DEPARTMENT OF ACCOUNTING AND FINANCE

Understanding copula transforms: a review of dependence properties

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Understanding copula transforms: a review of dependence properties

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1 Introduction

A copula is a flexible modeling tool which contributes substantially to the study of dependencies among random variables. A broad copula class with many nice properties is the Archimedean copula class. Usually, one works with the classical bivariate models, e.g. as summarized in Nelsen (2006), which are one-parametric models. However, in many cases when practitioners want to model dependencies by means of copulas, it would be more rational to work with multi-parametric models. Indeed, multi-parametric models would allow to better harmonize empirical information with the model, as it would be possible to directly import more than one characteristic into the model, e.g. measures of concordance, tail dependence and so on. Various ways exist and have been explored to define multi-parameter Archimedean models. This paper intends to elaborate on one particular method, namely the technique of transforms. More specifically, the contribution of this article is threefold:

1. Genest et al. (1998) sum up five feasible transformations applicable on the Archimedean generator φ. In this note we present an overview of these transformations by generalizing tail dependence properties and limiting cases.

2. In an earlier paper, see Michiels et al. (2008), we showed that it can be advantageous to work with the λ-function instead of with the generator function. We investigate here the effect of transforms on this λ-function.

3. We introduce a new type of transform which is *concordance invariant*. As such, this type of transform has practical use as it allows to create comparable test spaces (see Michiels and De Schepper (2008)) from a particular copula family.

The paper is organised as follows. In section two the most important copula properties are discussed, with the focus on the Archimedean class. Next, section three reviews generally known copula transforms and generator transforms. In section four the effect of the transforms on the λ-level is being discussed, which allows the derivation of general tail dependence properties and limiting cases. We also introduce a concordance invariant transform and illustrate its use through simulations. Finally, section five concludes.

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2 Basic Concepts

We start with the definition and the most important theorem for copulas. We use the symbol **I** to denote the unit interval [0, 1].

Definition 1 *A bivariate copula is a function* $C : \mathbf{I^2} \to \mathbf{I}$ *with the following properties:*

1. *C is 2-increasing, or for all* $u_1 \leq u_2, v_1 \leq v_2 \in \mathbf{I}$ *it is true that* $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$.
2. *C is grounded, or* $C(u, 0) = C(0, v) = 0$ *for all* $(u, v) \in \mathbf{I^2}$.
3. *C has uniform [0, 1] margins, or* $C(u, 1) = u$ *and* $C(1, v) = v$ *for all* $(u, v) \in \mathbf{I^2}$.

In fact a copula represents the link between the marginal distribution functions and their joint aggregate. This link can be formalized through the following theorem:

Theorem 1 (Sklar's theorem) *Let H be a bivariate joint distribution function with margins F and G. Then there exists a copula C in such a way that* $H(x, y) = C(F(x), G(y))$ *for all* $(x, y) \in \bar{\mathbb{R}}$.

If F and G are defined continuously, then C is unique. If not, then C is unique on $\text{im}F \times \text{im}G$. Conversely, if C is a copula and F and G are distribution functions, then H is defined as indicated above.

Some well-known examples of copula functions are the Fréchet lower bound $W(u, v) = \max(u + v - 1, 0)$ and upper bound $M(u, v) = \min(u, v)$ and also the independence copula $\Pi(u, v) = uv$. The first and second copulas refer to the case where perfect positive respectively negative dependence exists. The latter refers to the independence case.

The most important dependence property of a copula is, in the first place, the general degree of dependence it depicts. This can be studied through the relationship between Kendall's τ and the copula C:

$$\tau = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1 = 1 - 4 \int_0^1 \int_0^1 \frac{\partial C}{\partial u}(u, v) \frac{\partial C}{\partial v}(u, v) du dv. \quad (1)$$

A second important aspect of the dependence structure is the dependence in the tails. In this respect, the coefficients of tail dependence (see Joe (1997)) are a powerful tool.

Let X and Y be continuous random variables with margins F and G, respectively.

Definition 2 *The upper tail dependence parameter* λ_U *is the limit (provided it exists) of the conditional probability that Y is greater than the t-th percentile of G given that X is greater than the t-th percentile of F as t approaches 1, i.e.* $\lambda_U = \lim_{t \to 1^-} P[Y > G^{[-1])}(t) | X > F^{[-1]}(t)]$.
The lower tail dependence parameter λ_L *is the limit (provided it exists) of the conditional probability that Y does not exceed the t-th percentile of G given that X does not exceed the t-th percentile of F as t approaches 0, i.e.* $\lambda_L = \lim_{t \to 0^+} P[Y \leq G^{[-1]}(t) | X \leq F^{[-1]}(t)]$.

In this paper we will mainly focus on a well-known class of copula families, the Archimedean copula class. This class is solely characterized by a generator φ, a function which enables the construction of a copula from this class. The copula generator φ and its pseudo-inverse are defined in the following way:

Definition 3 *A generator* φ *is a continuous, strictly decreasing convex function defined on* ***I*** *and image* $[0, \infty)$. *If* $\varphi(0) = \infty$ *then the generator is called strict. The pseudo-inverse of* φ *is the function* $\varphi^{[-1]}$ *with support [0,* ∞ *) and image* ***I****, given by*

$$\varphi^{[-1]} = \begin{cases} \varphi^{-1}(t), & 0 \leq t \leq \varphi(0) \\ 0, & \varphi(0) \leq t < \infty \end{cases} . \quad (2)$$

With this generator the Archimedean copula can be defined as follows:

Definition 4 *A bivariate Archimedean copula with generator φ is the function $C : \mathbf{I^2} \to \mathbf{I}$ defined as:*

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)). \tag{3}$$

For dependence measures and coefficients, the following results hold:

Lemma 1 *For an Archimedean copula with generator φ, the concordance measure Kendall's τ can be written as*

$$\tau = 1 + 4 \int_0^1 \frac{\varphi(t)}{\varphi'(t)}\, \mathrm{d}t. \tag{4}$$

For a proof, see Genest and Mackay (1986).

Lemma 2 *For an Archimedean copula with generator φ, the coefficients of tail dependence can be calculated as*

$$\lambda_L = \lim_{t \to 0+} \frac{\varphi^{[-1]}(2\varphi(t))}{t} = \lim_{x \to \infty} \frac{\varphi^{[-1]}(2x)}{\varphi^{[-1]}(x)} \tag{5}$$

and

$$\lambda_U = \lim_{t \to 1-} \frac{1 - \varphi^{[-1]}(2\varphi(t))}{1 - t} = 2 - \lim_{x \to 0} \frac{1 - \varphi^{[-1]}(2x)}{1 - \varphi^{[-1]}(x)}. \tag{6}$$

For a proof, see Nelsen (2006).
An alternative way for working with Archimedean copulas is by using the λ function, defined as follows:

Definition 5 *For an Archimedian copula with generator function φ, the λ function is defined as*

$$\lambda : [0, 1] \to [-1, 0] : t \mapsto \lambda(t) = \frac{\varphi(t)}{\varphi'(t)}. \tag{7}$$

From (7) we can recover φ by solving the differential equation. This yields

$$\varphi(t) = \varphi(t_0) e^{\int_{t_0}^{t} \frac{1}{\lambda(z)} \mathrm{d}z} \tag{8}$$

for $0 < t_0 < 1$. This function is properly defined, see e.g. Genest and Rivest (1993). In order to define a feasible φ some restrictions need to be imposed on λ.

Property 1 *The function φ as in (8) will be a well-defined generator function, if the function λ is a continuously differentiable function with domain [0, 1] for which is true that*

LR1 $\lambda(0) \in$ *[-1, 0]*

LR2 $\lambda(1) =$ *0*

LR3 $\lambda(t) <$ *0*, $t \in$ *(0, 1)*

LR4 $\lambda'(t) <$ *1*, $t \in$ *(0, 1).*

As the corresponding generator function can only be strict if $\lambda(0) = 0$, we will limit ourselves to λ-functions starting in zero. Defining an Archimedean copula in this way is advantageous, as there exists a straightforward relationship between the λ function and dependence measures and coefficients.

Lemma 3 *For an Archimedean copula with generator φ, the concordance measure Kendall's τ can be written as*

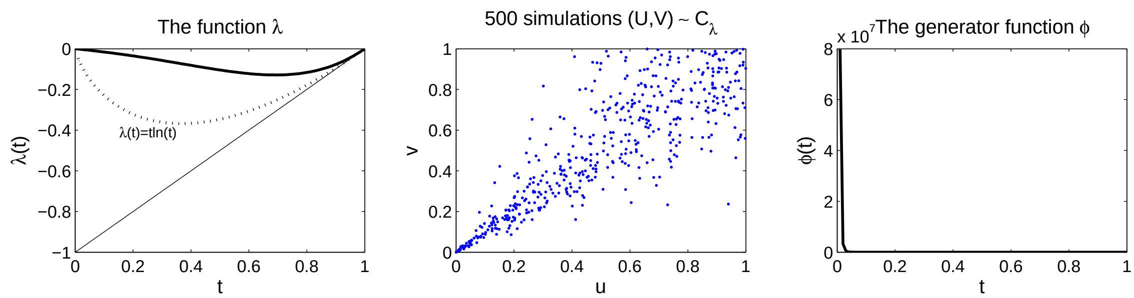
$$\tau = 1 + 4\int_0^1 \lambda(t)\,\mathrm{d}t. \tag{9}$$

Lemma 4 *For an Archimedean copula with generator φ, the coefficients of tail dependence can be calculated as*

$$\lambda_L = 2^{\lambda'(0^+)} \qquad \lambda_U = 2 - 2^{\lambda'(1^-)}. \tag{10}$$

Another advantage of the λ-approach is the fact that it is visually more informative than the common Archimedean generator φ. In figure 1 the visual interpretation of λ is shown, compared to that of the generator φ. In figure 1 (left) a graph of the λ function is shown in what we will call the "Kendall's τ space", which is the region bounded by the λ versions of the two Fréchet bounds. Note that from (9) it follows that the (negative) area between the λ curve and the horizontal line is straightforwardly linked to the value of τ. When comparing this area to the independence case $\lambda(t) = t\ln t$ it can be seen that the λ function displays strong positive dependence. Furthermore, when comparing the λ function (figures 1 (left)) and the generator function φ (figure 1 (right)) to simulations from the pair $(U, V) \sim C_\lambda$ (figure 1 (central)) it becomes clear that the function λ acts as a better representative of the affiliated copula than φ. Indeed, notice how the relative distance from the Fréchet upper bound indicates the level of dependence per quantile unit. For a more detailed discussion about the λ-function, see Michiels et al. (2008).

Figure 1: Illustration of a copula with $\lambda(t) = t(t-1)(0.63t^2 + 0.2180t + 0.1520)$ and $\tau = 70\%$, $\lambda_L = 90\%$ and $\lambda_U = 0\%$.



Finally, we will use the notation φ_Π, φ_W and $\varphi_{\frac{\Pi}{\Sigma-\Pi}}$ for the generator functions and λ_Π, λ_W and $\lambda_{\frac{\Pi}{\Sigma-\Pi}}$ for the λ-functions of the independent copula (Π), the countermonotone copula (W) and the copula function $C(u,v) = \dfrac{uv}{u+v-uv}$ $(\frac{\Pi}{\Sigma-\Pi})$, which is a member of the well-known Clayton family. See Table 1 for an overview.

	generator $\varphi(t)$	λ-function $\lambda(t)$
Π	$-\ln(t)$	$t\ln(t)$
W	$1-t$	$t-1$
$\frac{\Pi}{\Sigma-\Pi}$	$\frac{1}{t}-1$	t^2-t

Table 1: Overview φ and λ for particular dependencies.

3 Copula transforms: review

A copula transform is a transformation on an existing copula or copula generating function with the intention to

- create a new copula model with interesting properties;
- combine several existing copula models;
- add additional parameters to the copula function to increase modeling freedom.

The advantages of this approach are obvious, as this allows us to work with much more powerful models that can provide a much better fit.
Applications of transforms can, amongst others, be found in:

- Genest et al. (1998): construction of a three parameter family including Frank's family, Clayton's family and Gumbel's family.
- Nelsen (2006): the α and β two-parameter families of Archimedean copulas,
- Michiels and De Schepper (2008): diversifying copula test spaces, with the aim at increasing the possibility of finding a good fit.

3.1 Transformations on C

For the sake of completeness we start by recalling some well-known copula transforms together with a discussion about their effect on both concordance and tail dependence. There are four feasible transforms on the copula level:

- The convex combination:
  $C_{12}^{\alpha}(u,v) = \alpha C_1(u,v) + (1-\alpha)C_2(u,v), \alpha \in [0,1]$,
- The first associated transform:
  $C'(u,v) = u - C(u, 1-v)$,
- The second associated transform:
  $C''(u,v) = v - C(1-u, v)$,
- The third associated transform (survival copula):
  $\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$.

In Table 2 we summarize the effect of the transformations on the overall dependence, in terms of Kendall's τ, as well as on the local dependence in the tails, given by λ_L and λ_U.

	C'	C''	$\hat{C}$
τ	$-\tau_C$	$-\tau_C$	τ_C
λ_L	$1-\lim_{t\to 0}\frac{dC(t,1-t)}{dt}$	$1-\lim_{t\to 0}\frac{dC(1-t,t)}{dt}$	$\lambda_{U,C}$
λ_U	$1-\lim_{t\to 0}\frac{dC(1-t,t)}{dt}$	$1-\lim_{t\to 0}\frac{dC(t,1-t)}{dt}$	$\lambda_{L,C}$

	C_{12}^{α}
τ	$1+\alpha^2(\tau_{C_1}-1)+(1-\alpha)^2(\tau_{C_2}-1)-4\alpha(1-\alpha)\cdot \left(\int_0^1\int_0^1 \frac{dC_1(u,v)}{du}\frac{dC_2(u,v)}{dv}+\frac{dC_1(u,v)}{dv}\frac{dC_2(u,v)}{du}dudv\right)$
λ_L	$\alpha\lambda_{L,C_1}+(1-\alpha)\lambda_{L,C_2}$
λ_U	$\alpha\lambda_{U,C_1}+(1-\alpha)\lambda_{U,C_2}$

Table 2: Overview transforms properties.

Note that for exchangeable copulas the last term of $\tau_{C_{12}^{\alpha}}$ can be simplified. The effect on the tails of the copula can be understood as a linear combination of the tail dependence values of the initial copulas.
Associated copula families have tau values with the same magnitude as the original family. For the tails of the associated copulas the following is true: $\lambda_{U,C'} = \lambda_{L,C''}$ and $\lambda_{L,C'} = \lambda_{U,C''}$, $\lambda_{U,\hat{C}} = \lambda_{L,C}$ and $\lambda_{L,\hat{C}} = \lambda_{U,C}$.

3.2 Transformations on φ

Genest et al. (1998) introduce right and left composition rules for Archimedean generators, next to the scaling rule, exponentiation and linear combination. Given the importance of these transforms we repeat them here and give a proof. Afterwards, we show that all copula families mentioned in Nelsen (2006) can be written as a transformation of three basis copulas of Table 1. In order to determine whether the transformed generator φ^t is a feasible generator in the bivariate case, three aspects need to be checked:

GR1 φ^t is decreasing, or $\varphi'(t) < 0, t \in (0,1)$;

GR2 φ^t is convex, or $\varphi''(t) > 0, t \in (0,1)$;

GR3 $\varphi^t(1) = 0$.

A. Right composition

Lemma 5 (right composition (RC)) *If φ is a generator and $f : [0,1] \mapsto [0,1]$ is an increasing concave bijection with $f(0) = 0$ and $f(1) = 1$, the function defined by*

$$\varphi^t(t) = (\varphi \circ f)(t) \tag{11}$$

is a well defined Archimedean generator function. Furthermore, φ^t is strict iff φ is strict.

Proof: GR1 follows from the fact that $f'(t) > 0$ and GR2 follows from the fact that $f''(t) < 0$ (increasing concavity). GR3 follows from the fact that $f(1) = 1$ (increasing bijection on $[0,1]$).

B. Left composition

Lemma 6 (left composition (LC)) *If φ is a generator and $f : [0,+\infty) \mapsto [0,+\infty)$ is an increasing convex function with $f(0) = 0$, the function defined by*

$$\varphi^t(t) = (f \circ \varphi)(t) \tag{12}$$

is a well-defined Archimedean generator function. Furthermore, φ^t is strict iff φ is strict.

Proof: GR1 follows from the fact that $f'(t) > 0$ and GR2 follows from the fact that $f''(t) > 0$ (increasing convexity). GR3 follows from the fact that $f(0) = 0$.

C. Scaling

Lemma 7 (scaling rule (SC)) *If $\alpha \in (0,1)$ and φ is a generator, the function defined by*

$$\varphi^t(t) = \varphi(\alpha t) - \varphi(\alpha) \tag{13}$$

is a well-defined Archimedean generator function. Furthermore, φ^t is strict iff φ is strict.

Proof: GR1 and GR2 follow from the fact that $\alpha \in (0,1)$. GR3 is satisfied since $\varphi^t(1) = \varphi(\alpha) - \varphi(\alpha) = 0$.

D. Exponentiation

Lemma 8 (exponentation rule (EX)) *If φ_1 and φ_2 are two generators with $(\varphi_2')^2 \leq \varphi_2''$, the function defined by*

$$\varphi^t(t) = \varphi_1(e^{-\varphi_2(t)}) \tag{14}$$

is a well-defined Archimedean generator function. Furthermore, φ^t is strict iff φ_1 and φ_2 are strict.

Proof: GR1 follows readily from the proper definition of φ_1 and φ_2, and the fact that $f(x) = e^x > 0, x \in [0, \infty)$. As such $\varphi^t(t)' = \varphi_1'(e^{-\varphi_2(t)}) \cdot (-\varphi_2(t)' e^{-\varphi_2(t)}) < 0$. GR2 can be rewritten as $\varphi^t(t)'' = \varphi_1''(e^{-\varphi_2(t)}) \left[e^{-\varphi_2(t)}\varphi_2'(t)\right]^2 + \varphi_1'(e^{-\varphi_2(t)})e^{-\varphi_2(t)} \left[\varphi_2'(t)^2 - \varphi_2''(t)\right]$. As such, a sufficient condition to assure $\varphi^{t''} > 0$ is that $(\varphi_2')^2 \leq \varphi_2''$. GR3 follows from the proper definition of φ_1.

E. Linear combination

Lemma 9 (linear combination (LI)) *Let α and β be two positive constants, and φ_1 and φ_2 are two generators, the function defined by*

$$\varphi^t(t) = \alpha\varphi_1(t) + \beta\varphi_2(t) \tag{15}$$

is a well-defined Archimedean generator function. Furthermore, φ^t is strict iff φ_1 and/or φ_2 is strict.

Proof: GR1, GR2 and GR3 follow straightforwardly from the proper definition of φ_1 and φ_2.

In order to illustrate the possibilities of these transformations, we show in Table 3 how the Archimedean copula families from Nelsen (2006) can be retrieved from a limited number of basic copulas by means of a transformation[(1)]. Three different transforms are important: the left composition, the right composition and the linear combination. The column 'base generator' entails the starting generator on which the transform is applied; we recognize the three copulas from Table 1. Note that for some families two composition rules have to be combined. For example, copula family 4.2.14 is constructed by using first a right composition and then a left composition. This is indicated by 'RC-LC'. As also mentioned in Nelsen (2005), it is remarkable that next to the independent and countermonotone copula, the copula based on $\frac{\Pi}{\Sigma - \Pi}$ is rather prominent in this table.

4 Copula transforms and the function λ

In this section, we want to show that the effect of the generator transform can be studied more efficiently using the λ-function. We start by rewriting the transforms in such a manner that they can be used on the level of the λ-function, afterwards we will discuss tail dependence properties, and we also pay attention to a number of limiting cases which can be interesting for modeling purposes. Finally, we discuss the effect of the transforms on the degree of dependence (through Kendall's τ) and we discuss a τ preserving transform.

[(1)] Some of the copulas of Table 3 are known by their names rather than by their number: 4.2.1 (Clayton), 4.2.3 (Ali-Mikhail-Haq), 4.2.4 (Gumbel-Hougaard), 4.2.5 (Frank), 4.2.6 (Joe), 4.2.9 (Gumbel-Barnett), 4.2.15 (Genest-Ghoudi).

4.2.#	$\varphi_\theta(t)$	θ range	base generator	transform type	transform(s)
1	$\frac{1}{\theta}(t^{-\theta}-1)$	$[-1,\infty)\setminus\{0\}$	φ_Π	LC	$f_\theta(t)=\frac{e^{\theta t}-1}{\theta}$
2	$(1-t)^\theta$	$[1,\infty)$	φ_W	LC	$f_\theta(t)=t^\theta$
3	$\ln\frac{1-\theta(1-t)}{t}$	$[-1,1]$	φ_Π	RC	$f_\theta(t)=\frac{t}{1+\theta(t-1)}$
4	$(-\ln t)^\theta$	$[1,\infty)$	φ_Π	LC	$f_\theta(t)=t^\theta$
5	$-\ln\frac{e^{-\theta t}-1}{e^{-\theta}-1}$	$(-\infty,\infty)\setminus\{0\}$	φ_Π	RC	$f_\theta(t)=\frac{e^{-\theta t}-1}{e^{-\theta}-1}$
6	$-\ln[1-(1-t)^\theta]$	$[1,\infty)$	φ_Π	RC	$f_\theta(t)=1-(1-t)^\theta$
7	$-\ln[\theta t+(1-\theta)]$	$(0,1]$	φ_Π	RC	$f_\theta(t)=\theta t+(1-\theta)$
8	$\frac{1-t}{1+(\theta-1)t}$	$[1,\infty)$	φ_W	LC	$f_\theta(t)=-\frac{t(t-\theta(t-1))}{\theta}$
9	$\ln(1-\theta\ln t)$	$(0,1]$	φ_Π	RC	$f_\theta(t)=\frac{1}{1-\theta\ln t}$
10	$\ln(2-t^{-\theta}-1)$	$(0,1]$	φ_Π	RC	$f_\theta(t)=\frac{1}{2t^{-\theta}-1}$
11	$\ln(2-t^\theta)$	$(0,1/2]$	φ_Π	RC	$f_\theta(t)=\frac{1}{2-t^\theta}$
12	$\left(\frac{1}{t}-1\right)^\theta$	$[1,\infty)$	$\varphi_{\frac{\Pi}{\Sigma-\Pi}}$	LC	$f_\theta(t)=t^\theta$
13	$(1-\ln t)^\theta-1$	$(0,\infty)$	φ_Π	LC	$f_\theta(t)=(1+t)^\theta-1$
14	$(t^{-1/\theta}-1)^\theta$	$[1,\infty)$	$\varphi_{\frac{\Pi}{\Sigma-\Pi}}$	RC-LC	$f_{\theta,R}(t)=t^\theta, f_{\theta,L}(t)=t^\theta$
15	$(1-t^{1/\theta})^\theta$	$[1,\infty)$	φ_W	RC-LC	$f_{\theta,R}(t)=t^\theta, f_{\theta,L}(t)=t^\theta$
16	$\left(\frac{\theta}{t}+1\right)(1-t)$	$\theta_1=1,\theta_2\in[0,\infty)$	$\varphi_W,\varphi_{\frac{\Pi}{\Sigma-\Pi}}$	LI	$\varphi^t(t)=\theta_1\varphi_W+\theta_2\varphi_{\frac{\Pi}{\Sigma-\Pi}}$
17	$-\ln\frac{(1+t)^{-\theta}-1}{2^{-\theta}-1}$	$(-\infty,\infty)\setminus\{0\}$	φ_Π	RC	$f_\theta(t)=\frac{(1+t)^{-\theta}-1}{2^{-\theta}-1}$
18	$e^{\theta/(t-1)}$	$[2,\infty)$	φ_W	LC	$f_\theta(t)=-\frac{t^2}{\theta}$
19	$e^{\theta/t}-e^\theta$	$(0,\infty)$	φ_Π	LC	$f_\theta(t)=e^{\theta e^t}-e^\theta$
20	$\exp(t^{-\theta})-e$	$(0,\infty)$	φ_Π	LC	$f_\theta(t)=e^{e^{\theta t}}-e$
21	$1-[1-(1-t)^\theta]^{1/\theta}$	$[1,\infty)$	φ_W	RC-RC	$f_{\theta,1}(t)=t^\theta, f_{\theta,2}(t)=1-(1-t)^\theta$
22	$\arcsin(1-t^\theta)$	$(0,1]$	φ_W	RC-LC	$f_{\theta,R}(t)=t^\theta, f_{\theta,L}(t)=\arcsin(t)$

Table 3: Overview of transforms needed for the copulas in Nelsen (2006)

4.1 Transformations on λ

In general, a transformation on φ cannot be rewritten exclusively in terms of λ. Instead, there will be a remainder which we will specify as δ, i.e.

$$\varphi^t = f(\varphi) \Rightarrow \lambda^t = f(\lambda, \delta).$$

We will denote this remainder as δ_{RC} and δ_{LC} for respectively the right and left composition, δ_{SC} for the scaling and δ_{EX} and δ_{LI} for respectively the exponentiation and linear combination. The proofs are omitted as they are straightforward.

A. Right composition

Lemma 10 (right composition (RC)) *If $f : [0,1] \mapsto [0,1]$ is an increasing concave bijection with $f(0) = 0$ and $f(1) = 1$, the λ-function corresponding to the transformed generator is given by*

$$\lambda^t(t) = \delta_{RC}(t) \cdot (\lambda \circ f)(t) \tag{16}$$

with $\delta_{RC}(t) = \frac{1}{f'(t)}$.

B. Left composition

Lemma 11 (left composition (LC)) *If $f : [0,+\infty) \mapsto [0,+\infty)$ is an increasing convex function with $f(0) = 0$, the λ-function corresponding to the transformed generator is given by*

$$\lambda^t(t) = \delta_{LC}(t)\lambda(t) \tag{17}$$

with $\delta_{LC}(t) = \frac{(f\circ\varphi)(t)}{\varphi(t)(f'\circ\varphi)(t)}$.

C. Scaling

Lemma 12 (scaling rule (SC)) *If $\alpha \in (0,1)$, the λ-function corresponding to the transformed generator is given by*

$$\lambda^t(t) = \delta_{SC}(t) \cdot \lambda(\alpha t) \tag{18}$$

with $\delta_{SC}(t) = \frac{1}{\alpha}\left(1 - \frac{\varphi(\alpha)}{\varphi(\alpha t)}\right)$.

D. Exponentiation

Lemma 13 (exponentation rule (EX)) *If φ_1 and φ_2 are two generators with $(\varphi_2')^2 \leq \varphi_2''$, the λ-function corresponding to the transformed generator is given by*

$$\lambda^t(t) = \delta_{EX}(t) \cdot \lambda_1\left(e^{-\varphi_2(t)}\right) \tag{19}$$

with $\delta_{EX}(t) = -\frac{e^{\varphi_2(t)}}{\varphi_2'(t)}$ *and* $\lambda_1 = \lambda$ *corresponding to* φ_1.

E. Linear combination

Lemma 14 (linear combination (LI)) *If α and β are two positive constants, the λ-function corresponding to the transformed generator is given by*

$$\lambda^t(t) = \lambda_2(t) + \delta_{LI}(t)\left(\lambda_1(t) - \lambda_2(t)\right) \tag{20}$$

with $\delta_{LI}(t) = \frac{\alpha\varphi_1'(t)}{\alpha\varphi_1'(t)+\beta\varphi_2'(t)}$.

Note that the visual advantage of the λ-function disappears, because in most cases the function δ is dependent on the generator function φ. Nevertheless, working with transformations on λ still provides other advantages compared to the generator function. This will be illustrated in the following subsections.

4.2 Tail dependence properties

In Charpentier and Segers (2008) the effect on tail dependence for four particular transform cases is outlined. In this subsection we extend these results for general transforms. Note that the effect on the tails can be determined using (10).

A. Right composition

Lemma 15 *Let $f : [0,1] \mapsto [0,1]$ be an increasing concave bijection. The effect of the right composition on the tails of the transformed copula is given by:*

- $\lambda_L^t = \lambda_L \cdot 2^{\lim_{t\to 0}\left\{\lambda(f(t))\delta_{RC}'(t)\right\}}$,
- $\lambda_U^t = 2 - (2 - \lambda_U) \cdot 2^{\lim_{t\to 1}\left\{\lambda(f(t))\delta_{RC}'(t)\right\}}$.

Corollary 1 *If $\lim_{t\to 0}\left\{\lambda(f(t))\delta_{RC}'(t)\right\} = 0$ then the right composition provides a lower tail preserving transformation.*
If $\lim_{t\to 1}\left\{\lambda(f(t))\delta_{RC}'(t)\right\} = 0$ then the right composition provides an upper tail preserving transformation.

Example 1 *:*
Take the transform $f_\alpha(t) = \frac{\ln(\alpha t+1)}{\ln(\alpha+1)}$, $\alpha \in [0,+\infty)$. The following can a-priori be deduced:

- $\delta_{RC,\alpha}(t) = \left(t + \frac{1}{\alpha}\right)\ln(\alpha+1)$.
- $\delta'_{RC,\alpha}(0) = \delta'_{RC,\alpha}(1) = \ln(\alpha+1)$.
- $\lim_{t\to 0}\left\{\lambda(f_\alpha(t))\delta'_{RC,\alpha}(t)\right\} = 0$.
- $\lim_{t\to 1}\left\{\lambda(f_\alpha(t))\delta'_{RC,\alpha}(t)\right\} = 0$.

It follows that $\lambda_L^t = \lambda_L$; $\lambda_U^t = \lambda_U$.
As such the function $f_\alpha(t) = \frac{\ln(\alpha t+1)}{\ln(\alpha+1)}$, $\alpha \in [0,+\infty)$, provides a tail preserving transform through the right composition.

Example 2 *:*
Take the well-known transform $f_\alpha(t) = t^\alpha$, $0 < \alpha < 1$. The following can a-priori be deduced:

- $\delta_{RC,\alpha}(t) = \frac{1}{\alpha}t^{1-\alpha}$
- $\delta'_{RC,\alpha}(0) = \infty$; $\delta'_{RC,\alpha}(1) = \frac{1-\alpha}{\alpha}$.
- $\lim_{t\to 0}\left\{\lambda(f_\alpha(t))\delta'_{RC,\alpha}(t)\right\} = \left(\frac{1-\alpha}{\alpha}\right)\frac{\ln(\lambda_L)}{\ln(2)}$.
- $\lim_{t\to 1}\left\{\lambda(f_\alpha(t))\delta'_{RC,\alpha}(t)\right\} = 0$.

It follows that $\lambda_L^t = \lambda_L^{\frac{1}{\alpha}}$; $\lambda_U^t = \lambda_U$.
As such the function $f_\alpha(t) = t^\alpha, 0 < \alpha < 1$ provides only an upper tail preserving transform through the right composition.

B. Left composition

Lemma 16 *Let $f : [0,+\infty) \mapsto [0,+\infty)$ be an increasing convex function with f(0)=0. The effect of the left composition on the tails of the transformed copula is given by:*

- $\lambda_L^t = 2^{\lim_{t\to 0}\left\{\delta'_{LC}(t)\lambda(t)+\delta_{LC}(t)\lambda'(t)\right\}}$,
- $\lambda_U^t = 2 - 2^{\lim_{t\to 1}\left\{\delta'_{LC}(t)\lambda(t)+\delta_{LC}(t)\lambda'(t)\right\}}$.

Corollary 2 *If $\lim_{t\to 0}\{\delta'_{LC}(t)\lambda(t) + \delta_{LC}(t)\lambda'(t)\} = \lambda'(0+)$ then the left composition is a lower tail preserving transformation.*
If $\lim_{t\to 1}\{\delta'_{LC}(t)\lambda(t) + \delta_{LC}(t)\lambda'(t)\} = \lambda'(1-)$ then the left composition is an upper tail preserving transformation.

Example 3 *:*
Take the transform $f_\alpha(t) = te^{\alpha t}$, $\alpha \in [0,\infty]$. The following can a-priori be deduced:

- $\delta_{LC,\alpha}(t) = \frac{1}{1+\alpha\varphi(t)}$.
- $\delta'_{LC,\alpha}(t) = -\frac{\alpha\varphi'(t)}{[1+\alpha\varphi(t)]^2}$.
- $\lim_{t\to a}\left\{\delta'_{LC,\alpha}(t)\lambda(t) + \delta_{LC,\alpha}(t)\lambda'(t)\right\} = \lim_{t\to a}\left\{\frac{\lambda'(t)}{1+\alpha\varphi(t)} - \frac{\alpha\varphi(t)}{[1+\alpha\varphi(t)]^2}\right\}$, $a = \{0,1\}$.

It follows that $\lambda_L^t = 2^{\lim_{t\to 0}\left\{\frac{\lambda'(t)}{1+\alpha\varphi(t)} - \frac{\alpha\varphi(t)}{[1+\alpha\varphi(t)]^2}\right\}}$; $\lambda_U^t = \lambda_U$.
As such the transformation is upper tail preserving and the lower tail measure is generator specific. Take e.g. $\varphi(t) = -\ln t$ (independence), then $\lambda_L^t = 2^{-\frac{1}{\alpha}}$.

Example 4 *:*
Take the well-known transform $f_\alpha(t) = t^\alpha$, $\alpha > 1$. The following can a-priori be deduced:

- $\delta_{LC,\alpha}(t) = \frac{1}{\alpha}$.
- $\delta'_{LC,\alpha}(t) = 0$.
- $\lim_{t\to a}\left\{\delta'_{LC,\alpha}(t)\lambda(t) + \delta_{LC,\alpha}(t)\lambda'(t)\right\} = \frac{\lambda'(a)}{\alpha}$, $a = \{0, 1\}$.

It follows that $\lambda_L^t = \lambda_L^{1/\alpha}$; $\lambda_U^t = 2 - (2 - \lambda_U)^{1/\alpha}$.
As such both upper and lower tail dependence are generator specific. Take again $\varphi(t) = -\ln t$, then $\lambda_L^t = 0$, $\lambda_U^t = 2 - 2^{1/\alpha}$.

Example 5 *:*
Take the transform $f_\alpha(t) = \alpha t$, $\alpha > 0$. The following can a-priori be deduced:

- $\delta_{LC,\alpha}(t) = t$.
- $\delta'_{LC,\alpha}(t) = 1$.
- $\lim_{t\to a}\left\{\delta'_{LC,\alpha}(t)\lambda(t) + \delta_{LC,\alpha}(t)\lambda'(t)\right\} = \lambda'(a)$, $a = \{0, 1\}$.

It follows that $\lambda_L^t = \lambda_L$; $\lambda_U^t = \lambda_U$.
As such, the transformation is upper and lower tail preserving.

C. Scaling

Lemma 17 *The effect of the scaling rule on the tails of the transformed copula is given by:*

- $\lambda_L^t = 2^{\frac{\varphi(\alpha)}{\varphi(0+)}}\lambda_L^{1-\frac{\varphi(\alpha)}{\varphi(0+)}}$,
- $\lambda_U^t = 0$.

If the basic copula is strict, then $\lambda_L^t = \lambda_L$.

Corollary 3 *The scaling rule induces no upper tail dependence and is a lower tail preserving transformation for strict copulas.*

Example 6 *:*
Take $\varphi(t) = \left(\frac{1}{t} - 1\right)^2$. The following can readily be seen:

- $\lambda_L = \dfrac{1}{\sqrt{2}}$, $\lambda_U = 2 - \sqrt{2}$.
- φ is strict.

It follows that $\lambda_L^t = \lambda_L$; $\lambda_U^t = 0$.
As such the transform is lower tail preserving.

D. Exponentiation

Lemma 18 *The effect of the exponentiation rule on the tails of the transformed copula is given by:*

- $\lambda_L^t = 2^{\lim_{t\to 0}\left\{\lambda_1'(e^{-\varphi_2(t)})\right\}} \cdot 2^{\lim_{t\to 0}\left\{\delta_{EX}'(t)\lambda_1(e^{-\varphi_2(t)})\right\}}$,
- $\lambda_U^t = 2 - (2 - \lambda_{U,1}) \cdot 2^{\lim_{t\to 1}\left\{\delta_{EX}'(t)\lambda_1(e^{-\varphi_2(t)})\right\}}$.

If φ_2 is strict, then the expression for the lower tail dependence reduces to $\lambda_L^t = \lambda_{L,1} \cdot 2^{\lim_{t\to 0}\left\{\delta_{EX}'(t)\lambda_1(e^{-\varphi_2(t)})\right\}}$.

Corollary 4 *If $\lim_{t\to 0}\left\{\delta_{EX}'(t)\lambda_1(e^{-\varphi_2(t)})\right\} = 0$ and φ_2 is strict, the exponentiation transform is lower tail preserving with respect to λ_1.*
If $\lim_{t\to 1}\left\{\delta_{EX}'(t)\lambda_1(e^{-\varphi_2(t)})\right\} = 0$, the exponentiation transform is upper tail preserving with respect to λ_1.

Example 7 *:*
Take $\varphi_2(t) = -\ln(t)$, the independence case. The following can readily be seen:

- $(\varphi_2')^2 = \varphi_2''$, φ_2 is strict.
- $\delta_{EX}(t) = 1$.
- $\delta_{EX}'(t) = 0$.
- $\lim_{t\to a}\left\{\delta_{EX}'(t)\lambda_1(e^{-\varphi_2(t)})\right\} = 0$, $a = \{0, 1\}$.

It follows that $\lambda_L^t = \lambda_{L,1}$; $\lambda_U^t = \lambda_{U,1}$.
As such the transform is tail preserving with respect to φ_1 for any choice of φ_1.

Example 8 *:*
Take $\varphi_2(t) = \frac{1}{t} - 1$ (copula $\frac{\Pi}{\Sigma - \Pi}$). The following can readily be seen:

- $(\varphi_2')^2 > \varphi_2''$ and as such it cannot be used in the exponentiation transform.

Example 9 *:*
Take $\varphi_2(t) = \ln(2 - \sqrt{t})$. The following holds:

- $(\varphi_2')^2 \leq \varphi_2''$, φ_2 is not strict.
- $\delta_{EX}(t) = 2\sqrt{t}(2 - \sqrt{t})^2$.
- $\delta_{EX}'(t) = \dfrac{4 - 8\sqrt{t} + 3t}{\sqrt{t}}$.
- $\lim_{t\to 1}\left\{\delta_{EX}'(t)\lambda_1(e^{-\varphi_2(t)})\right\} = 0$.
- $\lim_{t\to 0}\left\{\lambda_1'(e^{-\varphi_2(t)}) + \delta_{EX}'(t)\lambda_1(e^{-\varphi_2(t)})\right\} = -\infty$.

It follows that $\lambda_L^t = 0$; $\lambda_U^t = \lambda_{U,1}$.
As such the transform is only upper tail preserving with respect to φ_1 for any choice of φ_1.

E. Linear combination

Lemma 19 *The effect of the linear combination on the tails of the transformed copula is given by:*

- $\lambda_L^t = \lambda_{L,2} \cdot 2^{\lim_{t\to 0}\delta'_{LI}(t)(\lambda_1(t)-\lambda_2(t))+\lim_{t\to 0}\delta_{LI}(t)\left(\lambda'_1(t)-\lambda'_2(t)\right)}$
  $= \lambda_{L,1} \cdot 2^{\lim_{t\to 0}(1-\delta'_{LI}(t))(\lambda_2(t)-\lambda_1(t))+\lim_{t\to 0}(1-\delta_{LI}(t))\left(\lambda'_2(t)-\lambda'_1(t)\right)}$,
- $\lambda_U^t = 2-(2-\lambda_{U,2}) \cdot 2^{\lim_{t\to 1}\delta'_{LI}(t)(\lambda_1(t)-\lambda_2(t))+\lim_{t\to 1}\delta_{LI}(t)\left(\lambda'_1(t)-\lambda'_2(t)\right)}$
  $= 2-(2-\lambda_{U,1}) \cdot 2^{\lim_{t\to 1}(1-\delta'_{LI}(t))(\lambda_2(t)-\lambda_1(t))+\lim_{t\to 1}(1-\delta_{LI}(t))\left(\lambda'_2(t)-\lambda'_1(t)\right)}$.

Corollary 5 *If* $\lim_{t\to 0}\delta'_{LI}(t)\,(\lambda_1(t)-\lambda_2(t))+\lim_{t\to 0}\delta_{LI}(t)\,(\lambda'_1(t)-\lambda'_2(t))=0$*, the linear combination is lower tail preserving with respect to* φ_2*; if* $\lim_{t\to 0}(1-\delta'_{LI}(t))\,(\lambda_2(t)-\lambda_1(t))+\lim_{t\to 0}(1-\delta_{LI}(t))\,(\lambda'_2(t)-\lambda'_1(t))=0$*, the linear combination is lower tail preserving with respect to* φ_1*.*

If $\lim_{t\to 1}\delta'_{LI}(t)\,(\lambda_1(t)-\lambda_2(t))+\lim_{t\to 1}\delta_{LI}(t)\,(\lambda'_1(t)-\lambda'_2(t))=0$*, the linear combination is upper tail preserving with respect to* φ_2*; if* $\lim_{t\to 1}(1-\delta'_{LI}(t))\,(\lambda_2(t)-\lambda_1(t))+\lim_{t\to 1}(1-\delta_{LI}(t))\,(\lambda'_2(t)-\lambda'_1(t))=0$*, the linear combination is upper tail preserving with respect to* φ_1*.*

Example 10 *:*
Take $\varphi_1(t)=[-\ln(t)]^2$ (Gumbel, $\theta=2$) and $\varphi_2(t)=\frac{1}{t^2}-1$ (Clayton, $\theta=2$). The following can a priori be deduced:

- $\lambda_{L,1}=0$, $\lambda_{U,1}=2-\sqrt{2}$.
- $\lambda_{L,2}=\dfrac{1}{\sqrt{2}}$, $\lambda_{U,2}=0$.
- $\delta_{LI}(t)=\frac{\alpha t^2\ln t}{\alpha t^2\ln t-\beta}$.
- $\delta'_{LI}(t)=\frac{-\alpha\beta t(\alpha\ln t+1)}{(\alpha t^2\ln t-\beta)^2}$.
- $\lim_{t\to a}\delta'_{LI}(t)\,(\lambda_1(t)-\lambda_2(t))=0, a=\{0,1\}$.
- $\lim_{t\to a}\delta_{LI}(t)\,(\lambda'_1(t)-\lambda'_2(t))=0, a=\{0,1\}$.

It follows that $\lambda_L^t=\lambda_{L,2};\lambda_U^t=\lambda_{U,2}$.
As such, the transformation is upper and lower tail preserving with respect to φ_2.

Example 11 *:*
Take $\varphi_1=-\ln(t)$ (independence) and $\varphi_2=\frac{1}{t}-1$ (copula $\frac{\Pi}{\Sigma-\Pi}$). The following can a priori be deduced:

- $\lambda_{L,1}=0$, $\lambda_{U,1}=0$.
- $\lambda_{L,2}=\dfrac{1}{2}$, $\lambda_{U,2}=0$.
- $\delta_{LI}(t)=\frac{\alpha t}{\alpha t+\beta}$.
- $\delta'_{LI}(t)=\frac{\alpha\beta}{(\alpha t+\beta)^2}$.
- $\lim_{t\to a}\delta'_{LI}(t)\,(\lambda_1(t)-\lambda_2(t))=0, a=\{0,1\}$.
- $\lim_{t\to 0}\delta_{LI}(t)\,(\lambda'_1(t)-\lambda'_2(t))=0;\lim_{t\to 1}\delta_{LI}(t)\,(\lambda'_1(t)-\lambda'_2(t))=\frac{\alpha}{\alpha+\beta}\,(\lambda'_1(t)-\lambda'_2(t))$.

It follows that $\lambda_L^t=\lambda_{L,2};\lambda_U^t=2-(2-\lambda_{U,1})^{\frac{\alpha}{\alpha+\beta}}(2-\lambda_{U,2})^{\frac{\beta}{\alpha+\beta}}$.
As such, the transformation is lower tail preserving with respect to φ_2.
Because $\lambda_{U,1}=\lambda_{U,2}=0$, also $\lambda_U^t=0$, which implies that in this case the transform is upper tail preserving with respect to φ_1 and φ_2.

4.3 Limiting situations

Apart from the effect of the generator transforms on the tails of the distribution, other important dependence information can a priori be deduced. From a construction point of view it is interesting to look at the limiting behaviour of the transform parameter θ, because this information is closely related to the limiting cases the transformed copula possesses. As such, this also gives more information about the dependence range that is covered by a particular transform. Limiting situations which appear naturally for several Archimedean copulas are Π, M, W and $\frac{\Pi}{\Sigma-\Pi}$. In what follows, the limiting behaviour of the transform parameter is studied for the right and left composition rules and illustrated by means of some examples taken from Table 3.

A. Right composition

Lemma 20 *Consider a parametric function* $f_\theta : [0,1] \to [0,1]$, *which is an increasing concave bijection for each* $\theta \in G \subseteq \mathbb{R}$.
If $\lim_{\theta\to\theta^*} f_\theta(t) \equiv 1$ *for a value* θ^*, *then* $\lambda^t_{\theta^*}(t) = \frac{\ln(2-\lambda_U)}{\ln 2} \lim_{\theta\to\theta^*} \left\{ \frac{\frac{d}{d\theta} f_\theta(t)}{\frac{d}{d\theta} f'_\theta(t)} \right\}$.
If $\lim_{\theta\to\theta^*} f_\theta(t) \equiv t$ *for a value* θ^*, *then* $\lambda^t_{\theta^*} = \lambda$.
If $\lim_{\theta\to\theta^*} f_\theta(t) \equiv 0$ *for a value* θ^*, *then* $\lambda^t_{\theta^*} = \frac{\ln \lambda_L}{\ln 2} \lim_{\theta\to\theta^*} \left\{ \frac{\frac{d}{d\theta} f_\theta(t)}{\frac{d}{d\theta} f'_\theta(t)} \right\}$ *for* $\lambda_L \neq 0$ *and* $\lambda^t_{\theta^*} = \lim_{\theta\to\theta^*} \left\{ \frac{\lambda'(f_\theta(t)\frac{d}{d\theta} f_\theta(t)}{\frac{d}{d\theta} f'_\theta(t)} \right\}$ *for* $\lambda_L = 0$.

This immediately follows from the result in lemma 10.

In order to illustrate these results, consider some of the transforms of Archimedean copulas as mentioned in Table 3.

4.2.3. $f_\theta(t) = \frac{t}{1+\theta(t-1)}, \theta \in [-1, 1)$

- For the derivatives, we have
  - $f'_\theta(t) = \frac{1-\theta}{(1+\theta(t-1))^2}$,
  - $\frac{d}{d\theta} f_\theta(t) = -\frac{t(t-1)}{(1+\theta(t-1))^2}$,
  - $\frac{d}{d\theta} f'_\theta(t) = \frac{1-\theta+t(\theta-2)}{(1+\theta(t-1))^3}$.
- For $\theta \to 1$ it is true that
  - $\lim_{\theta\to 1} f_\theta(t) \equiv 1$ and
  - $\lim_{\theta\to 1} \lambda^t_\theta(t) = \frac{\ln(2-\lambda_U)}{\ln 2} \cdot \lambda_{\frac{\Pi}{\Sigma-\Pi}}(t)$.

  As such, this transform entails a rescale of the well-known special case $\frac{\Pi}{\Sigma-\Pi}$, based on the upper tail dependence coefficient value.

  Clearly, if $\lambda_U = 0$, we have $C^t_{\theta=1} = \frac{\Pi}{\Sigma-\Pi}$ and if $\lambda_U = 1$ we have $C^t_{\theta=1} = M$. Hence we can conclude that $C^t = \frac{\Pi}{\Sigma-\Pi}$ appears as a natural limiting case in this transform.
- For $\theta \to 0$ it is true that
  - $\lim_{\theta\to 0} f_\theta(t) \equiv t$ and
  - $\lim_{\theta\to 0} \lambda^t_\theta(t) = \lambda(t)$.

  As such, the transformed copula equals the base copula (whatever the choice of φ).

4.2.5 $f_\theta(t) = \frac{e^{-\theta t}-1}{e^{-\theta}-1}, \theta \in (-\infty, \infty) \setminus \{0\}$

- For the derivatives, we have
  - $f'_\theta(t) = \frac{-\theta e^{-\theta t}}{e^{-\theta}-1}$,
  - $\frac{d}{d\theta} f_\theta(t) = \frac{-te^{-\theta t}}{e^{-\theta}-1} + \frac{(e^{-\theta t}-1)e^{-\theta}}{(e^{-\theta}-1)^2}$,
  - $\frac{d}{d\theta} f'_\theta(t) = \frac{(\theta t-1)e^{-\theta t}}{e^{-\theta}-1} - \frac{\theta e^{-\theta(t+1)}}{(e^{-\theta}-1)^2}$.
- For $\theta \to -\infty$ it is true that
  - $\lim_{\theta\to-\infty} f_\theta(t) \equiv 0$.
  - For base generators with $\lambda_L \neq 0$ we have $\lim_{\theta\to-\infty} \lambda^t_\theta(t) = 0$; the limit entails the comonotonic copula M.
  - For base generators with $\lambda_L = 0$, e.g. the independence case, we get $\lim_{\theta\to-\infty} \lambda^t_\theta(t) = t-1$; the limit entails the countermonotonic copula W.
- For $\theta \to 0$ it is true that
  - $\lim_{\theta\to 0} f_\theta(t) \equiv t$ and
  - $\lim_{\theta\to 0} \lambda^t_\theta(t) = \lambda(t)$.

  As such, the transformed copula equals the base copula (whatever the choice of φ).
- For $\theta \to \infty$ it is true that
  - $\lim_{\theta\to\infty} f_\theta(t) \equiv 1$ and
  - $\lim_{\theta\to\infty} \lambda^t_\theta(t) = 0$.

  This transform entails the comonotonic copula M as a natural limit.

4.2.6 $f_\theta(t) = 1 - (1-t)^\theta, \theta \in [1, \infty)$

- For the derivatives, we have
  - $f'_\theta(t) = \theta(1-t)^{\theta-1}$,
  - $\frac{d}{d\theta} f_\theta(t) = -(1-t)^\theta \ln(1-t)$,
  - $\frac{d}{d\theta} f'_\theta(t) = -(1-t)^{\theta-1}(\theta \ln(1-t) + 1)$.
- For $\theta \to \infty$ it is true that
  - $\lim_{\theta\to\infty} f_\theta(t) \equiv 1$ and
  - $\lim_{\theta\to\infty} \lambda^t_\theta(t) = 0$.

  This transform entails the comonotonic copula M as a natural limit.

4.2.7 $f_\theta(t) = \theta t + (1-\theta), \theta \in (0, 1]$

- For the derivatives, we have
  - $f'_\theta(t) = \theta$,
  - $\frac{d}{d\theta} f_\theta(t) = t - 1$,
  - $\frac{d}{d\theta} f'_\theta(t) = 1$.

- For $\theta \to 0$ it is true that
  - $\lim_{\theta\to 0} f_\theta(t) \equiv 1$ and
  - $\lim_{\theta\to 0} \lambda^t(t) = \frac{\ln(2-\lambda_U)}{\ln 2}\lambda_W(t)$.

  This transform entails the countermonotonic copula W as a natural limit for choices of base generator with $\lambda_U = 0$.

- For $\theta \to 1$ it is true that
  - $\lim_{\theta\to 1} f_\theta(t) \equiv t$ and
  - $\lim_{\theta\to 1} \lambda^t(t) = \lambda(t)$.

  As such, the transformed copula yields the base copula (whatever the choice of φ).

B. Left composition

Lemma 21 *Consider a parametric function $f_\theta : [0,1] \to [0,1]$, which is an increasing convex function with $f_\theta(0) = 0$ for each $\theta \in G \subseteq \mathbb{R}$, and define $\delta_{LC,\theta}(t)$ as in lemma 11.*
If $\lim_{\theta\to\theta^*} \delta_{LC,\theta}(t) \equiv 1$ for a value θ^*, then $C^t_{\theta^*} = C_\varphi$.
If $\lim_{\theta\to\theta^*} \delta_{LC,\theta}(t) \equiv 0$ for a value θ^*, then $C^t_{\theta^*} = C_M$.

This immediately follows from the result in lemma 11.

Again, we illustrate these results by means of some of the transforms of Archimedean copulas as mentioned in Table 3.

4.2.1 $f_\theta(t) = \frac{e^{\theta t}-1}{\theta}, \theta \in (0,\infty)$

- For the remainder, we have $\delta_{LC,\theta}(t) = \frac{e^{\theta\varphi(t)}-1}{\theta\varphi(t)e^{\theta\varphi(t)}}$.
- For $\theta \to 0^+$ it is true that $\lim_{\theta\to 0^+} \delta_{LC,\theta}(t) = 1$.

  As such the transform equals the base copula (whatever choice of φ) in the lower limit.
- For $\theta \to \infty$ it is true that $\lim_{\theta\to\infty} \delta_{LC,\theta}(t) = 0$.

  As such the transform equals the comonotonic copula in the upper limit.

4.2.12 $f_\theta(t) = t^\theta, \theta \in (1,\infty)$

- For the remainder, we have $\delta_{LC,\theta}(t) = \frac{1}{\theta}$.
- For $\theta \to 1$ the transform equals the base copula.
- For $\theta \to \infty$ the transform yields the comonotonic case.

4.2.13 $f_\theta(t) = (1+t)^\theta - 1, \theta \in (0,\infty)$

- For the remainder, we have $\delta_{LC,\theta}(t) = \frac{(1+\varphi(t))^\theta-1}{\theta\varphi(t)(1+\varphi(t))^{\theta-1}}$.
- For $\theta \to 0^+$ it is true that $\lim_{\theta\to 0^+} \delta_{LC,\theta}(t) = \frac{\ln(1+\varphi(t))}{\varphi(t)}$.

  As such, a particular base copula choice, e.g. φ_Π yields[(2)] $\varphi^t = e^{Li(1+\varphi_\Pi)}$
- For $\theta \to \infty$ it is true that $\lim_{\theta\to\infty} \delta_{LC,\theta}(t) = 0$.

  As such the transform equals the comonotonic copula in the upper limit.

[(2)] The notation Li is used to denote the logaritmic integral, i.e. $Li(x) = \int_0^x \frac{dt}{\ln(t)}$.

4.2.19 $f_\theta(t) = e^{\theta e^t} - e^\theta, \theta \in (0, \infty)$

- For the remainder, we have $\delta_{LC,\theta}(t) = \frac{e^{\theta e^{\varphi(t)}} - e^\theta}{\theta\varphi(t)e^{\varphi(t)}e^{\theta e^{\varphi(t)}}}$.
- For $\theta \to 0^+$ it is true that $\lim_{\theta\to 0^+} \delta_{LC}(\varphi(t), \theta) = \frac{e^{\varphi(t)}-1}{\varphi(t)e^{\varphi(t)}}$.

  Note this yields the same as for case 4.2.1 with $\theta \to 1$. Indeed, for base copula choice φ_Π this yields $\varphi^t = \varphi_{\frac{\Pi}{\Sigma - \Pi}}$.

4.2.20 $f_\theta(t) = e^{e^{\theta t}} - e^1, \theta \in (0, \infty)$

- For the remainder, we have $\delta_{LC,\theta}(t) = \frac{e^{e^{\theta\varphi(t)}} - e}{\theta\varphi(t)e^{e^{\theta\varphi(t)}}e^{\theta\varphi(t)}}$.
- For $\theta \to 0^+$ it is true that $\lim_{\theta\to 0^+} \delta_{LC,\theta}(t) = 1$.

  As such, the lower limiting value equals the base copula.
- For $\theta \to \infty$ it is true that $\lim_{\theta\to\infty} \delta_{LC,\theta}(t) = 0$.

  As such the transform equals the comonotonic copula in the upper limit.

4.4 Transition transforms

One of the advantages of the investigation of the limiting cases in the previous subsection is the fact that this gives more information about the modeling power of the transformed copula in terms of the dependence range it can deal with. E.g., a copula having with limiting cases W and Π can be used for the whole negative dependence range, and a copula with $\frac{\Pi}{\Sigma-\Pi}$ and M can model situations in the τ range $[\frac{1}{3}, 1]$.

It is also possible to construct transforms that include some particular copulas, specified in advance. As such, this 'transition' transform can be used as a starting point to create a new copula family including particular copula cases.

In this subsection we investigate such transition transforms for the right and left composition rules and we provide some examples.

A. Right composition

- The case $\varphi = \varphi_\Pi$ and $\varphi^t = \varphi_W$.

  We have $f(t) = \exp\left(e^{\int \frac{1}{\lambda^t_W(t)}dt}\right) = e^{t-1}$.

  Based on this information one can easily compose a one parameter family including both Π and W. Take for example $f_\theta(t) = t^{(1-\theta)}e^{\theta(t-1)}, \theta \in [0, 1]$. The corresponding generator, calculated as in lemma 5, is strict, it equals $\varphi^t_\theta(t) = -\ln\left(t^{(1-\theta)}e^{\theta(t-1)}\right)$, and it includes Π for $\theta \to 1$ and W for $\theta \to 0$.
- The case $\varphi = \varphi_\Pi$ and $\varphi^t = \varphi_{\frac{\Pi}{\Sigma-\Pi}}$.

  We have $f(t) = \exp\left(e^{\int \frac{1}{\lambda^t_{\frac{\Pi}{\Sigma-\Pi}}(t)}dt}\right) = e^{\frac{t-1}{t}}$.

  Based on this information one can easily compose a one parameter family including both $\frac{\Pi}{\Sigma-\Pi}$ and Π or W. Take for example $f_\theta(t) = e^{\frac{t-1}{t^\theta}}, \theta \in [0, 1]$. The corresponding strict generator equals $\varphi^t_\theta(t) = -\ln\left(e^{\frac{t-1}{t^\theta}}\right)$ and includes $\frac{\Pi}{\Sigma-\Pi}$ for $\theta \to 1$ and W for $\theta \to 0$.

B. Left composition

- The case $\varphi = \varphi_\Pi$ and $\varphi^t = \varphi_W$.

  We have $\delta_{LC} = \frac{\lambda_W}{\lambda_\Pi}$ and thus $f(t) = 1 - e^{-t}$.

  The only left composition Π based model from Table 3 which includes W is the Clayton model (4.2.1), but this model becomes non-strict for negative dependence.

  A strict example is $f(t) = \frac{e^{1-\theta}(1-e^{-t})}{\theta^t}, \theta \in [1,\infty)$ which yields $\varphi_\theta^t(t) = \frac{e^{1-\theta}(1-t)}{t^{\ln\theta}}$, calculated as in lemma 6 and $C = W$ for $\theta \to 1$, $C = \Pi$ for $\theta \to \infty$.

- The case $\varphi = \varphi_{\frac{\Pi}{\Sigma-\Pi}}$ and $\varphi^t = \varphi_W$.

  We have $\delta_{LC} = \frac{\lambda_W}{\lambda_{\frac{\Pi}{\Sigma-\Pi}}}$ and thus $f(t) = t(1+t)$.

  Take e.g. the parametrization $f_\theta(t) = t(1+t)^{1-e^\theta}$, $\theta \in (-\infty, 0]$. This yields the strict generator $\varphi_\theta(t) = \frac{1-t}{t^{2-e^\theta}}$, which includes $\frac{\Pi}{\Sigma-\Pi}$ for $\theta \to 0$ and W for $\theta \to -\infty$.

4.5 Kendall's τ

The previous sections allow us to more clearly predict the effect of the transforms from Genest et al. (1998) in terms of tail dependence properties and limiting cases. The effect on the concordance, expressed in terms of Kendall's τ, is given by (4).

Notice that it is difficult to determine a functional transform (except the most trivial one, e.g. $f(t) = t$) which allows to preserve the τ-value from the original copula. If this would be possible, however, it could be advantageous when in a dependence modeling application a practitioner would like to test different copulas which all correspond to the same concordance value. The practitioner can use for example the well-known test space entailing the Gumbel-Hougaard, Frank and Clayton families to start from. Next a collection of transformed copulas can be added to the test space, using different transforms which assure $\tau = \hat{\tau}$, but increase the diversity on the local dependence level.

One possibility is to take a convex combination of two λ functions having the same τ level.

Proposition 1 *Let $\lambda^t(t) = \alpha\lambda_1(t) + (1-\alpha)\lambda_2(t)$, $\alpha \in [0,1]$ be a feasible convex combination of two lambda functions and let $\tau_{\lambda_1} = \tau_{\lambda_2}$. Then $\lambda^t(t)$ is a τ-preserving transform.*

Proof: It is straightforward to check from (4) that $\tau_{\lambda^t} = \alpha\tau_{\lambda_1} + (1-\alpha)\tau_{\lambda_2}$.

Lemma 22 *The τ-preserving transform given by $\lambda^t(t) = \alpha\lambda_1(t) + (1-\alpha)\lambda_2(t)$,$\alpha \in [0,1]$, where $\tau_{\lambda_1} = \tau_{\lambda_2}$ has the following tail dependence properties:*

- $\lambda_L^t = \lambda_{L,1}^\alpha \cdot \lambda_{L,2}^{1-\alpha}$;
- $\lambda_U^t = 2 - (2-\lambda_{U,1})^\alpha \cdot (2-\lambda_{U,2})^{1-\alpha}$.

This simple but useful transform allows to combine properties of two or more copula families. From a practical point of view, this transform allows to search for improvements of the fit, assuming the best fit lies between two or more copulas. If this is not the case, another type of transform needs to be used.

Proposition 2 *Let $f_\alpha : [0,1] \mapsto [0,1]$ be an increasing bijection.*
A τ-preserving transform is given by $\lambda^t(t) = f'_\alpha(t) \cdot \lambda(f_\alpha(t))$, where range α is determined using conditions LR1-LR4.

Proof: From (4) we have $\tau^t = 1 + 4\int_0^1 \lambda^t(t)dt = 1 + 4\int_0^1 f'_\alpha(t) \cdot \lambda(f_\alpha(t))dt$. A substitution $u = f_\alpha(t)$ leads to the desired result.

Lemma 23 *The τ-preserving transform given by $\lambda^t(t) = f'_\alpha(t) \cdot \lambda(f_\alpha(t))$ has the following tail dependence properties:*

- $\lambda_L^t = 2^{\lim_{t\to 0}\left\{f''_\alpha(t)\lambda(f_\alpha(t)) + [f'_\alpha(t)]^2\lambda'(f_\alpha(t))\right\}}$;
- $\lambda_U^t = 2 - (2 - \lambda_U)^{[f'_\alpha(1)]^2}$.

Corollary 6 *Let $f_\alpha : [0,1] \mapsto [0,1]$ be an increasing bijection. An increase in concavity increases the upper tail dependence, while an increase in convexity will reduce the upper tail dependence.*

To illustrate the use of this concordance invariant transform we diversify the well-known test space consisting of the Frank, Clayton and Gumbel-Hougaard families. Suppose the observed dependence structure exhibits a general degree of dependence of $\tau = 50\%$.
We will consider the transform $f(t) = \frac{e^{-\alpha t}-1}{e^{-\alpha}-1}$ which is increasing for $\alpha \in [-\infty, \infty] \setminus \{0\}$. The feasible parameter range, together with the tail dependence properties of the τ-preserving transform can be found in table 4.

	Clayton	Gumbel-Hougaard	Frank
α-range	$(0, 3.2625]$	$[-0.7355, 2.5271]$	$(0, 3.2224]$
λ_L^t	$2^{\frac{-1/2\alpha^2}{(e^{-\alpha}-1)^2}}$	0	0
λ_U^t	$2 - 2^{\frac{\alpha^2 e^{-2\alpha}}{(e^{\alpha}-1)^2}}$	$2 - (1.4142)^{\frac{\alpha^2 e^{-2\alpha}}{(e^{\alpha}-1)^2}}$	$2 - 2^{\frac{\alpha^2 e^{-2\alpha}}{(e^{\alpha}-1)^2}}$

Table 4: Properties for the τ-preserving transform

The transform based on the Clayton family allow to increase the diversity of the test space by including copulas having higher upper tail dependence and less lower tail dependence. This is shown in figure 2 (a), for $\alpha \in \{0, 0.5, 1\}$.

Notice that $f_\alpha(t)$ for the Gumbel-Hougaard family can be both convex and concave. As such, we can include transforms which will have a lower or a higher upper tail dependence structure, while the lower tail dependence remains zero. As an illustration we show three particular cases, $\alpha \in \{-0.5, 0, 0.5\}$ in figure 2 (b), where $\lambda_G^t = \lambda_G$ for $\alpha \to 0$.

The transform based on the Frank family allow to increase the diversity of the test space by including copulas having higher upper tail dependence and no lower tail dependence. This is shown in figure 2 (c), for $\alpha \in \{0, 0.5, 1\}$.

5 Conclusion

In this paper we discussed the five proposed Archimedean copula generator transforms from Genest et al. (1998), with the focus on tail dependence properties and limiting cases, and with an extension to lambda transforms. Several conditions are identified in order to create a transform that is *lower or upper tail dependence preserving*. We also show how to use transforms in order to include particular copulas into a family and we illustrate the possibilities of limiting situations.
A concordance preserving transform, however, cannot be found when relying on these known transforms. As such, we define a *Kendall's τ-preserving transform* and show its use by applying it to the well-known Gumbel-Hougaard, Clayton and Frank families. This

approach can straightforwardly be combined with the theory of *comparable test spaces* as defined in Michiels and De Schepper (2008). In a forthcoming paper we will illustrate the practical use and advantages of this τ-preserving transform for dependence modeling problems.

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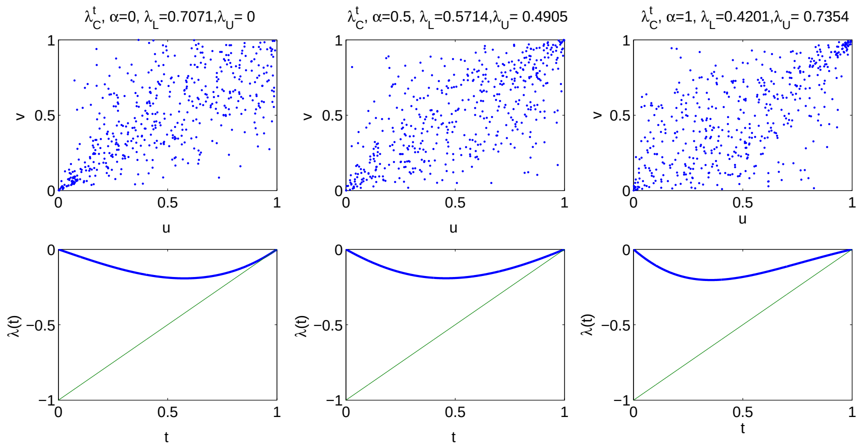
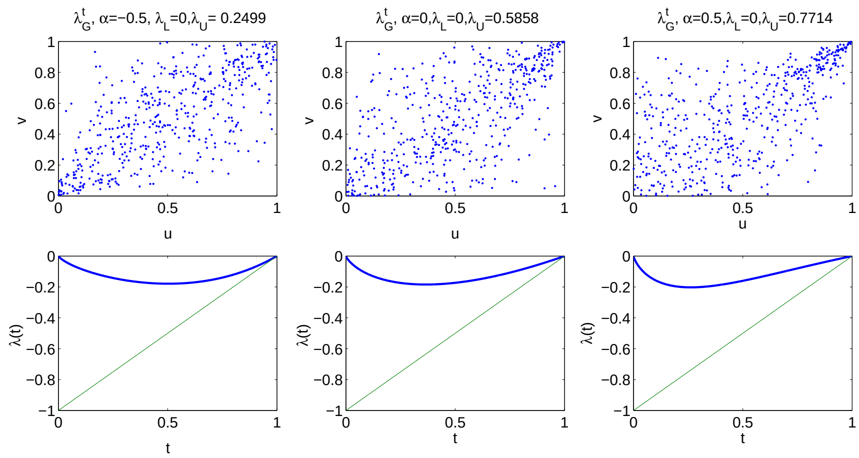
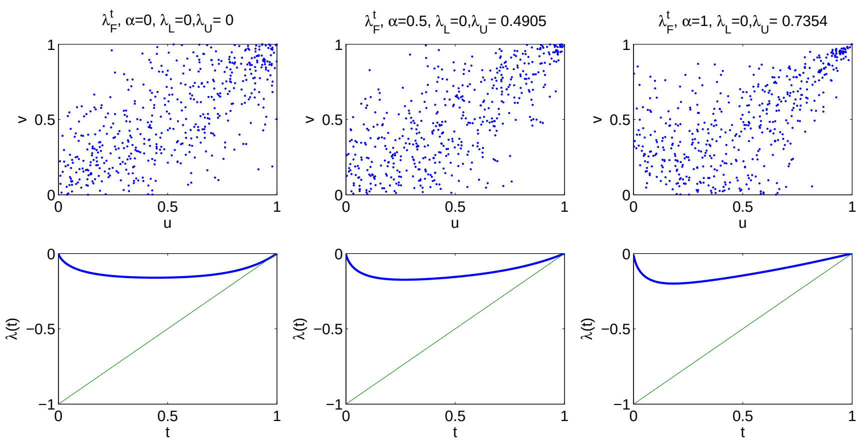
Figure 2: Illustration of the τ-preserving transform on the Clayton, Gumbel-Hougaard and Frank families with $f_\alpha(t) = \frac{e^{-\alpha t}-1}{e^{-\alpha}-1}$, $\tau = 50\%$.



(a) $\lambda_C^t = f'_\alpha(t) \cdot \lambda_C(f_\alpha(t))$, $\alpha \in \{0, 0.5, 1\}$.



(b) $\lambda_G^t = f'_\alpha(t) \cdot \lambda_G(f_\alpha(t))$, $\alpha \in \{-0.5, 0, 0.5\}$.



(c) $\lambda_F^t = f'_\alpha(t) \cdot \lambda_F(f_\alpha(t))$, $\alpha \in \{0, 0.5, 1\}$.