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On the multitude of monoidal closed structures on **UAP**

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Dedicated to Professor Bernhard Banaschewski on the occasion of his 75th birthday

Abstract

In this note, we prove that all compact Hausdorff topological spaces are exponential objects in the category **UAP** of uniform approach spaces and contractions as introduced in R. Lowen, Approach Spaces: the Missing Link in the Topology-Uniformity-Metric Triad, Oxford University Press, 1997. As a consequence, we show that **UAP** admits at least as many monoidal closed structures as there are infinite cardinals. We also prove that under the assumption that no measurable cardinals exist, there exists a proper conglomerate of these monoidal closed structures on **UAP**.

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1. Introduction and preliminaries

The topological category **AP** of approach spaces and contractions was introduced by the first author as a resolution of some problems arising in (categorical) topology, mainly the non-canonical metrizability of arbitrary products of metric spaces. These approach spaces form a common supercategory of both topological and metric spaces, where products of arbitrary (set-indexed) families of metric spaces now possess a canonical product still

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concordant with the product of the underlying metric topologies. In this paper we will mainly be concerned with the full subcategory **UAP** of **AP**, which is the epireflective hull of $p\mathbf{MET}^{\infty}$ in **AP**. We only note that **UAP** is initially closed in **AP**, whence also a topological construct. For any information and notations concerning approach theory and its wide range of applications to other fields of general topology or functional analysis, we refer to [6]. Because **AP** contains **TOP** in the nicest possible way (both reflectively and coreflectively), it also inherits some deficiencies of the latter with respect to algebraic properties like Cartesian closedness. Also **UAP** fails to be Cartesian closed. It is well known that all compact Hausdorff spaces are exponentiable in **TOP**, i.e., their associated product functors admit a right adjoint. In a first part of the paper, we will show that they also are exponentiable in **UAP** and we will give a nice internal characterization of the corresponding natural function space structure building this right adjoint. On the other hand, it was proved in [4], that non-symmetric monoidal closed structures can be used very efficiently to adapt a category for algebraic purposes while extracting the maximum of good exponential behaviour it has. We refer to the standard literature on categories for any information on monoidal closed structures.

2. Main results

As proved in [5,7] the construct **PSAP** is the quasitopos hull of both **AP** and **UAP**, whence surely Cartesian closed (we refer to [5] for further information, definitions and notations). If for $Y, Z \in |\mathbf{PSAP}|$, the convergence approach limit λ_c on $\mathbf{PSAP}(Y, Z)$ is defined by

$$\lambda_c(\Psi)(f) := \inf\bigl\{\alpha \geqslant 0 \mid \forall \mathcal{F} \in \mathbf{F}(\underline{Y})\colon\ \lambda_Z\bigl(\Psi(\mathcal{F})\bigr) \circ f \leqslant \lambda_Y(\mathcal{F}) \vee \alpha\bigr\}$$

for all $f \in \mathbf{PSAP}(Y, Z)$ and $\Psi \in \mathbf{F}(\mathbf{PSAP}(Y, Z))$ (where for a set X, $\mathbf{F}(X)$ stands for the set of all filters on X), then λ_c is a **PSAP**-limit (throughout the paper, $\underline{\cdot}$ always denotes the underlying set and $\mathbb{R}^+$ (respectively $\mathbb{R}_0^+$) denotes the set of positive real numbers including (respectively excluding) 0). To simplify notations, put

$$[Y, Z]_c := \bigl(\mathbf{PSAP}(Y, Z), \lambda_c\bigr).$$

Let us also recall the following, which is a special case of Theorem 3.3 from [8], since it was shown in [7] that **UAP** is finally dense in **PSAP**:

Theorem 2.1 (Schwarz [8]). *For every $Y \in |\mathbf{UAP}|$ the following assertions are equivalent*:

(1) Y is exponential in **UAP**,
(2) $\forall Z \in |\mathbf{UAP}|\colon\ [Y, Z]_c \in |\mathbf{UAP}|$.

It is our aim to show the following claim: *Every compact Hausdorff topological space, viewed as a uniform approach object, is exponential in* **UAP**.

In the sequel **Comp** (respectively $\mathbf{Comp}_2$) denotes the category of all compact (respectively compact Hausdorff) topological spaces and continuous maps, sometimes viewed as a full subcategory of **AP** (respectively **UAP**). Fix $Y \in |\mathbf{Comp}_2|$ and $Z \in |\mathbf{UAP}|$.

We denote the corresponding convergence structure on $\underline{Y}$ by q_Y. First we note that $\mathbf{UAP}(Y,Z)$ in fact consists of all continuous functions from Y to the topological coreflection of Z, which is completely regular. We now have to verify that the pseudo-approach limit λ_c is in fact a uniform approach limit. Let $f \in \mathbf{UAP}(Y,Z)$, $\Psi \in \mathbf{F}(\mathbf{UAP}(Y,Z))$. Then (note that, because Hausdorffness implies uniqueness of limits, we can use a notation like $y_{\mathcal{F}}$ for the unique limit point of a convergent filter $\mathcal{F}$)

$$\begin{aligned}\lambda_c(\Psi)(f) &= \inf\bigl\{\alpha \in [0,\infty] \mid \forall \mathcal{F} \in \mathbf{F}(\underline{Y})\colon\ \lambda_Z\bigl(\Psi(\mathcal{F})\bigr) \circ f \leqslant \alpha \vee \theta_{\lim_{q_Y} \mathcal{F}}\bigr\}\\ &= \sup_{(\mathcal{F}, y_{\mathcal{F}}) \in q_Y} \lambda_Z\bigl(\Psi(\mathcal{F})\bigr)\bigl(f(y_{\mathcal{F}})\bigr)\\ &= \sup_{y \in \underline{Y}} \lambda_Z\bigl(\Psi\bigl(\mathcal{N}_Y(y)\bigr)\bigr)\bigl(f(y)\bigr).\end{aligned}$$

(Here, for any subset A of a given set X the function $\theta_A : X \to [0,\infty]$ is defined by $\theta_A(x) := 0$ if $x \in A$ and $\theta_A(x) := \infty$ if $x \notin A$.)

From this formula, it is immediately clear that λ_c inherits the (PRAL) property from Z. It remains to verify (AL) for λ_c, and that λ_c can be generated by ∞p-metrics.

Instead of doing this directly, we will try to find an alternative description for the convenient exponential approach structure on $\mathbf{UAP}(Y,Z)$, which then has to be identical to λ_c. We write $\mathcal{G}_Z^s$ for the symmetric gauge of Z as introduced in [6], i.e., $\mathcal{G}_Z^s$ is the largest set $\mathcal{D}$ of ∞p-metrics on $\underline{Z}$ such that

$$\forall x \in \underline{Z},\ \forall A \subset \underline{Z}\colon\ \delta_Z(x,A) = \sup_{d \in \mathcal{D}} \inf_{a \in A} d(x,a).$$

For every $d \in \mathcal{G}_Z^s$,

$$\tilde{d} : \mathbf{UAP}(Y,Z) \times \mathbf{UAP}(Y,Z) \to [0,\infty]\colon\ (f,g) \mapsto \sup_{y \in \underline{Y}} d\bigl(f(y), g(y)\bigr)$$

is an ∞p-metric on $\mathbf{UAP}(Y,Z)$, and it is clear that

$$\bigl\{\tilde{d} \mid d \in \mathcal{G}_Z^s\bigr\}$$

is a symmetric gauge basis for a uniform approach structure λ_{uc}, which we call the structure of uniform convergence on the function space $\mathbf{UAP}(Y,Z)$, where

$$\lambda_{\mathrm{uc}}(\Psi)(f) = \sup_{d \in \mathcal{G}_Z^s} \inf_{\mathcal{F} \in \Psi} \sup_{g \in \mathcal{F}} \tilde{d}(f,g)$$

for all $f \in \mathbf{UAP}(Y,Z)$, $\Psi \in \mathbf{F}(\mathbf{UAP}(Y,Z))$. We also use the subscript ‘uc’ for the other equivalent representations of the same uniform approach structure, like the distance etc.. To abbreviate notations, we put

$$Z^Y := \bigl(\mathbf{UAP}(Y,Z), \lambda_{\mathrm{uc}}\bigr).$$

In order to have some more flexibility to work, let us also introduce a numerified counterpart to another important topology in the context of exponentiability: the compact-open topology. For all $B \subset \underline{Z}$ (note that it follows from the definition below that it makes no difference, taking all $B \in 2^{\underline{Z}}$ or only all $B \in 2^{\underline{Z}}$ that are closed with respect to the topological bicoreflection $\mathcal{T}_Z$ of Z) and all $K \subset \underline{Y}$ that are compact with respect to $\mathcal{T}_Y$, we need to look at functionals of the following form:

$$\mathbf{UAP}(Y, Z) \to [0, \infty]: \ f \mapsto \inf\bigl\{\varepsilon \geqslant 0 \mid f(K) \cap B^{(\varepsilon)} \neq \emptyset\bigr\}.$$

Note that this infimum in fact is a minimum and that for each $f \in \mathbf{UAP}(Y, Z)$ it equals $\inf_{f(K)} \delta_Z(\cdot, B)$. This motivates for all $K \subset \underline{Y}$ $\mathcal{T}_Y$-compact and $\rho \in \mathcal{R}_Z$,

$$\Gamma(K, \rho) : \mathbf{UAP}(Y, Z) \to [0, \infty]: \ f \mapsto \inf_{f(K)} \rho.$$

If $Y \in |\mathbf{Comp_2}|$ and $Z \in |\mathbf{UAP}|$, then

$$\mathcal{R}_{\mathrm{cr}} := \bigl(\bigl\{\Gamma(K, \rho) + \alpha \mid \alpha \in [0, \infty], \rho \in \mathcal{R}_Z, K \subset \underline{Y} \ \mathcal{T}_Y\text{-compact}\bigr\}^{\wedge}\bigr)^{\vee}$$

is a regular function frame on $\mathbf{UAP}(Y, Z)$. We call $\mathcal{R}_{\mathrm{cr}}$ the compact-regular structure. Note that at the moment we only know that

$$\bigl(\mathbf{UAP}(Y, Z), \mathcal{R}_{\mathrm{cr}}\bigr) \in |\mathbf{AP}|!$$

Now take $X, Z \in |\mathbf{UAP}|$ and $Y \in |\mathbf{Comp_2}|$. We write

$$\mathrm{ev}_{Y,Z} : \mathbf{UAP}(Y, Z) \times \underline{Y} \to \underline{Z}: \ (h, y) \mapsto h(y)$$

and if $f : \underline{X} \times \underline{Y} \to \underline{Z}$, its transpose is given by

$$\hat{f} : \underline{X} \to \underline{Z}^{\underline{Y}}: \ x \mapsto f(x, \cdot).$$

For $y \in \underline{Y}$, we write $\mathrm{ev}_y := \mathrm{ev}_{Y,Z}(\cdot, y)$.

Lemma 2.2. *Let $Y \in |\mathbf{Comp_2}|$. Take $Z \in |\mathbf{UAP}|$. Then the structure of uniform convergence is coarser than the compact-regular structure.*

Proof. We denote the regular function frame representing the structure of uniform convergence by $\mathcal{R}_{\mathrm{uc}}$. By construction, it is now clear that $\mathcal{R}_{\mathrm{uc}} = \bigwedge_{d \in \mathcal{G}_Z^s} \mathcal{R}_{\tilde{d}}$ (the infimum is taken in the fibre in the sense of [1]). So it suffices to show $\forall d \in \mathcal{G}_Z^s : \mathcal{R}_{\mathrm{cr}} \leqslant \mathcal{R}_{\tilde{d}}$. To do so, fix $d \in \mathcal{G}_Z^s$ and $\mathcal{H} \subset \mathbf{UAP}(Y, Z)$. It now suffices to prove that $\gamma := \delta_{\tilde{d}}(\cdot, \mathcal{H}) \in \mathcal{R}_{cr}$. So pick $f \in \mathbf{UAP}(Y, Z)$. We are sure that $h_{\mathrm{cr}}(\gamma)(f) \leqslant \gamma(f)$ so the converse remains to be shown. If $\gamma(f) = 0$ we are done, so assume that $\gamma(f) > 0$. We treat the case where $\gamma(f) < \infty$. (The case where $\gamma(f) = \infty$ is treated in an analogous way.) Fix $\alpha \in \mathbb{R}_0^+$ with

$$\gamma(f) > \alpha.$$

We are done if we show that

$$h_{\mathrm{cr}}(\gamma)(f) \geqslant \alpha.$$

Therefore, pick $\varepsilon \in \mathbb{R}_0^+$ arbitrary such that

$$\alpha + 2\varepsilon < \gamma(f).$$

By definition, we have that

$$\mathcal{H} \cap B_{\tilde{d}}(f, \alpha + 2\varepsilon) = \emptyset.$$

For all $y \in \underline{Y}$, we now can find a compact $\mathcal{T}_Y$-neighbourhood N_y of y for which $f(N_y) \subset B_d(f(y), \varepsilon)$. By compactness of Y, there exist $y_1, \ldots, y_n \in \underline{Y}$ such that $\underline{Y} = \bigcup_{j=1}^n N_{y_j}$. Because $\mathcal{R}_Z = \bigwedge_{e \in \mathcal{G}_Z^s} \mathcal{R}_e$, it is obvious, that $\rho_j := \delta_d(\cdot, \underline{Z} \setminus B_d(f(y_j), \alpha)) \in \mathcal{R}_Z$ for $1 \leqslant j \leqslant n$, whence $\psi := \bigwedge_{j=1}^n \Gamma(N_{y_j}, \rho_j) \in \mathcal{R}_{\mathrm{cr}}$. It is also immediately clear from the metric triangle inequality that $\psi(f) \geqslant \alpha + 2\varepsilon$. To see this, fix $j \in \{1, \ldots, n\}$ and assume that $\Gamma(N_{y_j}, \rho_j)(f) < \alpha + 2\varepsilon$. Then there exists $y \in N_{y_j}$ with $\rho_j(f(y)) \leqslant \alpha + 2\varepsilon$, so we can pick $z \in \underline{Z}$ with $d(f(y), z) < \alpha + 2\varepsilon$ and $d(f(y_j), z) \geqslant \alpha$, which leads to a contradiction because $d(f(y_j), f(y)) < \varepsilon$. We are now finished if we prove that $\gamma + 2\varepsilon \geqslant \psi$. Suppose on the contrary that

$$\psi(g) > \gamma(g) + 2\varepsilon$$

for some $g \in \mathbf{UAP}(Y, Z)$. This now means that

$$\forall j \in \{1, \ldots, n\}\colon \inf_{y \in N_{y_j}} \delta_d\big(g(y), \underline{Z} \setminus B_d\big(f(y_j), \alpha\big)\big) > \gamma(g) + 2\varepsilon$$

and some computation then would result in $\gamma(g) \geqslant \gamma(g) + 2\varepsilon$, a contradiction.

We have now shown that $h_{\mathrm{cr}}(\gamma)(f) \geqslant \alpha$ and this completes the proof. □

Proposition 2.3. *Let $X, Z \in |\mathbf{UAP}|$ and $Y \in |\mathbf{Comp}_2|$. Then*

$$\mathrm{ev}_{Y,Z} \in \mathbf{UAP}\big(Z^Y \times Y, Z\big).$$

Proof. First note that

$$\mathrm{ev}_{Y,Z} \in \mathbf{UAP}\big(Z^Y \times Y, Z\big) \iff \mathrm{id}_{\mathbf{UAP}(Y,Z)} \in \mathbf{PSAP}\big(Z^Y, [Y, Z]_c\big).$$

Fix $f \in \mathbf{UAP}(Y, Z)$, Ψ a filter on $\mathbf{UAP}(Y, Z)$. Assume that $\lambda_c(\Psi)(f) \in \mathbb{R}^+$. (If it is $+\infty$ a similar reasoning does the trick.) Fix $\varepsilon \in \mathbb{R}_0^+$. Take $y \in \underline{Y}$ with $\lambda_c(\Psi)(f) \leqslant \varepsilon + \lambda_Z(\Psi(\mathcal{N}_Y(y))(f(y))$. For every $d \in \mathcal{G}_Z$, there exists $N_d \in \mathcal{N}_Y(y)$ such that $f(N_d) \subset B_d(f(y), \varepsilon)$. We then have that

$$\begin{aligned}\lambda_c(\Psi)(f) &\leqslant \varepsilon + \sup_{d \in \mathcal{G}_Z^s} \inf_{\mathcal{F} \in \Psi} \sup_{g \in \mathcal{F}} \sup_{t \in N_d} \big(d\big(f(y), f(t)\big) + d\big(f(t), g(t)\big)\big) \\ &\leqslant 2\varepsilon + \sup_{d \in \mathcal{G}_Z^s} \inf_{\mathcal{F} \in \Psi} \sup_{g \in \mathcal{F}} \tilde{d}(f, g) = \lambda_{\mathrm{uc}}(\Psi)(f) + 2\varepsilon. \qquad \square\end{aligned}$$

Lemma 2.4. *Take $X \in |\mathbf{AP}|$, $Y \in |\mathbf{Comp}|$ and denote*

$$\mathrm{pr}_X : X \times Y \to X\colon (x, y) \mapsto x.$$

Then

$$\forall \rho \in \mathcal{R}_{X \times Y}\colon \mathrm{pr}_X(\rho) \in \mathcal{R}_X$$

where for every $x \in \underline{X}$, $\mathrm{pr}_X(\rho)(x) := \inf_{y \in \underline{Y}} \rho(x, y)$.

Proof. It is equivalent to show that $h_X(\mathrm{pr}_X(\rho)) = \mathrm{pr}_X(\rho)$ or equivalently, that $h_X(\mathrm{pr}_X(\rho)) \geqslant \mathrm{pr}_X(\rho)$. So fix $x \in \underline{X}$ and assume that $\alpha \in \mathbb{R}^+$ with $\mathrm{pr}_X(\rho)(x) > \alpha$. Then obviously, $\forall y \in \underline{Y}\colon \rho(x, y) > \alpha$, so for all $y \in \underline{Y}$, there exist $\varphi_y \in \mathcal{A}_X(x)$ and $V_y \in \mathcal{N}_Y(y)$ with

$$\inf_{(s,t) \in \underline{X} \times \underline{Y}} \big(\rho(s, t) + \varphi_y(s) \vee \theta_{V_y}(t)\big) > \alpha.$$

By compactness of Y, we can find $y_1, \dots, y_n \in \underline{Y}$ for which $\underline{Y} = \bigcup_{j=1}^{n} V_{y_j}$. Now $\varphi := \bigvee_{j=1}^{n} \varphi_{y_j} \in \mathcal{A}_X(x)$ and obviously $\rho(s,t) + \varphi(s) > \alpha$ for all $s \in \underline{X}$, $t \in \underline{Y}$, whence $\mathrm{pr}_X(\rho)(s) + \varphi(s) \geqslant \alpha$, yielding $h_X(\mathrm{pr}_X(\rho))(x) \geqslant \alpha$. □

Lemma 2.5. *Let* $X, Z \in |\mathbf{UAP}|$ *and* $Y \in |\mathbf{Comp}_2|$. *Then*

$$\forall f \in \mathbf{UAP}(X \times Y, Z)\colon\ \hat{f} \in \mathbf{UAP}\big(X, Z^Y\big).$$

Proof. Take $K \subset \underline{Y}$ compact, $\rho \in \mathcal{R}_Z$; it suffices to show that $\Gamma(K,\rho) \circ \hat{f} \in \mathcal{R}_X$. Because f is a contraction, we know that $\rho \circ f \in \mathcal{R}_{X \times Y}$ so applying the previous lemma yields that $\Gamma(K,\rho) \circ \hat{f}(\cdot) = \inf_{\{\cdot\} \times K} \rho \circ f \in \mathcal{R}_X$. This proves that

$$\hat{f} : X \to \big(\mathbf{UAP}(Y, Z), \mathcal{R}_{\mathrm{cr}}\big)$$

is a contraction and since we have proved in 2.2 that λ_{uc} is coarser than $\mathcal{R}_{\mathrm{cr}}$, we are done. □

Summarizing the previous, we obtain

Theorem 2.6. *All compact Hausdorff topological objects are exponential in* **UAP**, *and for all* $Y \in |\mathbf{Comp}_2|$, $Z \in |\mathbf{UAP}|$, $[Y, Z]_c = Z^Y$.

Since we have shown there are “enough” exponential objects in **UAP**, we can now use a standard technique from Greve [4] to derive something about the number of monoidal closed (or MC) structures that the category **UAP** admits. For more information on related results about the category **TOP**, we refer to [2–4].

Theorem 2.7. *There exists at least a proper class of non-naturally isomorphic MC-structures on* **UAP**, *namely as many as there are infinite cardinals.*

Proof. For every infinite cardinal α, let $\mathbf{C}_\alpha := \{Y \in |\mathbf{Comp}_2| \mid \mathrm{Card}(\underline{Y}) \leqslant \alpha\}$, which is finitely productive (in **TOP** or **UAP**). Use the construction which is given in [4] to yield a monoidal closed structure $(-\square_\alpha-, H_\alpha(-,-))$ on **UAP** such that $- \times Y = -\square_\alpha Y$ for all $Y \in \mathbf{C}_\alpha$. Let us recall the definition of the inner hom functor $H_\alpha(-,-)$ from [4] (the explicit description of the tensorproduct $-\square_\alpha-$ will not be needed here): for very $X, Z \in |\mathbf{UAP}|$, $H_\alpha(X, Z)$ is taken to consist of the underlying set $\mathbf{UAP}(Y, Z)$, equipped with the initial **UAP**-structure for the source

$$\big(\mathbf{UAP}(f,g) : \mathbf{UAP}(Y,Z) \to A^B\big)_{A \in |\mathbf{UAP}|, B \in \mathbf{C}_\alpha, f \in \mathbf{UAP}(B,Y), g \in \mathbf{UAP}(Z,A)},$$

where for all $A \in |\mathbf{UAP}|$, $B \in \mathbf{C}_\alpha$, $f \in \mathbf{UAP}(B, Y)$ and $g \in \mathbf{UAP}(Z, A)$

$$\mathbf{UAP}(f,g) : \mathbf{UAP}(Y,Z) \to \mathbf{UAP}(B,A)\colon\ h \mapsto g \circ h \circ f$$

is the usual hom-functor.

Now take $\alpha < \beta$ and let Y_β be a compact Hausdorff space of cardinality β. Let $\mathbb{R}$ stand for the real line with the Euclidean metric d_E. We are done if we show that

$H_\beta(Y_\beta,\mathbb{R}) \neq H_\alpha(Y_\beta,\mathbb{R})$. Clearly $H_\beta(Y_\beta,\mathbb{R})$ is finer than $H_\alpha(Y_\beta,\mathbb{R})$. We have to prove it is not coarser, or equivalently, that

$$\mathrm{id}_{\mathbf{UAP}(Y_\beta,\mathbb{R})} : H_\alpha(Y_\beta,\mathbb{R}) \to H_\beta(Y_\beta,\mathbb{R})$$

is not a contraction, or again equivalently, that there exist $S \in \mathbf{C}_\beta$, $T \in |\mathbf{UAP}|$, $f \in \mathbf{UAP}(S, Y_\beta)$ and $g \in \mathbf{UAP}(\mathbb{R}, T)$ such that

$$\mathbf{UAP}(f,g) : H_\alpha(Y_\beta,\mathbb{R}) \to T^S : h \mapsto g \circ h \circ f$$

is non-contractive. We intend to prove that

$$\mathrm{id}_{\mathbf{UAP}(Y_\beta,\mathbb{R})} : H_\alpha(Y_\beta,\mathbb{R}) \to \mathbb{R}^{Y_\beta}$$

is not a contraction. By definition of the MC-structure $(-\Box_\alpha-, H_\alpha(-,-))$, the source

$$\bigl(\mathbf{UAP}(f,g) : H_\alpha(Y_\beta,\mathbb{R}) \to A^B\bigr)_{A\in|\mathbf{UAP}|,\, B\in\mathbf{C}_\alpha,\, f\in\mathbf{UAP}(B,Y_\beta),\, g\in\mathbf{UAP}(\mathbb{R},A)}$$

is $\mathbf{UAP}$-initial. If for all $A \in |\mathbf{UAP}|$, $B \in \mathbf{C}_\alpha$, $f \in \mathbf{UAP}(B, Y_\beta)$, $g \in \mathbf{UAP}(\mathbb{R}, A)$ and $d \in \mathcal{G}^s_A$, we put

$$\rho_{f,g,d} := \tilde{d} \circ \bigl(\mathbf{UAP}(f,g) \times \mathbf{UAP}(f,g)\bigr)$$

then

$$\bigl\{\rho_{f,g,d} \mid A \in |\mathbf{UAP}|,\ B \in \mathbf{C}_\alpha,\ f \in \mathbf{UAP}(B,Y_\beta),\ g \in \mathbf{UAP}(\mathbb{R},A),\ d \in \mathcal{G}^s_A\bigr\}$$

is a symmetric gauge basis for $H_\alpha(Y_\beta,\mathbb{R})$.

We claim that the topology of the topological coreflection of $\mathbb{R}^{Y_\beta}$ is finer than the compact-open topology on $\mathbf{UAP}(Y_\beta,\mathbb{R})$. To see that this is true, take $K \subset \underline{Y_\beta}$ $\mathcal{T}_{Y_\beta}$-compact, $O \in \mathcal{T}_{d_E}$ and fix

$$f \in \bigl\{g \in \mathbf{UAP}(Y_\beta,\mathbb{R}) \mid g(K) \subseteq O\bigr\} =: \langle K, O\rangle.$$

By compactness of Y_β, there exists $\varepsilon > 0$ such that

$$\inf_{y\in K} \inf_{z\in\mathbb{R}\setminus O} \bigl|f(y) - z\bigr| > \varepsilon$$

and it is now easy to verify that $B_{\widetilde{d_E}}(f, \frac{\varepsilon}{2}) \subset \langle K, O\rangle$.

We therefore have that

$$\bigl\langle \underline{Y_\beta},\ ]-1,1[\bigr\rangle := \bigl\{g \in \mathbf{UAP}(Y_\beta,\mathbb{R}) \mid g(\underline{Y_\beta}) \subset]-1,1[\bigr\}$$

is open in the topology of the topological coreflection of $\mathbb{R}^{Y_\beta}$, so we are done if we show that it is not open in the topology of the topological coreflection of $H_\alpha(Y_\beta,\mathbb{R})$. Suppose on the contrary that $\langle \underline{Y_\beta},\ ]-1,1[\,\rangle$ would be open with respect to the last mentioned topology. Since $k_0 : Y_\beta \to \overline{\mathbb{R}:}\ y \mapsto 0$ belongs to $\langle \underline{Y_\beta},\ ]-1,1[\,\rangle$, this would imply that there exist $\varepsilon > 0$, $A \in |\mathbf{UAP}|$, $B \in \mathbf{C}_\alpha$, $f \in \mathbf{UAP}(B, \overline{Y_\beta})$, $g \in \mathbf{UAP}(\mathbb{R}, A)$ and $d \in \mathcal{G}^s_A$ such that

$$k_0 \in B_{\rho_{f,g,d}}(k_0, \varepsilon) \subset \bigl\langle \underline{Y_\beta},\ ]-1,1[\bigr\rangle.$$

Now $f(\underline{B})$ is a compact, whence closed, subset of Y_β of cardinality at most α, so we can pick a point $y \in \underline{Y_\beta} \setminus f(\underline{B})$. Because compact Hausdorff topological spaces are completely regular there exists a continuous map $\varphi : Y_\beta \to \mathbb{R}$ (so $\varphi \in \mathbf{UAP}(Y_\beta,\mathbb{R})$) such that $\varphi(f(\underline{B})) = \{0\}$ and $\varphi(y) = 2$. Then obviously $\varphi \in B_{\rho_{f,g,d}}(k_0,\varepsilon) \setminus \langle \underline{Y_\beta},\ ]-1,1[\,\rangle$, which is a contradiction. □

Lemma 2.8. *For every $Y \in |\mathbf{Comp}_2|$ infinite, $\mathbb{R}^Y = [Y,\mathbb{R}]_c \neq [Y,\mathbb{R}]_p$, (where $\mathbb{R}$ stands for the real line with the Euclidean metric and $[Y,\mathbb{R}]_p$ denotes $\mathbf{UAP}(Y,\mathbb{R})$ equipped with the* **UAP***-product structure*).

Proof. This is shown using [4, 2.2], since the topology of the topological coreflection of $\mathbb{R}^Y = [Y,\mathbb{R}]_c$ is finer than the compact-open topology (this is shown in the same way as in the proof of 2.7) and since concrete coreflectors preserve initiality. □

Let us recall a theorem from [10], which will enable us to strengthen our result under the set-theoretical condition that no measurable cardinals exist (which is consistent with ZFC).

Theorem 2.9 (Trnková [10]). *If there exist no measurable cardinals, then one can construct a rigid proper class of compact Hausdorff spaces.*

Note that because of the fullness of the embedding of **TOP** in **AP** every **TOP**-rigid class is **AP**-rigid.

Theorem 2.10. *Under the assumption that no measurable cardinals exist, one can construct a proper conglomerate* (*i.e., one which is not codable by a class*) *of not naturally isomorphic MC-structures on* **UAP**.

Proof. Take $\mathbf{M}$ to be a rigid proper class of infinite compact Hausdorff spaces. Take

$$\emptyset \neq \mathbf{D} \subsetneq \mathbf{E} \subset \mathbf{M}.$$

We now apply the construction form [4] (see 2.7 for a recollection of the definition of the inner hom functor) to $\mathbf{D}^\times$ (respectively $\mathbf{E}^\times$), being the saturation of $\mathbf{D}$ (respectively $\mathbf{E}$) with respect to finite products, yielding MC structures $(-\Box_{\mathbf{D}}-, H_{\mathbf{D}}(-,-))$ (respectively $(-\Box_{\mathbf{E}}-, H_{\mathbf{E}}(-,-))$ on **UAP** for which $-\Box_{\mathbf{D}} D = - \times D$ for all $D \in \mathbf{D}$ (respectively $-\Box_{\mathbf{E}} E = - \times E$ for all $E \in \mathbf{E}$). Now take $E \in \mathbf{E} \setminus \mathbf{D}$. We are done if we show that $H_{\mathbf{D}}(E,\mathbb{R}) \neq H_{\mathbf{E}}(E,\mathbb{R})$. This is obtained using the previous lemma together with a completely analogous proof as the one in the pre-approach case given in [9]. □

References

[1] J. Adámek, H. Herrlich, G. Strecker, Abstract and Concrete Categories, Wiley, New York, 1990.

[2] J. Činčura, Tensor products in the category of topological spaces, Comment. Math. Univ. Carolin. 20 (3) (1979) 431–446.

[3] G. Greve, How many monoidal closed structures are there in **TOP**?, Arch. Math. 34 (1980) 538–539.

[4] G. Greve, Monoidal closed structures in categories of topological, in: Uniform and Nearness Spaces, in Category Theory, in: Lecture Notes in Math., vol. 962, Springer, Berlin, 1982, pp. 100–114.

[5] E. Lowen, R. Lowen, Topological quasitopos hulls of categories containing topological and metric objects, Cahiers Topologie Géom. Différentielle Catégoriques 30 (1989) 213–228.

[6] R. Lowen, Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad, Oxford Univ. Press, Oxford, 1997.

[7] M. Nauwelaerts, The hulls of the category of uniform approach spaces, submitted for publication.

[8] F. Schwarz, Powers and exponentials in initially structured categories and applications to categories of limit spaces, Quaestiones Math. 6 (1983) 227–254.
[9] M. Sioen, Symmetric monoidal closed structures in **PRAP**, Rend. Istit. Mat. Univ. Trieste Suppl. 30 (1999) 1–13.
[10] V. Trnkovà, Non-constant coninuous maps of metric or compact Hausdorff spaces, Comment. Math. Univ. Carolin. 13 (1972) 283–295.