DEPARTMENT OF ENGINEERING MANAGEMENT

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On the counterintuitive behaviour of the economic order quantity in the presence of backorders

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Abstract

The basic deterministic stationary inventory problem, in which backorders are allowed, is analyzed. Instead of considering the backorder cost to be a cost per unit and per time, we also suppose the presence of a fixed backorder cost per unit. The latter generates a supplementary dependence of the economic order quantity on the demand, which happens to be of a different polynomial degree in the popular square root formula. Sensitivity analysis with respect to changes in the demand allows us to conclude that for some values of the holding cost, backorder cost, average yearly demand, etc. a counterintuitive behaviour might occur. This behaviour, being the decrease of the economic order quantity incurred by an increase of the demand, is due to the presence of multiple competing (nonlinear) dependencies of the economic order quantity on this demand.

1 Introduction

The basic economic order quantity (EOQ) model, dating back to Harris (1913), bearing several names throughout the past century (for an overview see e.g. Erlenkotter (1990)), has for quite some time been considered to be a standard topic in operations management/research courses (see e.g. standard

OR/MS books such as Lawrence and Pasternack (2002), Wagner (1969) and many others). The simplicity of the Harris EOQ formula has contributed to its popularity, but has unfortunately (together with lack of knowledge) also induced the misuse of this model in practice. Moreover, from sensitivity analysis with respect to the different parameters of the model, it appears that the use of a suboptimal order quantity generates a rather minor impact at the level of the total cost (see e.g. Simchi-Levi et al. (2008)). Hence, the reluctance in practice to use models better suited for the specific business context for inventory management is strengthened by this fact.
In Sphicas (2006) new attention was paid to the EOQ model with linear and fixed backorder costs, as previously analyzed by Johnson and Montgomery (1974). However, in most textbooks including the EOQ model, extended for the presence of backorders, the fixed backorder costs are not or rarely included (see e.g. Zipkin (2000); Axsäter (2006)). However, unlike the backorder costs proportional to the waiting time of the customers, which depend on intangible costs such as e.g. the customer's goodwill, these fixed backorder costs per unit are much easier to evaluate. For example the cost of transporting goods ex post, eventually using express delivery services, is perfectly known. Merely to introduce the notations we present the corresponding E.O.Q. in section 2. Next we perform a sensitivity analysis on the demand and we end with some conclusions.

2 Model formulation and optimization

As already indicated the hypotheses, under which this model is applicable, are the same as for Harris' model. The only change occurs with respect to the allowance of backorders. This implies that after a time interval T_1 in the order interval T, the inventory $I(t)$ will be depleted. Backorders $B(t)$ will start to accumulate until a value b, i.e. the maximal backorder level, during an interval of length $T_2 = T - T_1$ after which an order replenishes the inventory. It is assumed in this model that first backorders are filled and the remaining items of the replenishment batch are added to the inventory. In order to take into account backorders, the evolution of net inventory $I_n(t)$ is considered (see figure 1). For a derivation of this evolution we refer e.g. to (Zipkin, 2000) p.40ff.
Similar to Harris' model the cost function for an order period consists of a fixed order cost C_o; a purchase cost per unit C_p times the order quantity q; and a long-term average inventory cost involving the holding cost per unit per time unit C_h. Additionally, the long-term average backorder cost consists of a 'fixed' backorder cost and a backorder cost dependent on the

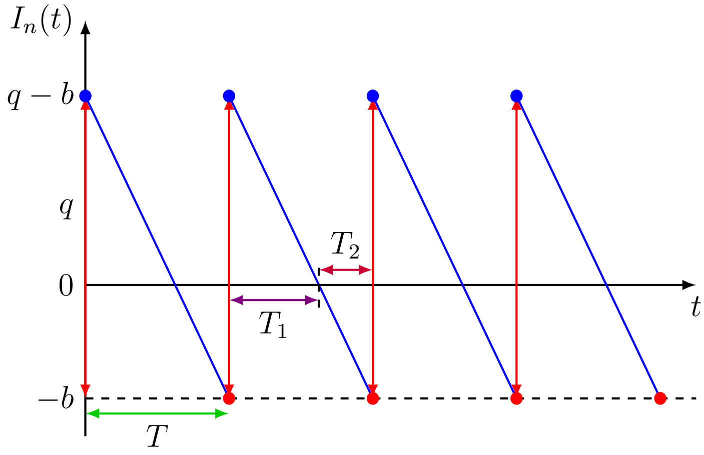

Figure 1: Evolution of the net inventory in a deterministic stationary inventory model with backorders

average backorder level and the time a customer has to wait and related to the customer's goodwill C_g. The former is given by the 'fixed' backorder cost per unit C_f multiplied by the amount of backorder units to be delivered b. This result in the following cost function for a single order interval:

$$C(q,b) \equiv C_o + C_p q + C_h \frac{(q-b)}{2} T_1 + C_f b + C_g \frac{b}{2} T_2. \tag{1}$$

Using the hypotheses of the model the expressions for T_1 and T_2:

$$T_1 = \frac{q-b}{D} \qquad T_2 = \frac{b}{D} \tag{2}$$

with D the average yearly demand, the cost (1) can be rewritten as

$$C(q,b) \equiv C_o + C_p q + C_h \frac{(q-b)^2}{2D} + C_f b + C_g \frac{b^2}{2D} \tag{3}$$

with the following conditions on the variables:

$$q \geqslant b \geqslant 0. \tag{4}$$

To further analyse the model, it is convenient to use the long-term average yearly cost c, obtained through division of the cost (3) by T, expressed in terms of $T = \frac{q}{D}$ and $\alpha = \frac{T_1}{T} = \frac{q-b}{q}$:

$$c(T,\alpha) = \frac{C_o}{T} + C_p D + C_h \frac{DT\alpha^2}{2} + C_f D(1-\alpha) + C_g \frac{DT(1-\alpha)^2}{2}, \tag{5}$$

while the conditions (4) on the variables become

$$T \geqslant 0 \text{ and } 0 \leqslant \alpha \leqslant 1. \tag{6}$$

Minimization of $c(T, \alpha)$ without taking into account the conditions (6) results in:

$$\widetilde{\alpha} = \frac{C_g + \dfrac{C_f}{\widetilde{T}}}{C_g + C_h} \tag{7}$$

$$\widetilde{T} = \sqrt{\frac{2C_o(C_g + C_h) - DC_f^2}{DC_gC_h}} \tag{8}$$

Expressions (7, 8) only represent the solution of the constrained optimization problem, if they exist i.e. if the following condition holds:

$$2C_o(C_g + C_h) - DC_f^2 \geqslant 0 \tag{9}$$

and if the constraints (6) are satisfied. Given expressions (7, 8) the conditions $\widetilde{T} \geqslant 0$ and $\widetilde{\alpha} \geqslant 0$ are automatically satisfied under the assumption that (9) holds. However, $\widetilde{\alpha} \leqslant 1$ together with (8) implies the necessary condition:

$$2C_oC_h - DC_f^2 \geqslant 0 \tag{10}$$

being more restrictive than condition (9) since it is implicitely assumed that $C_g > 0$. Hence, it can be concluded that under condition (10), the E.O.Q. for this model is given by

$$q^* = \sqrt{\frac{2C_o(C_g + C_h)D - D^2C_f^2}{C_gC_h}}. \tag{11}$$

In the same manner the optimal maximum level of backorders can be obtained:

$$b^* = \frac{C_hq^* - DC_f}{C_g + C_h}. \tag{12}$$

Using eqs.(11, 12) it is easy to see that the optimal yearly cost can be expressed as

$$c(q^*, b^*) = C_pD + C_h(q^* - b^*). \tag{13}$$

3 Sensitivity analysis

In this section a sensitivity analysis is performed on the yearly demand D. In order to do this the situation is considered in which the yearly demand has changed to a value $D' = \lambda D$. Hence, the optimal values for the E.O.Q. and the maximal backorder level have changed to the values resulting from the corresponding expressions (11, 12) in which D' was substituted for D. Condition (10) for the new situation can be rewritten as

$$2C_oC_h - D'C_f^2 \geqslant 0 \Leftrightarrow \frac{2C_oC_h}{DC_f^2} \geqslant \lambda. \tag{14}$$

Starting from expression (11) and its analogue in the new situation, the economic order quantities can be compared. Without loss of generality the squares of the economic quantities are compared:

$$(q'^*)^2 - (q^*)^2 = \frac{D(\lambda - 1)}{C_gC_h}\left[2C_o(C_h + C_g) - DC_f^2(\lambda + 1)\right] \overset{?}{\gtrless} 0. \tag{15}$$

Further in this paper, the notations $(15)_>$ and $(15)_<$ are used to refer respectively to the condition with a $>$ or $<$ sign.

3.1 The case of increasing demand

An increase in yearly demand is modelled by considering $D' = \lambda D$ with $\lambda > 1$. In this case the sign of $(q'^*)^2 - (q^*)^2$ is determined by the second factor in inequality (15).
Starting with the situation in which $(15)_>$ holds, the following constraints on λ can be derived:

$$\frac{2C_o(C_h + C_g)}{DC_f^2} - 1 > \lambda > 1 \tag{16}$$

which, together with the necessary condition (14), results in the domain:

$$\lambda \in \left]1, \min\left(\frac{2C_oC_h}{DC_f^2}, \left\{\frac{2C_o(C_h + C_g)}{DC_f^2} - 1\right\}^-\right)\right] \tag{17}$$

where we have used the notations:

$$\min(a, \{b\}^-) = \begin{cases} a & \text{if } a < b \\ \{b\}^- & \text{if } a > b \end{cases} \quad \text{and }]a, \{b\}^-] =]a, b[.$$

Every value of λ belonging to this domain, corresponding to an increasing demand, results into an increase of the economic order quantity.
In case $(15)_<$ must hold,

$$\frac{2C_o(C_h + C_g)}{DC_f^2} - 1 < \lambda \tag{18}$$

together with $\lambda > 1$ and condition (14) are considered, resulting in the domain:

$$\lambda \in \left]\max\left(1, \frac{2C_o(C_h + C_g)}{DC_f^2} - 1\right), \frac{2C_oC_h}{DC_f^2}\right]. \tag{19}$$

Every value of λ in this interval corresponds to a situation in which yearly demand increases, however the economic order quantity will decrease since we are considering $(15)_<$. This is a counterintuitive result, since a rather linear reasoning is often made in practical situations. This counterintuitive behaviour is due to the presence of two competing terms in the expression of the economic order quantity (11) depending on the average yearly demand.

At the level of the order interval, using expression (8), the difference

$$(T'^*)^2 - (T^*)^2 = \frac{2C_o(C_h + C_g)}{C_gC_hD}\left(\frac{1}{\lambda} - 1\right) < 0 \tag{20}$$

implies $T'^* < T^*$ for every $\lambda > 1$. Consequently, the increase in demand will always result in an increase of the optimal order frequency $\nu^* = (T^*)^{-1}$.
Since the maximal backorder level can be rewritten as

$$b = q(1 - \alpha) \tag{21}$$

it is clear from expression (7) that, in case of a decreasing order quantity, its value will decrease.
Using a monte carlo simulation, it is possible to show that an increasing order quantity as a result of an increasing demand, does not lead to a univoke change of the maximal backorder level b^*. Hence there exist parameter settings for which $b'^* > b^*$ and other settings resulting in $b'^* < b^*$.

3.2 The case of decreasing demand

Again this case will be modelled by setting $D' = \lambda D$ with the sole difference that $0 < \lambda < 1$. This results in a sign inversion in the condition related to the second factor of the l.h.s. of (15) since $\lambda - 1 < 0$.

Starting with the situation where $(15)_<$ holds, λ must be subject to the following conditions:

$$\frac{2C_o(C_h + C_g)}{DC_f^2} - 1 > \lambda > 0 \tag{22}$$

of which the upper bound should be compared to the value 1. This leads to the domain:

$$\lambda \in \left]0, \min\left(1, \frac{2C_o(C_h + C_g)}{DC_f^2} - 1\right)\right[\tag{23}$$

Hence, every λ in this domain results in a decreasing yearly demand and a corresponding decreasing economic order quantity.
The case $(15)_>$ corresponding to an increasing value of the economic order quantity, imposes the constraint:

$$\frac{2C_o(C_h + C_g)}{DC_f^2} - 1 < \lambda \tag{24}$$

Based on necessary condition (10), the l.h.s. of this constraint is always positive. Hence, the domain in this situation is reduced to:

$$\lambda \in \left] \frac{2C_o(C_h + C_g)}{DC_f^2} - 1, 1 \right[\tag{25}$$

meaning that there exist parameter values for which a decrease in the yearly demand incurs an increase of the economic order quantity. Again this is a rather counterintuitive situation from a managerial point of view.

The difference (20) implies in all cases for which $0 < \lambda < 1$ that $T'^* > T^*$, hence the order frequency will decrease. Again from relation (21) one may conlude that, if (25) holds, the maximal backorder level will increase: $b'^* > b^*$. In the other situation (23), one may show, using monte carlo simulation for a broad class of parameter values, that either an increase or a decrease is possible.

3.3 Summary

The results of the different situations are summarised in figure 2. The intervals for λ in which counterintuitive results are obtained, are indicated by a boldface line.

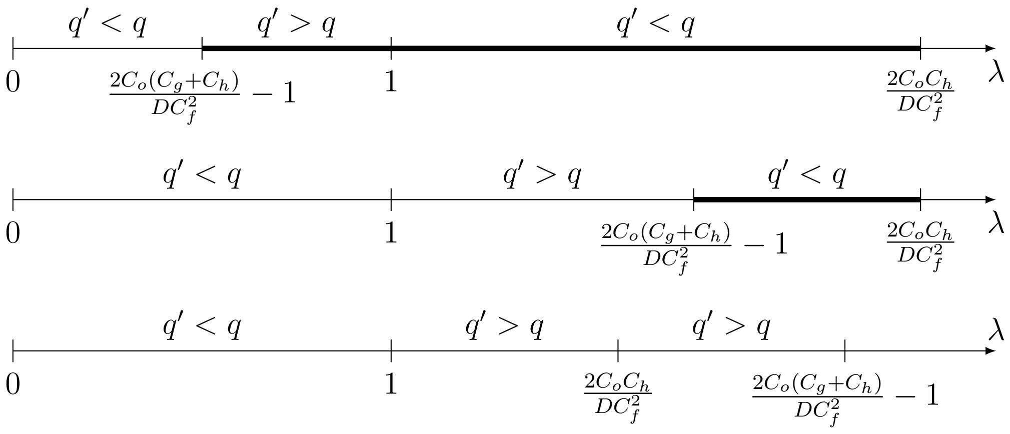

Figure 2: Summary of the different situations resulting from the sensitivity analysis

This enables us to conclude that the counterintuitive results will only occur if the following condition holds:

$$\frac{2C_oC_g}{DC_f^2} < 1. \tag{26}$$

Combining this result with (10) shows that counterintuitive decisions for changing the EOQ might be appropriate in case of high holding costs in comparison with the fixed backorder costs and with goodwill costs being smaller than the former.

3.4 Sensitivity on the goodwill cost

Still one of the major challenges in inventory management is the determination of the goodwill cost related to the customers' waiting time in case of a stockout. Since this cost, although present in several inventory models, is very hard to estimate, sensitivity analysis on this parameter is in order.
Starting with expression (11) and the corresponding one for a perturbed goodwill cost $C_g' = \lambda C_g$, it is easy to show that

$$(q'^*)^2 - (q^*)^2 = \frac{(2C_oC_h - DC_f^2)D}{C_gC_h}\left(\frac{1}{\lambda} - 1\right). \tag{27}$$

As a result, taking into account rel. (10), an increase in goodwill cost ($\lambda > 1$) will lead to a decrease of the economic order quantity: $q'^* < q^*$ and an increase of the order frequency. In order to compensate for the higher backorder cost, the maximal backorder level will also have decreased ($b'^* < b^*$). A similar reasoning for a decreasing goodwill cost leads to an increase of the economic order quantity and an increase of the maximal backorder level.

4 Conclusions

In this paper we have performed a sensitivity analysis for the economic order quantity in case of the deterministic stationary inventory model with planned backorders to analyze the impact of a changing demand. Explicitation of the backorder cost into a term independent of the waiting time of the customers and 'goodwill'-term leads to some remarkable conclusions. On the one hand there exist parameter values for which an increase of the yearly demand leads to a decrease of the optimal value of the order quantity. On the other hand, in the converse situation of decreasing demand there exist parameter settings for which the optimal value of the order quantity increases. Moreover, we derived a necessary condition on the parameters for this counterintuitive behaviour to occur.
Furthermore, a sensitivity analysis was performed on the goodwill cost, resulting in a risk reducing behavior of the economic order quantity and the maximal backorder quantity.

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