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Relative Hermitian Morita Theory
I. Morita Equivalences of Algebras with Involution

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INTRODUCTION

Hermitian Morita Theory for algebras with involution was introduced and studied by Fröhlich and McEvett in [3], as part of a general theory of forms over a not necessarily commutative ring with involution. The set-up in [3] was later slightly modified by Hahn in [9], in order to be applicable to the more general notion of algebras with antistructure, as well as to make the underlying idea of a hermitian Morita context more transparant. Hahn's formalism clearly indicates how hermitian Morita theory parallels ordinary Morita theory. In particular, [9] contains a full hermitian version of Morita I (i.e., Morita equivalence data yield equivalences of categories). On the other hand, we refer to [12] for a hermitian version of Morita II (i.e., equivalences of categories arise from Morita equivalence data).

In the present paper, which is the first of a series of notes on relative hermitian Morita theory, see also [19], we combine the previous set-up with the theory of relative invariants, as developed in [17], essentially to introduce hermitian Morita theory for arbitrary involutive Grothendieck categories.

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Indeed, first recall that the Gabriel–Popescu Theorem, cf. [5, 15], essentially states that any Grothendieck category is, up to isomorphism, of the form (A, σ)-mod. Here, (A, σ)-mod denotes the quotient category of A-mod, the category of left modules over some not necessarily commutative ring A, by T_σ, the localizing subcategory of A-mod, consisting of left A-modules, which are torsion with respect to some idempotent kernel functor σ in A-mod, cf. [6] or Section 1, for details. So, in order to study Grothendieck categories, it actually suffices to study (seemingly) more concrete categories of the form (A, σ)-mod, embedded into A-mod.

This point of view is the starting point of the theory of relative invariants, cf. [17]. As an example, it is well known that the group Pic(A-mod) of isomorphism classes of autoequivalences of A-mod may be identified with the (module-theoretic) Picard group Pic(A) of A. Relativizing this, it appears that the group Pic((A, σ)-mod), defined similarly, coincides with the so called relative Picard group Pic(A, σ) of A with respect to σ, consisting of isomorphism classes of "σ-invertible" A-bimodules, a rather technical notion, which generalizes that of an invertible A-bimodule within (A, σ)-mod, cf. [17, 18].

This set-up has two advantages: (i) it permits, in principle at least, to calculate the Picard group of any Grothendieck category, using the Gabriel–Popescu Theorem, and (ii) it allows, by specializing σ, to give a new interpretation for "classical" invariants, generalizing these and putting known (and new) results about them in a more natural context. (For example, if A is a commutative Krull domain and if σ is associated to the height one primes, cf. [17], then Pic$(A, \sigma) = Cl(A)$, the divisor class group of A, and several traditional results, like functoriality aspects, follow from general results for relative Picard groups.)

As indicated by the title, in this note we introduce and study relative hermitian Morita theory for algebras with involution. It has already been proved in [17] that ordinary Morita theory, which links equivalences between categories of modules A-mod and B-mod to Morita equivalence data, based upon invertible bimodules, may be generalized to study equivalences between quotient categories (A, σ)-mod and (B, τ)-mod. Essentially, this completely settles the problem of describing equivalences between Grothendieck categories. So, the present text complements both [9] and [17], by studying the relative hermitian case. Note however, that, in order to avoid technicalities in this first paper, we have not worked in complete generality, i.e., we restrict to algebras with involution, instead of the more general algebras with antistructure.

Let us now briefly sketch the contents of this note. In the Section 1, we recall some of the fundamentals of abstract localization theory. We do not give any proofs, nor do we strive for any form of completeness, but we refer to the literature for details.

In Section 2, we study localizations of rings and algebras with involution. It appears that a ring with involution does, in general, not localize to another ring with involution. Instead, if σ is an idempotent kernel functor in A-mod, then any involution $\alpha: A \to A$ extends to an anti-isomorphism $\hat{\alpha}: Q_\sigma(A) \to Q_{\alpha(\sigma)}(A)$, where $Q_{\alpha(\sigma)}(A)$ is the ring of fractions of A at some suitably constructed idempotent kernel functor $\alpha(\sigma)$ in the category mod-A. of *right* A-modules.

In order to introduce the notion of a relative hermitian Morita equivalence, we have to consider so-called involutive (relatively) invertible bimodules. These are introduced in the third section, where, somewhat surprisingly, it appears that the existence of an involutive invertible A–B-bimodule, for another ring with involution B, implies the localizations $Q_\sigma(A)$ and $Q_{\alpha(\sigma)}(A)$ to coincide, thus eliminating most of the problems sketched in the previous Section.

The Sections 3 and 4 are then properly concerned with relative hermitian Morita theory. First, we show how relative hermitian Morita equivalences induce equivalences between suitable categories of sesquilinear and hermitian modules in quotient categories (Morita I). In Section 5, on the other hand, we show how both Morita I and Morita II hold, in the somewhat more general context of involutive relative Morita equivalences, but restricted to modules with some (harmless) extra finiteness assumptions.

For a geometric interpretation, a general study of relative hermitian Morita *contexts* and a full treatment within the framework of monoidal categories, we refer to work in progress.

1. Generalities on Abstract Localization

(1.1) Throughout, R denotes a commutative ring and A a fixed R-algebra, (both with unit). We denote by A-mod resp. mod-A the category of (unitary) left resp. right A-modules and by ${}_A[M, N]$ resp. $[M, N]_A$ the corresponding sets of morphisms in these categories. If B is a second R-algebra, we also work in the category A-mod-B of A–B-bimodules, i.e., left A-right B-modules M with compatible actions, and such that the R-module structures on M defined through the action of A and B coincide.

The corresponding morphism sets are denoted by ${}_A[M, N]_B$. Although we assume the reader is familiar with the fundamentals of abstract localization theory, such as expounded in [4, 6–8, 15], for example, we briefly recall some relevant definitions and basic properties here.

We define an idempotent kernel functor σ over A to be a left exact subfunctor of the identity in A-mod (or in mod-A, if we wish to work on the right) with the property that $\sigma(M/\sigma M) = 0$, for any $M \in A$-mod. For example, we might assume σ to be an idempotent kernel functor in R-mod, and consider the idempotent kernel functor $\bar{\sigma}$ in A-mod, induced by σ. When no ambiguity arises, we usually just write σ instead of $\bar{\sigma}$. Any idempotent kernel functor σ determines a torsion class $\mathbf{T}_\sigma$ resp. a torsionfree class $\mathbf{F}_\sigma$, consisting of all $M \in A$-mod with $\sigma M = M$ resp. $\sigma M = 0$. Conversely, each of these classes completely determines σ. Recall also that σ is completely determined by its Gabriel filter $\mathbf{L}(\sigma)$, which consists of all left A-ideals L, with the property that $A/L \in \mathbf{T}_\sigma$. Indeed, it is easy to see that for any $M \in A$-mod, we have $m \in \sigma M$ if and only if $\text{Ann}(m) \in \mathbf{L}(\sigma)$.

(1.2) Let σ be an idempotent kernel functor in A-mod and let E be a σ-torsionfree left A-module. We say that E is σ-closed, if for any σ-isomorphism $N \to M$, i.e., a left A-linear map with σ-torsion kernel and cokernel, the induced map $\text{Hom}_A(M, E) \to \text{Hom}_A(N, E)$ is bijective. Of course, it actually suffices to verify that for any $L \in \mathbf{L}(\sigma)$ and any left A-linear map $f: L \to E$, there exists a unique $g: A \to E$ extending f. One denotes by (A, σ)-mod the full subcategory of A-mod consisting of σ-closed left A-modules and one usually calls this the quotient category of A-mod at (or with respect to) σ.

(1.3) To any A-mod and any idempotent kernel functor σ, we may canonically associate a σ-closed left A-module $Q_\sigma(M)$, endowed with a natural morphism $j_\sigma = j_{\sigma, M}: M \to Q_\sigma(M)$ with σ-torsion kernel and cokernel, and with obvious universal properties. One calls $Q_\sigma(M)$ the module of quotients of M at σ and the left exact functor $Q_\sigma: A\text{-mod} \to A\text{-mod}$ the associated localization functor. It may also be viewed as an exact functor $A\text{-mod} \to (A, \sigma)\text{-mod}$.

The left A-module $Q_\sigma(M)$ may be constructed in many different ways, cf. [4, 7, 8, 10, 13, 15], one of them being as follows. Let E be an injective hull of $\bar{M} = M/\sigma M$, then $Q_\sigma(M)$ may be defined to be the set of all $e \in E$, such that $Le \subseteq \bar{M}$ for some $L \in \mathbf{L}(\sigma)$. In particular, $Q_\sigma(M)/\bar{M} \in \mathbf{T}_\sigma$. Alternatively, let

$$Q_\sigma(M) = \lim\{{}_A[L, M/\sigma M]; L \in \mathbf{L}(\sigma)\}.$$

In this case, the left A-module structure on $Q_\sigma(M)$ is given as follows. If $q \in Q_\sigma(M)$ is represented by some left A-linear $f: L \to M/\sigma M$, with $L \in \mathbf{L}(\sigma)$ and $\sigma \in A$, then ${}_A(L : a) = \{b \in A; ba \in L\} \in \mathbf{L}(\sigma)$. We then let $aq \in Q_\sigma(A)$ be represented by

$$a.f\colon {}_A(L : a) \xrightarrow{\cdot a} L \xrightarrow{f} M/\sigma M,$$

i.e., $(a.f)(b) = f(ba)$, for any $a \in A$ and any $b \in {}_A(L : a)$. On the other hand, if $M \in A$-mod-A, then $Q_\sigma(M)$ also possesses a right A-module structure, for which, with the same data, qa is represented by

$$f.a : L \xrightarrow{f} M/\sigma M \xrightarrow{a} M/\sigma M,$$

i.e., $(f.a)(b) = f(b)a$, for any $a \in A$ and any $b \in L$.

(1.4) It is easy to see that $Q_\sigma(A)$ is endowed with a unique ring-structure, extending that of A and that for any $M \in A$-mod, the localization $Q_\sigma(M)$ is a left $Q_\sigma(A)$-module, in a canonical way. It is also fairly straightforward to check that for any idempotent kernel functor σ in R-mod with induced idempotent kernel functor $\bar{\sigma}$ in A-mod, the functors Q_σ and Q_σ coincide on left A-modules. In particular, $Q_\sigma(A)$ is an R-algebra, in a canonical way. On the other hand, it is also easy to verify, cf. [16, 17], that if M is an A–B-bimodule for some R-algebras A resp. B, then for any idempotent kernel functor σ in A-mod, the localization $Q_\sigma(M)$ is an A–B-bimodule in a canonical way and the canonical map $j_\sigma : M \to Q_\sigma(M)$ is a morphism in A-mod-B. This clearly makes Q_σ into a functor in A-mod-B.

2. Algebras with Involution and Localization

(2.1) Throughout, (A, σ) (or A, when no ambiguity arises) will be an *R-algebra with involution*. This means that $\alpha\colon A \to A$ is an R-involution of A, i.e., for any $r \in R$ and any a, $a' \in A$, we have $\alpha(aa') = \alpha(a')\,\alpha(a)$ resp. $\alpha(ra) = r\alpha(a)$ and $\alpha^2 = id_A$. In particular, this shows that A and its opposite R-algebra A^{opp} may be identified through α. If $R = \mathbf{Z}$, the ring of all integers, then we will simply speak of a *ring with involution*. Let σ be an idempotent kernel functor in R-mod. Since $\alpha\colon A \to A$ is R-linear, it extends to an R-linear map $\hat{\alpha} = Q_\sigma(\alpha)\colon Q_\sigma(A) \to Q_\sigma(A)$. Then we have:

(2.2) Proposition. *Let (A, α) be an R-algebra with involution and let σ be an idempotent kernel functor in R-mod. Then, with the above notations, $(Q_\sigma(A), \hat{\alpha})$ is an R-algebra with involution.*

Proof. Since $\alpha(\sigma A) = \sigma A$, the involution α extends to $\bar{\alpha}\colon A/\sigma A \to A/\sigma A$, so we may assume A to be σ-torsionfree. Since α is R-linear, it extends to an R-linear map $\hat{\alpha}\colon Q_\sigma(A) \to Q_\sigma(A)$. Let a, $b \in Q_\sigma(A)$, then there exist I, $J \in \mathrm{L}(\sigma)$, with $Ia \subseteq A$ and $Jb \subseteq A$. But then, for any $i \in I$ and $j \in J$, we have

$$ij\hat{\alpha}(ab) = \hat{\alpha}((ij)(ab)) = \hat{\alpha}((ia)(jb)) = \alpha(jb)\,\alpha(ia) = j\hat{\alpha}(b)\,i\hat{\alpha}(a) = ij\hat{\alpha}(b)\,\hat{\alpha}(a).$$

So $IJ(\hat{\alpha}(ab) - \hat{\alpha}(b)\,\hat{\alpha}(a)) = 0$. Now, $IJ \in \mathrm{L}(\sigma)$, so, since $A \in \mathbf{F}_\sigma$, it follows that $\hat{\alpha}(ab) = \hat{\alpha}(b)\,\hat{\alpha}(a)$, and $\hat{\alpha}$ is an involution, indeed.

(2.3) More generally, let us now consider an arbitrary idempotent kernel functor σ in A-mod with Gabriel filter $\mathbf{L}(\sigma)$. We may define an idempotent kernel functor $\alpha(\sigma)$ in mod-A, the category of right A-modules, by its Gabriel filter $\mathbf{L}(\alpha(\sigma))$, consisting of all right A-ideals $\alpha(L)$ with $L \in \mathbf{L}(\sigma)$. In particular, for any right A-module M, this means that $m \in M$ belongs to $\alpha(\sigma)M$ if and only if $m\alpha(L) = 0$, for some $L \in \mathbf{L}(\sigma)$.

Let $M \in A$-mod, then the right A-module M^α has the same underlying additive group as M and its right multiplication is given by $ma = \alpha(a)m$, for any $m \in M^\alpha$ and any $a \in A$. Obviously, any left A-linear map $f\colon M \to N$ yields a right A-linear map $f^\alpha\colon M^\alpha \to N^\alpha$, thus defining a covariant functor $(-)^\alpha\colon A\text{-mod} \to \text{mod-}A$. If $M \in \text{mod-}A$, then the left A-module ${}^\alpha M$ is defined similarly, thus yielding an exact (covariant) functor ${}^\alpha(-)\colon \text{mod-}A \to A\text{-mod}$. Finally, if $M \in A\text{-mod-}A$, then we define the A-bimodule M_α by letting $a.m.b = \alpha(b)\, m\alpha(a)$, for any $a, b \in A$ and $m \in M$. It is then fairly straightforward to see that for any $M \in A$-mod resp. $N \in \text{mod-}A$, we have equalities $\alpha(\sigma)M^\alpha = (\sigma M)^\alpha$ resp. $\sigma({}^\alpha N) = {}^\alpha(\alpha(\sigma)N)$.

(2.4) Lemma. *For any σ-closed left A-module M, the right A-module M^α is $\alpha(\sigma)$-closed.*

Proof. First, note that $\alpha(\sigma)(M^\alpha) = (\sigma M)^\alpha = 0$. So, to prove that M^α is $\alpha(\sigma)$-closed, consider a right A-linear morphism $f\colon \alpha(L) \to M^\alpha$, where $L \in \mathbf{L}(\sigma)$. We want to show that f extends to A. The morphism $f'\colon L \to M\colon j \to f(\alpha(j))$ is obviously left A-linear, so f' extends to some left A-linear $g'\colon A \to M$. It is now clear that $g\colon A \to M^\alpha\colon a \to g'(\alpha(a))$ is the desired right A-linear extension of f. This proves the assertion.

(2.5) Corollary. *For any A-bimodule M the A-bimodules $Q_{\alpha(\sigma)}(M)$ and $Q_\sigma(M_\alpha)_\alpha$ are isomorphic.*

Proof. The remarks made in (2.3) show that, applying the functor $(-)_\alpha$ to the σ-isomorphism (of A-bimodules) $j_{\sigma,M}\colon M_\alpha \to Q_\sigma(M_\alpha)$, clearly yields an $\alpha(\sigma)$-isomorphism (of A-bimodules) $(j_{\sigma,M})_\alpha\colon M \to Q_\sigma(M_\alpha)_\alpha$. On the other hand, the previous lemma shows that $Q_\sigma(M_\alpha)_\alpha$ is $\alpha(\sigma)$-closed. This proves the assertion.

Concretely, since we may obviously assume M and M_α to be torsionfree, the isomorphism $Q_{\alpha(\sigma)}(M) \to Q_\sigma(M_\alpha)_\alpha$ may also be given by the map

$$\phi\colon Q_{\alpha(\sigma)}(M) = \lim_{L \in \mathbf{L}(\sigma)} [\alpha(L), M]_A \to Q_\sigma(M_\alpha) = \lim_{L \in \mathbf{L}(\sigma)} {}_A[L, M_\alpha],$$

which sends the class of $f \in [\alpha(L), M]_A$ in $Q_{\alpha(\sigma)}(M)$ to the class of $\phi_L(f)$ in $Q_\sigma(M_\alpha)_\alpha$. We leave this as a straightforward exercise to the reader.

(2.6) *Note.* In a completely similar way, one proves that the right A-modules $Q_{\alpha(\sigma)}(M)$ and $Q_\sigma({}^\alpha M)^\alpha$ are isomorphic for any right A-module M. Applying the functor ${}^\alpha(—)$, this also yields an isomorphism ${}^\alpha Q_{\alpha(\sigma)}(M) = Q_\sigma({}^\alpha M)$ of left A-modules. Somewhat more generally, if (B, β) is a second R-algebra with involution, and if $M \in B$-mod-A, then the B–A-bimodules ${}^\alpha Q_{\alpha(\sigma)}(M)^\beta$ and $Q_\sigma({}^\alpha M^\beta)$ are canonically isomorphic.

(2.7) PROPOSITION. *Let (A, α) be a ring with involution and let σ and $\alpha(\sigma)$ be as above, then $\alpha: A \to A$ uniquely extends to an anti-isomorphism $\hat{\alpha}: Q_\sigma(A) \to Q_{\alpha(\sigma)}(A)$, i.e., an additive isomorphism with $\hat{\alpha}(ab) = \hat{\alpha}(b)\,\hat{\alpha}(a)$, for any $a, b \in Q_\sigma(A)$.*

Proof. First note that the above definitions imply that $\alpha(\sigma A) = \alpha(\sigma)A$. So, α extends to an anti-isomorphism $\bar{\alpha}: A/\sigma A \to A/\alpha(\sigma)A$, i.e., we may assume A to be torsionfree for σ and $\alpha(\sigma)$. Let $a \in Q_\sigma(A)$, then a is represented by some left A-linear $f: I \to A$, where $I \in \mathbf{L}(\sigma)$. Define the map $g: \alpha(I) \to A$ by sending $j \in \alpha(I)$ to $\alpha(f(\alpha(j))) \in A$. Clearly, g is right A-linear and since $\alpha(I) \in \mathbf{L}(\alpha(\sigma))$, the morphism g defines some element $\hat{\alpha}(a) \in Q_{\alpha(\sigma)}(A)$. We leave it as an easy verification to the reader to check that this construction yields a well-defined anti-isomorphism $\hat{\alpha}: Q_\sigma(A) \to Q_{\alpha(\sigma)}(A)$, making the following diagram commutative:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ {\scriptstyle j_\sigma}\downarrow & & \downarrow{\scriptstyle j_{\alpha(\sigma)}} \\ Q_\sigma(A) & \xrightarrow[\hat{\alpha}]{} & Q_{\alpha(\sigma)}(A). \end{array}$$

In particular, this shows that $\hat{\alpha}$ induces a ring isomorphism $Q_{\alpha(\sigma)}(A) \cong Q_\sigma(A)^{\mathrm{opp}}$ and an A-bimodule isomorphism $Q_\sigma(A)_\alpha \cong Q_{\alpha(\sigma)}(A)$.

(2.8) *Note.* The involution $\alpha: A \to A$ defines an isomorphism $\gamma: A \to A_\alpha$ of A-bimodules by putting $\gamma(q) = \alpha(q)$ for any $q \in A$. Hence, localizing at σ, clearly γ extends to an A-bimodule isomorphism $\hat{\gamma}: Q_\sigma(A) \to Q_\sigma(A_\alpha)$. Using (2.4), it is easy to see that, up to twisting by α, this map essentially reduces to the previous one.

Let us conclude this section, with the following remarks, which are used in (3.7).

(2.9) LEMMA. *Let A be as before, let B be another R-algebra and let τ be an idempotent kernel functor in* mod-B. For any $M \in A$-mod *and any (right)* τ-closed *A–B-bimodule E, the right B-module ${}_A[M, E]$ is τ-closed.*

Proof. Consider an exact sequence in A-mod of the form

$$A^{(J)} \to A^{(I)} \to M \to 0.$$

Applying ${}_A[-, E]$ to this exact sequence yields an exact sequence of right B-modules

$$0 \to {}_A[M, E] \to {}_A[A, E]^I \to {}_A[A, E]^J.$$

So, since arbitrary products of τ-closed modules remain τ-closed, it clearly suffices to verify that the right B-module ${}_A[A, E]$ is τ-closed. But, as ${}_A[A, E]$ and E are canonically isomorphic in mod-B, this proves the assertion.

(2.10) COROLLARY. *For any $M \in A$-mod and any $\alpha(\sigma)$-closed A-bimodule E, the left A-module ${}_A^{\alpha}[M, E]$ is σ-closed.*

Proof. This follows immediately from (2.5) and the previous lemma.

3. HERMITIAN MORITA EQUIVALENCES

(3.1) If (A, α) is an R-algebra with involution and if σ is an idempotent kernel functor in A-mod, then we call (A, α, σ) a *torsion triple* over R. On the other hand, let B, C be other R-algebras and let M be an A–B-bimodule. If N is a left B-module resp. a B–C-bimodule, then we denote by $M \perp_\sigma N$ the left A-module resp. A–C-bimodule $Q_\sigma(M \otimes_B N)$, at least, when no ambiguity arises. If $m \in M$ and $n \in N$, and if $j_\sigma\colon M \otimes_B N \to M \perp_\sigma N$ denotes the canonical morphism, then we sometimes write $m \perp n$ for the element $j_\sigma(m \otimes n) \in M \perp_\sigma N$.

(3.2) Let (A, α, σ) and (B, β, τ) be torsion triples. Recall from [17, 18] that an A–B-bimodule Q is said to be (σ, τ)-flat, if it possesses the following two properties:

(3.2.1) if $M'' \in B$-mod is τ-torsion, then $Q \otimes_B M'' \in A$-mod is σ-torsion;

(3.2.2) if $i\colon M' \to M$ is a monomorphism in B-mod, then $\mathrm{Ker}(Q \otimes_B i) \in A$-mod is σ-torsion.

If Q is a (σ, τ)-flat A–B-bimodule, then it follows, in particular, that for any left B-module M there is a canonical isomorphism of left A-modules $Q \perp_\sigma M \cong Q \perp_\sigma Q_\tau(M)$. Applying this with $M = B$, it follows that any σ-closed (σ, τ)-flat A–B-bimodule Q is automatically $Q_\sigma(A)$–$Q_\tau(B)$-bimodule.

Recall also that a σ-closed A–B-bimodule Q is said to be (σ, τ)-invertible, if it is (σ, τ)-flat and if there exists another τ-closed (τ, σ)-flat B–A-bimodule P together with bimodule isomorphisms $\mu: P \perp {}_\tau Q \to Q_\tau(B)$ resp. $\nu: Q \perp {}_\sigma P \to Q_\sigma(A)$. (When no ambiguity arises, we then also say that P is *relatively* invertible.)

For details about this notion and its applications in the theory of relative invariants, we refer to [17]. Let us only recall that, given Q, the "inverse" P is unique up to bimodule isomorphism and may be given, e.g., by $P = {}_A[Q, Q_\sigma(A)]$. Moreover, the canonical morphism $B \to {}_A[Q, Q]$ (right multiplication) extends to a B-bimodule isomorphism $Q_\tau(B) \cong {}_A[Q, Q]$, which may also be viewed as a ring isomorphism between $Q_\tau(B)$ and $({}_A[Q, Q])^{\text{opp}}$.

(3.3) Lemma. *Let (A, α, σ) and (B, β, τ) be torsion triples. Then an A–B-bimodule Q is (σ, τ)-flat if and only if the B–A-bimodule ${}^\beta Q^\alpha$ is $(\alpha(\sigma), \beta(\tau))$-flat.*

Proof. This follows immediately by applying the obvious equivalences $(\ \ )^\alpha$, ${}^\alpha(\text{—})$: A-mod $\approx$ mod-A resp. $(\text{—})^\beta$, ${}^\beta(\text{—})$: B-mod $\approx$ mod-B.

(3.4) Corollary. *Let (A, α, σ) and (B, β, τ) be torsion triples. Then an A–B-bimodule Q is (σ, τ)-invertible if and only if the B–A-bimodule ${}^\beta Q^\alpha$ is $(\alpha(\sigma), \beta(\tau))$-invertible.*

Proof. Indeed, assume that Q is (σ, τ)-invertible, then ${}^\beta Q^\alpha$ is $\alpha(\sigma)$-closed by (2.4) and $(\alpha(\sigma), \beta(\tau))$-flat by the foregoing result. Moreover, there is an isomorphism of A-bimodules

$$Q_{\alpha(\sigma)}({}^\alpha P^\beta \otimes_B {}^\beta Q^\alpha) = Q_{\alpha(\sigma)}((Q \otimes_B P)_\alpha) = Q_\sigma(Q \otimes_B P)_\alpha = Q_\sigma(A)_\alpha \cong Q_{\alpha(\sigma)}(A),$$

the last isomorphism being induced by $\hat{\alpha}: Q_\sigma(A) \to Q_{\alpha(\sigma)}(A)$. A similar argument yields a B-bimodule isomorphism $Q_{\beta(\tau)}({}^\beta Q^\alpha \otimes_A {}^\alpha P^\beta) \cong Q_{\beta(\tau)}(B)$. So, ${}^\beta Q^\alpha$ is $(\alpha(\sigma), \beta(\tau))$-invertible, indeed. The inverse implication may be verified in an analogous way.

(3.5) The foregoing remarks have a category theoretic interpretation. Indeed, recall that any equivalence of categories (A, σ)-mod $\approx (B, \tau)$-mod given by a couple of adjoint functors[1] $F: (A, \sigma)\text{-mod} \to (B, \tau)\text{-mod}$ resp. $G: (B, \tau)\text{-mod} \to (A, \sigma)\text{-mod}$, is completely determined by $P = F(Q_\sigma(A))$ and $Q = G(Q_\tau(B))$, cf. [17, 18], for example. Here, P and Q are then (τ, σ)- resp. (σ, τ)-invertible bimodules and $F = (P \perp {}_\tau \text{—})$ resp. $G = (Q \perp {}_\sigma \text{—})$ (up to isomorphism). Conversely, any pair of "mutually inverse" relatively invertible bimodules P and Q thus determines an equivalence of categories (A, σ)-mod $\approx (B, \tau)$-mod.

[1] In this note, all functors are assumed to be additive, at least.

With the same notations, let us now consider two torsion triples (A, α, σ) and (B, β, τ). The functors

$$F'\colon \text{mod-}(A, \alpha(\sigma)) \to \text{mod-}(B, \beta(\tau))\colon M \to F({}^{\alpha}M)^{\beta} = (P \perp {}_{\tau}{}^{\alpha}M)^{\beta}$$

resp.

$$G'\colon \text{mod-}(B, \beta(\tau)) \to \text{mod-}(A, \alpha(\sigma))\colon N \to G({}^{\beta}N)^{\alpha} = (Q \perp {}_{\sigma}{}^{\beta}N)^{\alpha}$$

then define an equivalence of categories $\text{mod-}(A, \alpha(\sigma)) \approx \text{mod-}(B, \beta(\tau))$. Indeed, for any $M \in \text{mod-}(A, \alpha(\sigma))$, we then have

$$\begin{aligned} G'F'M &= G'((P \perp {}_{\tau}{}^{\alpha}M)^{\beta}) = (Q \perp {}_{\sigma}{}^{\beta}((P \perp {}_{\tau}{}^{\alpha}M)^{\beta}))^{\alpha} \\ &= (Q \perp {}_{\sigma}(P \perp {}_{\tau}{}^{\alpha}M))^{\alpha} = Q_{\sigma}({}^{\alpha}M)^{\alpha} = ({}^{\alpha}M)^{\alpha} = M. \end{aligned}$$

It thus follows that $F' = Q_{\beta(\tau)}(-\otimes_A P')$ for some $(\beta(\tau), \alpha(\sigma))$-invertible B–A-bimodule P'. Since P' is (up to isomorphism) completely determined by F', it follows from

$$Q_{\beta(\tau)}(M \otimes_A P') = Q_{\tau}(P \otimes_A {}^{\alpha}M)^{\beta} = Q_{\tau}({}^{\beta}M \otimes_A {}^{\alpha}P^{\beta}))^{\beta} = Q^{\beta(\tau)}(M \otimes_A {}^{\alpha}P^{\beta}),$$

for any $M \in \text{mod-}(A, \alpha(\sigma))$, that we may take $P' = {}^{\alpha}P^{\beta}$.

Of course, it also follows, directly from (3.4), that ${}^{\alpha}P^{\beta}$ is relatively invertible, with "inverse" ${}^{\beta}Q^{\alpha}$.

(3.6) We have already pointed out before that the (σ, τ)-invertible A–B-bimodule Q defines a canonical B-bimodule isomorphism $Q_{\tau}(B) \cong {}_A[Q, Q]$ and a ring isomorphism $Q_{\tau}(B) \cong ({}_A[Q, Q])^{\text{opp}}$. Similarly, with the above notations, $Q' = {}^{\beta}Q^{\alpha}$ yields a B-bimodule isomorphism $Q_{\beta(\tau)}(B) \cong [Q', Q']_A$ as well as a ring isomorphism $Q_{\beta(\tau)}(B) \cong [Q', Q']_A$ (induced by *left* multiplication).

In particular, since the rings ${}_A[Q, Q]$ and $[{}^{\beta}Q^{\alpha}, {}^{\beta}Q^{\alpha}]_A$ are isomorphic (through the map which sends $f \in {}_A[Q, Q]$ to $\tilde{f} \in [{}^{\beta}Q^{\alpha}, {}^{\beta}Q^{\alpha}]_A$, with $\tilde{f}(q) = f(q)$, for any $q \in {}^{\beta}Q^{\alpha}$), this yields a ring isomorphism $Q_{\tau}(B) \cong Q_{\beta(\tau)}(B)^{\text{opp}}$, which, of course, is just the map $\hat{\beta}\colon Q_{\tau}(B) \to Q_{\beta(\tau)}(B)$!

On the other hand, since the B-bimodules $[{}^{\beta}Q^{\alpha}, {}^{\beta}Q^{\alpha}]_A$ and $({}_A[Q, Q])_{\beta}$ may be identified as well, we also recover the B-bimodule isomorphism $Q_{\tau}(B)_{\beta} \to Q_{\beta(\tau)}(B)$, induced by $\hat{\beta}$.

Let us call a (σ, τ)-invertible A–B-bimodule Q *involutive* (with respect to α and β), if there is an A–B-bimodule isomorphism $Q \cong {}_A^{\alpha}[Q, Q_{\sigma}(A)]^{\beta}$. The associated equivalence of categories $G = Q_{\sigma}(Q \otimes_B -)\colon (B, \tau)\text{-mod} \to (A, \sigma)\text{-mod}$ is then called *involutive* as well. In this case, clearly $P = {}^{\beta}Q^{\alpha}$ may be chosen as an "inverse" for Q. Of course, it then follows that

$$P \cong {}_A[Q, Q_{\sigma}(A)] \cong {}^{\beta}Q^{\alpha} \cong {}_B^{\beta}[P, Q_{\tau}(B)]^{\alpha},$$

so the (τ, σ)-invertible B–A-bimodule P is involutive as well. Hence, so is the "inverse" $F = Q_\tau(P \otimes_A -)$: (A, σ)-mod $\to$ (B, τ)-mod.

We then have:

(3.7) PROPOSITION. *Let (A, α, σ) and (B, β, τ) be torsion triples. If there exists an involutive (σ, τ)-invertible A–B-bimodule (or, equivalently, an involutive (τ, σ)-invertible B–A-bimodule), then the A-bimodules $Q_\sigma(A)$ and $Q_{\alpha(\sigma)}(A)$ resp. the B-bimodules $Q_\tau(B)$ and $Q_{\beta(\tau)}(B)$ are canonically isomorphic.*

Proof. Let Q be an involutive (σ, τ)-invertible A–B-bimodule, then $Q = {}^{\alpha}_{A}[Q, Q_\sigma(A)]^{\beta}$, up to isomorphism. Since $P = {}_A[Q, Q_\sigma(A)]$ is τ-closed, by (2.4) the A–B-bimodule $Q = {}^{\alpha}P^{\beta}$ is $\beta(\tau)$-closed (on the right). From (2.9), it then follows that $Q_\tau(B) = {}_A[Q, Q]$ is $\beta(\tau)$-closed as well. In a similar way, it follows that $Q_{\beta(\tau)}(B) = [Q', Q']_A = [{}^{\beta}Q^{\alpha}, {}^{\beta}Q^{\alpha}]_A = [P, P]_A$ is τ-closed.

Denote by $u: B \to Q_\tau(B)$ resp. $v: B \to Q_{\beta(\tau)}(B)$ the canonical maps, then u and v are B-bimodule morphisms. From the remarks made in (1.4), it follows that there exist B-bimodule morphisms $p: Q_\tau(B) \to Q_{\beta(\tau)}(B)$ resp. $q: Q_{\beta(\tau)}(B) \to Q_\tau(B)$ with $v = pu$ resp. $u = qv$. Since it follows that $u = qv = qpu$ and $v = pu = pqv$, a unicity argument shows that $qp = \mathrm{id}$ resp. $pq = \mathrm{id}$ (on $Q_\tau(B)$ resp. $Q_{\beta(\tau)}(B)$). So, $Q_\tau(B)$ and $Q_{\beta(\tau)}(B)$ are isomorphic B-bimodules, indeed.

The statement about A follows by symmetry.

We have pointed out before that $Q_\sigma(A)$ (and similarly $Q_{\alpha(\sigma)}(A)$) possesses a unique ringstructure, canonically extending that on A. So:

(3.8) COROLLARY. *Under the same assumptions, the rings $Q_\sigma(A)$ and $Q_{\alpha(\sigma)}(A)$ resp. $Q_\tau(B)$ and $Q_{\beta(\tau)}(B)$ are canonically isomorphic.*

In the presence of an involutive relatively invertible bimodule, we may thus identify $Q_\sigma(A)$ and $Q_{\alpha(\sigma)}(A)$, as well as $Q_\tau(B)$ and $Q_{\beta(\tau)}(B)$. With this identification, it follows from (2.7):

(3.9) COROLLARY. *Let (A, α, σ) be a torsion triple. If there exists a torsion triple (B, β, τ) and an involutive (σ, τ)-invertible A–B-bimodule, then the involution $\alpha: A \to A$ extends to an involution $\hat{\alpha}: Q_\sigma(A) \to Q_\sigma(A)$.*

(3.10) *Note.* Of course, if $Q_\sigma(A)$ is $\alpha(\sigma)$-closed, then $Q_{\alpha(\sigma)}(A)$ is σ-closed too, by (2.4) and the remarks following (2.7). As in the previous results, this proves that $Q_\sigma(A)$ and $Q_{\alpha(\sigma)}(A)$ may be identified, hence $\alpha: A \to A$ extends to an involution $\hat{\alpha}: Q_\sigma(A) \to Q_\sigma(A)$. But then, the isomorphism $Q_\sigma(A) \cong Q_{\alpha(\sigma)}(A)_\alpha \cong Q_\sigma(A)_\alpha$, induced by $\hat{\alpha}$, shows that $Q_\sigma(A)$

is itself an involutive (σ, σ)-invertible A-bimodule. So the conditions in (3.8) are not only sufficient, but necessary as well.

(3.11) Mimicking an analogous definition in the absolute case, given in [9], we now define a *relative hermitian Morita equivalence* between torsion triples (A, α, σ) and (B, β, τ) to be a tuple

$$(P, Q, \theta\colon P \to Q, \mu\colon P \perp {}_\tau Q \to Q_\tau(B), \nu\colon Q \perp {}_\sigma P \to Q_\sigma(A)),$$

where P resp. Q is a (τ, σ)-invertible B–A-bimodule resp. a (σ, τ)-invertible A–B-bimodule defined over R, where θ is an additive bijection and where μ resp. ν is an isomorphism of B- resp. A-bimodules. Moreover, these data should satisfy the following requirements for any $p, p' \in P$, $q, q' \in Q$, $a \in A$ and $b \in B$:

(3.11.1) $\theta(bpa) = \alpha(a)\,\theta(p)\,\beta(b)$;

(3.11.2) $\mu(p \perp q)\,p' = p\nu(q \perp p')$ resp. $\nu(q \perp p)\,q' = q\mu(p \perp q')$;

(3.11.3) $\mu(p \perp \theta(p')) = \hat{\beta}(\mu(p' \perp \theta(p)))$ resp.
$\nu(\theta(p) \perp p') = \hat{\alpha}(\nu(\theta(p') \perp p))$.

Since we may obviously also view θ as an isomorphism of B–A-bimodules $\tilde{\theta}\colon P \to {}^\beta Q^\alpha$, it follows that P and Q are involutive. In particular, the left A-module ${}^\alpha P$ is σ-closed, so the right A-module P is $\alpha(\sigma)$-closed, hence a right $Q_{\alpha(\sigma)}(A) = Q_\sigma(A)$-module. It thus follows that the identities (3.11.2) make sense. Note also that (3.9) implies the identities (3.11.3) to make sense as well.

Finally, note that the identities (3.11.2)' are equivalent to the commutativity of the diagrams

$$\begin{array}{ccc} P \perp Q \perp P & \xrightarrow{\mu \perp P} & B \perp P \\ {\scriptstyle P \perp \nu}\downarrow & & \downarrow \\ P \perp A & \longrightarrow & P \end{array}$$

resp.

$$\begin{array}{ccc} Q \perp P \perp Q & \xrightarrow{\nu \perp Q} & A \perp Q \\ {\scriptstyle Q \perp \mu}\downarrow & & \downarrow \\ Q \perp B & \longrightarrow & Q \end{array}$$

and similarly for (3.11.3).

(3.12) LEMMA. *Let* $\psi: M \to N$ *be an additive morphism from the* $Q_\tau(B)$-$Q_{\alpha(\sigma)}(A)$*-bimodule* M *to the* A-B*-bimodule* N, *which is torsionfree with respect to* σ *and* $\beta(\tau)$. *If* $\psi(bma) = \alpha(a)\,\psi(m)\,\beta(b)$, *for any* $a \in A$ *and* $b \in B$, *then, for any* $a \in Q_{\alpha(\sigma)}(A)$, $b \in Q_\tau(B)$ *and* $m \in M$, *we have* $\psi(bma) = \hat{\alpha}^{-1}(a)\,\psi(m)\,\hat{\beta}(b)$.

Proof. First note that M is a $B/\tau B$-$A/\alpha(\sigma)A$-bimodule, with $\bar{b}m\bar{a} = bma$, for any $a \in A$ resp. $b \in B$ and that $\psi(\bar{b}m\bar{a}) = \bar{\alpha}^{-1}(\bar{a})\,\psi(m)\,\beta(\bar{b}) = \alpha^{-1}(a)\,\psi(m)\,\beta(b)$.

Let $b \in Q_\tau(B)$, then $Jb \subseteq B/\tau B$ for some $J \in \mathbf{L}(\tau)$. If $j \in J$, then $\psi(j(bm)) = \psi(bm)\,\beta(j)$ and $\psi((jb)m) = \psi(m)\,\bar{\beta}(jb) = \psi(m)\,\hat{\beta}(b)\,\beta(j)$. So we get that $(\psi(bm) - \psi(m)\,\hat{\beta}(b))\,\beta(J) = 0$, hence $\psi(bm) = \psi(m)\,\hat{\beta}(b)$, since $\beta(J) \in \mathbf{L}(\beta(\tau))$ and the right B-module N is $\beta(\tau)$-torsionfree.

Similarly, if $a \in Q_{\alpha(\sigma)}(A)$, then $a\alpha(I) \subseteq A/\alpha(\sigma)A$, for some $I \in \mathbf{L}(\sigma)$. If $i \in I$, then we obtain that $\psi((ma)\,\alpha(i)) = i\psi(ma)$ and that $\psi(m(a\alpha(i))) = \bar{\alpha}^{-1}(a\alpha(i))\,\psi(m) = i\hat{\alpha}^{-1}(a)\,\psi(m)$. So, $I(\psi(ma) - \hat{\alpha}^{-1}(a)\,\psi(m)) = 0$, hence $\psi(ma) = \hat{\alpha}^{-1}(a)\,\psi(m)$, since $I \in \mathbf{L}(\sigma)$ and the left A-module N is σ-torsionfree.

Of course, after identifying $Q_{\alpha(\sigma)}(A)$ and $Q_\sigma(A)$ for a relative hermitian Morita context as in (3.11), this just says that θ induces an isomorphism of $Q_\sigma(A)$-$Q_\tau(B)$-bimodules ${}^{\hat{\alpha}}P^{\hat{\beta}} \to Q$.

In a similar way, one proves:

(3.13) LEMMA. *In any relative hermitian Morita equivalence*

$$(P, Q, \theta: P \to Q,\ \mu: P \perp {}_\tau Q \to Q_\tau(B),\ \nu: Q \perp {}_\sigma P \to Q_\sigma(A)),$$

the maps μ *resp.* ν *are morphisms of* $Q_\tau(B)$ *resp.* $Q_\sigma(A)$*-bimodules.*

4. SESQUILINEAR MODULES

(4.1) As before, let us fix an R-algebra with involution (A, α). We start by recalling some definitions and elementary properties of sesquilinear and hermitian forms over A, cf. [9, 11, 12, 14] for more details.

Let M, $N \in A$-mod and let P be an A-bimodule. A map $h: M \times N \to P$ is said to be a *sesquilinear morphism* (with respect to α) if it is biadditive and if we have $h(am, n) = ah(m, n)$ resp. $h(m, an) = h(m, n)\,\alpha(a)$, for any $a \in A$, $m \in M$, and $n \in N$. If $P = A$, then we speak of a *sesquilinear form*. Finally, if σ is an idempotent kernel functor in A-mod, and if $P = Q_\sigma(A)$, endowed with its canonical A-bimodule stucture, then we call h a σ*-sesquilinear form*. If (B, β) is a second ring with involution and if M and N are A-B-bimodules, then we say that h is compatible with B or B-compatible if, moreover, $h(mb, n) = h(m, n\beta(b))$. In this case, it is easy to see that h extends to an A-bimodule morphism $h': M \otimes_B {}^{\beta}N^{\alpha} \to P$. Conversely, any A-bimodule morphism $h': M \otimes_B {}^{\beta}N^{\alpha} \to P$ uniquely determines a B-compatible sesquilinear morphism $h: M \times N \to P$, by putting $h(m, n) = h'(m \otimes n)$, for any $m \in M$ and $n \in N$.

(4.2) Example. The standard example of a sesquilinear form is the product $h: A \times A \to A: (a, a') \to a\alpha(a')$. Another typical example is given as follows. Let M be an A–B-bimodule and let M^* be the A–B-bimodule ${}^{\alpha}_{A}[M, A]^{\beta}$ with left A-action defined through α and right B-action defined through β, i.e., with $(a.f)(m) = f(m)\,\alpha(a)$ resp. $(f.b)(m) = f(m\beta(b))$, for any $a \in A$, $b \in B$, $f \in {}_A[M, A]$ and $m \in M$. The map $h: M \times M^* \to A: (m, f) \to f(m)$ is then easily verified to be a sesquilinear form, which is compatible with B.

(4.3) Let M and N be left A-modules and let $h: M \times N \to P$ be a sesquilinear morphism, then h induces a left A-linear morphism $h^a: N \to {}^{\alpha}_{A}[M, P]$, the *adjoint* of h, by putting $h^a(n)(m) = h(m, n)$ for any $m \in M$ and $n \in N$. It is easy to see that this defines a bijective correspondence between sesquilinear morphisms $M \times N \to P$ and left A-linear morphisms $N \to {}^{\alpha}_{A}[M, P]$. Moreover, if M and N are A–B-bimodules for some ring with involution (B, β), then this bijection restricts to a bijection between sesquilinear forms $M \times N \to P$ which are compatible with B and A–B-linear morphisms $N \to {}^{\alpha}_{A}[M, P]^{\beta}$.

Let us call a couple (M, h) with $M \in A$-mod and $h: M \times M \to A$ a sesquilinear form a *sesquilinear* left A-module. The foregoing then shows that we then may alternatively view h as a left A-linear morphism $M \to {}^{\alpha}_{A}[M, A]$. If M is σ-closed and if $h: M \times M \to Q_\sigma(A)$ is a σ-sesquilinear form, then we call (M, h) a σ-sesquilinear left A-module.

(4.4) Let us define a morphism of (σ-)sesquilinear modules $(M, h) \to (N, k)$ to be a left A-linear map $f: M \to N$ with the property that $(f \times f)k = h$. This is clearly equivalent to requiring the diagram

$$\begin{array}{ccc} M & \xrightarrow{h^a} & {}^{\alpha}_{A}[M, B] \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle {}^{\alpha}_{A}[f, B]} \\ N & \xrightarrow[k^a]{} & {}^{\alpha}_{A}[N, B] \end{array}$$

(where $B = A$ or $Q_\sigma(A)$) to be commutative. For example, let $f: M \to N$ be a left A-linear morphism and for any map $h: N \times N \to P$, define $f^{-1}(h): M \times M \to P$ by putting $f^{-1}(h)(m, m') = h(f(m), f(m'))$, for any m, $m' \in M$. It is clear that if P is an A-bimodule and if h is a sesquilinear morphism, then so is $f^{-1}(h)$. In particular, for any (σ-) sesquilinear module (N, h), this yields a (σ-) sesquilinear module $(M, f^{-1}(h))$.

Since morphisms of (σ-) sesquilinear modules may be composed in the obvious way, this allows us to introduce the category $\mathrm{S}(A, \alpha)$ resp. $\mathrm{S}(A, \alpha, \sigma)$ of sesquilinear resp. σ-sesquilinear left A-modules.

We need the following result:

(4.5) LEMMA. *Let $f\colon M \to N$ be a σ-isomorphism in A-*mod. *Assume that P is a (left) σ-closed A-bimodule, with the property that ${}^{\alpha}_{A}[N, P]$ is σ-closed. Then f^{-1} defines a bijective correspondence between sesquilinear morphisms $N \times N \to P$ and sesquilinear morphisms $M \times M \to P$.*

Proof. We have already pointed out that $f^{-1}(h)\colon M \times M \to P$ is a sesquilinear morphism if $h\colon N \times N \to P$ is. Conversely, assume that $k\colon M \times M \to P$ is a sesquilinear morphism, then the adjoint $k^a\colon M \to {}^{\alpha}_{A}[M, P]$ yields a commutative diagram of left A-modules

$$\begin{array}{ccc} M & \xrightarrow{\;k^a\;} & {}^{\alpha}_{A}[M, P] \\ {\scriptstyle f}\big\downarrow & & \big\uparrow{\scriptstyle {}^{\alpha}_{A}[f,P]} \\ N & \xrightarrow[\;t\;]{} & {}^{\alpha}_{A}[N, P] \end{array}$$

(where t is the obvious map), since our assumptions also imply $f' = {}^{\alpha}_{A}[f, P]$ to be an isomorphism. But then t uniquely defines a sesquilinear morphism $h\colon N \times N \to P$, with $h^a = t$. However, for any m, $m' \in M$, we have

$$h(f(m), f(m')) = t(f(m'))(f(m)) = f'(t)(f(m'))(m) = k^a(m')(m) = k(m, m'),$$

i.e., $k = f^{-1}(h)$, indeed. This proves the assertion.

(4.6) Let us now consider torsion triples (A, α, σ) and (B, β, τ) and a relative hermitian Morita equivalence $(P, Q, \theta\colon P \to Q,\ P \perp {}_{\tau}Q \to Q_\tau(B),\ \nu\colon Q \perp {}_{B}P \to Q_\sigma(A))$ between them. We may define an A-compatible sesquilinear form $h\colon P \times P \to Q_\tau(B)$ (with respect to β), by putting $h(p, p') = \mu(p \perp \theta(p'))$, for any $p, p' \in P$.

Note that it is then easy to see that the adjoint $h^a\colon P\,{}^{\beta}_{B}[P, Q_\tau(B)]^\alpha$ of h is just the isomorphism $\tilde{\theta}$ in B-mod-A. Indeed, since $Q = {}_{B}[P, Q_\tau(B)]$ as A–B-bimodules, the isomorphism being given by sending $q \in Q$ to the left B-linear map $\hat{q}\colon P \to Q_\tau(B)$, with $\hat{q}(p) = \mu(p \perp q)$ for any $p \in P$, cf. [17]. Hence ${}^{\beta}_{B}[P, Q_\tau(B)]^\alpha$ may be identified with ${}^{\beta}Q^{\alpha}$ and so h^a coincides with $\tilde{\theta}$, indeed.

With these notations, we may now prove the following relative counterpart of the main result in [9]:

(4.7) THEOREM (Morita I, cf. [2]). *Let* (A, α, σ) *and* (B, β, τ) *be torsion triples. Then any relative hermitian Morita equivalence*

$$(P, Q, \theta\colon P \to Q, \mu\colon P \perp {}_\tau Q \to Q_\tau(B), \nu\colon Q \perp {}_\sigma P \to Q_\sigma(A)),$$

defines an equivalence of categories $\mathbf{S}(A, \alpha, \sigma) \approx \mathbf{S}(B, \beta, \tau)$.

Proof. Let (M, h) be a σ-sequilinear left A-module, then we let $F_P M = P \perp {}_\tau M$. On the other hand, define a τ-sequilinear form

$$F^1_P h\colon P \otimes_A M \times P \otimes_A M \to Q_\tau(B)$$

by $(F^1_P h)(p \otimes m,\ p' \otimes m') = \mu(ph(m, m') \perp \theta(p'))$. By (4.5), applied to $Q_\tau(B)$ and the τ-isomorphism $P \otimes_A M \to P \perp_\tau M$, this yields a τ-sesquilinear form $F_P h\colon (P \perp {}_\tau M) \times (P \perp {}_\tau M) \to Q_\tau(B)$. Of course, essentially, $F_P h$ may also be defined by $F_P h(p \perp m,\ p' \perp m') = \mu(ph(m, m') \perp \theta(p'))$. It is now clear that the previous construction produces a functor

$$F_P\colon \mathbf{S}(A, \alpha, \sigma) \to \mathbf{S}(B, \beta, \tau)\colon (M, h) \to (F_P M, F_P h).$$

We claim that F_P yields an equivalence of categories, with "inverse" $F_Q\colon \mathbf{S}(B, \beta, \tau) \to \mathbf{S}(A, \alpha, \sigma)$, defined similarly. Indeed, first note that for any $(M, h) \in \mathbf{S}(A, \alpha, \sigma)$, we have $F_Q F_P M = Q \perp {}_\sigma(P \perp {}_\tau M) = M$, up to canonical isomorphism. On the other hand,

$$\begin{aligned}
&(F_Q F_P h)(q \perp (p \perp m),\ q' \perp (p' \perp m'))\\
&\quad = \nu(q(F_P h)(p \perp m,\ p' \perp m') \perp \theta^{-1}(q'))\\
&\quad = \nu(q\mu(ph(m, m') \perp \theta(p')) \perp \theta^{-1}(q'))\\
&\quad = \nu(\nu(q \perp ph(m, m'))\, \theta(p') \perp \theta^{-1}(q'))\\
&\quad = \nu(q \perp ph(m, m'))\, \nu(\theta(p') \perp \theta^{-1}(q'))\\
&\quad = \nu(q \perp p)\, h(m, m')\, \hat{\alpha}(\nu(q' \perp p')).
\end{aligned}$$

Let $x \in Q \perp {}_\sigma P$ be defined by $\nu(x) = 1 \in Q_\sigma(A)$, then we may choose $I \in \mathbf{L}(\sigma)$ with $Ix \subseteq \overline{Q \otimes_B P} = Q \otimes_B P/\sigma(Q \otimes_B P)$. Pick $j \in I$, then $jx = \sum_i q_i \perp p_i$, so $j = \nu(jx) = \nu(\sum_i q_i \perp p_i)$. Hence, identifying M and $Q \perp {}_\sigma(P \perp {}_\tau M)$ through $q \perp (p \perp m) \to \nu(q \perp p)m$, we have $jm = \sum_i q_i \perp (p_i \perp m)$, so the foregoing calculation shows that

$$\begin{aligned}
j(F_Q F_P h)(m,\ q' \perp (p' \perp m')) &= (F_Q F_P h)\left(\sum_i q_i \perp (p_i \perp m),\ q' \perp (p' \perp m')\right)\\
&= \sum_i \nu(q_i \perp p_i)\, h(m, m')\, \hat{\alpha}(\nu(q' \perp p'))\\
&= jh(m, m')\, \hat{\alpha}(\nu(q' \perp p')).
\end{aligned}$$

So, $(F_Q F_P h)(m, q' \perp (p' \perp m')) = h(m, m')\,\hat{\alpha}(v(q' \perp p'))$, as $Q_\sigma(A)$ is σ-torsionfree. Similarly,

$$\begin{aligned}(F_Q F_P h)(m, m')\,\alpha(j) &= (F_Q F_P h)(m, jm') \\ &= (F_Q F_P h)\left(m, \sum_i q_i \perp (p_i \perp m')\right) \\ &= \sum_i h(m, m')\,\hat{\alpha}(v(q_i \perp p_i)) = h(m, m')\,\alpha(j).\end{aligned}$$

Hence, $(F_Q F_P h)(m, m') = h(m, m')$, since $Q_\sigma(A)$ is $\alpha(\sigma)$-torsionfree and $\alpha(I) \in \mathbf{L}(\alpha(\sigma))$. This shows that F_P and F_Q are inverse to each other, which finishes the proof.

(4.8) Let us now assume, for simplicity's sake, that A is σ-closed, (hence A also is $\alpha(\sigma)$-closed). Assume that $\lambda \in R$ satisfies $\lambda^2 = 1$. We call a (σ-) sesquilinear left A-module *λ-hermitian* if for any $m, m' \in M$, we have

$$h(m, m') = \lambda\alpha(h(m', m)).$$

(If A is not necessarily σ-closed, assume the existence of an involutive relatively invertible bimodule, so that $Q_\sigma(A) = Q_{\alpha(\sigma)}(A)$, and use $\hat{\alpha}$, instead of α, in the definition). Denote by $\mathbf{H}_\lambda(A, \alpha, \sigma)$ the full subcategory of $\mathbf{S}(A, \alpha, \sigma)$, consisting of λ-hermitian objects.
(Still assuming A to be σ-closed and, similarly, B to be τ-closed), we then have:

(4.9) COROLLARY. *Let (A, α, σ) and (B, β, τ) be torsion triples. Then any relative hermitian Morita equivalence*

$$(P, Q, \theta\colon P \to Q, \mu\colon P \perp_\tau Q \to B, v\colon Q \perp_\sigma P \to A),$$

defines an equivalence of categories $\mathbf{H}_\lambda(A, \alpha, \sigma) \approx \mathbf{H}_\lambda(B, \beta, \tau)$.

Proof. It suffices to verify that the equivalence $\mathbf{S}(A, \alpha, \sigma) \approx \mathbf{S}(B, \beta, \tau)$ in (4.7) restricts to an equivalence $\mathbf{H}_\lambda(A, \alpha, \sigma) \approx \mathbf{H}_\lambda(B, \beta, \tau)$. We leave this straightforward verification to the reader.

5. INVOLUTIVE MORITA EQUIVALENCES

(5.1) In this section, for simplicity's sake (and for notational convenience), we assume throughout A to be σ-closed and B to be τ-closed. In particular, this implies that A resp. B is $\alpha(\sigma)$-closed resp. $\beta(\tau)$-closed, as well. (The reader may easily verify that the results below remain valid, if we just assume the existence of a suitable involutive relatively bimodule, and work with $Q_\sigma(A) = Q_{\alpha(\sigma)}(A)$, etc.)

Let P be an involutive (τ, σ)-invertible B–A-bimodule, i.e., assume we are given an isomorphism $\theta: P \to {}^{\beta}_{B}[P, B]^{\alpha}$ of B–A-bimodules. Let $Q = {}_{B}[P, Q_\tau(B)]$, then the pairing $P \times Q \to B: (p, f) \to f(p)$, defines an isomorphism of B-bimodules $\mu: P \perp {}_{\tau}Q \to B$. It follows from results in [17], that we may then also find and (essentially unique) isomorphism of A-bimodules $\nu: Q \perp {}_{\sigma}P \to A$, such that the relations (3.11.2) are satisfied. We call the resulting tuple

$$(P, Q, \theta: P \to Q, \mu: P \perp {}_{\tau}Q \to B, \nu: Q \perp {}_{\sigma}P \to A),$$

a *relative involutive Morita equivalence*. Of course, structurally, there is no difference with a relative hermitian Morita context. However, although it is clear that we may still define a τ-sesquilinear form

$$h: P \times P \to B: (p, p') \to \mu(p \perp \theta(p')) = \theta(p')(p),$$

in general, the identities (3.11.3) are not necessarily valid anymore. Indeed, these identities are equivalent to h being 1-*hermitian*, i.e., to $h(p, p') = \alpha(h(p', p))$, for any $p, p' \in P$, cf. (4.8). So, as a consequence, the conclusions of (4.7) and (4.9) no longer hold for arbitrary involutive Morita contexts.

In this section, we show, however, how (4.7) may be generalized to relative involutive Morita equivalences, modulo some natural finiteness conditions on the sesquilinear modules we consider.

(5.2) Let us keep notations as before. Recall, from [17], for example, that a left A-module M is said to be σ-finitely generated resp. presented, if there exists a σ-isomorphism $u: N \to M$, with N finitely generated resp. finitely presented in A-mod. For example, if A is σ-noetherian, i.e., if $A = Q_\sigma(A)$ is a noetherian object in (A, σ)-mod, then any σ-finitely generated left A-module is also σ-finitely presented. Denote by $(A, \sigma)\text{-mod}^{\mathrm{f}}$ the full subcategory of (A, σ)-mod, consisting of σ-finitely presented left A-modules. Similarly, denote by $\mathrm{S}^{\mathrm{f}}(A, \alpha, \sigma)$ the subcategory of $\mathrm{S}(A, \alpha, \sigma)$, consisting of σ-sesquilinear left A-modules (M, h), with $M \in (A, \sigma)\text{-mod}^{\mathrm{f}}$. A functor $F: (A, \sigma)\text{-mod}^{\mathrm{f}} \to (B, \tau)$-mod is said to be *duality preserving*, if for any $M \in (A, \sigma)\text{-mod}^{\mathrm{f}}$, there is a functorial isomorphism of B-bimodules $\eta_M: F({}^{\alpha}_{A}[M, A]) \to {}^{\beta}_{B}[F(M), B]$, i.e., if there exists a functorial isomorphism

$$\eta: F({}^{\alpha}_{A}[-, A]) \to {}^{\beta}_{B}[F(-), B]$$

on $(A, \sigma)\text{-mod}^{\mathrm{f}}$. Note that it does not make sense to impose this condition on functors $F: (A, \sigma)\text{-mod}^{\mathrm{f}} \to (B, \tau)\text{-mod}^{\mathrm{f}}$, since, unless A is σ-noetherian, e.g., ${}^{\alpha}_{A}[-, A]$ does not necessarily transform σ-finitely presented left A-modules to A-modules of the same type. Note also, that when M is an

A–C-bimodule, for some R-algebra with involution (C, γ), then η_M may be viewed (by naturality) as an isomorphism $F({}^{\alpha}_{A}[M, A]^{\gamma}) \cong {}^{\beta}_{B}[F(M), B]^{\gamma}$ of B–C-bimodules.

Finally, a functor $F: (A, \sigma)\text{-mod} \to (B, \tau)\text{-mod}$ is said to be duality preserving, if this property is satisfied by the restriction $F \mid (A, \sigma)\text{-mod}^{\mathrm{f}}$.

(5.3) PROPOSITION. *Let (A, α, σ) and (B, β, τ) be torsion triples. Then any duality preserving functor $F: (A, \sigma)$-mod $\to (B, \tau)$-mod canonically induces a functor $F^{\mathrm{s}}: \mathbf{S}^{\mathrm{f}}(A, \alpha, \sigma) \to \mathbf{S}^{\mathrm{f}}(B, \beta, \tau)$.*

Proof. If $(M, h) \in \mathbf{S}^{\mathrm{f}}(A, \alpha, \sigma)$, then the σ-sesquilinear form h induces a left A-linear morphism

$$h^{\mathrm{a}}: M \to {}^{\alpha}_{A}[M, A].$$

Applying F and composing with η_M yields a left B-linear morphism

$$\eta_M F(h^{\mathrm{a}}): F(M) \to F({}^{\alpha}_{A}[M, A]) \to {}^{\beta}_{B}[F(M), B].$$

Since $\eta_M F(h^{\mathrm{a}})$ may be viewed as the adjoint $F(h)^{\mathrm{a}}$ of a τ-sesquilinear form $F(h): F(M) \times F(M) \to B$, it is easy to check that this defines a functor

$$F^{\mathrm{s}}: \mathbf{S}^{\mathrm{f}}(A, \alpha, \sigma) \to \mathbf{S}^{\mathrm{f}}(B, \beta, \tau),$$

mapping $(M, h) \in \mathbf{S}^{\mathrm{f}}(A, \alpha, \sigma)$ to $(F(M), F(h)) \in \mathbf{S}^{\mathrm{f}}(B, \beta, \tau)$, indeed.

Let us call an equivalence $(A, \sigma)\text{-mod} \approx (B, \tau)\text{-mod}$ *duality preserving*, if it is defined by a duality preserving functor $F: (A, \sigma)\text{-mod} \to (B, \tau)\text{-mod}$. It follows that the "inverse" $G: (B, \tau)\text{-mod} \to (A, \sigma)\text{-mod}$ is then duality preserving too.

(5.4) COROLLARY. *Let (A, α, σ) and (B, β, τ) be torsion triples. Then any duality preserving equivalence $F: (A, \sigma)$-mod $\approx (B, \tau)$-mod induces an equivalence $F^{\mathrm{s}}: \mathbf{S}^{\mathrm{f}}(A, \alpha, \sigma) \approx \mathbf{S}^{\mathrm{f}}(B, \beta, \tau)$.*

The link between duality preserving equivalences and involutive relatively invertible bimodules is given by the following results, which, combined with (5.4), may be viewed as an involutive version of Morita I and II, cf. [4]:

(5.5) THEOREM. *Let (A, α, σ) and (B, β, τ) be torsion triples. Any involutive equivalence of categories $F: (A, \sigma)$-mod $\to (B, \tau)$-mod is duality preserving.*

Proof. Let $F: (A, \sigma)\text{-mod} \to (B, \tau)\text{-mod}$ be given by the involutive (τ, σ)-invertible B–A-bimodule P, i.e., assume $F = (P \perp_{\tau} -)$. Let

$$(P, Q, \theta: P \to Q, \mu: P \perp_{\tau} Q \to B, \nu: Q \perp_{\sigma} P \to A)$$

be the associated involutive relative Morita equivalence and let $h: P \times P \to B$ be given as before, i.e., $h(p, p') = \mu(p \perp \theta(p'))$, for any p, $p' \in P$. We want to exhibit, for any $M \in (A, \sigma)\text{-mod}^{f}$, a functorial isomorphism

$$\eta_M: P \perp_{\tau}({}^{\alpha}_{A}[M, A]) \to {}^{\beta}_{B}[P \perp_{\tau} M, B].$$

Define

$$\eta^1_M: P \otimes_A ({}^{\alpha}_{A}[M, A]) \to {}^{\beta}_{B}[P \otimes_A M, B]$$

by mapping $p' \otimes f \in P \otimes_A ({}^{\alpha}_{A}[M, A])$ to $\eta^1_M(p' \otimes f) \in {}^{\beta}_{B}[P \otimes_A M, B]$, with

$$\eta^1_M(p' \otimes f): P \otimes_A M \to B: p \otimes m \to h(pf(m), p').$$

It is easy to see that any left A-linear morphism $u: M \to N$ yields a commutative diagram

$$\begin{array}{ccc} P \otimes {}^{\alpha}_{A}[N, A] & \xrightarrow{P \otimes {}^{\alpha}_{A}[u, A]} & P \otimes {}^{\alpha}_{A}[M, A] \\ \downarrow{\scriptstyle \eta^1_N} & & \downarrow{\scriptstyle \eta^1_M} \\ {}^{\beta}_{B}[P \otimes N, B] & \xrightarrow[{}^{\beta}_{B}[P \otimes u, B]]{} & {}^{\beta}_{B}[P \otimes M, B] \end{array}$$

Let $\eta_M: P \perp_{\tau}({}^{\alpha}_{A}[M, A]) \to {}^{\beta}_{B}[P \perp_{\tau} M, B]$ be associated to η^1_M, by localizing at τ, then this clearly defines a natural morphism

$$\eta: P \perp_{\tau}({}^{\alpha}_{A}[-, A]) \to {}^{\beta}_{B}[P \perp_{\tau} -, B].$$

Let us finish the proof by showing that η is actually *isomorphic* on (A, σ)-mod^{f}.

First, let us assume $M = A$, then $P \otimes_A ({}^{\alpha}_{A}[A, A])$ may be identified with P through

$$P \otimes_A ({}^{\alpha}_{A}[A, A]) \to P: p \otimes f \to p\alpha(f(1)).$$

and $P \otimes_A A = P$, so ${}^{\beta}_{B}[P \otimes_A A, B]$ may be identified with ${}^{\beta}_{B}[P, B]$. So, η^1_A reduces to

$$\eta^1_A: P \to {}^{\beta}_{B}[P, B]: p' \to (p \to \mu(p \perp \theta(p')) = \theta(p')(p)),$$

i.e., $\eta_A = \eta^1_A = \theta$. Hence, η_A is an isomorphism, indeed.

Next, if $M = A^n$, for some positive integer n, then

$$\eta^1_{A^n}: P \otimes_A ({}^{\alpha}_{A}[A^n, A]) \to {}^{\beta}_{B}[P \otimes_A A^n, B]$$

may be identified with

$$(\eta^1_A)^n \colon (P \otimes_A ({}^{\alpha}_{A}[A, A]))^n \to {}^{\beta}_{B}[P \otimes_A A, B]^n,$$

so $\eta_{A^n} = \eta^1_{A^n}$ is again isomorphic, by the foregoing.

If we assume the left A-module M to be finitely presented, then M fits into an exact sequence of the form

$$A^q \xrightarrow{\pi} A^p \to M \to 0,$$

for some positive integers p, q. Since our assumptions imply $P \perp_\tau —\colon (A, \sigma)\text{-mod} \to (B, \tau)\text{-mod}$ to be exact, and since ${}^{\alpha}_{A}[—, A]$ and ${}^{\beta}_{B}[—, B]$ are left exact, this yields an exact diagram

$$\begin{array}{ccccc}
P \perp {}^{\alpha}_{A}[A^p, A] & \xrightarrow{P \otimes {}^{\alpha}_{A}[\pi, A]} & P \perp {}^{\alpha}_{A}[A^q, A] & \to & P \perp {}^{\alpha}_{A}[M, A] \\
\eta_{A^p} \downarrow & & \downarrow \eta_{A^q} & & \\
{}^{\beta}_{B}[P \perp A^p, B] & \xrightarrow[{}^{\beta}_{B}[P \perp \pi, B]]{} & {}^{\beta}_{B}[P \perp A^q, B] & \to & {}^{\beta}_{B}[P \perp M, B].
\end{array}$$

But, since both vertical arrows are isomorphic, this commutative diagram yields that the canonical map

$$\begin{aligned}
\eta_M \colon P \perp {}^{\alpha}_{A}[M, A] &= \operatorname{Ker}(P \perp {}^{\alpha}_{A}[\pi, A]) \to \operatorname{Ker}({}^{\beta}_{B}[P \perp \pi, B]) \\
&= {}^{\beta}_{B}[P \perp M, B]
\end{aligned}$$

is an isomorphism, as well.

Finally, if M is σ-finitely presented, then there exists some σ-isomorphism $u\colon N \to M$, where N is a finitely presented left A-module. Since u induces isomorphisms ${}^{\alpha}_{A}[u, A]$ and ${}^{\beta}_{B}[P \perp {}_\tau u, B]$ in A-mod resp. B-mod, the previous case yields that η_M is again isomorphic. This finishes the proof.

Conversely:

(5.6) Proposition. *Let* (A, α, σ) *and* (B, β, τ) *be torsion triples. Any duality preserving equivalence of categories* $F\colon (A, \sigma)\text{-mod} \to (B, \tau)\text{-mod}$ *is involutive.*

Proof. Let $F = (P \perp_\tau —)$, for some (τ, σ)-invertible B–A-bimodule P. If F is duality preserving, then, applying the definition to the A–A-bimodule A, we obtain an isomorphism

$$\eta_A \colon P \perp {}_\tau({}_A[A, A])_\alpha \to {}^{\beta}_{B}[P \perp {}_\tau A, B]^\alpha.$$

But, as we have pointed out in the previous proof, $P \perp_{\tau} A$ and $P \perp_{\tau} ({}_{A}[A, A])_{\alpha}$ may both be identified with P, so η_{A} actually yields an isomorphism $P \rightarrow {}^{\beta}_{B}[P, B]^{\alpha}$. Hence P is involutive, indeed.

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