DEPARTMENT OF ENGINEERING MANAGEMENT

Optimal capacitated ring trees

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Abstract. We study a new network design model combining ring and tree structures under capacity constraints. The solution topology of this *capacitated ring tree problem* (CRTP) is based on *ring trees* which are the union of trees and 1-trees. The objective is the minimization of edge costs but could also incorporate other types of measures. This overall problem generalizes prominent capacitated vehicle routing and Steiner tree problem variants. Two customer types have to be connected to a distributor ensuring single and double node connectivity, respectively, while installing optional Steiner nodes. The number of ring trees and the number of customers supplied by such a single structure are bounded. After embedding this combinatorial optimization model in existing network design concepts, we develop a mathematical formulation and introduce several valid inequalities for the CRTP that are separated in our exact algorithm. Additionally, we use local search techniques to tighten the obtained upper bounds. For a set of literature-derived instances we consider various reliability scenarios and present computational results.

Keywords: capacitated ring tree problem, Steiner tree, ring tree, vehicle routing, survivable network design, integer programming

1 Introduction

In supply network design as well as telecommunications, the graph class of trees is widely used as a base structure to model optimization problems. Typically, a set of specified customer nodes has to be connected to a central distributor by a selection of supply edges for which individual installation costs apply. In a natural way there exists a tree that minimizes the overall connection costs when considering this basic setting. The determination of such a tree is well known as the *spanning tree problem* (SPTP). However, many real-world networks would allow the establishment of links that do not necessarily connect two customers directly, but utilize optional intermediate nodes. The usage of these *Steiner nodes* might be either essential for the network connectivity, result in an overall cost reduction or be non-advantageous. Although providing a broader applicability, the resulting well-known *Steiner tree problem* (STP) [32] is more challenging as its complexity is known to be NP-hard.

A crucial requirement for the design of networks in various applications is the ability to provide reliable service to the customers. Even after a link failure due to technological or environmental reasons the customer connectivity might be highly desirable in the remaining network. Since trees can be characterized as graphs in which two nodes are connected by a unique path, any missing link disconnects the network. To overcome this weakness, the ring structure has proven to be a suitable option because of its 2-connectivity property: after the removal of any single edge the graph is still connected.

The model that we introduce in this work fills a gap in the existing literature. We bring together the tree structure and the ring structure under additional capacity constraints. In this *capacitated ring tree problem* (CRTP) we are given two categories of customers that have to be connected to a central distributor by using optional Steiner nodes. We say customers are of *type 2* if they require a link-failure reliability with respect to the distributor, sometimes also called 1+1 protection. The remaining customers are labeled as of *type 1* and need simple connectivity. Albeit, the latter might be equipped with additional reliability if this is favorable in terms of the overall network cost. We want to find a set of rings that intersect in the distributor node and contain all type 2 customers. At the same time the remaining type 1 customers have to be connected to these ring structures by forming trees or be ring nodes themselves. Such an individual structure that is connected to the distributor, either a pure tree or a ring with its attached trees, is called a *ring tree*. We impose two capacity limits on the resulting network: the number of these ring trees as well as the number of customers in such a ring tree are bounded. We allow these ring trees to be pure trees that are directly attached to the distributor but count each incident non-ring edge as one ring tree. The objective is to minimize the overall costs for the installed edges. Fig. 1 illustrates a solution for the CRTP implementing four ring trees.

To the best of our knowledge the modeling of this advanced ring extension structure under capacity constraints has not been explicitly considered in the literature so far. By allowing the assignment of trees to rings in the CRTP we present the first research approach into this direction. A major strategical planning feature of these models is the anterior two-type categorization of the customers. Ring tree problems find applications in the design of telecommunication networks. The CRTP can be used in local access network design, for modeling backbone networks or even combining these levels. It can be used to integrate ring-based reliability in recent real world applications which are based on Steiner trees [12]. In transportation network planning we can represent ship routes by rings and simultaneously model the inter-modal freight distribution networks from their ports of call. Here we see another particular strength of the model in the ability of linking two strategical levels.

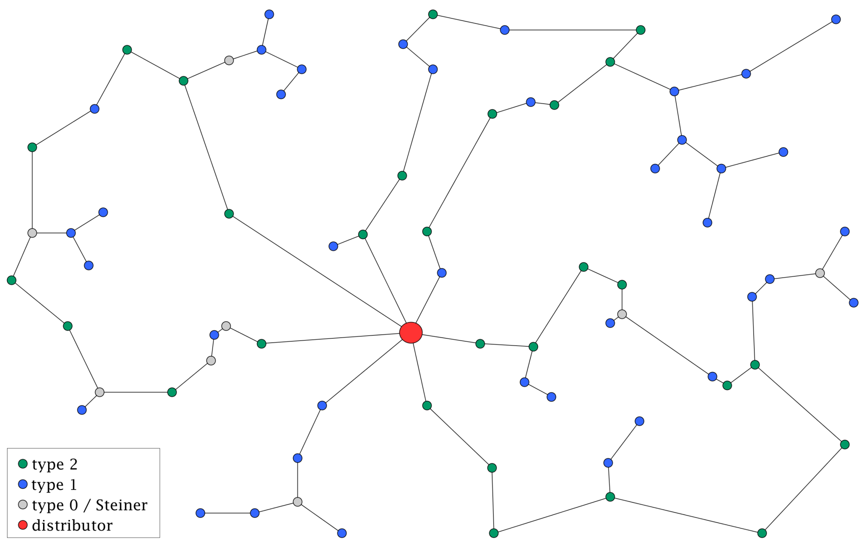

Fig. 1: A solution for a capacitated ring tree problem based on 4 ring trees (3 of which are 1-trees and one is a tree).

Given the above exposition, the idea of this work is to provide a new type of problem, the CRTP, which allows for generic treatment and extended understanding of developing new algorithmic approaches for the CRTP as well as some of the arising subproblems. This paper is structured as follows. In Section 2 we relate our new model to existing concepts in the greater network design literature. A formal definition of the CRTP is presented in Section 3 with the notation used throughout this work. In Section 4 we present our mathematical formulation on which the exact algorithm is based. Along the description of our exact algorithm in Section 5 we develop valid inequalities and show in detail how these can be separated. In our computational study we apply our algorithms on literature-derived test instances for different reliability scenarios. The results and an analysis of the impact of reliability variation are provided in 6. We close with our conclusions in Section 7.

2 Related models

In this section we show the originality of the CRTP by summarizing relationships to existing related network design models. We focus on the models with an overall edge cost minimization objective function and do not address various extensions such as price-collecting problems or revenue maximization. Figure 5 illustrates the relationships between the models mentioned in the following. In addition to the 1-connectivity required for type 1 nodes and the 2-connectivity for the type 2 nodes, we denote the optional Steiner node usage as a 0-connectivity requirement.

2.1 Ring models

The ring component of the CRTP is used in classical *capacitated vehicle routing problems* (VRPs) to represent vehicle routes. The CRTP reduces to a VRP with a homogeneous vehicle fleet and constant customer demands in the case of no type 1 customer presence. As a consequence, the CRTP generalizes the prominent *travelling salesman problem* (TSP) asking for a Hamiltonian cycle of minimal total edge costs. The *Steiner travelling salesman problem* (STSP) [21] asks for a cost minimal tour in which an edge may be traversed multiple times as illustrated in Figure 2. Moreover, we pay the edge cost for each of the edges in a solution network, which is generally not a simple graph. For a set of predefined clusters the *generalized travelling salesman problem* (GTSP) [6] asks for a ring that just includes one node of each cluster rather than all of them. Figure 2 illustrates such a non-spanning tour. In contrast to most routing models we allow Steiner nodes when designing ring trees in the CRTP. A collection of related vehicle routing models and existing exact algorithms can be found in [4].

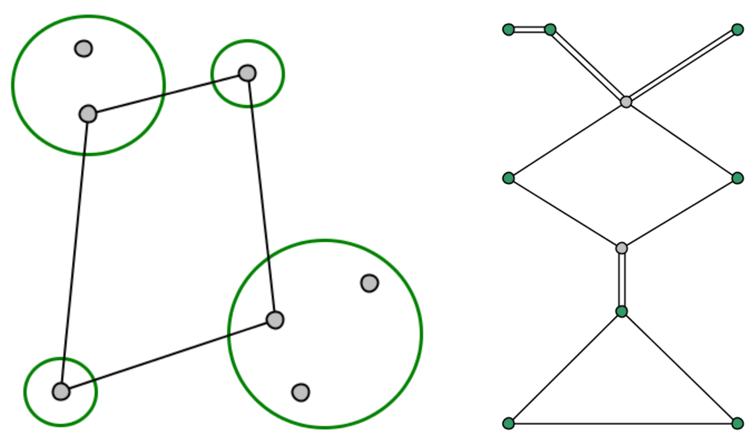

Fig. 2: A generalized travelling salesman tour (left) and a solution for the Steiner travelling salesman problem (right).

2.2 Tree models

The CRTP generalizes the *capacitated minimum spanning tree problem* (CSPTP) with unit node demands. The CSPTP asks for a minimum spanning tree in which the sum of given node demands in each subtree induced by an edge incident to the distributor is bounded by σ. A CSPTP can be formulated as a CRTP when setting the limit on the number of ring trees to infinity, bounding the customers per ring tree by σ and labelling all its non-distributor nodes as type 1 customers.

A survey on heuristics for related problems can be found in [2]. The *minimum capacitated Steiner problem* (CSTP) shares the cardinality constraints but allows the usage of Steiner nodes in the network. We note in passing that an explicit consideration of the CSTP is somewhat lacking in literature. When even relaxing the ring tree capacity constraints, this problem is equivalent to the STP.

2.3 Ring star models

A ring that is extended by single node assignments is known to follow the *ring star* pattern ([19]) as illustrated in Figure 3. Each node either belongs to a ring or is a leaf node of degree 1. An efficient layout then usually means the interlinkage of customers to a central distributor by (disjoint) ring stars such that the overall edge costs are minimized. Due to practical requirements capacity limits may apply to the number of customers per ring star or the number of installed ring stars ([3]). In this *capacitated ring star problem* (CRSP) the customers that are allowed to be assigned to rings are given in advance and are therefore of type 1. The CRTP goes beyond this idea by replacing single customer assignments by assignments of trees but does not generalize the CRSP. In ring star problems the allowed assignments of customers to the rings are commonly the result of a previous optimization-based modeling step. Once a solution is at hand, the actual assignment is realized by the installation of a shortest path from the assigned type 1 customer to its chosen ring supplier. Multiple such paths are possibly implemented by a combining tree structure as illustrated in Figure 3. Hence, the optimization potential is fully utilized in the CRTP by the integration of the design of the type 1 customer assignment structures into the overall model. With an increasing rate of the latter customers we magnify the overall cost saving potential compared to the described two-step approach.

In the *travelling purchaser problem* (TPP) [28] a cost-efficient tour has to be designed to purchase required products at selected markets. These products can be obtained from various markets at different prices. A decision to purchase a certain product at a market on the route can be interpreted as a product assignment to a route that includes this market, resulting in a ring star structure. In [11] an extension is considered in which the tour length as well as the number of assignments per market are restricted. However, in the TPP the assignable products cannot be tour nodes whereas a type 1 customer can be a ring node in the CRTP and the CRSP, respectively.

2.4 Survivable network design

Requiring a certain degree of connectivity between network nodes is the basic concept in *survivable network design problems* (SNDPs). The *survivability* of a node is either measured by the number of edge-disjoint paths to the remaining network or the stronger node-disjoint paths. In the CRTP these underlying connectivity requirements with respect to the distributor are of order 0, 1 and 2, depending on the node type. They are typical for *low connectivity constrained* survivable network design problems ([31],[8]). However, the CRTP

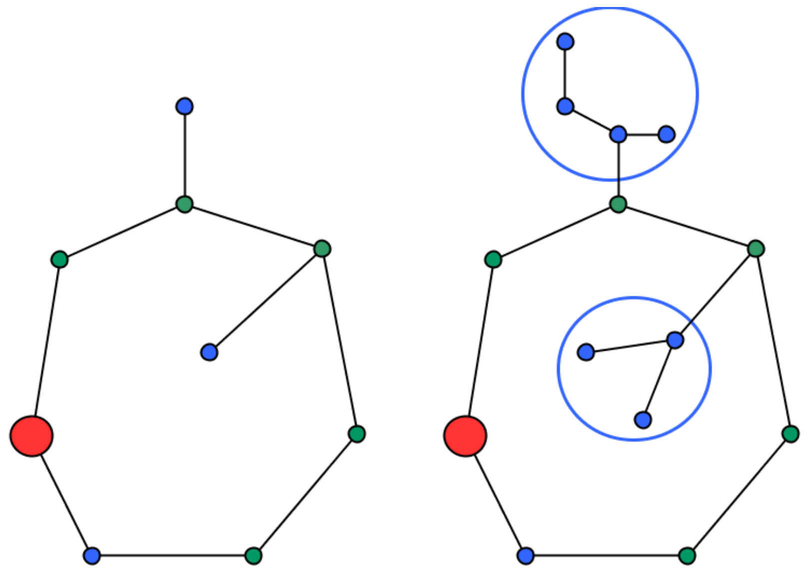

Fig. 3: A CRTP approximating ring star network (left) and its realization using the ring tree structure (right).

enforces a ring tree topology whereas SNDP models do not restrict the obtained network structure as long as the connectivity requirements are fulfilled. Figure 4 gives examples for optimal SNDP topologies that result from the given survivability requirements and the edge cost structure. Related models, polyhedral results and solution methods can be found in [31] and [17]. Due to its rather generic survivability requirement, special cases including regular survivability and bounded survivability got particular attention. Some results with a special focus on low redundancy are summarized in [7]. The numerous suitable applications for SNDP-based models motivated their extensions to design networks that satisfy various supplementary requirements. These additional restrictions are largely of capacity-bounding type which reflect technological or business limitations. Well-established representatives are node degree constraints, hop constraints, diameter constraints, node/edge supply capacity constraints, cardinality constraints, mesh constraints and their combinations. In [8] the authors introduce a capacity constraint on the number of customers on the rings in a two-connected network to bound the rerouting distances in the case of a link failure. Several network design models can be considered as SNDPs with imposed capacity constraints. Figure 5 summarizes the major problems and problem classes discussed in this section. It also puts the CRTP into perspective.

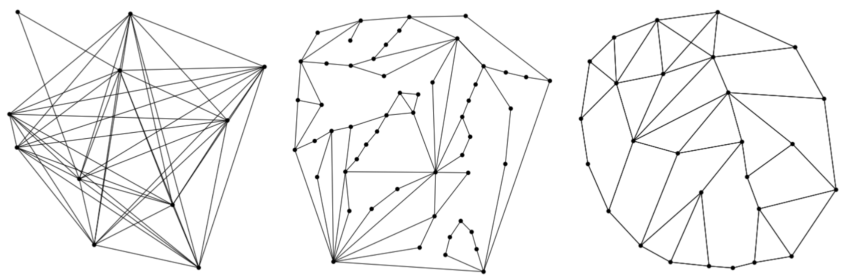

Fig. 4: SNDP solution topologies for SNDlib instances ZIB54, DFN-GWIN and SUN [27].

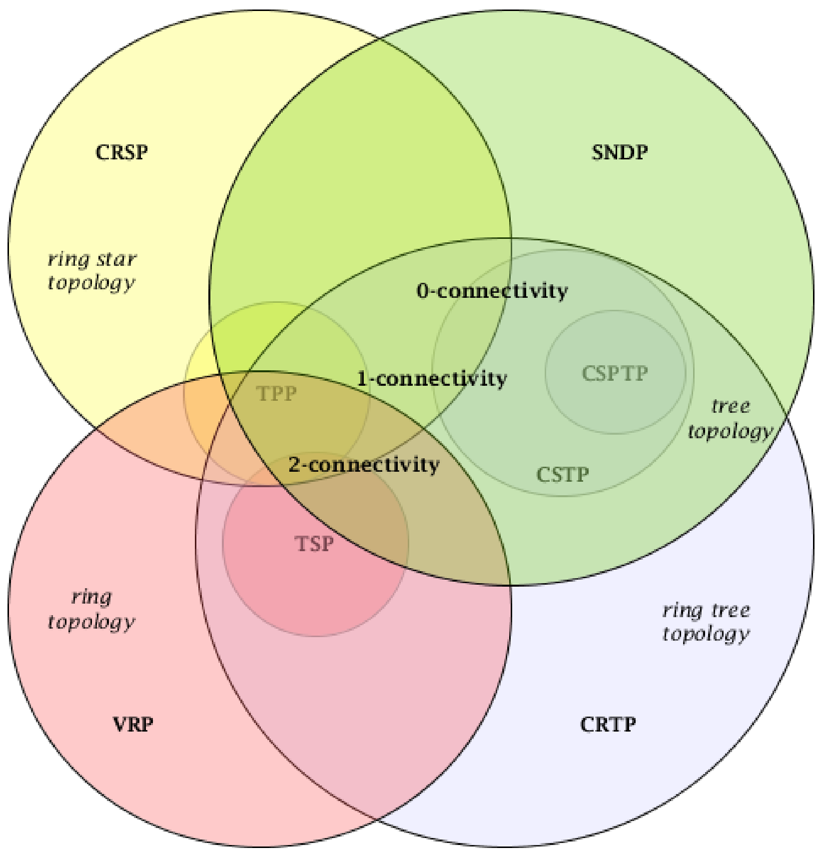

Fig. 5: The capacitated ring tree problem and related literature network design models.

3 The capacitated ring tree problem

Before giving a formal definition of the capacitated ring tree problem we introduce the base topology of the CRTP in a graph theoretic manner. Throughout this work we denote the node set of a graph G as $V[G]$, its set of edges by $E[G]$ and the arc set by $A[G]$ if G is directed. Recall that a *1-tree* can be characterized as a connected undirected graph containing a unique cycle.

Definition 1. *A* ring tree *is a connected graph containing at most one cycle.*

In other words, a ring tree is a connected graph Q with at most $|V[Q]|$ edges. Therefore, the graph class of ring trees is the disjoint union of trees and 1-trees. Since the class of cycle graphs is included in 1-trees, ring trees generalize both, rings and trees. We recall that 1-trees have been proven useful for deriving lower bounds and solution techniques for the classical TSP [13].

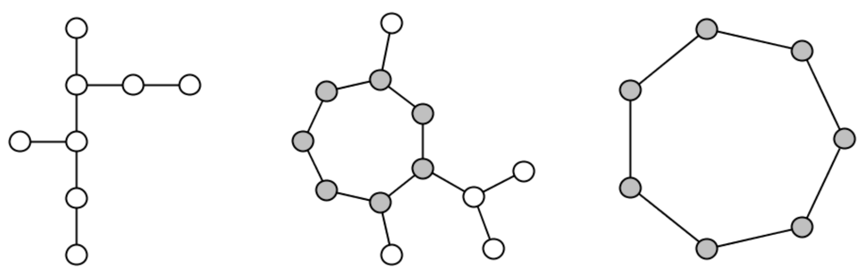

Fig. 6: Some ring trees and their fundamental subcycles.

Given a tree T we can create $(|V[T]|^2 - |V[T]|)/2 - |E[T]|$ distinct subcycles in T by the insertion of single chords which are called the *fundamental cycles*. Figure 6 depicts examples for the ring tree structure and fundamental cycles. Similarly, we can define a *directed ring tree* as a directed graph that is either an arborescence or the union of a *directed (fundamental) cycle* C and arborescences rooted in $V[C]$.

Definition 2. *We are given an undirected complete simple graph G. Its node set is the disjoint union of type 1 customers, type 2 customers and Steiner nodes, complemented by a distributor node: $V[G] = U_1 \,\dot\cup\, U_2 \,\dot\cup\, W \,\dot\cup\, \{d\}$. Each edge $e \in E[G]$ is associated with a non-negative weight c_e. Let a ring tree limit m and a customer per ring tree limit q be given. For a set of ring trees $S = \{Q_1 \subseteq G, ..., Q_k \subseteq G\}$ we denote the network graph by $N_S = (\bigcup_{Q \in S} V[Q], \bigcup_{Q \in S} E[Q])$. S represents a solution for the CRTP if*

- *each type 1 customer is contained in exactly one ring tree,*
- *each type 2 customer is contained in exactly one ring tree's fundamental cycle,*

- *each Steiner node is contained in at most one ring tree,*
- *the number of ring trees k is at most m,*
- *the number of customers in a ring tree does not exceed q, and*
- *for each ring tree, d is either a degree-two cycle node or a leaf if no fundamental cycle is present.*

The CRTP asks for a solution of minimal total cost, i.e. minimized sum of edge costs $\sum_{e \in E[N_S]} c_e$.

Note that following our definition of the CRTP we allow the direct assignment of trees to the distributor. It is easy to see that requiring every distributor-outbound structure to link back to it would favor solutions containing Steiner rings which we want to avoid here. We assume that the distributor has the same capacity consumption through a tree serving a certain number of customers as it has by a ring (tree) with equally many customers. When applying the customer limit we consider each tree induced by an edge incident to d individually. Some ring-based models require m to be met exactly (e.g. [3]), which we relax here for the sake of overall cost efficiency. We define U to be the set of customers $U_1 \cup U_2$ that require to be contained in a solution and assume that $mq \geqslant |U|$ since the CRTP instance is obviously infeasible. The NP-hardness of the CRTP follows from its reducibility to the TSP, for instance. Fig. 1 above illustrates a solution for the CRTP.

4 Mathematical formulation

We present a mathematical model for the CRTP that is based on a directed network representation. Since non-compact formulations were shown to be computationally more efficient than flow-based formulations in many cases (e.g.[3]) we propose a 2-index cut set formulation. Advanced branch & cut techniques for an efficient algorithm are developed in the next section. As concluded in [22], the LP lower bounds obtained by a directed formulation of the Steiner tree problem are at least as good as their counterparts from the undirected case. Similarly, this holds for directed formulations of vehicle routing problems. Therefore, we formulate the CRTP based on the complete orientation of G, denoted by H. The resulting forward and backward arcs are assigned the cost of the corresponding edge in $E[G]$. We search for a solution based on directed ring trees which can be transformed into a solution of the CRTP by definition. A binary variable x_a indicates whether an arc a is used in such a directed representation. The installation of a forces a corresponding binary edge variable y_e to take value 1. A continuous flow variable $f_a \in [0, 1]$ takes value 1 if the arc a is part of a directed ring and 0 otherwise. Our directed formulations might also be used for an *asymmetric capacitated ring tree problem* (ACRTP) that we will not further investigate in this paper. The CRTP can be formulated as a Steiner arborescence problem with additional side constraints. To achieve this, artificial sink nodes have to be introduced that represent terminals for the arborescence rooted in

the distributor whenever a directed path is closed to a ring. However, we decided to develop a separate model for the CRTP without such a reformulation to underline its importance in its own right.
In our mathematical formulation we occasionally use ij to denote an arc (i, j) for the sake of simplified notation. For two disjoint node sets $X, Y \subset V[H]$ in a directed graph H, we define $\delta^-_Y(X) = \{(i, j) \in A[H] : i \in X, j \in Y\}$ and $\delta^+_Y(X) = \{(i, j) \in A[H] : i \in Y, j \in X\}$. If clear from context we may omit to mention Y in the case that $V[H] \setminus X \subseteq Y$. For $X = \{i\}$, $i \in V[H]$, we may use $\delta^-_Y(i)$ and $\delta^+_Y(i)$, respectively. We also use $X(Y) = X \cap Y$ for denoting intersecting sets, as for example the customers $U(S)$ in a node set $S \subseteq V[G]$.

$$\min \quad \sum_{e\in E[G]} c_e y_e \tag{1}$$

$$s.\,t. \quad \sum_{a\in\delta^-(S)} x_a \;\geqslant\; \frac{|U(S)|}{q} \quad \forall\, S \subset V[H] \setminus d, \tag{2}$$

$$\sum_{a\in\delta^-(i)} x_a \;=\; 1 \quad \forall\, i \in U, \tag{3}$$

$$\sum_{a\in\delta^-(i)} x_a \;\leqslant\; 1 \quad \forall\, i \in W, \tag{4}$$

$$\sum_{a\in\delta^-(d)} x_a \;\leqslant\; m, \tag{5}$$

$$x_{ij} + x_{ji} \;=\; y_{ij} \quad \forall\, \{i, j\} \in E[G], \tag{6}$$

$$\sum_{a\in\delta^-(i)} f_a \;=\; \sum_{a\in\delta^+(i)} f_a. \quad \forall\, i \in V[H], \tag{7}$$

$$\sum_{a\in\delta^-(i)} f_a \;=\; 1, \quad \forall\, i \in U_2, \tag{8}$$

$$0 \;\leqslant\; f_a \;\leqslant\; x_a \quad \forall\, a \in A[H], \tag{9}$$

$$x_a \in \{0, 1\} \quad \forall\, a \in A[H], \tag{10}$$

$$y_e \in \{0, 1\} \quad \forall\, e \in E[G]. \tag{11}$$

Our cut set formulation is based on binary arc variables x_a for the arcs in $A[H]$. The assignment constraints (3) ensure an in-degree equal to one for each customer, whereas the capacity constraints (4) limit the inbound arcs to one for each Steiner node. The *capacitated connectivity constraints* (2) bound the number of customers per ring tree to q. We model an underlying *single commodity flow* (SCF) structure by *arc flow variables* f_a and (in)equalities (7), (8) and (9). Since we consider directed ring trees, inequality (5) is sufficient to limit the number of ring trees to m. To obtain a simple undirected solution network and identify its edges we implement the variable linking equalities (6). When $q = |U|$ and $U_2 = \emptyset$ the right hand sides of (2) is bounded by 1, leading to a well-known cut set formulation for the STP. If $q = |U|$ and $U_1 = W = \emptyset$ then we obtain a corresponding model for the TSP. Although we are just dealing with a total of $3|E[G]|$ variables we are faced with an exponential number of constraints of type (2). The objective 1 measures the network cost of by summing up the costs of installed edges.

A CRTP variant that considers a different cost function for edges on fundamental cycles than for edges of attached trees can be modeled by modifying the objective. Let c_e^r be the cost of a ring edge $e \in E[G]$ and $c_{e'}^t$ the cost of an edge that is connected to d by a unique path in a solution. Then the total cost of a ring tree design can be measured by replacing (1) by the following objective function.

$$\sum_{e=\{i,j\}\in E[G]} \left[c_e^r(f_{ij} + f_{ji}) + c_e^t(y_e - f_{ij} - f_{ji}) \right]. \tag{12}$$

5 Exact solution techniques

In this section we develop an efficient branch & bound algorithm based on our non-compact mathematical formulation in Section 4. An emphasis is put on bound-tightening, which we achieve by CRTP specific cutting techniques and solution polishing. These two matters are crucial for the efficiency of a mathematical programming based approach as extensively discussed in the literature (e.g. [23]). For various hard combinatorial optimization problems the most competitive algorithms rely on the application of sophisticated cutting planes combined with efficient primal heuristics.

5.1 Strengthening the lower bounds

In the following we present valid inequalities and corresponding separation techniques in order to improve the lower bounds during the branch & cut algorithm. Due to the specific CRTP topology we combine cutting planes based on ideas from network design models for trees and vehicle routing. In the special cases that $U_2 = \emptyset$ or $U_1 = \emptyset$ some of our valid inequalities collapse to equivalent ones for the STP or the VRP, respectively. Let LP denote the linear program obtained after relaxing the integrality of variables x and y in our formulation. We consider an optimal fractional arc solution for the LP-relaxed subproblem in the branch & bound tree as the assignment of values $x^* : a \in A[H] \to [0, 1]$ and f^* for the ring flow, respectively. Such a typical solution combines characteristics from the Steiner tree problem with VRP typical subtours as depicted in Figure 7. For a more convenient formulation of the inequalities we introduce a continuous auxiliary *ring node* variable z_i for each node $i \in V[H] \setminus \{d\}$ that identifies i as a fundamental cycle node. These variables are linked to the node's total *inbound* ring flow as follows.

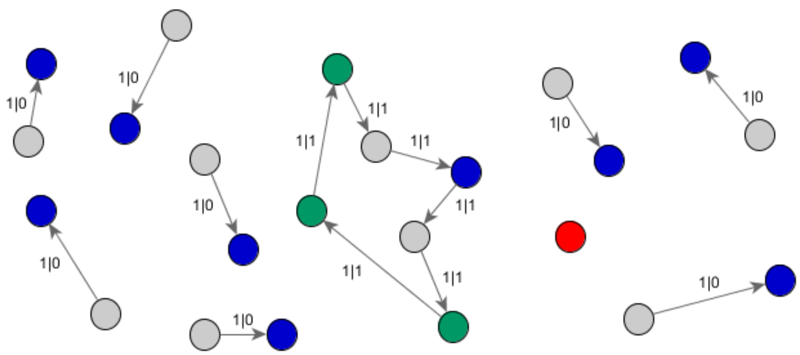

Fig. 7: A typical solution of a LP-relaxed CRTP in the directed formulation $(x_a^*|f_a^*)$.

$$z_i \;=\; \sum_{a\in\delta^-(i)} f_a \quad \forall\, i \in V[H]\setminus\{d\}. \tag{13}$$

Inasmuch as $z_i = 1$ holds $\forall\, i \in U_2$ we are more interested in the connectivity of type 1 customers and Steiner nodes with respect to d. Optimal ring node values complementing x^* and f^* are denoted by z^*. The different node types in the CRTP give rise to various cut arc configurations for a given node subset. We refer to the illustration in Figure 8 along our descriptions.

Ring flow inequalities The inequalities (9) that link the ring flow to the ring arc variables can be further tightened for mandatory cycle nodes through (14).

$$f_a \;=\; x_a \quad \forall\, a \in \delta^-(i),\; i \in U_2 \cup \{d\}. \tag{14}$$

In the CRTP we even allow non-type-2 nodes to obtain double connectivity by being a ring node. In terms of our formulation such a node $i \in U \cup W$ is equipped with reliability if there is ring flow entering i, i.e. $z_i > 0$. If this is the case then the cycle structure requires a unique subsequent ring node on i's ring. Thus there is at most one natural ring node (type 2 customer or distributor) connected by an arc from i.

$$\sum_{a\in\delta^+_{U_2\cup\{d\}}(i)} x_a \;\leqslant\; z_i \quad \forall\, i \in V[H]\setminus d. \tag{15}$$

To avoid reverse ring flow we can require the outbound ring flow from j to nodes in $V[H]\setminus i$ to be at least the ring flow f_{ij} on each arc $(i,j) \in A[H]$.

$$f_{ij} \;\leqslant\; \sum_{a\in\delta^+_{V[H]\setminus\{i\}}(j)} f_a \quad \forall\, (i,j) \in A[H]. \tag{16}$$

Since there are $|A[H]|^2$ such inequalities (16) we separate them dynamically by a straightforward arc search.

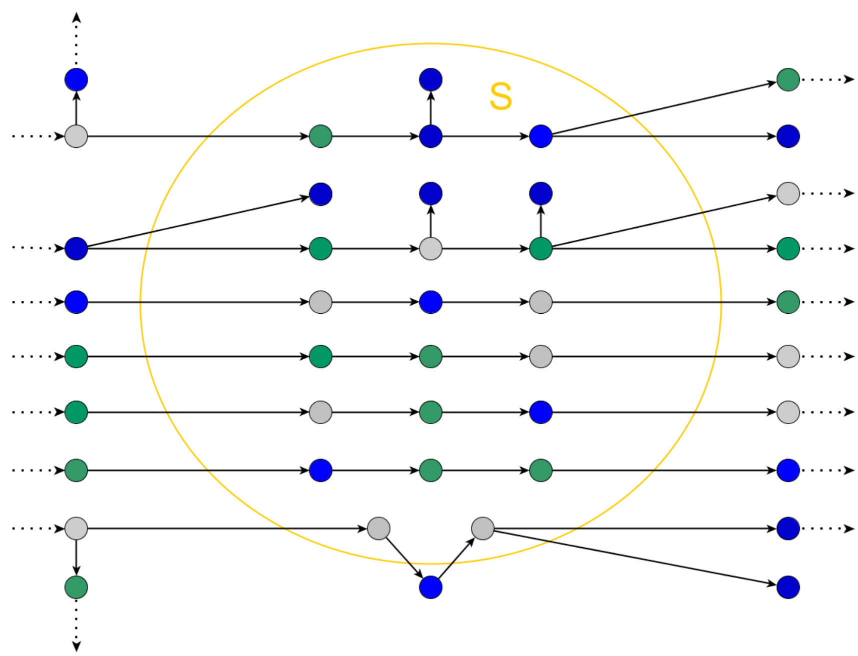

Fig. 8: A CRTP cut set and examples for the various types of intersections with ring tree structures considered by cutting planes.

Connectivity inequalities The following inequalities are also well-known as *subtour elimination constraints* and impose a unitary lower bound on the right hand side of (2).

$$\sum_{a \in \delta^-(S)} x_a \geqslant 1 \quad \forall\, S \subset V[H] \setminus d : i \in S,\ \forall\, i \in U. \tag{17}$$

To separate (17) for a customer i we compute a directed $d-u$ cut (D, S) of minimal weight w in H with respect to arc weights x^*. If $w < 1$ then we add an inequality (17) for the cut set S.

Capacitated connectivity inequalities Instead of inequalities (2) we add the stronger *capacitated connectivity inequalities.*

$$\sum_{a \in \delta^-(S)} x_a \geqslant \frac{|U(S)|}{q} \quad \forall\, S \subset V[H] \setminus d. \tag{18}$$

In fact, we can even assume that the sum of the inbound arc variable values is integer to get dominating rounded versions.

$$\sum_{a \in \delta^-(S)} x_a \geqslant \left\lceil \frac{|U(S)|}{q} \right\rceil \quad \forall\, S \subset V[H] \setminus d. \tag{19}$$

The separation of (18) requires the computation of a minimal $d-s$ cut (R, S) in a directed auxiliary graph $H^-_{1,2}$ with node set $V[H^-_{1,2}] = V[H] \cup \{s\}$ and arc set $A[H^-_{1,2}] = A[H] \cup \{(i, s) : i \in U\}$. An arc (i, s) has weight $1/q$ $\forall i \in U$ and the remaining arcs have weight f^*_{ij} . S is a violating cut set if the obtained cut weight is less than $|U|/q$.

Capacitated ring tree multi-star inequalities Furthermore, we introduce several *capacitated ring tree multi-star inequalities* for the CRTP which generalize (2). For a set of nodes S not including d, we additionally estimate the number of *distinct* customers in $U \setminus S$ that are connected to a node in S to ensure a sufficient number of arcs entering S. Due to the (ring) tree topology such a customer can be incident to multiple arcs in $\delta^-_{U \setminus S}(S)$. Hence counting all these *inbound* customer arcs generally results in an overestimation of the number of inbound customers. Nevertheless, we can give a lower bound on the inbound customers for a given a subset X of S by calculating the inbound customer ring flow $\sum_{a \in \delta^-(X)} f_a$. Moreover, for each type 2 customer $i \in X(S)$ we can replace an in-flow variable f_{ji} by the in-arc variable x_{ji}. At the same time we obtain a lower approximation by using the fact that the out-degree of a customer node is at most q. Thus we can sum over all the inbound customer arc variables while correcting by dividing through this decremented ring tree customer capacity $q-1$. These two arguments are combined in the following inequalities on $X \subseteq S$

and its complementary set $S \setminus X$.

$$\sum_{a\in\delta^-(S)} x_a \geqslant \frac{1}{q}\left(|U(S)| + \sum_{a\in\delta^-_{U\setminus S}(X(U_2(S)))} x_a + \sum_{a\in\delta^-_{U\setminus S}(X(S\setminus U_2)))} f_a + \frac{1}{q}\sum_{a\in\delta^-_{U\setminus S}(S\setminus X)} x_a\right) \quad \forall\ X,\ S\subset V[H]\setminus d \tag{20}$$

These inequalities are similar to partial multi-star inequalities known for the VRP. We are able to efficiently separate these CRTP specific inequalities for a fixed set $X \subset V[H]\backslash d$. The separation of inequalities (20) is based on the minimal cut computation for (18) with modified arc costs in $H^-_{1,2}$. We set the weight for an arc $a \in U\times U_2(X)$ to $(1-1/q)x^*_a$ and for $a\in U\times(X\setminus U_2)$ to $x^*_a - f^*_a/q$ using the fact that $f^*_a \leqslant x^*_a$. The arc weight for $a\in U\times(V[H]\setminus X)$ is $(1-1/q^2)x^*_a$. The sets we selected are inspired by the cut arc node type combinations illustrated in Figure 8. More precisely, we enforce (20) for $X \subseteq \{\bigcup_{L\in P} L : P\in\mathcal{P}(\{W,U_1,U_2\})\}$ resulting in at most eight different types of inequalities. When adding such a cut we can replace the maximal out-degree q by $\min\{q,|S|\}$ and if $S\cap W=\emptyset$ by $\min\{q-1,|S|\}$.
An alternative way to strengthen (2) is to count arcs leaving S towards customers not in S since they consume capacity. Actually, even a non-ring arc (i,j) in $\delta^+_W(S)$ implies at least one more customer since there exists an optimal solution without Steiner leave nodes. However, this customer might be already incorporated as a node in S. Such potential *ears* with respect to S are the reason that we cannot relate inbound and outbound arcs at the same time. In contrast to (20), every customer that is reached from S can be counted without approximation as follows.

$$\sum_{a\in\delta^-(S)} x_a \ \geqslant\ \frac{1}{q}\left(|U(S)| + \sum_{a\in\delta^+_{U\setminus S}(S)} x_a\right) \quad \forall\ S\subset V[H]\setminus d \tag{21}$$

Note that the separation of inequalities (21) is NP-hard since it is equivalent to finding a directed cut of maximal weight.

Rounded ring tree multi-star inequalities Although rounding the right hand side of (21) results in further dominating valid inequalities, the constraint linearity would be violated. So we use the techniques from [3] to derive linear inequalities through an estimate as follows. Lemma 1 of [3] states that for integers $(\alpha,\beta,\gamma)\in\mathbb{N}^3$ with $\alpha>\gamma>0$ and $\alpha \mod \gamma \neq 0$ the inequality $\lceil\frac{\alpha-\beta}{\gamma}\rceil \geqslant \lceil\frac{\alpha}{\gamma}\rceil - \frac{\beta}{\alpha \mod \gamma}$ holds. We use this after rewriting the summation terms for the case that $|U|>q$ and $|U| \mod q\neq 0$. Note that if $|U|\leqslant q$ then we deal with an instance that is effectively uncapacitated.

$$\begin{aligned}
\sum_{a\in\delta^-(S)} x_a \;&\geqslant\; \left\lceil \frac{1}{q}\Big(|U(S)| + \sum_{i\in U_1(S)} z_i + \sum_{a\in\delta^+_{U\setminus S}(S)} x_a \Big) \right\rceil \\
&\geqslant\; \left\lceil \frac{1}{q}\Big(|U(S)| + \Big[|U\setminus S| - \sum_{i\in U\setminus S} \sum_{a\in\delta^-_{V[H]\setminus S}(i)} x_a \Big] \Big) \right\rceil \\
&\geqslant\; \left\lceil \frac{1}{q}\Big(|U| - \sum_{i\in U\setminus S} \sum_{a\in\delta^-_{V[H]\setminus S}(i)} x_a \Big) \right\rceil \\
&\geqslant\; \left\lceil \frac{|U|}{q} \right\rceil - \frac{1}{|U| \mod q} \sum_{i\in U\setminus S} \sum_{a\in\delta^-_{V[H]\setminus S}(i)} x_a \quad \forall\, S\subset V[H]\setminus d : S\neq\emptyset.
\end{aligned} \tag{22}$$

The partial multi-star inequalities (21) and (22) cannot be separated polynomially [20]. Therefore, we check whether any of these is violated by any cut set identified in a previous separation procedure and eventually add it.

Ring closure inequalities Compared to (2) the following inequalities ensure type 2 customer connectivity towards d in our directed formulation.

$$\sum_{a\in\delta^+(S)} f_a \;\geqslant\; 1 \quad \forall\, S\subset V[H]\setminus d : i\in S,\ \forall\, i\in U_2. \tag{23}$$

Inequalities (23) can be adapted to be applicable to nodes of type 0 and 1. Since such a node i is not a ring node of necessity, we express the constraint based on the optional ring flow z_i through i.

$$\sum_{a\in\delta^+(S)} f_a \;\geqslant\; z_i \quad \forall\, S\subset V[H]\setminus d : i\in S,\ \forall\, i\in U_1\cup W. \tag{24}$$

The separation of (23) and (24) is done by minimal $i-d$ cut computations in H using arc weights f^*. The violation of the first inequality is detected as for (17) and we add (24) if the obtained cut weight is lower than z_i^*.

Capacitated ring closure inequalities The connectivity requirement in inequalities (23) can be extended to *capacitated ring closure inequalities* that take into account the ring tree capacity q when imposing necessary rings.

$$\sum_{a\in\delta^+(S)} f_a \;\geqslant\; \frac{|U_2(S)|}{q} \quad \forall\, S\subset V[H]\setminus d. \tag{25}$$

After rounding the constant term as in(19) we obtain *rounded capacitated ring closure inequalities.*

$$\sum_{a\in\delta^+(S)} f_a \;\geqslant\; \left\lceil \frac{|U_2(S)|}{q} \right\rceil \quad \forall\, S\subset V[H]\setminus d. \tag{26}$$

Inequality (25) is separated by the computation of a minimal $s-d$ cut S, D on the directed auxiliary graph H_2^+ with $V[H_2^+] := V[H] \cup \{s\}$ and additional arcs from s to all the type 2 customers: $A[H_2^+] := A[H] \cup \{(s,i) : i \in U_2\}$. The weight of an arc $(i,j) \in A[H_2^+]$ is $1/q$ if $i = s$ and else x^*_{ij}. The cut set S violates (25) if the cut weight is less than $|U_2|/q$. Furthermore, we can take into account type 1 ring customers in S since they consume ring tree capacity, too. They can be identified by the conveyed ring flow z_i $\forall i \in U_1$. Therefore, inequalities (25) are generalized by stronger inequalities (27) that count the number of type 1 ring nodes based on the ring flow.

$$\sum_{a \in \delta^+(S)} f_a \;\geqslant\; \frac{1}{q}\left(|U_2(S)| + \sum_{i \in U_1(S)} z_i\right) \quad \forall\, S \subset V[H] \setminus d. \tag{27}$$

We separate them on the graph $H_{1,2}^+$ which is obtained from H_2^+ by extending the arc set to $A[H_{1,2}^+] = A[H_2^+] \cup \{(s,i) : i \in U_1\}$ with arc weights z^*_j $\forall (s,j) \in s \times U_1$. A $s-d$ cut weight less than $(|U_2| + \sum_{j \in U_1} z^*_j)/q$ indicates that S is a cut set that violates the inequality the most. We note that an alternative separation technique can be derived by expressing z_i as $\sum_{a \in \delta^+(i)}$ using (7) and modifying the corresponding arc costs in H_2^+.

Capacitated ring closure multi-star inequalities To ensure ring-to-distributor connectivity we take into account a unitary capacity consumption for each arc from a ring node in S to a customer in $V[H] \setminus S$. This does not hold for an arbitrary node in S since the connected customer outside of S is not necessarily part of a ring that intersects with S. However, we can tighten the capacitated ring closure inequalities (27) by a similar counting argument. We utilize the ring flow information to count the number of customers outside of S that are connected from ring nodes in S as follows.

$$\sum_{a \in \delta^+(S)} f_a \;\geqslant\; \frac{1}{q}\left(|U_2(S)| + \sum_{i \in U_1(S)} z_i + \sum_{a \in \delta^+_{U \setminus S}(S)} f_a\right) \quad \forall\, S \subset V[H] \setminus d \tag{28}$$

To derive an even tighter version of (28) we first rewrite the introduced outbound customer ring flow term as

$$\sum_{a \in \delta^+_{U_1 \setminus S}(S \setminus U_2)} f_a + \sum_{a \in \delta^+_{U \setminus S}(U_2(S))} f_a + \sum_{a \in \delta^+_{U_2 \setminus S}(S \setminus U_2)} f_a. \tag{29}$$

We observe that values of the ring flow variables in the last summation term will be equal to the corresponding arc variable values by (14). The second term counts the customers in $U \setminus S$ that are connected from type 2 customers in S by ring arcs. In fact every customer that is connected from a type 2 node consumes capacity of a ring tree that requires a fundamental cycle. Thus we can exchange the summation flow variables f to arc variables x for this term as well. Unfortunately, this argument can just be applied conditionally to the first sum. More precisely, we cannot count an outbound arc (i,j) since we do not know whether the originating node $i \in S$ is connected to a ring. This lifting procedure on the right hand side of (28) yields the following right hand side.

$$\frac{1}{q}\Bigg(|U_2(S)| + \sum_{a\in\delta^+_{U_1\setminus S}(S\setminus U_2)} f_a + \sum_{a\in\delta^+_{U\setminus S}(U_2(S))} x_a + \sum_{a\in\delta^+_{U_2\setminus S}(S\setminus U_2)} x_a\Bigg) \tag{30}$$

The separation procedure for inequalities (28) can be deduced from (23) and (27). An arc $a \in A[H^+_{1,2}]$ has weight $(1-1/q)f^*_a$ if $a \in (U \cup W) \times U$ and f^*_a otherwise. We extend this by include an approximating component to reflect (30). Thereby, the weight of an arc $a \in U_2 \times U$ of $H^+_{1,2}$ is set to $\max\{0, f^*_a - x^*_a/q\}$ based on the suggested variable exchange. So far we tried to enforce connectivity from ring nodes to the depot. Conversely, we are able to formulate capacitated inequalities that ensure sufficient inbound ring flow for a cut set as follows.

$$\sum_{a\in\delta^-(S)} f_a \;\geqslant\; \frac{1}{q}\Bigg(|U_2(S)| + \sum_{i\in U_1(S)} z_i + \sum_{a\in\delta^-_{U\setminus S}(S\setminus U_2)} f_a + \sum_{a\in\delta^-_{U\setminus S}(U_2(S))} x_a\Bigg) \quad \forall\ S \subset V[H]\setminus d \tag{31}$$

The separation procedure can be adapted from (28) including (30) on $H^-_{1,2}$ which we will not elaborate here. The obtained types of ring closure inequalities all together ensure a certain outbound connectivity of S whereas the various ring tree inequalities target sufficient inbound connectivity in a similar way. However, we remind that due to the ring tree structure we will not be able to match the number of arcs entering such a cut set S with the order of the leaving arcs in general.

5.2 Strengthening the upper bounds

In a branch & bound algorithm it is crucial to generate tight upper bounds that are used for pruning. During the initial branching process integer feasible CRTP solutions are found scarcely and are at best of moderate quality. Therefore, we compute an integer-feasible start solution in our algorithm using a multi-start local search heuristic as described in [14]. Based on several construction strategies various single and multi ring tree exchange neighborhoods are explored to identify potential improvements. Ties can be broken by reusing these techniques for a CRTP specific solution polishing to optimize solutions found during the exact method. Consequently, each time an integer feasible solution is found we perform local search and if this results in an improved solution we replace the incumbent.

5.3 Cut management

In our algorithm we add (14) and (15), separate (16), (17), (20) and (28). Inequalities incorporating (30) are separated heuristically as explained in the previous

subsection whereas (21), (19), (26) and (22) are added if violated for any of the obtained cut sets. In addition to these inequalities we include constraints in our initial model which do not improve the theoretical lower bounds computed by solving the LP but in practice speed up the overall solution process.
Since customers of type 2 are required to be ring nodes and Steiner leave nodes cannot improve a solution, we add (32). These inequalities are implied by (23) and (3).

$$\sum_{a \in \delta^+(i)} x_a \geqslant \sum_{a \in \delta^-(i)} x_a \quad \forall\ i \in U_2 \cup W. \tag{32}$$

We additionally add inequality (28) for $S = U_2$ to the initial model. Furthermore, we add inequalities (19) for $S = V[G] \setminus \{d\}$ and $S = \{v\}\ \forall\ v \in V(G) \setminus d$. Besides our own CRTP-specific cutting techniques we activated the solver's internal cutting routines that implement common cuts. Various experiments with the different branching strategies using different prioritizations of arc and edge variables have shown that the pseudo cost branching is most effective for our instances.
We let the CPLEX-internal cut management decide whether to purge added cuts if convenient. However, integrality enforcing cuts of type 19 and capacitated ring closure cuts 28 are forced to stay in the model.

6 Computational study

In our computational study we follow two objectives. On the one hand we give results of our exact branch & cut algorithm and compare them to the results of our heuristic solution approach from [14]. On the other hand we consider various reliability scenarios and draw some conclusions about the cost of reliability in terms of overall costs and computational effort.

6.1 Implementation details

The algorithms was implemented in C++ using the CPLEX 12.6 branch & cut framework. Computations were done on an Intel i7-3667U 2.00 GHz processor unit. CPLEX was set to run in the single thread mode. We searched for an optimal LP-feasible solution at the root node and generated inequalities for all violated cuts. In our experiments it turned out that our algorithm performed better when additionally utilizing the solver cutting techniques. Among the various branching strategies suggested in the literature, we decided to use a strategy based pseudo costs which is implemented in the solver.

6.2 Scenarios

Our 675 CRTP instances[1] are derived from the 45 class A random instances generated for the CRSP in [3]. These TSPLib-based instances with $12 \leqslant |U| \leqslant$

[1] The instances can be obtained from the corresponding author.

100, $3 \leqslant q \leqslant 38$ and $m \in \{3,4,5\}$ also served for computational studies in [15] and [25]. During our adaptation process we assigned customers to be of type 1 using the following strategies. We prioritized according to their closeness to d (DC), remoteness to d (DF), closeness to a random customer (RC), remoteness to a random customer (RF) or performed a random assignment (R). For each class of obtained instances we use five different type 1 customer rates: $r_1 \in \{0, 0.25, 0.5, 0.75, 1\}$. Note that $U_1 = \emptyset$ and $U_1 = U$ result in a VRP variant and a CSTP, respectively. The used random seed depends on the CRSP instance and is constant for its derived CRTP instances. For two instances I and I' that are constructed based on the same strategy with $r_1 < r'_1$ we have $U'_1 \subset U_1$. Hence the optimal values z and z' obey $z \geqslant z'$. The scenarios are illustrated in Figure 9 for the base instance A30.

6.3 Results

Tables 1, 2 and 3 shows the computational results for selected instances. The first 8 columns indicate the base CRSP instance, the type 1 customer rate r_1, graph details and the capacity bounds. Lower and upper bounds obtained by our exact method using a one hour time limit can be found in columns lb and ub, respectively. The root node relaxation objective is given in lb_0 and the primal bound resulting from the heuristic from [14] can be found in ub_0. The corresponding computation time (in seconds) and the number of explored nodes in the branch & bound tree can be found in columns $t(s)$ and *nodes*, whereas the lower bound computed in the root node can be found in column lb_0. The initial upper bound computed by our heuristic is given in column ub_0. The run time of the heuristic procedure never exceeded 25 seconds during our tests.

As expected, the pure tree or ring structured problems can usually be solved more efficiently in terms of optimality gap and number of explored nodes. We observed the instances with balanced customer reliability requirements as the most challenging. Even though we could solve 64% of the purely ring based instances and all the purely tree based instances to optimality, we proved optimality for just 31% of the problems with $r_1 = 0.5$. Our heuristic algorithm from [14] produced solutions that were optimal for 68% of the instances. Additionally, the local search techniques polished integer-feasible solutions during the branch & cut procedure in many cases for the remaining instances. For the entire test set we obtained an average optimality gap of 2.6%.

6.4 The cost of reliability

We are particularly interested in the effect of increased reliability requirements on the overall costs. Certainly, different cost functions as well as parameters such as the capacity limits m, q and the reliability distribution have a strong impact on solutions for CRTP instances. Nevertheless, we give some consequences of reliability parametrization in our different scenarios based on our solution approaches. As the ring tree structure suggests, the CRTP solutions can be quite different, from pure tree or ring based ones.

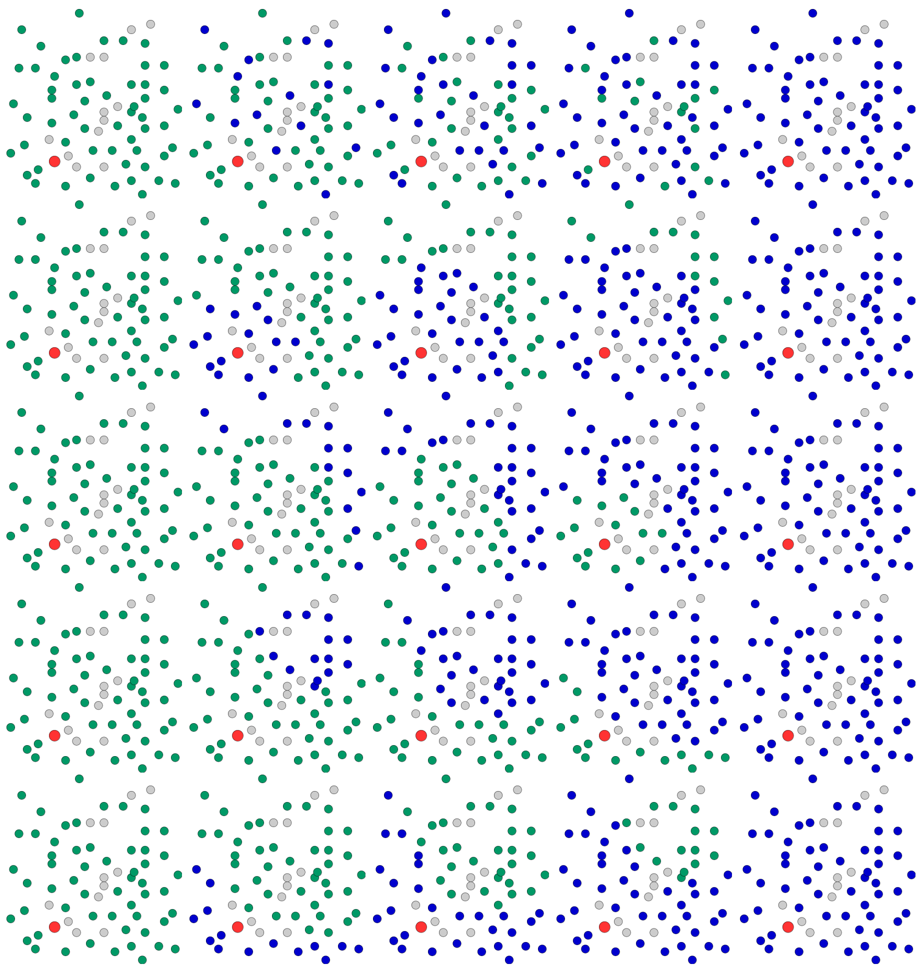

Fig. 9: CRTP random instances (left to right: 0, 0.25, 0.5, 0.75, 1 - type 1 customer rate; top to bottom: R, DC, DF, RC, RF - type 1 customer assignment strategy).

Table 1: Results for CRTP instances for reliability expansion scenario (R) and type 1 customer rates $r_1 \in \{0, 0.25, 0.5, 0.75, 1\}$.

Instance	r_1	$\|V\|$	$\|U_2\|$	$\|U_1\|$	$\|W\|$	m	q	lb_0	lb	ub	ub_0	Δ	$t(s)$	$nodes$
Q-1	1	26	0	12	13	3	5	157	157	**157**	**157**	0	2	0
	0.75		3	9				207	210	**210**	215	0	4	4
	0.5		6	6				221	227	**227**	**227**	0	8	45
	0.25		9	3				236	236	**236**	**236**	0	3	5
	0		12	0				241	242	**242**	**242**	0	1	0
Q-2	1	26	0	12	13	4	4	163	163	**163**	164	0	2	0
	0.75		3	9				207	207	**207**	**207**	0	2	0
	0.5		6	6				233	240	**240**	**240**	0	9	118
	0.25		9	3				247	249	**249**	**249**	0	3	2
	0		12	0				251	251	**251**	**251**	0	1	0
Q-3	1	26	0	12	13	5	3	170	170	**170**	173	0	1	0
	0.75		3	9				235	242	**242**	244	0	11	81
	0.5		6	6				245	251	**251**	**251**	0	4	2
	0.25		9	3				278	279	**279**	**279**	0	3	0
	0		12	0				279	279	**279**	**279**	0	1	0
Q-4	1	32	0	18	13	3	7	207	207	**207**	**207**	0	1	0
	0.75		4	14				249	256	**256**	**256**	0	10	97
	0.5		9	9				267	274	**274**	**274**	0	6	27
	0.25		13	5				284	292	**292**	**292**	0	12	161
	0		18	0				292	301	**301**	305	0	5	20
Q-5	1	26	0	18	7	4	5	217	217	**217**	220	0	1	0
	0.75		4	14				277	285	**285**	**285**	0	17	116
	0.5		9	9				304	313	**313**	318	0	27	128
	0.25		13	5				317	334	**334**	**334**	0	102	889
	0		18	0				334	339	**339**	**339**	0	5	26
Q-6	1	26	0	18	7	5	4	227	227	**227**	231	0	1	0
	0.75		4	14				276	278	**278**	**278**	0	5	22
	0.5		9	9				320	336	**336**	**336**	0	67	433
	0.25		13	5				353	361	**361**	**361**	0	13	73
	0		18	0				374	375	**375**	**375**	0	2	2
Q-7	1	26	0	25	0	3	10	245	245	**245**	248	0	0	0
	0.75		6	19				283	294	**294**	**294**	0	11	135
	0.5		13	12				296	313	**313**	**313**	0	55	1414
	0.25		18	7				312	327	**327**	**327**	0	20	501
	0		25	0				326	328	**328**	**328**	0	1	4
Q-8	1	26	0	25	0	4	7	252	252	**252**	267	0	0	0
	0.75		6	19				300	311	**311**	315	0	12	132
	0.5		13	12				319	345	**345**	**345**	0	769	5530
	0.25		18	7				342	357	**357**	**357**	0	60	754
	0		25	0				358	362	**362**	**362**	0	1	8
Q-9	1	26	0	25	0	5	6	254	254	**254**	262	0	0	0
	0.75		6	19				307	319	**319**	322	0	17	150
	0.5		13	12				352	369	**369**	372	0	326	2882
	0.25		18	7				369	378	**378**	379	0	20	296
	0		25	0				394	396	**396**	397	0	2	11
Q-10	1	51	0	12	38	3	5	156	156	**156**	**156**	0	16	0
	0.75		3	9				181	192	**192**	196	0	234	32
	0.5		6	6				203	215	**215**	**215**	0	340	365
	0.25		9	3				220	222	**222**	**222**	0	9	2
	0		12	0				238	242	**242**	**242**	0	7	0
Q-11	1	51	0	12	38	4	4	159	159	**159**	163	0	16	0
	0.75		3	9				199	209	**209**	**209**	0	89	54
	0.5		6	6				226	230	**230**	**230**	0	54	34
	0.25		9	3				238	238	**238**	**238**	0	7	0
	0		12	0				250	251	**251**	**251**	0	10	0
Q-12	1	51	0	12	38	5	3	170	170	**170**	172	0	15	1
	0.75		3	9				203	203	**203**	**203**	0	20	0
	0.5		6	6				240	251	**251**	**251**	0	508	116
	0.25		9	3				271	278	**278**	**278**	0	77	46
	0		12	0				279	279	**279**	**279**	0	11	0
Q-13	1	51	0	25	25	3	10	244	245	**245**	248	0	28	2
	0.75		6	19				279	293	302	305	3.1	3600	2686
	0.5		12	13				295	312	**312**	**312**	0	2760	2482
	0.25		18	7				310	322	**322**	**322**	0	858	796
	0		25	0				323	328	**328**	**328**	0	32	30
Q-14	1	51	0	25	25	4	7	250	252	**252**	267	0	17	3
	0.75		6	19				296	304	**304**	321	0	583	301
	0.5		12	13				327	341	352	352	3.1	3600	2050
	0.25		18	7				344	357	**357**	**357**	0	1795	1145
	0		25	0				355	362	**362**	**362**	0	55	29
Q-15	1	51	0	25	25	5	6	254	254	**254**	262	0	14	2
	0.75		6	19				320	331	335	339	1.1	3600	3035
	0.5		12	13				348	359	370	372	3	3600	1440
	0.25		18	7				360	372	387	387	3.9	3600	1012
	0		25	0				344	390	**390**	397	0	13	9

Table 2: Results for CRTP instances for reliability expansion scenario (R) and type 1 customer rates $r_1 \in \{0, 0.25, 0.5, 0.75, 1\}$.

Instance	r_1	$\|V\|$	$\|U_2\|$	$\|U_1\|$	$\|W\|$	m	q	lb_0	lb	ub	ub_0	Δ	$t(s)$	$nodes$
Q-16	1	51	0	37	13	3	14	303	304	**304**	**304**	0	7	0
	0.75		9	28				346	350	375	375	6.6	3600	3745
	0.5		18	19				356	364	376	378	3.2	3600	2878
	0.25		27	10				366	379	**379**	380	0	1428	4254
	0		37	0				376	380	**380**	381	0	32	21
Q-17	1	51	0	37	13	4	11	308	308	**308**	309	0	13	2
	0.75		9	28				351	363	**363**	369	0	3472	2867
	0.5		18	19				376	384	399	399	3.8	3600	2361
	0.25		27	10				384	396	404	404	1.9	3600	3852
	0		37	0				404	410	**410**	418	0	358	200
Q-18	1	51	0	37	13	5	9	311	314	**314**	**314**	0	11	24
	0.75		9	28				372	374	408	408	8.2	3600	1239
	0.5		18	19				397	401	431	431	7	3600	1700
	0.25		27	10				411	417	436	436	4.5	3600	1880
	0		37	0				435	446	**446**	452	0	359	1316
Q-19	1	51	0	50	0	3	19	376	376	**376**	377	0	1	0
	0.75		12	38				407	418	427	436	2.1	3600	5032
	0.5		25	25				425	435	445	447	2.3	3600	6217
	0.25		37	13				453	451	**451**	454	0	1953	2396
	0		50	0				462	462	**462**	473	0	1311	1068
Q-20	1	51	0	50	0	4	14	384	384	**384**	386	0	4	56
	0.75		12	38				418	423	458	458	7.7	3600	2236
	0.5		25	25				444	448	493	493	9.1	3600	2700
	0.25		37	13				464	471	502	502	6.2	3600	4800
	0		50	0				480	493	**493**	513	0	799	2042
Q-21	1	51	0	50	0	5	12	391	390	**390**	392	0	6	80
	0.75		12	38				488	447	491	501	8.9	3600	1474
	0.5		25	25				514	478	526	526	9.1	3600	2132
	0.25		37	13				489	497	525	525	5.3	3600	3233
	0		50	0				506	522	526	541	0.8	3600	5792
Q-22	1	76	0	18	57	3	7	213	214	**214**	**214**	0	77	2
	0.75		4	14				271	272	**272**	**272**	0	624	28
	0.5		9	9				282	288	318	318	9.6	3600	268
	0.25		13	5				294	303	318	318	4.8	3600	414
	0		18	0				320	332	**332**	**332**	0	1020	953
Q-23	1	76	0	18	57	4	5	229	233	**233**	235	0	97	139
	0.75		4	14				300	302	309	312	2.1	3600	396
	0.5		9	9				333	336	**336**	**336**	0	1869	144
	0.25		13	5				352	359	369	369	2.8	3600	0
	0		18	0				383	386	**386**	390	0	1710	339
Q-24	1	76	0	18	57	5	4	249	259	**259**	**259**	0	223	369
	0.75		4	14				314	325	**325**	**325**	0	1829	161
	0.5		9	9				357	368	379	379	2.9	3600	330
	0.25		13	5				395	397	**397**	**397**	0	345	3
	0		18	0				450	448	**448**	451	0	2663	370
Q-25	1	76	0	37	38	3	14	252	320	**320**	**320**	0	856	9
	0.75		9	28				360	363	390	390	6.8	3600	502
	0.5		18	19				369	372	402	402	7.4	3600	622
	0.25		27	10				384	390	403	403	3.3	3600	607
	0		37	0				410	409	**409**	413	0	2586	1556
Q-26	1	76	0	37	38	4	11	326	326	**326**	336	0	231	123
	0.75		9	28				378	382	402	402	5	3600	372
	0.5		18	19				408	410	455	455	9.8	3600	312
	0.25		27	10				415	418	460	460	9.2	3600	361
	0		37	0				434	446	458	458	2.6	3600	1176
Q-27	1	76	0	37	38	5	9	333	340	**340**	343	0	539	1379
	0.75		9	28				405	407	446	446	8.7	3600	240
	0.5		18	19				422	426	473	473	9.9	3600	149
	0.25		27	10				443	443	497	497	10.9	3600	223
	0		37	0				472	477	506	506	5.6	3600	1110
Q-28	1	76	0	56	19	3	21	382	383	**383**	395	0	21	7
	0.75		14	42				426	427	462	462	7.6	3600	869
	0.5		28	28				436	438	477	477	8.1	3600	659
	0.25		42	14				451	461	465	472	1	3600	4168
	0		56	0				467	476	**476**	495	0	3600	2353
Q-29	1	76	0	56	19	5	13	388	389	**389**	402	0	28	14
	0.75		14	42				437	441	488	488	9.7	3600	396
	0.5		28	28				462	466	520	520	10.4	3600	316
	0.25		42	14				487	492	532	532	7.4	3600	610
	0		56	0				500	514	535	543	4	3600	1725
Q-30	1	76	0	56	19	5	13	396	399	**399**	414	0	38	148
	0.75		14	42				468	469	533	533	11.9	3600	253
	0.5		28	28				492	493	554	554	11	3600	234
	0.25		42	14				509	512	558	558	8.2	3600	545
	0		56	0				534	546	557	561	1.9	3600	1411

Table 3: Results for CRTP instances for reliability expansion scenario (R) and type 1 customer rates $r_1 \in \{0, 0.25, 0.5, 0.75, 1\}$.

Instance	r_1	$\|V\|$	$\|U_2\|$	$\|U_1\|$	$\|W\|$	m	q	lb_0	lb	ub	ub_0	Δ	$t(s)$	$nodes$
Q-31	1	76	0	75	0	3	28	473	473	**473**	478	0	2	0
	0.75		18	57				515	516	551	551	6.4	3600	1676
	0.5		37	38				532	537	564	564	4.9	3600	1399
	0.25		56	19				547	554	564	573	1.8	3600	2800
	0		75	0				567	572	**572**	584	0	230	463
Q-32	1	76	0	75	0	4	21	478	482	**482**	494	0	8	35
	0.75		18	57				530	531	573	573	7.4	3600	539
	0.5		37	38				550	552	612	612	9.8	3600	954
	0.25		56	19				581	586	618	618	5.2	3600	1640
	0		75	0				597	603	626	626	3.7	3600	3890
Q-33	1	76	0	75	0	5	17	482	488	**488**	495	0	88	456
	0.75		18	57				546	552	623	623	11.3	3600	178
	0.5		37	38				576	585	623	623	6.1	3600	343
	0.25		56	19				598	608	656	656	7.4	3600	522
	0		75	0				623	641	674	674	4.9	3600	2358
Q-34	1	101	0	25	75	3	10	274	277	**277**	282	0	450	20
	0.75		6	19				308	314	**314**	327	0	1760	114
	0.5		12	13				332	337	353	353	4.6	3600	323
	0.25		18	7				351	356	363	363	2	3600	180
	0		25	0				365	366	**366**	**366**	0	121	0
Q-35	1	101	0	25	75	4	7	288	289	**289**	293	0	333	24
	0.75		19	6				344	344	367	367	6.2	3600	34
	0.5		12	13				367	367	405	405	9.3	3600	60
	0.25		18	7				385	385	416	416	7.5	3600	27
	0		25	0				407	409	425	425	3.8	3600	362
Q-36	1	101	0	25	75	5	6	295	299	**299**	**299**	0	330	47
	0.75		19	6				362	361	393	393	8.1	3600	10
	0.5		12	13				377	378	403	403	6.2	3600	15
	0.25		18	7				406	407	429	429	5.1	3600	17
	0		25	0				435	440	452	452	2.7	3600	48
Q-37	1	101	0	50	50	3	19	409	411	**411**	**411**	0	410	10
	0.75		12	38				457	457	492	492	7.1	3600	9
	0.5		25	25				472	473	499	499	5.3	3600	70
	0.25		37	13				482	483	503	503	3.9	3600	45
	0		50	0				492	493	508	523	2.9	3600	645
Q-38	1	101	0	50	50	4	14	415	415	**415**	420	0	380	0
	0.75		12	38				460	460	480	480	4.1	3600	117
	0.5		25	25				484	484	517	517	6.5	3600	43
	0.25		37	13				501	501	531	531	5.7	3600	76
	0		50	0				521	525	537	537	2.3	3600	223
Q-39	1	101	0	50	50	5	12	422	426	**426**	443	0	790	67
	0.75		12	38				479	481	505	505	4.8	3600	128
	0.5		25	25				569	495	527	527	6.1	3600	65
	0.25		37	13				523	523	564	564	7.3	3600	49
	0		50	0				551	553	574	574	3.6	3600	126
Q-40	1	101	0	75	25	3	28	498	511	**511**	516	0	840	168
	0.75		18	57				554	555	594	594	6.6	3600	223
	0.5		37	38				569	570	592	592	3.8	3600	159
	0.25		56	19				586	588	612	612	4	3600	220
	0		75	0				600	606	**606**	622	0	2098	916
Q-41	1	101	0	75	25	4	21	500	516	**516**	519	0	780	112
	0.75		18	57				560	558	595	595	6.2	3600	40
	0.5		37	38				583	582	607	607	4.2	3600	141
	0.25		56	19				600	603	619	619	2.6	3600	177
	0		75	0				623	624	639	642	2.3	3600	532
Q-42	1	101	0	75	25	5	17	521	522	**522**	529	0	93	85
	0.75		18	57				584	584	653	653	10.6	3600	20
	0.5		37	38				597	598	645	645	7.3	3600	189
	0.25		56	19				623	622	670	670	7.1	3600	123
	0		75	0				648	649	689	689	5.8	3600	223
Q-43	1	101	0	100	0	3	38	554	555	**555**	**555**	0	1	0
	0.75		25	75				612	611	652	652	6.2	3600	260
	0.5		50	50				623	624	657	660	5	3600	532
	0.25		75	25				639	644	648	656	0.7	3600	2170
	0		100	0				660	663	**663**	683	0	292	578
Q-44	1	101	0	100	0	4	28	561	564	**564**	568	0	2	50
	0.75		25	75				624	624	663	663	5.9	3600	207
	0.5		50	50				642	644	690	690	6.7	3600	455
	0.25		75	25				661	665	683	691	2.7	3600	1523
	0		100	0				681	684	700	700	2.3	3600	993
Q-45	1	101	0	100	0	5	23	570	570	**570**	576	0	2	1
	0.75		25	75				625	629	695	695	9.5	3600	100
	0.5		50	50				670	674	717	717	6	3600	203
	0.25		75	25				687	689	730	730	5.6	3600	206
	0		100	0				708	709	743	743	4.6	3600	952

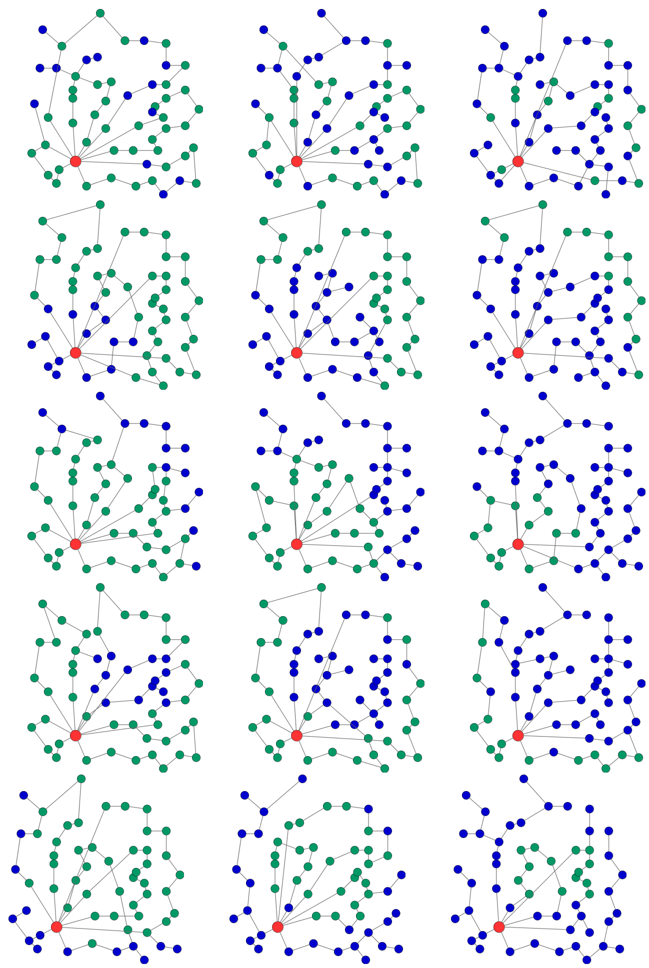

Fig. 10: Solutions for CRTP random instances (left to right: 0.25, 0.5, 0.75 - type 1 customer rate; top to bottom: R, DC, DF, RC, RF - type 1 customer assignment strategy).

The series of optimal solutions for the different type 1 customer scenarios in Figure 10 give an impression of the topological spectrum covered by the CRTP. With an increasing type 1 customer rate, we expect a smaller number of fundamental cycles as it is the case the exemplary evolution for instance Q-1 in Figure 11.

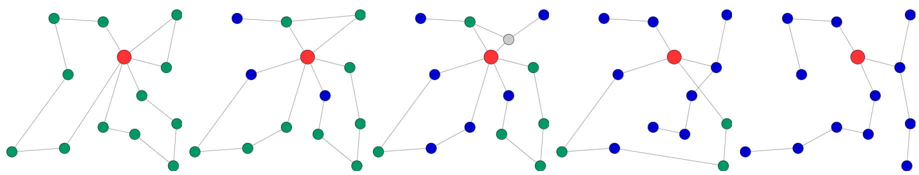

Fig. 11: Optimal solutions for the CRTP instances derived from Q01 ($m = 3$, $q = 5$) with type 1 customer rates 0.00, 0.25, 0.50, 0.75, and 1.00 using random type 1 customer assignments (R).

For the increasing type 1 customer rates, we extended the type 1 customer set incrementally in our scenarios. Therefore, we can assume that the function of optimal network costs is monotonically increasing for decreasing type 1 customers. Depending on the distribution of the reliability requirements among the customers this results in different correlations between the optimal network cost and the type 1 customer rate, as shown in Figure 12. The curves show the relative cost increase with a decreasing type 1 customer rate averaged over all the instances for our scenarios. We observe that providing additional reliability to all the customers increased the overall network costs by 35 to 65% for our instances. More precisely, installing initial reliability is costly, whereas it gets less expensive the more reliability is already implemented. This is intuitive due to the fact that the rerouting of an existing ring is more efficient than the implementation of a ring structure on a widely tree-spanned customer domain. Therefore, the reliability cost function tends to be concave. In the scenario that assigns reliability to customers closer to the distributor first (DF), we see this function to be less curved on average than when randomly turning customers into type 2 (R). In turn, providing reliability in remote areas (DC) requires to close rings towards a distant d which is more elaborate when the ring tree capacity limits are tight. We expect this effect to become even stronger when reducing the ring tree customer limit since the number of required ring trees increases.

In Figure 13 we show the average relative optimality gaps for different reliability scenarios. It can be seen that our algorithm achieves tighter results for instances of type DF compared to DC. However, type R instances are even harder to solve.

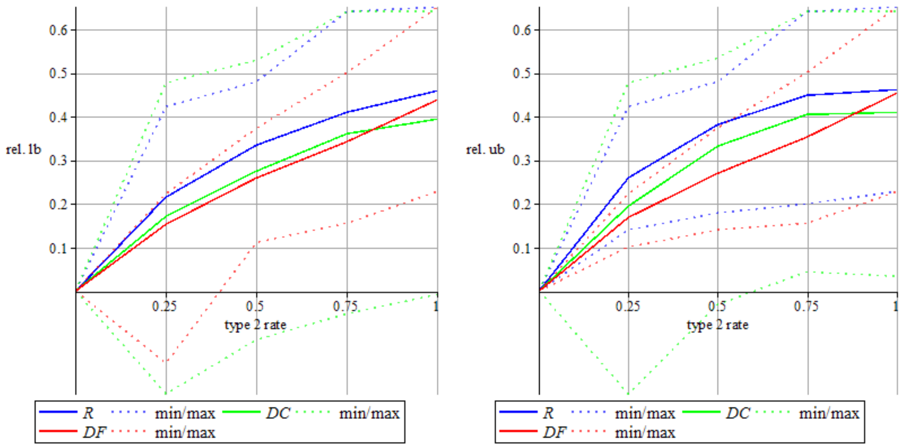

Fig. 12: The relative cost of reliability for different reliability expansion scenarios based on the best upper bounds for the network costs. Upper bounds on the left and lower bounds on the right.

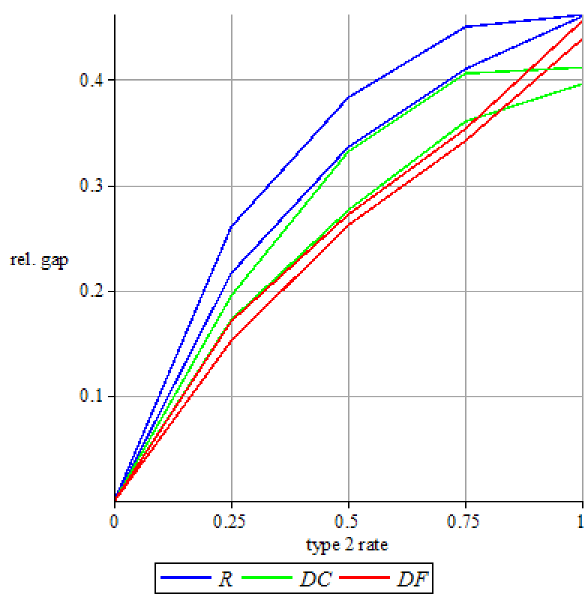

Fig. 13: The average relative optimality gaps for different reliability scenarios.

7 Conclusions

We presented a novel model for designing cost-optimized capacitated networks. This capacitated ring tree problem (CRTP) combines ring and tree structures that are common models in telecommunication applications and in logistics. Our approach generalizes existing optimization models and allows a broader use due to its capability of embracing problems that were previously modeled independently. We related the resulting ring tree topology to tree-based, ring-based, ring-star-based and survivable network design concepts previously studied in the literature. The presented mathematical formulation for the problem was used to elaborate an efficient branch & cut algorithm based on mathematical programming. Therefore, we developed cutting techniques that are tailored to the capacitated ring tree structure. We showed how to separate valid inequalities exactly and explained our heuristic addition of violated inequalities with hard separation problems. A local search based heuristic was used to produce starting solutions that support the solver's search and to polish integer-feasible solutions during the branch & bound method. For a set of small and medium sized capacity-tight literature derived instances we gave computational results for our algorithms. Using different reliability scenarios we observed that a balanced type 1 and type 2 reliability ratio yields the most difficult instances for our methods. After studying different reliability distributions we obtained an indication that instances with uniformly distributed customers with an additional reliability tend to be of increased difficulty. Nevertheless, we were able solve instances with up to 50 nodes to optimality. When considering existing scenarios that imply a purely tree or ring structure we could even solve instances up to 100 nodes.

We suggest further research on the CRTP in terms of heuristics and model extensions. It seems to be a fruitful model for the application of efficient metaheuristics or matheuristics that take advantage of the specific solution network structure. Corresponding efficient solution techniques could either be integrated in our exact methods or could be used to tackle bigger problem sizes. We are also aware that a column generation based algorithm could improve the presented results, especially in the case of an increasing number of ring trees. A model extension that could be of practical use concerns the integration of lower bounds on the number of customers served by a ring tree. Another balancing measure could be the introduction of separate lower and upper bounds q_1, q_2, q_r for type 1 customers, type 2 customers or ring customers, respectively.

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