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Capacity investment choices under cost heterogeneity and output flexibility in oligopoly

Benoît Chevalier-Roignant[a,*], Christoph M. Flath[b], Peter M. Kort[c,d], Lenos Trigeorgis[e]

[a] *emlyon business school, 23 Avenue Guy de Collongue, 69130 Ecully, France*
[b] *University of Würzburg, Josef-Stangl-Platz 2, 97070 Würzburg, Germany*
[c] *Tilburg University, Warandelaan 2, 5037 AB Tilburg, The Netherlands*
[d] *University of Antwerp, Prinsstraat 13, 2000 Antwerpen, Belgium*
[e] *University of Cyprus, Kallipoleos 75, Nicosia 1678, Cyprus*

Abstract

We study capacity investment decisions among oligopoly firms under conditions of cost heterogeneity and output flexibility within capacity constraints. Output flexibility causes the value of the firm to be convex in the state of demand, which implies that the firm invests in larger capacity when the economic environment is more uncertain. Under cost heterogeneity among oligopoly firms, a lower-cost firm invests in larger capacity, while a less efficient rival chooses lower capacity as capacities are strategic substitutes. Consequently, higher uncertainty leads to more dispersion of equilibrium capacities and greater industry concentration. More competition thus induces a welfare loss when uncertainty and cost heterogeneity are high.

JEL: G31, G32

Keywords: Investment analysis, real options, capacity choices, output flexibility, firm asymmetry.

*Corresponding author

Email addresses: benoit.chevalier-roignant@outlook.com (Benoît Chevalier-Roignant), christoph.flath@uni-wuerzburg.de (Christoph M. Flath), kort@uvt.nl (Peter M. Kort), lenos@ucy.ac.cy (Lenos Trigeorgis)

1. Introduction

Capacity investment decisions present a challenge under demand uncertainty when future capacity adjustment is infeasible or too costly. A more uncertain economic environment is generally believed to have a depressing effect on capacity investment. First, a firm is likely to defer a capital investment decision when the economic environment is more uncertain as the value of the option to defer increases with the level of uncertainty (see, e.g., McDonald and Siegel 1986, Dixit and Pindyck 1994). Further, when firms do not adjust their output levels, they may collectively end up in a situation of under or oversupply compared to the industry equilibrium that would prevail under the current market circumstances. Larger deviations from the temporary optimal output rate become more likely under market uncertainty. This second depressing effect can be mitigated if firms have flexibility to adjust their production outputs.

Hubbard (1994, p. 1828) notes that the literature on real options (see also Trigeorgis 1996) "do[es] not offer specific predictions about the *level* of investment." Some progress has been made since. Dangl (1999) characterizes the optimal exercise of an *expansion option* deciding on the capacity size upon investment, while Bensoussan and Chevalier-Roignant (2018) analyze the *sequential* exercise of such expansion options. Despite such progress, the treatment of capacity choices in real options models remains somewhat sparse according to a recent review by Trigeorgis and Tsekrekos (2018). Authors addressing such matters in oligopoly settings typically make two restrictive assumptions: (i) oligopoly firms always produce *at* capacity (see, e.g., Grenadier 2002, Huisman and Kort 2015) and (ii) firms are symmetric in terms of investment costs (see, e.g. Grenadier 2002, Aguerrevere 2003, 2009).[1] We show that relaxing these two standard assumptions leads to different and important insights regarding the impact of uncertainty on oligopoly firms' capacity investment decisions, industry concentration and social welfare.

We focus on the interplay between (ex-ante) capacity investment choices and (ex-post) constrained output decisions in an asymmetric oligopoly, abstracting away from timing considerations.[2] Our framework generalizes the analysis of McDonald and Siegel (1985) who consider a monopoly firm's option to shut down production *temporarily* (when demand is too low to recoup marginal costs) but ignore investment timing. In

[1] Huisman and Kort (2015) acknowledge that "relaxing this constraint [that the firms produce at capacity] complicates the analysis considerably" (p. 378)].

[2] Accounting for expansion timing would give rise to either an *optimal stopping game* if the firms decide once on capacity expansion or an *impulse/singular control game* if firms can expand gradually. In the case of an optimal stopping game, one needs to solve a capacity choice game similar to ours to determine the investment lumps *at* the investment time. In case of an impulse/singular control game, the value functions for the oligopoly firms will involve at the investment times a capacity choice game for which the objectives depend on the value functions. This recursivity presents a mathematical challenge. Allowing for timing considerations would greatly increase the technical nature of our contribution (because of the generality of our problem with k firms), yet would not impact the main economic insights from our model substantially: in an oligopoly with asymmetric firms and output flexibility within constraints, uncertainty leads to a greater industry concentration *at* the time of capacity expansion. Optimal stopping or switching-option models, e.g., Adkins and Paxson (2019), generally consider payoff functions at the exercise times that are simpler than the expression we have in our model.

contrast, we consider a k-firm oligopoly where firms make output decisions up until their capacity constraint becomes binding and treat capacity constraints as equilibrium choices in a static game among oligopoly firms facing heterogeneous costs.

We adopt the less common assumption that firms can produce below capacity and relax the assumption that firms are symmetric in costs. We derive an explicit expression for the value of a firm that embeds future output flexibility and accounts for industry dynamics when rival firms face a capacity constraint. We prove the existence and uniqueness of a Nash-Equilibrium capacity vector whereby firms' production capacities are ranked in line with the ranking of their unit capacity costs. We then perform comparative statics with respect to demand volatility and cost asymmetry using an efficient iterative scheme to obtain equilibrium capacity choices in oligopoly.

Our analysis allows us to generate several interesting *insights*. For any firm, output flexibility leads to a profit function that is weakly convex in the state of demand (because one can react to increased demand by expanding output up until one faces a capacity constraint). Given output flexibility, the value of capacity is thus strictly convex. Given output flexibility, it is therefore optimal for a firm to invest in greater capacity when demand is more uncertain. Further, in an oligopoly where firms hold heterogeneous capacities, an unconstrained firm will share residual demand among fewer rivals when a smaller rival gets constrained (and so the denominator in its Cournot profit function decreases). This makes the value of larger capacity "more convex" in the state of demand. A low investment-cost oligopoly firm therefore has an incentive to undertake a larger capacity investment to benefit from greater value convexity. Because capacity decisions are strategic substitutes (see Tirole 1988), less cost-efficient firms will in turn limit their capacity investment. Consequently, when asymmetric firms have output flexibility, higher uncertainty leads to higher capacity dispersion in equilibrium and greater industry concentration. This result arises from the joint assumption of output flexibility and asymmetry among oligopoly firms. If there is no output flexibility, then volatility has no (direct) effect on capacity investment. If there is no original asymmetry among firms, the industry concentration will solely be linked to the number of entrants. Whereas demand uncertainty has no impact on (expected) social welfare when firms are symmetric (beliefs are homogenous), more competition may induce a welfare loss when cost heterogeneity is high. This provides another context in which oligopolistic competition may generate welfare losses (see, e.g., Guesnerie and Hart 1985, Ritz 2014). The welfare loss resulting from greater industry concentration here, however, may be offset as the average unit capacity cost decreases with concentration.

Our approach helps us identify logical implications that policy makers should consider when designing an *industrial policy*. First, antitrust authorities should take care when challenging industry concentration as concentration can have beneficial effects on upfront cost efficiency, especially when demand is uncertain.

Further, providing subsidies to industry front-runners may enlarge asymmetry and in turn lead to larger production capacity in the industry.

2. Literature review

According to Dick (1994), Uzunca and Cassiman (2018) and Murphy and Smeers (2005), capacity decisions are key value levers in the semiconductor and electricity industries. For instance, Uzunca and Cassiman (2018) remark that "empirical evidence from the global semiconductor manufacturing industry shows that incumbents respond to the threat of entry by expanding their capacity."

Dangl (1999) and Bensoussan and Chevalier-Roignant (2013) study a setup where a *monopoly* firm decides on the timing and size of its capital investment. In recent works by Hagspiel et al. (2016a), Wen et al. (2017) and De Giovanni and Massabò (2018), a monopoly firm that can produce below capacity decides on the timing and size of its capacity investment. Output flexibility raises the value of capacity units, which incentivizes the firm to undertake larger investments. Consequently, investment is delayed compared to the case without output flexibility. In contrast to these papers focusing on monopoly and investment timing, we consider an asymmetric oligopoly where firms invest upfront in production capacity. We are also able to account for a greater dispersion of production capacities (in an asymmetric oligopoly) in the face of uncertainty.

The study of real options in *symmetric oligopoly* settings has led to interesting economic insights (see also the review in Chevalier-Roignant et al. 2011). Huisman and Kort (2015) study a market-entry game among two firms that decide on the timing and size of their capacity investments. The leader has in Huisman and Kort (2015) two incentives to invest more in capacity: (i) to induce lower capacity investment by the rival and (ii) to force the follower to delay entry thereby enjoying monopoly rents over a longer period, an incentive which increases with uncertainty. Our results are different: the value of capacity in Huisman and Kort (2015) is linear in the stochastic variable, while it is strictly convex in our case (owing to output flexibility). Consequently, we obtain that uncertainty directly affects the firms' capacity choices, while it does so in Huisman and Kort (2015) indirectly via the impact on the firms' investment thresholds. In contrast, we also abstract away from timing considerations for simplicity.

Another stream of the literature on investment under uncertainty examines the gradual buildup of capacity. Pindyck (1988) considers a *monopoly* setup where the firm expands gradually. Because higher volatility enhances the value of the defer option, the firm delays investment and reduces capacity commitment. Abel and Eberly (1994, 1996) allow for divestment also: the firm invests if the marginal revenue product of capital exceeds the marginal cost, divests if it falls below the resale price, and stays put otherwise. Bensoussan and

Chevalier-Roignant (2018) consider a setting where the firm faces an uncertain commodity price and incurs a fixed cost upon capacity installment (of endogenous size). The underlying tradeoff is modeled as an impulse control problem of dimension two. If the fixed adjustment cost vanishes, then the problem is one of singular control.

Grenadier (2002) and Aguerrevere (2003, 2009) analyze the gradual expansion and contraction of production capacities in a *symmetric oligopoly*. Grenadier (2002) assumes firms produce at capacity, while in Aguerrevere (2003, 2009) firms can produce below capacity. Grenadier (2002) argues that oligopolistic competition drastically erodes the value of the option to defer (and hence the incentive to delay investment). Aguerrevere (2003) finds that the market-clearing price exhibits mean reversion over time due to discretion in the capacity utilization, with significant spikes during periods of high capacity utilization. Aguerrevere (2003) and Aguerrevere (2009) differ in the specifications for the demand function and inclusion of "time to build." As in Aguerrevere (2003), we obtain that uncertainty encourages a firm to increase capacity. Grenadier and Aguerrevere employ techniques akin to singular control. Yet, the use of such techniques in a game setting is controversial (see Back and Paulsen 2009, Guo and Xu 2019).

Our focus here is on the impact of *output flexibility* on the upfront capacity choices of *asymmetric oligopoly* firms. Doing so allows us to examine the direct effect of uncertainty on industry concentration independently of the effect that uncertainty may have on timing decisions. Because we set aside timing considerations, we also do not have to deal with the technical challenges implied by preemption games (if the firms invest only once) or by singular or impulse control games (if the firms expand gradually).

3. Model Setup

Consider k asymmetric oligopoly firms that have the choice to make an upfront investment in production capacities (at fixed time 0). We assume capacities once set are held fixed thereafter and that there is no flexibility with respect to timing. This setting is analogous to a situation where firms disregard the "option to defer" due to strategic, operational or other considerations. These may include learning-curve or network effects as well as technological obsolescence in high-tech industries (e.g., semiconductors or cell phones). Furthermore, the timeframe for investment is short-lived following the opening of a "window of opportunity" as a new technology emerges (see Christensen et al. 1998, Klepper 2002, Lukas et al. 2017). Consequently, oligopoly firms are pressed to make upfront capacity investment decisions that might determine their eventual success and risk exposure.[3] For instance, semiconductor manufacturers generally build one large facility that

[3]Beyond emerging industries, the decisions to invest at the outset may relate to a decision by a central planner to, e.g., deregulate and open up a market for competition or by a government to grant a license (e.g., in telecommunications or transportation) to several rivals.

lasts across the technology cycle, because the investment cost for a modern Fab are prohibitive.[4] Firm i's total upfront investment outlay to install $\bar{q}_i$ units of capacity is $C_i(\bar{q}_i)$. We assume a linear investment cost function of the form $q \mapsto c_i q$. The unit capacity cost c_i can embed perpetual fixed operating costs and is possibly distinct for the different oligopoly firms owing to economies of scope, financing or lower fixed maintenance costs. Such costs are typically known to key market participants: they may be available in annual reports or internal management accounting reports or obtainable via competitive market intelligence. We rank firms in terms of *decreasing capacity costs*, with firm k being the "cost leader" and $c_k \leq c_{k-1} \leq \cdots \leq c_1$. Firm i selects capacity $\bar{q}_i \in \mathbb{R}_+$, with all other rivals holding combined capacity $\bar{Q}_{-i} \in \mathbb{R}_+^{k-1}$. Given this cost ranking, in the unique Nash equilibrium (obtained later), firm capacities are also weakly ranked, with

$$0 =: \bar{q}_0 < \bar{q}_1 \leq \cdots \leq \bar{q}_i \leq \cdots \leq \bar{q}_k < \bar{q}_{k+1} := \infty. \tag{1}$$

Following their upfront capacity choices (at fixed time $t = 0$), firms compete *repeatedly* à la Cournot subject to stochastic demand. Kreps and Scheinkman (1983) interpret Cournot profits as reduced-form rents in a setting where firms decide first on capacity and then on price. Our Cournot assumptions apply best to an industry with nonstorable commodity goods whose price fluctuates as the supply and demand dynamics evolve. These features hold reasonably well in the semiconductor industry (Dick 1994, Uzunca and Cassiman 2018) and the power/energy sector (Murphy and Smeers 2005). Our assumptions about demand dynamics and the market structure hold best in a deregulated market. At each time $t \in \mathbb{R}_+$, firm i chooses the output quantity $q_i(t)$ to produce subject to a capacity constraint, viz.

$$0 \leq q_i(t) \leq \bar{q}_i. \tag{2}$$

The firms in the industry collectively produce $Q_t \cdot \mathbf{1}_k = \sum_{i=1}^k q_i(t)$. As in Aguerrevere (2003) and Hackbarth and Miao (2012), we assume firms face a linear inverse demand function of the form:

$$p(X_t, Q_t) := X_t - bQ_t \cdot \mathbf{1}_k, \quad b > 0. \tag{3}$$

The demand intercept $(X_t; t \geq 0)$ follows geometric Brownian motion of the form

$$\mathrm{d}X_t = \mu X_t \, \mathrm{d}t + \sigma X_t \, \mathrm{d}B_t, \quad X_0 = x_0 \text{ a.s.}, \tag{4}$$

[4] https://www.eetimes.com/semi-industry-fab-costs-limit-industry-growth/

where μ is the drift rate (capturing expected growth or decline), σ (> 0) is volatility, and $(B_t; t \geq 0)$ is a Brownian motion. This process is defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ augmented for the Brownian filtration $\mathbb{F} := (\mathscr{F}_t; t \geq 0)$. Process $(X_t; t \geq 0)$ can take an arbitrary value $x > 0$—representing the state of demand—at time $t\,(\geq 0)$. The *initial* demand state is $x_0\,(> 0)$. The k oligopoly firms (and a presumed social planner) are risk-neutral and discount future payoffs at the risk-free interest rate $r\,(> 0)$. We assume homogeneous marginal production costs c. In a homogeneous market, the variable costs incurred by firms to procure inputs should not differ much—especially if the supplier base is itself a competitive market. Over time, marginal cost asymmetry may not sustain owing to process standardization and convergence toward the most efficient production technology. Firm i's profit is given by

$$\pi_i(X_t, Q_t) := [p(X_t, Q_t) - c]\;\; q_i(t). \tag{5}$$

Firm i's payoff given capacities $\bar{Q} = (\bar{q}_i, \bar{Q}_{-i}) \in \mathbb{R}^k_+$ and output policies $Q(\cdot) = \big(q_i(\cdot), Q_{-i}(\cdot)\big)$ consists of the stream of expected profits (discounted at the risk-free rate r), net of capital investment costs:

$$J_i\big(x, Q(\cdot), \bar{Q}\big) := \mathbb{E}\left[\int_0^\infty e^{-rt}\pi_i(X_t, Q_t)\,\mathrm{d}t\right] - C_i(\bar{q}_i). \tag{6}$$

Here, $\mathbb{E}$ denotes the expectation operator conditional on initial demand state being x. The firms' strategies are nonanticipative and allow for flexibility to shut down or adjust production level up to the capacity constraint $\bar{q}_i$.

We proceed backwards to solve the game. To this end, we first derive the Cournot equilibrium outputs and profits and then the firms' values as a function of industry capacity (vector). From these payoff functions we then obtain the upfront equilibrium capacity choices.

4. Output and Profits in Constrained Cournot Oligopoly

We next identify a profile of output policies from which no firm has a unilateral incentive to deviate. Let $Q^C(\cdot) := (q_i^C, Q_{-i}^C)$ be the Markov equilibrium output policies. Given an arbitrary capacity vector $\bar{Q}$, firm i's equilibrium Cournot profit, $\pi_i^C(X_t, \bar{Q})$, satisfies

$$\pi_i^C(X_t, \bar{Q}) := \pi_i\left(X_t, (q_i^C, Q_{-i}^C)\right) \geq \pi_i\left(X_t, (q_i(t), Q_{-i}^C)\right)$$

for all output decisions $q_i(t)$ satisfying the capacity constraint (2). The lemma below provides the equilibrium output policies for a k-firm Cournot oligopoly with asymmetric capacity constraints. All proofs are provided in the appendices.

Lemma 1 (Cournot output policy). *Consider a capacity vector* $\bar{Q}$ *satisfying the ranking in* (1). *Firm i's state-dependent output policy is given by*

$$q_i^C(X_t, \bar{Q}) = \begin{cases} 0, & X_t \in (0, c) \quad \{\text{firm } i \text{ is idle}\} \\ \frac{X_t - \Sigma_t}{b(1+K_t)}, & X_t \in [c, \bar{x}_i) \quad \{\text{firm } i \text{ is unconstrained}\} \\ \bar{q}_i, & X_t \in [\bar{x}_i, \infty) \quad \{\text{firm } i \text{ is constrained}\}, \end{cases} \tag{7a}$$

where

$$\bar{x}_i := c + b\left[\sum_{j=0}^{i-1} \bar{q}_j + (k - i + 2)\,\bar{q}_i\right], \quad i = 1, \ldots, k, \tag{7b}$$

is a scalar depending on initial capacity choices $\bar{Q}$, $\Sigma_t = \Sigma\left(X_t, \bar{Q}\right)$ *and* $K_t = K\left(X_t, \bar{Q}\right)$ *are state-dependent. The functions* Σ *and* K *are given by*

$$\Sigma\big(x, \bar{Q}\big) := c + b\left[\sum_{m=0}^{k} \bar{q}_m \mathbb{1}_{\{x \geq \bar{x}_m\}}\right], \tag{7c}$$

$$K\big(x, \bar{Q}\big) := \sum_{m=0}^{k} \mathbb{1}_{\{x \leq \bar{x}_m\}}, \tag{7d}$$

with $\mathbb{1}_{\{\cdot\}}$ *being the indicator function. (We set* $\bar{x}_{-1} := 0$ *and* $\bar{x}_0 := c$ *by convention.)*

According to (7a), firm i is "idle" in the low demand region $(0, c)$ as it cannot recover the marginal cost c. This corresponds to McDonald and Siegel's (1985) model which deals with the special (monopoly) case without capacity constraints and without the "option to expand output." Parameter $\bar{x}_i$ in (7b) is the demand threshold above which firm i's capacity constraint becomes binding. This threshold increases in the number of rivals, k, because, as the market pie is shared among a larger number of firms (with firm i supplying a smaller slice), the total market size has to get larger for firm i to get constrained *ceteris paribus.* At intermediate demand $[c, \bar{x}_i)$, the output of an unconstrained firm also depends on the number of constrained firms, $k - K_t$, and their aggregate output, $\sum_{m=0}^{k} \bar{q}_m \mathbb{1}_{\{X_t \geq \bar{x}_m\}} = [\Sigma_t - c]/b$, because these firms exert competitive pressure by taking away a given slice of the total demand pie X_t. Here, the K_t unconstrained firms vie for residual demand in a similar fashion as k Cournot oligopolists vie for total demand in an unconstrained oligopoly. Ignoring capacity constraints, the equilibrium Cournot output policy is given by $\tilde{q}(x, k) := [x - c]^+/[b(1 + k)]$. Equality $q_i^C(x, \bar{Q}) = \tilde{q}(x, K(x, \bar{Q}))$ holds if $x \in (c, \bar{x}_i)$, while $q_i^C(x, \bar{Q}) = \tilde{q}(x, k)$ holds if $x \in (c, \bar{x}_1)$. For large demand, where $X_t \geq \bar{x}_i$, firm i will reach a binding constraint and produce at full capacity $\bar{q}_i$. Compared to the standard symmetric Cournot model, firms with heterogeneous capacities produce more when unconstrained, but less so once their capacity limit is reached.

Note that functions $x \mapsto K(x, \bar{Q})$ and $x \mapsto \Sigma(x, \bar{Q})$ are discontinuous at each threshold $\bar{x}_m$, $m = 0, \dots, k$. Lemma 2 determines firm i's equilibrium Cournot profit given the capacity constraints.

Lemma 2 (Cournot profit). *In a k-firm Cournot oligopoly with asymmetric capacity constraints facing demand X_t, firm i's equilibrium profit, π_i^C, is given by*

$$\pi_i^C\left(X_t, \bar{Q}\right) = \begin{cases} 0 & \text{if } X_t \in (0, c) \qquad \{\text{firm } i \text{ is idle}\} \\ \frac{1}{b}\left[\frac{X_t - \Sigma_t}{1+K_t}\right]^2 & \text{if } X_t \in [c, \bar{x}_i) \qquad \{\text{firm } i \text{ is unconstrained}\} \\ \bar{q}_i\left[\frac{X_t - \Sigma_t}{1+K_t}\right] & \text{if } X_t \in [\bar{x}_i, \infty) \qquad \{\text{firm } i \text{ is constrained}\}. \end{cases} \tag{8}$$

The profit function $x \mapsto \pi_i^C\left(x, \bar{Q}\right)$ is continuous, nil on $(0, c)$, strictly convex increasing on $(c, \bar{x}_i)$ and linear increasing on $(\bar{x}_i, \infty)$. The function $x \mapsto \pi_i^C\left(x, \bar{Q}\right)$ is not continuously differentiable but rather exhibits kinks at each $\bar{x}_m$ with $m = 1, \dots, k$. The function $\bar{q}_i \mapsto \pi_i^C\left(x, \bar{Q}\right)$ is weakly concave.

Owing to its capacity constraint, firm i cannot expand output for large demand though it still benefits from a higher market-clearing price. For low to intermediate demand, the profit is (weakly) convex in the state of demand x. The latter property holds for other functional forms for the inverse demand function. For instance, in case of iso-elastic demand of the form $p(X_t, Q_t) = X_t Q_t^{-\gamma}$, the equilibrium profit is also convex (in X_t) if the constant elasticity coefficient $\gamma < 1$. If the demand function is $p(X_t, Q_t) = X_t(1 - bQ_t)$, the equilibrium profit will not be convex over the entire state space but only on a subset. Our key results on capacity investment behaviors are robust as long as the equilibrium profit is convex at intermediate demand.

5. Firm Value in Constrained Cournot Oligopoly

Firm i decides on its output over time, with $q_i(\cdot)$ being a $\mathbb{F}$-adapted controlled process. The value of the firm—representing the expected discounted sum of Cournot profits π_i^C in (8)—is given by

$$V_i(x, \bar{Q}) := \sup_{q_i(\cdot)} \mathbb{E}\left[\int_0^\infty e^{-rt} \pi_i\Big(X_t, \Big(q_i(t), Q_{-i}^C(t)\Big)\Big) \, \mathrm{d}t\right]$$

Due to the adaptedness of the controlled process $q_i(\cdot)$ and the absence of adjustment costs, we can write

$$\begin{aligned} V_i(x, \bar{Q}) =& \mathbb{E}\left[\int_0^\infty e^{-rt} \sup_{q_i(t) \in \mathbb{R}_+} \pi_i\Big(X_t, \big(q_i(t), Q_{-i}^C(t)\big)\Big) \, \mathrm{d}t\right], \\ =& \mathbb{E}\left[\int_0^\infty e^{-rt} \pi_i^C(X_t, \bar{Q}) \, \mathrm{d}t\right]. \end{aligned} \tag{9}$$

Cournot profit π_i^C changes over time depending on the processes $(X_t; t \geq 0)$, $(K_t; t \geq 0)$ and $(\Sigma_t; t \geq 0)$.

A basic component of firm value is the *perpetuity value of Cournot profits* given by

$$v_i\left(x,\bar{Q}\right) := \begin{cases} 0, & x \in (0,c), \\ \frac{1}{b\left[1+K(x,\bar{Q})\right]^2}\left[\frac{x^2}{r-2\mu-\sigma^2} - \frac{2x\Sigma(x,\bar{Q})}{r-\mu} + \frac{\Sigma(x,\bar{Q})^2}{r}\right], & x \in [c,\bar{x}_i), \\ \frac{\bar{q}_i}{1+K(x,\bar{Q})}\left[\frac{x}{r-\mu} - \frac{\Sigma(x,\bar{Q})}{r}\right], & x \in [\bar{x}_i,\infty). \end{cases} \tag{10}$$

The above value takes different forms depending on whether firm i is *currently* idle ($x \in (0,c)$), operating ($x \in [c,\bar{x}_i)$) or constrained ($x \in [\bar{x}_i,\infty)$). When the firm shuts down production to avoid selling below the variable cost c, the value is nil. When demand is larger but the capacity constraint is not (yet) binding, for $x \in [c,\bar{x}_i)$, firm value v_i reflects the perpetuity value of quadratic Cournot profits $(x-\Sigma(x,\bar{Q}))^2/[b(1+K(x,\bar{Q}))^2]$ forever. For large demand ($x > \bar{x}_i$) when firm i is constrained and produces at full capacity $\bar{q}_i$ hence earning linear profits, v_i is the perpetuity value of the linear profits. We decompose perpetuity value v_i in (10) into two terms ν_i^A and ν_i^B as follows:

$$v_i(x,\bar{Q}) = \nu_i^A(x,\bar{Q}) - \nu_i^B(x,\bar{Q}) \tag{11a}$$

where ν_i^A and ν_i^B are

$$\nu_i^{A/B}\left(x,\bar{Q}\right) := \begin{cases} 0, & x \in (0,c), \\ \frac{1}{b[1+K(x,\bar{Q})]^2}\left\{\frac{2-\gamma_{B/A}}{\gamma_A-\gamma_B}\frac{x^2}{r-2\mu-\sigma^2} - \frac{1-\gamma_{B/A}}{\gamma_A-\gamma_B}\frac{2x\Sigma(x,\bar{Q})}{r-\mu} - \frac{\gamma_{B/A}}{\gamma_A-\gamma_B}\frac{\Sigma(x,\bar{Q})^2}{r}\right\}, & x \in [c,\bar{x}_i), \\ \frac{\bar{q}_i}{1+K(x,\bar{Q})}\left\{\frac{1-\gamma_{B/A}}{\gamma_A-\gamma_B}\frac{x}{r-\mu} + \frac{\gamma_{B/A}}{\gamma_A-\gamma_B}\frac{\Sigma(x,\bar{Q})}{r}\right\}, & x \in [\bar{x}_i,\infty), \end{cases} \tag{11b}$$

with γ_A and γ_B given by

$$\gamma_A, \gamma_B := -\frac{\mu-\sigma^2/2}{\sigma^2} \pm \sqrt{\left(\frac{\mu-\sigma^2/2}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \tag{11c}$$

Subscript "A/B" used in (11b) shall read "A" or "B:" Expressions ν_i^A involves γ_B in the numerator, whereas ν_i^B involves γ_A. We have $\gamma_B < 0 < 1 < \gamma_A$. The terms $\nu_i^A(x,\bar{Q})$ and $-\nu_i^B(x,\bar{Q})$ in (11b) distinguish respectively the values on the right of x (the upside) and the values on the left (the downside). In other words, $\nu_i^A(x,\bar{Q})$ is the portion of firm i's perpetuity profit value attributable to scenarios where future demand exceeds current demand level x, (quasi) production costs remain fixed at $\Sigma(x,\bar{Q})$, and $K(x,\bar{Q})$ firms compete on residual demand. Term $-\nu_i^B(x,\bar{Q})$ is the perpetuity portion for scenarios where future demand is lower than current demand x. This decomposition of perpetuity value v_i helps distinguish among cases of demand upsurge or demand contraction. Interestingly, $x \mapsto v_i(x, \bar{Q})$ and $x \mapsto \nu_i^{A/B}(x, \bar{Q})$ are discontinuous at each threshold $\bar{x}_m$ given by (7b). If the capacity constraint of yet another rival firm becomes binding (resp. gets relaxed), this effectively reduces (resp., increases) the number of firms exerting market power and hence greatly impacts the price-setting mechanism; this change explains the lack of regularity of $x \mapsto v_i(x, \bar{Q})$ and $x \mapsto \nu_i^{A/B}(x, \bar{Q})$. We note the jump of $x \mapsto \nu_i^{A/B}(x, \bar{Q})$ at $\bar{x}_m$ by

$$\Delta_i^{A/B}(\bar{x}_m, \bar{Q}) := \nu_i^{A/B}(\bar{x}_m+, \bar{Q}) - \nu_i^{A/B}(\bar{x}_m-, \bar{Q}). \tag{12}$$

with "$-$" and "$+$" denoting the left and right limits, respectively.

As demand builds up or contracts, the firms' capacity constraints become either binding (in demand growth scenarios) or get relaxed (in demand contraction scenarios). Such changes are not accounted for in the perpetuity value v_i in (10) as v_i depends on $\Sigma_0 = \Sigma(x, \bar{Q})$ and $K_0 = K(x, \bar{Q})$ but not on later values of the processes $(K_t; t > 0)$ and $(\Sigma_t; t > 0)$. Owing to output flexibility, total firm value will exceed the above perpetuity value v_i because the firm has flexibility to shut-down and reduce or raise production if it is optimal to do so in the future. Before providing a closed-form expression for total firm value, V_i, in Proposition 1, we set two additional conditions, $\lim_{x \downarrow 0} \frac{V_i(x, \bar{Q})}{v_i(x, \bar{Q})} = 1$ and $\lim_{x \uparrow \infty} \frac{V_i(x, \bar{Q})}{v_i(x, \bar{Q})} = 1$, that ensure the perpetuity value component dominates at either extreme of the demand values.

Proposition 1 (Firm value). *Given a capacity vector $\bar{Q} \in \mathbb{R}_+^k$ and demand state x, firm i's value is*

$$V_i(x, \bar{Q}) = v_i(x, \bar{Q}) + A_i(x, \bar{Q}) x^{\gamma_A} + B_i(x, \bar{Q}) x^{\gamma_B}, \tag{13a}$$

where v_i is given in (10) *and the expansion and loss-reduction benefits are*

$$A_i(x, \bar{Q})\ x^{\gamma_A} := \sum_{m=0}^{k} \mathbb{1}_{\{x \le \bar{x}_m\}} \Delta_i^A(\bar{x}_m, \bar{Q}) \left(\frac{x}{\bar{x}_m}\right)^{\gamma_A}, \tag{13b}$$

$$B_i(x, \bar{Q})\ x^{\gamma_B} := \sum_{m=0}^{k} \mathbb{1}_{\{x \ge \bar{x}_m\}} \Delta_i^B(\bar{x}_m, \bar{Q}) \left(\frac{x}{\bar{x}_m}\right)^{\gamma_B}, \tag{13c}$$

with $\Delta_i^{A/B}(\bar{x}_m, \bar{Q})$ as in (12). *The function V_i is continuously differentiable and convex in x and strictly concave in $\bar{q}_i$.*

Firm value in (13a) consists of three components: the perpetuity value of current profits [v_i in (10)] and the expansion and loss-reduction benefits. *Expansion adjustments* captured by $A_i(x, \bar{Q})\ x^{\gamma_A}$ in (13a) account for scenarios in which demand rises and unconstrained firms scale up production. This value component relates to various changes in the industry structure arising when yet another previously unconstrained firm is suddenly limited by its capacity. The term $\Delta_i^A(\bar{x}_m, \bar{Q})$ corrects for the jump in firm i's perpetuity value at demand threshold $\bar{x}_m$; at once, the number of firms exerting market power (on residual demand) is reduced from $K(\bar{x}_m-, \bar{Q})$ to $K(\bar{x}_m+, \bar{Q})$, whereas unconstrained firms face a (quasi) production cost increase from $\Sigma(\bar{x}_m-, \bar{Q})$ to $\Sigma(\bar{x}_m+, \bar{Q})$. We are able to predict that this change in value will take place at a time $\tau_m^A := \inf\{t \geq 0 \mid X_t \geq \bar{x}_m\}$, which is a stochastic time at which a firm's constraint becomes binding; it would be relatively straightforward to study the probability distribution of this time τ_m^A together with comparative statics. To obtain firm i's current value $V_i(x, \bar{Q})$, one must therefore discount back at a stochastic time. This explains the presence of the term $(x/\bar{x}_m)^{\gamma_A}$ in expression (13b), where $\mathbb{E}[e^{-r\tau_m^A}] = (x/\bar{x}_m)^{\gamma_A}$ (see Dixit and Pindyck 1994, Chapter 9).

The third term, $B(x, \bar{Q})\, x^{\gamma_B}$, in (13a) captures *downside loss-reduction benefits* in scenarios for which demand shrinks and firms contract or shut-down production. McDonald and Siegel (1985) discuss a related case with a single threshold on the downside corresponding to the monopolist's marginal cost. The term in (13a) is linked to various changes in the industry structure. At each stochastic time $\tau_m^B = \inf\{t \geq 0 \mid X_t \leq \bar{x}_m\}$ when future demand falls below demand threshold $\bar{x}_m$ $(< x)$, the capacity constraint of yet another smaller rival gets relaxed. Firm perpetuity value drops—as reflected in $\Delta_i^B(\bar{x}_m, \bar{Q})$—as (quasi) production costs decline and more firms exert market power on residual demand. Such changes in values are discounted back with stochastic discount factor $\mathbb{E}[e^{-r\tau_m^B}] = (x/\bar{x}_m)^{\gamma_B}$ in (13c). This corresponds to the present value of a bond that pays \$1 in the future event that demand falls below $\bar{x}_m$. The above stochastic discount factors provide a clear link between the upfront capacity decision taken at time $t = 0$ (discussed in the next section) and future state payoffs, which is another key benefit of our approach.

Table 1 decomposes firm value V_i in various scenarios for demand (x), the industry structure (k) and demand volatility (σ). More demand x leads to larger firm output and profit, although these amounts are reduced if a firm faces more competitors (k). The perpetuity values also increase with demand x and decline with the number of competitors. Demand volatility influences the perpetuity value through the growth rate of the quadratic term $\left(X_t^2; t \geq 0\right)$. The perpetuity values are inflated when a firm is not constrained because it is assumed the firm will not face a capacity constraint—hence, the use of an "expansion adjustment." The firm is not constrained to produce at a loss when demand falls below the marginal cost—hence the "loss-reduction benefits."

6. Upfront Capacity Choices

We next examine firms' capacity choices at the outset (fixed time 0), which are coupled with market-entry decisions. We first discuss the monopoly case and then we provide the Nash-equilibrium solution for the general asymmetric oligopoly case.

Demand	$x = 10$						$x = 25$					
# of firms	$k = 1$		$k = 2$		$k = 3$		$k = 1$		$k = 2$		$k = 3$	
Output q_k^C	4.5		3.0		2.3		10.0		8.5		6.3	
Profit π_k^C	20.3		9.0		5.1		140.0		72.3		40.1	
Volatility	$\sigma = .1$	.3	.1	.3	.1	.3	.1	.3	.1	.3	.1	.3
Perpetuity v_k	216	2438	96	1084	54	610	1133	1133	723	14612	407	6580
Adjustment $A_k x^{\gamma_A}$	0	-1994	0	-804	0	-402	0	0	-16	-13168	22	-5440
Adjustment $B_k x^{\gamma_A}$	0	-1	0	0	0	0	371	825	68	-17	27	5
Total V_k	216	444	96	279	54	208	1504	1958	775	1426	455	1145

Table 1: **Decomposition of firm value.** $b = 1, \mu = -0.025, \sigma = 0.3, c = 1, r = 0.05, \bar{Q} = 10$ when $k = 1$, $\bar{Q} = (7, 10)^\top$ in case $k = 2$ and $\bar{Q} = (5, 7, 10)^\top$ when $k = 3$.

6.1. Upfront capacity choice in monopoly

We show in Proposition 2 that a monopoly firm faces diminishing marginal returns. Higher capacity $\bar{q}_1$ widens the range of demand values $(c, \bar{x}_1)$ for which the monopoly profit is convex (vs. being linear when the capacity constraint is binding). When future demand X_t is lognormally distributed it remains fairly concentrated in the intermediate demand range (unless demand volatility rises). Raising capacity ($\bar{q}_1$) adds less firm value when capacity is already high as the extra range of demand for which monopoly profits are convex is less likely to be reached. This explains the concavity of firm value V_1 in capacity $\bar{q}_1$ in monopoly.

Proposition 2 (Capacity choice in monopoly). *Monopoly value $\bar{q}_1 \mapsto V_1(x, \bar{q}_1)$ is concave increasing and continuous. If*

$$\varphi(r, \mu, \sigma) := \frac{2}{r - \mu} \frac{1}{(2 - \gamma_B)\gamma_B \kappa_B} > 1, \tag{14}$$

then a monopoly firm benefiting from output flexibility invests at the outset in $\bar{q}_1^C(x_0) = \hat{q}(x_0)\mathbb{1}_{\{x_0 \geq \bar{x}_\star\}}$ units of capacity, where

$$\bar{x}_\star = \begin{cases} \left(\frac{2c_1}{c(\gamma_A - 2)\kappa_A}\right)^{1/\gamma_A} c, & 0 \leq f(0) \\ \{x \mid \frac{x}{r-\mu} - \frac{c}{r} - (2 - \gamma_B)\frac{c^{1-\gamma_B}}{2}\kappa_B x^{\gamma_B} = c_1\}, & 0 > f(0) \end{cases} \tag{15a}$$

$$\hat{q}(x) = \begin{cases} \left\{\bar{q}_1 \mid \frac{\kappa_A}{2}(\gamma_A - 2)x^{\gamma_A}(c + 2b\bar{q}_1)^{1-\gamma_A} = c_1\right\}, & f\left(\frac{x-c}{2b}\right) \geq 0, \\ \{\bar{q}_1 \mid \frac{x}{r-\mu} - \frac{c+2b\bar{q}_1}{r} + \frac{\kappa_B}{2}(\gamma_B - 2)x^{\gamma_B}(c + 2b\bar{q}_1)^{1-\gamma_B} = c_1\}, & f\left(\frac{x-c}{2b}\right) < 0. \end{cases} \tag{15b}$$

$$f(\bar{q}_1) := \frac{\kappa_A}{2}(\gamma_A - 2)(c + 2b\bar{q}_1) - c_1 \tag{15c}$$

$$\kappa_{A/B} := \frac{1}{\gamma_A - \gamma_B}\left[\frac{2 - \gamma_{B/A}}{r - 2\mu - \sigma^2} - 2\frac{1 - \gamma_{B/A}}{r - \mu} - \frac{\gamma_{B/A}}{r}\right]. \tag{15d}$$

The above analytic expression for the monopoly firm's capacity choice at $t = 0$ allows us to decouple (i) the market-entry decision (no investment if demand is lower than a certain threshold, $\bar{x}_\star$) and (ii) the capacity "investment intensity" policy prescribing how much capacity to install provided investment takes place. The monopolist's optimal capacity choice depends on initial demand x_0 because the latter drives the distribution of future demand. The monopolist firm is better off not entering the market—with $\bar{q}_1^C(x_0) = 0$—if initial demand is low ($x_0 < \bar{x}_\star$).[5] A larger capacity cost (c_1) or unit production cost (c) raises the market-entry threshold $\bar{x}_\star$. If the firm enters (i.e., if $x_0 > \bar{x}_\star$), it installs a larger capacity $\hat{q}(x_0)$ when demand x_0 is larger, production costs (c) are lower or the unit capacity cost c_1 is lower.

Figure 1 illustrates the effect of demand volatility σ on the monopolist's market-entry and capacity choices assuming the capacity cost function is linear. The marginal value ($\partial V_1/\partial \bar{q}_1$) decreases less steeply under high demand volatility ($\sigma = 0.3$ vs. $\sigma = 0.1$ in panel a) because large demand states are more likely to be reached. Increased demand volatility σ leads to larger capacity investment $\bar{q}_1^C(x_0)$ because the marginal value, $\partial V_1/\partial \bar{q}_1$, increases with volatility σ, whereas marginal capacity cost remains fixed at c_1.

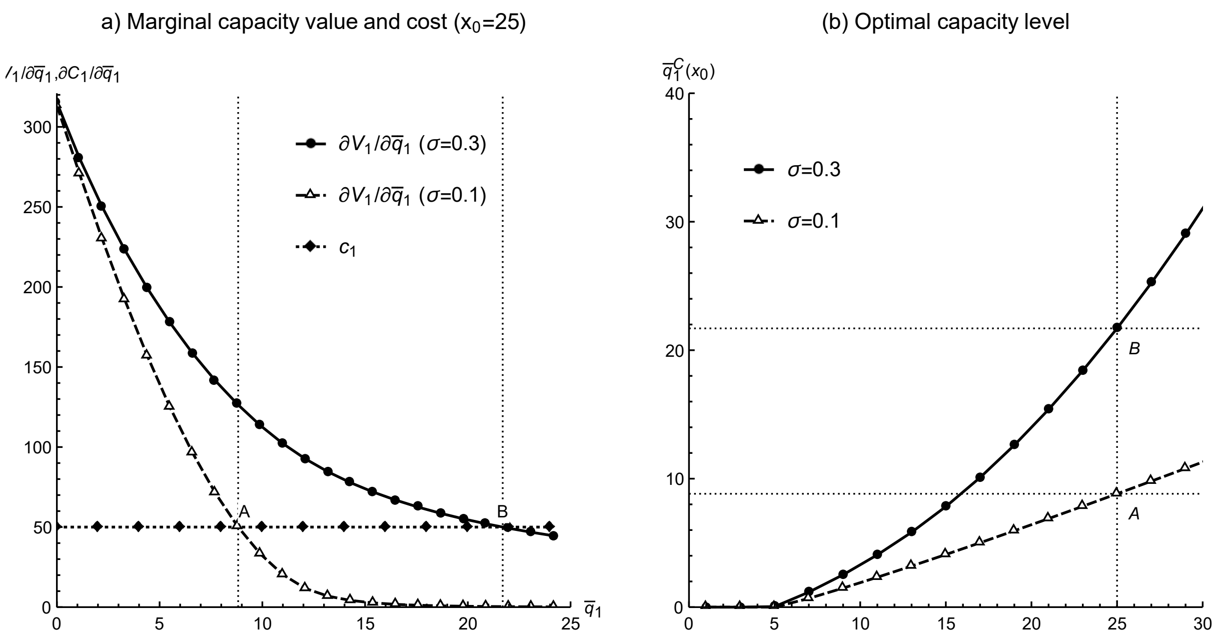

Figure 1: **Monopolist's marginal value of capacity $\partial V_1/\partial \bar{q}_1$ and optimal capacity choice $\bar{q}_1^C(x_0)$ at varying volatility levels.** $b = 1, c = 1, \mu = -0.025, r = 0.05, c_1 = 50$

As the net present value embeds the flexibility to expand output, which is more valuable at higher demand volatility, the firm invests more in capacity $\bar{q}_1$ until the "uplifted" marginal value ($\partial V_1/\partial \bar{q}_1$) just equals marginal cost c_1. This occurs at a higher capacity level $\bar{q}_1$, for instance at point B for $\sigma = 0.3$ vs. point A for $\sigma = 0.1$, as illustrated in panels (a) and (b) of Figure 1. In this illustrative case, which precludes a defer option, demand volatility has a limited impact on the monopolist's market-entry decision ($\bar{x}_\star$ in panel

[5] This effect arises because, given our assumption of linear demand, the Inada condition is not satisfied at 0, so the monopoly's (global) optimal capacity choice can be zero.

b).[6]

Our finding on the beneficial impact of demand volatility on initially installed capacity differs from Huisman and Kort (2015) [equation 5] where the capacity choice is independent of demand volatility σ. This is because the demand shock affects the market-clearing price in a multiplicative fashion and firms are assumed to produce at full capacity ($Q_t = \bar{q}_1$) at all times.[7] In our case, output flexibility gives an incentive to install larger upfront capacity when demand volatility is high, so the firm enjoys more payoff convexity because it can expand output further (within set limits) and can wield extra market power (quadratic vs. linear profit).

The dependence of the upfront capacity choice on initial demand x_0 renders the monopolist's optimal output choice and value path-dependent. We note $\hat{q}_1(x, x_0) := q_1^C\left(x, \bar{q}_1^C(x_0)\right)$ and $\hat{V}_1(x, x_0) := V_1\left(x, \bar{q}_1^C(x_0)\right)$, with V_1 given in (13) for $k = 1$. Figure 2 depicts firm value $(x, x_0) \mapsto \hat{V}_1(x, x_0)$ under distinct drift assumptions (μ). When demand grows at a higher rate ($\mu = 0.01$ in panel c vs. $\mu = -0.025$ in panel a), the monopoly firm has more incentive to enter ($\bar{x}_\star$ is lower) and install a larger capacity, which makes its value steeper in x and x_0. The case depicted in panel a—which corresponds to the situation faced by rivals in a fast-paced industry where demand is on the decline (e.g., due to technological obsolescence as in Hagspiel et al. 2016b, 2020)—will be used as base case in the subsequent illustrations.

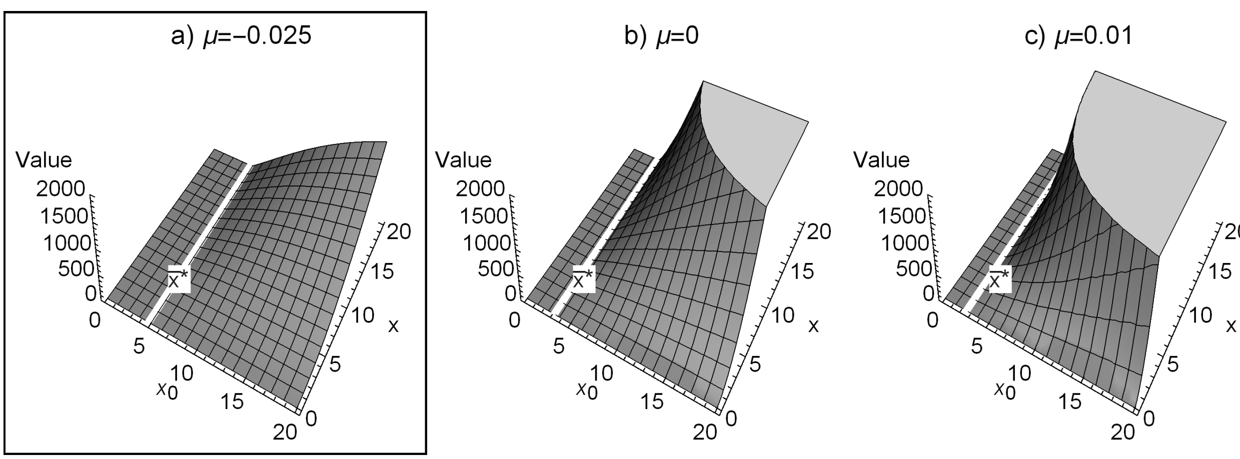

Figure 2: **Monopoly value $\hat{V}_1$ for different levels of initial demand (x_0), current demand (x) and drift term μ.** $b = 1, \sigma = 0.3, c = 1, r = 0.05, c_1 = 50$.

[6]We do not claim that demand volatility has no impact, but that its impact is insignificant *relative* to the impact on the optimal lump-sum capacity installments.

[7]Putting the expectation operator within the integral in the monopolist's objective function of Huisman and Kort (2015, equation 3) suggests that optimal capacity will depend on expected demand but not on demand volatility (terms in X_t have power 0 or 1, not 2).

6.2. Upfront capacity choices in oligopoly

In Proposition 1 we obtained the values of the firms for given capacities $\bar{Q}$ and demand size x. Proposition 3 specifies properties of the best-reply functions and proves the existence and uniqueness of a Nash-Equilibrium capacity vector $\bar{Q}^C(x_0)$. These results were obtained using analytic techniques.

Proposition 3 (Capacity choices in oligopoly). *Let $\bar{x}^i_\star(\bar{q}_{-i})$ denote the unique solution of $\frac{\partial V_i}{\partial \bar{q}_i}\big(x_0, (0, \bar{q}_{-i})^\top\big) = c_i$ in x_0. Further, if $x_0 > \bar{x}^i_\star(\bar{q}_{-i})$, let $\mathcal{R}_i(x_0, \bar{q}_{-i})$ denote the unique solution of $\frac{\partial V_i}{\partial \bar{q}_i}\big(x_0, \bar{Q}\big) = c_i$ in $\bar{q}_i$. Firm i's best-reply function is then given by*

$$\bar{q}_{-i} \mapsto \mathrm{R}_i\big(x_0, \bar{q}_{-i}\big) = \mathcal{R}_i(x_0, \bar{q}_{-i}) \mathbb{1}_{\{x_0 > \bar{x}^i_\star(\bar{q}_{-i})\}}, \quad i \in \{1, \ldots, k\}.$$

If $c_i < c_j$, the thresholds and investment amounts satisfy the inequalities $\bar{x}^i_\star(\cdot) < \bar{x}^j_\star(\cdot)$ and $\mathcal{R}_i(x_0, \cdot) > \mathcal{R}_j(x_0, \cdot)$, respectively. A firm $i \in \{1, \ldots, k\}$ is less inclined to invest and would install less capacity if rival firms invest more, in the sense that $\partial \bar{x}^i_\star / \partial \bar{q}_j > 0$ and $\partial \mathcal{R}_i / \partial \bar{q}_j < 0$, respectively.
The Nash-Equilibrium (NE) capacity vector $\bar{Q}^C(x_0) \in \mathbb{R}^k_+$ is given by

- *Case $k = 2$. The pair $\bar{Q}^C(x_0) = (0, 0)^\top$ is the unique NE iff $x_0 \leq \bar{x}^2_\star(0)$. The pair $\bar{Q}^C(x_0) = \big(0, \mathcal{R}_2(x_0, 0)\big)^\top$ is the unique NE iff $\bar{x}^2_\star\big(0\big) < x_0 < \bar{x}^1_\star(\mathcal{R}_2(x_0, 0))$. The fixed point $\bar{Q}^C(x_0) \in \mathbb{R}^2_+$ of the map $\bar{q} \mapsto \big(\mathcal{R}_1(x_0, \bar{q}_2), (\mathcal{R}_2(x_0, \bar{q}_1)\big)$ is the unique NE iff $x_0 > \bar{x}^1_\star(\bar{Q}^C_2(x_0))$. The equilibrium capacities satisfy $\bar{Q}^C_2(x_0) \geq \bar{Q}^C_1(x_0)$.*

- *Case $k \in \{3, \ldots\}$. If $\sum_{j \neq i} \left| \frac{\partial \mathcal{R}_j}{\partial \bar{q}_j}(q) \right| < 1$ for all $q \in \mathbb{R}^k$ and all $i \in \{1, \ldots, k\}$, then the fixed point $\bar{Q}^C(x_0)$ of the map $\bar{q} \mapsto \big(\mathcal{R}_1(x_0, \bar{q}_{-1}), \ldots, \mathcal{R}_k(x_0, \bar{q}_{-k})\big)$ is the unique NE iff $x_0 > \bar{x}^1_\star(\bar{Q}^C_{-1}(x_0))$. When demand is lower, it may be that some firms decide not to install capacity as is the case when $k = 2$. The NE satisfies the ranking $\bar{x}^k_\star\big(\bar{Q}^C_{-k}(x_0)\big) \leq \cdots \leq \bar{x}^1_\star\big(\bar{Q}^C_{-1}(x_0)\big)$ and $\bar{Q}^C_k(x_0) \geq \cdots \geq \bar{Q}^C_1(x_0)$.*

Further, the equilibrium output policy $x \mapsto \hat{Q}(x, x_0) := Q^C\big(x, \bar{Q}^C(x_0)\big)$ and value function $x \mapsto \hat{V}_i(x, x_0) := V_i\big(x, \bar{Q}^C(x_0)\big)$ are path-dependent (as they depend on x_0).

Several properties are noteworthy. First, capacity choices are "strategic substitutes" (i.e., $\partial \mathcal{R}_i / \partial \bar{q}_j < 0$). Output choices are known to be strategic substitutes in one-shot unconstrained Cournot oligopoly (see Tirole 1988). As shown above, this property generalizes to an asymmetric oligopoly setup where firms face capacity constraints over time.[8] Second, because the function $\bar{q}_i \mapsto \bar{x}^i_\star\big(\mathcal{R}_{-i}(x_0, \bar{q}_i)\big)$ is monotone decreasing,

[8] The economic intuition for this result is the following. Consider a duopoly ($k = 2$) where two firms are endowed with capacities $\bar{q}_1$ and $\bar{q}_2$ ($> \bar{q}_1$). If the smaller firm (firm 1) raises capacity $\bar{q}_1$ by a small amount ε, the range of demand values $[\bar{x}_1 + 2b\varepsilon, \bar{x}_2 + b\varepsilon]$ at which the larger rival (firm 2) exerts monopoly market power on residual demand narrows. Further, firm 1 supplies more when both firms are constrained for $x \geq \bar{x}_2$, which dampens firm 2's profit. Consequently, firm 2's marginal value, $\partial V_2 / \partial \bar{q}_2$, being a "weighted average" over the various demand regions, declines—whereas the marginal capacity cost remains constant—and the larger firm reacts by reducing its capacity $\bar{q}_2$. The smaller firm would similarly reduce its capacity if the larger rival raises it.

the cost leader benefits from a "strategic effect"—in the sense of Fudenberg and Tirole (1984)—from raising its production capacity: by raising it, rivals will decide to invest less (because capacities are strategic substitutes), so the cost leader's threshold $\bar{x}^i_\star\big(\mathcal{R}_{-i}(x_0, \bar{q}_i)\big)$ is reduced while the rivals' thresholds are increased. This effect may preclude entry by less efficient firms. Finally, the ranking of the unit capacity costs $c_1 \geq \cdots \geq c_k$ maps into a ranking of entry thresholds $\bar{x}^k_\star\big(\bar{Q}^C_{-k}(x_0)\big) \leq \cdots \leq \bar{x}^1_\star\big(\bar{Q}^C_{-1}(x_0)\big)$ and of capacity installments $\bar{Q}^C_k(x_0) \geq \cdots \geq \bar{Q}^C_1(x_0)$. This result is less straightforward in our case because we account for the possibility of some firms getting capacity constrained in the future.

To illustrate the equilibrium capacity choices, consider a scenario in which the unit cost of capacity is

$$c_i = 50 + \mathbb{1}_{\{k>1\}} \frac{k-i}{k-1} \chi. \tag{16}$$

Here, the parameter χ (≥ 0) captures the degree of cost heterogeneity among the oligopoly firms. We note that the Nash Equilibrium is unique. We can construct a Cauchy sequence $\{\bar{Q}^n(x_0); n \in \mathbb{N}\}$ of capacity choices: the capacities $\bar{Q}^n(x_0)$ start at an arbitrary vector, say $\bar{Q}^0(x_0) = (0, \ldots, 0)^\top$, and are replaced in each round by the best replies to the capacity vector from the previous round, viz. with the recursive equation $\bar{Q}^{n+1}(x_0) = \big(\mathrm{R}_1(x_0, \bar{Q}^n_{-1}(x_0)), \ldots, \mathrm{R}_k(x_0, \bar{Q}^n_{-k}(x_0))\big)^\top$. In line with the contraction-mapping theorem, this sequence will converge to the unique NE capacity choice $\bar{Q}^C(x_0)$ as n goes to ∞. This numerical procedure is known as "Cournot tâtonnement process" (see Tirole 1988). Figure 3 illustrates this process for an oligopoly with $k = 3$ firms facing cost heterogeneity of increasing degree χ. We use this procedure to compute the NE capacity choices.

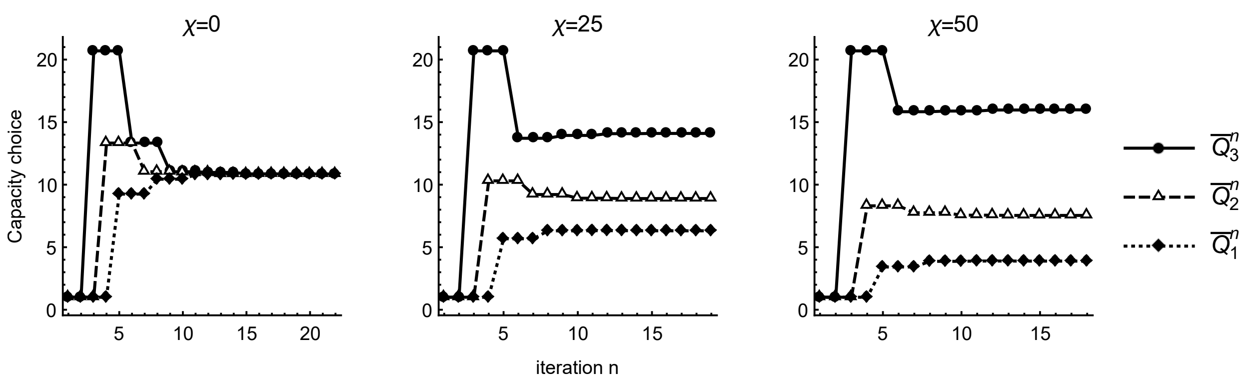

Figure 3: **Illustration of "Cournot tâtonnement process."** $b = 1$, $c = 1$, $\mu = -0.025$, $r = 0.05$, $c_3 = 50$, $x_0 = 25$.

The payoff expressions in Proposition 1 embed changes to the industry structure with some capacity constraints becoming binding or relaxed as demand declines or builds up over time. Based on an oligopoly with three firms deciding on their capacity choices (see Proposition 3), Figure 4 depicts the industry changes along two exemplary sample paths $t \mapsto X_t(\omega_m), m = 1, 2$. Whenever the demand process hits a threshold

$\bar{x}_k,\ k = 1,2,3$ from below, another firm becomes capacity constrained. If it falls below such a threshold, a capacity constraint is relaxed. A discrete-time model would not allow such a representation of the probabilistic evolution of an industry.

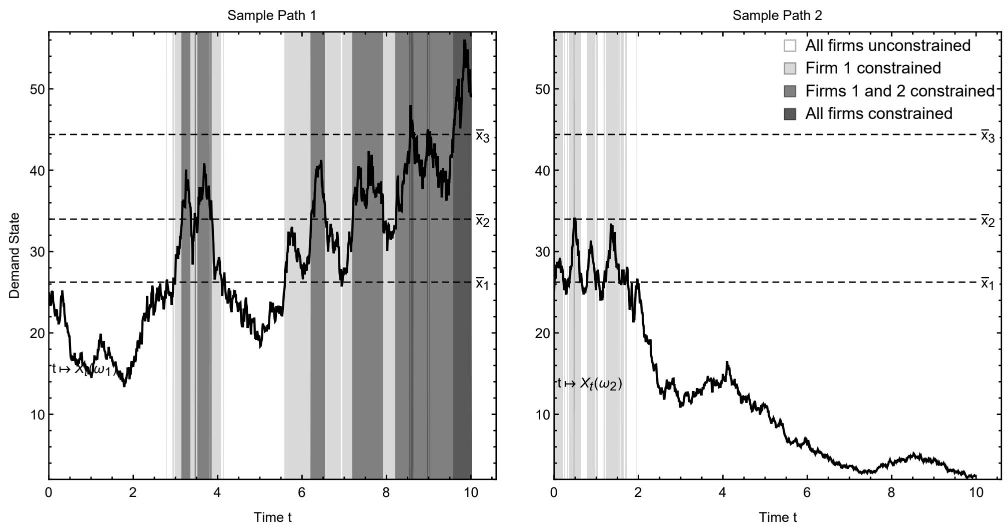

Figure 4: **Industry evolution along two exemplary (demand) sample paths.** $b = 1,\ c = 1,\ \mu = -0.025,\ r = 0.05,$ $C_3(\bar{q}_3) = 50 \times \bar{q}_3,\ \chi = 25,\ x_0 = 25.$ Capacity levels selected according to the Nash equilibrium condition from Proposition 3.

Figure 5 depicts firm k's initial capacity choice in Nash equilibrium depending on the number of rivals $(k = 1, , \ldots, 7)$ and the demand volatility (σ) it faces.

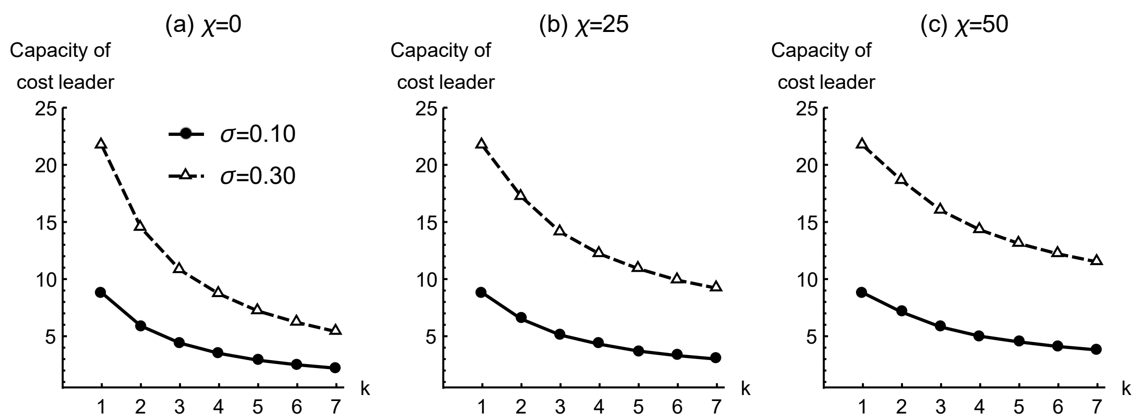

Figure 5: **Effect of cost heterogeneity χ and demand volatility σ on firm k's capacity choice for different industry structures.** $x_0 = 25, b = 1,\ c = 1,\ c_k = 50, \mu = -0.025, r = 0.05$

Panel (a) with no heterogeneity ($\chi = 0$) shows the symmetric benchmark, while panels (b) and (c) with $\chi = 25$ and $\chi = 75$, respectively, represent settings where firm k is the cost leader. A number of observations are noteworthy. First, as the number of operating firms increases, the cost leader gradually loses market power on the output market and, consequently, invests less. Further, a symmetric firm ($\chi = 0$) invests in greater capacity when demand volatility is higher ($\sigma = 0.3$ vs. $\sigma = 0.1$). Greater asymmetry leads to greater investment by the cost leader (panel c vs. b), which gets reinforced as demand volatility gets higher. The cost leader invests more to benefit both from the enhanced firm value convexity arising from output expansion flexibility and from greater market power in the intermediate demand regions.

7. Industry concentration and social welfare implications

Industry concentration. We next examine industry concentration measured by the Herfindahl-Hirschman Index (HHI), given by

$$\mathcal{H}(\bar{Q}) := \sum_{i=1}^{k} \left(\frac{\bar{q}_i}{\bar{Q} \cdot \mathbf{1}_k} \right)^2 .$$

Figure 6 depicts the impact of cost heterogeneity (panels a, b, and c with $\chi = 25$, 50 and 75 respectively) and demand volatility ($\sigma = 0.1$ vs. 0.3) on industry concentration with $k = 3$ firms. Ex-ante cost symmetry (dashed horizontal line) leads to identical capacity choices by all firms: the HHI is constant at $1/k$ irrespective of the degree of demand volatility. Cost heterogeneity leads to a more concentrated industry, especially when demand is limited and uncertainty is substantial. Industry concentration evolves in stages driven by rival firms' market-entry decisions (thresholds $\bar{x}^i_\star$). Firm $k = 3$ and its less efficient rivals (firms 1 and 2) stay put if demand is below $\bar{x}^3_\star$. For demand above this level and below $\bar{x}^2_\star$, firm 3, the cost leader, will be the only market entrant. For larger demand, rivals also invest but the cost leader invests significantly more. Less efficient rivals (firms 1 and 2) invest less because capacities are strategic substitutes, but also because their marginal investment costs increase (with cost heterogeneity χ). As demand builds up, industry concentration is reduced and stabilizes at a level greater than the symmetric benchmark. Firm asymmetry is furthermore exacerbated by demand volatility.

Social welfare. We next analyze whether the above strategic interactions among k firms competing in a constrained Cournot oligopoly yield results beneficial to society. The gross *producer surplus* (PS), given by

$$\mathcal{PS}\big(x, Q(\cdot)\big) := \mathbb{E}\left[\int_0^\infty e^{-rt}\left[p\left(X_t, Q_t\right) - c\right] Q_t \ \mathrm{d}t\right], \tag{17}$$

represents the expected aggregate value of all k firms for a given output policy $Q(\cdot)$. We here define producer surplus and social welfare in gross terms, i.e., not net of investment costs. Figure 8 depicts net social welfare in different scenarios as of time $t = 0$ where $x = x_0$. In a constrained oligopoly, producer surplus is

$$\mathcal{PS}\big(x, \hat{Q}(\cdot, x_0)\big) = \sum_{i=1}^{k} \hat{V}_i\left(x, x_0\right). \tag{18}$$

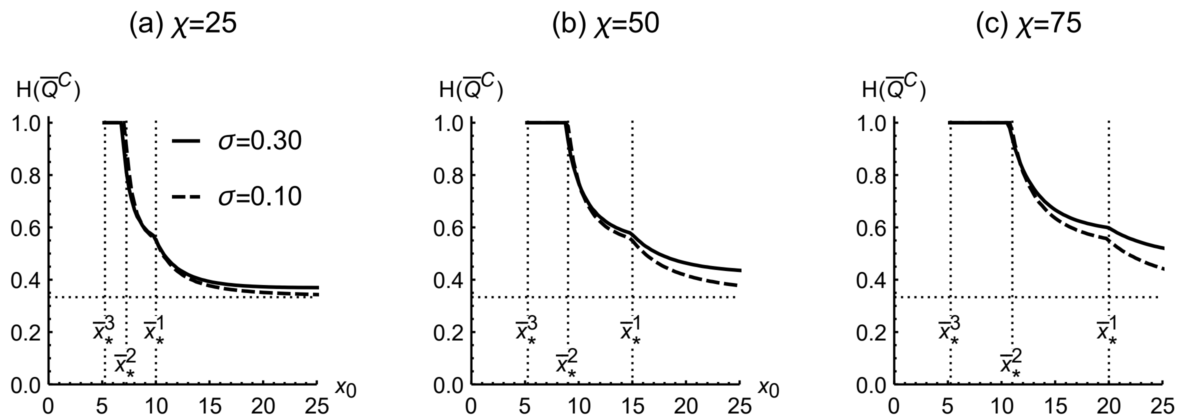

Figure 6: **Herfindahl-Hirschmann Index $\mathcal{H}(\bar{Q}^C)$ in a triopoly for different levels of cost heterogeneity χ and demand volatility σ as initial demand x_0 varies.** $c = 1$, $b = 1, c_3 = 50, \mu = -0.025, r = 0.05$

with $\hat{Q}(\cdot, x_0)$ and $\hat{V}_i(x, x_0)$ as given in Proposition 3.

We define the demand function $p \mapsto D(x, p)$ as the inverse of the function $q \mapsto p(x, q)$ in (3). For a given output policy $Q(\cdot)$ satisfying (2), *consumer surplus* (CS) is

$$\mathcal{CS}(x, Q(\cdot)) := \mathbb{E}\left[\int_0^\infty e^{-rt}\left\{\int_{p(X_t,Q_t)}^{X_t} D(X_t, p)\,\mathrm{d}p\right\}\,\mathrm{d}t\right], \tag{19}$$

i.e., the expected discounted "sum" of the consumer surpluses achieved in the states (X_t, Q_t). Proposition 4 below describes the consumer surplus attained in this constrained Cournot oligopoly.

Proposition 4 (Consumer surplus). *In a constrained oligopoly, consumer surplus is*

$$\mathcal{CS}\big(x, \hat{Q}(\cdot, x_0)\big) = s(x, x_0) + A_s(x, x_0)\ x^{\gamma_A} + B_s(x, x_0)\ x^{\gamma_B},$$

with $\hat{Q}(\cdot, x_0)$ defined in Proposition 3. The perpetuity, upside potential benefits and downside loss reduction value terms are, respectively,

$$s(x, x_0) = \frac{\rho_m^2 x^2}{r - 2\mu - \sigma^2} + \frac{\rho_m^1 x}{r - \mu} + \frac{\rho_m^0}{r}, x \in [\bar{x}_{m-1}^C, \bar{x}_m^C], m = 0, \cdots, k+1, \tag{20a}$$

$$A_s(x, x_0)\ x^{\gamma_A} = \sum_{m=0}^{k} \mathbb{1}_{\{x \le \bar{x}_m^C\}} \alpha_s^m \left(\frac{x}{\bar{x}_m^C}\right)^{\gamma_A} \tag{20b}$$

$$B_s(x, x_0)\ x^{\gamma_B} = \sum_{m=0}^{k} \mathbb{1}_{\{x \ge \bar{x}_m^C\}} \beta_s^m \left(\frac{x}{\bar{x}_m^C}\right)^{\gamma_B} \tag{20c}$$

with

$$\alpha_s^m, \beta_s^m := \eta_{A/B}\left(\bar{x}_m^C, \rho_{m+1}\right) - \eta_{A/B}\left(\bar{x}_m^C, \rho_m\right), \quad m = 0, \cdots, k, \tag{20d}$$

$$\eta_{A/B}(x,\rho) := \frac{1}{\gamma_A - \gamma_B}\left\{\frac{2-\gamma_{B/A}}{r-2\mu-\sigma^2}\rho^2 x^2 + \frac{1-\gamma_{B/A}}{r-\mu}\rho^1 x - \frac{\gamma_{B/A}}{r}\rho^0\right\}, \tag{20e}$$

$\rho_0 = (0,0,0)$*;* $\rho_m = (\rho_m^2, \rho_m^1, \rho_m^0)$ *for* $m \geq 1$ *is a vector with components*

$$\rho_m^j := \begin{cases} \frac{1}{2b}\left[\frac{K_m}{K_{m-1}}\right]^2, & j = 2, \\ \frac{K_m}{2K_{m-1}}\left[\frac{\Sigma_m}{K_{m-1}} - c\right], & j = 1, \\ \frac{1}{2b}\left[\frac{\Sigma_m}{K_{m-1}} - c\right]^2, & j = 0. \end{cases} \tag{20f}$$

Social welfare is the sum of producer and consumer surplus, namely

$$\mathcal{W} := \mathcal{PS} + \mathcal{CS}, \tag{21}$$

with $\mathcal{PS}$ and $\mathcal{CS}$ as given in (17) and (19), respectively. While consumer surplus increases with demand x in a constrained oligopoly, it does so at a lower rate than producer surplus, with the share of consumer surplus to social welfare decreasing for larger demand x. This is because consumer surplus is negatively affected by an increase in the output price that occurs when the demand intercept, x, goes up. The producer surplus is more convex, with the exercise of market power being a lever of value convexity for oligopoly firms. The *social optimum*, $W(x, x_0)$, is the optimal social welfare achieved when the social planner maximizes over capacity at the outset and subsequently over output subject to capacity constraint (2).

Social optimum. Proposition 5 describes the gross welfare in social optimum. We assume investment cost $C(\bar{Q}) = \sum_{i=1}^k C_i(\bar{q}_i)$ to determine $\bar{Q}^W(x_0)$. Assuming asymmetric investment costs results in the social planner putting all the "weight" on the cost leader. This assumption is reasonable if the social planner is interested to promote the most efficient production technology.

Proposition 5 (Social optimum). *The optimum (gross) social welfare is*

$$W(x, x_0) = w\left(x, x_0\right) + A_w(x, x_0)\ \ x^{\gamma_A} + B_w(x, x_0)\ \ x^{\gamma_B}, \tag{22a}$$

where w is the perpetuity value to society of sustaining the (optimal) status quo, given by

$$w\left(x, x_0\right)=\begin{cases}0, & x \in(0, c), \\ \frac{1}{2 b}\left[\frac{x^2}{r-2 \mu-\sigma^2}-\frac{2 c x}{r-\mu}+\frac{c^2}{r}\right], & x \in\left[c, \bar{x}^W\right], \\ \frac{x \bar{Q}^W\left(x_0\right)}{r-\mu}-\frac{\bar{Q}^W\left(x_0\right)}{r}\left[c+\frac{b}{2} \bar{Q}^W\left(x_0\right)\right], & x \in\left(\bar{x}^W, \infty\right),\end{cases} \tag{22b}$$

with $\bar{x}^W := c + b\bar{Q}^W(x_0)$ the demand threshold above which the social planner is constrained. The amounts $A_w(x, x_0)$ and $B_w(x, x_0)$ reflect upside value potential and downside loss reduction:

$$A_w(x, x_0)\ x^{\gamma_A} = \frac{\kappa_A}{2b}\left[\mathbb{1}_{\{x \leq c\}} c^2 \left(\frac{x}{c}\right)^{\gamma_A} - \mathbb{1}_{\{x \leq \bar{x}^W\}} \bar{x}^{W2} \left(\frac{x}{\bar{x}^W}\right)^{\gamma_A}\right] \tag{22c}$$

$$B_w(x, x_0)\ x^{\gamma_B} = \frac{\kappa_B}{2b}\left[\mathbb{1}_{\{x \geq c\}} c^2 \left(\frac{x}{c}\right)^{\gamma_B} - \mathbb{1}_{\{x \geq \bar{x}^W\}} \bar{x}^{W2} \left(\frac{x}{\bar{x}^W}\right)^{\gamma_B}\right]. \tag{22d}$$

Parameters κ_A and κ_B are given in Equation (15d). *The socially optimal capacity $\bar{Q}^W(x_0)$ is obtained numerically by first-order considerations.*

Table 2 leverages on this social optimum characterization to assess numerically the efficiency loss from cost-heterogeneous oligopolistic competition. The investment size increases in the initial demand level x_0 and in demand volatility σ and decreases in the number of oligopoly firms k. A monopolist ($k = 1$) installs exactly 50% of the socially optimal capacity independently of the demand level x_0 and demand volatility σ. Under other (asymmetric) oligopoly structures, the shares to the socially optimal capacity increase with initial demand x_0 and decrease with demand volatility σ. Figure 7 illustrates aggregate industry capacity in oligopoly as a share of the social optimum capacity for various degrees of cost heterogeneity (χ) and demand volatility (σ) as the industry configuration (k) varies. As expected, increased competition (i.e., larger k) brings aggregate capacity closer to the social optimum capacity. Under symmetric investment costs [panel (a) with $\chi = 0$], the share of aggregate industry *capacity* (to socially optimal capacity) is the same as the share of aggregate *output* (to the socially optimal output) in the classical static Cournot model. This share does not depend on the level of demand volatility σ because the social planner and the oligopolistic firms have homogeneous expectations about demand prospects. Under cost heterogeneity [panel (b) with $\chi = 25$ or panel (c) with 75], aggregate industry capacity represents a smaller share of the socially optimal capacity. This share gets smaller when demand volatility σ is high as it leads to more dispersion in firm capacities—additional investment by the cost leader does not fully compensate the lower amounts invested by rivals.

Panel (a) in Figure 8 shows the impact of industry structure (i.e., the number of firms k) on *net* social welfare. Although a monopolist ($k = 1$) invests 50% of the social optimum benchmark (as per Figure 7), its welfare contribution is larger (75%). Each capacity unit in monopoly has a greater welfare contribution than

		Cost leader's capacity $\bar{q}_k^c(x_0)$			Oligopoly capacity			Share of social optimum capacity		
k	σ	$x_0 = 20$	$x_0 = 25$	$x_0 = 30$	$x_0 = 20$	$x_0 = 25$	$x_0 = 30$	$x_0 = 20$	$x_0 = 25$	$x_0 = 30$
1	0.1	6.4	8.8	11.3	6.4	8.8	11.3	50.0%	50.0%	50.0%
	0.3	14.0	21.7	31.0	14.0	21.7	31.0	50.0%	50.0%	50.0%
2	0.1	4.9	6.5	8.2	8.0	11.2	14.4	62.2%	63.3%	63.9%
	0.3	11.1	17.2	24.5	16.8	26.2	37.5	60.2%	60.4%	60.5%
3	0.1	3.9	5.1	6.4	9.0	12.5	16.2	69.9%	71.1%	71.8%
	0.3	9.1	14.1	20.1	34.1	52.8	75.3	67.3%	67.5%	67.6%

Table 2: **Capacity choices under asymmetric oligopolistic competition and under social optimum depending on industry structure (k), initial demand x_0 and demand volatility σ.** $\mu = -0.025$, $r = 0.05$, $c_k = 50$, $\chi = 25$, $b = 1$, $c = 1$.

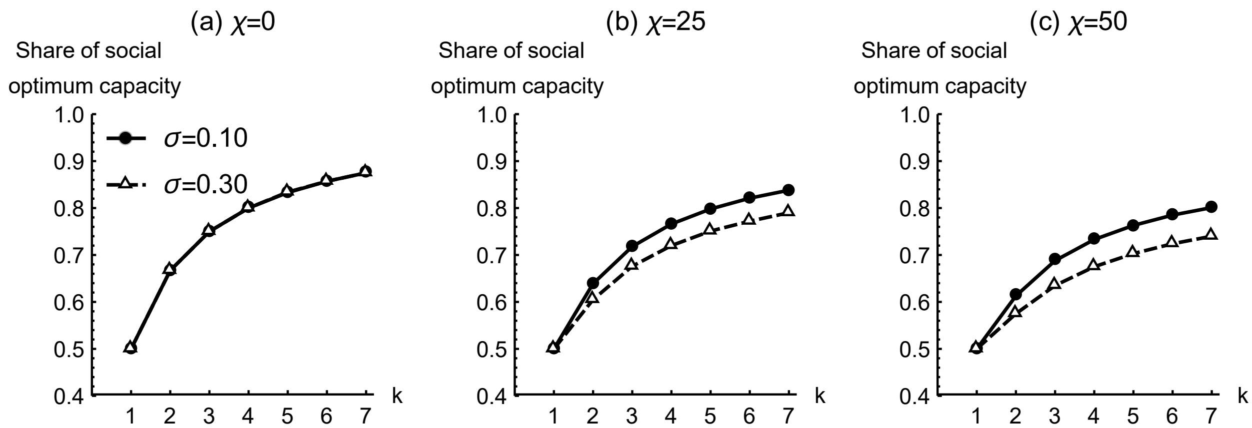

Figure 7: **Total industry capacity relative to the social optimum level for varying cost heterogeneity χ, demand volatility σ and industry structure (number of firms k).** $x_0 = 25, b = 1, c = 1, c_k = 50, \mu = -0.025, r = 0.05$.

the average unit in the social optimum. This result (known from standard microeconomics) remains valid here because the social planner and monopolist have homogeneous expectations. Under cost symmetry ($\chi = 0$), promoting competition (a larger k) is unsurprisingly always beneficial with respect to social welfare. Under cost heterogeneity, however, more competition can reduce welfare depending on the degree of heterogeneity. In panel (b) the net welfare effect of encouraging a duopoly (over a monopoly) is positive if cost heterogeneity is limited (for $\chi < \chi_2^\star$), but becomes negative (or nil at best) if cost heterogeneity is more pronounced (for $\chi \geq \chi_2^\star$), with point C being below the monopoly value (75% of the social optimum). The term $\chi_2^\star$ denotes the cost heterogeneity level at which society is indifferent between monopoly ($k = 1$) and duopoly ($k = 2$). Above the level $\chi_2^{\star\star}$, (ex-ante) duopoly yields monopoly welfare. The set $[\chi_2^\star, \chi_2^{\star\star})$ specifies the range of cost heterogeneity values for which the negative (net welfare) effect dominates. Duopoly welfare falls below monopoly welfare here because a capacity installment by the less efficient firm entails a larger average unit investment cost. Although cost heterogeneity induces larger industry concentration and may restrict consumer surplus, it may lead to a lower average investment cost (per capacity unit), which is beneficial to total welfare, because the large-cost firm invests less than the low-cost firm. When cost heterogeneity is very large ($\chi \geq \chi_2^{\star\star}$), the less efficient firm is discouraged from investing altogether, with the industry becoming a virtual monopoly. Panel (b) also illustrates the effect of uncertainty on net welfare. When heterogeneity is limited, increased volatility (from $\sigma = 0.1$ to 0.3) reduces welfare. This is because higher volatility leads to larger industry concentration and depresses consumer surplus *ceteris paribus*. Threshold $\chi_2^{\star}$ shifts to the left, whereas the set of heterogeneity values $[\chi_2^{\star}, \chi_2^{\star\star})$ for which duopoly leads to a welfare loss (compared to monopoly) broadens. Policy makers should thus be more cautious and refrain from merely promoting competition (increasing k) in industries where firms are highly heterogeneous and face an uncertain environment. Industry concentration may have beneficial effects on upfront cost efficiency, especially when demand is uncertain. Further, providing subsidies to industry frontrunners may enlarge asymmetry and thus increase production capacity in the whole industry.

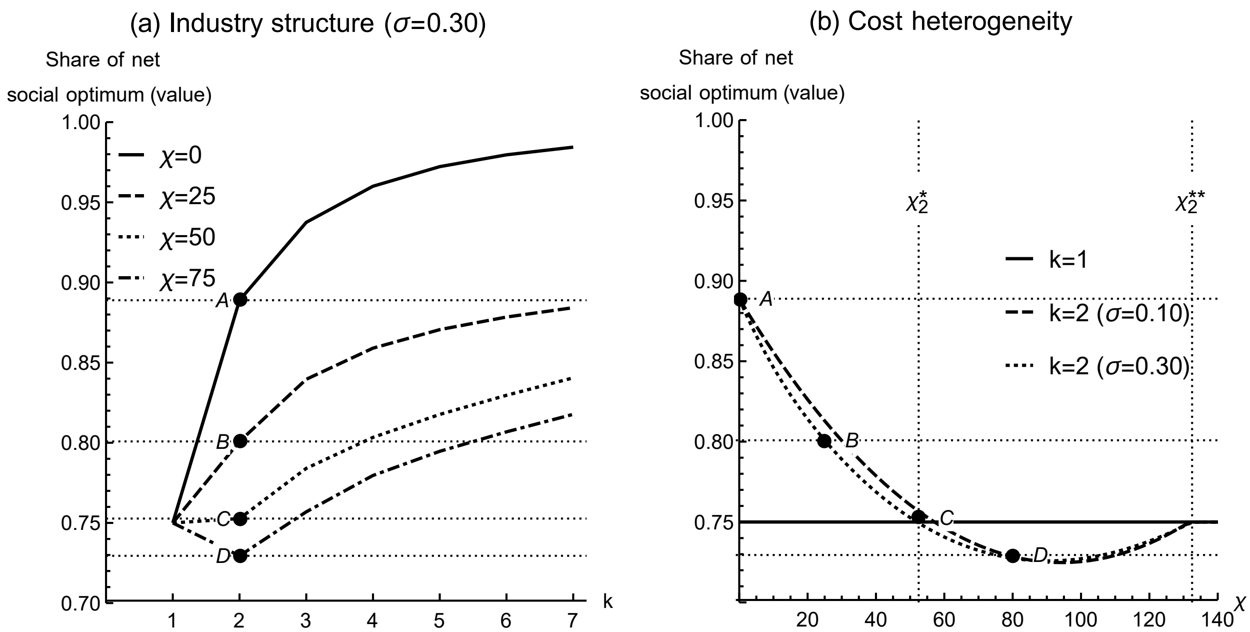

Figure 8: **Net social welfare relative to the social optimum for varying cost heterogeneity χ and industry structure.** $x_0 = 25, b = 1, c = 1, c_k = 50, \mu = -0.025, r = 0.05$, and $\sigma = 0.30$.

8. Conclusions

We have examined upfront capacity investment choices for two or more oligopoly firms, relaxing the common assumption that firms are symmetric while allowing firms to contract or expand their outputs within capacity constraints. The combination of these features lead to a set of different but insightful results, which we believe are of interest to the theory and practice of management.

We summarize a few results. First, the presence of output flexibility within capacity constraints induces a firm to choose larger upfront capacity in order to relax the constraint when future demand is more dispersed and more likely to attain extreme values. Higher uncertainty makes firm value more convex as it enhances the value of the firm's expansion potential, providing a greater incentive to invest. The result that higher uncertainty encourages more upfront capacity investment is not standard as the theory of investment under uncertainty (see, among many others, Dixit and Pindyck 1994) emphasizes that uncertainty depresses investment as it increases the value of the option to defer. Second, when demand is large, a firm with low capacity will likely become constrained and marginalized. By contrast, a firm with large (unconstrained) capacity wields extra market power as it can expand output. Higher uncertainty makes the value of a capacity-advantaged firm more convex as it increases its market power advantage over smaller rivals who get marginalized when their capacity constraint becomes binding. As a consequence, high demand uncertainty encourages larger upfront capacity investment by a more cost-efficient firm to benefit from greater market power in high demand states. Finally, given the combined incentives arising from cost asymmetry and output flexibility in oligopoly for a firm to produce below capacity, a low-cost firm (or cost leader) facing demand uncertainty will invest in larger upfront capacity, whereas less cost-efficient rivals will invest less as capacities are strategic substitutes. Thus, when demand uncertainty is high, industry concentration rises as capacities are more dispersed and social welfare likely suffers. Whereas uncertainty has little impact on welfare when firms are symmetric, it results in substantial (net) welfare loss when firms face heterogeneous costs. Although from a welfare perspective encouraging a larger number of competitors is considered desirable, we find that high uncertainty in conjunction with cost asymmetry in oligopoly reduces net welfare when firms face highly heterogeneous costs in uncertain environments. The above findings have different implications for normative investment theory. One implication is that antitrust authorities should be careful when deciding about (potentially) cost-reducing mergers if the economic environment is uncertain.

Our analysis has a number of limitations, which also present opportunities for future research. Different sources of cost asymmetry (e.g., fixed vs. marginal production cost) and the presence of economies of scale (with the larger firm enjoying lower marginal cost) may lead to somewhat different results. The assumption that capacity is installed once at the outset and stays fixed thereafter—which seems reasonable for certain situations discussed above—may be relaxed to investigate other industry settings. In cases where demand builds up gradually, firms may have some leeway in timing or staging their capacity investments. The sequential buildup of capacity has been considered by Bensoussan and Chevalier-Roignant (2018) though in a monopoly setting and for a restrictive profit function choice which does not account for output flexibility. Capacity timing games are known to give rise to preemption [see Huisman and Kort (2015) for a duopoly setting where firms produce at capacity]. In practice firms may incur certain adjustment costs when altering their output, such as hiring and firing costs. Considering such adjustment costs would give rise to a further partitioning of the state space with situations arising where firms stay put (hysteresis) even though they may achieve greater profits by adjusting output. We also assume that parameter values and states are common knowledge, while in some real contexts some information may remain private or evolve over time. Alternatively, one could assume that firms have prior beliefs which they update over time, leading to a "partially observed system" (see, e.g, Bensoussan et al. 2009). We also (implicitly) assume that capacity investments are financed by the firm's shareholders, while firms may issue various types of securities to finance their operations (see Shibata and Nishihara 2015). We have, nevertheless, relaxed the common restrictive assumption of symmetric oligopoly firms always producing at capacity. Incorporating cost asymmetry and output flexibility in the context of oligopoly firms facing capacity constraints allows larger firms to expand output and take advantage of enhanced market power as smaller rivals drop out. Our analysis has led to nontraditional implications concerning the role and impact of uncertainty on firm investment decisions, industry concentration, and social welfare.

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Appendices

Appendix A. Proof of Lemma 1

Firm i's Markov output policy is defined as a mapping $x \mapsto \bar{q}_i(x)$. For a given $x \in \mathbb{R}_+$, consider firm i's best-reply correspondence, $\mathcal{T}_i : \bigotimes_{j \neq i}[0, \bar{q}_j] \to [0, \bar{q}_i]$, given by

$$\mathcal{T}_i Q_{-i}(x) = \arg\max_{0 \leq q_i \leq \bar{q}_i} \left\{ \left[p\left(x, q_i + Q_{-i} \cdot \mathbf{1}_{k-1}\right) - c \right] q_i \right\}. \tag{A.1}$$

We define $\mathcal{T} = \bigotimes_{i=1}^{k} \mathcal{T}_i$ as a mapping from $\mathbb{R}_+^k$ to $\mathbb{R}_+^k$. The MPE, $Q^C(\cdot)$, is a solution of $Q^C(\cdot) \in \mathcal{T} Q^C(\cdot)$. The MPE is unique if $Q^C(\cdot) = \mathcal{T} Q^C(\cdot)$.

The objective function of Equation (A.1) is concave, while the feasible region is convex. The KKT conditions are thus both necessary and sufficient. Standard techniques yield firm i's best-response correspondence,

$$\mathcal{T}_i Q_{-i}(x) = \begin{cases} 0 & x \leq \underline{x}_i(Q_{-i}), \\ \frac{x - c - bQ_{-i} \cdot \mathbf{1}_{k-1}}{2b} & \underline{x}_i(Q_{-i}) < x < \bar{x}_i(Q_{-i}), \\ \bar{q}_i, & x \geq \bar{x}^i(Q_{-i}), \end{cases} \tag{A.2}$$

which is single-valued (hence, is a best-reply function). Here, we defined

$$\underline{x}_i(Q_{-i}) := c + bQ_{-i} \cdot \mathbf{1}_{k-1}, \quad \text{and} \quad \bar{x}^i(Q_{-i}) := c + 2b\bar{q}_i + 2bQ_{-i} \cdot \mathbf{1}_{k-1}.$$

We characterize four cases and check whether we can pin down the MNE:

1. If $x \leq c$, $q_i^C(x) = 0$ for all i as per Equation (A.2).
2. If no firm is capacity-constrained, i.e., if $c < x < c + b(k+1)\bar{q}_1$, each firm i produces $[x - c - bQ_{-i} \cdot \mathbf{1}_{k-1}]/2b$. By summing over all k firms, it follows the collective output $\frac{k}{k+1}\frac{x-c}{b}$. Because firms are symmetric (no capacity constraint binds), a firm's output is $\frac{1}{k+1}\frac{x-c}{b}$. To check that no firm is capacity constrained we only need to ensure that $\frac{1}{k+1}\frac{x-c}{b} < \bar{q}_1$, i.e., that $x < c + b(k+1)\bar{q}_1 =: \bar{x}_1$ according to the definition of $\bar{x}_1$ in Equation (7b).
3. Suppose the firms $1, \ldots, m$ are constrained by their capacity (and none other) and denote by $Q_{1,\ldots,m} = \sum_{i=1}^{m} \bar{q}_i$ their collective output and by $Q_{m+1,\ldots,k} = \sum_{j=m+1}^{k} q_j$ the collective output of the firms that are not constrained. From Equation (A.2) an unconstrained firm i produces

$$\frac{1}{2b}\left[x - c - b\Big(Q_{1,\ldots,m} + Q_{\{m+1,\ldots,k\} \setminus \{i\}} \Big) \right] \tag{A.3}$$

Summing (A.3) over $(k-m)$ terms yields the aggregate output of nonconstrained firms:

$$Q_{m+1,\dots,k} = \frac{k-m}{k-m+1}\left(\frac{x-c}{b} - Q_{1,\dots,m}\right). \tag{A.4}$$

So each unconstrained firm produces the output $\left[\frac{x-c}{b} - Q_{1,\dots,m}\right]/[k-m+1]$. It remains to check that firm $m+1$ is not constrained, i.e., that

$$\frac{1}{k-m+1}\left[\frac{x-c}{b} - Q_{1,\dots,m}\right] \le \bar{q}_{m+1} \iff x \le \bar{x}_m,$$

where $\bar{x}_i$ is given in (7b).

4. Finally, all firms will be constrained producing at capacity $\bar{q}_i$ if

$$\frac{1}{k-(k-1)+1}\left[\frac{x-c}{b} - Q_{1,\dots,k-1}\right] > \bar{q}_k \iff x > \bar{x}_k.$$

The MNE in (7a) now obtains from considering the terms defined in (7c)–(7d). It is unique for each state x.

Appendix B. Proof of Lemma 2

Lemma 2 can be viewed as a corollary of Lemma 1. From (A.4), the total industry output obtains to be

$$\frac{k-m}{k-m+1}\frac{x-c}{b} + \frac{1}{k-m+1}\sum_{i=1}^{m}\bar{q}_i$$

if the firms $1,\dots,m$ are constrained, i.e., if $\bar{x}_m < x < \bar{x}_{m+1}$. So, we have

$$\|Q(X_t)\| = \begin{cases} 0 & \text{if } X_t \in (0,c), \\ \frac{k-m+1}{b(k-m+2)}\left(X_t - \Sigma_m\right) + \sum_{i=1}^{m-1}\bar{q}_i & \text{if } X_t \in [\bar{x}_{m-1}, \bar{x}_m), m = 1, ..., k+1. \end{cases} \tag{B.1}$$

Substituting (B.1) into (3) yields

$$p(X_t) = \begin{cases} X_t & \text{if } X_t \in (0,c), \\ \frac{X_t + (k-m+1)c - b\sum_{i=1}^{m-1}\bar{q}_i}{(k-m+2)} & \text{if } X_t \in [\bar{x}_m, \bar{x}_{m+1}), m = 1, ..., k. \end{cases} \tag{B.2}$$

Firm profits then obtain as

$$\pi_i = \begin{cases} 0 & \text{if } X_t \in (0, c), \\ q_i^C(X_t)\frac{X_t - \Sigma_m}{(k-m+2)} & \text{if } X_t \in [\bar{x}_m, \bar{x}_{m+1}), m = 1, ..., k. \end{cases} \tag{B.3}$$

The proposition obtains by substituting the output (A.2) in the above.

We investigate the regularity of the profit function (8) in x. Continuity is immediate. We have

$$\frac{\partial \pi_i^C}{\partial x}(x, k) = \begin{cases} 0, & x \in (0, c), \\ \frac{2(x-\Sigma_m)}{b(k-m+2)^2}, & x \in [\bar{x}_{m-1}, \bar{x}_m), m = 1, \dots, i, \\ \frac{\bar{q}_i}{k-m+2}, & x \in [\bar{x}_{m-1}, \bar{x}_m), m = i+1, \dots, k+1. \end{cases}$$

So the function $x \mapsto \frac{\partial \pi_i^C}{\partial x}(x, k)$ is continuous at c but not at the thresholds $\bar{x}_m$.

We now study the function $\bar{q}_i \mapsto \pi_i^C(x, \bar{Q})$. We re-write (8), so

$$\pi_i^C(x, \bar{Q}) = \begin{cases} 0 & x, c \\ \frac{1}{b}\left[\frac{x-\Sigma(x,\bar{Q})}{1+K(x,\bar{Q})}\right]^2 & c \le x < \bar{x}_i \\ \bar{q}_i \left[\frac{x-\Sigma(x,\bar{Q})}{1+K(x,\bar{Q})}\right] & x \ge \bar{x}_i, \end{cases}$$

with the functions Σ and K defined in (7c) and (7d) respectively. We have

$$\frac{\partial \Sigma}{\partial \bar{q}_i}(x, \bar{Q}) = b\mathbb{1}_{\{x \ge \bar{x}_i\}}, \quad \text{and} \quad \frac{\partial K}{\partial \bar{q}_i}(x, \bar{Q}) = 0.$$

It obtains after differentiation and simplification that

$$\frac{\partial \pi_i^C}{\partial \bar{q}_i}(x, \bar{Q}) = \frac{x - \Sigma(x, \bar{Q}) - b\bar{q}_i}{1 + K(x, \bar{Q})}\mathbb{1}_{\{x \ge \bar{x}_i\}} \quad \text{and} \quad \frac{\partial^2 \pi_i^C}{\partial \bar{q}_i^2}(x, \bar{Q}) = -\frac{2b}{1 + K(x, \bar{Q})}\mathbb{1}_{\{x \ge \bar{x}_i\}} \le 0. \tag{B.4}$$

Appendix C. Proof of Proposition 1

Using a verification argument (à la Feynman-Kac), it can be established that the function $x \mapsto V_i(x, \bar{Q})$ in (9) is the probabilistic interpretation of the *weak*/distributional solution to the second-order ordinary differential equation (ODE)

$$\mathcal{L}V_i(x, \bar{Q}) = \pi_i^C(x, \bar{Q}), \tag{C.1}$$

where the second-order differential operator $\mathcal{L}$ is defined by $\mathcal{L}f(x) := rf(x) - \mu x f'(x) - \frac{1}{2}\sigma^2 x^2 f''(x)$, subject to boundary conditions at 0 and ∞. Note that we do not claim a priori that the term at the LHS of (C.1) exists (i.e., that $x \mapsto V_i(x, \bar{Q})$ is at least continuously differentiable with locally integrable second-order derivatives). Proving this regularity property using abstract arguments is particularly challenging because the function $x \mapsto \pi_i^C(x, \bar{Q})$ in (8) fails to be continuously differentiable (see Lemma 2). By the Lax-Milgram theorem, we can however claim that the ODE (C.1) has a unique weak solution in the Sobolev space of square-integrable functions with square-integrable first-order weak derivatives.

The approach below is to construct a function that solves the ODE "piece by piece" and is continuously differentiable. We can then claim that this (constructed) function coincides with the unique weak solution [and thus to the function $x \mapsto V_i(x, \bar{Q})$ in (9)]. Our approach is different from the classical "smooth fit" argument, also because this terminology is used exclusively in the context of optimal stopping (to be best of our knowledge). The "smooth-fit approach" consists in proving using abstract arguments that the value function of optimal stopping is continuously differentiable and in establishing the structure of the continuation set. [The next step is generally to use the regularity property to derive a free-boundary problem and obtain the free boundary explicitly.] Again, the difference is that we do not establish a priori that the function $x \mapsto V_i(x, \bar{Q})$ in (9) is sufficiently regular to apply the operator $\mathcal{L}$, but obtain this property "at the end" once we construct a continuously differentiable solution to the ODE (C.1).

The functions $x \mapsto x^{\gamma_A}$ and $x \mapsto x^{\gamma_B}$ are two independent solutions of $\mathcal{L}f(x) = 0$, where $\gamma_{A/B}$ in (11c) are the roots of $\mathscr{Q}(\gamma) = r - \beta\gamma - \gamma(\gamma - 1)\sigma^2/2$. We have $\mathscr{Q}(2) = r - 2\mu - \sigma^2 (> 0)$, $\mathscr{Q}(1) = r - \mu$ (> 0) and $\mathscr{Q}(0) = r$ (> 0). The general solution to the second-order ODE is

$$V_i(x, \bar{Q}) = \begin{cases} A_0 x^{\gamma_A} + B_0 x^{\gamma_B}, & x \in (0, x_0), \\ \frac{1}{b(1+K_m)^2}\left[\frac{x^2}{\mathscr{Q}(2)} - \frac{2\Sigma_m x}{r-\mu} + \frac{\Sigma_m^2}{r}\right] + A_m x^{\gamma_A} + B_m x^{\gamma_B}, & x \in [\bar{x}_{m-1}, \bar{x}_m), m = 1, \ldots, i, \\ \frac{\bar{q}_i}{(1+K_m)}\left[\frac{x}{r-\mu} - \frac{\Sigma_m}{r}\right] + A_m x^{\gamma_A} + B_m x^{\gamma_B}, & x \in [\bar{x}_{m-1}, \bar{x}_m), m = i+1, \ldots, k+1, \end{cases}$$

where $\Sigma_m = c + b\sum_{j=0}^{m-1} \bar{q}_j$, $K_m := k - m + 1$, $\bar{x}_m := \Sigma_m + b(k - m + 2)\bar{q}_m$, and A_m and B_m, $m = 0, \ldots, k+1$ are constants we must determine. Note (i) $x \mapsto K(x, \bar{Q})$ and $x \mapsto \Sigma(x, \bar{Q})$ are discontinuous at $\bar{x}_m$ with $K(\bar{x}_m-, \bar{Q}) = K_m$, $K(\bar{x}_m+, \bar{Q}) = K_{m+1}$, $\Sigma(\bar{x}_m-, \bar{Q}) = \Sigma_m$ and $\Sigma(\bar{x}_m+, \bar{Q}) = \Sigma_{m+1}$ and (ii) that $x \mapsto V_i(x, \bar{Q})$ is piecewise C^2. We set A_m and B_m, $m = 0, \ldots, k+1$, such that $x \mapsto V_i(x, \bar{Q})$ is C^1 globally.

To ensure that $V_i(0; \bar{Q}) = 0$ and avoid bubble solutions, we set $B_0 = A_{k+1} = 0$. Let's define $\Delta A_m := A_m - A_{m+1}$ and $\Delta B_m := B_m - B_{m+1}$. The smoothness conditions at $\bar{x}_0 = c$ read

$$\begin{aligned} \Delta A_0\ \bar{x}_0^{\gamma_A} + \Delta B_0\ \bar{x}_0^{\gamma_B} =& \frac{1}{bK_0^2}\left[\frac{\bar{x}_0^2}{\mathscr{Q}(2)} - \frac{2c\bar{x}_0}{r-\mu} + \frac{c^2}{r}\right] \\ \gamma_A \Delta A_0\ \bar{x}_0^{\gamma_A - 1} + \gamma_B \Delta B_0\ \bar{x}_0^{\gamma_B - 1} =& \frac{1}{bK_0^2}\left[\frac{2\bar{x}_0}{\mathscr{Q}(2)} - \frac{2c}{r-\mu}\right]. \end{aligned}$$

It obtains for $\underline{m=0}$ $A_0 = A_1 + \bar{x}_0^{-\gamma_A}\Delta_i^A(c,\bar{Q})$ and $B_1 = B_0 + \bar{x}_0^{-\gamma_B}\Delta_i^B(c,\bar{Q})$, with $\Delta_i^{A/B}$ defined by (12). The smoothness conditions at $\bar{x}_m$, $m = 1, \ldots, i-1$, are

$$\begin{aligned}\Delta A_m\ \bar{x}_m^{\gamma_A} + \Delta B_m\ \bar{x}_m^{\gamma_B} =& \frac{1}{bK_m^2}\left[\frac{\bar{x}_m^2}{\mathscr{Q}(2)} - \frac{2\Sigma_{m+1}\bar{x}_m}{r-\mu} + \frac{\Sigma_{m+1}^2}{r}\right] \\ &- \frac{1}{bK_{m-1}^2}\left[\frac{\bar{x}_m^2}{\mathscr{Q}(2)} - \frac{2\Sigma_m\bar{x}_m}{r-\mu} + \frac{\Sigma_m^2}{r}\right] \\ \gamma_A\Delta A_m\ \bar{x}_m^{\gamma_A-1} + \gamma_B\Delta B_m\ \bar{x}_m^{\gamma_B-1} =& \frac{1}{bK_m^2}\left[\frac{2\bar{x}_m}{\mathscr{Q}(2)} - \frac{2\Sigma_{m+1}}{r-\mu}\right] - \frac{1}{bK_{m-1}^2}\left[\frac{2\bar{x}_m}{\mathscr{Q}(2)} - \frac{2\Sigma_m}{r-\mu}\right]\end{aligned}$$

We thus have for $\underline{m = 1, \ldots, i-1}$ $A_m = A_{m+1} + \bar{x}_m^{-\gamma_A}\Delta_i^A(\bar{x}_m,\bar{Q})$ and $B_{m+1} = B_m + \bar{x}_m^{-\gamma_B}\Delta_i^B(\bar{x}_m,\bar{Q})$. The smoothness conditions at $\bar{x}_i$ are

$$\begin{aligned}\Delta A_i\ \bar{x}_i^{\gamma_A} + \Delta B_i\ \bar{x}_i^{\gamma_B} =& \frac{\bar{q}_i}{K_i}\left[\frac{\bar{x}_i}{r-\mu} - \frac{\Sigma_{i+1}}{r}\right] - \frac{1}{bK_{i-1}^2}\left[\frac{\bar{x}_i^2}{\mathscr{Q}(2)} - \frac{2\Sigma_i\bar{x}_i}{r-\mu} + \frac{\Sigma_i^2}{r}\right], \\ \gamma_A\Delta A_i\ \bar{x}_i^{\gamma_A-1} + \gamma_B\Delta B_i\ \bar{x}_i^{\gamma_B-1} =& \frac{\bar{q}_i}{K_i}\frac{1}{r-\mu} - \frac{1}{bK_{i-1}^2}\left[\frac{2\bar{x}_i}{\mathscr{Q}(2)} - \frac{2\Sigma_i}{r-\mu}\right].\end{aligned}$$

Similarly, we obtain for $m = i$ $A_i = A_{i+1} + \bar{x}_i^{-\gamma_A}\Delta_i^A(\bar{x}_i,\bar{Q})$ and $B_{i+1} = B_i + \bar{x}_i^{-\gamma_B}\Delta_i^B(\bar{x}_i,\bar{Q})$. Finally, the smoothness conditions at $\bar{x}_m$, $m = i+1, \ldots, k$, read

$$\begin{aligned}\Delta A_m\ \bar{x}_m^{\gamma_A} + \Delta B_m\ \bar{x}_m^{\gamma_B} =& \frac{\bar{q}_i}{K_m}\left[\frac{\bar{x}_m}{r-\mu} - \frac{\Sigma_{m+1}}{r}\right] - \frac{\bar{q}_i}{K_{m-1}}\left[\frac{\bar{x}_m}{r-\mu} - \frac{\Sigma_m}{r}\right] \\ \gamma_A\Delta A_m\ \bar{x}_m^{\gamma_A-1} + \gamma_B\Delta B_m\ \bar{x}_m^{\gamma_B-1} =& \frac{\bar{q}_i}{K_m}\frac{1}{r-\mu} - \frac{\bar{q}_i}{K_{m-1}}\frac{1}{r-\mu}.\end{aligned}$$

It thus obtains for $m = i+1, \cdots, k$, $A_m = A_{m+1} + \bar{x}_m^{-\gamma_A}\Delta_i^A(\bar{x}_m,\bar{Q})$ and $B_{m+1} = B_m + \bar{x}_m^{-\gamma_B}\Delta_i^B(\bar{x}_m,\bar{Q})$. In summary, sequences $\{A_n\}$ and $\{B_n\}$ are given by

$$A_n = \begin{cases}\sum_{m=n}^k \bar{x}_m^{-\gamma_A}\Delta_i^A(\bar{x}_m,\bar{Q}), & n = 0, \cdots, k \\ 0, & n = k+1,\end{cases}$$

and

$$B_n = \begin{cases}0, & n = 0, \\ \sum_{m=0}^{n-1} \bar{x}_m^{-\gamma_B}\Delta_i^B(\bar{x}_i,\bar{Q}), & n = 1, \cdots, k+1.\end{cases}$$

Expression (13a) obtains from re-writing the above.

We proved that the function V_i in (9) is C^1; therefore, we can interpret the linear ODE (C.1) in the strong sense (almost everywhere). Using standard techniques on linear PDE, we can re-write V_i in the form

$$V_i\big(x,\bar{Q}\big) = \int_{\mathbb{R}_+} \Gamma(x,\xi)\pi_i^C(\xi,\bar{Q})\mathrm{d}\xi, \tag{C.2}$$

where

$$\Gamma: \quad (x,\xi) \mapsto \Gamma(x,\xi) := \begin{cases} \frac{2}{(\gamma_A-\gamma_B)\sigma^2} x^{\gamma_A}\xi^{-\gamma_A-1}, & \xi > x, \\ \frac{2}{(\gamma_A-\gamma_B)\sigma^2} x^{\gamma_B}\xi^{-\gamma_B-1}, & \xi < x \end{cases} \tag{C.3}$$

is the Green's function of the operator $\mathcal{L}$ and π_i^C is given in (8). First, it follows from $\partial^2\Gamma/\partial x^2 > 0$ and $\pi_i^C \geq 0$ that $x \mapsto V_i(x,\bar{Q})$ is strictly convex. Because $\Gamma \geq 0$, it follows from (B.4) that

$$\begin{aligned} \frac{\partial^2 V_i}{\partial \bar{q}_i^2}\big(x,\bar{Q}\big) &= -\int_{\mathbb{R}_+} \Gamma(x,\xi)\frac{2b}{1+K(\xi,\bar{Q})}\mathbb{1}_{\{\xi \geq \bar{x}_i\}}\mathrm{d}\xi \\ &= -\int_{\bar{x}_i}^{\infty} \Gamma(x,\xi)\frac{2b}{1+K(\xi,\bar{Q})}\mathrm{d}\xi < 0. \end{aligned} \tag{C.4}$$

It follows that $\bar{q}_i \mapsto V_i\big(x,\bar{Q}\big)$ is strictly concave. This concludes the proof of Proposition 1.

Appendix D. Proof of Proposition 2

Let $\bar{q}_\star$ denote an exogenous lower bound on capacity (e.g., $\bar{q}_\star = 0$). We are now looking for $\bar{q}_1^C(x,\bar{q}_\star) \in \arg\max_{\bar{q}_1 \geq \bar{q}_\star} \{V_1(x,\bar{q}_1) - c_1\bar{q}_1\}$. In monopoly, we have from (7b)-(7d)

$$\bar{x}_1 = c + 2b\bar{q}_1 \tag{D.1a}$$

$$(\Sigma, K)\big(x,\bar{Q}\big) = \begin{cases} (0,1), & x \in (0,c) \\ (c,1), & x \in [c,\bar{x}_1) \\ (c+b\bar{q}_1, 0)\,, & x \in [\bar{x}_1,\infty) \end{cases} \tag{D.1b}$$

We obtain from (11b) and (12) that $\Delta_1^{A/B}(c,\bar{q}_1) = c^2\kappa_{A/B}/4b$, with $\kappa_{A/B}$ given in (15d). Besides, we also have from (11b) and (12)

$$\begin{aligned} (\gamma_A-\gamma_B)\Delta_1^{A/B}(\bar{x}_1,\bar{q}_1) =& (1-\gamma_{B/A})\frac{\bar{x}_1\bar{q}_1}{r-\mu} + \gamma_{B/A}\frac{c\bar{q}_1+b\bar{q}_1^2}{r} \\ & - \frac{2-\gamma_{B/A}}{\mathcal{Q}(2)}\frac{\bar{x}_1^2}{4b} + \frac{1-\gamma_{B/A}}{r-\mu}\frac{2\bar{x}_1 c}{4b} + \frac{\gamma_{B/A}}{r}\frac{c^2}{4b} \end{aligned}$$

Equation $\Delta_1^{A/B}(\bar{x}_1, \bar{q}_1) = -\bar{x}_1^2 \kappa_{A/B}/4b$ now follows from (D.1a). In monopoly, (13b)-(13c) simplify to

$$A_1(x, \bar{q}_1) = \frac{c^{2-\gamma_A}}{4b}\kappa_A \mathbb{1}_{\{x \le c\}} - \frac{\bar{x}_1^{2-\gamma_A}}{4b}\kappa_A \mathbb{1}_{\{x \le \bar{x}_1\}} \tag{D.2a}$$

$$B_1(x, \bar{q}_1) = \frac{c^{2-\gamma_B}}{4b}\kappa_B \mathbb{1}_{\{x \ge c\}} - \frac{\bar{x}_1^{2-\gamma_B}}{4b}\kappa_B \mathbb{1}_{\{x \ge \bar{x}_1\}}. \tag{D.2b}$$

Besides, from (10) and (D.1a),

$$\frac{\partial v_1}{\partial \bar{q}_1}(x, \bar{q}_1) = \left\{ \frac{x}{r-\mu} - \frac{\bar{x}_1}{r} \right\} \mathbb{1}_{\{x \ge \bar{x}_1\}}. \tag{D.2c}$$

We now want to investigate the curvature and continuity of V_1. We note that $x \le \bar{x}_1 \Leftrightarrow \bar{q}_1 \ge (x-c)/2b$. It follows from (13a) and (D.2c) that

$$\frac{\partial V_1}{\partial \bar{q}_1}(x, \bar{q}_1) = \begin{cases} -\frac{1}{2}(2-\gamma_A)\kappa_A x^{\gamma_A} \bar{x}_1^{1-\gamma_A}, & x \le \bar{x}_1 \text{ or } \bar{q}_1 \ge \frac{x-c}{2b}, \\ \frac{x}{r-\mu} - \frac{\bar{x}_1}{r} - \frac{1}{2}(2-\gamma_B)\kappa_B x^{\gamma_B} \bar{x}_1^{1-\gamma_B}, & x \ge \bar{x}_1 \text{ or } \bar{q}_1 \le \frac{x-c}{2b} \end{cases}$$

$$\frac{\partial^2 V_1}{\partial x \partial \bar{q}_1}(x, \bar{q}_1) = \begin{cases} -\frac{1}{2}(2-\gamma_A)\gamma_A \kappa_A x^{\gamma_A - 1} \bar{x}_1^{1-\gamma_A}, & x \le \bar{x}_1 \text{ or } \bar{q}_1 \ge \frac{x-c}{2b}, \\ \frac{1}{r-\mu} - \frac{1}{2}(2-\gamma_B)\gamma_B \kappa_B x^{\gamma_B - 1} \bar{x}_1^{1-\gamma_B}, & x \ge \bar{x}_1 \text{ or } \bar{q}_1 \le \frac{x-c}{2b}. \end{cases}$$

$$\frac{\partial^2 V_1}{\partial \bar{q}_1^2}(x, \bar{q}_1) = \begin{cases} -(2-\gamma_A)(1-\gamma_A) b \kappa_A x^{\gamma_A} \bar{x}_1^{-\gamma_A}, & x \le \bar{x}_1 \text{ or } \bar{q}_1 \ge \frac{x-c}{2b}, \\ -\frac{2b}{r} - (2-\gamma_B)(1-\gamma_B) b \kappa_B x^{\gamma_B} \bar{x}_1^{-\gamma_B}, & x \ge \bar{x}_1 \text{ or } \bar{q}_1 \le \frac{x-c}{2b}. \end{cases}$$

$$\frac{\partial^3 V_1}{\partial x \partial \bar{q}_1^2}(x, \bar{q}_1) = \begin{cases} -(2-\gamma_A)(1-\gamma_A)\gamma_A b \kappa_A x^{\gamma_A - 1} \bar{x}_1^{-\gamma_A}, & x \le \bar{x}_1 \text{ or } \bar{q}_1 \ge \frac{x-c}{2b}, \\ -(2-\gamma_B)(1-\gamma_B)\gamma_B b \kappa_B x^{\gamma_B - 1} \bar{x}_1^{-\gamma_B}, & x \ge \bar{x}_1 \text{ or } \bar{q}_1 \le \frac{x-c}{2b}. \end{cases}$$

From (15d),

$$\begin{aligned} \operatorname{sgn}\{\kappa_{A/B}\} &= \operatorname{sgn}\left\{(2-\gamma_{B/A})(r-\mu)r - 2(1-\gamma_{B/A})(r-2\mu-\sigma^2)r - \gamma_{B/A}(r-2\mu-\sigma^2)(r-\mu)\right\} \\ &= \operatorname{sgn}\left\{r(r-\mu)\left[2-\gamma_{B/A} - 2 + 2\gamma_{B/A} - \gamma_{B/A}\right] + (\mu+\sigma^2)\left[r(2-\gamma_{B/A}) - \mu\gamma_{B/A}\right]\right\} \\ &= \operatorname{sgn}\left\{(\mu+\sigma^2)\left[r(2-\gamma_{B/A}) - \mu\gamma_{B/A}\right]\right\} \end{aligned}$$

Besides, $\gamma_B < 0 < \gamma_A$. We have $\mathcal{Q}''(\gamma) = -\sigma^2 < 0$ and $\mathcal{Q}(2) > 0$ by assumption, $\gamma_A > 2$. Hence, $\kappa_A > 0 > \kappa_B$. Thus, $\frac{\partial^3 V_1}{\partial x \partial \bar{q}_1^2}(x, \bar{q}_1) < 0$, such that $x \mapsto \frac{\partial^2 V_1}{\partial \bar{q}_1^2}(x, \bar{q}_1)$ decreases from 0 to $-2b/r$, taking negative values, proving that $\bar{q}_1 \mapsto V_1(x, \bar{q}_1)$ is concave. Because $\gamma_A > 2$ and $\kappa_A > 0$, $\partial V_1/\partial \bar{q}_1 > 0$. We want to prove that $(x, \bar{q}_1) \mapsto \frac{\partial V_1}{\partial \bar{q}_1}(x, \bar{q}_1)$ is continuous. Continuity of $x \mapsto \frac{\partial V_1}{\partial \bar{q}_1}(x, \bar{q}_1)$ (resp. $\bar{q}_1 \mapsto \frac{\partial V_1}{\partial \bar{q}_1}(x, \bar{q}_1)$) is immediate on $(0,\bar{q}_1)$ and $(\bar{x}_1,\infty)$ (resp. $[0,\frac{x-c}{2b})$ and $(\frac{x-c}{2b},\infty)$). From the expression of $\partial V_1/\partial\bar{q}_1$,

$$\frac{\partial V_1}{\partial \bar{q}_1}(c+2b\bar{q}_1,\bar{q}_1)=\begin{cases}-(2-\gamma_A)\frac{c+2b\bar{q}_1}{2}\kappa_A, & x\le\bar{x}_1,\\ \frac{c+2b\bar{q}_1}{2}\left[\frac{2}{r-\mu}-\frac{2}{r}-(2-\gamma_B)\kappa_B\right], & x\ge\bar{x}_1.\end{cases}\tag{D.3}$$

But, from (15d),

$$-(2-\gamma_A)\kappa_A+(2-\gamma_B)\kappa_B-\frac{2}{r-\mu}+\frac{2}{r}=0,\tag{D.4}$$

which proves the continuity of $x\mapsto\frac{\partial V_1}{\partial\bar{q}_1}(x,\bar{q}_1)$ at $\bar{x}_1=c+2b\bar{q}_1$. Similarly, we can prove that $\bar{q}_1\mapsto\frac{\partial V_1}{\partial\bar{q}_1}(x,\bar{q}_1)$ is continuous at $(x-c)/2b$. We define

$$f(\bar{q}_1):=\frac{\partial V_1}{\partial\bar{q}_1}(c+2b\bar{q}_1,\bar{q}_1)-c_1 \quad\text{and}\quad g(x):=f\left(\frac{x-c}{2b}\right).$$

Equation (15c) results from (D.3).

Function $x\mapsto\frac{\partial V_1}{\partial\bar{q}_1}(x,\bar{q}_1)$ increases on $[0,\bar{x}_1)$ because $\gamma_A>2$ and $\kappa_A>0$. On $(\bar{x}_1,\infty)$, we have

$$\frac{\partial^3 V_1}{\partial x^2\partial\bar{q}_1}(x,\bar{q}_1)=-\frac{1}{2}(2-\gamma_B)(\gamma_B-1)\gamma_B\kappa_B x^{\gamma_B-1}\bar{x}_1^{1-\gamma_B}\ >0.$$

It follows that $x\mapsto\frac{\partial^2 V_1}{\partial x\partial\bar{q}_1}(x,\bar{q}_1)$ increases on $(\bar{x}_1,\infty)$ from $\frac{1}{r-\mu}-\frac{1}{2}(2-\gamma_B)\gamma_B\kappa_B$, to $1/(r-\mu)\ >0$. If we assume $\underline{\varphi>1}$ with φ given by (14), then $\frac{\partial^2 V_1}{\partial x\partial\bar{q}_1}(x,\bar{q}_1)>0$ on $(\bar{x}_1,\infty)$. If follows that $x\mapsto\frac{\partial V_1}{\partial\bar{q}_1}(x,\bar{q}_1)$ is continuous and increasing on $(0,\infty)$ from 0 to ∞. Here, there exists a unique $\bar{x}_\star$ (as a function of $\bar{q}_\star$) such that $\frac{\partial V_1}{\partial\bar{q}_1}(x,\bar{q}_\star)\lesseqqgtr c_1$ if $x\lesseqqgtr\bar{x}_\star$. We note that $0\lesseqqgtr f(\bar{q}_\star)\Leftrightarrow\bar{x}_\star\lesseqqgtr c+2b\bar{q}_\star$. It follows that:

$$\bar{x}_\star=\begin{cases}\left(\frac{2c_1}{(c+2b\bar{q}_\star)(\gamma_A-2)\kappa_A}\right)^{1/\gamma_A}(c+2b\bar{q}_\star), & f(\bar{q}_\star)\ge0\\ \{x\mid\frac{x}{r-\mu}-\frac{c+2b\bar{q}_\star}{r}-\frac{1}{2}(2-\gamma_B)(c+2b\bar{q}_\star)^{1-\gamma_B}\kappa_B x^{\gamma_B}=c_1\}, & f(\bar{q}_\star)<0\end{cases}$$

Equation (15a) now obtains by setting $\bar{q}_\star=0$; Note that $\bar{x}_\star>0$ if $\bar{q}_\star=0$.

Assume first $\underline{x<\bar{x}_\star}$. Because $\bar{q}_1\mapsto V_1(x,\bar{q}_1)$ is concave, $\bar{q}_1\mapsto\frac{\partial V_1}{\partial\bar{q}_1}(x,\bar{q}_1)$ is monotone decreasing on $[\bar{q}_\star,\infty)$ from $\frac{\partial V_1}{\partial\bar{q}_1}(x,\bar{q}_\star)$ to $\frac{\partial V_1}{\partial\bar{q}_1}(x_0,\infty)=0$; hence, $c_1>\frac{\partial V_1}{\partial\bar{q}_\star}(x,\bar{q}_\star)>\frac{\partial V_1}{\partial\bar{q}_1}(x,\bar{q}_1)$ for all $\bar{q}_1>\bar{q}_\star$. In this case, the maximum capacity level is thus attained at $\bar{q}_\star$. Assume now $\underline{x>\bar{x}_\star}$. We know that $\bar{q}_1\mapsto\frac{\partial V_1}{\partial\bar{q}_1}(x,\bar{q}_1)$ is monotone decreasing on $[\bar{q}_\star,\infty)$ from $\frac{\partial V_1}{\partial\bar{q}_1}(x,\bar{q}_\star)>c_1$ (by assumption) to 0. It follows that there exists a unique solution $\hat{q}(x)>\bar{q}_\star$ of equation

$$\frac{\partial V_1}{\partial\bar{q}_1}(x,\bar{q}_1)=c_1.\tag{D.5}$$

We can show that $g(x) \gtreqless 0 \Leftrightarrow \hat{q}(x) \gtreqless (x-c)/2b$. It follows equation (15b). In summary,

$$\bar{q}_1^C(x, \bar{q}_\star) = \bar{q}_\star, \mathbb{1}_{\{x \le \bar{x}_\star\}} + \hat{q}(x)\mathbb{1}_{\{x > \bar{x}_\star\}}.$$

Note that, if $\frac{\partial V_1}{\partial \bar{q}_1}(x, 0) > c_1$ or $f(0) > 0$, then there exists a unique solution $\hat{q}(x)$ of (D.5) in $\mathbb{R}_+$. Hence, $\bar{q}_1^C(x) = \max\{\bar{q}_\star, \hat{q}(x)\}$.

We summarize the results:

$$\hat{q}(x) = \begin{cases} \frac{\left(\frac{2c_1}{\kappa_A(\gamma_A-2)x^{\gamma_A}}\right)^{\frac{1}{1-\gamma_A}} - c}{2b}, & x \ge \frac{2c_1}{\kappa_A(\gamma_A-2)}, \\ \left\{\bar{q}_1 \mid \frac{x}{r-\mu} - \frac{c+2b\bar{q}_1}{r} + \frac{\kappa_B}{2}(\gamma_B-2)x^{\gamma_B}(c+2b\bar{q}_1)^{1-\gamma_B} = c_1\right\}, & x < \frac{2c_1}{\kappa_A(\gamma_A-2)}. \end{cases}$$

Assume $\bar{q}_\star = 0$. From (15c), $f(0) \lesseqgtr 0 \Leftrightarrow \frac{2c_1}{(\gamma_A-2)\kappa_A c} \lesseqgtr 1$. Hence,

$$\bar{x}_\star = \begin{cases} \left(\frac{2c_1}{(\gamma_A-2)c\kappa_A}\right)^{1/\gamma_A} c, & \frac{2c_1}{(\gamma_A-2)\kappa_A c} \le 1 \\ \{x \mid \frac{x}{r-\mu} - \frac{c}{r} + \frac{\kappa_B}{2}(\gamma_B-2)x^{\gamma_B}c^{1-\gamma_B} = c_1\}, & \frac{2c_1}{(\gamma_A-2)\kappa_A c} > 1. \end{cases}$$

Appendix E. Proof of Proposition 3

We prove the Proposition by use the Banach fixed-point theorem, first deriving properties of the best-reply functions.

Best-reply functions.. We recall the expression for V_i and $\frac{\partial \pi_i}{\partial \bar{q}_i}$ in eqs. (B.4) and (C.2) respectively. It follows that

$$\frac{\partial^3 V_i}{\partial x^2 \partial \bar{q}_i}(x, \bar{Q}) = \int_{\bar{x}_i}^{\infty} \frac{\partial^2 \Gamma}{\partial x^2}(x, \xi) \frac{\xi - \Sigma(\xi, \bar{Q}) - b\bar{q}_i}{1 + K(\xi, \bar{Q})} \mathrm{d}\xi,$$

with $\bar{x}_i$, Σ and K given in eqs. (7b) to (7d), respectively. It is immediate from (C.3) that $\frac{\partial^2 \Gamma}{\partial x^2}(x, \xi) > 0$. Further, it follows from (B.4) that the function $x \mapsto \frac{\partial \pi_i^C}{\partial \bar{q}_i}(x, \bar{Q})$ is strictly increasing on $(\bar{x}_i, \infty)$ from

$$\begin{aligned} \lim_{x \downarrow \bar{x}_i} \frac{\partial \pi_i^C}{\partial \bar{q}_i}(x, \bar{Q}) &= \frac{c + b\left[\sum_{m=0}^{i-1} \bar{q}_m + (k-i+2)\bar{q}_i\right] - c - b\sum_{m=0}^{i} \bar{q}_m - 2b\bar{q}_i}{1 + K(\bar{x}_i, \bar{Q})} \\ &= \frac{b(k-i-1)\bar{q}_i}{1 + K(\bar{x}_i, \bar{Q})} > 0 = \lim_{x \uparrow \bar{x}_i} \frac{\partial \pi_i^C}{\partial \bar{q}_i}(x, \bar{Q}), \end{aligned}$$

taking positive values on $(\bar{x}_i, \infty)$. It follows that $\frac{\partial^3 V_i}{\partial x^2 \partial \bar{q}_i} > 0$ and that $x \mapsto \frac{\partial^2 V_i}{\partial x \partial \bar{q}_i}(x, \bar{Q})$ is monotone increasing on $(0, \infty)$ from

$$\frac{\partial^2 V_i}{\partial x \partial \bar{q}_i}(0, \bar{Q}) = \int_{\bar{x}_i}^{\infty} \underbrace{\frac{\partial \Gamma}{\partial x}(0, \xi)}_{=0} \frac{\xi - \Sigma(\xi, \bar{Q}) - b\bar{q}_i}{1 + K(\xi, \bar{Q})} \mathrm{d}\xi = 0$$

to a value, necessarily positive. So the function $x \mapsto \frac{\partial V_i}{\partial \bar{q}_i}(x, \bar{Q})$ is monotone increasing on $(0, \infty)$ from

$$\frac{\partial V_i}{\partial \bar{q}_i}(0, \bar{Q}) = \int_{\bar{x}_i}^{\infty} \underbrace{\Gamma(0, \xi)}_{=0} \frac{\xi - \Sigma(\xi, \bar{Q}) - b\bar{q}_i}{1 + K(\xi, \bar{Q})} \mathrm{d}\xi = 0$$

to a value, necessarily positive. Because $x \mapsto \Gamma(x, \xi)$ is unbounded, the upper bound for $x \mapsto \frac{\partial V_i}{\partial \bar{q}_i}(x, \bar{Q})$ is necessarily ∞.

We assume that $C_i(\cdot)$ of the linear form $C_i(\bar{q}_i) = c_i \bar{q}_1$. Given the above, there exists a unique $\bar{x}^i_\star(\bar{q}_i) > 0$ such that

$$\frac{\partial V_i}{\partial \bar{q}_i}\Big(\bar{x}^i_\star(\bar{q}_{-i}), \big(0, \bar{q}_{-i}\big)^\top\Big) = c_i \quad \text{and} \quad \frac{\partial V_i}{\partial \bar{q}_i}\Big(x, \big(0, \bar{q}_{-i}\big)^\top\Big) > c_i \text{ for all } x > \bar{x}^i_\star\big(\bar{q}_{-i}\big). \tag{E.1}$$

It follows that

$$c_j > c_i \implies \bar{x}^j_\star(\bar{q}) > \bar{x}^i_\star(\bar{q}) \quad \text{for the same vector } \bar{q} \in \mathbb{R}^{k-1}; \tag{E.2}$$

in other words, the thresholds $\{\bar{x}^i_\star(\bar{q}); i = 1, \ldots, m\}$ are ranked as the unit costs $\{c_i; i = 1, \ldots, m\}$. We recall from Proposition 1 that $\bar{q}_i \mapsto V_i(x, \bar{Q})$ is strictly concave. So $\bar{q}_i \mapsto \frac{\partial V_i}{\partial \bar{q}_i}\big(x, \bar{Q}\big)$ is strictly decreasing on $(0, \infty)$ from

$$\frac{\partial V_i}{\partial \bar{q}_i}\Big(x, \big(0, \bar{q}_{-i}\big)^\top\Big) > c_i$$

to some value. It results from eqs. (B.4) and (C.2) that

$$\frac{\partial V_i}{\partial \bar{q}_i}\big(x, \bar{Q}\big) = \int_{\bar{x}_i}^{\infty} \Gamma(x, \xi) \frac{\xi - \Sigma(\xi, \bar{Q}) - b\bar{q}_i}{1 + K(\xi, \bar{Q})} \mathrm{d}\xi.$$

It follows from the definitions of Σ and K in eqs. (7c) and (7d) respectively that

$$\lim_{\bar{q}_i \uparrow \infty} V_i\big(x, \bar{Q}\big) = -\infty.$$

Consequently, if $x > \bar{x}^i_\star\big(\bar{q}_{-i}\big)$, then there exists a unique $\mathcal{R}_i\big(x, \bar{q}_{-i}\big)$ such that

$$\frac{\partial V_i}{\partial \bar{q}_i}\Big(x, \big(\mathcal{R}_i(x, \bar{q}_{-i}), \bar{q}_{-i}\big)^\top\Big) = c_i. \tag{E.3}$$

It follows that

$$c_j > c_i \implies \mathcal{R}_j(x, \bar{q}) > \mathcal{R}_i(x, \bar{q}) \quad \forall (x, \bar{q}) \in \mathbb{R} \times \mathbb{R}^{k-1}.$$

We define the best-response correspondence

$$\mathrm{R}_i\big(x, \bar{q}_{-i}\big) = \begin{cases} 0, & 0 < x \leq \bar{x}^i_\star\big(\bar{q}_{-i}\big), \\ \mathcal{R}_i(x, \bar{q}_{-i}), & x \geq \bar{x}^i_\star\big(\bar{q}_{-i}\big). \end{cases}$$

We drop the dependence on x when no confusion arises and define the maps

$$\begin{aligned} \bar{x}_\star : \quad & \mathbb{R}^k && \to \mathbb{R}^k \\ & \big(\bar{q}_1, \cdots, \bar{q}_k\big)^\top && \mapsto \big(\bar{x}^i_\star(\bar{q}_{-i}), \cdots, \bar{x}^i_\star(\bar{q}_{-i})\big)^\top \\ \mathcal{R} : \quad & \mathbb{R}^k && \to \mathbb{R}^k \\ & \big(\bar{q}_1, \cdots, \bar{q}_k\big)^\top && \mapsto \big(\mathcal{R}_1(\bar{q}_{-1}), \cdots, \mathcal{R}_k(\bar{q}_{-k})\big)^\top. \end{aligned}$$

Strategic substituability.. We recall the definition of $\bar{q}_j \mapsto \mathcal{R}_i\big(x, \bar{q}_{-i}\big)$ in (E.3) and obtain by implicit differentiation that

$$\frac{\partial \mathcal{R}_i}{\partial \bar{q}_j}\big(x, \bar{q}_{-i}\big) = -\frac{\frac{\partial^2 V_i}{\partial \bar{q}_j \partial \bar{q}_i}\Big(x, \big(\mathcal{R}_i(\bar{q}_{-i}), \bar{q}_{-i}\big)^\top\Big)}{\frac{\partial^2 V_i}{\partial \bar{q}_i^2}\Big(x, \big(\mathcal{R}_i(\bar{q}_{-i}), \bar{q}_{-i}\big)^\top\Big)}. \tag{E.4}$$

It follows from (B.4)

$$\frac{\partial^2 \pi^C_i}{\partial \bar{q}_i \partial \bar{q}_j}(x, \bar{Q}) = -\frac{b}{1 + K(x, \bar{Q})} \mathbb{1}_{\{x \geq \bar{x}_i\}} \mathbb{1}_{\{x \geq \bar{x}_j\}}$$

and from eq. (C.2) that

$$\begin{aligned} \frac{\partial^2 V_i}{\partial \bar{q}_i \partial \bar{q}_j}\big(x, \bar{Q}\big) &= \int_{\bar{x}_i}^\infty \Gamma(x, \xi) \frac{\partial^2 \pi^C_i}{\partial \bar{q}_i \partial \bar{q}_j}(\xi, \bar{Q}) \mathrm{d}\xi, \\ &= -\int_{\max\{\bar{x}_i; \bar{x}_j\}}^\infty \Gamma(x, \xi) \frac{b}{1 + K\big(\xi, \bar{Q}\big)} \mathrm{d}\xi < 0. \end{aligned} \tag{E.5}$$

It now follows from eqs. (C.4), (E.4) and (E.5) that

$$\frac{\partial \mathcal{R}_i}{\partial \bar{q}_j}\big(x,\bar{q}_{-i}\big) = -\frac{1}{2} \times \frac{\int_{\max\{\bar{x}_i;\bar{x}_j\}}^{\infty} \frac{\Gamma(x,\xi)}{1+K\left(\xi,(\mathcal{R}_i(\bar{q}_{-i}),\bar{q}_{-i})^\top\right)}\mathrm{d}\xi}{\int_{\bar{x}_i}^{\infty} \frac{\Gamma(x,\xi)}{1+K\left(\xi,(\mathcal{R}_i(\bar{q}_{-i}),\bar{q}_{-i})^\top\right)}\mathrm{d}\xi} \in \Big(-\frac{1}{2},0\Big). \tag{E.6}$$

A consequence is that the capacity choices are strategic substitutes. Further, by implicit differentiation of $\bar{x}^i_\star\big(\bar{q}_{-i}\big)$ defined in (E.1),

$$\frac{\partial \bar{x}^i_\star}{\partial \bar{q}_j}(\bar{q}_{-i}) = -\frac{\frac{\partial V_i}{\partial \bar{q}_i \partial \bar{q}_j}\Big(\bar{x}^i_\star(\bar{q}_{-i}),\big(0,\bar{q}_{-i}\big)^\top\Big)}{\frac{\partial V_i}{\partial x \partial \bar{q}_i}\Big(\bar{x}^i_\star(\bar{q}_{-i}),\big(0,\bar{q}_{-i}\big)^\top\Big)}. \tag{E.7}$$

It follows from eqs. (B.4), (C.2) and (E.5)

$$\frac{\partial \bar{x}^i_\star}{\partial \bar{q}_j}(\bar{q}_{-i}) = \frac{\int_{\max\{\bar{x}_i;\bar{x}_j\}}^{\infty} \Gamma(\bar{x}^i_\star,\xi)\frac{b}{1+K\left(\xi,(0,\bar{q}_{-i})^\top\right)}\mathrm{d}\xi}{\int_{\bar{x}_i}^{\infty} \frac{\partial \Gamma}{\partial x}(x,\xi)\frac{\xi-\Sigma(\xi,(0,\bar{q}_{-i})^\top)}{1+K(\xi,(0,\bar{q}_{-i})^\top)}\mathrm{d}\xi} > 0.$$

We apply the (multivariate) the chain rule to the function $\bar{q}_i \mapsto \bar{x}^i_\star\big(\mathcal{R}_{-i}(\bar{q}_i)\big)$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}\bar{q}_i}\Big(\bar{x}^i_\star\big(\mathcal{R}_{-i}(\bar{q}_i)\big)\Big) = \sum_{j\neq i} \frac{\partial \bar{x}^i_\star}{\partial \bar{q}_j}\big(\mathcal{R}_{-i}(\bar{q}_i)\big) \times \frac{\partial \mathcal{R}_i}{\partial \bar{q}_j}\big(\bar{q}_{-i}\big) < 0.$$

Contraction mapping. We consider the normed vector space $\big(\mathbb{R}^k,\|\cdot\|_k\big)$ with the taxicab distance

$$\|Q\|_k = \sum_{i=1}^{k} |q_i|.$$

- First, <u>assume that $k=2$</u> and that x is sufficiently large for $\mathcal{R}_i$ to be defined. We have

  $$\|\mathcal{R}(q')-\mathcal{R}(q)\|_2 = \big|\mathcal{R}_1(\bar{q}_2')-\mathcal{R}_1(\bar{q}_2)\big| + \big|\mathcal{R}_2(\bar{q}_1')-\mathcal{R}_2(\bar{q}_1)\big|$$

  By the mean value theorem,

  $$\mathcal{R}_1(\bar{q}_2')-\mathcal{R}_1(\bar{q}_2) = \int_0^1 \frac{\partial \mathcal{R}_1}{\partial \bar{q}_2}\Big(x,\theta q_2' + (1-\theta)q_2\Big)(q_2'-q_2)\mathrm{d}\theta.$$

  It now follows from (E.6) that

  $$\Big|\mathcal{R}_1(\bar{q}_2')-\mathcal{R}_1(\bar{q}_2)\Big| \leq \frac{1}{2}\big|q_2'-q_2\big|.$$

We proceed similarly for $\left|\mathcal{R}_2(\bar{q}_1') - \mathcal{R}_2(\bar{q}_1)\right|$, obtaining

$$\begin{aligned}\|\mathcal{R}(q') - \mathcal{R}(q)\|_2 &\leq \frac{1}{2}\Big(\left|q_1' - q_1\right| + \left|q_2' - q_2\right|\Big)\\ &\leq \frac{1}{2}\|q' - q\|_2.\end{aligned}$$

We conclude that the map $\mathcal{R}$ is a contraction when $k = 2$. Consequently, by the Banach fixed-point theorem, there exists a unique fixed point, noted $\hat{q} \in \mathbb{R}_+^2$ of the map $\mathcal{R}$. We summarize in case $k = 2$:

a) the pair $(0,0)^\top$ is the unique NE iff $x < \min\{\bar{x}_\star^1(0); \bar{x}_\star^2(0)\}$;

b) the pair $\big(0, \mathcal{R}_2(0)\big)^\top$ is the unique NE iff $0 < x < \bar{x}_1^\star(\mathcal{R}_2(0))$ and $x > \bar{x}_\star^2(0)$;

b') the pair $\big(\mathcal{R}_1(0), 0\big)^\top$ is the unique NE iff $x > \bar{x}_\star^1\big(0\big)$ and $0 < x < \bar{x}_2^\star(\mathcal{R}_1(0))$;

c) the fixed point $\hat{q} \in \mathbb{R}_+^2$ of the map $\mathcal{R}$ is the unique NE iff $x > \max\big\{\bar{x}_1^\star(\hat{q}_2); \bar{x}_2^\star(\hat{q}_1)\big\}$.

Because of the ranking $c_2 < c_1$ and the property (E.2), we have $\bar{x}_\star^2(0) < \bar{x}_\star^1(0)$; so the set $\{x > 0 | \bar{x}_\star^1(0) < x < \bar{x}_\star^2(0)\}$ is empty and the equilibrium described in b') does not arise. This ranking of thresholds also simplifies the expressions of the NE as per Proposition 3.

- Assume that $k \in \{3, 4, \dots\}$ and that x is sufficiently large. Then, by the mean-value theorem in normed vector space,

$$\|\mathcal{R}(q') - \mathcal{R}(q)\|_k \leq \|q' - q\|_k \times \sup_{\tilde{q} \in [q,q']} \Big\{\big\|D\mathcal{R}(\tilde{q})\big\|_k\Big\},$$

where $D\mathcal{R}$ is understood as the Fréchet derivative of $\mathcal{R}$, given by

$$D\mathcal{R}(q)h = \begin{pmatrix} 0 & \frac{\partial \mathcal{R}_1}{\partial \bar{q}_2}(q) & \dots & \frac{\partial \mathcal{R}_1}{\partial \bar{q}_{m-1}}(q) & \frac{\partial \mathcal{R}_1}{\partial \bar{q}_m}(q) \\ \frac{\partial \mathcal{R}_2}{\partial \bar{q}_1}(q) & 0 & \dots & \frac{\partial \mathcal{R}_1}{\partial \bar{q}_{m-1}}(q) & \frac{\partial \mathcal{R}_1}{\partial \bar{q}_m}(q) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \mathcal{R}_k}{\partial \bar{q}_1}(q) & \frac{\partial \mathcal{R}_k}{\partial \bar{q}_2}(q) & \dots & \frac{\partial \mathcal{R}_k}{\partial \bar{q}_{m-1}}(q) & 0 \end{pmatrix} h, \quad h \in \mathbb{R}^k.$$

The condition

$$\sum_{j \neq i} \left|\frac{\partial \mathcal{R}_j}{\partial \bar{q}_j}(q)\right| < 1, \quad \forall q \in \mathbb{R}^k \text{ and } \forall i \in \{1, \dots, k\}$$

is thus sufficient to establish that the mapping $\mathcal{R}$ is a contraction. Using the equation (E.6), we can re-write this condition in terms of model primitives, yet could not find a meaningful economic interpretation. Consequently, by the Banach fixed-point theorem, there exists a unique fixed point noted $\hat{q}$. This fixed point $\hat{q}$ will be unique NE if $x > \max\big\{\bar{x}_\star^i(\hat{q}_{-i}); i = 1, \dots, k\big\}$. If this condition is not met, then we can proceed as above to determine the unique NE in the lower demand intervals. Again, some cases are not relevant because of the ranking $c_1 > c_2 > c_3 > \dots > c_k$ and the property (E.2).

This concludes the proof.

Appendix F. Proof of Proposition 4

Consider an arbitrary capacity vector $\bar{Q}$ and an industry output Q_t that satisfies constraints (2). From (3), the demand function reads $D(x,p) = [x-p]/b$. The consumer surplus (rate) is here

$$\phi\left(x,q,\bar{Q}\right) = \int_{x-bq}^{x} D(x,p)\,\mathrm{d}p = \frac{1}{b}\left[xp - \frac{1}{2}p^2\right]_{x-bq}^{x} = \frac{b}{2}q^2. \tag{F.1}$$

Consider now the policy in (7a). We restate the aggregate industry output $Q^C(x)$ in (B.1) as:

$$Q^C(x) = \frac{1}{b}\left[\frac{K_m}{K_{m-1}}x + \left(\frac{\Sigma_m}{K_{m-1}} - c\right)\right]^+, \quad x \in (\bar{x}_{m-1}, \bar{x}_m), \tag{F.2}$$

for $m = 0,\dots,k+1$. Setting $Q^C(x)$ in (F.2) into $\phi\left(x, Q_t, \bar{Q}\right)$ in (F.1) yields the consumer surplus (rate) in Cournot oligopoly, given by

$$\varsigma(x,\bar{Q}) := \phi\left(x, Q^C(x), \bar{Q}\right) = \left[\sqrt{\rho_m^2}x + \sqrt{\rho_m^0}\right]^2 \mathbb{1}_{\{x\geq c\}} = \left[\rho_m^2 x^2 + \rho_m^1 x + \rho_m^0\right]\mathbb{1}_{\{x\geq c\}},$$

if $x \in (\bar{x}_{m-1}, \bar{x}_m)$, $m = 0,\dots,k+1$, with $\rho_m \in \mathbb{R}^3$ given in (20f). For $x \in (\bar{x}_k, \bar{x}_{k+1}) = (\bar{x}_k, \infty)$, $\varsigma(x,\bar{Q})$ simplifies to $b[\sum_{j=0}^{k}\bar{q}_j]^2/2$. We want to prove that $x \mapsto \varsigma\left(x;\bar{Q}\right)$ is continuous. Continuity is immediate in the open intervals $(\bar{x}_{m-1}, \bar{x}_m)$ but not at $\bar{x}_m$ for $m = 0,\dots,k$. We note the left and right limits at $\bar{x}_m$ by $\bar{x}_m^-$ and $\bar{x}_m^+$ respectively. Because $\Sigma_1 = c$, we have from (20f) $\sqrt{\rho_1^2} = \left(\frac{k}{k+1}\right)/\sqrt{2b}$ and $\sqrt{\rho_1^0} = -c\left(\frac{k}{k+1}\right)/\sqrt{2b}$. It follows that $\varsigma(c^+,\bar{Q}) = 0 = \varsigma(c^-,\bar{Q})$, proving continuity at $x = \bar{x}_0$. For $m = 1,\dots,k$, we have from (20f) $\sqrt{\rho_{m+1}^2} = \left(\frac{K_{m+1}}{K_m}\right)/\sqrt{2b}$ and $\sqrt{\rho_{m+1}^0} = \left(\frac{\Sigma_{m+1}}{K_m} - c\right)/\sqrt{2b}$. Besides, $\Sigma_{m+1} = \bar{x}_m - bK_m\bar{q}_m$ by definition of $\bar{x}_m$ and Σ_{m+1}. Hence, the right limit is

$$\varsigma(\bar{x}_m^+,\bar{Q}) = \frac{1}{2b}\left[\frac{K_{m+1}}{K_m}\bar{x}_m + \frac{\bar{x}_m - bK_m\bar{q}_m}{K_m} - c\right]^2 = \frac{1}{2b}\left[\bar{x}_m - b\bar{q}_m - c\right]^2.$$

Using the relationship $\bar{x}_m = \Sigma_m + bK_{m-1}\bar{q}_m$, the left limit obtains as

$$\varsigma(\bar{x}_m^-,\bar{Q}) = \frac{1}{2b}\left[\bar{x}_m + \frac{\Sigma_m - \bar{x}_m}{K_{m-1}} - c\right]^2 = \frac{1}{2b}\left[\bar{x}_m - b\bar{q}_m - c\right]^2/2b.$$

Hence, $\varsigma(\bar{x}_m^-,\bar{Q}) = \varsigma(\bar{x}_m^+,\bar{Q})$, proving continuity at $x = \bar{x}_m$, $m = 1,\dots,k$. In summary, $x \mapsto \varsigma(x,\bar{Q})$ is continuous on $\mathbb{R}_+$.

Applying the Feynmann-Kac formula to the consumer surplus (value) in Cournot oligopoly,

$$S(x,\bar{Q}) := \mathcal{CS}\big(x, Q^C(\cdot), \bar{Q}\big) = \mathbb{E}\left[\int_0^\infty e^{-rt}\varsigma\big(X_t,\bar{Q}\big)\ \mathrm{d}t\right]$$

yields $\mathcal{L}S(x,\bar{Q}) = \varsigma(x,\bar{Q})$, which has the following general solution $S(x,\bar{Q}) = s(x,\bar{Q}) + A_m x^{\gamma_A} + B_m x^{\gamma_B}$ for $x \in (\bar{x}_{m-1}, \bar{x}_m), m = 0,\ldots,k+1$, where $s(x,\bar{Q})$ is the perpetuity value in (20a). Because $x \mapsto \varsigma(x,\bar{Q})$ is continuous, we look for a solution $x \mapsto S(x,\bar{Q})$ that is C^1. Besides, we assume boundary conditions (i) $S(x,\bar{Q}) = 0$ and (ii) $\lim_{x\to\infty}\big\{S(x,\bar{Q}) - s(x,\bar{Q})\big\} = 0$. Because we set $\rho_0 = (0,0,0)$, $\eta_A(x,\rho_0) = \eta_B(x,\rho_0) = 0$. We proceed as in Appendix Appendix C. Using the notations in (20e), we obtain for $m = 0,\ldots,k$,

$$A_m = A_{m+1} + \bar{x}_m^{-\gamma_A}\left[\eta_A\left(\bar{x}_m,\rho_{m+1}\right) - \eta_A\left(\bar{x}_m,\rho_m\right)\right] \text{ and}$$

$$B_{m+1} = B_m + \bar{x}_m^{-\gamma_B}\left[\eta_B\left(\bar{x}_m,\rho_{m+1}\right) - \eta_B\left(\bar{x}_m,\rho_m\right)\right].$$

Besides, conditions (i) and (ii) yields $B_0 = 0$ and $A_{k+1} = 0$ respectively because $\gamma_B < 0 < \gamma_A$.

Define the sequences

$$\alpha_s^m(\bar{Q}), \beta_s^m(\bar{Q}) = \eta_{A/B}\left(\bar{x}_m,\rho_{m+1}\right) - \eta_{A/B}\left(\bar{x}_m,\rho_m\right), m = 0,\ldots,k. \tag{F.3}$$

Definition (20d) correspond to (F.3) specialized for $\bar{Q}^C(x_0)$. Sequences $\{A_n\}$ and $\{B_n\}$ are now expressed as

$$A_n = \begin{cases} \sum_{m=n}^k \bar{x}_m^{-\gamma_A}\alpha_s^m(\bar{Q}), & n = 0,\cdots,k \\ 0, & n = k+1, \end{cases}$$

$$B_n = \begin{cases} 0, & n = 0, \\ \sum_{m=0}^{n-1} \bar{x}_m^{-\gamma_B}\beta_s^m(\bar{Q}), & n = 1,\cdots,k+1. \end{cases}$$

Expressions (20b) and (20c) obtain from re-writing the above for capacity choice $\bar{Q}^C(x_0)$. Proposition 4 exhibits value $S(x,x_0) = S\left(x,\bar{Q}^C(x_0)\right)$ where $\bar{Q}^C(x_0)$ is the equilibrium capacity vector described in Proposition 3.

Appendix G. Proof of Proposition 5

The consumer surplus flow is given in (F.1), while producer surplus is $(x - bq - c)q$. It follows that the social welfare flow is $(x-c)q - \frac{1}{2}bq^2$. We are interested in discount values, not flows, i.e., in

$$W\big(x, \bar{Q}\big) := \mathbb{E}\left[\int_0^\infty e^{-rt}\xi\big(X_t, \bar{Q}\big)\,\mathrm{d}t\right] \quad \text{where} \quad \xi\big(x, \bar{Q}\big) := \max_{0 \le q \le \bar{Q}} \Big\{-\frac{1}{2}bq^2 + (x-c)q\Big\}. \tag{G.1}$$

Note that (i) $q \mapsto -\frac{1}{2}bq^2 + (x-c)q$ is concave and (ii) the feasibility region is convex, the KKT conditions are here both necessary and sufficient to find a solution to the constrained optimization problem defining ξ. Let's introduce the Lagrangian $\mathscr{L}\left(q, \lambda, x, \bar{Q}\right) = -bq^2/2 + (x-c)q + \lambda_3\left(\bar{Q} - q\right) + \lambda_4 q$, with $\lambda = (\lambda_3, \lambda_4)$. Several equations/inequalities must be satisfied:

$$-bq^\star + x - c - \lambda_3^\star + \lambda_4^\star = 0, \tag{G.2}$$

$\lambda_3^\star\left(\bar{Q} - q\right) = 0$, $\lambda_4^\star q^\star = 0$, $\bar{Q} - q^\star \geq 0$, $\lambda_3^\star \geq 0$, and $\lambda_4^\star \geq 0$. We analyze three cases in turn: case A ($\lambda_3^\star = \lambda_4^\star = 0$), case B ($q^\star = \lambda_3^\star = 0$), and case C ($q^\star = \bar{Q}$; $\lambda_4^\star = 0$). In case A, we have from (G.2), $q^\star = (x-c)/b$; this is feasible if $x \in [c, c + b\bar{Q}]$. In case B, we have from (G.2) $\lambda_4^\star = c - x$, which is feasible if $x \geq c$. In case C, (G.2) implies $\lambda_3^\star = x - c - b\bar{Q}$, which is feasible if $x \geq \bar{x}$ with $\bar{x} := c + b\bar{Q}$. Note that these cases are mutually exclusive. This proves the social optimality of the output strategy given by

$$Q^\star(x) = \left[\frac{x-c}{b}\right]\mathbb{1}_{\{x \in [c, \bar{x}]\}} + \bar{Q}\mathbb{1}_{\{x \geq \bar{x}\}}. \tag{G.3}$$

Substituting $Q^\star(x)$ in (G.3) yields

$$\xi\left(x, \bar{Q}\right) = \left[\frac{(x-c)^2}{2b}\right]\mathbb{1}_{\{x \in [c, \bar{x}]\}} + \left[-\frac{b}{2}\bar{Q}^2 + (x-c)\bar{Q}\right]\mathbb{1}_{\{x \geq \bar{x}\}}.$$

We can easily show that $x \mapsto \xi\left(x; \bar{Q}\right)$ is continuous.

The functional representation of W in (G.1) is the weak solution W of $\mathcal{L}W(x, \bar{Q}) = \xi\left(x, \bar{Q}\right)$. We look for a solution that is C^1. This boils down to solving:

$$W\left(x, \bar{Q}\right) = \begin{cases} A_1 x^{\gamma_A} & \text{if } x \leq c, \\ \frac{1}{2b}\left[\frac{x^2}{\mathscr{Q}(2)} - \frac{2cx}{r-\mu} + \frac{c^2}{r}\right] + A_2 x^{\gamma_A} + B_1 x^{\gamma_B} & \text{if } x \in [c, \bar{x}], \\ \frac{x\bar{Q}}{r-\mu} - \frac{\bar{Q}}{r}\left[c + \frac{b}{2}\bar{Q}\right] + B_2 x^{\gamma_B} & \text{if } x \geq \bar{x}, \end{cases} \tag{G.4a}$$

subject to two conditions at threshold c:

$$A_1 c^{\gamma_A} = \frac{c^2}{2b}\left[\frac{1}{\mathscr{Q}(2)} - \frac{2}{r-\mu} + \frac{1}{r}\right] + A_2 c^{\gamma_A} + B_1 c^{\gamma_B}, \tag{G.4b}$$

$$\gamma_A A_1 c^{\gamma_A - 1} = \frac{c}{2b}\left[\frac{2}{\mathscr{Q}(2)} - \frac{2}{r-\mu}\right] + \gamma_A A_2 c^{\gamma_A-1} + \gamma_B B_1 c^{\gamma_B - 1}, \tag{G.4c}$$

and two conditions at threshold $\bar{x}$:

$$\frac{1}{2b}\left[\frac{\bar{x}^2}{\mathscr{Q}(2)} - \frac{2\bar{x}c}{r-\mu} + \frac{c^2}{r}\right] + A_2\bar{x}^{\gamma_A} + B_1\bar{x}^{\gamma_B} = \frac{\bar{Q}\bar{x}}{r-\mu} - \frac{\bar{Q}}{r}\left[c + \frac{b}{2}\bar{Q}\right] + B_2\bar{x}^{\gamma_B}, \tag{G.4d}$$

$$\frac{1}{2b}\left[\frac{2\bar{x}}{\mathscr{Q}(2)} - \frac{2c}{r-\mu}\right] + \gamma_A A_2 \bar{x}^{\gamma_A-1} + \gamma_B B_1 \bar{x}^{\gamma_B - 1} = \frac{\bar{Q}}{r-\mu} + \gamma_B B_2 \bar{x}^{\gamma_B-1}. \tag{G.4e}$$

It follows from eqs. (G.4b) and (G.4c) that

$$A_1 - A_2 = \frac{c^{2-\gamma_A}}{2b}\frac{1}{\gamma_A-\gamma_B}\left\{\frac{2-\gamma_B}{\mathcal{Q}(2)} - 2\frac{1-\gamma_B}{r-\mu} - \frac{\gamma_B}{r}\right\},$$

$$B_1 = \frac{c^{2-\gamma_B}}{2b}\frac{1}{\gamma_A-\gamma_B}\left\{\frac{2-\gamma_A}{\mathcal{Q}(2)} - 2\frac{1-\gamma_A}{r-\mu} - \frac{\gamma_A}{r}\right\}.$$

Recalling the definitions of κ_A and κ_B in (15d), we can now write

$$A_1 = A_2 + \frac{c^{2-\gamma_A}}{2b}\kappa_A \quad \text{and} \quad B_1 = \frac{c^{2-\gamma_B}}{2b}\kappa_B.$$

Further, from eqs. (G.4d) and (G.4e),

$$\begin{aligned}
A_2 &= -\frac{\bar{x}^{-\gamma_A}}{\gamma_A-\gamma_B}\left\{\frac{\bar{x}^2}{2b}\frac{2-\gamma_B}{\mathcal{Q}(2)} - \left[\bar{Q}\bar{x} + \frac{\bar{x}c}{b}\right]\frac{1-\gamma_B}{r-\mu} - \left[\frac{c^2}{2b} + \bar{Q}\big(c + \frac{b}{2}\bar{Q}\big)\right]\frac{\gamma_B}{r}\right\} \\
&= -\frac{\bar{x}^{-\gamma_A}}{\gamma_A-\gamma_B}\left\{\frac{\bar{x}^2}{2b}\frac{2-\gamma_B}{\mathcal{Q}(2)} - \underbrace{\left[c + b\bar{Q}\right]}_{=:\bar{x}}\frac{\bar{x}}{b}\frac{1-\gamma_B}{r-\mu} - \frac{1}{2b}\underbrace{\left[c^2 + 2b\bar{Q}c + b^2\bar{Q}^2\right]}_{\bar{x}^2}\frac{\gamma_B}{r}\right\}, \\
&= -\frac{\bar{x}^{2-\gamma_A}}{2b}\frac{1}{\gamma_A-\gamma_B}\left\{\frac{2-\gamma_B}{\mathcal{Q}(2)} - 2\frac{1-\gamma_B}{r-\mu} - \frac{\gamma_B}{r}\right\}, \\
&= -\frac{\bar{x}^{2-\gamma_A}}{2b}\kappa_A, \\
B_2 - B_1 &= -\frac{\bar{x}^{-\gamma_B}}{\gamma_A-\gamma_B}\left\{\frac{\bar{x}^2}{2b}\frac{2-\gamma_A}{\mathcal{Q}(2)} - \underbrace{\left[\bar{Q}\bar{x} + \frac{\bar{x}c}{b}\right]}_{=\frac{\bar{x}^2}{b}}\frac{1-\gamma_A}{r-\mu} - \underbrace{\left[\frac{c^2}{2b} + \bar{Q}\big(c + \frac{b}{2}\bar{Q}\big)\right]}_{=\frac{\bar{x}^2}{2b}}\frac{\gamma_A}{r}\right\} \\
&= -\frac{\bar{x}^{2-\gamma_B}}{2b}\kappa_B.
\end{aligned}$$

We conclude that

$$A_1 = \frac{\kappa_A}{2b}\left[c^{2-\gamma_A} - \bar{x}^{2-\gamma_A}\right], \quad A_2 = -\frac{\bar{x}^{2-\gamma_A}}{2b}\kappa_A, \quad B_1 = \frac{c^{2-\gamma_B}}{2b}\kappa_B, \quad \text{and} \quad B_2 = \frac{\kappa_B}{2b}\left[c^{2-\gamma_B} - \bar{x}^{2-\gamma_B}\right].$$

This concludes the proof.