General restrictions on tail probabilities

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Abstract

When limited information on the distribution of a positive random variable X (continuous or discrete) is known (e.g., mode, mean, variance), the tail probability $P(X \geq t)$ cannot be chosen independently. In this paper supremum and infimum for $P(X \geq t)$ will be calculated over the set of positive random variables with unique mode, mean and/or variance given.

Keywords: Tail probability; Incomplete information; Mode; Moments

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1. Introduction

Because of the developments in the field of computer technology (i.e., reduction of cost, increase of speed, etc.) analytical work has lost a great deal of its interest. However, ignoring some general patterns can be very dangerous and can lead to problems unexplainable by computer logic.

A characteristic example of this phenomenon can be found in [4], where numerical best bounds are derived on a Riemann–Stieltjes integral with respect to a certain distribution function. The underlying random variable X is not known, and only the unique mode, mean, variance, etc. are available. Goovaerts et al. [4] point out, without any explication, that the method fails in case the mode ($= m$) equals 1, the mean ($= \mu_1$) equals 10 and the tail probability $P(X \geq 100)$ equals 5%. In the tables with numerical illustrations only an “overflow” message is mentioned.

The reason for this failure is rather fundamental. In fact, the tail probability of 5% simply is not compatible with $m = 1$ and $\mu_1 = 10$ for a positive random variable. One can prove that, in this situation, $P(X \geq 100) < 4.77\%$!

On the one hand, everybody knows that moments of an unknown distribution function cannot be chosen completely freely. Of course, e.g., $E(X^2)$ may not be smaller than $E(X)^2$. On the other hand,

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however, also mode, moments in combination with tail probabilities cannot be chosen arbitrarily, and this is essentially what happens in [4].

In this paper, the interval which must contain a certain probability $P(X \geq t)$ – where t is a given constant in the range of X – is determined, if the unique mode m , the first and second moments μ_1 and μ_2 and their range $[0, b]$ are known. In other words, we are looking for the best analytical upper and lower bound for the probability $P(X \geq t)$, given m, μ_1, μ_2 and $[0, b]$. The results remain valid for b going to infinity.

In the following paragraph, we briefly discuss the method used to derive the extreme values for $P(X \geq t)$. Afterwards we give the results in the different cases of known mode and moments. For the detailed derivation of all results, we would like to refer to [2].

2. Method

2.1. Formulation of the problem

If one wants to determine the boundary values that restrict all possible outcomes for tail probabilities $P(X \geq t) = E[1_{[t, +\infty)}(X)]$, the problem in fact is to find

$$\sup_{F \in \Phi} \int_0^b 1_{[t, +\infty)}(x) dF(x) \quad \text{and} \quad \inf_{F \in \Phi} \int_0^b 1_{[t, +\infty)}(x) dF(x), \quad (1)$$

where Φ is the class of all distribution functions with range $[0, b]$ and with mode m and moments μ_1 and μ_2 if known.

2.2. Use of the knowledge of the mode

Lemma 1. *If a unimodal random variable X has range $[0, b]$, mode m and moments μ_1 and μ_2 , then there exists a random variable Y with same range $[0, b]$ and with moments $\nu_1 = 2\mu_1 - m$ and $\nu_2 = 3\mu_2 - 2m\mu_1$, such that for any function $g(x)$ the following equality holds:*

$$E[g(X)] = E[g(Y)], \quad (2)$$

where

$$f(x) = \frac{1}{x - m} \int_0^{x-m} g(\xi + m) d\xi \quad (\text{“Khinchine transform”}).$$

Proof. The proof of this result can be found in Khinchine’s famous characterization of unimodality [3]. $\square$

In the case of the tail probabilities, one has $g(x) = 1_{[t, b]}(x)$, which implies a Khinchine transform

$$f(x) = \begin{cases} \frac{x - t}{x - m} 1_{[t, b]}(x) & t > m, \\ \frac{t - m}{x - m} 1_{[0, t]}(x) + 1_{[t, b]}(x) & t \leq m, \end{cases} \quad (3)$$

and as a consequence the problem now is to find

$$\sup_{F \in \Psi} \int_0^b f(x) dF(x) \quad \text{and} \quad \inf_{F \in \Psi} \int_0^b f(x) dF(x), \quad (4)$$

where Ψ is the class of all distribution functions with range $[0, b]$ and moments v_1 and v_2 if known.

2.3. Basic reasoning

If $F \in \Psi$ and if $P(x)$ is a polynomial of degree 1 (resp. 2) or less, then the value of $\int_0^b P(x) dF(x)$ only depends on the first moment of F (resp. the first and second moments of F), and so it is the same for each F and Ψ in case these moments are known.

We now look for such polynomials $P(x)$ greater (resp. smaller) or equal to $f(x)$ on $[0, b]$ and for some distribution G of Ψ for which

$$\int_0^b P(x) dG(x) = \int_0^b f(x) dF(x). \quad (5)$$

As distribution G we will use one, two or three point distributions of Ψ , as polynomial $P(x)$ we will use that polynomial that matches $f(x)$ in the mass-points of G , such that Eq. (5) holds.

2.4. Generation of few points distributions

Suppose that X is a random variable with range $[0, b]$ and moments v_1 and v_2 . Define $r' = (v_2 - v_1 r)/(v_1 - r)$ for $r \in [0, b]$, $r \neq v_1$, then $0 < b' < v_1 < 0' < b$, as suggested in [5].

Two point distribution

If $r \in [0, b']$, then there exists a unique two point distribution in $\{r, r'\}$ with masses

$$q_r = \frac{v_1 - r'}{r - r'} \quad \text{and} \quad q_{r'} = \frac{v_1 - r}{r' - r}. \quad (6)$$

Three point distribution

If $r \in [b', 0']$, then there exists a unique three point distribution in $\{0, r, b\}$ with masses

$$q_r = \frac{bv_1 - v_2}{r(b - r)}, \quad q_b = \frac{v_2 - v_1 r}{b(b - r)} \quad \text{and} \quad q_0 = 1 - q_r - q_b. \quad (7)$$

2.5. Essential conditions

To guarantee the existence of a distribution function on $[0, b]$ with given b , m , μ_1 and μ_2 , this “known” parameters cannot be chosen completely arbitrarily. We have to take into consideration

the following essential conditions, which can be deduced from elementary conditions on distribution functions:

$$\begin{aligned} b\mu_1 &\geq \mu_2, \quad \mu_1^2 \leq \mu_2, \quad 2\mu_1 \geq m, \\ 4\mu_1^2 - 2\mu_1 m + m^2 &\leq 3\mu_2, \quad 2b\mu_1 + 2\mu_1 m - mb \geq 3\mu_2. \end{aligned} \quad (8)$$

2.6. Calculation of the extreme values

After having determined the few point distribution for which Eq. (5) holds, we can deduce (after (sometimes) tedious calculations) the supremum or infimum as follows:

- one point distribution in $\{r\}$:

$$f(r) \cdot q_r; \quad (9)$$

- two point distribution in $\{r, r'\}$:

$$f(r) \cdot q_r + f(r') \cdot q_{r'}; \quad (10)$$

- three point distribution in $\{0, r, b\}$:

$$f(0) \cdot q_0 + f(r) \cdot q_r + f(b) \cdot q_b. \quad (11)$$

3. Resulting restrictions on the tail probabilities

3.1. Only the mode m known

The restrictions for this case are given in Table 1.

Table 1

Conditions	Upper bound	Lower bound
$t \leq m$	1	$(m - t)/m$
$t > m$	$(b - t)/(b - m)$	0

3.2. Only the first moment μ_1 known

The restrictions for this case are given in Table 2.

Table 2

Conditions	Upper bound	Lower bound
$t \leq \mu_1$	1	$(\mu_1 - t)/(b - t)$
$t > \mu_1$	μ_1/t	0

Table 3

Conditions	Upper bound	Lower bound
$0 \leq t \leq b'$	1	$\frac{(\mu_1 - t)^2}{(\mu_1 - t)^2 + \mu_2 - \mu_1^2}$
$b' < t \leq 0'$	$\frac{(b + t)\mu_1 - \mu_2}{bt}$	$\frac{\mu_2 - \mu_1 t}{b(b - t)}$
$0' < t \leq b$	$\frac{\mu_2 - \mu_1^2}{\mu_2 - \mu_1^2 + (\mu_1 - t)^2}$	0

3.3. The two moments μ_1 and μ_2 known

The restrictions for this case are given in Table 3, where

$$b' = (\mu_2 - \mu_1 b)/(\mu_1 - b) \quad \text{and} \quad 0' = \mu_2/\mu_1. \quad (12)$$

3.4. Mode m and mean μ_1 known

(a) $t \leq m$. Define

$$c_1 \equiv t - \sqrt{(m - t)(b - t)}. \quad (13)$$

Table 4 gives the relevant restrictions.

Table 4

Conditions	Upper bound	Conditions	Lower bound
$t < v_1$	1	$c_1 \leq 0$	$\frac{b(m - t) + v_1 t}{bm}$
$v_1 \leq t$	$\frac{m + v_1 - t}{m}$	$0 < c_1 \leq v_1$	$\frac{(t - c_1)v_1 + (b - c_1)m - bt + c_1^2}{(b - c_1)(m - c_1)}$
		$v_1 < c_1 \leq t$	$\frac{m - t}{m - v_1}$

(b) $t > m$. Define

$$c_2 \equiv t + \sqrt{t(t - m)}. \quad (14)$$

Table 5 gives the relevant restrictions.

Table 5

Conditions	Upper bound	Conditions	Lower bound
$t \leq c_2 \leq v_1$	$\frac{v_1 - t}{v_1 - m}$	$t < v_1$	$\frac{v_1 - t}{b - m}$
$v_1 < c_2 \leq b$	$\frac{v_1(c_2 - t)}{c_2(c_2 - m)}$	$v_1 \leq t$	0
$b \leq c_2$	$\frac{v_1(b - t)}{b(b - m)}$		

Table 6

Conditions	Upper bound
$0 \leq t \leq b'$	1
$b' < t \leq 0'$	$\frac{b(m - t) + (b + t)v_1 - v_2}{bm}$
$0' < t \leq b$	$\frac{1}{t - t'} \left((v_1 - t') + \frac{(t - v_1)(m - t)}{m - t'} \right)$

3.5. Mode m and moments μ_1 and μ_2 known

(a) $t \leq m$. Upper bounds for this case are given in Table 6, where

$$b' = (v_2 - v_1 b)/(v_1 - b), \quad 0' = v_2/v_1 \quad \text{and} \quad t' = (v_2 - v_1 t)/(v_1 - t). \quad (15)$$

Lower bounds in case $\mu_1 > m$ are to be treated now. Consider $b' = (v_2 - v_1 b)/(v_1 - b)$ and $0' = v_2/v_1$. Define

$$s \equiv \frac{bt + mt - bm - \sqrt{mb(b - t)(m - t)}}{t}, \quad (16)$$

and calculate r as the unique root in $[0, \min(b', t)]$ of $r^3 + Ar^2 + Br + C = 0$ with

$$A = -\frac{1}{2}(2v_1 + m + 3t), \quad (17)$$

$$B = 2tv_1 + tm, \quad (18)$$

$$C = \frac{1}{2}(v_2 m - v_2 t - 2v_1 tm). \quad (19)$$

The lower bounds obtained are given in Table 7.

Table 7

Conditions	Lower bound
$t < \frac{0'm}{2m + 0'}$	$\frac{v_1^2 t + v_2(m - t)}{mv_2}$
$\frac{0'm}{2m + 0'} \leq t \leq m$ and $b' \geq m$	
or	$\frac{m - t}{m - r} \frac{v_2 - v_1^2}{v_2 - 2v_1 r + r^2} + \frac{(v_1 - r)^2}{v_2 - 2v_1 r + r^2}$
$\frac{0'm}{2m + 0'} \leq t < \frac{mb + mb' - 2b'^2}{2m - 3b' + b}$ and $b' < m$	
$\frac{mb + mb' - 2b'^2}{2m - 3b' + b} \leq t < \frac{bm(m - 2b' + b)}{(m - b')^2 + b(m - 2b' + b)}$	$\frac{1}{b - b'} \left((v_1 - b') + \frac{(b - v_1)(m - t)}{m - b'} \right)$
and $b' < m$	
$\frac{bm(m - 2b' + b)}{(m - b')^2 + b(m - 2b' + b)} \leq t \leq m$	$\frac{m - t}{m} \frac{v_2 - v_1(b + s) + bs}{bs}$
and $b' < m$	$+ \frac{m - t}{m - s} \frac{bv_1 - v_2}{s(b - s)} + \frac{v_2 - sv_1}{b(b - s)}$

Table 8

Conditions	Lower bound
$0 \leq t \leq b'$	$\frac{(v_1 - t)^2}{(v_1 - m)(v_1 - t) + v_2 - v_1^2}$
$b' < t \leq 0'$	$\frac{v_2 - v_1 t}{b(b - m)}$
$0' < t \leq b$	0

Lower bounds in case $\mu_1 \leq m$ are to be dealt with now. Define

$$m^G \equiv b - m, \quad t^G \equiv b - t, \quad \mu_1^G \equiv b - \mu_1 \quad \text{and} \quad \mu_2^G \equiv b^2 - 2b\mu_1 + \mu_2. \quad (20)$$

Determine the upper bound for this new distribution (for which $\mu_1^G > m^G$). The lower bound for the old distribution can then be found by subtracting this result from 1.

(b) $t > m$. Lower bounds for this case are provided in Table 8, where

$$b' = (v_2 - v_1 b)/(v_1 - b) \quad \text{and} \quad 0' = v_2/v_1. \quad (21)$$

Upper bounds in case $\mu_1 > m$ are to be derived now. Consider $b' = (v_2 - v_1 b)/(v_1 - b)$ and $0' = v_2/v_1$. Define

$$s \equiv \frac{t(b-m) + \sqrt{bt(b-m)(t-m)}}{b-t} \quad \text{and} \quad c_3 \equiv t + \sqrt{t(t-m)}, \quad (22)$$

and calculate r as the unique root in $[\max(0', t), b]$ of $r^3 + Ar^2 + Br + C = 0$ with

$$A = -\frac{1}{2}(2v_1 + m + 3t), \quad (23)$$

$$B = 2tv_1 + tm, \quad (24)$$

$$C = \frac{1}{2}(v_2 m - v_2 t - 2v_1 tm). \quad (25)$$

Now upper bounds given in Table 9 can be obtained.

Table 9

Conditions	Upper bound
$m < t \leq \frac{bb'^2}{(b'-m)^2 + b(2b'-m)}$ and $b' > c_3$	$\frac{1}{b-b'} \left(\frac{(v_1 - b')(b-t)}{b-m} + \frac{(b-v_1)(b'-t)}{b'-m} \right)$
$m < t \leq \frac{b0'^2}{(0'-m)^2 + b(20'-m)}$ and $b' \leq c_3$	$\frac{1}{b-s} \left(\frac{bv_1 - v_2(s-t)}{s(s-m)} + \frac{(v_2 - v_1 s)(b-t)}{b(b-m)} \right)$
or	
$\frac{bb'^2}{(b'-m)^2 + b(2b'-m)} < t \leq \frac{b0'^2}{(0'-m)^2 + b(20'-m)}$	
and $b' > c_3$	
$\frac{b0'^2}{(0'-m)^2 + b(20'-m)} < t \leq \frac{20'^2 - m0'}{30' - 2m}$	$\frac{v_1^2(v_2 - tv_1)}{v_2(v_2 - mv_1)}$
$\frac{20'^2 - m0'}{30' - 2m} < t \leq \frac{2b^2 - bm - b'm}{3b - b' - 2m}$	$\frac{r-t}{r-m} \frac{v_2 - v_1^2}{r^2 - 2v_1 r + v_2}$
$\frac{2b^2 - bm - b'm}{3b - b' - 2m} < t \leq b$	$\frac{b-t}{b-m} \frac{v_2 - v_1^2}{b^2 - 2v_1 b + v_2}$

Fig. 1. Restrictions on tail probabilities in case moment μ_1 is known. Fig. 2. Restrictions on tail probabilities in case mode m is known.

Upper bounds in case $\mu_1 \leq m$ are to be dealt with. For this define

$$m^G \equiv b - m, \quad t^G \equiv b - t, \quad \mu_1^G \equiv b - \mu_1 \quad \text{and} \quad \mu_2^G \equiv b^2 - 2b\mu_1 + \mu_2. \quad (26)$$

Determine the lower bound for this new distribution (for which $\mu_1^G > m^G$). The upper bound for the old distribution can then be found by subtracting this result from 1.

Fig. 3. Restrictions on tail probabilities in case moments μ_1 and μ_2 are known.

Fig. 4. Restrictions on tail probabilities in case mode m and moment μ_1 are known.

4. Numerical examples

In this section, we calculate the extreme values for tail probabilities for two different portfolios; one is meant to show the different possibilities, while the other concerns the problem already mentioned in the Introduction.

The first portfolio has range $b = 50$, mode $m = 7$, and moments $\mu_1 = 10$ and $\mu_2 = 220$. Figs. 1–5 show the range that contains all values that can be obtained by the tail probability $P(X \geq t)$ as

Fig. 5. Restrictions on tail probabilities in case mode m , moments μ_1 and μ_2 are known.

Fig. 6. Restrictions on tail probabilities (first part).

a function of t , for the different cases of knowledge of the characteristics for the unknown distribution.

It is interesting to remark that the region in Fig. 5 cannot be found as a simple intersection of the regions in Figs. 2 and 3, because the various conditions are not mutually independent.

For the second example, we look at the portfolio considered in [4]. Here $b = 1000$, $m = 1$ and $\mu_1 = 10$. Figs. 6 and 7 deal with the situation of knowledge of all these three parameters.

Fig. 7. Restrictions on tail probabilities (second part).

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