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Algebraically Birational Extensions

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INTRODUCTION

In [5], a class of birational algebras has been introduced as a generalization of Zariski central rings. These classes of rings received some interest on one hand because all semiprime PI rings are birational algebras over their centre, and on the other hand because birationality properties determine interesting classes within the classes of fully left bounded Noetherian rings, PI rings, HNP-rings, fully idempotent rings like regular rings à la Von Neuman, V-rings, and biregular rings. Birationality is defined in purely topological terms, that is, in terms of open sets in the Zariski topologies of the prime ideal specture of the rings involved, very similar to the commutative case of birationality in algebraic geometry. In the latter situation however the birationality is defined between rings that are contained in their common ring of fractions which forms the rational function field of the algebraic varieties associated to the rings. In the non-commutative situation, the topological definition does not entail any result in the direction of a relation between the rings of fractions; in fact the latter may even not exist at all. Nevertheless, since M. Artin defined non-commutative Proj in connection with the geometry of quantum spaces in a category theoretical way there is some interest in knowing which non-commutative rings define isomorphic Proj's or, more generally, for which rings we can introduce birationality of their Proj in a suitable way. For graded PI algebras, birationality of Proj's becomes expressible in prime ideals, Zariski topologies (as in [6]), and their sheaves.

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All of this has prompted us to reconsider the theory of birational extensions and to add algebraic considerations stemming from the philosophy that the topological conditions have to be completed with the corresponding sheaf information in order to extend the geometrical situation to the non-commutative case. The algebraically birational extensions we introduce are birational extensions $A \hookrightarrow B$ such that $B = AC_B(A)$, where $C_B(A)$ is the commuting ring of A in B, and $C_B(A)$ is an Azumaya algebra or more generally a relative Azumaya algebra in the sense of [7]. This extra information is intrinsically non-commutative, it adds nothing to birationality when A, B are commutative, and as it turns out it will imply a lot about the extension $AZ(B) \hookrightarrow B$ but nothing extra about $Z(A) \hookrightarrow Z(B)$. The method of ab-extensions combined with properties of the extension $Z(A) \hookrightarrow Z(B)$, for example, properties like the "arithmetical situation" appearing in connection with the study of maximal orders over Noetherian integrally closed domains (cf. [6, 7]), provides a useful tool for obtaining non-commutative information from central information.

In Section 2 we introduce the "global case," that is, the case where $C_B(A)$ is an Azumaya algebra, and study the behavior under localization. In Section 3 we focus on the "Zariski local case," that is, the case where $C_B(A)$ is a relative Azumaya algebra and in fact the case fitting in the best way the generality of birational extensions.

The aim of this paper is to introduce the basic ideas and tools essential in describing the ring theoretical structure, or one could say: the ideal theory of birational extensions; the actual description of the effect of the existence of an ab-extension within certain concrete classes of non-commutative rings is the topic of forthcoming work.

1. Preliminaries

All rings are associative with unit. A ring homomorphism $f\colon A \to B$ is an *extension* in the sense of C. Procesi [4] when $B = f(A)C_B(f(A))$. In this paper we shall only consider inclusions $A \hookrightarrow B$, so the condition reduces to $B = AC_B(A)$. For any ring R the set of prime ideals $X = \operatorname{Spec}(R)$ may be equipped with the Zariski topology given by the open sets $X(I) = \{P \in \operatorname{Spec}(R),\ I \not\subset P\}$ associated to (two-sided) ideals I of R. An extension $f\colon A \to B$ is a *birational extension* if there exist non-empty open sets $U \subset Y = \operatorname{Spec}(B)$ and $V \subset X = \operatorname{Spec}(A)$ such that a $P \in Y$ and $f^{-1}(P) \in V$ then $P \in U$ and the correspondence $P \mapsto f^{-1}(P)$ restricts to a topological isomorphism $U \cong V$. The fact that f is an extension is important for $f^{-1}(P) \in X$. It is easily verified that there exists a unique maximal pair of open sets satisfying the foregoing birationality condition and we may assume from now on that U and V are maximal with respect to that property. There are uniquely determined semiprime ideals $I \subset B$, $I' \subset A$ such that $Y(I) = U$, $X(I') = V$ and we say that (I, I') *determines* the birationality $U \cong V$ for $A \to B$.

1.1. Lemma. *Let f: $A \to B$ be a birational extension determined by (I, I'); then $I = \mathrm{rad}(BI')$. To an ideal J of B there corresponds an ideal J' of A such that $Y(JI) \cong X(J'I')$ and $\mathrm{rad}(JI) = \mathrm{rad}(J'I)$.*

1.2. Proposition. *For a birational extension f: $A \to B$ determined by (I, I') the following statements are equivalent:*

a. *To any ideal J of B there corresponds an ideal J' of A such that $J' \subset J$ and $X(I'J') \cong Y(IJ)$ under f^{-1}.*

b. *For each ideal J of B we have $Y(IJ) \simeq X(I'(J \cap A))$ under f^{-1}.*

Note that the statement in (b) is always true if J is semiprime.

1.3. *Observation.* A birational extension f: $A \to B$ automatically satisfies the properties of Proposition 1.2 in each of the following cases:

i. A is commutative.

ii. B is Noetherian. (cf. [5, Sect. 1]).

A birational extension f: $A \to B$ is *globally birational* if $U = \mathrm{Spec}(B)$; a globally birational extension is determined by (B, I') with $BI' = B$.

A birational extension f: $A \to B$ having one of the equivalent properties of Proposition 1.2 is called a Zariski extension.

A globally birational extension $A \hookrightarrow B$ is a globally Zariski extension if and only if for all ideals J of B, $\mathrm{rad}(J) = \mathrm{rad}(B(J \cap A))$.

1.4. Proposition. *Let $A \subset B$ be a Zariski extension.*

a. *If B is semiprime then every nonzero ideal J of B contained in I has a non-trivial intersection with A.*

b. *If B is semiprime and U is dense in Y then every nonzero ideal of B has a non-trivial intersection with A.*

c. *If B is prime then* b *applies.*

d. *If B is semiprime and A is simple then B is simple.*

e. *If B is a Zariski extension of A and A is a Zariski extension (U', V') of C; assuming that U' is dense in X, then B is a Zariski extension of C.*

It is clear that the ideal theory connected to birational extensions is expressed in terms of radical ideals, i.e., ideals equal to their prime radical. This property is essential in the study of relations between localizations of A and B, respectively. For generalities on localization, kernel functors (or torsion theories) we refer to [2, 6]. A kernel functor κ on R-mod is said to be *radical* if $\mathscr{L}(\kappa)$ has a cofinal subset consisting of ideals and moreover $J \in \mathscr{L}(\kappa)$, for an ideal J of R, if and only if $\mathrm{rad}(J) \in \mathscr{L}(\kappa)$.

If R is left Noetherian, the latter condition follows from the first; hence in this case a radical kernel functor is just a symmetric one. In the absence of the Noetherian condition it is still true that every symmetric κ that is perfect (having property T or equivalently such that the corresponding localization functor Q_κ is exact on R-mod) is also radical. Note that here, we will always deal with idempotent kernel functors whereas in [5] kernel functors were not assumed to be idempotent from the start.

1.5. PROPOSITION. *Let $A \subset B$ be a Zariski extension and let κ be a radical kernel functor on B-mod such that $I \in \mathscr{L}(\kappa)$. Then the set $\{H'$ ideal of A such that $Y(H \cap I) \cong X(H')$ under restriction for some ideal $H \in \mathscr{L}(\kappa)\}$ is a filter basis for $\mathscr{L}(\kappa')$ determining a radical kernel functor κ' on A-mod. If κ' is idempotent then so is κ and moreover κ is induced by κ' under restriction of scalars, $\kappa = \bar{f}(\kappa')$, that is, a B-module M is κ-torsion if and only if ${}_AM$ is κ'-torsion.*

To an ideal J of a ring R we associate a radical kernel functor κ_J by putting $\mathscr{L}(\kappa_J) = \{L$ left ideal of R, L contains an ideal H or R such that $\mathrm{rad}(H) \supset J\}$. To a prime ideal P of R we associate κ_{R-P} given by $\mathscr{L}(\kappa_{R-P}) = \{L$ left ideal of R, L contains an ideal H of R such that $H \not\subset P\}$.

1.6. THEOREM. *Let $A \subset B$ be a Zariski extension. Let H be an ideal of B such that $I \in \mathscr{L}(H)$, that is, $H \subset I$ since I is semiprime; then κ_H is a radical kernel functor such that $(\kappa_H)' = \kappa_{H'}$, where H' is any ideal of A such that $X(H') \cong Y(H)$ under restriction. If P is a prime ideal of B such that $I \not\subset P$ then κ_{B-P} is a radical kernel functor such that $(\kappa_{B-P})' = \kappa_{A-p}$, where $p = P \cap A \in X$.*

In order to relate the localization functors Q_κ and $Q_{\kappa'}$ acting on B-mod, we could evoke Theorem 2.7 of [5]. However, we now have a much more general result of that type, which we include here. The rings that one usually hopes to apply birationality arguments to are never too far away from the class of PI rings, and in any case one wants them to have a classical ring of fractions (usually semisimple Artinian) that is at least a two-sided ring of fractions. So we let γ be a kernel functor such that it is true that Q^l_γ, and Q^r_γ coincide, e.g., the kernel functor corresponding to a classical ring of fractions, and consider kernel functors $\sigma < \gamma$.

1.7. PROPOSITION. *Let $A \subset B$ be any extension and let κ be a symmetric kernel functor on B-mod such that $\kappa < \gamma$ and $B \subset Q_\gamma(B)$; then $Q_\kappa(B) \cong Q_{\kappa'}(B)$, i.e., $Q_{\kappa'}(B)$ is a B-module and the isomorphism is one of B-modules.*

Proof. Since ideals of A extend to ideals of B we have $\kappa'(B) = \kappa(B)$ and so we may assume that B is κ-torsionfree.

Since $Q_{\kappa'}(B) = \varinjlim_{J' \in \mathscr{L}(\kappa')} \mathrm{Hom}_A(J', B)$ we may represent a $q \in Q_{\kappa'}(B)$ by an A-linear f_q: $J' \to B$: $j \mapsto jq$, where $J' \in \mathscr{L}(\kappa')$ is such that $J'q \subset B$. It is clear that $Q_{\kappa'}(B)$ is a right B-module in the obvious way. But we may also define the structure of a left B-module as $(b \cdot q)$: $J' \to B$, $j \mapsto (jb)q$, where $jb \in BJ'$ yields $jb = \Sigma_\alpha b_\alpha j_\alpha$ with $b_\alpha \in B$, $j_\alpha \in J'$, and $(jb)q = \Sigma b_\alpha (j_\alpha q) \in B$. It remains to check that $(b \cdot q)$ is well-defined; indeed $jb = \Sigma b_\alpha j_\alpha$ may arrive in several ways and in each case $\Sigma b_\alpha (j_\alpha q)$ should define the same element of B. In other words, if $\Sigma b_\alpha j_\alpha = 0$ then is $\Sigma b_\alpha (j_\alpha q) = 0$ in B?

Now $q \in Q_{\kappa'}(B) \subset Q_\gamma(B)$ yields that $qH_\gamma \subset B$ for some $H_\gamma \in \mathscr{L}(\gamma)$ and then we calculate

$$0 = \left(\sum b_\alpha j_\alpha\right)(qH_\gamma) = \sum b_\alpha (j_\alpha q H_\gamma) = \sum b_\alpha (j_\alpha q) H_\gamma.$$

This yields that $\Sigma b_\alpha (j_\alpha q)$ is (right) γ-torsion but since $B \subset Q_\gamma(B)$ it then follows that $\Sigma b_\alpha (j_\alpha q) = 0$, as desired.

Now $Q_{\kappa'}(B)/B$ is a B-module and κ'-torsion, hence κ-torsion as a B-module, consequently $Q_{\kappa'}(B) \subset Q(B)$. On the other hand $Q_\kappa(B)/B$ is κ-torsion, hence κ'-torsion as an A-module, so $Q_\kappa(B) \cong Q_{\kappa'}(B)$ follows.

1.8. COROLLARY. *Let $A \subset B$ be an extension and assume that $B \subset Q_{cl}(B)$, where the latter is a left and a right ring of quotients; then for any symmetric κ on B-mod, $Q_{\kappa'}(B) \simeq Q_\kappa(B)$.*

In particular we may apply this in the situation of Theorem 1.6 for κ_H and κ_{B-P}, e.g., $Q_{A-P}(B) \cong Q_{B-P}(B)$ for every prime ideal P of B such that $I \not\subset P$.

For more results on birational extensions we refer to [5]. In the next section we turn to the added condition that they be algebraically birational.

2. ALGEBRAICALLY BIRATIONAL EXTENSIONS

A birational (Zariski) extension $A \hookrightarrow B$ is said to be algebraically birational (Zariski) (ab-extension) whenever $B = AC_B(A)$ and $C_B(A)$ is an Azumaya algebra. By definition it is clear that $Z(A) \subset Z(B)$ and $Z(B) = Z(C_B(A))$. For detail on Azumaya algebras, cf. [1].

2.1. LEMMA. *Let $A \hookrightarrow B$ be an ab-extension; then $C_B(C_B(A)) = AZ(B)$.*

Proof. It is obvious that $AZ(B) \subset C_B(C_B(A))$. Since $C_B(A)$ is a projective $Z(B)$ module we have a commutative diagram:

$$\begin{array}{ccccc} 0 \longrightarrow & C_B(A) \otimes_{Z(B)} AZ(B) & \overset{j}{\hookrightarrow} & C_B(A) \otimes_{Z(B)} C_B(C_B(A)) \\ & \searrow & & \swarrow_{i}\ \cong \\ & & B & \end{array}$$

Clearly $ij(C_B(A) \otimes_{Z(B)} AZ(B)) = C_B(A)A = B$, hence j is an isomorphism. By the faithful flatness of $C_A(B)$ over $Z(B)$ it then follows that $C_B(C_B(A)) = AZ(B)$.

2.2. COROLLARIES. 1. *B is faithfully projective as an $AZ(B)$-module.*

2. *There is a Noetherian subring $Z(B)'$ in $Z(B)$ and over $Z(B')$ there is a Noetherian Azumaya algebra Λ in $C_B(A)$ such that $B = AZ(B)\Lambda$, $C_B(A) = Z(B)\Lambda$.*

3. *If M is any $(C_B(A), Z(B))$-bimodule over B then we have $M = C_B(A) \otimes_{Z(B)} C_{C_B(A)}(M)$, where $C_{C_B(A)}(M) = \{m \in M, \lambda m = m\lambda$ for $\lambda \in C_B(A)\}$. In particular when T is an ideal of B then $T = C_B(A)(T \cap AZ(B))$.*

4. *In view of 3, there is bijective correspondence between ideals of B and ideals of $AZ(B)$. In particular if $T \subset Z(B)$ is such that $BT = B$ then $AT = AZ(B)$.*

5. *If $AZ(B)$ is a Zariski extension, globally birational or Zariski, of A then B is resp. a Zariski extension, globally birational or Zariski, of A too. This follows directly from 3 and 4.*

6. *If p is a prime ideal of $AZ(B)$ then Bp is a prime ideal of B.*
Indeed, if I and J are ideals of B such that $IJ \subset Bp$ then $(I \cap AZ(B))(J \cap AZ(B)) \subset p$ yields $I \cap AZ(B) \subset p$, say, and thus $I = C_B(A)(I \cap AZ(B)) \subset Bp$. Therefore restriction defines a homeomorphism Spec $B \cong$ Spec $AZ(B)$.

7. *If $p \in X(I')$ consider $pZ(B) \subset P \cap AZ(B)$.*
Since $P =$ rad $Bp \in Y(I)$ it follows that for a prime ideal Q of $AZ(B)$ such that $Q \supset pZ(B)$, we have BQ (see 6) $\supset P$, hence $Q \supset P \cap Z(B)$. This yields $P \cap AZ(B) =$ rad $pZ(B)$.
Moreover if $p \in X(I')$ then this is clearly equivalent to $P \not\supset I$ and also to $P \cap AZ(B) \not\supset I \cap AZ(B)$. Consequently restriction also defines an ab-extension structure $A \hookrightarrow AZ(B)$ given by $(I', I \cap AZ(B))$, where of course the Azumaya algebra part is now trivial in the sense that it is commutative.

2.3. LEMMA. *Let $A \hookrightarrow B$ be an ab-extension. Then this extension decomposes in an ab-extension $A \hookrightarrow AZ(B)$ with $I', I \cap AZ(B))$ and an ab-exten-*

sion $AZ(B) \hookrightarrow B$ that is a global Zariski extension. Hence the study of ab-extensions reduces to central ab-extensions plus global Zariski extensions. Moreover $I \cap AZ(B) = \mathrm{rad}(I'Z(B))$ (from Lemma 1.1).

The extension $AZ(B) \hookrightarrow_g B$ is a very nice one; in fact it is clear that the restriction of scalars functor corresponding to g is a separable functor in the sense of [3]. As an immediate consequence we obtain the following.

2.4. *Properties* (cf. [3]). (i). If P is a projective, resp. injective, B-module then ${}_AP$ is projective, resp. injective.

ii. ψ: $B \otimes_{AZ(B)} B \to B$, $b \otimes b' \mapsto bb'$, splits as an $AZ(B)$-bimodule map.

iii. There is an idempotent $e \in B \otimes_{AZ(B)} B$ such that $\psi(e) = 1$ and $eb = be$ for all $b \in B$.

2.5. *Note.* If H is an ideal of A then $HZ(B) \subset BH \cap AZ(B)$. The restriction of the isomorphism $C_B(A) \otimes_{Z(B)} (BH \cap AZ(B)) \cong BH$ to $C_B(A) \otimes_{Z(B)} HZ(B)$ still has image BH because $C_B(A)H = C_B(A)AH = BH$. The faithful projectivity of $C_B(A)$ over $Z(B)$ yields $HZ(B) = BH \cap AZ(B)$.

For any ideal J of B we will write $J^c = J \cap AZ(B)$. The bijective correspondence of ideals between B and $AZ(B)$ entails that $(HJ)^c = H^cJ^c$, $(\mathrm{rad}(H))^c = \mathrm{rad}(H^c)$, etc. In particular I^c is semiprime and $I^c = \mathrm{rad}(I'Z(B))$. If κ is a symmetric kernel functor on B-mod then κ_c denotes the symmetric kernel functor on $AZ(B)$-mod obtained by taking $\mathscr{L}(\kappa^c) = \{H^c, H \in \mathscr{L}(\kappa)\}$. Clearly κ^c defines the same κ' on A-mod if $I \in \mathscr{L}(\kappa)$; indeed $\{H'$ ideal of A such that $Y(H \cap I) \cong X(H')$ under restriction, for some $H \in \mathscr{L}(\kappa)\} = \{H''$ ideal of A such that $Z(H_1 \cap I^c) \simeq X(H'')$ under restriction, for some $H_1 \in \mathscr{L}(\kappa^c)\}$, where $Z = \mathrm{Spec}(AZ(B))$ and $I^c \in \mathscr{L}(\kappa^c)$ because $I \in \mathscr{L}(\kappa)$.

2.6. Proposition. *Let $A \hookrightarrow B$ be an ab-extension and let κ be as above, i.e., $I \in \mathscr{L}(\kappa)$; then $Q_{\kappa'}(AZ(B)) \cong Q_{\kappa^c}(AZ(B))$ and also $Q_{\kappa'}(B) \cong Q_{\kappa^c}(B) \cong Q_\kappa(B)$.*

Proof. The first statement follows from Theorem 2.7(1) in [5] up to the remark $C_{AZ(B)}(A) = Z(B)$, that is, $A \hookrightarrow AZ(B)$ is a central extension.

The second statement is a variation on Theorem 2.7(2) in [5] (basis is to be understood as a set of generators). Note that this result also follows from Proposition 1.7 if we assume that B has a left and right ring of fractions.

2.7. Proposition. *Let $A \hookrightarrow B$ be a Zariski ab-extension and suppose that π: $B \to B_1$ is a surjective ring homomorphism such that $I \not\subset$* rad Ker(π)

(or $\operatorname{Ker} \pi \not\subset \mathscr{L}(\kappa_I)$*). Then* $A_1 \hookrightarrow B_1$ *is a Zariski ab-extension, where* $A_1 = \pi(A)$.

Proof. By Proposition 1.1 of [5], or an easy straightforward argument, it is clear that $A_1 \hookrightarrow B_1$ is a Zariski extension. So it remains to verify that $C_{B_1}(A_1)$ is an Azumaya algebra such that $B_1 = A_1 C_{B_1}(A_1)$.

It is clear that $B_1 = A_1 \pi(C_B(A))$ and $\pi(C_B(A))$ is an Azumaya algebra. Since $T_1 = \pi(C_B(A))Z(B_1)$ is an epimorphic image of the Azumaya algebra $\pi(C_B(A)) \otimes_{\pi(Z(B))} Z(B_1)$ it follows that T_1 is an Azumaya algebra such that $Z(T_1) = Z(B_1)$.

Now $A_1 Z(B_1) \subset C_{B_1}(T_1)$ and $A_1 Z(B_1) \otimes_{Z(B_1)} T_1 \hookrightarrow C_{B_1}(T_1) \otimes_{Z(B_1)} T_1 \simeq B_1$. From $A_1 T_1 = B_1$ and the faithful projectivity of T_1 over $Z(B_1)$ we obtain $A_1 Z(B_1) = C_{B_1}(T_1)$. On the other hand:

$C_{B_1}(A_1 Z(B_1)) = C_{B_1}(T_1))$ contains T_1 and it is a $T_1 - Z(B_1)$-bimodule, therefore $S_1 = C_{B_1}(A_1 Z(B_1)) = T_1 C_{S_1}(T_1)$.

If $x \in C_{S_1}(T_1)$ then x commutes with $A_1 Z(B_1)$ since it is in S_1 and it commutes with T_1 by definition, hence x commutes with $A_1 T_1 = B_1$ or $x \in Z(B_1) \subset T_1$. Consequently $C_{S_1}(T_1) = Z(B_1)$ and $T_1 = S_1 = C_{B_1}(A_1 Z(B_1)) = C_{B_1}(A_1)$, i.e., $B_1 = A_1 C_{B_1}(A_1)$ and $C_{B_1}(A_1)$ is an Azumaya algebra.

2.8. Corollary. *If* $A \hookrightarrow B$ *is a Zariski ab-extension then for* κ *on* B*-mod such that* $I \in \mathscr{L}(\kappa)$ *we have that* $A/\kappa'(A) \hookrightarrow B/\kappa(B)$ *is a Zariski ab-extension. So in studying properties of the localized extension* $Q_{\kappa'}(A) \hookrightarrow Q_{\kappa'}(B) = Q_\kappa(B)$ *we may assume that* A *and* B *are without* κ'*—resp.* κ*-torsion.*

In the sequel we assume that B is a left Noetherian ring; then $A \hookrightarrow B$ being an ab-extension is the same as its being a Zariski ab-extension and it simplifies terminology (but results can be extended for non-Noetherian rings if one absolutely desires to do so). By the separability of the restriction of scalars with respect to $AZ(B) \hookrightarrow B$ it follows that $AZ(B)$ is a left Noetherian ring too (but A need not be, as one can see by taking for A a prime PI ring and for $B = Q_{cl}(A) = AZ(B)$, where $Z(B)$ is a field and B is a central simple algebra).

2.9. Theorem. *Let* $A \hookrightarrow B$ *be an ab-extension such that* B *is a left Noetherian ring and let* $I \in \mathscr{L}(\kappa)$ *for some kernel functor* κ *on* B*-mod. Then* $Q_{\kappa'}(A) \hookrightarrow Q_\kappa(B)$ *need not be an extension but nevertheless* $C_{Q_\kappa(B)}(Q_{\kappa'}(A)) = T$ *is an Azumaya algebra over* $Z(Q((B))$ *and we have the equalities*

$$T = C_B(A)Z(Q_\kappa(B)) = C_{Q_\kappa(B)}(Q_{\kappa'}(AZ(B))),$$

$$C_{Q_\kappa(B)}(T) = Q_{\kappa'}(AZ(B))$$

and

$$Q_\kappa(B) = C_B(A)Q_{\kappa'}(AZ(B)) = TQ_{\kappa'}(AZ(B)).$$

Proof. First note that $x \in Q_\kappa(B)$ commutes with B if and only if $x \in Z(Q_\kappa(B))$; indeed if x is such an element then for any $y \in Q_\kappa(B)$ there is an ideal $H \in \mathscr{L}(\kappa)$ such that $Hy \subset B$ (note that we have assumed that B is κ-torsionfree, A is κ'-torsionfree since B is left Noetherian so Corollary 2.8 applies, and $0 = xhy - hyx = k(xy - yx)$ for all $h \in H$, yields $H(xy - yx) = 0$ or $xy - yx = 0$ because $Q_L(B)$ is κ-torsionfree. Now we consider $S = Q_{\kappa'}(A)Z(Q_\kappa(B))$ and $T = C_{Q_\kappa(B)}(S) \supset C_B(A)Z(Q_\kappa(B))$. The latter is an epimorphic image of the Azumaya algebra $C_B(A) \otimes_{Z(B)} Z(Q_\kappa(B))$; hence it is an Azumaya algebra over $Z(Q_\kappa(B))$. Therefore, $T = C_B(A)Z(Q_\kappa(B))C_T(C_B(A)Z(Q_\kappa(B)))$.

However, if $z \in C_T(C_B(A)Z(Q_\kappa(B)))$ then $z \in T$, hence z commutes with $Q_{\kappa'}(A)$ and z commutes with $C_B(A)$ by definition. Thus z commutes with $AC_B(A) = B$ and thus $z \in Z(Q_\kappa(B))$ by the opening remark. Consequently $T = C_B(A)Z(Q_\kappa(B))$ follows. Next consider an element $\xi \in Q_\kappa(B)$ that commutes with $C_B(A)Z(Q_\kappa(B))$; then $H'\xi \subset B$ for some $H' \in \mathscr{L}(\kappa')$ (we use the fact that $Q_{\kappa'}(B) \cong Q_\kappa(B)$). Obviously we have, for all $h' \in H'$ and for all $c \in C_B(A)$, that $ch'\xi = h'\xi c$; hence $H'\xi \subset C_B(C_B(A)) = AZ(B)$. Then $\xi \subset Q_{\kappa'}(AZ(B))$. This establishes $C_{Q_\kappa(B)}(T) \subset Q_{\kappa'}(AZ(B))$. The latter obviously commutes with $C_B(A)$, so in fact $C_{Q_\kappa(B)}(T) = Q_{\kappa'}(AZ(B))$ and thus $Q_\kappa(B) = TQ_{\kappa'}(AZ(B))$.

From $T \subset C_{Q_\kappa(B)}(Q_{\kappa'}(AZ(B))) = V$ it follows that $C_{Q_\kappa(B)}(Q_{\kappa'}(AZ(B))) = TC_V(T)$. Now note that $\eta \in C_V(T)$ if and only if η commutes with T, hence with $C_B(A)$, and η commutes with $Q_{\kappa'}(AZ(B))$, hence with A, and thus η commutes with B or $\eta \in Z(Q_\kappa(B))$. Therefore we arrive at $C_{Q_\kappa(B)}(Q_{\kappa'}(AZ(B))) = T$, finishing the proof of all claims.

2.10. THEOREM. *If in the situation of Theorem 2.9, κ' is moreover a perfect kernel functor then $C_{Q(B)}(Q_{\kappa'}(A))$ is an Azumaya algebra and $Q_\kappa(B) = Q_{\kappa'}(A)C_{Q_\kappa(B)}(Q_{\kappa'}(A)) = Q_{\kappa'}(A)C_B(A)$. In particular $Q_{\kappa'}(A) \hookrightarrow Q_\kappa(B)$ is a global Zariski ab-extension.*

Proof. As in the foregoing proof, consider a $y \in Q_\kappa(B)$ commuting with $C_B(A)$; then for some $H' \in \mathscr{L}(\kappa)$ we have $H'y \subset C_B(C_B(A)) = AZ(B)$, hence $Q_{\kappa'}(A)H'y = Q_{\kappa'}(A)y \subset Q_{\kappa'}(A)Z(B)$ and $y \in Q_{\kappa'}(A)Z(B)$.

It follows that $C_{Q_\kappa(B)}(C_B(A)Z(Q_\kappa(B))) = Q_{\kappa'}(A)Z(B) = Q_{\kappa'}(A)Z(Q_\kappa(B))$ and $Q_\kappa(B) = Q_{\kappa'}(A)C_B(A)$ follows because $Q_{\kappa'}(A)Z(B)$ contains $Z(Q_\kappa(B))$ by the foregoing equality.

2.11. *Note.* 1. Without the assumption on the perfectness of κ' we still have that $Z(Q_\kappa(B)) \subset Q_{\kappa'}(AZ(B))$ but is is not necessarily true that $Z(Q_\kappa(B))$ may be obtained as a localization of $Z(B)$. Note that the essential problem in trying to prove that $C_{Q_\kappa(B)}(T) = Q_{\kappa'}(A)Z(Q_\kappa(B))$ is that we do not know in general whether it is true that

$$Q_{\kappa'}(A)C_B(A)Z(Q_\kappa(B)) = Q_\kappa(B). \qquad (\Delta)$$

However, if κ is a central localization, or indeed any localization such that $BZ(Q_\kappa(B)) = Q_\kappa(B)$ then the equality (Δ) holds and again it will then be true that $Q_\kappa(B)$ is a global ab-extension of $Q_{\kappa'}(A)$, $Q_\kappa(B) = Q_{c'}(A) \cdot T$.

2. Note that κ' is perfect if and only if κ is perfect $(I \in \mathscr{L}(\kappa))$. This follows from $\mathrm{rad}(JI) = \mathrm{rad}(J'I)$ in Lemma 1.1, so it is easy to see that $Q_{\kappa'}(A)J' = Q_{\kappa'}(A)$ for all $J' \in \mathscr{L}(\kappa')$ is equivalent to $Q_\kappa(B)J = Q_\kappa(B)$ for all $J \in \mathscr{L}(\kappa)$.

3. Instead of $Q_\kappa(B)$ one could consider Q_κ^{bi} (see [6]). In [6] it was shown that $Q_\kappa^{bi}(B) = Z(Q_\kappa(B))B$ (in particular $B \hookrightarrow Q_\kappa^{bi}(B)$ is a central ab-extension). In this case we do obtain $Q_\kappa^{bi}(B) = Q_{\kappa'}^{bi}(A)C_B(A)Z(Q_\kappa(B)) = Q_{\kappa'}^{bi}(A)C_{Q_\kappa^{bi}(B)}(Q_{\kappa'}^{bi}(A))$, where $C_{Q_\kappa^{bi}(B)}(Q_{\kappa'}^{bi}(A))$ is an Azumaya algebra equal to $C_B(A)Z(Q_\kappa^{bi}(B))$.

4. If κ and κ' are as in the theorem then κ is geometric if and only if κ' is geometric (in the sense of [6]), this follows from 2 and the obvious property of ideals extending to ideals of the extension. This allows one to use the ab-extensions in connection with the theory of affine PI algebras and geometric localizations appearing in the sheaf theory of the non-commutative geometry of PI algebras. We call $A \hookrightarrow B$ a geometrically birational extension if it is an ab-extension such that B and A have a common classical quotient ring. Then a gb-extension is a central extension $A \hookrightarrow AZ(B)$ such that $Z(A) \to Z(B)$ is classically birational.

The rings A_1 and A_2 are said to be algebraically birational if they have a common extension B such that $A_1 \hookrightarrow B$ and $A_2 \hookrightarrow B$ are ab-extensions such that the intersection of the open sets of birationality in B is non-empty. Note that, if B is Noetherian, these ab-extensions are Zariski extensions (cf. 1.3).

2.12. Example. Witten's Gauge algebras for SU_2 are given as $\mathbb{C}\langle X, Y, Z\rangle$ modulo the relations

$$\begin{aligned} XY + \alpha YX + \beta Y &= 0 \\ YZ + \gamma ZY + \delta X^2 + \varepsilon X &= 0 \\ ZX + \xi XZ + \eta Z &= 0, \end{aligned} \qquad (*)$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \xi, \eta$ are constants.

On the other hand the quantum enveloping algebra $W_q(sl_2)$ may be defined as $\mathbb{C}\langle X, Y, Z\rangle$ modulo relations

$$\begin{aligned} \sqrt{q}\,XZ - \sqrt{q}^{-1} ZX &= \sqrt{q+q}^{-1} Z \\ \sqrt{q}^{-1} XY - \sqrt{q}\,YX &= -\sqrt{q+q^{-1}}\,Y \\ YZ - ZY &= \left(\sqrt{q} - \sqrt{q}^{-1}\right) X^2 - \sqrt{q+q^{-1}}\,X \end{aligned} \qquad (**)$$

(where classically $q = \exp(2\pi i/(k+2))$, k = Chern coupling constant).

The deformed Casimir operator $C = \sqrt{q}^{-1} ZY + \sqrt{q}\,YZ + X^2$ is central in $W_q(sl_2)$ and so is $A = 1 - C(\sqrt{q} - \sqrt{q}^{-1})(\sqrt{q+q^{-1}})^{-1}$. If we put

$$\begin{aligned} x &= \left(X - \left(\sqrt{q} + \sqrt{q}^{-1}\right)\sqrt{q+q^{-1}}^{-1} C\right)\sqrt{q+q}^{-1} A^{-1} \\ y &= Y\left(\sqrt{q+q^{-1}}^{-1}\right)(\sqrt{A})^{-1} \\ z &= Z\left(\sqrt{q+q^{-1}}\right)^{-1}(\sqrt{A})^{-1} \end{aligned} \qquad (\mathrm{T})$$

then the relations $(**)$ become (up to adding $(\sqrt{A})^{-1}$ to the ring)

$$\begin{aligned} \sqrt{q}\,xz - \sqrt{q}^{-1} zx &= z \\ \sqrt{q}^{-1} xy - \sqrt{q}\,yx &= y \\ q^{-1} zy - qyz &= x. \end{aligned} \qquad (**)'$$

Now $\mathbb{C}\langle x, y, z\rangle$ modulo $(**)'$ is of type $(*)$ up to choosing $\alpha, \beta, \gamma, \ldots$. For that choice of constants the algebra given by $(*)$ localized at $\sqrt{A}$ yields the localization of the algebra given by $(**)'$ at $\sqrt{A'}$, where A' is obtained by writing A in a function of x, y, z via (T).

3. Relative ab-Extensions

It is self-evident that the combination of a Zariski local condition like birationality with a global condition like $C_B(A)$ is an Azumaya algebra, calls for a generalization in such a way that both parts of the definition become Zariski local conditions. The notion that prompts itself here is the notion of a relative Azumaya algebra with respect to a localization or, in particular, with respect to an ideal of the centre.

Throughout, R is a commutative ring and Λ is an R-algebra. We say that Λ is a κ-Azumaya algebra with respect to some kernel functor κ on R-mod such that R is κ-closed (that is, $Q_\kappa(R) = R$) if the canonical R-morphism $\mu, \mu: \Lambda \otimes_R \Lambda^\circ \to \mathrm{End}_R(\Lambda)$, $\lambda_1 \otimes \lambda_2 \mapsto (x \mapsto \lambda_1 x \lambda_2)$, induces an isomorphism $Q_\kappa(\Lambda \otimes_R \Lambda^\circ) \cong \mathrm{End}_R(\Lambda)$ and Λ is a κ-progenerator in the sense of [7]. In particular a κ-Azumaya algebra Λ is κ-closed, that is, $Q_\kappa(\Lambda) = \Lambda$.

The following criterion, which stems from [7], is useful.

3.1. Proposition. *Let Λ be a κ-closed R-algebra and assume that Λ is finitely presented as an R-module; then the following statements are equivalent:*

1. *Λ is a κ-Azumaya algebra with centre R.*

2. *For every $P \in C(\kappa)$, i.e., $P \notin \mathscr{L}(\kappa)$ and maximal as such, $Q_{R-P}(\Lambda)$ is an Azumaya algebra over $Q_{R-P}(R)$.*

3.2. Examples. a. Put $\kappa = \kappa_1 = \inf\{\kappa_P, p \in \mathrm{Spec}\, R, htp \leq 1\}$ and R is a Krull domain. Then for any R-lattice L we have $Q_{\kappa_1}(L) = L^{**}$, where $L^* = \mathrm{Hom}_R(L, R)$, A κ_1-Azumaya algebra is then exactly a reflexive Azumaya algebra; cf. [6, 7].

In [6] it was pointed out that reflexive graded Azumaya algebras turn up as algebras representing elements of the Brauer group of low dimensional projective schemes. On the other hand reflexive Azumaya algebras appear in the theory of maximal orders over Krull domains (in central simple algebras).

b. Put $\kappa = \kappa_I$ associated to the Zariski open subset $X(I)$ of Spec$C = X$, and let R be ring of sections of the structure sheaf $\underline{O}_C$ over $X(I)$ (that is, $R = Q_I(C)$). Then κ_I-Azumaya algebras appear as rings of sections over $X(I)$ of locally separable sheaves over X.

For a basic open set, i.e., $X(f)$, $f \in R$, a κ_f-Azumaya algebra is nothing but an Azumaya algebra over $Q_f(R)$.

3.3. Proposition [7]. *If Λ is a κ-Azumaya algebra over R and M is a κ-closed (Λ, R)-bimodule then $M = Q_\kappa(\Lambda \otimes_R M^\Lambda)$, where $M^\Lambda = \{m \in M, \lambda m = m\lambda$ for $\lambda \in \Lambda\}$.*

Now we shall look at a birational extension $A \hookrightarrow B$ and a central localization κ on R-mod, where $R = Z(B)$ such that $I \in \mathscr{L}(\kappa)$.

Throughout we use notation and conventions of Section 2. Looking at a central κ on B-mod we have also a central κ' on A-mod and in this situation $Q_{\kappa'}(B) \cong Q_\kappa(B)$ holds without further condition on the extension $A \hookrightarrow B$. When B is κ-closed it follows that $C_B(A)$ is κ-closed too.

Indeed, if $\xi \in B$ satisfies $H\xi \subset C_B(A)$ for some $H \in \mathscr{L}(\kappa)$ (and we may assume $H = B(H \cap R)$ with $H \cap R \in \mathscr{L}(\kappa)$ if κ is considered as a functor on $Z(B)$-mod) for all $h \in H \cap R$ and $a \in A$ we have $h\xi a - ah\xi = h(\xi a - a\xi) = 0$, hence $\xi a - a\xi = 0$ because there is no κ-torsion in B. We say that a birational extension $A \hookrightarrow B$ is a relative algebraically birational (r.a.b.-extension) whenever B is κ-closed and $B = AC_B(A)$, where $C_B(A)$ is a κ-Azumaya algebra over $Z(C_B(A)) = Z(B)$. In view of Proposition 3.3. we obtain $B = Q_\kappa(C_B(A) \otimes_{Z(B)} C_B(C_B(A)))$.

One has to distinguish two cases.

3.4. *Case* 1. $I' \in \mathscr{L}(\kappa')$ or $I \in \mathscr{L}(\kappa)$. If $p \in \operatorname{Spec}(A)$ is such that $p \notin \mathscr{L}(\kappa')$ then $I' \not\subset p$, that is, $p \in X(I')$. Hence there is a unique $P \in \operatorname{Spec}(B)$ such that $p = P \cap A$, $I \not\subset P$, that is, $P \in Y(I)$ such that $Q_{A-p}(B) = Q_{B-P}(B)$. We write $Y(\kappa)$ for the prime ideals of B not in $\mathscr{L}(\kappa)$ and $C(\kappa)$ for the maximal elements of $Y(\kappa)$, and similarly, $X(\kappa')$ and $C(\kappa')$. This case corresponds therefore to the situation $X(\kappa') \subset X(I')$, $Y(\kappa) \subset Y(I)$.

3.5. *Case* 2. $I' \notin \mathscr{L}(\kappa')$ or $I \notin \mathscr{L}(\kappa)$. In this case $X(\kappa') \not\subset X(I')$, $Y(\kappa) \not\subset Y(I)$. Now only prime ideals of $X(\kappa') \cap X(I')$ correspond nicely to prime ideals of $Y(\kappa) \cap Y(I)$, such that localization at these ideals yields Azumaya algebras. A minor problem in this case (in fact also in Case 1) is that $Y(\kappa) \cap Y(I)$ is not open in the Zariski topology.

In the Noetherian case we can eliminate this drawback.

3.6. LEMMA. *Let $Z(B)$ be a Noetherian domain and Λ a κ-Azumaya algebra over $Z(B)$; then Λ is a κ_J-Azumaya algebra for some ideal J of $Z(B)$.*

Proof. We have $\kappa = \Lambda\{\kappa_q, q \in \operatorname{Spec}(Z(B)) - \mathscr{L}(\kappa)\}$, where we write κ, κ_q for the kernel functors on $Z(B)$-mod.

For $q \in \operatorname{Spec}(Z(B)) - \mathscr{L}(\kappa)$ we know that Λ_q is an Azumaya algebra and this property is characterized by the existence of a separability idempotent e_q in $\Lambda_q \otimes_{Z(B)_q} \Lambda_q^\circ$. If $c_q \in Z(B) - q$ is a denominator for e_q then $\Lambda_{c_q} \otimes_{Z(B)_{c_q}} \Lambda_{c_q}^\circ$ is also an Azumaya algebra and then for every prime q' of $Z(B)$, $c_q \notin q'$, we have that Λ_q is an Azumaya algebra. So we may enlarge $\operatorname{Spec}(Z(B)) - \mathscr{L}(\kappa)$ to the open set $U_{c_q} = \{q', q' \not\ni c_q\}$, which corresponds to some ideal J of $Z(B)$. It is clear that $Z(B)$ is κ_J-closed since it is κ-closed and Λ remains a κ_J-progenerator (note that the relative progenerator property for Λ yields that Λ_q is a progenerator $Z(B)_q$-module and again Λ_c is a progenerator $Z(B)_c$-module for some $c \notin q$. This yields a J' such that Λ is a $\kappa_{J'}$-progenerator and we may replace J and J' by $J \cap J'$ and call this ideal J again). From $(A \otimes_{Z(B)} A^\circ)_q = (\mathrm{End}_{Z(B)}(A))_q$ for all q, $J \not\subset q$, it follows that $Q_{\kappa_J}(A \otimes_{Z(B)} A^\circ = \bigcap_{q, J \not\subset q}(A \otimes_{Z(B)} A^\circ)_q = \bigcap_{q, J \not\subset q}(\mathrm{End}_{Z(B)}(A))_q = Q_{\kappa_J}(\mathrm{End}_{Z(B)}(A)) = \mathrm{nd}_{Z(B)}(A)$ (the latter because A is a κ_J-closed κ_J-progenerator) and therefore A is a κ_J-Azumaya algebra.

3.7. COROLLARY. *Let $A \hookrightarrow B$ be a r.a.b.-extension such that $Z(B)$ is a Noetherian domain. Then there exists a Zariski open set $Y(J)$ such that $Y(J) \subset Y(I)$ and $C_B(A)$ is a κ_J-Azumaya algebra; i.e., for all $q \in$* Spec $Z(B)$, *$J \cap Z(B) \not\subset q$ we have that $C_B(A)_q$ is Azumaya.*

Proof. Easy from the lemma combined with 3.4 and 3.5.

3.8. *Note.* If $Q \in Y$ is such that $Q \notin \mathscr{L}(\kappa)$ then Q is κ-closed because it is a prime ideal of B and B is κ-closed. Put $q = Q \cap Z(B)$.

In view of Proposition 3.3 we have $Q = Q_\kappa(C_B(A)(Q \cap C_B C_B(A)))$ and $Q \cap C_B(A) = Q_\kappa(C_B(A)q)$.

3.9. THEOREM. *Let $A \hookrightarrow B$ be a r.a.b.-extension. Then this extension decomposes as $A \hookrightarrow AZ(B)$, which is an ab-extension, and $AZ(B) \hookrightarrow B$, which is a r.a.b.-extension such that the κ-closed ideals of $Q_\kappa(AZ(B))$ and B correspond bijectively.*

Proof. In view of Proposition 3.3 we have $B \cong Q_\kappa(C_B(A) \otimes_{Z(B)} C_B(C_B(A)))$ and B is a κ-finitely generated $Q_\kappa(AZ(B))$-module. It is obvious that $Q_\kappa(AZ(B)) \subset C_B(C_B(A))$. Since $C_B(A)$ is a κ-progenerator over $Z(B)$ and because $Q_\kappa(C_B(A) \otimes_{Z(B)} Q_\kappa(AZ(B))) \cong B$ too, we derive from $Q_\kappa(C_B(A) \otimes C_B(C_B(A))/Q_\kappa(AZ(B))) = 0$ that $C_B(C_B(A)) = Q_\kappa(AZ(B))$.

If J is a κ-closed ideal of B then again from Proposition 3.3 we have $J = Q_\kappa(C_B(A) \otimes_{Z(B)} C_{C_B(A)}(J)) = Q_\kappa(C_B(A)(J \cap Q_\kappa(AZ(B))))$.

Conversely, if H is a κ-closed ideal of $Q_\kappa(AZ(B))$ then $C_B(A)H$ is an ideal of B and $H^e = Q_\kappa(C_B(A)H)$ is a κ-closed ideal because κ is a central localization on B-mod. It is clear that $H \subset H^e \cap Q_\kappa(AZ(B))$, and $Q_\kappa(C_B(A) \otimes (H^e \cap Q_\kappa(AZ(B)))/H) = 0$. Thus $H = H^e \cap Q_\kappa(AZ(B))$ again follows from the fact that $C_B(A)$ is a κ-progenerator (in particular κ-faithfully flat; cf. [6]). Hence, the bijective correspondence between the sets of κ-closed ideals in B and in $Q_\kappa(AZ(B))$, resp. B.

For any ideal $H' \subset I' \subset A$, with $H = \mathrm{rad}\, BH' \subset I \subset B$, we have $Q_\kappa(H'Z(B)) = Q_\kappa(H) \cap Q_\kappa(AZ(B))$.

Of course for H' and H as before, the birationality property alone leads to $X(H') \simeq Z(H'Z(B)) \cong Z(H \cap AZ(B)) \cong Y(H)$, i.e., $\mathrm{rad}(H'Z(B)) = H \cap AZ(B)$ and also $\mathrm{rad}(H'Z(B)) \subset Q_\kappa(H'Z(B))$.

So if $\mathrm{rad}(H'Z(B))$ is κ-closed then $\mathrm{rad}(H'Z(B)) = Q_\kappa(H'Z(B))$.

3.10. EXAMPLE. It is well known that a semiprime PI ring is a birational extension of the centre and moreover for the primes in the open set of birationality of the centre localization yields an Azumaya algebra. Therefore maximal orders over an integrally closed Noetherian domain are close to being r.a.b.-extensions of their centre but they are not unless they are relative Azumaya algebras (the catch is in the κ-progenerator property!).

3.11. *Remark*. For every perfect τ on $Z(B)$-mod such that $\tau \geq \kappa$, i.e., $\mathscr{L}(\kappa) \subset \mathscr{L}(\tau)$, an r.a.b.-extension $A \hookrightarrow B$ localizes to a global Zariski ab-extension $Q_\tau(A) \hookrightarrow Q_\tau(B)$.

Proof. First, it is clear that $B_1 = B/\tau B, C_B(A)/\tau C_B(A), Z(B)/\tau Z(B)$ are κ-closed as well as $Z(B_1)$. That $C_B(A)/\tau C_B(A) \cong Q_\kappa(C_B(A) \otimes_{Z(B)} Z(B)/\tau Z(B))$ is a κ-Azumaya algebra is clear and similarly for

$$T_1 = Q_\kappa((C_B(A)/\tau C_B(A))Z(B_1)) \cong Q_\kappa(C_B(A) \otimes_{Z(B)} Z(B_1))$$

From the κ-faithful projectivity of T_1 over $Z(B_1)$ we obtain $C_{B_1}(T_1) = Q_\kappa(A_1 Z(B_1))$, where $A_1 = A/\tau A \subset B_1$. Moreover $C_{B_1}(A_1) = C_{B_1}(Q_\kappa(A_1 Z(B_1))) = C_{B_1}(C_{B_1}(T_1))$. Since $C_{B_1}(C_{B_1}(T_1))$ is a κ-closed $(T_1, Z(B_1))$-bimodule it follows that $S_1 = C_{B_1}(A_1) = T_1 C_{S_1}(T_1)$. If $x \in C_{S_1}(T_1)$ then x commutes with A_1 since $x \in S_1$ and x commutes with T_1 by definition, hence x commutes with $A_1(C_B(A)/\tau C_B(A)) = B_1$; therefore $C_{S_1}(T_1) \subset Z(B_1)$ and hence $C_{B_1}(A_1) = T_1$, $B_1 = A_1 T_1$ and T_1 is a κ-Azumaya algebra.

From here on the proof follows the lines of Theorem 2.10 and in this case we have that $Q_\tau(B) = BZ(Q_\tau(B)) = BQ_\tau(Z(B))$ because of the properties of τ; hence we may use the argumentation of Note 2.11(1).

At this point we have obtained a good part of the desired "ideal theory" for birational extensions. This will allow us to study the transfer of properties from A to B and vice versa, much like the theory obtained in [7], but there it was necessary to assume A commutative in order to obtain stringent results. In the present case, if A is commutative then B is a relative Azumaya algebra and this is relatively well understood; the non-commutative situation of a r.a.b.-extension $A \hookrightarrow B$ presents us with a restriction of scalars functor that is "locally separable" on a Zariski open set. Again we are able to study rings A_1 and A_2 having a common r.a.b-extension B such that the open sets of birationality used in B have a non-empty intersection and use the same κ over B for both $A_1 \hookrightarrow_{i_1} B$ and $A_2 \hookrightarrow_{i_2} B$. Properties relating A_1 and A_2 are studied by going up i_1 and descending i_2; we do not go more deeply into this here.

References

1. F. De Meyer and E. Ingraham, "Separable Algebras over Commutative Rings," Lecture Notes in Mathematics, Vol. 181, Springer-Verlag, Berlin, 1971.
2. J. Golan, "Localization of Non-commutative Rings," Monographs in Mathematics, Dekker, New York, 1975.
3. C. Năstăsescu, M. Van den Bergh, and F. Van Oystaeyen, Separable functors applied to graded rings, *J. Algebra* **123**, No. 2 (1989), 397–413.
4. C. Procesi, "Rings with Polynomial Identities," Monographs in Mathematics, Dekker, New York, 1977.
5. F. Van Oystaeyen, Birational extensions of rings, *in* "Ring Theory" (Proceedings, 1978 Antwerp Conference), pp. 287–328, Lecture Notes in Pure and Applied Mathematics, Vol. 51, Dekker, New York, 1979.
6. F. Van Oystaeyen and A. Verschoren, "Non-commutative Algebraic Geometry—An Introduction," Lecture Notes in Mathematics, Vol. 887, Springer-Verlag, Berlin, 1980.
7. F. Van Oystaeyen and A. Verschoren, "Relative Invariants of Rings, The Commutative Theory" Monographs in Mathematics, Dekker, New York, 1982.