Convergence of the Hundsdorfer–Verwer Scheme for Two-dimensional Convection-diffusion Equations with Mixed Derivative Term

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Abstract. Alternating Direction Implicit (ADI) schemes are popular in the numerical solution of multidimensional time-dependent partial differential equations (PDEs) arising in various contemporary application fields such as financial mathematics. The Hundsdorfer–Verwer (HV) scheme is an often used ADI scheme. A structural analysis of its fundamental properties, notably convergence, is of main interest. Up to now, however, a convergence result is only known in the literature relevant to the case of one-dimensional PDEs. In this paper we prove that, under natural stability and smoothness conditions, the HV scheme has a temporal order of convergence equal to two, uniformly in the spatial mesh width, whenever it is applied to two-dimensional convection-diffusion equations with mixed derivative term.

Keywords: ADI splitting schemes, Hundsdorfer–Verwer scheme, Convection-Diffusion equations, Convergence analysis.
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INTRODUCTION

We consider the time discretization of initial value problems for large systems of stiff ordinary differential equations,

$$U'(t) = F(t,U(t)) \qquad (0 \le t \le T), \qquad U(0) = U_0, \tag{1}$$

with given vector-valued function F and given vector U_0, arising from semidiscretization of initial-boundary value problems for multidimensional convection-diffusion equations. For such systems (1) standard application of classical implicit time stepping methods, e.g. the Crank–Nicolson scheme, can be computationally very intensive. In order to overcome this, various operator splitting techniques have been studied in the literature. A prominent class of operator splitting methods are the Alternating Direction Implicit (ADI) schemes. ADI schemes employ a splitting of the semidiscrete PDE operator in the different spatial directions. This leads to a major computational advantage in each time step as it turns out that the implicitness is usually much easier when the suboperators are handled individually, instead of treating the full operator all at once. A modern ADI scheme is the Hundsdorfer–Verwer (HV) scheme [5, 6, 7]. The HV scheme is often used, for example, in financial applications. It has been adapted in [2] so as to deal with mixed spatial derivative terms, which are pervasive in finance. Let the semidiscrete function F be split as

$$F(t,\xi) = F_0(t,\xi) + F_1(t,\xi) + \ldots + F_k(t,\xi),$$

where F_0 represents all mixed spatial derivative terms and F_j, for $1 \le j \le k$, represents all derivatives in the j-th spatial direction. Let N denote the number of time steps, set $\Delta t = T/N$ and $t_n = n\Delta t$. Then, for a given parameter $\theta > 0$, the *HV scheme* generates approximations U_n to $U(t_n)$ for $n = 1,2,\ldots,N$ by

$$\begin{cases} Y_0 = U_{n-1} + \Delta t F(t_{n-1}, U_{n-1}), \\ Y_j = Y_{j-1} + \theta \Delta t \left(F_j(t_n, Y_j) - F_j(t_{n-1}, U_{n-1})\right), & j = 1,2,\ldots,k, \\ \widetilde{Y}_0 = Y_0 + \frac{1}{2}\Delta t \left(F(t_n, Y_k) - F(t_{n-1}, U_{n-1})\right), \\ \widetilde{Y}_j = \widetilde{Y}_{j-1} + \theta \Delta t (F_j(t_n, \widetilde{Y}_j) - F_j(t_n, Y_k)), & j = 1,2,\ldots,k, \\ U_n = \widetilde{Y}_k. \end{cases}$$

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Recently, various positive *stability* results for the HV scheme have been proved relevant to multidimensional convection-diffusion equations with mixed derivative terms, see e.g. [1, 2, 3]. A rigorous *convergence* analysis appears to be lacking at this moment, however. To the best of our knowledge, a convergence result is only known in the literature up to now for the case $k = 1$, see [5]. In this paper, we shall derive a convergence theorem for $k = 2$. This result is directly relevant, for example, to two-dimensional PDEs from financial mathematics.

CONVERGENCE THEOREM

Assume (1) stems from semidiscretization of a linear convection-diffusion problem with mixed derivative terms and

$$F(t,\xi) = A\xi + g(t) \;\text{ and }\; F_j(t,\xi) = A_j\xi + g_j(t) \quad \text{whenever } 0 \le t \le T,\ \xi \in \mathbb{R}^m,\ 0 \le j \le k,$$

with given real $m \times m$-matrices A and A_j $(0 \le j \le k)$ and given real m-vector valued functions g and g_j $(0 \le j \le k)$. Here m denotes the number of spatial grid points. To simplify notation, define the matrices

$$Z = \Delta t A, \quad Z_j = \Delta t A_j, \quad Q_j = I - \theta Z_j, \quad P = Q_1 Q_2 \cdots Q_k, \quad \text{for } 1 \le j \le k.$$

We consider the norm $\|\cdot\|_2$ induced by the (naturally scaled) inner product $(v,w) = \frac{1}{m} v^{\mathrm{T}} w$ on $\mathbb{R}^m$ and assume that the semidiscretization satisfies $(A_j v, v) \le 0$ for all $v \in \mathbb{R}^m$ and $1 \le j \le k$. This implies that the Q_j and P are invertible and

$$\|Q_j^{-1}\|_2 \le 1, \quad \|Z_j Q_j^{-1}\|_2 \le 1, \quad \|P^{-1}\|_2 \le 1, \quad \text{for } 1 \le j \le k. \tag{2}$$

It can be verified, cf. [5], that the *global (temporal) discretization error* $e_n = U(t_n) - U_n$ satisfies

$$e_n = R e_{n-1} + d_n, \tag{3}$$

with *stability matrix*

$$R = I + 2P^{-1}Z - P^{-2}Z + \tfrac{1}{2}(P^{-1}Z)^2,$$

and *local discretization error*

$$\begin{aligned} d_n \;=\; & (I - P^{-1} + \tfrac{1}{2}P^{-1}Z)(P^{-1}\rho_0 + \sum_{j=1}^{k} Q_k^{-1} Q_{k-1}^{-1} \cdots Q_j^{-1} \rho_j) \\ & + P^{-1}(\rho_0 + \widetilde{\rho}_0) + \sum_{j=1}^{k} Q_k^{-1} Q_{k-1}^{-1} \cdots Q_j^{-1} \widetilde{\rho}_j, \end{aligned} \tag{4}$$

with

$$\rho_0 = \tfrac{1}{2}(\Delta t)^2 U''(t_{n-1}) + \tfrac{1}{6}(\Delta t)^3 U'''(t_{n-1}) + \mathscr{O}\left((\Delta t)^4\right), \tag{5}$$

$$\rho_j = -\theta(\Delta t)^2 \varphi_j'(t_{n-1}) + \mathscr{O}\left((\Delta t)^3\right), \qquad j = 1,\ldots,k, \tag{6}$$

$$\widetilde{\rho}_0 = -\tfrac{1}{2}(\Delta t)^2 U''(t_{n-1}) - \tfrac{1}{4}(\Delta t)^3 U'''(t_{n-1}) + \mathscr{O}\left((\Delta t)^4\right), \tag{7}$$

$$\widetilde{\rho}_j = 0, \qquad j = 1,\ldots,k, \tag{8}$$

where $\varphi_j(t) = F_j(t, U(t))$ for $0 \le j \le k$. We assume that U and the φ_j are sufficiently often differentiable and that their derivatives are bounded on $[0,T]$ uniformly in the spatial mesh width. *With the notation $\mathscr{O}((\Delta t)^p)$ we always mean that the norm $\|\cdot\|_2$ of the pertinent term is bounded by a positive constant times $(\Delta t)^p$, with the constant independent of the spatial mesh width, the time step size Δt and the time step number n. If $p = 0$, then we write $\mathscr{O}(1)$ for short.*

The recursion (3) yields $e_n = \sum_{j=1}^{n} R^{n-j} d_j$. From this it is clear that one can distinguish two important steps in proving convergence. First one wishes to show *stability*, i.e., there exists a moderate constant M such that $\|R^n\|_2 \le M$ uniformly in the spatial mesh width, the time step size Δt and n. Secondly, one wishes to prove *consistency*, i.e., the local discretization errors d_n tend to zero as the time step Δt tends to zero, uniformly in the spatial mesh width and n. For the analysis in the present paper it will be assumed that the HV is stable.

We consider the case $k = 2$ and, to simplify the analysis, assume that the matrices A_j $(0 \leq j \leq k)$ commute. All of the foregoing assumptions in this section were made in [5] as well. For convenience we also assume in the following that A is invertible.

By using the expressions (4)–(8), one immediately finds

$$\begin{aligned} d_n &= (I - P^{-1} + \tfrac{1}{2}P^{-1}Z)[\tfrac{1}{2}(\Delta t)^2 P^{-1}U''(t_{n-1}) - \theta(\Delta t)^2 P^{-1}\varphi_1'(t_{n-1}) - \theta(\Delta t)^2 Q_2^{-1}\varphi_2'(t_{n-1})] \\ &\quad + (I - P^{-1} + \tfrac{1}{2}P^{-1}Z)\mathscr{O}\left((\Delta t)^3\right) - \tfrac{1}{12}(\Delta t)^3 P^{-1}U'''(t_{n-1}) + P^{-1}\mathscr{O}\left((\Delta t)^4\right). \end{aligned}$$

In general, when the local discretization errors are of second-order, one would only expect first-order convergence. Often, however, an order of convergence can be recovered through the following lemma [4].

Lemma 1 *Suppose that $e_n = \sum_{j=1}^{n} R^{n-j} d_j$. If the local discretization error can be written as $d_j = (R - I)\xi_j + \eta_j$ with $\xi_j = \mathscr{O}\left((\Delta t)^2\right)$ and $\xi_j - \xi_{j-1} = \mathscr{O}\left((\Delta t)^3\right), \eta_j = \mathscr{O}\left((\Delta t)^3\right)$, then for the global error one has that $e_n = \mathscr{O}\left((\Delta t)^2\right)$.*

As a first attempt, one could check whether the matrix $(R - I)^{-1}(I - P^{-1} + \frac{1}{2}P^{-1}Z)$ is bounded uniformly in the spatial mesh width. Unfortunately, as it turns out, this is not always the case. Therefore, we consider splitting the local discretization error. First, rewrite d_n as

$$\begin{aligned} d_n &= (I - P^{-1} + \tfrac{1}{2}P^{-1}Z)P^{-1}[\tfrac{1}{2}(\Delta t)^2\varphi_0'(t_{n-1}) + (\tfrac{1}{2} - \theta)(\Delta t)^2\Sigma_{j=1}^2\varphi_j'(t_{n-1})] \\ &\quad + (I - P^{-1} + \tfrac{1}{2}P^{-1}Z)P^{-1}\theta(\Delta t)^2(I - Q_1)\varphi_2'(t_{n-1}) \\ &\quad - \tfrac{1}{12}(\Delta t)^3 P^{-1}U'''(t_{n-1}) + (I - P^{-1} + \tfrac{1}{2}P^{-1}Z)\mathscr{O}\left((\Delta t)^3\right) + P^{-1}\mathscr{O}\left((\Delta t)^4\right). \end{aligned}$$

Then, we split the error into four parts: $d_n = d_n^{(1)} + d_n^{(2)} + d_n^{(3)} + d_n^{(4)}$ with

$$\begin{aligned} d_n^{(1)} &= (R - I)(R - I)^{-1}(I - P^{-1} + \tfrac{1}{2}P^{-1}Z)P^{-1}[\tfrac{1}{2}(\Delta t)^2\varphi_0'(t_{n-1}) + (\tfrac{1}{2} - \theta)(\Delta t)^2\Sigma_{j=1}^2\varphi_j'(t_{n-1})], \\ d_n^{(2)} &= (R - I)Z^{-1}\theta^2(\Delta t)^2 Z_1\varphi_2'(t_{n-1}), \\ d_n^{(3)} &= -(R - I)(R - I)^{-1}P^{-1}\theta^2(\Delta t)^2 Z_1\varphi_2'(t_{n-1}), \\ d_n^{(4)} &= -\tfrac{1}{12}(\Delta t)^3 P^{-1}U'''(t_{n-1}) + (I - P^{-1} + \tfrac{1}{2}P^{-1}Z)\mathscr{O}\left((\Delta t)^3\right) + P^{-1}\mathscr{O}\left((\Delta t)^4\right). \end{aligned}$$

It is clear that the second part meets the requirements for application of Lemma 1 if $A^{-1}A_1 = \mathscr{O}(1)$. Further, as P^{-1} is bounded in $\|\cdot\|_2$ by 1, the fourth part satisfies the requirements if $I - P^{-1} + \frac{1}{2}P^{-1}Z = \mathscr{O}(1)$. For the first part it is sufficient if the matrix

$$Z^{-1}P(2I - P^{-1} + \tfrac{1}{2}P^{-1}Z)^{-1}(I - P^{-1} + \tfrac{1}{2}P^{-1}Z)P^{-1} \tag{9}$$

is uniformly bounded. By using that the A_j commute, it follows that this is equivalent to the matrix

$$Z^{-1}(-\theta Z_1 - \theta Z_2 + \theta^2 Z_1 Z_2 + \tfrac{1}{2}Z)P^{-1}(2I - P^{-1} + \tfrac{1}{2}P^{-1}Z)^{-1}$$

being uniformly bounded. As for the second part, we assume that $A^{-1}A_1 = \mathscr{O}(1)$. Analogously, the assumption is made that $A^{-1}A_2 = \mathscr{O}(1)$. Next, since $P^{-1} = Q_2^{-1}Q_1^{-1}$ we have

$$Z^{-1}Z_1 Z_2 P^{-1} = Z^{-1}Z_1 Z_2 Q_2^{-1}Q_1^{-1}.$$

By (2) it holds that $\|Z_2 Q_2^{-1}\|_2$ and $\|Q_1^{-1}\|_2$ are both bounded by 1. From this, we obtain that the matrix (9) is uniformly bounded if $(2I - P^{-1} + \frac{1}{2}P^{-1}Z)^{-1} = \mathscr{O}(1)$. Finally, the third part fulfils the requirements if the matrix

$$Z^{-1}P(2I - P^{-1} + \tfrac{1}{2}P^{-1}Z)^{-1}P^{-1}Z_1,$$

is uniformly bounded. Using that the A_j commute, this matrix can be rewritten as

$$Z^{-1}Z_1(2I - P^{-1} + \tfrac{1}{2}P^{-1}Z)^{-1},$$

and uniform boundedness follows from the assumptions made above. Summarizing, we proved the next theorem.

Theorem 1 *Let $k = 2$. Assume U and the φ_j $(j = 0,1,2)$ are sufficiently often differentiable and their derivatives are bounded on $[0,T]$ uniformly in the spatial mesh width. Assume A is invertible, the A_j $(j = 0,1,2)$ commute and $(A_j v, v) \leq 0$ whenever $v \in \mathbb{R}^m$ and $j = 1,2$. Assume the HV scheme is stable, the matrix $2I - P^{-1} + \frac{1}{2}P^{-1}Z$ is invertible and the four matrices $A^{-1}A_1$, $A^{-1}A_2$, $I - P^{-1} + \frac{1}{2}P^{-1}Z$, $(2I - P^{-1} + \frac{1}{2}P^{-1}Z)^{-1}$ are all $\mathscr{O}(1)$. Then the global discretization errors satisfy $e_n = \mathscr{O}\left((\Delta t)^2\right)$.*

The above theorem extends the result of [5] from the one-dimensional ($k = 1$) to the two-dimensional case ($k = 2$) . The crucial step in our derivation is the splitting of the local discretization error into four convenient parts. The uniform boundedness of $A^{-1}A_1$ and $A^{-1}A_2$, which is often fulfilled, was also assumed in [5]. The uniform boundedness of $I - P^{-1} + \frac{1}{2}P^{-1}Z$ was tacitly assumed in there as well. Our assumption $(2I - P^{-1} + \frac{1}{2}P^{-1}Z)^{-1} = \mathscr{O}(1)$ is new. It may be regarded as replacing the condition (29) in [5].

For a theoretical result on the conditions that $I - P^{-1} + \frac{1}{2}P^{-1}Z$ and $(2I - P^{-1} + \frac{1}{2}P^{-1}Z)^{-1}$ are $\mathscr{O}(1)$, we consider here a two-dimensional constant coefficient convection-diffusion equation provided with periodic boundary condition. In this case the analysis reduces, see [2], to bounding from above

$$|1 - \tfrac{1}{p} + \tfrac{1}{2}\tfrac{z_0+z_1+z_2}{p}| \quad \text{and} \quad |2 - \tfrac{1}{p} + \tfrac{1}{2}\tfrac{z_0+z_1+z_2}{p}|^{-1}$$

with $p = (1 - \theta z_1)(1 - \theta z_2)$ for all complex numbers z_0, z_1, z_2 satisfying

$$\mathscr{R}z_1 \leq 0, \quad \mathscr{R}z_2 \leq 0, \quad |z_0| \leq 2\sqrt{\mathscr{R}z_1 \mathscr{R}z_2}. \tag{10}$$

Theorem 2 *If (10) and $\theta > 1/2$, then*

$$|1 - \tfrac{1}{p} + \tfrac{1}{2}\tfrac{z_0+z_1+z_2}{p}| \leq 2 \quad \textit{and} \quad |2 - \tfrac{1}{p} + \tfrac{1}{2}\tfrac{z_0+z_1+z_2}{p}|^{-1} \leq \tfrac{2\theta}{2\theta-1}.$$

Proof By [2, Lemmas 2.1, 2.3] we have $|1 + \frac{z_0+z_1+z_2}{p}| \leq 1$. From this and $|p| \geq 1$ it follows that

$$|1 - \tfrac{1}{p} + \tfrac{1}{2}\tfrac{z_0+z_1+z_2}{p}| \leq \tfrac{1}{2}|1 + \tfrac{z_0+z_1+z_2}{p}| + \tfrac{1}{2} + |\tfrac{1}{p}| \leq 2,$$

which proves the first part of the theorem. Next, by [2, Lemma 2.3] there holds

$$|\tfrac{z_0}{p}| + |\tfrac{1}{2\theta} + \tfrac{z_1+z_2}{p}| \leq \tfrac{1}{2\theta}.$$

Consequently,

$$|2 - \tfrac{1}{p} + \tfrac{1}{2}\tfrac{z_0+z_1+z_2}{p}| \geq |2 - \tfrac{1}{4\theta} - \tfrac{1}{p}| - \tfrac{1}{2}\left(|\tfrac{z_0}{p}| + |\tfrac{1}{2\theta} + \tfrac{z_1+z_2}{p}|\right) \geq 1 - \tfrac{1}{4\theta} - \tfrac{1}{4\theta} = \tfrac{2\theta-1}{2\theta},$$

which yields the second part of the theorem.

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CONCLUSION

ADI schemes are highly effective in the numerical solution of multidimensional time-dependent convection-diffusion equations with mixed derivative terms. Such equations are widespread, for example, in financial mathematics. A popular ADI scheme is the HV scheme. Recently, various positive stability results have been derived in literature for this scheme. Also, a convergence result has been obtained pertinent to the special case of one-dimensional PDEs. Clearly, obtaining convergence results relevant to multidimensional PDEs is of much interest. In this paper we studied the convergence of the HV scheme relevant to two-dimensional convection-diffusion equations with mixed derivative term. We proved that, under natural stability and smoothness conditions, the HV scheme is convergent of order two uniformly in the spatial mesh width. In future research we wish to extend the convergence analysis to, for example, higher-dimensional PDEs and to nonsmooth initial functions.

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