General restrictions on tail probabilities

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Abstract

When limited information on the distribution of a positive random variable $X$ (continuous or discrete) is known (e.g., mode, mean, variance), the tail probability $P(X \geq t)$ cannot be chosen independently. In this paper supremum and infimum for $P(X \geq t)$ will be calculated over the set of positive random variables with unique mode, mean and/or variance given.

Keywords: Tail probability; Incomplete information; Mode; Moments

1. Introduction

Because of the developments in the field of computer technology (i.e., reduction of cost, increase of speed, etc.) analytical work has lost a great deal of its interest. However, ignoring some general patterns can be very dangerous and can lead to problems unexplainable by computer logic.

A characteristic example of this phenomenon can be found in [4], where numerical best bounds are derived on a Riemann–Stieltjes integral with respect to a certain distribution function. The underlying random variable $X$ is not known, and only the unique mode, mean, variance, etc. are available. Goovaerts et al. [4] point out, without any explication, that the method fails in case the mode ($= m$) equals 1, the mean ($= \mu_1$) equals 10 and the tail probability $P(X \geq 100)$ equals 5%. In the tables with numerical illustrations only an “overflow” message is mentioned.

The reason for this failure is rather fundamental. In fact, the tail probability of 5% simply is not compatible with $m = 1$ and $\mu_1 = 10$ for a positive random variable. One can prove that, in this situation, $P(X \geq 100) < 4.77\%$!

On the one hand, everybody knows that moments of an unknown distribution function cannot be chosen completely freely. Of course, e.g., $E(X^2)$ may not be smaller than $E(X)^2$. On the other hand,
however, also mode, moments in combination with tail probabilities cannot be chosen arbitrarily, and this is essentially what happens in [4].

In this paper, the interval which must contain a certain probability \( P(X \geq t) \) – where \( t \) is a given constant in the range of \( X \) – is determined, if the unique mode \( m \), the first and second moments \( \mu_1 \) and \( \mu_2 \) and their range \([0, b]\) are known. In other words, we are looking for the best analytical upper and lower bound for the probability \( P(X \geq t) \), given \( m, \mu_1, \mu_2 \) and \([0, b]\). The results remain valid for \( b \) going to infinity.

In the following paragraph, we briefly discuss the method used to derive the extreme values for \( P(X \geq t) \). Afterwards we give the results in the different cases of known mode and moments. For the detailed derivation of all results, we would like to refer to [2].

2. Method

2.1. Formulation of the problem

If one wants to determine the boundary values that restrict all possible outcomes for tail probabilities \( P(X \geq t) = E[1_{[t, +\infty)}(X)] \), the problem in fact is to find

\[
\sup_{\Phi} \int_0^b 1_{[t, +\infty)}(x) \, dF(x) \quad \text{and} \quad \inf_{\Phi} \int_0^b 1_{[t, +\infty)}(x) \, dF(x),
\]

where \( \Phi \) is the class of all distribution functions with range \([0, b]\) and with mode \( m \) and moments \( \mu_1 \) and \( \mu_2 \) if known.

2.2. Use of the knowledge of the mode

Lemma 1. If a unimodal random variable \( X \) has range \([0, b]\), mode \( m \) and moments \( \mu_1 \) and \( \mu_2 \), then there exists a random variable \( Y \) with same range \([0, b]\) and with moments \( v_1 = 2\mu_1 - m \) and \( v_2 = 3\mu_2 - 2m\mu_1 \), such that for any function \( g(x) \) the following equality holds:

\[
E[g(X)] = E[f(Y)],
\]

where

\[
f(x) = \frac{1}{x - m} \int_0^{x-m} g(\xi + m) \, d\xi \quad \text{("Khinchine transform")}.
\]

Proof. The proof of this result can be found in Khinchine's famous characterization of unimodality [3].

In the case of the tail probabilities, one has \( g(x) = 1_{[t, b]}(x) \), which implies a Khinchine transform

\[
f(x) = \begin{cases} 
\frac{x-t}{x-m} 1_{[t,b]}(x) & t > m, \\
\frac{t-m}{x-m} 1_{[0,t]}(x) + 1_{[t,b]}(x) & t \leq m,
\end{cases}
\]
and as a consequence the problem now is to find

\[
\sup_{F \in \Psi} \int_0^b f(x) \, dF(x) \quad \text{and} \quad \inf_{F \in \Psi} \int_0^b f(x) \, dF(x),
\]

(4)

where \( \Psi \) is the class of all distribution functions with range \([0, b]\) and moments \(v_1\) and \(v_2\) if known.

2.3. Basic reasoning

If \( F \in \Psi \) and if \( P(x) \) is a polynomial of degree 1 (resp. 2) or less, then the value of \( \int_0^b P(x) \, dF(x) \) only depends on the first moment of \( F \) (resp. the first and second moments of \( F \)), and so it is the same for each \( F \) and \( \Psi \) in case these moments are known.

We now look for such polynomials \( P(x) \) greater (resp. smaller) or equal to \( f(x) \) on \([0, b]\) and for some distribution \( G \) of \( \Psi \) for which

\[
\int_0^b P(x) \, dG(x) = \int_0^b f(x) \, dF(x).
\]

(5)

As distribution \( G \) we will use one, two or three point distributions of \( \Psi \), as polynomial \( P(x) \) we will use that polynomial that matches \( f(x) \) in the mass-points of \( G \), such that Eq. (5) holds.

2.4. Generation of few points distributions

Suppose that \( X \) is a random variable with range \([0, b]\) and moments \(v_1\) and \(v_2\). Define \( r' = (v_2 - v_1 r)/(v_1 - r) \) for \( r \in [0, b] \), \( r \neq v_1 \), then \( 0 < b' < v_1 < 0' < b \), as suggested in [5].

**Two point distribution**

If \( r \in [0, b'] \), then there exists a unique two point distribution in \( \{r, r'\} \) with masses

\[
q_r = \frac{v_1 - r'}{r - r'} \quad \text{and} \quad q_{r'} = \frac{v_1 - r}{r' - r}.
\]

(6)

**Three point distribution**

If \( r \in [b', 0'] \), then there exists a unique three point distribution in \( \{0, r, b\} \) with masses

\[
q_r = \frac{b v_1 - v_2}{r(b - r)} \quad q_b = \frac{v_2 - v_1 r}{b(b - r)} \quad \text{and} \quad q_0 = 1 - q_r - q_b.
\]

(7)

2.5. Essential conditions

To guarantee the existence of a distribution function on \([0, b]\) with given \( b, m, \mu_1 \) and \( \mu_2 \), this "known" parameters cannot be chosen completely arbitrarily. We have to take into consideration
the following essential conditions, which can be deduced from elementary conditions on distribution functions:

\[ \begin{align*}
    & b \mu_1 \geq \mu_2, \quad \mu_1^2 \leq \mu_2, \quad 2\mu_1 \geq m, \\
    & 4\mu_1^2 - 2\mu_1 m + m^2 \leq 3\mu_2, \quad 2b\mu_1 + 2\mu_1 m - mb \geq 3\mu_2.
\end{align*} \]

(8)

2.6. Calculation of the extreme values

After having determined the few point distribution for which Eq. (5) holds, we can deduce (after (sometimes) tedious calculations) the supremum or infimum as follows:

- one point distribution in \{r\}:
  \[ f(r) \cdot q_r; \]

- two point distribution in \{r, r'\}:
  \[ f(r) \cdot q_r + f(r') \cdot q_r'; \]

- three point distribution in \{0, r, b\}:
  \[ f(0) \cdot q_0 + f(r) \cdot q_r + f(b) \cdot q_b. \]

3. Resulting restrictions on the tail probabilities

3.1. Only the mode \( m \) known

The restrictions for this case are given in Table 1.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Upper bound</th>
<th>Lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t \leq m )</td>
<td>1</td>
<td>( (m - t)/m )</td>
</tr>
<tr>
<td>( t &gt; m )</td>
<td>( (b - t)/(b - m) )</td>
<td>0</td>
</tr>
</tbody>
</table>

3.2. Only the first moment \( \mu_1 \) known

The restrictions for this case are given in Table 2.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Upper bound</th>
<th>Lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t \leq \mu_1 )</td>
<td>1</td>
<td>( (\mu_1 - t)/(b - t) )</td>
</tr>
<tr>
<td>( t &gt; \mu_1 )</td>
<td>( \mu_1/t )</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 3

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Upper bound</th>
<th>Lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ≤ t ≤ b'</td>
<td>1</td>
<td>\frac{(\mu_1 - t)^2}{(\mu_1 - t)^2 + \mu_2 - \mu_1^2}</td>
</tr>
<tr>
<td>b' &lt; t ≤ 0'</td>
<td>\frac{(b + t)\mu_1 - \mu_2}{bt}</td>
<td>\frac{\mu_2 - \mu_1 t}{b(b - t)}</td>
</tr>
<tr>
<td>0' &lt; t ≤ b</td>
<td>\frac{\mu_2 - \mu_1^2}{\mu_2 - \mu_1^2 + (\mu_1 - t)^2}</td>
<td>0</td>
</tr>
</tbody>
</table>

3.3. The two moments \( \mu_1 \) and \( \mu_2 \) known

The restrictions for this case are given in Table 3, where

\[
\begin{align*}
    b' &= (\mu_2 - \mu_1 b)/(\mu_1 - b) \quad \text{and} \quad 0' = \mu_2/\mu_1.
\end{align*}
\] (12)

3.4. Mode \( m \) and mean \( \mu_1 \) known

(a) \( t \leq m \). Define

\[
c_1 \equiv t - \sqrt{(m - t)(b - t)}.
\] (13)

Table 4 gives the relevant restrictions.

Table 4

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Upper bound</th>
<th>Conditions</th>
<th>Lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t &lt; v_1 )</td>
<td>1</td>
<td>( c_1 &lt; 0 )</td>
<td>\frac{b(m - t) + v_1 t}{bm}</td>
</tr>
<tr>
<td>( v_1 \leq t )</td>
<td>\frac{m + v_1 - t}{m}</td>
<td>0 &lt; ( c_1 \leq v_1 )</td>
<td>\frac{(t - c_1)v_1 + (b - c_1)m - bt + c_1^2}{(b - c_1)(m - c_1)}</td>
</tr>
<tr>
<td>( v_1 &lt; c_1 \leq t )</td>
<td>\frac{m - t}{m - v_1}</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) \( t > m \). Define

\[
c_2 \equiv t + \sqrt{t(t - m)}.
\] (14)

Table 5 gives the relevant restrictions.
Table 5

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Upper bound</th>
<th>Conditions</th>
<th>Lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t \leq c_2 \leq v_1$</td>
<td>$\frac{v_1 - t}{v_1 - m}$</td>
<td>$t &lt; v_1$</td>
<td>$\frac{v_1 - t}{b - m}$</td>
</tr>
<tr>
<td>$v_1 &lt; c_2 \leq b$</td>
<td>$\frac{v_1 (c_2 - t)}{c_2 (c_2 - m)}$</td>
<td>$v_1 \leq t$</td>
<td>0</td>
</tr>
<tr>
<td>$b \leq c_2$</td>
<td>$\frac{v_1 (b - t)}{b(b - m)}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; t &lt; b'$</td>
<td>$1$</td>
</tr>
<tr>
<td>$b' &lt; t \leq 0'$</td>
<td>$\frac{b(m - t) + (b + t)v_1 - v_2}{bm}$</td>
</tr>
<tr>
<td>$0' &lt; t \leq b$</td>
<td>$\frac{1}{t - t'} \left( (v_1 - t') + \frac{(t - v_1)(m - t)}{m - t'} \right)$</td>
</tr>
</tbody>
</table>

3.5. Mode $m$ and moments $\mu_1$ and $\mu_2$ known

(a) $t \leq m$. Upper bounds for this case are given in Table 6, where

$$b' = (v_2 - v_1 b)/(v_1 - b), \ 0' = v_2/v_1 \quad \text{and} \quad t' = (v_2 - v_1 t)/(v_1 - t). \quad (15)$$

Lower bounds in case $\mu_1 > m$ are to be treated now. Consider $b' = (v_2 - v_1 b)/(v_1 - b)$ and $0' = v_2/v_1$. Define

$$s \equiv \frac{bt + mt - bm - \sqrt{mb(b - t)(m - t)}}{t}, \quad (16)$$

and calculate $r$ as the unique root in $[0, \min(b', t)]$ of $r^3 + Ar^2 + Br + C = 0$ with

$$A = -\frac{1}{2} (2v_1 + m + 3t), \quad (17)$$

$$B = 2tv_1 + tm, \quad (18)$$

$$C = \frac{1}{2} (v_2 m - v_2 t - 2v_1 tm). \quad (19)$$

The lower bounds obtained are given in Table 7.
Table 7

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t \leq \frac{0'm}{2m + 0'} )</td>
<td>( v_1^2 t + v_2(m - t) )</td>
</tr>
<tr>
<td>( \frac{0'm}{2m + 0'} \leq t \leq m ) and ( b' \geq m )</td>
<td>( \frac{m - t}{m - r} \frac{v_2 - v_1^2}{v_2 - 2v_1 r + r^2} + \frac{(v_1 - r)^2}{v_2 - 2v_1 r + r^2} )</td>
</tr>
<tr>
<td>or ( \frac{0'm}{2m + 0'} \leq t &lt; \frac{mb + mb' - 2b'^2}{2m - 3b' + b} ) and ( b' &lt; m )</td>
<td>( \frac{1}{b - b'} (v_1 - b') + \frac{(b - v_1)(m - t)}{m - b'} )</td>
</tr>
</tbody>
</table>

and \( b' < m \)

\( \frac{bm(m - 2b' + b)}{(m - b')^2 + b(m - 2b' + b)} \leq t < m \)

\( v_2 - v_1 (b + s) + bs \)

\( \frac{m - t}{m} \frac{v_2 - v_1 (b + s) + bs}{m - s} + \frac{v_2 - v_1}{s(b - s)} + \frac{b(b - s)}{s} \)

Table 8

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 \leq t \leq b' )</td>
<td>( \frac{(v_1 - t)^2}{(v_1 - m)(v_1 - t) + v_2 - v_1^2} )</td>
</tr>
<tr>
<td>( b' &lt; t \leq 0' )</td>
<td>( \frac{v_2 - v_1 t}{b(b - m)} )</td>
</tr>
<tr>
<td>( 0' &lt; t \leq b )</td>
<td>0</td>
</tr>
</tbody>
</table>

Lower bounds in case \( \mu_1 \leq m \) are to be dealt with now. Define

\[ m^G = b - m, \quad t^G = b - t, \quad \mu_1^G = b - \mu_1 \quad \text{and} \quad \mu_2^G = b^2 - 2b\mu_1 + \mu_2. \] (20)

Determine the upper bound for this new distribution (for which \( \mu_1^G > m^G \)). The lower bound for the old distribution can then be found by subtracting this result from 1.
(b) $t > m$. Lower bounds for this case are provided in Table 8, where
\[ b' = (v_2 - v_1 b)/(v_1 - b) \quad \text{and} \quad 0' = v_2/v_1. \]  

Upper bounds in case $\mu_1 > m$ are to be derived now. Consider $b' = (v_2 - v_1 b)/(v_1 - b)$ and $0' = v_2/v_1$. Define
\[ s \equiv \frac{t(b - m) + \sqrt{bt(b - m)(t - m)}}{b - t} \quad \text{and} \quad c_3 \equiv t + \sqrt{t(t - m)}. \]  

and calculate $r$ as the unique root in $[\max(0', t), b]$ of $r^2 + Ar^2 + Br + C = 0$ with
\[ A = -\frac{1}{2}(2v_1 + m + 3t), \]  
\[ B = 2tv_1 + tm, \]  
\[ C = \frac{1}{2}(v_2 m - v_2 t - 2v_1 tm). \]

Now upper bounds given in Table 9 can be obtained.

**Table 9**

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m &lt; t \leq \frac{bb^2}{(b'-m)^2 + b(2b'-m)}$ and $b' &gt; c_3$</td>
<td>$\frac{1}{b - b'} \left( \frac{(v_1 - b')(b - t)}{b - m} + \frac{(b - v_1)(b' - t)}{b' - m} \right)$</td>
</tr>
<tr>
<td>$m &lt; t \leq \frac{b0^2}{(0'-m)^2 + b(20'-m)}$ and $b' \leq c_3$</td>
<td>$\frac{1}{b - s} \left( \frac{bv_1 - v_2(s - t)}{s(s - m)} + \frac{(v_2 - v_1 s)(b - t)}{b(b - m)} \right)$</td>
</tr>
<tr>
<td>$bb^2 \frac{(b'-m)^2 + b(2b'-m)}{(b'-m)^2 + b(2b'-m)} &lt; t \leq \frac{b0^2}{(0'-m)^2 + b(20'-m)}$ and $b' &gt; c_3$</td>
<td>$\frac{v_1^2(v_2 - tv_1)}{v_2(v_2 - mv_1)}$</td>
</tr>
<tr>
<td>$\frac{b0^2}{(0'-m)^2 + b(20'-m)} &lt; t \leq \frac{20^2 - m0'}{30' - 2m}$</td>
<td>$\frac{v_2^2(v_2 - v_1)}{v_2^2(r - mr^2 - 2v_1 r + v_2)}$</td>
</tr>
<tr>
<td>$20^2 \frac{m0'}{30' - 2m} &lt; t \leq \frac{2b^2}{3b - b' - 2m}$</td>
<td>$\frac{r t v_2 v_1^2}{r - mr^2 - 2v_1 r + v_2}$</td>
</tr>
<tr>
<td>$2b^2 - bm - b'm \frac{2b^2 - bm}{3b - b' - 2m} &lt; t \leq b$</td>
<td>$\frac{b - t}{b - m} \frac{v_2 - v_1^2}{b^2 - 2v_1 b + v_2}$</td>
</tr>
</tbody>
</table>
Upper bounds in case $\mu_1 \leq m$ are to be dealt with. For this define

$$m^G \equiv b - m, \quad t^G \equiv b - t, \quad \mu_1^G \equiv b - \mu_1 \quad \text{and} \quad \mu_2^G \equiv b^2 - 2b\mu_1 + \mu_2. \quad (26)$$

Determine the lower bound for this new distribution (for which $\mu_1^G > m^G$). The upper bound for the old distribution can then be found by subtracting this result from 1.
4. Numerical examples

In this section, we calculate the extreme values for tail probabilities for two different portfolios; one is meant to show the different possibilities, while the other concerns the problem already mentioned in the Introduction.

The first portfolio has range $b = 50$, mode $m = 7$, and moments $\mu_1 = 10$ and $\mu_2 = 220$. Figs. 1–5 show the range that contains all values that can be obtained by the tail probability $P(X \geq t)$ as
a function of $t$, for the different cases of knowledge of the characteristics for the unknown distribution.

It is interesting to remark that the region in Fig. 5 cannot be found as a simple intersection of the regions in Figs. 2 and 3, because the various conditions are not mutually independent.

For the second example, we look at the portfolio considered in [4]. Here $b = 1000$, $m = 1$ and $\mu_1 = 10$. Figs. 6 and 7 deal with the situation of knowledge of all these three parameters.
Fig. 7. Restrictions on tail probabilities (second part).

References