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**Discrete annuities  
using truncate stochastic interest rates:  
the case of a Vasicek and Ho-Lee model**

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# Discrete annuities using truncate stochastic interest rates: the case of a Vasicek and Ho-Lee model

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## Abstract

*A subject often recurring in the recent financial and actuarial research, is the investigation of present value functions with stochastic interest rates. Only in the case of uncomplicated payment streams and rather basic interest rate models, an exact analytical result for the distribution function is available. In the present contribution, we introduce the concept of truncate stochastic interest rates, useful to adapt general stochastic models to specific financial requirements, and we show how to obtain in that case analytical results for bounds for the present value. We employ our method in extension for the Ho-Lee model and the Vasicek model. We illustrate the accuracy of the approximations graphically, and we use the bounds to estimate the Value-at-Risk.*

## 1 Introduction

A substantial portion of the problems in the current financial and actuarial research deal with the problem of finding the distribution of the present value of a cash-flow in the form

$$V(t) = \sum_{i=1}^n \xi(t_i) e^{-X(t_i)}, \quad (1)$$

where  $0 < t_1 < t_2 < \dots < t_n = t$ , where  $\xi(t_i)$  is a (positive or negative) payment at time  $t_i$ , and where  $X = \{X(t)\}$  is a stochastic process with  $X(t_i)$  denoting the compounded rate of return for the period  $[0, t_i]$ .

A broad range of stochastic processes has been suggested and reviewed in order to answer the question of how to model the stochastic interest rates. In many cases, the proposed models would be more realistic if the interest rates are not completely free, but restricted to some range of acceptable values. If for example the interest rates appearing in the present value are nominal interest rates, they can not become negative. If an insurance contract guarantees a minimal return, the interest rate model should be adapted in order to meet this warranty. In other situations, also the enforcement of an upper limit for the yield can be necessary.

In this paper, we want to contribute to these last considerations. We show how common models can be adapted to several types of restrictions, and we show the influence on the present value of classical actuarial functions such as annuities. Since the exact distribution of the present value in most cases cannot be calculated analytically, we will make use of an approximation by means of convex bounds, as introduced by Goovaerts et al. (2001), and generalized in Dhaene et al. (2002a, 2002b).

## 2 Methodology

We start this section by defining what is meant by truncate interest rates. Afterwards, we briefly recall the most important results about the construction of convex bounds for present value functions. Finally, we explain why we focus on the Ho-Lee and Vasicek model for the description of the interest rate process.

### 2.1 Truncation

In some cases, the classic stochastic processes used for modelling interest rates, in their natural form generate outcomes in a way that is too general to fit specific requirements. In such situations, restrictions on the outcome of these processes can be a useful tool. A very common condition is a need for positive interest rates, which can be solved by utilizing the Heaviside function. We will use this basic solution to create more general truncating functions.

#### A. NON-NEGATIVE INTEREST RATES

When the interest rates under investigation refer to nominal rates, negative values should be avoided as much as possible. A feasible solution to this problem can be reached by multiplying the compounded rate of return with the Heaviside-function  $\mathcal{U}$ , defined by

$$\mathcal{U}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases} \quad (2)$$

such that the discount factor in the present value becomes  $e^{-X(t)\mathcal{U}(X(t))}$ . See e.g. De Schepper et al. (1997). With this adjustment, the compounded interest rate is kept equal to zero as long as the value of  $X(t)$  is negative.

#### B. TRUNCATE INTEREST RATES WITH FIXED FLOOR AND CAP

A somewhat more general solution consists of a truncate interest rate, by defining a cap and a floor for the interest rate – with the previous restriction as a special case. This can be done by mapping  $X(t)$  on  $c \in \mathbb{R}$  whenever  $X(t)$  exceeds  $c$ , and by mapping  $X(t)$  on  $f \in \mathbb{R}$  whenever  $X(t)$  is smaller than  $f$ .

**Definition 2.1** Let  $f, c \in \mathbb{R}$  with  $f < c$ . The truncation function  $\mathcal{S}_f^c : \mathbb{R} \rightarrow [f, c]$  then is defined by

$$\mathcal{S}_f^c(x) = \begin{cases} f & \text{if } x < f \\ x & \text{if } f \leq x \leq c \\ c & \text{if } c < x. \end{cases} \quad (3)$$

Figure 1 (a) shows a possible realisation of such a truncate interest rate. With this truncation function applied on the stochastic interest rate, the discount factor in the present value becomes  $e^{-\mathcal{S}_f^c(X(t))}$ . See also Koch & De Schepper (2004).

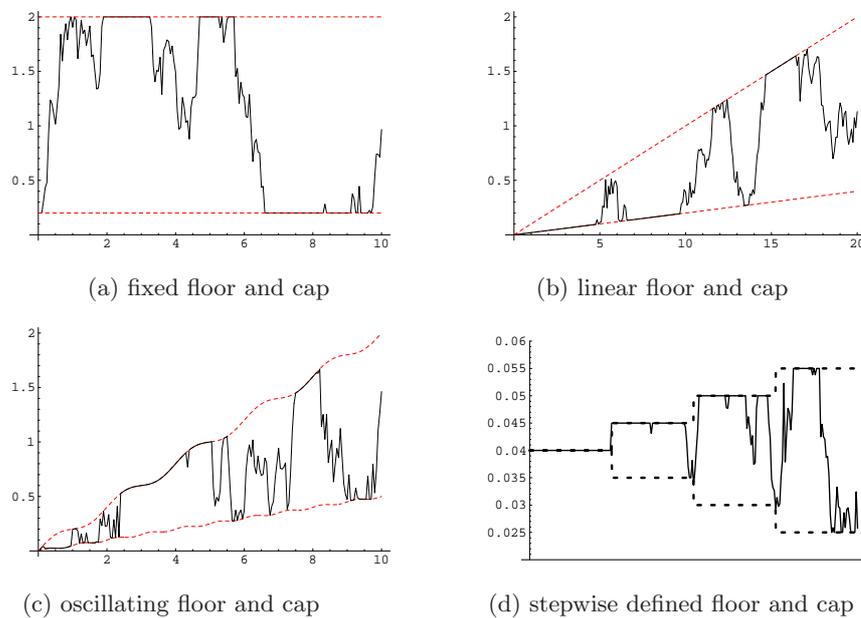


Figure 1: Examples of stochastic truncate interest rates

### C. GENERAL TRUNCATE INTEREST RATES

Since the stochastic variable  $X(t)$  corresponds to the *cumulative* interest rate for the period  $[0, t]$ , it seems more appropriate to use a floor and cap depending on time. This can be done by using any deterministic function of time for the floor and cap.

**Definition 2.2** Let  $f$  and  $c$  be two real-valued functions, defined in all points in the observed timeperiod, with  $\forall t \in \mathbb{R}^+ : f(t) \leq c(t)$ . The general truncation function  $\tilde{S}_f^c : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$\tilde{S}_f^c(t, x) = \begin{cases} f(t) & \text{if } x < f(t) \\ x & \text{if } f(t) \leq x \leq c(t) \\ c(t) & \text{if } c(t) < x. \end{cases} \quad (4)$$

Some interesting cases for the floorfunction and capfunction are described in the following examples:

### Example 2.1 Linear functions

A natural choice for the floor and cap is a linear function of time, since this can be interpreted as a floor and cap per unit of time. In this case, we can specify both functions as

$$\begin{cases} f(t) & = & f \cdot t + f_0 \\ c(t) & = & c \cdot t + c_0 \end{cases} \quad (5)$$

with  $f_0 \leq c_0$  and  $f \leq c$ .

In figure 1 (b), a possible realisation of such a linear truncate interest rate is drawn.

### Example 2.2 Oscillating functions

The previous example could be seen as corresponding to a situation with a constant inflation. When we leave this assumption, e.g. by presupposing that the inflation follows a cyclic pattern, the floor and capfunctions could be described by means of oscillating functions of time:

$$\begin{cases} f(t) & = & \alpha_1 t + \beta_1 \sin(\gamma_1 t) \\ c(t) & = & \alpha_2 t + \beta_2 \sin(\gamma_2 t). \end{cases} \quad (6)$$

In figure 1 (c), a possible path of such a truncate stochastic interest rate is shown.

### Example 2.3 Amortization scheme

Also in the context of amortization schemes, a variable interest rate could be interesting. Suppose that over a small time period  $T$ , the interest rate is kept invariant, and that at the end of each period  $T$ , the interest rate is evaluated, and adapted upwards or downwards depending on the value of the stochastic interest rate at that moment. In such situations, limitations on the amount of change constitute a normal contractual specification. The following floor and capfunctions describe a situation in which, at the end of each period  $T$ , there is a possible jump  $u$  upwards and  $d$  downwards for the interest rate  $i$ , with an initial value of  $i_0$ .

$$\begin{cases} f(t) & = & \max(0, i_0 - [t/T]d) \\ c(t) & = & i_0 + [t/T]u. \end{cases} \quad (7)$$

The notation  $[z]$  indicates the integer part of  $z$ .

A possible path of the truncate stochastic interest rate, restricted as described in (7), is shown in figure 1 (d).

## 2.2 Convex bounds

Present values as in (1) are built up as a sum of rather dependent terms. Indeed, the compounded rates of return  $X(t_i)$  for successive periods only differ for the last part of the period, and the effect of this difference diminishes for more distant payments. As a consequence, it is extremely difficult to derive an exact analytical expression for the distribution of such a present value. In order to solve this problem, Goovaerts et al. (2001) and Dhaene et al. (2002b) present bounds in convexity order. Following their approach, the original sum  $V(t)$  is replaced by a new sum with special characteristics: the components have the same marginal distributions as the components in the original sum, the dependence structure is the most “dangerous” that is possible, and the calculation of the distribution is much more easy.

In this subsection, we only briefly recall definitions and most important results about this approximation method. For details, we refer to Dhaene et al. (2002b).

**Definition 2.3** *Let  $X$  and  $Y$  be two random variables, then  $X$  is said to be smaller than  $Y$  in convex order sense, (notation  $X \leq_{cx} Y$ ), if and only if*

$$E[v(X)] \leq E[v(Y)]$$

for all real convex functions  $v : \mathbb{R} \rightarrow \mathbb{R}$ , provided the expectations exist.

In fact this ordering means that the variable  $Y$  is more likely to reach extreme values than it is the case for  $X$ , or, that the variable  $Y$  is more dangerous than  $X$ . Note that for such variables it is true that  $E[X] = E[Y]$  and  $Var[X] \leq Var[Y]$ .

**Theorem 2.1** *Let  $X_1, X_2, \dots, X_n$  be random variables with marginal distribution functions known as  $F_{X_1}, F_{X_2}, \dots, F_{X_n}$ , then*

$$X_1 + X_2 + \dots + X_n \leq_{cx} F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \dots + F_{X_n}^{-1}(U), \quad (8)$$

and

$$X_1 + X_2 + \dots + X_n \geq_{cx} E[X_1|\Lambda] + E[X_2|\Lambda] + \dots + E[X_n|\Lambda], \quad (9)$$

with  $U$  a uniform(0,1) distributed random variable, and with  $\Lambda$  an arbitrary variable for which the conditional distributions of  $X_i$  given  $\Lambda$  are known.

The upper bound of (8) can be improved to a closer bound

$$X_1 + X_2 + \dots + X_n \leq_{cx} F_{X_1|\Lambda}^{-1}(U) + F_{X_2|\Lambda}^{-1}(U) + \dots + F_{X_n|\Lambda}^{-1}(U), \quad (10)$$

with  $U$  and  $\Lambda$  as before.

This theorem can easily be extended to a sum of functions of the variables,  $\sum_{i=1}^n \psi_i(X_i)$ , where  $\psi_i$  can be any real-valued function. Note that the lower bound of (9) and the improved upper bound of (10) perform better the more  $\Lambda$  resembles the original sum.

## 2.3 Stochastic interest rate models

As mentioned in the introduction, the list of stochastic processes suggested to model interest rates is extensive. The simplest model in this context is the well known and frequently used Brownian motion with drift, defined by the stochastic differential equation:

$$dX(t) = \mu dt + \sigma dW(t), \quad (11)$$

with  $W = \{W(t)\}$  a standard Brownian motion.

This model benefits from the fact that it is not too difficult in calculations. It is very appropriate for situations with rather great variation; a disadvantage however is the fact that for long periods, a very large value (both positive and negative) could be reached, which causes the possibility of instability. However, by implementing a restriction as suggested in subsection 2.1, this disadvantage can be perfectly avoided.

For more information and analytical results according to this model in the setting of a truncate stochastic interest rate, we like to refer to earlier results, summarized in Koch & De Schepper (2004).

In the sequel, we will focus on more involved stochastic models for the interest rate. As in Vyncke et al. (2001), we will concentrate on the Ho-Lee model and the Vasicek model.

Both models will be used to describe the instantaneous interest rate,  $r(t)$ , instead of the accumulated interest rate,  $X(t) = \int_0^t r(s)ds$ . The most important consequence of this approach, which makes it worthwhile to proceed as such, is the fact that the conditional expectation of  $X(t)$  given  $X(s)$  and  $r(s)$  will depend on the value of  $r(s)$ , which is not the case when a model is applied directly for the accumulative interest rate. More information on this topic can be found in Parker (1994).

Let us first consider the Ho-Lee model as introduced in their seminal paper Ho & Lee (1986).

The advantage of this model over the Brownian model lies in the fact that it has a deterministic, but time-dependent drift term. Unfortunately, as it is the case for the simple Brownian motion, for long time periods, the interest rate becomes unbounded and therefore instable. However, this can be countered by imposing restrictions on the (accumulated) interest rate by means of a truncation function.

As mentioned before, we write the accumulated interest rate as  $X(t) = \int_0^t r(s)ds$  where the instantaneous interest rate  $r(s)$  is driven by the Ho-Lee model:

$$dr(t) = \alpha(t)dt + \gamma dW(t), \quad (12)$$

with  $\alpha(t)$  a non-negative function of time and  $\{W(t)\}$  a standard Brownian motion. Common calculation techniques lead to the result that the cumulative interest  $X(t)$  is normally distributed with mean

$$\mu_{\text{HL}}(t) = r(0)t + \int_0^t \alpha(u)(t-u)du \quad (13)$$

and variance

$$\sigma_{\text{HL}}^2(t) = \frac{\gamma^2 t^3}{3}. \quad (14)$$

The second model for which we will elaborate our method, is the Vasicek model (see Vasicek (1977)). The great advantage of this model over the Ho-Lee model is the mean-reverting property, the result being that the Vasicek model avoids explosive paths. As a disadvantage –just as for the Ho-Lee model– one can argue that negative values are still possible. Although there exist many other and more sophisticated models with only positive values (e.g. the Cox-Ingersoll-Ross model), the Vasicek model has an extra advantage over many of these models, in that the volatility does not disappear when the interest rate reaches zero. Therefore, it seems reasonable to remain with the Vasicek model and to use a truncation in order to avoid the problem of negative values.

Again, we write the accumulated interest rate as  $X(t) = \int_0^t r(s)ds$  where now the instantaneous interest rate  $r(s)$  is driven by the Vasicek model:

$$dr(t) = (\alpha - \beta r(t))dt + \gamma dW(t), \quad (15)$$

with  $\alpha, \beta$  and  $\gamma$  non-negative constants and  $\{W(t)\}$  a standard Brownian motion. As it was the case for the Ho-Lee model, the cumulative interest  $X(t)$  is normally distributed. Mean and variance functions can be calculated as

$$\mu_{\text{VA}}(t) = \frac{\alpha t}{\beta} + \frac{1}{\beta}(r(0) - \frac{\alpha}{\beta})(1 - e^{-\beta t}) \quad (16)$$

and

$$\sigma_{\text{VA}}^2(t) = \frac{\gamma^2}{\beta^2} \left( t - \frac{2}{\beta}(1 - e^{-\beta t}) + \frac{1}{2\beta}(1 - e^{-2\beta t}) \right). \quad (17)$$

If we use the notation  $F(t, x)$  for the cumulative distribution function for the variable  $X(t)$ , then for any  $x \in \mathbb{R}$

$$F(t, x) = \Phi \left( \frac{x - \mu(t)}{\sigma(t)} \right), \quad (18)$$

and for any  $p \in [0, 1]$  the quantile can be found as

$$F^{-1}(t, p) = \mu(t) + \sigma(t)\Phi^{-1}(p), \quad (19)$$

where we change  $[\mu(t), \sigma(t)]$  for  $[\mu_{\text{HL}}(t), \sigma_{\text{HL}}(t)]$  or  $[\mu_{\text{VA}}(t), \sigma_{\text{VA}}(t)]$ , according to the model involved.

### 3 Constant annuities

In this section, we present our results for constant annuities, i.e. for present values as introduced in (1) with  $\xi_i \equiv 1$ . We start by generating expressions for stochastic bounds for the present value without specifying the stochastic process used to model the interest rate. Afterwards, we show how for each of these bounds analytical results can be obtained in the case of a Ho-Lee model and a Vasicek model.

#### 3.1 General case

Consider a discrete annuity over the time-interval  $[0, t]$ , with general truncate stochastic interest rate with floorfunction  $f(t)$  and capfunction  $c(t)$  as defined in (4). The present value can be written as

$$V(t) = \sum_{i=1}^n e^{-\tilde{S}_f^c(t_i, X(t_i))}, \quad (20)$$

where  $0 < t_1 < t_2 < \dots < t_n = t$ , and where  $X = \{X(t)\}$  is a stochastic process with  $X(t_i)$  denoting the compounded rate of return for the period  $[0, t_i]$ .

Applying the methodology of convex bounds (see subsection 2.2), the following result can be obtained straightforwardly:

**Theorem 3.1** *The present value of annuity of equation (20) can be bounded in convex ordering sense as*

$$V_{low}(t) \leq_{cx} V(t) \leq_{cx} V_{imupp}(t) \leq_{cx} V_{upp}(t), \quad (21)$$

where the stochastic bounds are determined by

$$\begin{cases} V_{upp}(t) &= \sum_{i=1}^n e^{-\tilde{S}_f^c(t_i, F_{X(t_i)}^{-1}(1-U))} \\ V_{low}(t) &= \sum_{i=1}^n \mathbf{E}[e^{-\tilde{S}_f^c(t_i, X(t_i))} | \Lambda] \\ V_{imupp}(t) &= \sum_{i=1}^n e^{-\tilde{S}_f^c(t_i, F_{X(t_i)|\Lambda}^{-1}(1-U))}. \end{cases} \quad (22)$$

In these expressions,  $U$  is a uniform(0,1) distributed random variable, and  $\Lambda$  is an arbitrary variable such that the distribution of  $X(t_i)|\Lambda$  is known.

In the present contribution, we confine to discrete cash flow streams. However, note that by taking limits, the case of a continuous annuity  $V(t) = \int_0^t e^{-\tilde{S}_f^c[\tau, X(\tau)]} d\tau$  can be solved in a similar way.

### 3.2 The case of the Ho-Lee model and the Vasicek model

Recall from subsection 2.3 that for both models under investigation, the accumulated interest rate is normally distributed,  $X(\tau) \sim N[\mu(\tau), \sigma^2(\tau)]$ . For the Ho-Lee model, we change the mean  $\mu(t)$  and variance  $\sigma^2(t)$  for  $\mu_{\text{HL}}(t)$  and  $\sigma_{\text{HL}}^2(t)$  (see equations (13) and (14)); for the Vasicek model we take  $\mu_{\text{VA}}(t)$  and  $\sigma_{\text{VA}}^2(t)$  (see equations (16) and (17)).

In order to compute the lower bound and improved upper bound as stated in theorem 3.1, we have to decide on  $\Lambda$ , the conditioning random variable. Therefore, as proposed in Vyncke et al. (2001), we first define the variable

$$Z(\delta) = - \int_0^\delta X(\tau) d\tau, \quad (23)$$

which is also normally distributed, particularly with mean  $\mu_\delta$  and variance  $\sigma_\delta^2$ , both depending on the mean and variance of  $X(t)$ . Starting from this variable  $Z(\delta)$ , we choose  $\Lambda(\delta)$  as the standardized counterpart, or

$$\Lambda(\delta) = \frac{Z(\delta) - \mu_\delta}{\sigma_\delta}, \quad (24)$$

for which the distribution function can be easily written as  $F_{\Lambda(\delta)}(\lambda) = \Phi(\lambda)$ .

Note that  $\delta$  has to be fixed later on; the concrete choice for  $\delta$  is related to the amount of information that is useful to be incorporated in the conditioning variable.

Since both  $X(\tau)$  and  $\Lambda(\delta)$  are normally distributed, the conditional variable  $X(\tau)|\Lambda(\delta)$  is also normally distributed. Making use of the notation

$$k(\tau, \delta) = \text{Cov}[-X(\tau), \Lambda(\delta)] = \frac{1}{\sigma_\delta} \int_0^\delta \text{Cov}[X(\tau), X(\nu)] d\nu, \quad (25)$$

the mean and variance of this variable  $X(\tau)|\Lambda(\delta)$  can be found as

$$\begin{cases} \bar{\mu}(\tau, \delta, \lambda) &= \text{E}[X(\tau)|\Lambda(\delta) = \lambda] &= \mu(\tau) - k(\tau, \delta) \cdot \lambda \\ \bar{\sigma}^2(\tau, \delta, \lambda) &= \text{Var}[X(\tau)|\Lambda(\delta) = \lambda] &= \sigma^2(\tau) - k^2(\tau, \delta). \end{cases} \quad (26)$$

In the Ho-Lee case, variance and covariance can be calculated as

$$\begin{cases} \sigma_\delta &= \frac{\gamma \delta^2}{2} \sqrt{\delta/5} \\ k(\tau, \delta) &= \frac{\gamma^2 \tau^2}{2\sigma_\delta} \left( \frac{\tau^2}{12} - \frac{\tau\delta}{3} + \frac{\delta^2}{2} \right); \end{cases} \quad (27)$$

for the Vasicek model, one obtains

$$\begin{cases} \sigma_\delta &= \frac{\gamma}{\beta^2} \left\{ \beta \delta^2 \left( \frac{\beta \delta}{3} - 1 \right) - \delta (2e^{-\beta \delta} - 1) - \frac{1}{2\beta} (e^{-2\beta \delta} - 1) \right\}^{1/2} \\ k(\tau, \delta) &= \frac{1}{\sigma_\delta} \frac{\gamma^2}{\beta^2} \left\{ \tau \delta - \frac{\tau^2}{2} + \frac{1}{\beta} \left( \delta + \frac{e^{-\beta \delta}}{\beta} \right) (e^{-\beta \tau} - 1) - \frac{1}{2\beta^2} (e^{-2\beta \tau} + 1) \right\}. \end{cases} \quad (28)$$

With these specifications, bounds on the present value (20) as introduced in theorem 3.1 can be expressed analytically. In order not to complicate the formulas, we define the values

$$\begin{aligned} z_i^{(f)}(\delta, \lambda) &= \frac{f(t_i) - \bar{\mu}(t_i, \delta, \lambda)}{\bar{\sigma}(t_i, \delta, \lambda)} \\ \text{and } z_i^{(c)}(\delta, \lambda) &= \frac{c(t_i) - \bar{\mu}(t_i, \delta, \lambda)}{\bar{\sigma}(t_i, \delta, \lambda)}, \end{aligned} \quad (29)$$

as well as the functions

$$\begin{aligned} G_i &: [0, 1] \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^+ : (p, \delta, \lambda) \mapsto G_i(p, \delta, \lambda) \\ \text{and } H_i &: \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^+ : (\delta, \lambda) \mapsto H_i(\delta, \lambda) \end{aligned} \quad (30)$$

with

$$G_i(p, \delta, \lambda) = \begin{cases} e^{-c(t_i)} & \text{if } p \in \left[0, \Phi\left(-z_i^{(c)}(\delta, \lambda)\right)\right[ \\ e^{-f(t_i)} & \text{if } p \in \left[1 - \Phi\left(z_i^{(f)}(\delta, \lambda)\right), 1\right] \\ e^{-\bar{\mu}(t_i, \delta, \lambda) + \bar{\sigma}(t_i, \delta, \lambda)\Phi^{-1}(p)} & \text{if } p \in \left[\Phi\left(-z_i^{(c)}(\delta, \lambda)\right), 1 - \Phi\left(z_i^{(f)}(\delta, \lambda)\right)\right[ \end{cases} \quad (31)$$

and

$$\begin{aligned} H_i(\delta, \lambda) &= e^{-c(t_i)}\Phi\left(-z_i^{(c)}(\delta, \lambda)\right) + e^{-f(t_i)}\Phi\left(z_i^{(f)}(\delta, \lambda)\right) \\ &+ e^{-\bar{\mu}(t_i, \delta, \lambda) + \frac{1}{2}\bar{\sigma}^2(t_i, \delta, \lambda)} \\ &\cdot \left[\Phi\left(z_i^{(c)}(\delta, \lambda) + \bar{\sigma}(t_i, \delta, \lambda)\right) - \Phi\left(z_i^{(f)}(\delta, \lambda) + \bar{\sigma}(t_i, \delta, \lambda)\right)\right]. \end{aligned} \quad (32)$$

The following result then holds:

**Theorem 3.2** *If the interest rate is modelled by means of a Ho-Lee model or Vasicek model as described in section 2.3, the present value of a discrete annuity certain as in equation (20) can be bounded by*

$$V_{low}(t) \leq_{cx} V(t) \leq_{cx} V_{imupp}(t) \leq_{cx} V_{upp}(t), \quad (33)$$

where

$$V_{upp}(t) = \sum_{i=1}^n e^{-\tilde{S}_f^c[t_i, (\mu(t_i) + \sigma(t_i)\Phi^{-1}(1-U))]}, \quad (34)$$

$$V_{low}(t) = \sum_{i=1}^n H_i(\delta, \Lambda(\delta)), \quad (35)$$

and

$$V_{imupp} = \sum_{i=1}^n G_i(U, \delta, \Lambda(\delta)). \quad (36)$$

Note that for the Ho-Lee model, we change the mean  $\mu(t)$  and variance  $\sigma^2(t)$  for  $\mu_{HL}(t)$  and  $\sigma_{HL}^2(t)$  (see equations (13) and (14)); for the Vasicek model we take  $\mu_{VA}(t)$  and  $\sigma_{VA}^2(t)$  (see equations (16) and (17)).

Due to comonotonicity properties, distribution functions of these bounds can be found straightforwardly.

**Theorem 3.3** *The cumulative distribution functions of the convex bounds of theorem 3.2 can be calculated as*

$$\begin{cases} F_{V_{upp}}(x) &= 1 - \Phi(\nu_x), \\ F_{V_{low}}(x) &= \Phi(\lambda_{x,\delta}), \\ F_{V_{imupp}}(x) &= \int_{-\infty}^{+\infty} \kappa(x, \delta, \lambda) d\Phi(\lambda), \end{cases} \quad (37)$$

with  $\nu_x$  and  $\lambda_{x,\delta}$  defined implicitly by

$$\begin{cases} \sum_{i=1}^n e^{-\tilde{S}_f^c[t_i, \mu(t_i) + \sigma(t_i)\nu_x]} = x, \\ V_{low}(t)|_{\Lambda(\delta)=\lambda_{x,\delta}} = x, \end{cases} \quad (38)$$

and with  $\kappa(x, \delta, \lambda)$  defined explicitly as

$$\kappa(x, \delta, \lambda) = \sup \left\{ p \in [0, 1] \left| \sum_{i=1}^n G_i(p, \delta, \lambda) \leq x \right. \right\}. \quad (39)$$

## 4 Applications

Instead of a constant annuity, we now consider general discrete annuities over the time-interval  $[0, t]$ , with a truncate (stochastic) interest rate as defined in definition 2.2, and with a deterministic payment function  $\xi(t)$ :

$$V^*(t) = \sum_{i=1}^n \xi(t_i) e^{-\tilde{S}_f^c(t_i, X(t_i))}. \quad (40)$$

As before,  $X = \{X(t)\}$  is a stochastic process with  $X(t_i)$  denoting the compounded rate of return for the period  $[0, t_i]$ .

Theorem 3.1 can be extended as follows: (see also Kaas et al. (2000))

**Theorem 4.1** *The present value of equation (40) can be bounded in convex ordering sense as*

$$V_{low}^*(t) \leq_{cx} V^*(t) \leq_{cx} V_{imupp}^*(t) \leq_{cx} V_{upp}^*(t), \quad (41)$$

where the stochastic bounds are determined by

$$\left\{ \begin{array}{l} V_{upp}^*(t) = \sum_{i=1}^n \max(0, \xi(t_i)) e^{-\tilde{S}_f^c(t_i, F_{X(t_i)}^{-1}(1-U))} \\ \quad \quad \quad - \sum_{i=1}^n \max(0, -\xi(t_i)) e^{-\tilde{S}_f^c(t_i, F_{X(t_i)}^{-1}(U))} \\ V_{low}^*(t) = \sum_{i=1}^n \xi(t_i) E[e^{-\tilde{S}_f^c(t_i, X(t_i))} | \Lambda] \\ V_{imupp}^*(t) = \sum_{i=1}^n \max(0, \xi(t_i)) e^{-\tilde{S}_f^c(t_i, F_{X(t_i)|\Lambda}^{-1}(1-U))} \\ \quad \quad \quad - \sum_{i=1}^n \max(0, -\xi(t_i)) e^{-\tilde{S}_f^c(t_i, F_{X(t_i)|\Lambda}^{-1}(U))}, \end{array} \right. \quad (42)$$

where  $U$  is a uniform(0,1) distributed random variable, and where  $\Lambda$  is an arbitrary variable such that the distribution of  $X(t_i)|\Lambda$  is known.

As in the previous section, these bounds can be expressed analytically in the case the interest rate is described by means of a Ho-Lee or Vasicek model.

Applications of this result are obvious, e.g.

- for an indexed payment, use can be made of  $\xi(t) = (1 + d_t)^t$ , with  $d_t$  the indexing factor for the period  $[t_{i-1}, t_i]$ ;
- for a life annuity,  $\xi(t) = {}_t p_x$ , where  ${}_t p_x$  is the classical notation used for the probability of a person of age  $x$  to be still alive after  $t$  years;
- for an indexed life annuity:  $\xi(t) = (1 + d_t)^t \cdot {}_t p_x$ ;
- for a life assurance policy:  $\xi(t) = {}_t p_x \cdot \mu_{x+t}$ , where  $\mu_x$  is the mortality intensity at age  $x$ .

## 5 Numerical examples and conclusions

In this section we investigate the accuracy of the bounds for the present value function  $V(t)$  by comparing their cumulative distribution function to the empirical cumulative distribution function obtained with a Monte Carlo simulation. In order to visualize the goodness-of-fit, we construct a Q-Q-plot: we compare the upper and lower bound to the empirical distribution<sup>(1)</sup>. Finally, we calculate the Value-at-Risk (VaR) for some specified high percentages. We will consider the 90%, 95%, 97,5% and 99% quantiles<sup>(2)</sup>. In order to quantify the error of the simulated VaR, we use the variation

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<sup>(1)</sup>The simulation is based on 10000 randomly generated paths.

<sup>(2)</sup>We calculate these quantiles analytically for the distribution of the lower bound and upper bound, and compare them with the corresponding values for the real distribution by means of a simulation (20 times 5000 paths). Taking for each quantile the mean of these 20 values, gives us the simulated VaR.

coefficient (v.c.), defined as the standardized deviation, or the standard deviation divide by the mean. Since this standardized deviation is independent of the magnitude of the observed variable, it is more adequate to compare different situations.

For the numerical illustrations and the concrete examples, we have to choose a large number of parameters and functions. First, we consider two models for the description of the stochastic interest rate. In the case of a Ho-Lee model we then need to decide on the drift function  $\alpha(t)$ , the volatility  $\gamma$  and the starting point  $r_0$ ; in the case of a Vasicek model we have to determine  $\alpha, \beta, \gamma$  and  $r_0$ . Then, there is the timehorizon and the period (monthly, yearly,...) of the payments as well as the amount of each payment  $\xi_i$ . Finally, we have a wide range of possibilities for the definition of floor and capfunctions.

Out of this large set, we have chosen 6 combinations, which we summarized for convenience in the tables 1 and 2 below.

Let us start by visualizing the influence of the truncing operation on the present value function. In Figure 2.a and 2.b. we plotted the cumulative distribution function (cdf) of the upper bound (dashed line), lower bound (grey line) and the simulated cdf (solid line). In case there is no truncation of the interest rate, we obtain the well known results as in Dhaene et al. (2002a, 2002b). The second plot (2.b) is meant to show what happens when we impose a truncation function.

As we consider a floorfunction and a capfunction, the influence of truncing is twofold. The kink that can be observed on the right side of the cdf, is caused by the floorfunction. The influence of the capfunction is manifested on the left side of the cdf<sup>(3)</sup>.

Let us focus now on the accuracy of the approximations. In general both upper and lower bounds are quite good, the lower bound performing even better than the upper bound, especially in case  $\delta$  is rather large<sup>(4)</sup>.

If the floor and cap are significant, which means that they can be frequently reached by the stochastic process, then the lower bound is a better approximation of the real present value than the upper bound. This is nicely illustrated in Figure 4 and Figure 6.

Next to the truncation, the choices for the other parameters also affect the goodness-of-fit. A major factor of influence is the volatility of the process: the lower the volatility, the better the approximations.

We especially want to mention the interaction effect of the volatility and the truncation functions. Indeed, a high volatility implies a wide range of outcomes for the stochastic interest rate and hence a higher probability that the floor and cap will be reached.

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<sup>(3)</sup>A clear example of this phenomenon can be seen also in Figure 4.a.

<sup>(4)</sup>The larger  $\delta$ , the better the accuracy of the lower bound, which can be checked in Figure 2.b. This is completely due to the fact that large values of  $\delta$  correspond to the use of more information.

Table 1: Parameters used in the numerical examples (1)

Examples involving a Vasicek model				
	Figure 2.a	Figure 2.b	Figure 3	Figure 4
$\alpha$	0.2	0.2	0.03	0.03
$\beta$	0.1	0.1	0.2	0.2
$\gamma$	0.2	0.2	0.1	0.1
$r_0$	$\ln(1.04)$	$\ln(1.04)$	$\ln(1.04)$	$\ln(1.04)$
t	1	1	10	10
n	12	12	120	120
$\delta$	1	1 and 0.8	8	8
$f(t)$	$-\infty$	0.02	$0.01t + 0.005\sin(10\pi t)$	$\max(0, 0.03 - [t] \cdot 0.01)$
$c(t)$	$+\infty$	0.10	$0.3t + 0.005\sin(2\pi t)$	$0.03 + [t] \cdot 0.02$
$\xi_i$	1	1	1	$(1.02)^{i/12}$

Table 2: Parameters used in the numerical examples (2)

Examples involving a Ho-Lee model		
	Figure 5	Figure 6
$\alpha(t)$	$0.01 + 0.003e^{-0.01t} (3\cos(3t) - 0.01\sin(3t))$	$0.01 + 0.001[t]$
$\gamma$	0.01	0.1
$r_0$	0.02	$\ln(1.04)$
t	5	5
n	60	60
$\delta$	4	4
$f(t)$	$0.02t$	$0.02 + 0.01t$
$c(t)$	$0.08t$	$0.08 + 0.08t$
$\xi_i$	$(1.03)^{i/12}$	1

Value-At-Risk				
q	sim	(v.c.)	up	low
0.90	12.0656	(0.001269)	12.0785	12.0542
0.95	12.2746	(0.001461)	12.3000	12.2680
0.975	12.4620	(0.002057)	12.4971	12.4582
0.99	12.6896	(0.003523)	12.7321	12.6849

Table 3: No truncation

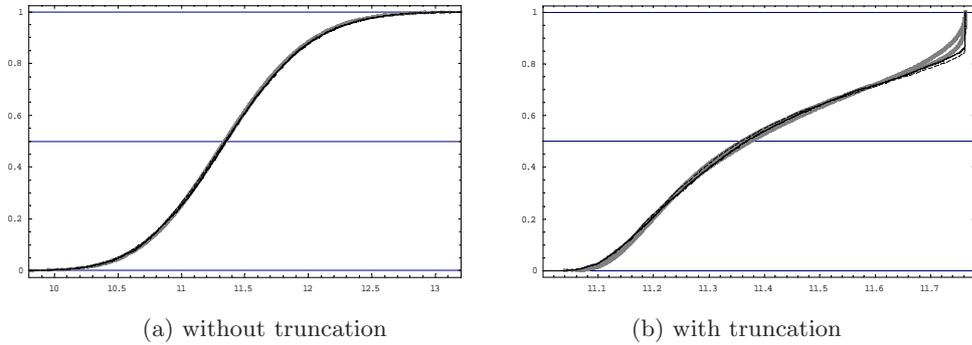


Figure 2: Influence of truncating the interest rate

Value-At-Risk					
q	sim	(v.c.)	up	low $\delta = 1$	low $\delta = 0.8$
0.90	11.7624	(0)	11.7624	11.7584	11.7465
0.95	11.7624	(0)	11.7624	11.7622	11.7597
0.975	11.7624	(0)	11.7624	11.7624	11.7620
0.99	11.7624	(0)	11.7624	11.7624	11.7624

Table 4: Truncation

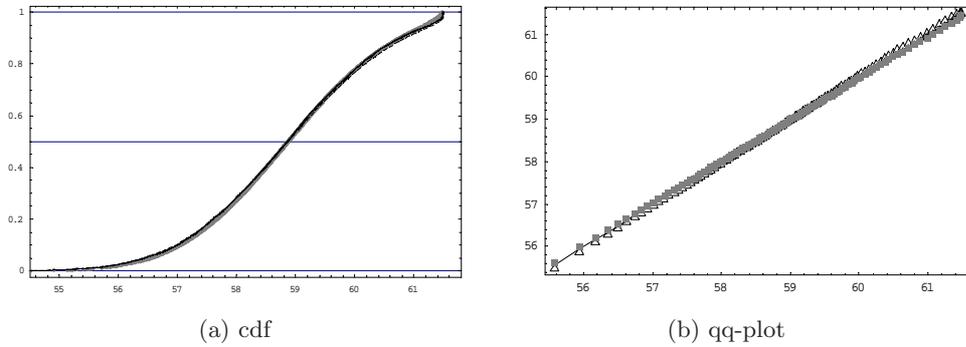


Figure 3: Vasicek model oscillating truncation functions

The QQ-plots give us an extra insight in the goodness-of-fit of the distributions, since they blow up the possible deviations. The grey boxes refer to the quantiles of the lower bound, the triangles refer to the upper bound. Again we can conclude that in

Value-At-Risk				
q	sim	(v.c.)	up	low
0.90	113.512	(0.0005908)	114.142	112.418
0.95	114.105	(0.0000590)	114.145	113.603
0.975	114.139	(9.596x10 <sup>-6</sup> )	114.146	113.926
0.99	114.145	(4.193x10 <sup>-6</sup> )	114.148	114.045

Table 5: Vasicek model oscillating truncation functions

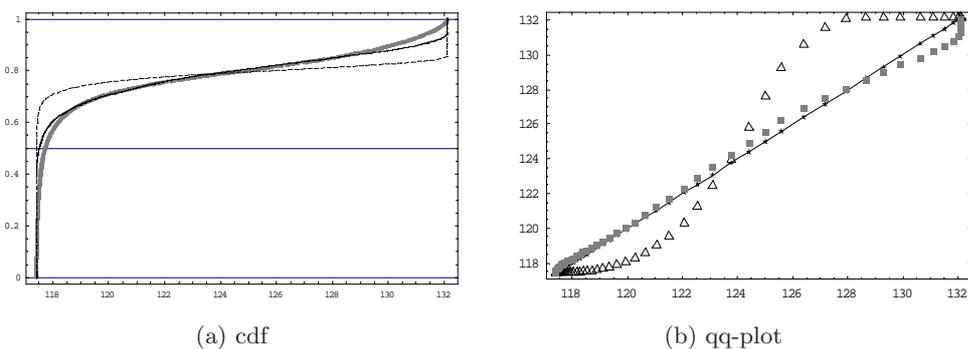


Figure 4: Vasicek model, amortization scheme

Value-At-Risk				
q	sim	(v.c.)	up	low
0.90	131.130	(0.0009523)	132.118	130.177
0.95	132.118	(0)	132.118	131.542
0.975	132.118	(0)	132.118	131.941
0.99	132.118	(0)	132.118	132.074

Table 6: Vasicek model, amortization scheme

general, the lower bound is a better approximation for the real distribution.

For each situation, we calculated the Value-at-Risk, following the procedure as described earlier. We can observe that the approximations are very close to the simulated VaR. In some cases, the upper bound gives identical results as the simulations, e.g. in Table 4, 6 and 8. This is completely due to the truncation. Indeed, when the interest rate reaches the floorfunction, it is at its minimum and therefore it maximizes the present value. If this happens frequently, it causes a vertical jump in the cdf of

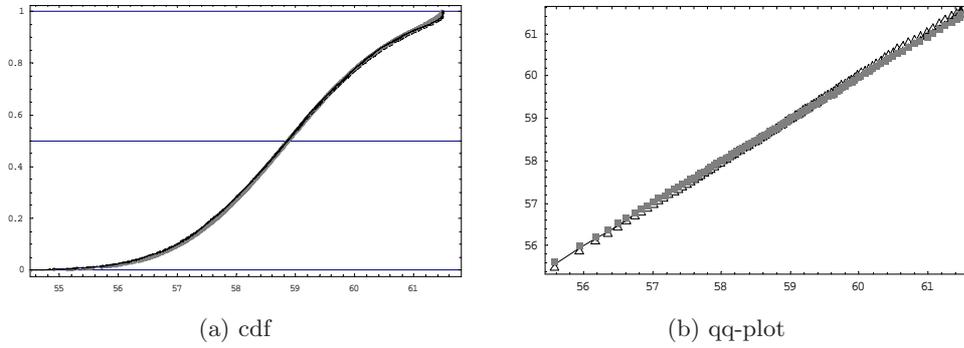


Figure 5: Ho-Lee model oscillating driftfunction, linear floor and cap

Value-At-Risk				
q	sim	(v.c.)	up	low
0.90	60.7707	(0.0004261)	60.8538	60.7542
0.95	61.2445	(0.0004295)	61.3135	61.1815
0.975	61.4482	(0.0001651)	61.4812	61.3699
0.99	61.4810	$(7.706 \times 10^{-6})$	61.4814	61.4551

Table 7: Ho-Lee model oscillating driftfunction, linear floor and cap

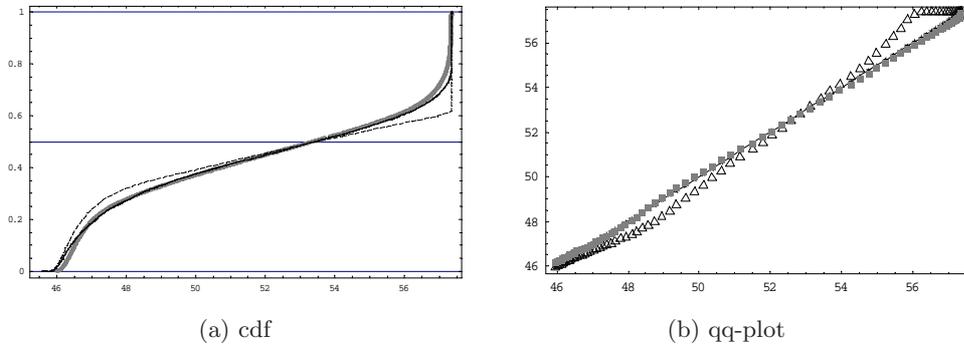


Figure 6: Ho-Lee model, linear floor and cap, stepfunction for drift

the present value function. The VaR, being a high quantile, will then coincide with this maximum value.

Value-At-Risk				
q	sim	(v.c.)	up	low
0.90	57.3419	(0)	57.3419	57.3270
0.95	57.3419	(0)	57.3419	57.3373
0.975	57.3419	(0)	57.3419	57.3401
0.99	57.3419	(0)	57.3419	57.3413

Table 8: Ho-Lee model, linear floor and cap, stepfunction for drift

Finally, in order to conclude, we take a look at the computation time of these quantiles. The upper VaR can be calculated ten times faster than the lower VaR, which on its turn is about 60000 times faster than the simulations! We are talking about less than a second versus more than ten hours. Combining these observations with the established precision, the upper and certainly the lower bound provide very good alternatives for the real VaR.

The differences in time are less drastic for the graphs of the cdf's of the present value function, since less simulations are needed to construct a rather accurate graph. In this setting, upper and lower bounds can be drawn about 600 times faster than a simulation. The improved upperbound, which is omitted in our examples here, is also rather time consuming, but as for the lower bound, the accuracy is very high.

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