Indag. Mathem., N.S., 19 (4), 493–506 December, 2008

Sheaves and sheafification on Q-sites ☆

by Judit Mendoza Aguilar [a], M. Victoria Reyes Sánchez [a] and Alain Verschoren [b]

[a] *Departamento de Matemática Fundamental, Universidad de La Laguna, Tenerife, Spain*
[b] *Department of Mathematics and Computer Science, University of Antwerp, Belgium*

Communicated by Prof. H.W. Broer

ABSTRACT

We introduce noncommutative sites over a quantale, the so-called Q-sites, and define appropriate presheaves and sheaves over these. We show how most of the technical machinery which allows to construct sheaves associated to arbitrary presheaves in the commutative cases transposes to this setting. This allows us to define and study sheafification in this new, noncommutative context.

INTRODUCTION

Topological spaces were generalized in several ways during the last decades, both from the geometric and the algebraic point of view. The best algebraic approach is probably that of locales (cf. [1], e.g.), as this also takes into account applications of topology within the strict domain of logic. Of course, topology is essentially a commutative matter: open sets and their generalizations behave decently with respect to operations like unions and intersections, which are inherently of commutative nature. But the world of mathematics does not stop at the commutative level, quite the contrary. In pure mathematics, in physics and (even) within logic it is clear that non-commutative behavior should be taken into account as a much more natural habitat. Within this framework, it became clear in the early eighties that the

MSC: 06F07, 14A22, 54B40

Key words and phrases: Sheaves, Quantales, Noncommutative topologies

☆ J. Mendoza Aguilar was supported by DGUI, Consejería de Educación – Gobierno de Canarias.

E-mails: jmendoza@ull.es (J. Mendoza Aguilar), mvreyes@ull.es (M.V. Reyes Sánchez), alain.verschoren@ua.ac.be (A. Verschoren).

notion of quantale provided a natural equivalent (or generalization) of locales in the noncommutative set-up, as well as providing a decent basis for non-commutative logic.

In this note, we will use quantales as an overall framework, which allows us to introduce and study a notion of *noncommutative* "site". The so-called *Q-sites* we use in this paper are a practical, natural and elegant setting in which to develop the sheafification techniques referred in the title. Let us stress from the start that although sheaves have been introduced on quantales in the literature, cf. [2] for example, our point of view is definitely different from previous approaches: we do not adapt the notion of a sheaf, inspired by requirements stemming from logic, e.g., to a structure over a quantale or locale but rather inspired our constructions directly by the roots of sheaf theory and its applications. More precisely we return to the pure concept of sheaves, as an efficient tool for providing local-global results and a fundamental ingredient in the construction of global concepts from local data.

In this way this paper should be viewed as a complement to [4], where the authors construct structure sheaves over non-commutative topologies by using localization techniques with respect to torsion theories (or their equivalent formulations, we refer to [5,7], for example). Actually, in this note we show that the abstract sheaves constructed in [4], may be constructed as well through the more down-to-earth local to global techniques, one tends to be familiar with in the traditional, commutative set-up. It should be clear, however, that the underlying non-commutativity of our "topology" does not permit for a "just as in the commutative case" approach, although the constructions themselves are, of course, directly inspired by the analogous ones in the usual, commutative situation. Actually, that the look-and-feel of the commutative situation has an equivalent in the non-commutative one even strengthened our belief in the applicability of our non-commutative approach.

The present note is organized as follows. We start by introducing and briefly studying so-called Q-presheaves, a natural generalization of traditional (commutative) presheaves on a non-commutative site Q. We show in detail, mimicking techniques usually applied in the commutative setting (cf. [7] and [6], for example), how to functorially construct for any such Q-presheaf P a suitable new Q-presheaf LP, on which our notions of sheaf and Q-sheafification will be based. Actually, in order to properly define the Q-sheafification functor, we have to prove two fundamental results which state that in our context for any Q-presheaf P the Q-presheaf LP is separated and that LP is a Q-sheaf, whenever P is already separated itself. In this way, just as might be expected from the usual, commutative situation, the Q-sheafification functor aimed at is just the composition $L \circ L$.

These results and the sheafification functor allow us to show that the category of Q-sheaves is a reflective (Grothendieck) subcategory of the corresponding category of Q-presheaves. Moreover, our constructions provide an alternative approach to the "abstract" sheaves constructed in [4] though non-commutative localization techniques. The abstract approach allows to prove that these sheaves behave nicely, in particular since this construction succeeds in associating in a decent functorial way sheaves to presheaves, just as in the commutative case. However the resulting sheaves lacked any concrete way to be used in practical applications:

our construction was more "existential" than "constructive". The present paper aims to remedy this problem, providing a much more intuitive interpretation of our constructions.

1. Q-SITES

The notion of a Q-site is based on the philosophy of quantales, whose main features we summarize in what follows; we refer to [2] for full details.

Definition 1.1. A *quantale* is a complete lattice $(\mathcal{Q}, \leqslant)$ endowed with an additional binary operation

$$\& : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q},$$

called *multiplication*, satisfying the following axioms:

(Q1) $U \& (V \& W) = (U \& V) \& W$;
(Q2) $U \& 1 = U$;
(Q3) $U \& U = U$ (*idempotent property*);
(Q4) $U \& (\bigvee_{i \in I} V_i) = \bigvee_{i \in I}(U \& V_i)$;
(Q5) $(\bigvee_{i \in I} U_i) \& V = \bigvee_{i \in I}(U_i \& V)$,

where I is a set, U, V, W, U_i, V_i are elements of $\mathcal{Q}$ and $1 = \vee \mathcal{Q}$ is the greatest element of $\mathcal{Q}$.

One easily verifies the following result:

Proposition 1.2. *Let $(\mathcal{Q}, \leqslant, \&)$ be a quantale. For all $U, V, W \in \mathcal{Q}$ the following relations hold:*

(1) *if $V \leqslant W$, then $U \& V \leqslant U \& W$;*
(2) *if $U \leqslant V$, then $U \& W \leqslant V \& W$;*
(3) $U \& 0 = 0 = 0 \& U$;
(4) $U \& V \leqslant U$;
(5) *if $U \leqslant W$ and $V \leqslant W$, then $U \& V \leqslant W$;*
(6) *if $U \leqslant V$, then $U = U \& V$;*
(7) $U \& V \& W = U \& W \& V$.

Throughout, $(\mathcal{Q}, \leqslant, \&)$ will denote a fixed quantale.

Definition 1.3. For any $U \in \mathcal{Q}$, a family $\{U_i\}_{i \in I}$ of elements of $\mathcal{Q}$ is said to be a *$\mathcal{Q}$-covering* of U if

(C1) $U = \bigvee_{i \in I} U_i$;
(C2) $U_i = U \& U_i$, for all $i \in I$.

We denote by $\text{Cov}(U)$ the set of all Q-coverings of U. The pair $(\mathcal{Q}, \{\text{Cov}(U)\}_{U\in\mathcal{Q}})$ will be referred to as a *Q-site.*

Taking into account the axioms and properties of a quantale, one easily verifies the following lemma.

Lemma 1.4. *Let $U \in \mathcal{Q}$ and let $\{U_i\}_{i\in I}$, $\{U'_j\}_{j\in J}$ be two Q-coverings of U. For all $i \in I$ and $j \in J$ we have:*

(1) $U_i \leqslant U$;
(2) $U_i \,\&\, U = U_i$;
(3) $U_i \,\&\, U'_j = U'_j \,\&\, U_i \leqslant U_i, U'_j$; *in particular, elements of the same Q-covering are &-commutative;*
(4) *if* $V \leqslant U$ *then* $V \,\&\, U_i \leqslant V, U_i$.

Example 1.5. A quantum space (in the sense of Borceux–Van den Bossche [3]) is a set X provided with a family of subsets $O(X)$ which has a quantale structure such that for all $U, V \in O(X)$ we have $U \cap V \subseteq U \,\&\, V$.

Let $(O(X), \subseteq, \&)$ be the quantale of open subsets of a quantum space X. For every $U \in O(X)$, the subset $\{U_i\}_{i\in I}$ of $O(X)$ is a *quantum covering* of U if

(i) $U = \bigcup_{i\in I} U_i$;
(ii) for all $i \in I$ we have $U_i = U \,\&\, U_i$.

Then the pair $(O(X), \{\text{Cov}(U)\}_{U\in O(X)})$ is a Q-site. In this case, note that $U \,\&\, U_i = U \cap U_i$, since $U_i \subseteq U$.

Example 1.6. Let R be an arbitrary ring with a unit. A right ideal I of R having the property that for all $a \in I$ there exists $e \in I$ such that $a \cdot e = a$, is called a *neat ideal.* The set of neat ideals of R with the multiplication given by the product of ideals is a quantale (cf. [2, Example 3]). For every neat ideal I of R, according to Definition 1.3, the family $\{I_a\}_{a\in A}$ in the quantale of neat ideals is a covering of I if

(i) $I = \sum_{a\in A} I_a$;
(ii) for all $a \in A$ we have $I_a = I \cdot I_a$.

Provided with such coverings, the quantale of neat ideals is a Q-site.

2. SHEAVES ON Q-SITES

Throughout the rest of this paper, $\mathcal{C}$ will denote a fixed Grothendieck category.

Definition 2.1. The *category of presheaves on $\mathcal{Q}$ with values in $\mathcal{C}$*, denoted by $\mathcal{P}(\mathcal{Q})$, is the functor category $\text{Fun}(\mathcal{Q}^{opp}, \mathcal{C})$, where $\mathcal{Q}^{opp}$ denotes the small category whose objects are the elements of $\mathcal{Q}$ and the set $\text{Hom}_{\mathcal{Q}^{opp}}(U, V)$ of morphisms is a singleton when $V \leqslant U$ and is empty otherwise. Thus, a *presheaf* P on $\mathcal{Q}$ with values in $\mathcal{C}$ consists of the data:

(i) for all $U \in \mathcal{Q}$ an object $P(U)$ in $\mathcal{C}$;
(ii) for every $V \leqslant U$ in $\mathcal{Q}$, a morphism in $\mathcal{C}$

$$P_{UV} : P(U) \to P(V); \quad s \mapsto P_{UV}(s) =: s|_V$$

subject to the conditions:

(1) $P_{UU} = \mathrm{id}_{P(U)}$, for all $U \in \mathcal{Q}$;
(2) if $W \leqslant V \leqslant U$ in $\mathcal{Q}$ then $P_{UW} = P_{VW} \circ P_{UV}$.

A *morphism* $f : P \to P'$ *of presheaves* on $\mathcal{Q}$ with values in $\mathcal{C}$ will just be a natural transformation between the corresponding functors.

Inspired by the classical definition of sheaves on a (commutative) topological space, we define:

Definition 2.2. The category ${}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$ of *separated Q-presheaves* is the full subcategory of $\mathcal{P}(\mathcal{Q})$ whose objects are the Q-presheaves P such that for every $U \in \mathcal{Q}$, and every $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$, the map

$$\xi : P(U) \to \prod_{i\in I} P(U_i); \quad s \mapsto (s|_{U_i})_{i\in I}$$

is injective. Equivalently, which satisfy

(Sh1) if $U \in \mathcal{Q}$ and $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$, then for every $s \in P(U)$ we have $s = 0$ whenever $s|_{U_i} = 0$ for all $i \in I$.

Definition 2.3. The category $\mathcal{S}h(\mathcal{Q})$ of *Q-sheaves with values in* $\mathcal{C}$ is the full subcategory of $\mathcal{P}(\mathcal{Q})$ whose objects are the Q-presheaves P which satisfy (Sh1) and the following *gluing condition*:

(Sh2) if $U \in \mathcal{Q}$, if $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$, and if for all $i \in I$ there is given $s_i \in P(U_i)$ verifying for all $i, j \in I$ that $s_i|_{U_i \& U_j} = s_j|_{U_i \& U_j}$, then there exists some $s \in P(U)$ such that $s|_{U_i} = s_i$, for all $i \in I$.

It may be checked (as in the classical case) that P is a Q-sheaf if and only if, for every open subset U and every Q-covering $\{U_i\}_{i\in I}$ of U, the sequence

$$0 \to P(U) \xrightarrow{\xi} \prod_{i\in I} P(U_i) \xrightarrow{\theta} \prod_{(i,j)\in I\times I} P(U_i \& U_j)$$

is exact, where $\xi(s) = (s|_{U_i})_{i\in I}$, for every $s \in P(U)$, and θ is given for all $(s_i)_{i\in I} \in \prod_{i\in I} P(U_i)$ by $(s_i|_{U_i \& U_j} - s_j|_{U_i \& U_j})_{(i,j)\in I\times I}$.

Definition 3.1. We define a functor $L : \mathcal{P}(\mathcal{Q}) \to \mathcal{P}(\mathcal{Q})$ given on every $P \in \mathcal{P}(\mathcal{Q})$ by $L(P) = LP$ where LP is the Q-presheaf given as follows.

For every $\mathcal{U} = \{U_i\}_{i \in I} \in \mathrm{Cov}(U)$, where $U \in \mathcal{Q}$, we denote by $P\mathcal{U}$ the system on $\mathcal{C}$ which consists of the diagrams

$$P(U_i) \xrightarrow{p_{U_i, U_i \& U_j}} P(U_i \& U_j) \xleftarrow{p_{U_j, U_i \& U_j}} P(U_j),$$

for all $i, j \in I$. It may be checked that $P\mathcal{U}$ is an inverse system on the quasi-ordered set $(\{U_i, U_i \& U_j\}_{i,j \in I}, \leqslant)$. Therefore, its *inverse limit* exists (i.e. the limit on the small category defined by I); it is of the form

$$\varprojlim P\mathcal{U} = \left\{ (s_i)_{i \in I} \in \prod_{i \in I} P(U_i) \;\middle|\; \forall i, j \in I, s_i|_{U_i \& U_j} = s_j|_{U_i \& U_j} \right\},$$

and comes equipped with projection maps $\pi_i : \varprojlim P\mathcal{U} \to P(U_i)$, for every $i \in I$.

Let us order Q-coverings as follows:

Definition 3.2. Let $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{U}' = \{U'_j\}_{j \in J}$ be two Q-coverings of U. We put $\mathcal{U}' \preceq \mathcal{U}$ if and only if there exists a map $\delta : J \to I$ such that for all $j \in J$ we have $U'_j \leqslant U_{\delta(j)}$. Then we say that $\mathcal{U}'$ is a *sub-Q-covering* of $\mathcal{U}$.

In this case, the universal property of the inverse limit guarantees the existence of a unique morphism

(1) $$P^U_{\mathcal{U}\mathcal{U}'} : \varprojlim P\mathcal{U} \to \varprojlim P\mathcal{U}'; \quad (s_i)_{i \in I} \mapsto (s_{\delta(j)}|_{U'_j})_{j \in J}.$$

Moreover, taking into account that $\mathrm{Cov}(U)$ is a directed set (with the inverse ordering of $\preceq$), there exists a direct system

$$\left(\{\varprojlim P\mathcal{U}\}_{\mathcal{U} \in \mathrm{Cov}(U)}, \{P^U_{\mathcal{U}\mathcal{U}'}\}_{\mathcal{U}' \preceq \mathcal{U} \in \mathrm{Cov}(U)}\right),$$

whose direct limit (colimit) belongs to $\mathcal{C}$. Thus, it makes sense to define LP on open subsets U by

(2) $$LP(U) = \varinjlim_{\mathcal{U} \in \mathrm{Cov}(U)} (\varprojlim P\mathcal{U}),$$

which comes equipped with maps

$$\eta_{\mathcal{U}} : \varprojlim P\mathcal{U} \to LP(U),$$

for all $\mathcal{U} \in \mathrm{Cov}(U)$. Hence, for every $s \in LP(U)$ there exists a Q-covering $\mathcal{U} = \{U_i\}_{i\in I}$ of U such that $s = \eta_{\mathcal{U}}(x)$ for some $x = (x_i)_{i\in I} \in \varprojlim P\mathcal{U}$ (where $x_i \in P(U_i)$ and for all $i, j \in I$ we have $x_i|_{U_i \,\&\, U_j} = x_j|_{U_i \,\&\, U_j}$).

In particular, $\eta_{\mathcal{U}}(x) = 0$ if and only if there exists a sub-Q-covering $\mathcal{U}' = \{U'_j\}_{j\in J}$ of $\mathcal{U}$ such that $P^U_{\mathcal{U}\mathcal{U}'}((x_i)_{i\in I}) = 0$, i.e., if and only if for all $j \in J$ we have $x_{\delta(j)}|_{U'_j} = 0$, where $\delta : J \to I$ is the map satisfying for all $j \in J$ that $U'_j \leqslant U_{\delta(j)}$.

On the other hand, let $V \leqslant U$ in $\mathcal{Q}$. In virtue of Lemma 1.4(4), and making use of the universal properties of limits and quantale properties, we may define the restriction morphism $(LP)_{UV} : LP(U) \to LP(V)$, for every $s \in LP(U)$, by

$$(LP)_{UV}(s) = \eta_{\mathcal{V}}\big((x_i|_{V \,\&\, U_i})_{i\in I}\big), \tag{3}$$

where $s = \eta_{\mathcal{U}}(x)$, for some $x = (x_i)_{i\in I} \in \varprojlim P\mathcal{U}$ and $\mathcal{U} = \{U_i\}_{i\in I} \in \mathrm{Cov}(U)$, and where $\mathcal{V} = \{V \,\&\, U_i\}_{i\in I} \in \mathrm{Cov}(V)$.

Finally, making use the universal property of limits, for every morphism $f : P \to P'$ in $\mathcal{P}(\mathcal{Q})$, the morphism of Q-presheaves $L(f) : L(P) \to L(P')$ is given by the collection of morphisms in $\mathcal{C}$,

$$L(f)(U) : LP(U) \to LP'(U); \quad s \mapsto \eta'_{\mathcal{U}}\big((f(U_i)(x_i))_{i\in I}\big), \tag{4}$$

where $s = \eta_{\mathcal{U}}(x)$ with $x = (x_i)_{i\in I} \in \varprojlim P\mathcal{U}$, and $\eta'_{\mathcal{U}}$ denotes the map $\varprojlim P'\mathcal{U} \to LP'(U)$. The reader may easily verify that the functor L is left exact.

Theorem 3.3. *If $P \in \mathcal{P}(\mathcal{Q})$, then $LP \in {}_{C}\mathcal{F}(\mathcal{Q})$.*

Proof. Let $U \in \mathcal{Q}$ and $\mathcal{U} = \{U_i\}_{i\in I} \in \mathrm{Cov}(U)$, and let us check that

$$\xi : LP(U) \to \prod_{i\in I} LP(U_i); \quad s \mapsto \big((LP)_{UU_i}(s)\big)_{i\in I}$$

is injective. If $s \in \mathrm{Ker}\,\xi$, then there exists a Q-covering $\mathcal{U}' = \{U'_a\}_{a\in A}$ of U such that $s = \eta_{\mathcal{U}'}(x)$, for some $x = (x_a)_{a\in A} \in \varprojlim P\mathcal{U}'$, and for all $i \in I$,

$$0 = LP_{UU_i}(s) = \eta_{\mathcal{U}'_i}\big((x_a|_{U_i \,\&\, U'_a})_{a\in A}\big),$$

where $\mathcal{U}'_i$ denotes the Q-covering $\{U_i \,\&\, U'_a\}_{a\in A}$ of U_i. Therefore, for all $i \in I$ there exists a sub-Q-covering $\mathcal{U}_i$ of $\mathcal{U}'_i$ such that

$$P^{U_i}_{\mathcal{U}'_i\mathcal{U}_i}\big((x_a|_{U_i \,\&\, U'_a})_{a\in A}\big) = 0.$$

Suppose $\mathcal{U}_i = \{U_{b_i}\}_{b_i \in B_i}$ and let $\delta_i : B_i \to A$ be the map such that for all $b_i \in B_i$ we have $U_{b_i} \leqslant U_i \,\&\, U'_{\delta_i(b_i)}$. Then, for all $b_i \in B_i$, we obtain by assumption that

$$(x_{\delta_i(b_i)}|_{U_i \,\&\, U'_{\delta_i(b_i)}})|_{U_{b_i}} = 0,$$

i.e., that $x_{\delta_i(b_i)}|_{U_{b_i}} = 0$. To conclude that $s = 0$ it is sufficient to find a sub-Q-covering $\mathcal{U}''$ of $\mathcal{U}'$ such that $P^U_{\mathcal{U}'\mathcal{U}''}((x_a)_{a\in A}) = 0$. We claim that

$$\mathcal{U}'' = \{U_{b_i}\}_{b_i \in B_i, i \in I}$$

does the trick. Indeed, clearly $\mathcal{U}''$ is a Q-covering of U. Moreover, $\mathcal{U}'' \preceq \mathcal{U}'$ by using $\delta : \coprod_{i\in I} B_i \to A$ given, for all $b_i \in B_i$ and $i \in I$, by $\delta(b_i) = \delta_i(b_i)$. Indeed, we have by assumption that $U_{b_i} \leqslant U_i \,\&\, U'_{\delta(b_i)}$, hence $U_{b_i} \leqslant U'_{\delta(b_i)}$ by Lemma 1.4(3). So, finally,

$$P^U_{\mathcal{U}'\mathcal{U}''}\big((x_a)_{a\in A}\big) = (x_{\delta(b_i)}|_{U_{b_i}})_{b_i \in B_i, i\in I} = 0. \qquad \square$$

Theorem 3.4. *If $P \in {}_C\mathcal{F}(\mathcal{Q})$ then $LP \in \mathcal{S}h(\mathcal{Q})$.*

Proof. Let $U \in \mathcal{Q}$ and $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$. We have to verify the exactness of the sequence

$$0 \to LP(U) \overset{\xi}{\to} \prod_{i\in I} LP(U_i) \overset{\theta}{\to} \prod_{(i,j)\in I\times I} LP(U_i \,\&\, U_j).$$

It remains to check that $\mathrm{Ker}\,\theta \subseteq \mathrm{Im}\,\xi$. Let $(s_i)_{i\in I} \in \mathrm{Ker}\,\theta$. Then for all $i \in I$ there exists $\mathcal{U}_i = \{U_{a_i}\}_{a_i \in A_i} \in \mathrm{Cov}(U_i)$ such that $s_i = \eta_{\mathcal{U}_i}(x_i)$, for some $x_i \in \varprojlim P\mathcal{U}_i$, where

$$\eta_{\mathcal{U}_i} : \varprojlim P\mathcal{U}_i \to LP(U_i),$$

and $x_i = (m_{a_i})_{a_i\in A_i}$ with $m_{a_i} \in P(U_{a_i})$ such that for all $a_i, b_i \in A_i$,

$$(5) \qquad m_{a_i}|_{U_{a_i} \,\&\, U_{b_i}} = m_{b_i}|_{U_{a_i} \,\&\, U_{b_i}}.$$

On the other hand, $\mathcal{U} = \{U_{a_i}\}_{a_i\in A_i, i\in I} \in \mathrm{Cov}(U)$. Let us now consider the element

$$x = (m_{a_i})_{a_i \in A_i, i\in I} \in \prod_{a_i\in A_i, i\in I} P(U_{a_i})$$

and let us verify that $x \in \varprojlim P\mathcal{U}$ (whence $\eta_{\mathcal{U}}(x) \in LP(U)$). It is sufficient to check for all $i, j \in I$, $a_i \in A_i$ and $b_j \in A_j$ that

$$(6) \qquad m_{a_i}|_{U_{a_i} \,\&\, U_{b_j}} = m_{b_j}|_{U_{a_i} \,\&\, U_{b_j}}.$$

When $i = j$ this equality is given by (5). Otherwise, let V denote the element $U_i \,\&\, U_j$. Then, by (3), the restriction of $s_i \in LP(U_i)$ to V coincides with

$$\eta_{V_1}\big((m_{a_i}|_{V \,\&\, U_{a_i}})_{a_i\in A_i}\big) \in LP(V),$$

where $\mathcal{V}_1$ denotes the Q-covering $\{V \& U_{a_i}\}_{a_i \in A_i}$ of V. On the other hand, denoting $\mathcal{U}_j = \{U_{b_j}\}_{b_j \in A_j} \in \mathrm{Cov}(U_j)$, the restriction of $s_j \in LP(U_j)$ to V is equal to

$$\eta_{\mathcal{V}_2}\big((m_{b_j}|_{V \& U_{b_j}})_{b_j \in A_j}\big) \in LP(V),$$

with $\mathcal{V}_2 = \{V \& U_{b_j}\}_{b_j \in A_j} \in \mathrm{Cov}(V)$.

Both restrictions coincide by assumption (since $(s_i)_{i \in I} \in \mathrm{Ker}\,\theta$), hence there exists a sub-Q-covering $\mathcal{V}_3 = \{V_k\}_{k \in K}$ of $\mathcal{V}_1$ and $\mathcal{V}_2$ such that for all $k \in K$

$$(m_{\delta_i(k)}|_{V \& U_{\delta_i(k)}})|_{V_k} = (m_{\delta_j(k)}|_{V \& U_{\delta_j(k)}})|_{V_k},$$

where $\delta_i : K \to A_i$ and $\delta_j : K \to A_j$ are the maps respectively satisfying for all $k \in K$ that $V_k \leqslant V \& U_{\delta_i(k)}$ and $V_k \leqslant V \& U_{\delta_j(k)}$. Consequently, for all $k \in K$:

$$m_{\delta_i(k)}|_{V_k} = m_{\delta_j(k)}|_{V_k}. \tag{7}$$

On the other hand, $\{U_{a_i} \& U_{b_j} \& V_k\}_{k \in K}$ is a Q-covering of $U_{a_i} \& U_{b_j}$, so the map

$$P(U_{a_i} \& U_{b_j}) \to \prod_{k \in K} P(U_{a_i} \& U_{b_j} \& V_k); \quad t \mapsto (t|_{U_{a_i} \& U_{b_j} \& V_k})_{k \in K}$$

is injective (since P is separated). Hence, in order to verify (6) it suffices to prove for all $k \in K$, that

$$m_{a_i}|_{U_{a_i} \& U_{b_j} \& V_k} = m_{b_j}|_{U_{a_i} \& U_{b_j} \& V_k}. \tag{8}$$

First of all, taking into account that $V_k \leqslant V \& U_{\delta_i(k)}$ and the properties listed in Proposition 1.2 and Lemma 1.4, we get that

$$\begin{aligned}(U_{a_i} \& U_{b_j}) \& V_k \leqslant U_{a_i} \& V_k &\leqslant (U_{a_i} \& U_i) \& U_j \& U_{\delta_i(k)} \\ &= U_{a_i} \& U_j \& U_{\delta_i(k)} = (U_{a_i} \& U_{\delta_i(k)}) \& U_j \\ &\leqslant U_{a_i} \& U_{\delta_i(k)}.\end{aligned}$$

Therefore,

$$m_{a_i}|_{U_{a_i} \& U_{b_j} \& V_k} = (m_{a_i}|_{U_{a_i} \& U_{\delta_i(k)}})|_{U_{a_i} \& U_{b_j} \& V_k},$$

which, by (5), coincides with

$$(m_{\delta_i(k)}|_{U_{a_i} \& U_{\delta_i(k)}})|_{U_{a_i} \& U_{b_j} \& V_k} = m_{\delta_i(k)}|_{U_{a_i} \& U_{b_j} \& V_k}.$$

On the other hand,

$$U_{a_i} \& U_{b_j} \& V_k \leqslant U_i \& U_j \& V_k = V \& V_k = V_k,$$

so

$$m_{a_i}|_{U_{a_i} \& U_{b_j} \& V_k} = (m_{\delta_i(k)}|_{V_k})|_{U_{a_i} \& U_{b_j} \& V_k}.$$

With a similar procedure, taking into account that

$$U_{a_i} \,\&\, V_k \leqslant U_i \,\&\, V \,\&\, U_{\delta_j(k)} = V \,\&\, U_{\delta_j(k)} = U_j \,\&\, U_i \,\&\, U_{\delta_j(k)}$$
$$= U_j \,\&\, U_{\delta_j(k)} \,\&\, U_i = U_{\delta_j(k)} \,\&\, U_i \leqslant U_{\delta_j(k)},$$

whence

$$U_{a_i} \,\&\, U_{b_j} \,\&\, V_k = (U_{a_i} \,\&\, V_k) \,\&\, U_{b_j} \leqslant U_{\delta_j(k)} \,\&\, U_{b_j},$$

we obtain that

$$m_{b_j}|_{U_{a_i} \& U_{b_j} \& V_k} = (m_{\delta_j(k)}|_{V_k})|_{U_{a_i} \& U_{b_j} \& V_k}.$$

Therefore, the equality (8) holds as a consequence of (7). Consequently $s = \eta_{\mathcal{U}}(x) \in LP(U)$, so it makes sense to consider $\xi(s)$. Finally, let us verify that $(s_i)_{i\in I} \in \operatorname{Im}\xi$ by checking that it coincides with $\xi(s)$.

By (3), clearly $\xi(s)$ coincides with $(\eta_{\mathcal{U}_i'}((m_{c_t}|_{U_i \& U_{c_t}})_{c_t\in A_t, t\in I}))_{i\in I}$, where $\mathcal{U}_i' = \{U_i \,\&\, U_{c_t}\}_{c_t\in A_t, t\in I} \in \operatorname{Cov}(U_i)$. On the other hand, $(s_i)_{i\in I}$ is equal to $(\eta_{\mathcal{U}_i}((m_{a_i})_{a_i\in A_i}))_{i\in I}$. Thus, it is sufficient to verify for all $i \in I$ that

$$\eta_{\mathcal{U}_i'}\big((m_{c_t}|_{U_i \& U_{c_t}})_{c_t\in A_t, t\in I}\big) = \eta_{\mathcal{U}_i}\big((m_{a_i})_{a_i\in A_i}\big).$$

Taking into account that $\mathcal{U}_i \preceq \mathcal{U}_i'$ (by letting δ_1 be the inclusion map $A_i \hookrightarrow \coprod_{t\in I} A_t$) and that $\mathcal{U}_i \preceq \mathcal{U}_i$, it is then sufficient to check for all $a_i \in A_i$ that

$$(m_{\delta_1(a_i)}|_{U_i \& U_{\delta_1(a_i)}})|_{U_{a_i}} = m_{a_i}.$$

Indeed,

$$(m_{\delta_1(a_i)}|_{U_i \& U_{\delta_1(a_i)}})|_{U_{a_i}} = (m_{a_i}|_{U_i \& U_{a_i}})|_{U_{a_i}} = (m_{a_i}|_{U_{a_i}})|_{U_{a_i}} = m_{a_i}.$$

Therefore, $(s_i)_{i\in I} = \xi(s)$. □

As an immediate consequence of Theorems 3.3 and 3.4 it follows:

Corollary 3.5. *If $P \in \mathcal{P}(\mathcal{Q})$, then $L^2P = L(LP) \in \mathcal{S}h(\mathcal{Q})$.*

Let us now establish some properties of L which will be useful in the next section.

Lemma 3.6. *To every $P \in \mathcal{P}(\mathcal{Q})$ corresponds a morphism of Q-pre sheaves $P \stackrel{\zeta_P}{\to} LP$. Moreover, if P is separated, then ζ_P is injective.*

Proof. For every $U \in \mathcal{Q}$ let $\mathcal{U}_0$ denote the trivial Q-covering $\{U\}$. Then $\varprojlim P\mathcal{U}_0$ coincides with $P(U)$, so its corresponding map $\eta_{\mathcal{U}_0}$ is the canonical morphism $P(U) \to LP(U)$. We define ζ_P to be given by the family

$$\{\zeta_P(U) : P(U) \to LP(U)\}_{U\in\mathcal{Q}}$$

of morphisms in $\mathcal{C}$, where $\zeta_P(U) = \eta_{\mathcal{U}_0}$. It follows in a straightforward way that this is indeed a morphism of presheaves. Moreover, if P is separated then, for every $U \in \mathcal{Q}$, we may easily check that $\zeta_P(U)$ is injective since sub-Q-coverings of $\mathcal{U}_0$ are Q-coverings of U. □

Corollary 3.7. *There exists a natural transformation*

$$\zeta : \mathrm{id}_{\mathcal{P}(\mathcal{Q})} \to i \circ L,$$

where i is the inclusion functor $_{\mathcal{C}}\mathcal{F}(\mathcal{Q}) \hookrightarrow \mathcal{P}(\mathcal{Q})$.

Lemma 3.8. *To every $P \in \mathcal{S}h(\mathcal{Q})$ corresponds an isomorphism of Q-sheaves $\varphi_P : LP \xrightarrow{\sim} P$.*

Proof. Let $U \in \mathcal{Q}$. In general, for every $s \in LP(U)$ there exists a Q-covering $\mathcal{U} = \{U_i\}_{i \in I}$ of U and $x = (x_i)_{i \in I} \in \varprojlim P\mathcal{U}$ such that $s = \eta_{\mathcal{U}}(x)$, with $x_i|_{U_i \,\&\, U_j} = x_j|_{U_i \,\&\, U_j}$ for all $i, j \in I$. In this case, since P is a Q-sheaf, we can go further and assert that there exists a unique $t^x \in P(U)$ such that $t^x|_{U_i} = x_i$, for all $i \in I$. Thus, we may define a morphism

$$\varphi_P(U) : LP(U) \to P(U); \quad s \mapsto t^x.$$

If there exists another $\mathcal{U}' = \{U'_j\}_{j \in J} \in \mathrm{Cov}(U)$ and $y = (y_j)_{j \in J} \in \varprojlim P\mathcal{U}'$ such that $s = \eta_{\mathcal{U}'}(y)$, then let us prove that $t^y = t^x$. Since $\eta_{\mathcal{U}}(x) = \eta_{\mathcal{U}'}(y)$, there exists a sub-Q-covering $\mathcal{U}'' = \{U''_k\}_{k \in K} \in \mathrm{Cov}(U)$ of $\mathcal{U}$ and $\mathcal{U}'$ such that for all $k \in K$ the restrictions of $x_{\delta_1(k)}$ and $y_{\delta_2(k)}$ to U''_k coincide (where $\delta_1 : K \to I$ and $\delta_2 : K \to J$ are the maps satisfying $U''_k \leqslant U_{\delta_1(k)}, U'_{\delta_2(k)}$). Thus, for all $k \in K$ we obtain the following sequence of equalities from which we derive that $t^x = t^y$, as P is separated:

$$t^x|_{U''_k} = \left(t^x|_{U_{\delta_1(k)}}\right)|_{U''_k} = x_{U_{\delta_1(k)}}|_{U''_k} = y_{U'_{\delta_2(k)}}|_{U''_k} = \left(t^y|_{U'_{\delta_2(k)}}\right)|_{U''_k} = t^y|_{U''_k}.$$

Therefore, $\varphi_P(U)$ is well defined. Besides if $s = \eta_{\mathcal{U}}(x) \in \mathrm{Ker}\,\varphi_P(U)$ then for all $i \in I$ we obtain from the very definition that $x_i = t^x|_{U_i} = 0$, and consequently $s = 0$. Hence, $\varphi_P(U)$ is injective. Finally, for every $t \in P(U)$ we may choose $s = \eta_{\mathcal{U}_0}(t)$ as the element in $LP(U)$ such that $t = \varphi_P(U)(s)$, so it is also surjective.

Thus, we assume φ_P to be given by the family of isomorphisms $\varphi_P(U)$ in $\mathcal{C}$. Let $V \leqslant U$ in $\mathcal{Q}$, $s = \eta_{\mathcal{U}}(x) \in LP(U)$ as before, and t^x its image by $\varphi_P(U)$. Then $(t^x|_V)|_{V \,\&\, U_i} = t^x|_{V \,\&\, U_i} = (t^x|_{U_i})|_{V \,\&\, U_i} = x_i|_{V \,\&\, U_i}$, for all $i \in I$. Therefore, the following diagram is commutative

$$\begin{array}{ccc} LP(U) & \xrightarrow{(LP)_{UV}} & LP(V) \\ {\scriptstyle \varphi_P(U)}\downarrow & & \downarrow{\scriptstyle \varphi_P(V)} \\ P(U) & \xrightarrow[P_{UV}]{} & P(V) \end{array}$$

proving that φ_P is an isomorphism of Q-sheaves, indeed. □

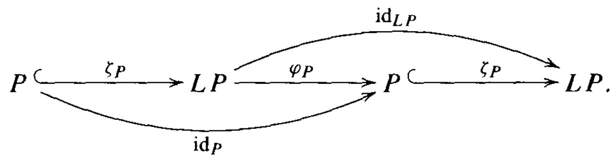
Corollary 3.9. *There exists a natural equivalence*

$$\varphi : L \circ j \to \mathrm{id}_{Sh(\mathcal{Q})},$$

where j is the inclusion functor $Sh(\mathcal{Q}) \hookrightarrow {}_{C}\mathcal{F}(\mathcal{Q})$, and L is considered to act from ${}_{C}\mathcal{F}(\mathcal{Q})$ to $Sh(\mathcal{Q})$.

Making use of Lemmas 3.6 and 3.8 we obtain the following lemma.

Lemma 3.10. *For every $P \in Sh(\mathcal{Q})$ we have the following commutative diagram*



Proof. Since φ_P is an isomorphism of Q-sheaves, it only remains to check that $\varphi_P \circ \zeta_P = \mathrm{id}_P$. Indeed, for all $U \in \mathcal{Q}$ and $x \in P(U)$,

$$\big(\varphi_P(U) \circ \zeta_P(U)\big)(x) = \varphi_P(U)\big(\eta_{\mathcal{U}_0}(x)\big)$$

is by definition the unique $t \in P(U)$ such that $t|_U = x$, thus $t = x$. □

4. AN ADJOINT PAIR

Definition 4.1 (*The Q-sheafification functor a*). At this point, in virtue of Definition 3.1 and Corollary 3.5, it becomes evident that we may define what we call the *Q-sheafification functor*,

$$a : \mathcal{P}(\mathcal{Q}) \to Sh(\mathcal{Q}),$$

as the composition $L \circ L$, where for every $P \in \mathcal{P}(\mathcal{Q})$, the Q-sheaf aP is given on $U \in \mathcal{Q}$ by $aP(U) = L(LP(U))$, with associated map

$$\eta^2_{\mathcal{U}} : \varprojlim(LP)\mathcal{U} \to aP(U).$$

On every morphism $f : P \to P'$ of Q-presheaves, the functor a is defined by $a(f) = L(L(f)) : aP \to aP'$.

What really allows us to call a the *Q-sheafification functor* is the fact that it is a left adjoint of the inclusion functor $Sh(\mathcal{Q}) \hookrightarrow \mathcal{P}(\mathcal{Q})$, i.e., that it is a reflector. In order to prove this, let us first include one more lemma:

Let P be a Q-presheaf. Then the image $L(\zeta_P)$ of the corresponding morphism ζ_P via the functor L is a morphism in $\mathrm{Hom}(LP, aP)$. On the other hand, we have another morphism $\zeta_{LP} \in \mathrm{Hom}(LP, aP)$ which is the one that corresponds to the Q-presheaf LP, as in Lemma 3.6.

Let us prove that they coincide:

Lemma 4.2. *For every $P \in \mathcal{P}(\mathcal{Q})$ we have $L(\zeta_P) = \zeta_{LP}$.*

Proof. Let $U \in \mathcal{Q}$. We have to verify that $L(\zeta_P)(U) = \zeta_{LP}(U)$. Let $s = \eta_{\mathcal{U}}(x) \in LP(U)$, where $\mathcal{U} = \{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ and $x = (x_i)_{i\in I} \in \varprojlim P\mathcal{U}$. By (4), we obtain

$$L(\zeta_P)(U)(s) = \eta^2_{\mathcal{U}}\big((\zeta_P(U_i)(x_i))_{i\in I}\big) = \eta^2_{\mathcal{U}}\big((\eta_{(\mathcal{U}_i)_0}(x_i))_{i\in I}\big),$$

where $\eta^2_{\mathcal{U}} : \varprojlim(LP)\mathcal{U} \to aP(U)$ and $\eta_{(\mathcal{U}_i)_0} : P(U_i) \to LP(U_i)$.

On the other hand,

$$\zeta_{LP}(U)(s) = \eta^2_{\mathcal{U}_0}(s),$$

where $\eta^2_{\mathcal{U}_0} : LP(U) \to aP(U)$.

Taking into account that $\mathcal{U} \preceq \mathcal{U}, \mathcal{U}_0$, in order to assert that both images coincide, it is sufficient to verify that $\eta_{(\mathcal{U}_i)_0}(x_i) = LP_{UU_i}(s)$, for all $i \in I$.

Indeed, by (3) we obtain $LP_{UU_i}(s) = \eta_{\mathcal{U}_i}((x_l|_{U_i \,\&\, U_l})_{l\in I})$, where $\mathcal{U}_i$ denotes the covering $\{U_i \,\&\, U_l\}_{l\in I}$. Thus, to prove that the latter element coincides with $\eta_{(\mathcal{U}_i)_0}(x_i)$, it is sufficient to check for all $l \in I$ that

$$x_l|_{U_i \,\&\, U_l} = x_i|_{U_i \,\&\, U_l},$$

(since $\mathcal{U}_i \preceq \mathcal{U}_i, (\mathcal{U}_i)_0$). In fact, this equality holds for all $i, l \in I$ just because $(x_i)_{i\in I} \in \varprojlim P\mathcal{U}$. □

Theorem 4.3. *The functor $a : \mathcal{P}(\mathcal{Q}) \to \mathcal{S}h(\mathcal{Q})$ is a left adjoint of the inclusion functor $i : \mathcal{S}h(\mathcal{Q}) \to \mathcal{P}(\mathcal{Q})$.*

Proof. First, in view of Corollary 3.7, we may define a natural transformation $\phi : \mathrm{id}_{\mathcal{P}(\mathcal{Q})} \to i \circ a$, given by the family of morphisms of Q-presheaves $\phi_P = \zeta_{LP} \circ \zeta_P$, for all $P \in \mathcal{P}(\mathcal{Q})$.

Secondly, by Corollary 3.9, we may define a natural equivalence $\psi : a \circ i \to \mathrm{id}_{\mathcal{S}h(\mathcal{Q})}$, by the family of isomorphisms of Q-sheaves $\varphi_P \circ \varphi_{LP}$, for all $P \in \mathcal{S}h(\mathcal{Q})$.

Thirdly, for all $P \in \mathcal{S}h(\mathcal{Q})$, making use of Lemma 3.10, one may easily check that the following diagram is commutative:

$$iP \xrightarrow{\ \phi_{iP}\ } iaiP \xrightarrow{\ i\psi_P\ } iP, \qquad i\psi_P \circ \phi_{iP} = \mathrm{id}_{iP}.$$

Finally, let $P \in \mathcal{P}(\mathcal{Q})$. By Lemma 4.2,

$$a(\zeta_P) = L\big(L(\zeta_P)\big) = L(\zeta_{LP}) = \zeta_{aP}.$$

In a similar way, $a(\zeta_{LP}) = \zeta_{a(LP)} = \zeta_{L(aP)}$ which is indeed equal to ζ_{aP} by Lemma 3.8. Hence, we conclude that $a(\phi_P) = (\zeta_{aP})^2$. On the other hand,

$$\psi_{aP} = \varphi_{aP} \circ \varphi_{L(aP)} = (\varphi_{aP})^2.$$

Consequently, $\psi_{aP} \circ a(\phi_P) = \mathrm{id}_{aP}$ by Lemma 3.10. Applying the previous four facts, the theorem now follows from [1, Theorem 3.1.5]. □

Note that the functor a is exact. Indeed, it is right exact being a reflector, and left exact by the left exactness of L, hence it preserves kernels, pullbacks and finite limits.

Corollary 4.4. *Let C be a Grothendieck category. The category $\mathcal{S}h(\mathcal{Q})$ of Q-sheaves on C is a Grothendieck category.*

Proof. The functor a is left exact and a left adjoint of the inclusion functor $\mathcal{S}h(\mathcal{Q}) \hookrightarrow \mathcal{P}(\mathcal{Q})$. Therefore, $\mathcal{S}h(\mathcal{Q})$ is a Giraud subcategory of the Grothendieck category $\mathcal{P}(\mathcal{Q})$, whence a Grothendieck category itself. □

REFERENCES

[1] Borceux F. – Handbook of Categorical Algebra 1, Basic Category Theory, Cambridge Univ. Press, Cambridge, 1994.

[2] Borceux F., Van den Bossche G. – Quantales and their sheaves, Order **3** (1986) 61–87.

[3] Borceux F., Van den Bossche G. – An essay on noncommutative topology, Topology and Its Applications **31** (1989) 203–223.

[4] Mendoza Aguilar J., Reyes Sánchez M.V., Verschoren A. – Noncommutative topologies, localization and sheaves, Communications in Algebra **36** (2008) 1289–1300.

[5] Stenström B. – Rings of Quotients: An Introduction to Methods of Ring Theory, Springer-Verlag, Berlin, 1975.

[6] Strooker J. – Introduction to Categories, Homological Algebra and Sheaf Cohomology, Cambridge Univ. Press, Cambridge, 1978.

[7] Van Oystaeyen F., Verschoren A. – Reflectors and Localization, Application to Sheaf Theory, Marcel Dekker, New York, 1979.

(Received September 2008)