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A PROCEDURE FOR TIMING TECHNOLOGY ADOPTIONS USING A GENERALIZED FORM OF TECHNOLOGICAL PROGRESS

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A procedure for timing technology adoptions using a generalized form of technological progress

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Abstract

In this paper we determine a procedure for the optimal timing of technology adoptions by a cost minimizing firm, when technological progress is a gradual and incremental process. Using the format of a stochastic rotation problem we generalize the baseline case in which the state of technology has an equal impact on operating costs and switching costs (discussed in earlier research), to allow for a less restrictive form of technological progress. Despite this additional complexity a remarkably clear recursive structure in the optimality conditions could be detected. Using these new conditions, we confirm an earlier explanation for the delay in the adoption of new technologies in a considerably more general setting.

1 Introduction

The rotation problem, in which the gain from periodic replacement of some asset by another is to be optimized, is a classic in economic theory. Its first appearance in economic theory was probably as the forestry problem (or Faustmann problem, after Martin Faustmann - See also Scorgie & Kennedy [12]). The forestry problem is about the optimal frequency of harvesting and replanting of forests, in order to maximize the profits of selling timber. The

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basic problem is simple, its solution often is not. The main difficulty is that in order to find the optimal timing of the first harvest, it is required to know future profits which depend on the timing of future harvesting. Therefore, the entire chain of replacements must be optimized simultaneously, which is a typical infinite horizon dynamic programming problem. Standard references for this problem are Samuelson [11] for a general overview, or Mitra & Wan [8][9] for a more technical approach.

Nowadays, the importance of the forestry problem is of course rather academic, but the general nature of a rotation problem isn't. Many contemporary problems in modern economics are related to or can be described as a rotation problem. Aside from numerous applications in agricultural economics, the entire problem of depreciation and replacement of capital equipment is a perfect example of a rotation problem. Wear and tear causes equipment to become more costly to operate as it ages, which requires periodic replacement with new equipment. The problem has many practical applications for all sorts of equipment, ranging from industrial installations to durable consumer goods (See e.g. Howe & McCabe [6] for an exploration of the problem in a deterministic setting or more recently Mauer & Ott [7] for the case in which wear is a stochastic process).

In this paper, we consider the replacement of existing equipment due to some other reason then wear and tear. Equipment is also frequently replaced because the state of the technology embodied in the equipment has become inferior to more recent equipment. A typical example would be the replacement of computer equipment. Computers usually are not replaced because they are worn, but because more efficient equipment has become available. The question when to adopt these new technologies when technological progress is stochastic, is the central topic of this paper. It has been observed that the adoption of new technologies often occurs quite late, i.e. much later then the traditional npv-rule used to evaluate investments in capital equipment suggests. Recent findings about real-options and uncertainty (Dixit & Pindyck [3], Trigeorgis [13]) point out that a probable explanation of this tardy adoption of new technologies could be the value of the option to postpone investment. Since this option is sacrificed at the time of replacement, it constitutes an additional cost of investment and hence warrants additional delay in the adoption of the new technology.

The timing of technology adoption when technology is subject to stochastic movement has been studied by several authors. In their paper on the replacement problem, Mauer & Ott [7] also allow for a single switch to a different technology. Their contribution to the issue of technology adoption is limited however, because they assume that the new technology will remain in operation perpetually. Technology adoption is thus treated as an optimal stopping problem, not as a rotation problem. A somewhat similar limitation occurs in Grenadier & Weiss [5], who consider a technology adoption model in which an operator has two options to switch to more efficient technology. An interesting feature of their model is that it allows for 'leapfrogging' (the operator skips the first option, but uses the second) and 'lagging' (the operator waits to adopt the first technological improvement until the second improvement has become available). Farzin, Huisman & Kort [4] on the other hand, present the problem in a true rotation model. They identify an option value of waiting for the last switch in a sequence, but fail to do so for earlier technology switches. Their model is also typically suited for situations in which technological progress is lumpy, i.e. improvements are sudden and sizable.

A paper in which technological progress is a continuous flow of marginal improvements is presented in Bethuyne [1]. The model also describes an entire rotation cycle and (contrary to Farzin et al.) identifies an option value at each technology switch. However, the model is still somewhat limited by the assumptions on the effect of technological progress on costs. In order to determine explicit analytic solutions for the optimal switching values, the model starts from the assumption that the ratio of operating costs of the new equipment over the net installation cost is unaffected by technological progress. Yet it is not unlikely that technological progress has a different effect on both types of costs. In this paper, we will generalize this earlier model to allow for an unequal effect of the state of technology on operating and installation costs. A procedure will be developed to determine implicit conditions for the optimal timing of technology adoptions using less restrictive assumptions. We will use the resulting conditions to explore the effect of different types of technological progress on the simulation results from previous research.

This paper is further organized as follows. In the next paragraph we sketch the main assumptions and briefly explain the basic setup of the solution method, referring to preliminary research results. The third section then explores the problem of sequential technology adoption the effect of different types of technological progress on the optimal switching value. We derive the optimality conditions for the entire sequence of switches. The fourth section contains the results from numerical simulations based on these optimality conditions. The paper is concluded with a summary of the main findings.

2 Technological progress

Technological progress can have a qualitative and a quantitative impact on production. In the first case, the characteristics (quality, durability, ...) of the output are altered. The latter case refers to changes in productivity. The present analysis focuses on the quantitative effect, measured as a reduction in the unit-cost of production. More specifically, we concentrate on technological progress characterized by a slow but steady process of cost reductions. The computer industry is a well-known example of a sector in which such gradual and continuous progress occurs. Although technology evolves rapidly in this sector, the difference between two successive generations of computer hardware is generally rather small[1].

We will describe this form of gradual technological progress by a cost index θ that follows a geometric Brownian motion $d\theta = -g\theta\, dt + \sigma\theta\, dz$, where dz is a Wiener process. Pindyck [10] uses a similar specification to describe the input cost uncertainty of technological progress. The negative sign of the drift-rate refers to the fact that technological progress reduces costs. Assume that at present, equipment is in operation and let $m\theta_0$ be the cost per unit of time of operating this equipment. We normalize $\theta_0 = 1$. We assume that the state of technological progress is embodied in the equipment, so that the operating cost of existing equipment does not change as a result of ongoing technological progress. Meanwhile, more efficient equipment becomes available. Let $\theta(t)$ is the value of the cost index at some point in time t (for simplicity, we will drop the argument t in further notation). We then assume that new equipment installed at t costs $m\theta$ per unit of time to operate and $P\theta^\alpha$ to install (α is a parameter). The central objective in this paper is to determine the optimal time to switch from the existing equipment of the older technology (the defender) to the new and more efficient technology (the challenger). We assume that a firm has n options to switch ($n = 1, 2, \ldots \infty$) and seeks to minimize costs over an infinite time horizon.

The special case in which technological progress has no bearing on the ratio of operating costs and installation costs ($\alpha = 1$) has been studied extensively in previous research (See Bethuyne [1]). However, the present setting allows for technological progress to have an unequal effect on both cost com-

[1] Of course, the difference between e.g. a 486 processor and a Pentium processor may seem considerable. However, of each processor type, different subtypes exist. Although the difference between an average 486 processor and an average pentium is considerable, the difference between two successive systems (the most efficient 486 and the earliest pentium) seems rather small.

ponents. Obviously when $\alpha < 1$ technological progress has a larger impact on operating costs then on installation costs (and vice versa for $\alpha > 1$). Unfortunately, this generalization increases the complexity of the optimization problem considerably. In what follows, a procedure will be developed to determine the optimal switching values of θ triggering technology adoption by a cost-minimizing firm.

In order to do so, consider an operator that currently uses equipment of technology θ_0. Assume that the cost of production per unit of time with this equipment can be expressed as:

$$m(\theta_0) = m\theta_0 \tag{1}$$

where m is a constant. Let $C(\theta;\theta_0)$ be the expected present value of the production cost at t over an infinite time-horizon, for an operator using technology θ_0, when the state of the most recent technology is θ (θ_0 is a parameter and will be mentioned only when useful). If the firm cannot switch to the more advanced technology, $C(\theta)$ will simply be $\frac{m\theta_0}{i}$ (in which i is the interest rate), i.e. the perpetual operating cost of the existing equipment. However, we will assume that the operator has one or more options to switch to a more efficient technology ($\theta < \theta_0$). Therefore $C(\theta)$ will be smaller than $\frac{m\theta_0}{i}$ because of potential gains from the technology switch.

If the state of technology θ follows a geometric Brownian motion, the cost of maintaining the present technology (the *continuation cost)* can be described by the following differential equation:

$$iC(\theta) = m\theta_0 + \frac{\varepsilon\left[dC\left(\theta\right)\right]}{dt} \tag{2}$$

The opportunity cost of operating the present equipment and its future challengers over an infinite time horizon can be split in immediate (operating) costs and the expected change in future costs. The source of $\varepsilon\left[dC\left(\theta\right)\right]$ is to be found in changes in the state of available technology. If there were no opportunity to replace the defending technology with a technologically more advanced challenger, $\varepsilon\left[dC\left(\theta\right)\right]$ would simply be zero[2]. For this standard type of technological progress, the solution of eq.(2) is of the form:

$$C(\theta) = \frac{m\theta_0}{i} + F\theta^{\lambda} \tag{3}$$

[2]Notice the analogy with the return on stocks and bonds (which of course is maximized instead of minimized). The expected return of stocks or bonds can also be split in an immediate return (dividends or coupons) and the expected change in value (capital gains).

in which:

$$\lambda = \frac{1}{2} + \frac{g}{\sigma^2} - \sqrt{\left(\frac{1}{2} + \frac{g}{\sigma^2}\right)^2 + \frac{2i}{\sigma^2}} \tag{4}$$

and F is a constant (See Dixit [2] or Dixit & Pindyck [3] for a general solution to optimal stopping problems). The first part of eq.(3) reflects the cost of perpetually operating the defending technology. The second term is negative and reflects the expected costs savings due to the option to switch to a more efficient technology. Notice that for $\theta = \theta_0 = 1$, the constant F immediately reflects the value of expected cost savings due to future technology switches, at the time the defending equipment is installed.

The option to switch technologies will be exercised when θ hits a lower barrier θ^*. The present value of all costs occurring from that moment an are represented by $\Omega(\theta^*)$ (referred to as the *terminal cost),* which includes the immediate switching cost as well as all costs related to operating the new technology afterwards. When $\theta = \theta^*$ the following boundary condition must hold (*value-matching condition*):

$$C(\theta^*) = F\theta^{*^{\lambda}} + \frac{m\theta_0}{i} = \Omega\left(\theta^*\right) \tag{5}$$

In addition, for the barrier θ^* to be an optimal stopping point, the continuation cost is required to be tangent to the terminal cost function (the *smooth-pasting condition*).

$$C'(\theta^*) = \lambda F\theta^{*^{(\lambda-1)}} = \Omega'\left(\theta^*\right) \tag{6}$$

Therefore F and θ^* can then be found solving eqs.(5) and (6) simultaneously. Notice that $F\theta^{*^{\lambda}}$ is the cost reduction due to the option to replace the defender. Obviously when the state of technology degenerates ($\theta \to \infty$), the option becomes worthless. A further comment on the endpoint function $\Omega\left(\theta\right)$ is in order now.

3 Sequential technology adoption

As a rule there is no absolute limit to the number of technology switches a firm can make over an infinite time horizon. For analytical purposes however, we will start with the assumption that an operator has a finite number (n) of options to switch to a new technology. Starting from the assumption that $n-1$ of these options have been used already, we determine the optimal timing to use the remaining option. The solution of this problem will then serve as a starting point for the problem in which the operator has two remaining options to adopt a new technology. This backward recursive solution method will allow to determine the optimal sequence of n switching values.

In this sequence, the first triggering value is of course by far the most important one, because we are primarily interested in the replacement timing of the technology that is currently in operation. All other switching values in the sequence are only relevant in so far that their value has an impact on the timing of the first replacement. Due to discounting, the marginal impact of an additional option to switch technologies on the timing of the first replacement decreases. As a result, for practical purposes a sufficiently large n may serve as an acceptable approximation for a chain of infinite technology switches.

3.1 The nth technology switch

Assume an operator with n options to switch to more efficient technologies has exercised $n-1$ options. When equipment of technology θ_{n-1} has just been installed, the operator has only one option to replace left. Then if the state of technology is θ, the general form of the present value of all costs is:

$$C_{n-1}(\theta) = F_{n-1}\theta^{\lambda} + \frac{m\theta_{n-1}}{i} \tag{7}$$

The terminal cost that occurs at the time of technology adoption is:

$$\Omega_{n-1}(\theta) = \frac{m\theta}{i} + P\theta^{\alpha} \tag{8}$$

At the switching value $\theta = \theta_n^*$, the following conditions must hold:

1. Value matching condition:

$$F_{n-1}\theta_n^{*\lambda} + \frac{m\theta_{n-1}}{i} = \Omega_{n-1}(\theta_n^*) \tag{9}$$

2. Smooth pasting condition:

$$\lambda F_{n-1}\theta_n^{*\lambda-1} = \Omega'_{n-1}(\theta_n^*) \tag{10}$$

The smooth pasting condition expresses that at the switching technology, the marginal reduction of the continuation cost due to technological progress equals the reduction of the terminal cost. From the smooth pasting condition we determine:

$$F_{n-1} = \frac{\Omega'_{n-1}\left(\theta_n^*\right)}{\lambda}\theta_n^{*^{1-\lambda}} \tag{11}$$

Substituting in the value matching condition leads to:

$$g^n\left(\theta_{n-1}, \theta_n\right) = \Omega_{n-1}\left(\theta_n^*\right) - \frac{\Omega'_{n-1}\left(\theta_n^*\right)}{\lambda}\theta_n^* - \frac{m\theta_{n-1}}{i} = 0 \tag{12}$$

For this particular terminal cost, the previous expression can be written as:

$$\frac{\lambda - 1}{\lambda}\frac{m\theta_n^*}{i} + \frac{\lambda - \alpha}{\lambda}P\theta_n^{*^{\alpha}} - \frac{m\theta_{n-1}}{i} = 0 \tag{13}$$

Since θ_{n-1} is a constant and the right-hand side of the previous expression is monotonically increasing in θ for $\theta > 0$, it implicitly determines a unique triggering value $\theta_n^*\left(\theta_{n-1}\right)$.

The minimal cost at the time the technology of level θ_{n-1} has just been installed is then:

$$C_{n-1}^*\left(\theta_{n-1}\right) = \frac{m\theta_{n-1}}{i} + \frac{\Omega'_{n-1}\left(\theta_n^*\right)}{\lambda}\theta_n^{*^{1-\lambda}}\theta_{n-1}^{\lambda} \tag{14}$$

which in this case is identical to:

$$C_{n-1}^*\left(\theta_{n-1}\right) = \frac{m\theta_{n-1}}{i} + \frac{1}{\lambda}\left(\frac{m}{i} + \alpha P\theta_n^{*^{\alpha-1}}\right)\theta_n^{*^{1-\lambda}}\theta_{n-1}^{\lambda} \tag{15}$$

3.2 The $(n-1)$th technology switch

Now consider an operator that uses equipment of technology θ_{n-2} that has two remaining options to switch to more efficient technologies. If the state of technology is θ, the general form of the present value of all costs is again:

$$C_{n-2}\left(\theta\right) = F_{n-2}\theta^{\lambda} + \frac{m\theta_{n-2}}{i} \tag{16}$$

The terminal cost now equals the cost of switching technology the first time $P\theta^{\alpha}$ plus the cost of operating the new technology, including the remaining option to switch to a more efficient technology in the future. The latter is exactly the optimal solution of the previous problem $C_{n-1}^*\left(\theta_{n-1}\right)$, so:

$$\Omega_{n-2}\left(\theta\right) = C_{n-1}^*\left(\theta_{n-1}\right) + P\theta^{\alpha} \tag{17}$$

At the time of replacement, the following equations must hold:

1. Value matching condition:

$$F_{n-2}\theta_{n-1}^{*\lambda} + \frac{m\theta_{n-2}}{i} = \Omega_{n-2}\left(\theta_{n-1}^{*}\right) \tag{18}$$

2. Smooth pasting condition :

$$\lambda F_{n-2}\theta_{n-1}^{*\lambda-1} = \Omega_{n-2}'\left(\theta_{n-1}^{*}\right) \tag{19}$$

As in the problem with only one replacement, the smooth pasting condition expresses the equal marginal effect of technological progress on continuation and terminal cost in the optimum. This time however, the effect on the terminal cost is somewhat more complicated, because waiting longer to make the first technology switch also shifts the second replacement. Therefore, the derivative $\Omega_{n-2}'\left(\theta_{n-1}^{*}\right)$ also contains a term in $\frac{d\theta_n^*}{d\theta_{n-1}}$, which can be determined by totally differentiating eq.(13) and rearranging:

$$\frac{d\theta_n^*}{d\theta_{n-1}} = \left(\frac{\lambda-1}{\lambda}\frac{m}{i} + \frac{\lambda-\alpha}{\lambda}\alpha P\theta_n^{*\alpha-1}\right)^{-1}\frac{m}{i} \tag{20}$$

As before, from the smooth pasting condition we determine F_{n-2} as:

$$F_{n-2} = \frac{\Omega_{n-2}'\left(\theta_{n-1}^{*}\right)}{\lambda}\theta_{n-1}^{*1-\lambda} \tag{21}$$

which we substitute in the value matching condition to find:

$$g^{n-1}\left(\theta_{n-2},\theta_{n-1},\theta_n\right) = \Omega_{n-2}\left(\theta_{n-1}^{*}\right) - \frac{\Omega_{n-2}'\left(\theta_{n-1}^{*}\right)}{\lambda}\theta_{n-1}^{*} - \frac{m\theta_{n-2}}{i} = 0 \tag{22}$$

Together with eq.(12), the previous expression implicitly determines both triggering values θ_{n-1} and θ_n conditional upon the starting value θ_{n-2}. The corresponding minimal expected cost at the beginning of the cycle is then[3]:

$$C_{n-2}^{*}\left(\theta_{n-2}\right) = \frac{m\theta_{n-2}}{i} + \frac{\Omega_{n-2}'\left(\theta_{n-1}^{*}\right)}{\lambda}\theta_{n-1}^{*1-\lambda}\theta_{n-2}^{\lambda} \tag{23}$$

[3]Strictly speaking, the cost C_{n-2} depends on all subsequent switching values (θ_{n-1},θ_n) and the initial starting value θ_{n-2}: $C_{n-2}\left(\theta_{n-2},\theta_{n-1},\theta_n\right)$. However, the optimal switching values $(\theta_{n-1}^*,\theta_n^*)$ depend on θ_{n-2}, so we can write C_{n-2} as a function of θ_{n-2} only: $C_{n-2}^*\left(\theta_{n-2}\right)$, the star indicating that all subsequent switches will be chosen optimally.

3.3 A general technology switch

Although the previous procedure remains largely unchanged for a general problem with n technology switches, the dimension of the problem tends to increase quickly. The operator who has n options to switch technology is of course mainly interested in the timing of the first replacement: when should he switch from the existing technology to a more efficient one? The timing of all subsequent replacements is only relevant in as much as it influences the timing of the first replacement. The procedure to determine the optimal switching value for the kth switch θ_k^* in such a chain, goes as follows.

Assume θ_{k-1} is the technological cost index of the technology presently in use. The operator still has $n-k+1$ options to switch to more efficient technologies which can be exercised freely. For this initial situation, the expected present value of future costs is again of the form:

$$C_{k-1}(\theta) = F_{k-1}\theta^{\lambda} + \frac{m\theta_{k-1}}{i} \tag{24}$$

The terminal cost now equals the sum of the switching cost $P\theta_k^{\alpha}$ and the remaining costs of operation including the $n-k$ future switches $C_k^*(\theta_k)$. Hence:

$$\Omega_{k-1}(\theta) = C_k^*(\theta) + P\theta^{\alpha} \tag{25}$$

Value matching and smooth pasting conditions determine the optimal value of the first technology switch θ_k^*:

1. Value matching condition:

$$F_{k-1}\theta_k^{*\lambda} + \frac{m\theta_{k-1}}{i} = \Omega_{k-1}(\theta_k^*) \tag{26}$$

2. Smooth pasting condition:

$$\lambda F_{k-1}\theta_k^{*\lambda-1} = \Omega_{k-1}'(\theta_k^*) \tag{27}$$

From the smooth pasting condition, the constant F_{k-1} can be determined as:

$$F_{k-1} = \frac{\Omega_{k-1}'(\theta_k^*)}{\lambda}\theta_k^{*1-\lambda} \tag{28}$$

and substituted in the value matching condition to find:

$$g^k(\theta_{k-1}, \theta_k, ..., \theta_{n-1}, \theta_n) = \Omega_{k-1}(\theta_k^*) - \frac{\Omega_{k-1}'(\theta_k^*)}{\lambda}\theta_k^* - \frac{m\theta_{k-1}}{i} = 0 \tag{29}$$

In terms of $C_k^*(\theta_k^*)$, this is identical to:

$$C_k^*(\theta_k^*) - \frac{1}{\lambda}\frac{dC_k^*(\theta_k^*)}{d\theta_k} + \frac{\lambda-\alpha}{\lambda}P\theta_k^{*\alpha} - \frac{m\theta_{k-1}}{i} = 0 \tag{30}$$

For practical purposes, the difficulty resides in the differential $\frac{dC_k^*(\theta_k^*)}{d\theta_k}$, since a marginal change in the switching value θ_k also affects $C_k^*(\theta_k^*)$ indirectly through shifts in subsequent switching values:

$$\frac{dC_k^*}{d\theta_k} = \frac{\partial C_k^*}{\partial\theta_k} + \sum_{j=k+1}^{n}\frac{\partial C_k^*}{\partial\theta_j}\frac{d\theta_j}{d\theta_k} \tag{31}$$

The direct effect of a marginal change in the switching value θ_k^* (reflected by the first term) measures the change in operating cost in the period immediately following replacement and the change in the value of the option to replace, for a given sequence of future triggering values θ_j^*. The second term captures the effect of changing θ_k through the timing of subsequent replacements. The derivative $\partial C_k^*/\partial\theta_j$ is straightforward (and equal to m/i for $j = k$); $d\theta_j/d\theta_k$ includes all direct and indirect effects of a change in the first switching value on future switching values and is harder to determine in terms of computational effort, especially for longer replacement chains.

The value of $d\theta_j/d\theta_k$ can be determined using the optimality conditions for the remaining $n-k$ switches. For each of the subsequent switches $j = k+1 \rightarrow n$, an implicit optimality condition $g^j(\theta_{j-1}, \theta_j, ..., \theta_{n-1}, \theta_n) = 0$ exists (similar to g^n and g^{n-1} determined earlier). Let J_{k+1} be the Jacobian matrix for the set of $n-k$ functions $g^j(\theta_{j-1}, \theta_j, ..., \theta_{n-1}, \theta_n)$:

$$J_{k+1} = \begin{bmatrix} g_{k+1}^{k+1} & g_{k+2}^{k+1} & \ldots & g_{n-1}^{k+1} & g_n^{k+1} \\ g_{k+1}^{k+2} & g_{k+2}^{k+2} & \ldots & g_{n-1}^{k+2} & g_n^{k+2} \\ 0 & g_{k+2}^{k+3} & \ldots & g_{n-1}^{k+3} & g_n^{k+3} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & g_{n-1}^{n} & g_n^n \end{bmatrix} \tag{32}$$

in which $g_k^j = \frac{\partial g^j}{\partial\theta_k}$. Notice $g_k^j = 0$ for $j-k > 1$, which means that the timing of all future technology switches is determined by the state of the technology presently in operation, but not by anterior technologies. By the implicit function theorem, the differentials $d\theta_j/d\theta_k$ then can be determined as:

$$\begin{bmatrix} \frac{d\theta_{k+1}}{d\theta_k} \\ \frac{d\theta_{k+2}}{d\theta_k} \\ \ldots \\ \frac{d\theta_n}{d\theta_k} \end{bmatrix} = -J_{k+1}^{-1}\begin{bmatrix} g_k^{k+1} \\ 0 \\ \ldots \\ 0 \end{bmatrix} \tag{33}$$

Following a similar line of reasoning, we can determine all cross-effects for the entire replacement chain.

3.4 A general solution

Although it is impossible to derive an analytic solution for the sequence θ_j, we can determine a series of implicit optimality conditions for each replacement in the sequence. As stated earlier, the $g^j(\theta_{j-1}, \theta_j, ..., \theta_{n-1}, \theta_n) = 0$ as expressed in eqs. (12), (22), ..., (29) determine such a sequence of optimality conditions. The general recursive procedure to determine such optimality conditions is described in the previous paragraphs and is summarized in figure 1. Applying this procedure manually requires cumbersome and tedious calculations, even for relatively small scale problems with three or four technology switches. Fortunately, modern mathematical software reduces the problem to editing relatively simple program code (an example of such program code for a problem of five technology switches developed for Mathematica IV can be found in appendix). Even more fortunate is that the general form of the optimality condition $g^j(\theta_{j-1}, \theta_j, ..., \theta_{n-1}, \theta_n) = 0$ that occurs after applying the procedure described above, is highly recurrent. Except for the optimality condition governing the last technology switch $g^n(\theta_{n-1}, \theta_n) = 0$, given in eq.(13), all optimality conditions for earlier technology switches are of the general form:

$$g^k(\theta_{k-1}, \theta_k, \theta_{k+1}) = \frac{\lambda - 1}{\lambda}\frac{m\theta_k}{i} + \frac{\lambda - \alpha}{\lambda}P\theta_k^\alpha + \frac{m}{i}\left(\frac{1}{\lambda}\theta_k^{1+\lambda}\theta_{k+1}^\lambda - \theta_{k-1}\right) = 0 \tag{34}$$

for $k = 1 \to n-1$. For known values of the parameters involved, determining the solution to this set of n optimality conditions is straightforward.

4 Some numerical simulations

In order to illustrate the characteristics of the solution resulting from the previous optimality conditions, consider the following numerical example. Assume a drift-rate of technological progress of $g = 3\%$ and a standard deviation of the Wiener-process $\sigma = 5\%$. The interest-rate is set at 5%. The operator uses the present technology at a cost of $m\theta_0$ ($m = 100$ and $\theta_0 = 1$) and has the option the switch to a new technology at a cost of $P\theta^\alpha$. The switching cost P is set at 0 , 1, 4 , 10 and 25 times the initial operating cost m. For simulations, the value of $\alpha = 1$ is used as a baseline reference. Special consideration goes to the effect of deviations for α ranging from 0

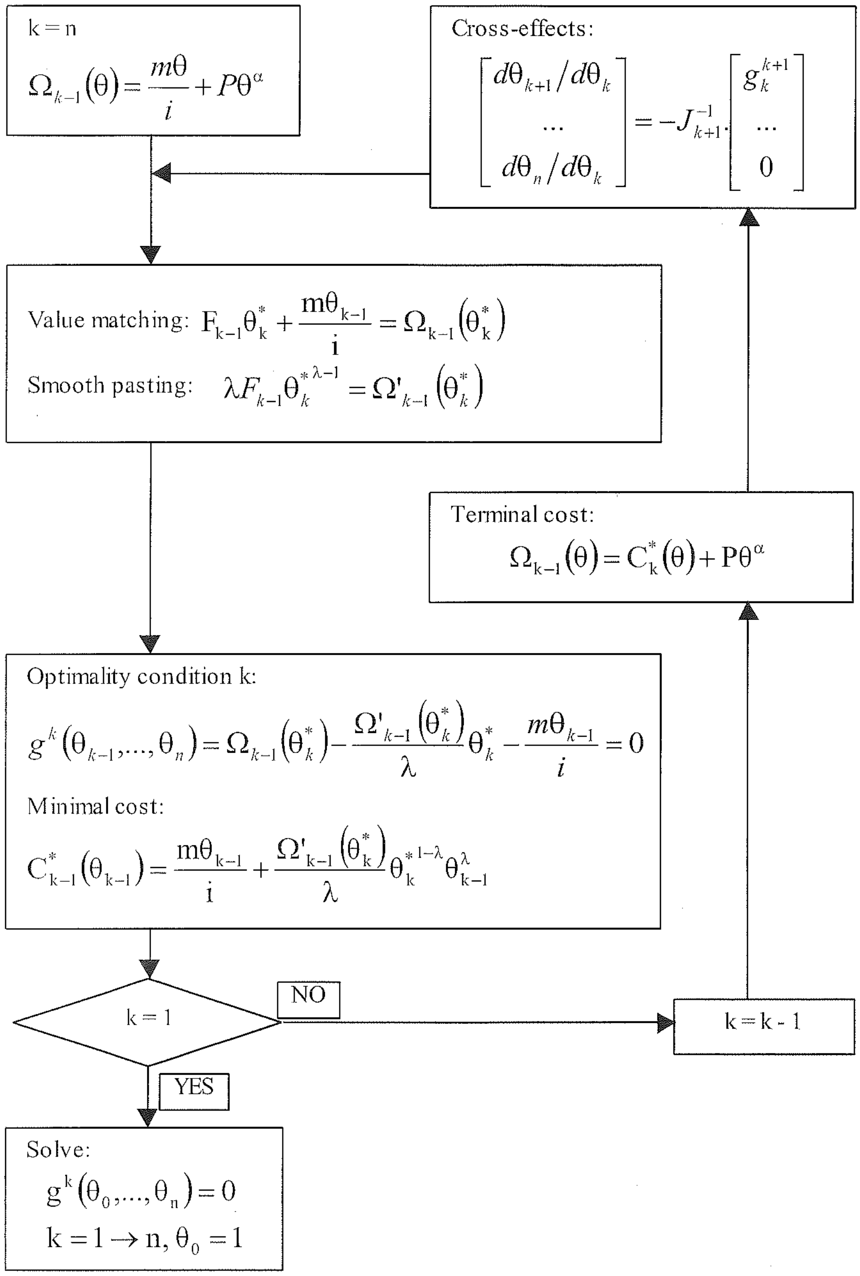

Figure 1: General solution procedure

to 1.5. We also consider several replacement cycles differing in the number of technology switches.

The case for $\alpha = 1$ is used as a baseline reference for several reasons. When $\alpha = 1$ the ratio of installation and operating costs remains unaffected by technological progress, which implies that the (expected) economic life of a certain technology is time-independent. Smaller values of α indicate a reduced impact of technological progress on the switching costs and are therefore expected to correspond to longer economic life. When operating costs are sufficiently high relative to the switching cost, this may even lead to firms not using all their options to switch to new technologies, because the remaining reductions in operating cost are insufficient to justify the switching cost. Similarly, higher values of α induce lower switching costs and therefore corresponds to shorter economic life.

Evidently, economic life as such is not determined explicitly in this model. The model only results in n optimal switching values θ_j. When the technology cost index θ hits a barriers, the firm switches technologies[4]. As an illustration, figure 2 expresses the relative switching values θ_j/θ_{j-1} for $P/m = 10$, $n = 1 \rightarrow 10$ and $\alpha = 0.50$, 1.00 and 1.50. Relative switching values θ_j/θ_{j-1} express the production cost of a new technology relative to the previous technology installed. For instance, the line ending in point A gives the relative switching values for a chain of 8 replacements when $\alpha = 1.50$ and $P/m = 10$. The first replacement in the chain takes place when the technological cost index θ is reduced to 47.74% of its original starting level, the second switch when θ is further reduced to 53.33% of the previous level, etc. Because in this example switching costs are affected more by θ then operating cost, switches initially occur sooner (at higher switching values) the further in the chain. Near the end of the chain however, when there is only a limited number of switching-options left, the option value of waiting forces subsequent switching values down and prolongs the expected period between the remaining technology switches. Similar results can be found for sequences of different lengths as well as for $\alpha = 1$. As could be expected, when $\alpha = 0.50$ the firm will only use a limited number of switches. In fact, for this case the firm will only execute two options to switch to better technologies (the first at 37.61% of the starting value, the second at 26.52% of the previous value). A sequence of 4 option will not be exhausted, since the gains from the last reduction in operating costs will never compensate the

[4]The expected economic life of a technology is can be derived as the solution of a first-hitting time problem, namely the expected period of time for a Brownian motion to move from an initial starting value θ_{j-1} to a predetermined barrier θ_j.

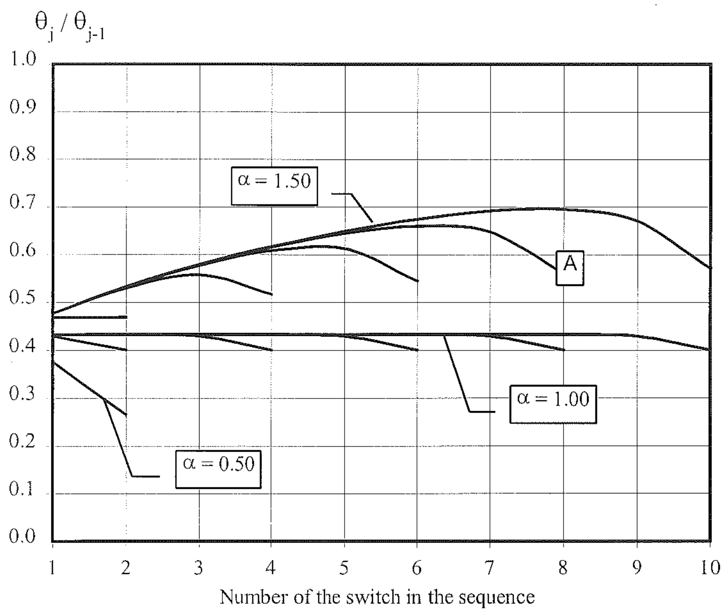

Figure 2: Relative switching values in a sequence of n for different α.

switching cost.

As argued earlier, the main issue is of course the first switching value in the sequence. Firms are interested mainly in the timing of the next technology switch to come. Other switching values are instrumental in determining the first switch but are not important as such. Figure 3 therefore focuses on the sensitivity of the first switching values to changes in the main parameters in the problem, notably the length of the sequence, the ratio P/m and the value of α. The previous figure represents five sets of curves ($P/m = 0$, 1, 4, 10 and 25 respectively). Within each set, the bold line represents the switching value for $\alpha = 1$. The two curves above represent $\alpha = 1.25$ and 1.50, curves below (if any) correspond to $\alpha = 0.75$, 0.50, 0.25 and 0. Point B for instance, indicates that in a sequence of 6 technology switches, with $P/m = 25$ and $\alpha = 1$, the first switch occurs when θ has reached the 34.04% of the initial starting value. Here also, we can see that for $\alpha < 1$, all options to switch may not be exercised. In the same case but for $\alpha = 0.75$ for instance, a sequence of maximum three options will be used.

Figure 3 illustrates the sensitivity of the first technology switch to the

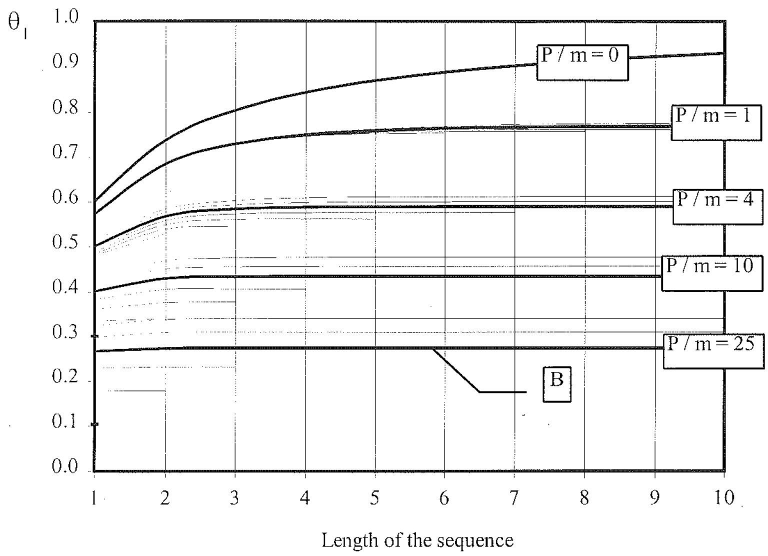

Figure 3: First technology switch in a sequence of n for different P/m and α.

length of the sequence, especially for lower values of P/m. An operator wishing to find the optimal timing of a technology switch, may find considerably biased results if the true number of switches in the sequence is underestimated. If for example $P/m = 4$ and the true number of options is relatively large (say e.g. 10), then the new technology will be adopted when its cost is at 58.93% of its starting level ($\alpha = 1$). The operator that optimizes the problem while considering only one technology switch would switch at 50.12%. Aside from the element of uncertainty and the option value of waiting, neglecting the impact of future technology switches in a rotation problem may therefore offer an additional explanation for the apparently tardy adoption of new technologies reported in economic literature. The optimal first switch is mainly sensitive for changes in α when switching costs are large relative to operating costs. Here, the length of the sequence seems to be a factor of less importance.

5 Summary and conclusion

In this paper a procedure is developed to determine necessary conditions for the optimal timing of technology adoptions in a rotation problem. The paper builds further on a previous model in which technological progress has an identical impact on the operating cost of the new equipment and on the switching cost itself. We relaxed the latter assumption and allow for technological progress to have a different impact on both cost components. Although this generalization impedes explicit analytic solutions for the optimal switching sequence, the procedure allows to derive analytically a set of implicit optimality conditions of a remarkable recursive structure. Once this structure is exposed, the further numerical solution of large scale rotation problems with long sequences of this nature is straightforward.

For the particular case where the ratio of operating cost and switching cost is unaffected by technological progress, we showed earlier that neglecting future technology switches in a sequence may be of great influence on the timing of the technology adoption. The optimality conditions derived in this paper for a much more general class of problems still confirms this result. In addition, the exact nature of technological progress (in terms of its impact on the ratio of switching cost and operating cost) is shown to have a serious effect on the timing of technology adoption when the switching cost is large with respect to the operating cost.

6 Appendix: Mathematica code for determining optimality conditions when $n = 5$

```
(* Meaning of symbols used: *)

(* oni = endpoint function n-i                                   *)
(* doni = differential of oni                                    *)
(* donis = idem, simplified                                      *)
(* fni = constant of integration n-i                             *)
(* gni = optimality condition n-i                                *)
(* cni = cost function n-i                                       *)
(* cnis = idem, simplified                                       *)

(* Initialisation *)

Clear[x, m, p, i, g, s, λ, a, tn0, tn1, tn2, tn3, tn4, tn5, n]

(* Terminal value *)

cn0[x_] := m*x/i

(* Replacement n *)

on1[x_] := cn0[x] + p*x^a
don1[x_] := Dt[on1[x], x, Constants→ {m, p, i, λ, a}]
don1[tn0];
don1s = Simplify[%]
gn0 = on1[tn0] - don1s*tn0 / λ - m* tn1 / i;
gn0s = FullSimplify[%]
fn1 = don1[tn0] * tn0^ (1- λ) / λ
cn1[x_] := fn1*x^ λ + m*x / i
cn1s = FullSimplify[cn1[tn1]]
```

```
(* Replacement n-1 *)

Clear[x, a, tn0, tn1, tn2, tn3, tn4]
on2[x_] := cn1[x] + p*x^a
dn0n1 = (-1) * D[gn0s, tn1] / D[gn0s, tn0];
don2[x_] := Dt[on2[x], x, Constants → {m, p, i, λ, a}]
don2[tn1] /. Dt[tn0, tn1, Constants → {a, i, m, p, λ}] → dn0n1;
don2s = FullSimplify[%]
gn1 = on2[tn1] - don2s * tn1 / λ - m * tn2 / i;
gn1s = FullSimplify[%]
fn2 = don2s * tn1^(1- λ) / λ
cn2[x_] := fn2 * x^λ + m * x / i
cn2s = FullSimplify[cn2[tn2]]

(* Replacement n-2 *)

on3[x_] := cn2[x] + p*x^a
Mn2 = ( D[gn1s, tn1]  D[gn1s, tn0] );
      ( D[gn0s, tn1]  D[gn0s, tn0] )
( dn1n2 )                  ( D[gn1s, tn2] )
( dn0n2 ) = -Inverse[Mn2].(       0       );
don3[x_] := Dt[on3[x], x, Constants → {m, p, i, λ, a}]
don3[tn2] /. {Dt[tn1, tn2, Constants → {a, i, m, p, λ}] → dn1n2,
   Dt[tn0, tn2, Constants → {a, i, m, p, λ}] → dn0n2};
don3s = Simplify[%]
gn2 = on3[tn2] - don3s * tn2 / λ - m * tn3 / i;
gn2s = Simplify[%]
fn3 = don3s * tn2^(1- λ) / λ
cn3[x_] := fn3 * x^λ + m * x / i
cn3s = Simplify[cn3[tn3]]

(* Replacement n-3 *)

on4[x_] := cn3[x] + p*x^a
      ( D[gn2s, tn2]  D[gn2s, tn1]  D[gn2s, tn0] )
Mn3 = ( D[gn1s, tn2]  D[gn1s, tn1]  D[gn1s, tn0] );
      (      0        D[gn0s, tn1]  D[gn0s, tn0] )
```

```
(dn2n3)                      (D[gn2s, tn3])
(dn1n3) = -Inverse[Mn3] .   (     0      ) ;
(dn0n3)                      (     0      )
don4[x_] := Dt[on4[x], x, Constants→ {m, p, i, λ, a}]
don4[tn3] /. {Dt[tn2, tn3, Constants→ {a, i, m, p, λ}] → dn2n3,
    Dt[tn1, tn3, Constants→ {a, i, m, p, λ}] → dn1n3,
    Dt[tn0, tn3, Constants→ {a, i, m, p, λ}] → dn0n3};
don4s = Simplify[%]
gn3 = on4[tn3] - don4s * tn3 / λ - m * tn4 / i;
gn3s = Simplify[%]
fn4 = don4s * tn3^(1- λ) / λ
cn4[x_] := fn4 * x^λ + m * x / i
cn4s = Simplify[cn4[tn4]]

(* Replacement n-4 *)

on5[x_] := cn4[x] + p * x^a
       (D[gn3s, tn3]  D[gn3s, tn2]  D[gn3s, tn1]  D[gn3s, tn0])
Mn4 =  (D[gn2s, tn3]  D[gn2s, tn2]  D[gn2s, tn1]  D[gn2s, tn0]) ;
       (     0        D[gn1s, tn2]  D[gn1s, tn1]  D[gn1s, tn0])
       (     0             0        D[gn0s, tn1]  D[gn0s, tn0])
(dn3n4)                      (D[gn3s, tn4])
(dn2n4) = -Inverse[Mn4] .   (     0      ) ;
(dn1n4)                      (     0      )
(dn0n4)                      (     0      )
don5[x_] := Dt[on5[x], x, Constants→ {m, p, i, λ, a}]
don5[tn4] /. {Dt[tn3, tn4, Constants→ {a, i, m, p, λ}] → dn3n4,
    Dt[tn2, tn4, Constants→ {a, i, m, p, λ}] → dn2n4,
    Dt[tn1, tn4, Constants→ {a, i, m, p, λ}] → dn1n4,
    Dt[tn0, tn4, Constants→ {a, i, m, p, λ}] → dn0n4};
don5s = Simplify[%]
gn4 = on5[tn4] - don5s * tn4 / λ - m * tn5 / i;
gn4s = Simplify[%]
fn5 = don5s * tn4^(1- λ) / λ
cn5[x_] := fn5 * x^λ + m * x / i
cn5s = Simplify[cn5[tn5]]
```

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