsemigroup  $S \subset \mathbb{N}$ , there is an  $\mathbb{k}$ -curve  $A$  with*

$$comp(A) = S$$

*as augmented semigroups.*

**Proof.** Suppose first that  $\gcd(S) = 1$ , that is the elements of  $S$  are coprime. By using results on polynomial and rational identities of matrices (see, for example, [21]) it was proved in [14] that there is an affine  $k$ -algebra with presentation

$$A = \frac{\mathbb{k}\langle x_1, \dots, x_a, y_1, \dots, y_b \rangle}{(1 - y_i p_i(x_1, \dots, x_a, y_1, \dots, y_{i-1}): 1 \leq i \leq b)}$$

(with each of the  $p_i \in k\langle x_1, \dots, x_a, y_1, \dots, y_{i-1} \rangle$ ) having the property that  $A$  has finite-dimensional representations of dimensions exactly the elements of  $S$ .  $A$  is a universal localization of  $\mathbb{k}\langle x_1, \dots, x_a \rangle$  and hence is an  $\mathbb{k}$ -curve (for example, use [22, Theorem 10.6] to prove that  $\Omega^1(A)$  is a projective  $A$ -bimodule). As such, for every  $n$ ,  $\text{rep}_n A$  is a Zariski open subset (possibly empty) of  $\text{rep}_n \mathbb{k}\langle x_1, \dots, x_a \rangle = M_n(\mathbb{k})^{\times a}$  and is therefore irreducible (when non-empty). Therefore,  $comp(A) = S \subset \mathbb{N}$  and consists precisely of those  $n \in \mathbb{N}$  for which none of the  $p_i$  (when expressed as a rational non-commutative function in  $x_1, \dots, x_a$ ) is a rational identity for  $n \times n$  matrices.

For the general case, assume that  $\gcd(S) = m$  and take  $S' = S/m$  with associated algebra (as above)  $A'$  for which  $comp(A') = S' \subset \mathbb{N}$ . But then,

$$comp(A' * M_m(\mathbb{k})) = S' \times_{\mathbb{N}} \mathbb{N}m = S$$

and  $A = A' * M_m(\mathbb{k})$  is again an  $\mathbb{k}$ -curve.  $\square$

### 5. Tits and Euler forms

In this section we will define bilinear forms on  $comp(A)$  (when  $A$  is an  $\mathbb{k}$ -curve) generalizing the Tits and Euler forms on the dimension vectors of a quiver. Let  $\text{rep } A$  be the abelian category of all finite-dimensional representations of  $A$ . If  $A$  is an affine  $\mathbb{k}$ -algebra, then  $\text{Hom}_A(M, N)$  and  $\text{Ext}_A^1(M, N)$  are finite-dimensional  $\mathbb{k}$ -spaces for all  $M, N \in \text{rep } A$ .

If  $A$  is hereditary (for example, if  $A$  is an  $\mathbb{k}$ -curve) we have that  $\chi_A(M, -)$  and  $\chi_A(-, N)$  are additive on short exact sequences in  $\text{rep } A$ , where

$$\chi_A(M, N) = \dim_{\mathbb{k}} \text{Hom}_A(M, N) - \dim_{\mathbb{k}} \text{Ext}_A^1(M, N).$$

For  $M \in \text{rep } A$  define its *semi-simplification*  $M^{ss}$  to be the semi-simple  $A$ -module obtained by taking the direct sum of the Jordan–Hölder components of  $M$ . From additivity on short exact sequences it follows for all  $M, N \in \text{rep } A$  that

$$\chi_A(M, N) = \chi_A(M^{ss}, N^{ss}).$$

For  $\alpha, \beta \in \text{comp}(A)$  it follows from [13] and [5, Lemma 4.3] that the functions

$$\text{rep}_{\alpha} A \times \text{rep}_{\beta} A \rightarrow \mathbb{Z}, \quad (M, N) \mapsto \begin{cases} \dim_{\mathbb{k}} \text{Hom}_A(M, N), \\ \dim_{\mathbb{k}} \text{Ext}_A^1(M, N) \end{cases}$$

are upper semicontinuous. In particular, there are Zariski open subsets (whence dense by irreducibility) of  $\text{rep}_{\alpha} A \times \text{rep}_{\beta} A$  where these functions attain a minimum. Following [23] we will denote these minimal values by  $\text{hom}(\alpha, \beta)$ , respectively  $\text{ext}(\alpha, \beta)$ .

The group  $GL_n$  acts on  $\text{rep}_n A$  by base-change and orbits  $\mathcal{O}(M)$  under this action are precisely the isomorphism classes of  $n$ -dimensional left  $A$ -modules. From [9] we recall that the semi-simplification  $M^{ss}$  belongs to the Zariski closure  $\overline{\mathcal{O}(M)}$  of the orbit and that  $\text{Ext}_A^1(M, M)$  can be identified to the *normal space* to the orbit  $\mathcal{O}(M)$  with respect to the scheme structure on  $\text{rep}_n A$ .

**Proposition 3.** *Let  $A$  be an affine  $\mathbb{k}$ -algebra.*

(1) *If  $\text{rep}_{\gamma} A$  is a smooth variety, then for all  $M \in \text{rep}_{\gamma} A$  we have*

$$|\gamma|^2 - \chi_A(M, M) = \dim \text{rep}_{\gamma} A$$

*and hence  $\chi_A(M, M)$  is constant on  $\text{rep}_{\gamma} A$ .*

(2) *If  $\text{rep}_{\alpha} A, \text{rep}_{\beta} A$  and  $\text{rep}_{\alpha+\beta} A$  are smooth varieties, then*

$$\chi_A(M, N) + \chi_A(N, M)$$

*is a constant function on  $\text{rep}_{\alpha} A \times \text{rep}_{\beta} A$ .*

**Proof.** If  $\text{rep}_{\gamma} A$  is smooth in  $M$ , it follows from the above remarks that

$$T_M \text{rep}_{\gamma} A = \text{Ext}_A^1(M, M) \oplus T_M \mathcal{O}(M), \quad \mathcal{O}(M) = GL_{|\gamma|} / \text{Stab}(M),$$

where  $\text{Stab}(M)$  is the stabilizer subgroup which by [13] has the same dimension as  $\text{Hom}_A(M, M)$ . Therefore,

$$\begin{aligned} \dim \operatorname{rep}_\gamma A &= \dim_{\mathbb{k}} T_M \operatorname{rep}_\gamma A \\ &= \dim_{\mathbb{k}} \operatorname{Ext}_A^1(M, M) + |\gamma|^2 - \dim_{\mathbb{k}} \operatorname{Hom}_A(M, M) \end{aligned}$$

whence (1). (2) follows from this by considering the point  $M \oplus N \in \operatorname{rep}_{\alpha+\beta} A$  and using bi-additivity of  $\chi_A$ .  $\square$

**Definition 6.** If  $A$  is an  $\mathbb{k}$ -curve, then for all  $\alpha \in \operatorname{comp}(A)$  the representation variety  $\operatorname{rep}_\alpha A$  is smooth. Therefore, the constant value

$$(\alpha, \beta)_A = \chi_A(M, N) + \chi_A(N, M)$$

on  $\operatorname{rep}_\alpha A \times \operatorname{rep}_\beta A$  defines a symmetric bilinear form

$$(-, -)_A : \operatorname{comp}(A) \times \operatorname{comp}(A) \longrightarrow \mathbb{Z}$$

which we call the *Tits-form* of the  $\mathbb{k}$ -curve  $A$ .

For general affine  $\mathbb{k}$ -algebras  $\chi_A(M, N) + \chi_A(N, M)$  does not have to be constant and the foregoing result can be used to deduce singularity of specific representation varieties.

**Example 1.** Let  $A = \mathbb{k}B_3$  be the group-algebra of the *third braid group*  $B_3 = \langle s, t \mid s^2 = t^3 \rangle$ . The one-dimensional representation variety is the cusp minus the singular origin:

$$\operatorname{rep}_1 A = \{(x, y) \in \bar{k}^2 \mid x^3 = y^2\} - \{(0, 0)\}$$

and hence is a smooth affine variety. As all points are simple  $A$ -modules we have that  $\dim_{\mathbb{k}} \operatorname{Hom}_A(-, -)$  is equal to zero on the open set  $\operatorname{rep}_1 A \times \operatorname{rep}_1 A - \Delta$  and is equal to one on the *diagonal*  $\Delta$ . As for  $\dim_{\mathbb{k}} \operatorname{Ext}_A^1(-, -)$  this is zero on  $\operatorname{rep}_1 A \times \operatorname{rep}_1 A - (\Delta_1 \sqcup \Delta_2)$ , where

$$\begin{cases} \Delta_1 = \{((x, y), (\rho x, -y)) \mid x^3 = y^2\}, \\ \Delta_2 = \{((x, y), (\rho^2 x, -y)) \mid x^3 = y^2\}, \end{cases}$$

for  $\rho$  a primitive third root of unity. As a consequence,  $\chi_A(M, N)$  is zero on the Zariski open subset  $\operatorname{rep}_1 A \times \operatorname{rep}_1 A - (\Delta_1 \sqcup \Delta_2)$  and is equal to  $-1$  on  $\Delta_1 \sqcup \Delta_2$ . Therefore,  $\mathbb{k}B_3$  is not an  $\mathbb{k}$ -curve. In fact,  $\operatorname{rep}_2 \mathbb{k}B_3$  is not smooth.

If  $\alpha$  is the dimension vector of a simple representation of  $A$ , then there is a Zariski open subset  $\operatorname{simp}_\alpha A$  of simple representations in  $\operatorname{rep}_\alpha A$ .

**Proposition 4.** If  $A$  is an  $\mathbb{k}$ -curve and  $\alpha, \beta$  are dimension vectors of simple representations, then the function

$$\chi_A(S, T)$$

is constant on  $\operatorname{simp}_\alpha A \times \operatorname{simp}_\beta A$ .

**Proof.** There is a Zariski open subset  $U \subset \text{simp}_\alpha A \times \text{simp}_\beta A$  consisting of couples  $(S', T')$  such that

$$\dim_{\mathbb{k}} \text{Ext}_A^1(S', T') = \text{ext}(\alpha, \beta) \quad \text{and} \quad \dim_{\mathbb{k}} \text{Ext}_A^1(T', S') = \text{ext}(\beta, \alpha).$$

Hence, for all  $(S, T) \in \text{simp}_\alpha A \times \text{simp}_\beta A$

$$\begin{cases} \dim_{\mathbb{k}} \text{Ext}_A^1(S, T) \geq \dim_{\mathbb{k}} \text{Ext}_A^1(S', T'), \\ \dim_{\mathbb{k}} \text{Ext}_A^1(T, S) \geq \dim_{\mathbb{k}} \text{Ext}_A^1(T', S'). \end{cases}$$

If  $\alpha \neq \beta$  (or if  $\alpha = \beta$  and  $S \not\cong T$ )  $\chi_A(S, T) = -\dim_{\mathbb{k}} \text{Ext}_A^1(S, T)$  and hence the above inequalities must be equalities by Proposition 3. Remains to prove for  $S, T \in \text{simp}_\alpha A$  with  $S \not\cong T$  that  $\chi_A(S, S) = \chi_A(S, T)$ . Consider the two semi-simple representations  $M = S \oplus S$  and  $N = S \oplus T$  in  $\text{rep}_{2\alpha} A$ . From Proposition 3(1) we get obtain that  $\chi_A(M, M) = \chi_A(N, N)$  whence

$$\begin{aligned} 4\chi_A(S, S) &= \chi_A(S, S) + \chi_A(T, T) + \chi_A(S, T) + \chi_A(T, S) \\ &= 2\chi_A(S, S) + 2\chi_A(S, T) \end{aligned}$$

(using Proposition 3(1) and the above fact that  $\chi_A(S, T) = \chi_A(T, S)$ ) whence  $\chi_A(S, S) = \chi_A(S, T)$ .  $\square$

If  $M \in \text{rep } A$ , its semi-simplification has as isotypical decomposition

$$M = S_1^{\oplus e_1} \oplus \dots \oplus S_k^{\oplus e_k}$$

with all  $S_i$  non-isomorphic. If  $S_i \in \text{rep}_{\beta_i} A$  we say that the *representation type* of  $M$  (which is determined upto permutation of the  $(e_i, \beta_i)$  terms).

$$\tau(M) = (e_1, \beta_1; \dots; e_k, \beta_k).$$

**Proposition 5.** *If  $A$  is an  $\mathbb{k}$ -curve, the Euler-form*

$$\chi_A(M, N) = \dim_{\mathbb{k}} \text{Hom}_A(M, N) - \dim_{\mathbb{k}} \text{Ext}_A^1(M, N)$$

*depends only on the representation types  $\tau(M)$  and  $\tau(N)$ .*

**Proof.** The proof follows from the foregoing result by observing that  $\chi_A(M, N) = \chi_A(M^{ss}, N^{ss})$ .  $\square$

In particular, there is a Zariski open subset in  $\text{rep}_\alpha A \times \text{rep}_\beta A$  of couples  $(M, N)$  on which the value of  $\chi_A(M, N)$  is constant and equal to the *Euler form*

$$\chi_A(\alpha, \beta) = \text{hom}(\alpha, \beta) - \text{ext}(\alpha, \beta).$$

Clearly, this open set contains all representations of *generic representation type*  $\tau_{\text{gen}}$ , see, for example, [18]. In fact, if  $\text{char}(\mathbb{k}) = 0$  the proof of Proposition 7 implies that  $\chi_A(M, N)$  is constant on  $\text{rep}_\alpha A \times \text{rep}_\beta A$ .

### 6. One quiver to rule them all

If  $A$  is an  $\mathbb{k}$ -curve, we will denote with  $\Sigma_A$  the minimal set of semigroup-generators of the component semigroup  $\text{comp}(A)$ . Observe that  $\Sigma_A$  is well defined as it follows from the Jordan–Hölder decomposition that

$$\Sigma_A = \{ \alpha \in \text{comp}(A) \mid \text{simp}_\alpha A = \text{rep}_\alpha A \}.$$

In particular, it follows from Proposition 5 that  $\chi_A(S, T) = \chi_A(\alpha, \beta)$  for all representations  $S \in \text{rep}_\alpha A$  and  $T \in \text{rep}_\beta A$  if  $\alpha, \beta \in \Sigma_A$ . In all examples known to us,  $\Sigma_A$  is a finite set.

**Definition 7.** If  $A$  is an  $\mathbb{k}$ -curve, we define its *one-quiver*  $Q_1(A)$  to be the quiver on the (possibly infinite) vertex set  $\{v_\alpha \mid \alpha \in \Sigma_A\}$  such that the number of directed arrows (loops) from  $v_\alpha$  to  $v_\beta$  is given by

$$\#\{ \textcircled{\alpha} \longrightarrow \textcircled{\beta} \} = \delta_{\alpha\beta} - \chi_A(\alpha, \beta).$$

The *one-dimension vector*  $\alpha_1(A)$  for  $A$  is the dimension vector for  $Q_1(A)$  having as its  $v_\alpha$ -component the total dimension  $|\alpha|$ .

If  $Q_1(A)$  is a quiver on finitely many vertices  $\{v_1, \dots, v_k\}$  and  $\alpha_1(A) = (n_1, \dots, n_k)$ , we can define the  $\mathbb{k}$ -algebra

$$B(Q_1(A), \alpha_1(A)) = \begin{bmatrix} B_{11} & \dots & B_{1k} \\ \vdots & & \vdots \\ B_{k1} & \dots & B_{kk} \end{bmatrix},$$

where  $B_{ij}$  is the  $n_i \times n_j$  block matrix having all its components equal to the subvector space of the path algebra  $\mathbb{k}Q_1(A)$  spanned by all oriented paths in  $Q_1(A)$  starting at vertex  $v_i$  and ending in  $v_j$ . Observe, that  $B(Q_1(A), \alpha_1(A))$  is Morita-equivalent to the path algebra  $\mathbb{k}Q_1(A)$  and as such is again an  $\mathbb{k}$ -curve.

**Example 2** (Deligne–Mumford curves). Recall from [4, Corollary 7.8] that a *smooth Deligne–Mumford curve* which is generically a scheme, determines (and is determine by) a smooth affine curve  $X$  and an *hereditary order*  $A$  over  $\mathbb{k}[X]$ . As such,  $A$  is an  $\mathbb{k}$ -curve with center  $\mathbb{k}[X]$  and is a subalgebra of  $M_n(\mathbb{k}(X))$  for some  $n$  called the p.i.-degree of  $A$ . If  $\mathfrak{m}_x$  is the maximal ideal of  $\mathbb{k}[X]$  corresponding to the point  $x \in X$  then for all but finitely many exceptions  $\{x_1, \dots, x_l\}$  we have that

$$A/\mathfrak{m}_x A \simeq M_n(\bar{k}).$$

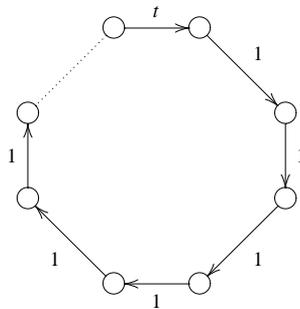
For the exceptional points (the *ramification locus* of  $A$ ) there are finitely many maximal ideals  $\{P_1(i), \dots, P_{k_i}(i)\}$  of  $A$  lying over  $\mathfrak{m}_{x_i}$  and

$$A/P_j(i) \simeq M_{n_j(i)}(\mathbb{k}) \quad \text{with } n_1(i) + \dots + n_{k_i}(i) = n.$$

As a consequence,  $\text{rep}_l A$  for all  $l < n$  consists of finitely many closed orbits each corresponding to a maximal ideal  $P_j(i)$  such that  $A/P_j(i) \simeq M_l(\mathbb{k})$ . Hence, the component semigroup  $\text{comp}(A)$  has generators  $\alpha_j(i)$  for all  $1 \leq i \leq l$  and  $1 \leq j \leq k_i$  and relations for all  $1 \leq i, j \leq l$

$$\alpha_1(i) + \dots + \alpha_{k_i}(i) = \alpha_1(j) + \dots + \alpha_{k_j}(j).$$

From direct calculation or using [16, Proposition 6.1] it follows that one quiver  $Q_1(A)$  is the disjoint union of  $l$  quivers of type  $\tilde{A}_{k_i}$ , that is the  $i$ th component is  $Q_1(A)(i)$  and is the quiver on  $k_i$  vertices



and the corresponding components for the one dimension vector  $\alpha_1(A)$  are  $\alpha_1(A)(i) = (n_1(i), \dots, n_{k_i}(i))$ . Therefore, the associated algebra

$$B(Q_1(A), \alpha_1(A)) = B_1 \oplus \dots \oplus B_l,$$

where  $B_i$  is the block-matrix algebra

$$\begin{bmatrix} M_{n_1(i) \times n_1(i)}(\mathbb{k}[t]) & M_{n_1(i) \times n_2(i)}(\mathbb{k}[t]) & \cdots & M_{n_1(i) \times n_{k_i}(i)}(\mathbb{k}[t]) \\ M_{n_2(i) \times n_1(i)}(t\mathbb{k}[t]) & M_{n_2(i) \times n_2(i)}(\mathbb{k}[t]) & \cdots & M_{n_2(i) \times n_{k_i}(i)}(\mathbb{k}[t]) \\ \vdots & \vdots & & \vdots \\ M_{n_{k_i}(i) \times n_1(i)}(t\mathbb{k}[t]) & M_{n_{k_i}(i) \times n_2(i)}(t\mathbb{k}[t]) & \cdots & M_{n_{k_i}(i) \times n_{k_i}(i)}(\mathbb{k}[t]) \end{bmatrix}.$$

It follows from [20, Chapter 9] or [16, Proposition 6.1] that in a neighborhood of  $x_i$  the  $\mathbb{k}$ -curve  $A$  is étale isomorphic to  $B_i$ .

Elsewhere, we will generalize this example by relating the  $\mathbb{k}$ -curve  $A$  with the algebra  $B(Q_1(A), \alpha_1(A))$  using the formal tubular neighborhood theorem [6, Section 6]. Here, we

will use the *one-quiver-setting*  $(Q_1(A), \alpha_1(A))$  to describe the  $GL_n$ -étale local structure of  $\text{rep}_n A$  in the neighborhood of a semi-simple representation. As this description uses the Luna slice result (see, for example, [16]), we will assume that  $\text{char}(\mathbb{k}) = 0$  in the remainder of this section. We recall the construction of the *local quiver* and refer to [15,16] for details and proofs.

**Definition 8.** Let  $M \in \text{rep}_\alpha A$  be a semi-simple  $A$ -module of representation type  $\tau_M = (e_1, \gamma_1; \dots; e_l, \beta_l)$ , that is

$$M = S_1^{\oplus e_1} \oplus \dots \oplus S_l^{\oplus e_l}$$

with all  $S_i$  non-isomorphic and of dimension vector  $\gamma_i$ .

The *local quiver*  $Q_M$  is the quiver on  $l$  vertices (corresponding to the distinct simple components of  $M$ ) such that the number of directed arrows from  $v_i$  to  $v_j$  is equal to  $\dim_{\mathbb{k}} \text{Ext}_A^1(S_i, S_j)$ .

The *local dimension vector*  $\alpha_M = (e_1, \dots, e_l)$  determined by the multiplicities  $e_i$  of the simple components of  $M$ .

Observe that we know already that the quiver  $Q_M$  only depends on the representation type  $\tau_M$  of  $M$  and not on the choice of the simple components  $S_i$ . The relevance of this *local quiver setting*  $(Q_M, \alpha_M)$  is that it determines the  $GL_n$ -equivariant étale structure of  $\text{rep}_\alpha A$  in a neighborhood of the closed orbit  $\mathcal{O}(M)$  by the results from [15].

As  $n = \sum_i e_i |\gamma_i|$  there is an embedding of  $GL(\alpha_M)$  into  $GL_n$  and with respect to this embedding there is a  $GL_n$ -equivariant étale isomorphism between

- $\text{rep}_\alpha A$  in a neighborhood of  $\mathcal{O}(M)$ , and
- $GL_n \times^{GL(\alpha_M)} \text{rep}_{\alpha_M} Q_M$  is a neighborhood of  $\mathcal{O}(1_n, 0)$ ,

where 0 is the zero representation. We will show that the one-quiver setting  $(Q_1(A), \alpha_1(A))$  contains enough information to describe all these local quiver settings  $(Q_M, \alpha_M)$  whenever  $A$  is an  $\mathbb{k}$ -curve.

$\Sigma_A = \{\beta_i \mid i \in I\}$  is the set of semigroup generators of  $\text{comp}(A)$  (possibly infinite). For any  $\alpha \in \text{comp}(A)$  we can write

$$\alpha = a_1 \beta_1 + \dots + a_k \beta_k, \quad a_i \in \mathbb{N},$$

(possibly in many several ways) with the  $\beta_i \in \Sigma_A$ . If the set of vertices  $V \leftrightarrow \Sigma_A$  is infinite, we can always replace the infinite one-quiver setting  $(Q_1(A), \alpha_1(A))$  by a finite quiver setting  $(\text{supp}(\alpha), \alpha_1(A)|_{\text{supp}(\alpha)})$  where  $\text{supp}(\alpha)$  is the *support* of  $\alpha$ , that is those vertices  $\beta_i \in V \leftrightarrow \Sigma_A$  such that  $a_i \in \mathbb{N}_+$  in a fixed description of  $\alpha$  in terms of the semigroup generators. For notational reasons, we denote this finite quiver setting again by  $(Q_1(A), \alpha_1(A))$ .

**Proposition 6.** *The one-quiver setting  $(Q_1(A), \alpha_1(A))$  contains enough information to determine  $\text{simp}(A)$  the set of all dimension vectors of simple finite-dimensional representations of  $A$ . More precisely, the following are equivalent:*

- (1)  $\alpha \in \text{simp}(A)$ ;
- (2) for all  $(a_1, \dots, a_k) \in \mathbb{N}^k$  such that  $\alpha = a_1\beta_1 + \dots + a_k\beta_k$  we have that  $(a_1, \dots, a_k)$  is a simple dimension vector of  $Q_1(A)$ .

**Proof.** If  $\alpha \in \text{comp}(A)$ , fix a description

$$\alpha = a_1\beta_1 + \dots + a_k\beta_k$$

with  $a_i \in \mathbb{N}_+$  and  $\{\beta_1, \dots, \beta_k\}$  among the semigroup generators of  $\text{comp}(A)$ . This implies that there are points in  $\text{rep}_\alpha A$  corresponding to semi-simple representations

$$M = S_1^{\oplus a_1} \oplus \dots \oplus S_k^{\oplus a_k},$$

where the  $S_i$  are distinct simple representations in  $\text{rep}_{\beta_i} A$ . But then the local quiver setting of  $M$  in  $\text{rep}_\alpha A$ ,  $(Q_M, \alpha_M)$  is just  $(Q_1(A), \epsilon)$  where  $\epsilon = (a_1, \dots, a_k)$ . Because  $\text{rep}_\alpha A$  is irreducible, it follows that  $\alpha \in \text{simp}(A)$  if and only if  $\epsilon$  is the dimension vector of a simple representation of  $Q_1(A)$ . These dimension vectors have been classified in [18] and we recall the result.

Let  $\chi$  be the Euler form of  $Q_1(A)$ , that is  $\chi = (c_{ij})_{i,j} \in M_k(\mathbb{Z})$  with  $c_{ij} = \delta_{ij} - \#\{i \rightarrow j\}$  and let  $\delta_i$  be the dimension vector of a vertex-simple concentrated in vertex  $v_i$ . Then,  $\epsilon$  is the dimension vector of a simple representation of  $Q_A$  if and only if the following conditions are satisfied:

- (1) the support  $\text{supp}(\epsilon)$  is a strongly connected subquiver of  $Q_A$  (there is an oriented cycle in  $\text{supp}(\epsilon)$  containing each pair  $(i, j)$  such that  $\{v_i, v_j\} \subset \text{supp}(\epsilon)$ );
- (2) for all  $v_i \in \text{supp}(\epsilon)$  we have the numerical conditions

$$\chi(\epsilon, \delta_i) \leq 0 \quad \text{and} \quad \chi(\delta_i, \epsilon) \leq 0$$

unless  $\text{supp}(\epsilon)$  in an oriented cycle of type  $\tilde{A}_l$  for some  $l$  in which case all components of  $\epsilon$  must be equal to one.

The statement follows from this.  $\square$

**Proposition 7.** *The one-quiver setting  $(Q_1(A), \alpha_1(A))$  contains enough information to compute the  $\mathbb{k}$ -dimension of  $\text{Ext}_A^1(S, T)$  for all finite-dimensional simple representations  $S$  and  $T$  of  $A$ .*

*If  $S \in \text{rep}_\alpha A$  where  $\alpha = \sum_i a_i \beta_i$  and  $T \in \text{rep}_\beta A$  where  $\beta = \sum_i b_i \beta_i$ , then*

$$\dim_{\mathbb{k}} \text{Ext}_A^1(S, T) = -\chi_{Q_1(A)}(\epsilon, \eta)$$

for  $\epsilon = (a_1, \dots, a_k)$  and  $\eta = (b_1, \dots, b_k)$ .

**Proof.** Let  $S_i$  and  $T_i$  be distinct simples in  $\text{rep}_{\beta_i} A$  and consider the semi-simple representations  $M$ , respectively  $N$ , in  $\text{rep}_{\alpha} A$ , respectively  $\text{rep}_{\beta} A$ :

$$M = S_1^{\oplus a_1} \oplus \dots \oplus S_k^{\oplus a_k} \quad \text{and} \quad N = T_1^{\oplus b_1} \oplus \dots \oplus T_k^{\oplus b_k}.$$

By the foregoing proposition, we have complete information on the local quiver setting of  $M \oplus N$  in  $\text{rep}_{\alpha+\beta} A$  from  $(Q_1(A), \alpha_1(A))$ . By assumption on  $\alpha$  and  $\beta$  there is a Zariski open subset of simples  $S' \in \text{rep}_{\alpha} A$  and simples  $T' \in \text{rep}_{\beta} A$  such that  $S' \oplus T'$  lies in a neighborhood of  $M \oplus N$ .

It follows from [18] that one can reconstruct the local quiver setting of  $S' \oplus T'$  from that of  $M \oplus N$ . This local quiver has two vertices  $\{v_1, v_2\}$  with  $-\chi_Q(\eta, \epsilon)$  arrows from  $v_1$  to  $v_2$  and  $-\chi_Q(\epsilon, \eta)$  arrows from  $v_2$  to  $v_1$ . In  $v_1$  there are  $1 - \chi_Q(\epsilon, \epsilon)$  loops and in  $v_2$  there are  $1 - \chi_Q(\eta, \eta)$  loops. The dimension vector is  $(1, 1)$ . From this we deduce that

$$\dim_{\mathbb{k}} \text{Ext}_A^1(S', T') = -\chi(\epsilon, \eta)$$

but we have seen before that the extension-dimension only depends on the representation type and not on the choice of simples, hence this number is also equal to  $\dim_{\mathbb{k}} \text{Ext}_A^1(S, T)$ .  $\square$

**Theorem 4.** *The one-quiver setting  $(Q_1(A), \alpha_1(A))$  contains enough information to construct the local quiver setting  $(Q_M, \alpha_M)$  for every semi-simple representation*

$$M = S_1^{\oplus e_1} \oplus \dots \oplus S_l^{\oplus e_l}$$

of  $A$ .

**Proof.** This is a direct consequence of the foregoing two propositions. To begin, we can determine the possible dimension vectors  $\alpha_i$  of the simple components  $S_i$ . Write  $\alpha_i = \sum_{j=1}^k a_j(i)\beta_j$  then  $\epsilon_i = (a_1(i), \dots, a_k(i))$  must be the dimension vector of a simple representation of the associated quiver  $Q_1(A)$ . Moreover, by the previous theorem we know that

$$\dim_{\mathbb{k}} \text{Ext}_A^1(S_i, S_j) = \delta_{ij} - \chi(\epsilon_i, \epsilon_j)$$

and hence we have full knowledge of the local quiver  $Q_M$ .  $\square$

### 7. The one-quiver for $\pi_1(\mathcal{S}_G)$

In this section we will construct the one-quiver setting for the fundamental algebra  $\pi_1(\mathcal{S}_G)$  of a graph  $\mathcal{S}_G$  of separable (that is, semi-simple)  $\mathbb{k}$ -algebras (recall that we fix a choice of maximal subtree). As an intermediary step we will construct a finite quiver  $Q_0(\mathcal{S}_G)$  such that finite-dimensional representations of  $\pi_1(\mathcal{S}_G)$  correspond to certain finite-dimensional representations of the path algebra  $\mathbb{k}Q_0(\mathcal{S}_G)$ .

We have decomposition of the vertex- and edge-algebras

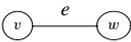
$$S_v = M_{d_v(1)}(\mathbb{k}) \oplus \cdots \oplus M_{d_v(n_v)}(\mathbb{k}), \quad \text{respectively}$$

$$S_e = M_{d_e(1)}(\mathbb{k}) \oplus \cdots \oplus M_{d_e(n_e)}(\mathbb{k}).$$

The embeddings  $S_e \hookrightarrow S_v$  are depicted via Bratelli-diagrams or, equivalently, by natural numbers  $a_{ij}^{(ev)}$  for  $1 \leq i \leq n_e$  and  $1 \leq j \leq n_v$  satisfying the numerical restrictions

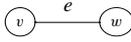
$$d_v(j) = \sum_{i=1}^{n_e} a_{ij}^{(ev)} d_e(i) \quad \text{for all } 1 \leq j \leq n_v \text{ and all } v \in V \text{ and } e \in E.$$

Remark that these numbers give the *restriction data*, that is, the multiplicities of the simple components of  $S_e$  occurring in the restriction  $V_j^{(v)} \downarrow_{S_e}$  for the simple components  $V_j^{(v)}$  of  $S_v$ . From these decompositions and Schur’s lemma it follows that for any edge

 in the graph  $G$  we have

$$\text{Hom}_{S_e}(V_i^{(v)}, V_j^{(w)}) = \sum_{k=1}^{n_e} a_{ki}^{(ev)} a_{kj}^{(ew)} = n_{ij}^{(e)}.$$

**Definition 9.** For a graph  $\mathcal{S}_G$  of separable  $\mathbb{k}$ -algebras we define a quiver  $Q_0(\mathcal{S}_G)$  as follows:

- vertices: for any vertex  $v \in V$  of  $G$  take  $n_v$  vertices  $\{\mu_1^{(v)}, \dots, \mu_{n_v}^{(v)}\}$ ;
- arrows: fix an orientation  $\vec{G}$  on all of the edges of  $G$ . For any edge  in  $G$  we add for each  $1 \leq i \leq n_v$  and each  $1 \leq j \leq n_w$  precisely  $n_{ij}^{(e)}$  arrows between the vertices  $\mu_i^{(v)}$  and  $\mu_j^{(w)}$  oriented in the same way as the edge  $e$  in  $\vec{G}$ .

We call  $Q_0(\mathcal{S}_G)$  the *Zariski quiver* of the graph of separable algebras  $\mathcal{S}_G$ .

The *representation space*  $\text{rep}_\alpha Q_0(\mathcal{S}_G)$  is the affine  $\mathbb{k}$ -space

$$\text{rep}_\alpha Q_0(\mathcal{S}_G) = \bigoplus_{\substack{v \\ \xrightarrow{e} \\ w}} \bigoplus_{i=1}^{n_v} \bigoplus_{j=1}^{n_w} M_{\alpha_j^{(w)} \times \alpha_i^{(v)}}(\mathbb{k})$$

and two  $\alpha$ -dimensional representations are said to be *isomorphic* if they are conjugated via the natural base-change action of  $GL(\alpha) = \prod_{v \in V} \prod_{i=1}^{n_v} GL(\alpha_i^{(v)})$ .

A dimension vector  $\alpha = (\alpha_i^{(v)}: v \in V, 1 \leq i \leq n_v)$  for  $Q_0(\mathcal{S}_G)$  is said to be an  $n$ -dimension vector if the following numerical conditions are satisfied:

$$\sum_{i=1}^{n_v} d_v(i) \alpha_i^{(v)} = n$$

for all  $v \in V$ . In addition, we require for every edge  $\textcircled{v} \xrightarrow{e} \textcircled{w}$  and all  $1 \leq j \leq n_e$  that the following numerical conditions are satisfied:

$$\sum_{i=1}^{n_v} \alpha_i^{(v)} a_{ji}^{(ev)} = \sum_{k=1}^{n_w} \alpha_k^{(w)} a_{jk}^{(ew)}.$$

For any edge  $\textcircled{v} \xrightarrow{e} \textcircled{w}$  we denote by  $Q_e$  the bipartite subquiver of  $Q_0(\mathcal{S}_G)$  on the vertices  $\{\mu_1^{(v)}, \dots, \mu_{n_v}^{(v)}\}, \{\mu_1^{(w)}, \dots, \mu_{n_w}^{(w)}\}$  and the  $n_{ij}^{(e)}$  arrows between  $\mu_i^{(v)}$  and  $\mu_j^{(w)}$  determined by the embeddings  $S_e \hookrightarrow S_v$  and  $S_e \hookrightarrow S_w$ .

**Definition 10.** Let  $\alpha$  be an  $n$ -dimension vector,  $M \in \text{rep}_\alpha Q_0(\mathcal{S}_G)$  and  $e \in E$ .

- $M$  is said to be  $e$ -semistable iff for all  $Q_e$ -subrepresentations  $N$  of  $M|_{Q_e}$  of dimension vector  $(u_1, \dots, u_{n_v}, u'_1, \dots, u'_{n_w})$  we have

$$\sum_{i=1}^{n_w} u'_i d_w(i) \geq \sum_{i=1}^{n_v} u_i d_v(i).$$

- $M$  is said to be  $e$ -stable iff for all proper  $Q_e$ -subrepresentations  $N$  of  $M|_{Q_e}$  of dimension vector  $(u_1, \dots, u_{n_v}, u'_1, \dots, u'_{n_w})$  we have

$$\sum_{i=1}^{n_w} u'_i d_w(i) > \sum_{i=1}^{n_v} u_i d_v(i).$$

- $M$  is said to be  $\mathcal{S}_G$ -semistable (respectively  $\mathcal{S}_G$ -stable) iff  $M$  is  $e$ -semistable (respectively  $e$ -stable) for all edges  $e \in E$ .

The relevance of the quiver  $Q_0(\mathcal{S}_G)$  and the introduced terminology is contained in the next result. It may be useful to consult [1, Section 6] for a work-out of the underlying idea in a special case.

**Proposition 8.** Every  $n$ -dimensional representation  $\pi_1(\mathcal{S}_G) \xrightarrow{\phi} M_n(\mathbb{K})$  determines (and is determined by) an  $\mathcal{S}_G$ -semistable representation  $M_\phi \in \text{rep}_\alpha Q_0(\mathcal{S}_G)$  for some  $n$ -dimension vector  $\alpha$ . Moreover, if  $\phi$  and  $\phi'$  are isomorphic representations of  $\pi_1(\mathcal{S}_G)$ , then  $M_\phi$  and  $M_{\phi'}$  are isomorphic as quiver representations.

**Proof.** Let  $N = \mathbb{k}_\phi^n$  be the  $n$ -dimensional module determined by  $\phi$ . For each vertex  $v \in V$  we have a decomposition by restricting  $N$  to the separable subalgebra  $S_v$

$$N \downarrow_{S_v} \simeq V_{1,v}^{\oplus \alpha_1^{(v)}} \oplus \cdots \oplus V_{n_v,v}^{\oplus \alpha_{n_v}^{(v)}},$$

where the  $V_{i,v}$  are the distinct simple modules of  $S_v$  of dimension  $d_v(i)$ . Choose an  $\mathbb{k}$ -basis  $\mathcal{B}_v$  of  $N \downarrow_{S_v}$  compatible with this decomposition. These decompositions determine an  $n$ -dimension vector  $\alpha$ . In fact, the numerical conditions required for  $\alpha$  to be an  $n$ -dimension vector encode that  $(N \downarrow_{S_v}) \downarrow_{S_e} \simeq (N \downarrow_{S_w}) \downarrow_{S_e}$  for every edge  $v \xrightarrow{e} w$ . The embeddings  $S_e \xrightarrow{\alpha} S_v$  and  $S_e \xrightarrow{\beta} S_w$  determine two  $n$ -dimensional  $S_e$ -representations

$$(N \downarrow_{S_v}) \downarrow_{S_e}^\alpha \quad \text{and} \quad (N \downarrow_{S_w}) \downarrow_{S_e}^\beta$$

which, by construction of  $\pi_1(\mathcal{S}_G)$  are isomorphic. That is, the basechange map  $\mathcal{B}_v \xrightarrow{\psi_{vw}} \mathcal{B}_w$  is an invertible element of

$$\text{Hom}_{S_e}(N \downarrow_{S_v}, N \downarrow_{S_w}) = \bigoplus_{i=1}^{n_v} \bigoplus_{j=1}^{n_w} M_{\alpha_j^{(w)} \times \alpha_i^{(v)}}(\text{Hom}_{S_e}(V_{i,v}, V_{j,w}))$$

and hence  $\psi_{vw}$  determines a representation of the bipartite quiver  $Q_e$  of dimension vector  $\alpha|_{Q_e}$ . Repeating this for all edges  $e \in E$  we obtain a representation  $M_\phi \in \text{rep}_\alpha Q_0(\mathcal{S}_G)$ . Invertibility of the map  $\psi_{vw}$  is equivalent to  $M_\phi$  being  $e$ -semistable, so  $M_\phi$  is  $\mathcal{S}_G$ -semistable. Isomorphic representations  $\phi$  and  $\phi'$  determine isomorphic vertex-decompositions whence, by Schur’s lemma, bases which are transferred into each other via an element of  $GL(\alpha)$  and hence the quiver representations  $M_\phi$  and  $M_{\phi'}$  are isomorphic. From the construction of the fundamental algebra  $\pi_1(\mathcal{S}_G)$  it follows that one can reverse this procedure to construct on  $n$ -dimensional representation of  $\pi_1(\mathcal{S}_G)$  from a  $\mathcal{S}_G$ -stable representation  $M \in \text{rep}_\alpha Q_0(\mathcal{S}_G)$  for some  $n$ -dimension vector  $\alpha$ .  $\square$

Under this correspondence simple  $\pi_1(\mathcal{S}_G)$ -representations correspond to  $\mathcal{S}_G$ -stable representations. If  $\alpha$  is an  $n$ -dimension vector such that  $\text{rep}_\alpha Q_0(\mathcal{S}_G)$  contains  $\mathcal{S}_G$ -stable representations (which then form a Zariski open subset), then  $\alpha$  is a *Schur root* of  $Q_0(\mathcal{S}_G)$  and consequently the dimension of the classifying variety is equal to  $1 - \chi_0(\alpha, \alpha)$  where  $\chi$  is the *Euler form* of the quiver  $Q_0(\mathcal{S}_G)$ . For this result and related material on Schur roots we refer to [23].

**Proposition 9.** *Isomorphism classes of simple  $n$ -dimensional representations of  $\pi_1(\mathcal{S}_G)$  are parametrized by the points of a smooth quasi-affine variety (possibly with several irreducible components):*

$$\text{isosimp}_n \pi_1(\mathcal{S}_G) = \bigsqcup_{\alpha} \text{isosimp}_\alpha \pi_1(\mathcal{S}_G),$$

where  $\alpha$  runs over all  $n$ -dimension vectors such that  $\text{rep}_\alpha Q_0(\mathcal{S}_G)$  contains  $\mathcal{S}_G$ -stable representations. These components have dimensions

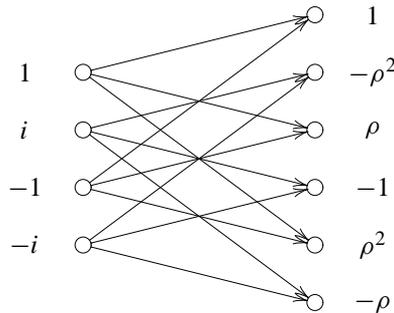
$$\dim \text{isosimp}_\alpha \pi_1(\mathcal{S}_G) = 1 - \chi_0(\alpha, \alpha),$$

where  $\chi_0$  is the Euler form of the quiver  $Q_0(\mathcal{S}_G)$ .

As an example consider the modular group  $SL_2(\mathbb{Z})$  which is the amalgamated product  $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ , see, for example, [7, I, Section 7]. If  $\text{char}(\mathbb{k}) \neq 2, 3$  the group-algebra  $\mathbb{k}SL_2(\mathbb{Z})$  is the fundamental algebra of the graph of separable  $\mathbb{k}$ -algebras

$$\textcircled{v} \xrightarrow{e} \textcircled{w} \quad \text{with } S_v = \mathbb{k}\mathbb{Z}_4, \quad S_w = \mathbb{k}\mathbb{Z}_6, \quad S_e = \mathbb{k}\mathbb{Z}_2.$$

As all simples are one-dimensional (determined by their eigenvalue), it is easy to verify that the zero quiver  $Q_0(\mathbb{k}SL_2(\mathbb{Z}))$  has the following form:



( $\rho$  is a primitive 3rd root of unity) which is the disjoint union of two copies of the quiver associated to  $PSL_2(\mathbb{Z})$  in [27].

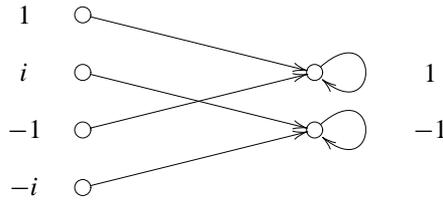
The congruence subgroup

$$\Gamma_0(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \text{ with } c \text{ even} \right\}$$

is the fundamental group of the graph of finite groups

$$\textcircled{v} \xrightarrow{e} \textcircled{w} \textcircled{f}, \quad G_w = G_e = G_f = \mathbb{Z}_2, \quad G_v = \mathbb{Z}_4.$$

If  $\text{char}(\mathbb{k}) \neq 2$ , the group algebra  $\mathbb{k}\Gamma_0(2)$  is the fundamental algebra of a graph of separable  $\mathbb{k}$ -algebras and the zero quiver  $Q_0(\mathbb{k}\Gamma_0(2))$  has the following form:



**Definition 11.** For a graph  $\mathcal{S}_G$  of separable  $\mathbb{k}$ -algebras we define a quiver  $Q_1(\mathcal{S}_G)$  as follows:

- *Vertices.* Let  $\{\alpha_1, \dots, \alpha_k\}$  be the minimal set of generators for the sub-semigroup of dimension vectors  $\alpha$  of  $Q_0(\mathcal{S}_G)$  which are  $n$ -dimension vectors for some  $n \in \mathbb{N}$  and such that  $\text{rep}_\alpha Q_0(\mathcal{S}_G)$  contains  $\mathcal{S}_G$ -semistable representations. The vertices  $\{\nu_1, \dots, \nu_k\}$  are in one-to-one correspondence with these generators  $\{\alpha_1, \dots, \alpha_k\}$ .
- *Arrows.* The number of directed arrows in  $Q_1(\mathcal{S}_G)$  from  $\nu_i$  to  $\nu_j$

$$\#\{(i \rightarrow j)\} = \delta_{ij} - \chi_0(\alpha_i, \alpha_j),$$

where  $\chi_0$  is the Euler-form of the Zariski quiver  $Q_0(\mathcal{S}_G)$ .

We call  $Q_1(\mathcal{S}_G)$  the one-quiver of the graph of separable algebras  $\mathcal{S}_G$ .

The one-quiver  $Q_1(\mathcal{S}_G)$  allows us to determine the components  $\text{rep}_\alpha \pi_1(\mathcal{S}_G)$  which contain (a Zariski open subset of) simple representations. Remark that the description of Schur roots is a lot harder than that of dimension vectors of simple representations.

**Proposition 10** ( $\text{char}(\mathbb{k}) = 0$ ). *If  $\alpha = c_1\alpha_1 + \dots + c_k\alpha_k \in \text{comp } \pi_1(\mathcal{S}_G)$  then the component  $\text{rep}_\alpha \pi_1(\mathcal{S}_G)$  contains simple representations if and only if:*

- $\chi_1(\gamma, \epsilon_i) \leq 0$  and  $\chi_1(\epsilon_i, \gamma) \leq 0$   
for all  $1 \leq i \leq k$  where  $\gamma = (c_1, \dots, c_k)$  and  $\epsilon_i = (\delta_{1i}, \dots, \delta_{ki})$  and where  $\chi_1$  is the Euler form of the one quiver  $Q_1(\mathcal{S}_G)$ ;
- $\text{supp}(\gamma)$  is a strongly connected subquiver of  $\pi_1(\mathcal{S}_G)$  and if  $\text{supp}(\gamma)$  is of extended Dynkin type  $\tilde{A}_l$  then all non-zero components of  $\gamma$  must be equal to one.

**Proof.** Follows from the proof of Proposition 6.  $\square$

If  $\text{char}(\mathbb{k}) = 0$  one can apply Luna slice machinery to construct a Zariski open subset of all simple representations in  $\text{rep}_\alpha \pi_1(\mathcal{S}_G)$  from the knowledge of low-dimensional simples. For example, suppose we have found simple representations

$$S_i \in \text{rep}_{\alpha_i} \pi_1(\mathcal{S}_G) \quad \text{for all } 1 \leq i \leq k$$

and consider the point  $M$  in the affine space  $\text{rep}_\alpha Q_0(\mathcal{S}_G)$  determined by the semi-simple representation of  $\pi_1(\mathcal{S}_G)$

$$M = S_1^{\oplus c_1} \oplus \dots \oplus S_k^{\oplus c_k}$$

then the normal space to the  $GL(\alpha)$ -orbit  $\mathcal{O}(M)$  is isomorphic to  $\text{Ext}_{\pi_1(\mathcal{S}_G)}^1(M, M)$  which we have seen can be identified to  $\text{rep}_\gamma Q_1(\mathcal{S}_G)$ .

**Proposition 11.** *Let  $\alpha = c_1\alpha_1 + \dots + c_k\alpha_k$  be a component such that  $\text{rep}_\alpha\pi_1(\mathcal{S}_G)$  contains simple representations. In the affine space  $\text{rep}_\alpha Q_0(\mathcal{S}_G)$  identify the normal space to the orbit  $\mathcal{O}(M)$  of the semi-simple representation  $M$  (as above) with*

$$N_M = \{M + V \mid V \in \text{rep}_\gamma Q_1(\mathcal{S}_G)\},$$

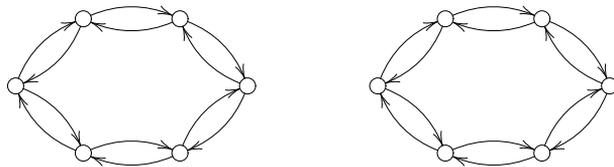
where  $\gamma = (c_1, \dots, c_k)$ . Then,  $GL(\alpha).N_M$  contains a Zariski open subset of all  $\alpha$ -dimensional simple representations of  $\pi_1(\mathcal{S}_G)$ .

**Proof.** This is a special case of the Luna slice result applied to the local quiver setting. In fact, one can generalize this result to other known semi-simple representations  $N$  of  $\pi_1(\mathcal{S}_G)$  but then one has to replace  $Q_1(\mathcal{S}_G)$  by the local quiver  $Q_N$  of  $N$ .  $\square$

In the  $SL_2(\mathbb{Z})$  example,  $\text{comp}(\mathbb{k}SL_2(\mathbb{Z}))$  is generated by the 12 components of two-dimensional representations of  $Q_0(\mathbb{k}SL_2(\mathbb{Z}))$

$$v_{ij} = (\delta_{1i}, \dots, \delta_{4i}; \delta_{1j}, \dots, \delta_{6j}), \quad 1 \leq i \leq 4, 1 \leq j \leq 6,$$

for which  $i$  and  $j$  are both even or both odd. From this the structure of the one quiver  $Q_1(\mathbb{k}SL_2(\mathbb{Z}))$  (corresponding to the 12 one-dimensional simples of  $\mathbb{k}SL_2(\mathbb{Z})$ ) can be verified to be



Here, the vertices of the first component correspond (in cyclic order) to  $v_{11}, v_{33}, v_{15}, v_{31}, v_{13}, v_{35}$  and those of the second component (in cyclic order) to  $v_{22}, v_{44}, v_{26}, v_{42}, v_{24}, v_{46}$ . Applications to the representation theory of the modular group  $SL_2(\mathbb{Z})$ , its central extension  $B_3$  (the third braid group) as well as other arithmetical groups will be given in [2].

In the  $\Gamma_0(2)$  example,  $\text{comp}(\mathbb{k}\Gamma_0(2))$  is generated by the 4-dimension vectors

$$(1, 0, 0, 0; 1, 0), \quad (0, 0, 1, 0; 1, 0), \quad (0, 1, 0, 0; 0, 1), \quad (0, 0, 0, 1; 0, 1)$$

and one verifies that the one-quiver  $Q_1(\mathbb{k}\Gamma_0(2))$  has the following form:



**Appendix. The component coalgebra  $\text{coco}(A)$**

Over an algebraically closed field  $\mathbb{k}$  we have seen that the component semigroup and Euler form contain useful information on the finite-dimensional representations of an  $\mathbb{k}$ -curve. Clearly, one can repeat all arguments verbatim for an arbitrary  $k$  by restricting to those components which contain  $k$ -rational points. However, this sub-semigroup  $\text{comp}(A)$  of  $\text{comp}(A \otimes \mathbb{k})$  is usually too small to be of interest.

**Example A.1.** Let  $k \subset L$  be a finite separable field extension of dimension  $k$ . As  $L$  is a simple algebra, all its finite-dimensional representations are of the form  $L^{\oplus a}$  and hence only components of  $\text{rep}_n L$  containing  $k$ -rational points exist when  $k|n$ . Over the algebraic closure we have

$$L \otimes \mathbb{k} = \underbrace{\mathbb{k} \times \cdots \times \mathbb{k}}_k$$

whence  $\text{comp}(L \otimes \mathbb{k}) \simeq \mathbb{N}^k$  generated by the factors of  $L \otimes \mathbb{k}$ . We have  $\text{comp}(A) \subset \text{comp}(A \otimes \mathbb{k})$  sending the generator  $k$  to  $(1, \dots, 1)$ .

We recall some standard facts from [10, Chapter 1] on unramified commutative algebras over an arbitrary basefield  $k$ . A commutative affine  $k$ -algebra  $C$  is said to be *unramified* whenever

$$C \otimes \mathbb{k} \simeq \mathbb{k} \times \cdots \times \mathbb{k}.$$

It is well known that all unramified  $k$ -algebras are of the form

$$C \simeq L_1 \times \cdots \times L_k,$$

where each  $L_i$  is a finite-dimensional separable field extension of  $k$ . From this it follows that subalgebras, tensorproducts and epimorphic images of unramified  $k$ -algebras are again unramified. As a consequence, an affine commutative  $k$ -algebra  $C$  has a unique *maximal unramified  $k$ -subalgebra*  $\pi_0(C)$ . In case  $C = k[X]$  is the coordinate algebra of an affine  $k$ -scheme  $X$ , the algebra  $\pi_0(C)$  contains all information about the connected components of  $X$ . Recall that an affine  $k$ -scheme  $X$  (or its coordinate algebra  $k[X]$ ) is said to be *connected* if  $k[X]$  contains no non-trivial idempotents and is called *geometrically connected* if  $k[X] \otimes \mathbb{k}$  is connected. We summarize [10, I.7] in

**Proposition A.1.** *For an affine  $k$ -scheme  $X$  we have:*

- (1)  $X$  is connected iff  $\pi_0(k[X])$  is a field;
- (2)  $X$  is geometrically connected iff  $\pi_0(k[X]) = k$ ;
- (3) if  $X$  is connected and has an  $k$ -rational point, then  $X$  is geometrically connected;
- (4) if  $\pi_0(k[X]) = L_1 \times \cdots \times L_k$  with all  $L_i$  separable field extensions of  $k$ , then  $X$  has exactly  $k$  connected components;

- (5) if  $Y$  is an affine  $k$ -scheme and  $X \rightarrow Y$  a morphism, then  $\pi_0(k[Y]) \xrightarrow{\pi}_0(k[X])$  is an  $k$ -algebra morphism;
- (6) if  $Y$  is an affine  $k$ -scheme, then the natural map

$$\pi_0(k[X]) \otimes \pi_0(k[Y]) \xrightarrow{\pi}_0(k[X] \otimes k[Y]) = \pi_0(k[X \times Y])$$

is an  $k$ -algebra isomorphism.

**Definition A.1.** For  $A$  an  $k$ -curve consider the sum-maps

$$\text{rep}_n A \times \text{rep}_m A \rightarrow \text{rep}_{m+n} A$$

which determine  $k$ -algebra morphisms

$$\Delta_{m,n} : \pi_0(k[\text{rep}_{m+n} A]) \rightarrow \pi_0(k[\text{rep}_n A]) \otimes \pi_0(k[\text{rep}_m A]).$$

Denote  $\pi_0(n) = \pi_0(k[\text{rep}_n A])$  and consider the graded  $k$ -vectorspace

$$\text{coco}(A) = \pi_0(0) \oplus \pi_0(1) \oplus \pi_0(2) \oplus \dots$$

Define a coalgebra structure by taking as the *comultiplication* map

$$\begin{aligned} \text{coco}(A) &\xrightarrow{\Delta} \text{coco}(A) \otimes \text{coco}(A), \\ \sum_{m+n=N} \Delta_{m,n} : \pi_0(N) &\rightarrow \sum_{n+m=N} \pi_0(n) \otimes \pi_0(m) \end{aligned}$$

and as the *counit*  $\text{coco}(A) \xrightarrow{\epsilon} \pi_0(0) = k$ . We call  $(\text{coco}(A), \Delta, \epsilon)$  the *component coalgebra* of the  $k$ -curve  $A$ .

It follows from Proposition A.1 that  $\text{coco}(A)$  is, in fact, a *mock bialgebra*, that is a bialgebra without a unit-map.

Recall that if  $G$  is a finite semigroup, its *function bialgebra*  $\text{func}(G)$  is the space of all  $k$ -valued functions on  $G$  with pointwise multiplication and co-multiplication induced by

$$\Delta(x_g) = \sum_{g' \cdot g'' = g} x_{g'} \otimes x_{g''},$$

where  $x_h$  is the function mapping  $h \mapsto 1$  and all other  $h' \in G$  to zero. If  $G$  is no longer finite,  $\text{func}(G)$  is still a mock bialgebra.

**Proposition A.2.** If  $A$  is an  $k$ -curve, then there is an isomorphism of  $\mathbb{k}$ -coalgebras

$$\text{coco}(A) \otimes \mathbb{k} \simeq \text{func}(\text{comp}(A \otimes \mathbb{k}))$$

and hence  $coco(A)$  contains enough information to reconstruct the component semigroup  $comp(A \otimes \mathbb{k})$ .

Alternatively, the Galois group  $Gal(\mathbb{k}/k)$  acts on  $A \otimes \mathbb{k}$  and hence on  $comp(A \otimes \mathbb{k})$  and the function coalgebra. The component coalgebra  $coco(A)$  can be obtained by Galois descent

$$coco(A) = func(comp(A \otimes \mathbb{k}))^{Gal(\mathbb{k}/k)}.$$

**Proof.** Follows immediately from the properties summarized in Proposition A.1, the properties of the sum-morphisms and the definition of the component semigroup.  $\square$

Clearly,  $coco(A)$  is a co-commutative coalgebra. We can bring in the Euler-form of  $A \otimes \mathbb{k}$  to encode also this information into a braided co-commutative coalgebra  $coco_\chi(A)$ . Let  $func_\chi(comp(A \otimes \mathbb{k}))$  be the braided function semi-group algebra with modified comultiplication

$$\Delta(x_\gamma) = \sum_{\alpha+\beta=\gamma} q^{-\chi(\alpha,\beta)} x_\alpha \otimes x_\beta,$$

where  $\chi$  is the Euler-form  $\chi_{A \otimes \mathbb{k}}$ . We then define the *twisted component coalgebra* of the  $k$ -curve  $A$  to be the fixed coalgebra

$$coco_\chi(A) = func_\chi(comp(A \otimes \mathbb{k}))^{Gal(\mathbb{k}/k)}.$$

The twisted component coalgebra  $coco_\chi(A)$  contains enough information to reconstruct the one-quiver of  $A \otimes \mathbb{k}$  and should be an interesting object to work out in explicit examples.

Our main motivation for considering  $k$ -curves over non-algebraically closed fields  $k$  comes from the (difficult) birational classification project of  $\mathbb{k}$ -curves. Possibly, this problem may be approached inductively by investigating function fields  $\mathbb{k}(C)$  of smooth affine curves  $C$  as subfields of universal localizations of a  $\mathbb{k}$ -curve. This localization can then be viewed as a  $\mathbb{k}(C)$ -curve over the obviously non-algebraically closed field  $\mathbb{k}(C)$  and the twisted component coalgebra may then provide useful information to iterate the process.

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