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OPTIMAL REPLACEMENT UNDER VARIABLE INTENSITY OF UTILIZATION AND TECHNOLOGICAL PROGRESS

by

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Abstract

This paper presents two modifications to the traditional model of optimal replacement of capital equipment. Starting from the concept of cost-minimization over an infinite planning-horizon, we will introduce a variable intensity of utilization in the traditional replacement model, to explain the concept of functional degradation, i.e. the reduction in intensity of utilization as equipment ages. Next, it will be demonstrated that technological progress can have a very substantial influence on the optimal time of replacement, which is neglected in the traditional model.

1 Introduction

The basic concept of an economic analysis of replacement investment goes back to the idea that it is not necessarily optimal to maintain a piece of equipment until it physically falls apart. Often, it is quite profitable to stop utilizing machinery long before this, so that the economic life can be much shorter than the physical life of equipment. The concept of replacement investments has been studied in the economic literature since the first half of this century, but it took economists a long time to come up with a more or less uniform criterium to evaluate the replacement decision. Pioneering research in this field has been done by Taylor[10], Hotelling[2], and Preinreich[9]. Although in these early studies, some very fundamental conclusions were found, it still took some time to let these ideas trickle down to practical applications in management situations. It was mainly Terborgh[11, p.176] who transformed the theory from a purely academic to a more practical level. His introduction of a number of assumptions and simplifications reduce the computational effort to determine the optimal time of replacement. However, modern computers and software allow us to stick to the more rigorous models. The argument of Terborgh that '*their* (i.e. Preinreichs e.a.) *speculations remain on a completely abstract and academic basis, and are of interest only to mathematical theorists*' is therefore -nowadays- somewhat overstated.

In modern replacement theory, the analysis is nearly always based on discounted cash-flows. The problem can be solved both in a profit-maximizing or a cost-minimizing framework. Within the concept of a cost-minimizing framework, we find both the minimization of total (discounted) costs (see e.g. Preinreich) or the minimization of equivalent yearly costs which will be used in the present analysis. Fortunately both methods are basically equivalent and can be translated into each other easily. The choice for one of these systems can be subscribed to subjective preference. For a brief overview of developments in replacement analysis, we can refer to Meyer [6] and Howe & McCabe [3].

In this work we will attempt to make some contributions and new interpretations in the theory of single-machine replacement. We will not extend the analysis to a multiple-machine environment, at least not in this study. The contribution to the theory of optimal replacement in this paper will be restricted to two complications of the replacement problem.

First, we will cast our view on the problem of the correct choice of a decision variable in the equipment replacement problem. Although in most theoretical analysis the optimal replacement moment is expressed in terms of age of the capital equipment, empirical applications of this theory often suffer from a certain ambivalence in this variable. The moment of replacement is by definition a time-variable, like age is, but in many cases this moment strongly depends on the output a certain piece of equipment has produced. A formal model will be constructed in which both time and outputrate determine the time of replacement. Furthermore, the model we use to demonstrate this will allow to explain another important observation in the practice of machine replacement, namely that machines are generally not suddenly replaced. On the contrary, in many cases the utilization of their services diminishes as their age grows; new equipment is mainly used for high-performance duties, whereas older equipment is often used for less demanding tasks. Or as Terborgh put it very eloquently: *'In the bloodless warfare of machines, life is taken, as a rule, by stages'*[11, p.17]. It will become clear that the problem of the correct variable choice and the principle of functional degradation can be formalized by one and the same model.

Second, once the problem of the choice of the correct decision variable is solved, we will focus on the effect of technological progress. Contrary to the practice of Terborgh, an exponential form of technological progress will be introduced. We will analyze the effect of technological progress on the optimal time of replacement. It will become clear that in the presence of ongoing technological progress, replacement will be postponed beyond the moment a better performing machine comes available. We will demonstrate that this is caused by the opportunity cost of lost future technological progress at the moment of replacement and reformulate the traditional replacement-problem in a way that demonstrates the value of future technological progress. It will be demonstrated that with ongoing technological progress, the optimal time of replacement can be influenced considerably. This concept is not entirely new -Massé [5, p. 66 - 68] and Perrin [8, p.62] for instance devoted some attention to it -, but it is often completely neglected in handbooks of managerial economics or engineering economics. Also, the misconception occurs that technological progress necessarily encourages the equipment to be replaced sooner than in a static situation (See e.g. Young [12, p.244], Constantinides [1, p.10] and Howe & McCabe [3, p.304]). It will be demonstrated with a simple example that this is not always the case.

Three different models with growing complexity will be developed to demonstrate the effect of technological progress. Their difference lies in the time-horizon taken into consideration in evaluating future technological progress. In the first model, we assume that progress ceases after the first replacement. Although this is in a sense an oversimplification, we base this assumption on the fact that decision-makers, like all other people, lack the ability of clairvoyance and therefore partially limit the time-horizon of their analysis. In the following models this myopia will be corrected. However, the important characteristics and conclusions of the former model remain virtually unchanged. In the process of developing these ideas, we also hope to present some refreshing interpretations of existing replacement models.

This paper is structured as follows. In a first paragraph, an introduction to standard replacement techniques is given. We explain the basic technique of minimizing equivalent costflows in the absence of technological progress. At the same time, we mention the basic terminology in this field. The second paragraph deals with the variable intensity problem. The third paragraph focuses on the effect of technological progress on the replacement decision.

We approach this problem from the standpoint of a decision-maker with very limited knowledge about the future and extend the problem to a case with an infinite horizon. Although the text may still be rather technical at some points, it has been attempted to move the purely mathematical derivations in appendix.

2 Terminology and basic model

The basic idea of replacement analysis is the distinction between the physical and economic life of capital equipment. In most cases, the operating costs to keep equipment running (i.e. maintenance and repair, energy, manpower, downtime, ...) gradually increase with time and will at a certain point in time be considered too high to continue operating the equipment. In a profit-maximizing firm for instance, one will never operate a piece of equipment which operating costs exceed the market-value of its production (See Preinreich[9] for a formal model). At the moment of replacement, the old equipment (called the defender) will be replaced by new equipment (the challenger). The challenger can be of the same type and technology of the defender (and thus differ only in age), but can also be quite different.

A standard textbook model of capital replacement determines the economic life by minimizing the equivalent costflow of a piece of equipment (or as mentioned earlier, the total discounted cost of an infinite stream of pieces of capital equipment). The implicit assumption of this elementary model is that each machine will be replaced at the end of its economic life by an identical piece of equipment. The objective is to minimize costs over an infinite planning horizon. The equivalent costflow is simply a transformation of total discounted costs over the entire economic life of the equipment into a continuous constant flow over the same span of time, with the same present value. In discrete time, we will refer to it as the equivalent annual cost. The equivalent costflow can be stated as:

$$k_e(T) = \frac{i}{1 - e^{-iT}} \int_0^T k(t).e^{-it}\, dt \qquad (1)$$

In which $k(t)$ is the total cost of the equipment at time t, consisting of operating costs and capital costs: $k(t) = m(t) + i.V(t) - v(t)$.

The operating cost $m(t)$ comprises costs of maintenance, energy costs, labor costs of operating staff, ... The capital cost consists of the opportunity-cost of the capital invested in the equipment $i.V(t)$, in which $V(t)$ represents the value of the equipment on the second-hand market, and the loss of value of the equipment at time t: $v(t) = dV(t)/dt = \dot{V}(t)$. Setting the first derivative of the objective function equal to zero and simplifying gives the following expression as the first order condition for the optimal replacement date T:

$$k(T) = k_e(T) \tag{2}$$

The second-order condition can be stated as:

$$k'(T) > k_e'(T) \tag{3}$$

Since the derivative of $k_e(t)$ is zero in the optimum, the second order condition implies that total costs must be rising in the optimum.

The intuition behind this basic problem can easily be demonstrated graphically. Even though it is possible that the cost of operating the equipment declines with age at the beginning of the economic life of the equipment, because of some kind of learning effect, we can assume that it starts increasing with age at a certain point in time. On the other hand, it also seems acceptable that the capital cost decreases with age ($dV/dt < 0$), and that also the rate of depreciation declines in absolute value with time: $dv/dt = d^2V/dt^2 > 0$. Furthermore, we can assume that the value of the equipment approaches a constant (the scrap value) as age goes to infinity. In this case, depreciation will tend to zero. Because of this, it is fair to postulate U-shaped total costs as a function of age. Under these circumstances, we can prove that the total cost curve passes through the minimum of the equivalent cost curve, as shown in figure 1.

The basic model can easily be generalized in a situation in which the challenging equipment differs from the defender. In that case replacement should take place whenever the cost of the defender exceeds the equivalent costflow of the challenger (as opposed to its own equivalent costflow in the basic model). Nonetheless there remain a number of apparent shortcomings. For a start, the replacement decision based solely on the age of the equipment seems an oversimplification. Output seems a worthy substitute for age to base a replacement decision on (See e.g. Nash [7] for an application in transport equipment).

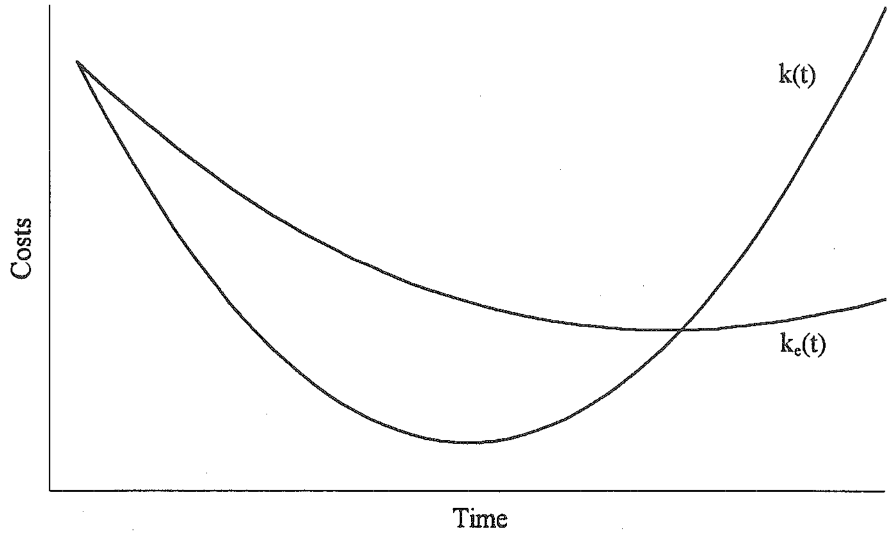

Figure 1: Basic model

Second, although the model can be used to distinguish the economic life of a defender with a non-identical challenger, it fails to predict the optimum if the challenger is continuously changing as a result of ongoing technological progress. Therefore we will extend the model to take this continuous evolution into account. Finally, the above model fails to explain why some pieces of equipment are replaced at a time where its operational costs are still falling. In the following paragraphs we will develop models which clarify these difficulties.

3 Model with variable intensity of utilization

Empirical research on the optimal time of replacement often struggles with the choice between age and output as the decision variable to determine the optimal replacement moment with. From intuition, it is obvious that both variables will play an important part in this problem. Too often, this problem is viewed as a matter of binary choice, as if only one variable at a time can be used in the replacement model. To resolve this problem, a new variable linking age and output will be introduced in the basic model.

Let $\psi\,(0 \leq \psi \leq 1)$ be the intensity of utilization of a certain piece of equipment. This variable is defined as the rate of output production as a percentage of the maximum outputrate. For simplicity, we will assume that the intensity of utilization is constant over the entire economic life. We will demonstrate that this parameter has a major influence on the economic life of equipment. Most types of equipment can be used with a variable intensity: locomotives, trucks, ... can perform variable mileage per day, or they can be used to haul heavy or light cargo, electrical power plants can vary the produced power, machines can produce at different speeds, etc. It is reasonable to assume that the intensity of utilization affects the cost of operating the equipment and its depreciation. Hence, the value of the equipment will be a function of the past intensity, or to put it differently, of the output produced. For the moment however, we will neglect the influence of intensity of utilization on the depreciation and the remaining value of the equipment. This is of course a simplification, but in cases where the potential buyers of second-hand equipment are uninformed about the past performance of the equipment, this simplification seems not to undermine the fundamentals of the problem.

In this case, total costs can be expressed as:

$$k(t,\psi) = m(t,\psi) + i.V(t) - v(t) \tag{4}$$

Minimizing the equivalent costflow using this cost function implies an optimality condition very similar to the one of the basic problem: $k(T,\psi) = k_e(T,\psi)$. Thus, the optimal time of replacement will be a function of the intensity of utilization: $T(\psi)$. The effect of ψ on the economic life can be clarified by taking the total differential of the optimality condition. Since $\partial\, k_e/\partial\, t = 0$ in the optimum, the effect of ψ on the economic life can be written as:

$$\frac{d\,T}{d\,\psi} = \frac{\frac{\partial\, k_e}{\partial\, \psi} - \frac{\partial\, k}{\partial\, \psi}}{\frac{\partial\, k}{\partial\, t}} \tag{5}$$

Since the second-order condition of the basic problem implied that total costs had to be rising in the optimum, the sign of the derivative is solely determined by the respective magnitude of $\partial\, k_e/\partial\, \psi$ and $\partial k/\partial\psi$. However, if we assume that $\frac{\partial^2 m}{\partial t \partial \psi} \geq 0$, then:

$$\frac{\partial k_e(t,\psi)}{\partial \psi} \tag{6}$$

$$= \frac{i}{1-e^{-iT}}\int_0^T \frac{\partial k(t,\psi)}{\partial \psi}.e^{-it}dt = \frac{i}{1-e^{-iT}}\int_0^T \frac{\partial m(t,\psi)}{\partial \psi}.e^{-it}dt$$

$$\leq \frac{i}{1-e^{-iT}}\int_0^T \frac{\partial m(T,\psi)}{\partial \psi}.e^{-it}dt = \frac{\partial m(T,\psi)}{\partial \psi} = \frac{\partial k(T,\psi)}{\partial \psi}$$

Hence, a decrease in the intensity of utilization will increase the economic life of the equipment,based on the assumption we made about the sign of the second cross-partial derivative of the operational cost function. Of course we have to wonder if this assumption is reasonable. The answer is yes. If we rewrite the derivative as follows:$\partial\left(\frac{\partial m}{\partial \psi}\right)/\partial t \geq 0$, we can see that our assumption implies that the influence of the intensity of utilization on operational costs has to increase in time. Provided that operational costs are proportional to the rate of intensity: $m(t,\psi) = f(\psi)M(t)$, where $M(t)$ is the operational cost at full intensity and $f(\psi)$ is a positive monotonic function, the second cross-partial derivative will be positive since $M(t)$ is a positive function of age. From the above, it is also obvious that a decrease in the intensity will reduce costs, thus: $\frac{\partial k}{\partial \psi} > 0$.The problem can be visualized as in figure 2. Suppose that in the initial situation, the intensity is 100%. In that case we return to the basic problem, where optimality implies equal total costs and equivalent costflow (point a). Reducing the intensity will increase optimal life and reduce costs, i.e. the minimum of the equivalent costflow will shift down and to the right as the intensity falls. We will call the path along the optimality condition when ψ is falling the cost-reduction path (CRP); it shows how optimal life and minimal costs evolve as intensity of utilization falls. Geometrically, the cost-reduction path can be considered as the intersection of the $k(t,\psi)$-surface and the $k_e(t,\psi)$-surface in a three-dimensional space. A three-dimensional representation can be found in figure 3. As explained for the basic model, the economic life can be found at the intersection of $k(t)$ and $k_e(t)$. If we consider the function $MIN\{k(t), k_e(t)\}$, optimal replacement age can be found as the point in time where this function is not smooth.

Now, consider the variable intensity case. The total cost and the equivalent costflow function depend in that case of two variables: the age of the

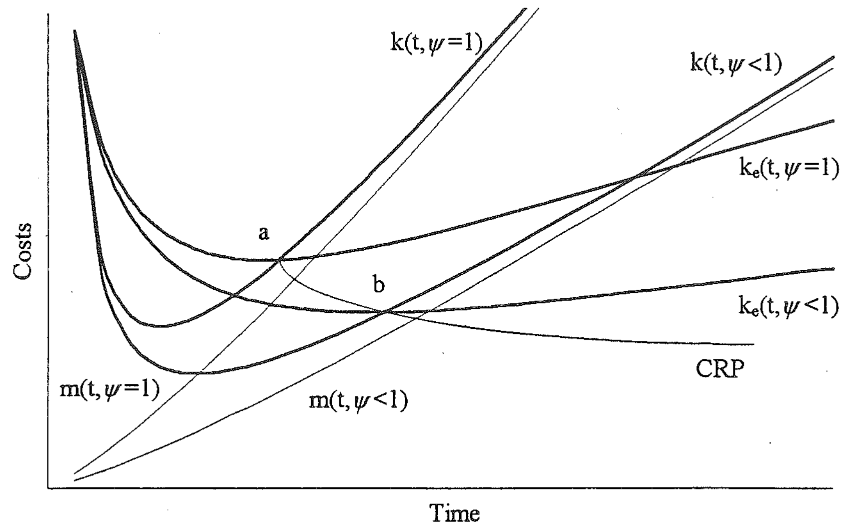

Figure 2: 2D-view of the variable intensity model

equipment and the intensity of utilization of the equipment. Thus: $k(t, \psi)$ and $k_e(t, \psi)$ can be represented by a surface in a three-dimensional space. The optimal time of replacement can again be found as the intersection of both surfaces or at the locus where the $MIN\{k(t,\psi), k_e(t,\psi)\}$-function is not smooth, represented by the line AB. On the left side of the line AB we can see the total cost as a function of age and intensity of utilization, on the right side the equivalent costflow is presented. The line AB is the locus of optimal replacement ages as a function of the intensity of utilization.

There are important advantages of this model-specification. First, it resolves the debate about the correct measure to determine the time of replacement. In the model, time is chosen as the main variable, but the solution to the problem is contingent upon the intensity of utilization. This new variable captures the influence that the rate of production exercises on the replacement moment. Second, the model can also be applied to illustrate the principle of functional degradation. This last point requires some further explanation.

Suppose a new machine were acquired that can be used at different de-

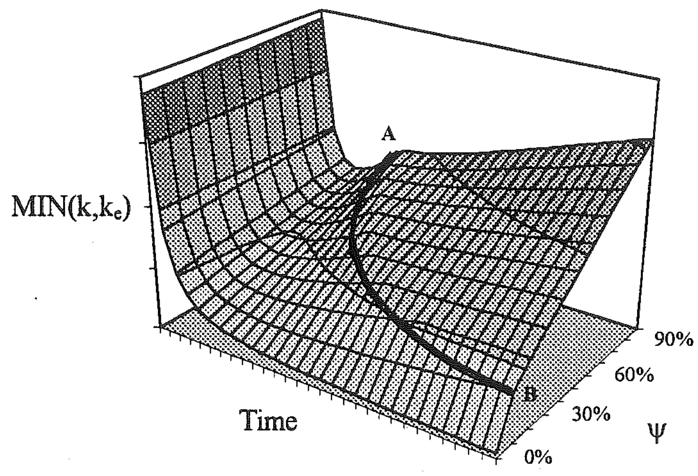

Figure 3: 3D-view of the variable intensity model

grees of intensity. If the new equipment is used at full intensity, its economic life will end at t_a (see figure 2). In the basic model, the equipment was necessarily abandoned at this age. However, in this model there is an alternative, since the equipment can be maintained for lighter duty. For instance, if the equipment is used with an intensity of $\psi < 1$, the economic life can be prolonged to t_b. This principle is often observed in real-life production applications and has been verbally described by Terborgh[11], as mentioned before. In formal models however, this principle has been neglected.

To further explore the idea of functional degradation, we should view the problem in a multiple-machine environment (See e.g. Malcomson[4]). However, the construction of a formal multiple-machine replacement model, incorporating the effect of the intensity of utilization on the economic life of equipment, is beyond the scope of this paper. For the moment, we will turn our attention to the concept of technological progress, to refine the model.

4 Model with technological progress

In the previous paragraph our attention was mainly focussed on the choice of a correct variable to determine the optimal time of replacement. We found the answer in the introduction of a new variable: the intensity of utilization. However, in order not to overload notation, in what follows we will omit the explicit reference to this variable.

We previously stated that the basic model of equipment replacement can easily be generalized to situations where the challenging equipment is not identical to the defending equipment. The decision rule becomes then: "*replace the defending equipment when total costs of the defending equipment exceed the minimal equivalent costflow of the challenging equipment*". This principle seems simple, but it is valid only under the very restrictive condition that nothing can be gained by postponing the replacement, as we will demonstrate. In a situation where there is ongoing technological progress, this is not the case since postponing the replacement decision will further reduce the cost of the challenging equipment.

We assume that the impact of technological progress consists of a reduction of all costs of newly installed equipment. Technological progress has no influence on capital and operating costs of equipment already installed; it only affects the costs of new equipment yet to come. The total cost of capital equipment is thus determined by its age (l) and the time it was first installed (t). For simplicity, we assume that the rate of technological progress (g) is constant in time. This implies:

$$k(l,t) = k(l,0).e^{-gt} \tag{7}$$

The basic model of replacement stated that existing equipment should be replaced whenever the equivalent costflow of the challenger became smaller than the operating cost of the defender. Since with technological progress the challenger will become less expensive, it could be argued that technological progress will lower the replacement-age. The decision-rule of the basic model is clear, but seems not entirely in accordance with the actual behavior of economic agents when faced with a replacement decision. We can not undo ourselves of the impression that quite often economic agents wait much longer to replace their equipment. Replacement often does not take place, although a challenger with lower equivalent costflow is available. In many cases where there is a strong ongoing technological progress, replacement is postponed because it is argued that the longer one waits, the more important the advantage of the challenger on the existing equipment. In view of the basic model of replacement, this kind of behavior is not rational, since it seems not to minimize costs.

The way in which this problem will be tackled is rather unconventional. We will start the analysis from the standpoint of a decision-maker with an infinite planning horizon, but with limited knowledge about technological progress in a more distant future, who makes simplified assumptions about future replacement decisions. We will denote this as the behavior of a partially myopic decision-maker. Although this line of reasoning is somewhat an oversimplification, it has the advantage of simplicity and graphical representation. Even more important, correction of this myopia does not fundamentally alter the conclusions of the more simple model.

4.1 Two-period model with technological progress

An important problem with replacement under technological progress is often that there is little or no information on the technological progress in the more distant future. The only information available is often the rate of ongoing technological progress, i.e. the rate at which the immediate challenger of existing equipment grows more efficient. For example, consider the market of computers. We can observe the rate at which the price of new computers is presently falling, but it is practically impossible to predict the price of computers in five years.

The decision-maker is thus confronted with a dilemma. Although he may want to use an infinite planning-horizon for replacement decisions, he lacks the correct information to base this decision on. Suppose however we try to find an approximation of the real situation. Assume therefore that the technological progress ceases to exist once the first machine has been replaced. After that, the new - more advanced and hence cheaper - equipment will be replaced at the end of its economic life by an infinite stream of identical machines.

In our new model, total costs of the challenger depend on both age and date of purchase. The later the equipment is purchased, the lower capital and operating costs. Once the equipment has been installed, total costs vary with age as described in the previous section. We use ΔT_j to denote the life of the j-th piece of equipment and T_j as the time the j-th replacement occurs. Thus: $T_1 = \Delta T_1$, $T_2 = T_1 + \Delta T_2$, ...

The economic life of the challenging technology can be found using the basic model, since then we fall back on a simple like-for-like replacement. Once the new technology has been installed, it will cause an infinite stream of costs. The objective is then to minimize the present value of the total cost of the equipment of present technology and the cost of the perpetuity induced by the new technology.

Hence, the minimization problem takes the following form:

$$\underset{\Delta T_1}{MIN} \quad K(\Delta T_1) = \int_0^{\Delta T_1} k(l,0).e^{-il}dl + e^{-i.\Delta T_1}\frac{k_e(\Delta T_2, T_1)}{i} \tag{8}$$

in which ΔT_2 satisfies $k(\Delta T_2, T_1) = k_e(\Delta T_2, T_1)$. Observe that the economic life of the challenger is independent of the time of the first replacement T_1, since in the optimality condition, the time of the first replacement can easily be canceled. Moreover, if the defending equipment were to be replaced with identical equipment, the replacement would also take place at the same age ΔT_2.

A minimum will be reached when $\partial K/\partial \Delta T_1 = 0$. Differentiating with respect to ΔT_1, and setting equal to zero gives:

$$k(\Delta T_1, 0) = k_e(\Delta T_2, T_1) + \frac{g}{i} k_e(\Delta T_2, T_1) \tag{9}$$

This expression can be interpreted as follows. If in the basic model - i.e. in the absence of a continuous technological progress - at a certain moment in time, the defending equipment were challenged by other equipment with an equivalent costflow that is equal or smaller than the total cost, it would be optimal to replace the defender by the challenger. However, if we introduce the concept of a continuous technological progress, which ceases when the existing equipment is replaced, the total cost $k(\Delta T_1, 0)$ of the defender at the time of replacement T_1 has to exceed the equivalent costflow of the challenger $k_e(\Delta T_2, T_1)$ at that time with the second term of equation 9, before it becomes optimal to replace. Hence, we can interpret this second term as the opportunity cost of lost future technological progress at the date of replacement, as it is perceived by a myopic decision-maker.

An other way to represent the same optimality condition is:

$$k(\Delta T_1, 0) = \varphi\, k_e(\Delta T_2, T_1) \tag{10}$$

with: $\varphi = \frac{i+g}{i}$. The total cost of the defender has to exceed the equivalent costflow of the challenger by a factor φ before replacement is due. It is important to see that this coefficient can easily attain relatively high values when the rate of technological progress is high relative to the interest rate.

Depending on the gradient of the cost curve, this could lead to substantial differences in economic life according to this model as opposed to the basic model.

A graphical representation can be found in figure 4. We start the analysis as in the basic problem. Consider the total cost of the defending equipment $k(l,0)$ and the corresponding equivalent costflow $k_e(l,0)$ as a function of age. The optimum replacement age for the basic problem is ΔT_2, the corresponding minimum equivalent costflow $k_e(\Delta T_2, 0)$. Observe that this is at the same time the optimal age of the challenging equipment, since the solutions of $k(l,T_1) = k_e(l,T_1)$ and $k(l,0) = k_e(l,0)$ coincide. Of course, although ΔT_2, does not depend on the time of the first replacement, the corresponding minimum cost does. Thus, the minimum costflow of the challenging equipment can be represented as a function of the time of replacement t of the defender: $k_e(\Delta T_2, t)$.

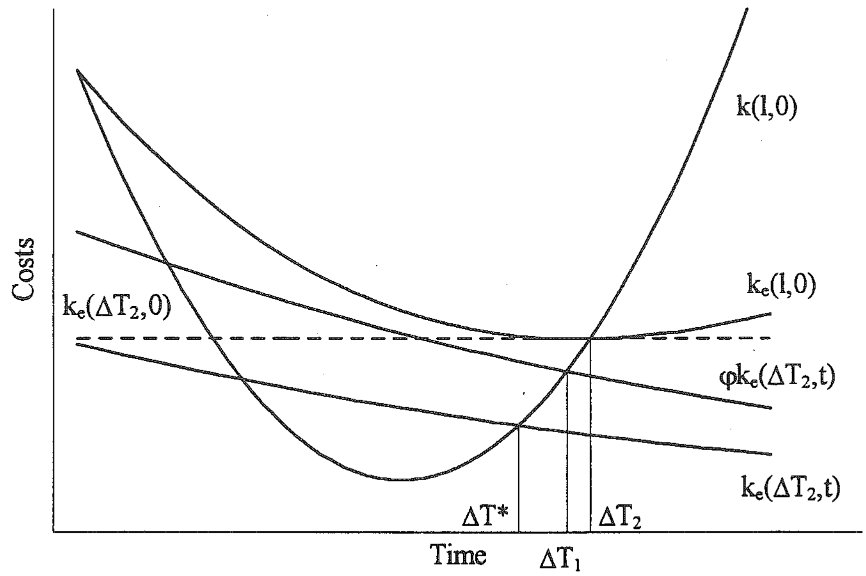

Figure 4: 2-period model with technological progress

Notice that in the absence of ongoing technological progress, the defender would be replaced at time ΔT^* , if a challenger with equivalent costflow $k_e(\Delta T_2, \Delta T^*)$ were to occur at that moment. The (perceived) value of future technological progress however, leads to postpone the replacement until ΔT_1, namely the time where the total cost curve $k(l,0)$ and $\varphi k_e(\Delta T_2,t)$ intersect.

Notice also that in this model, technological progress has an ambivalent influence on the cost of replacement. The progress reduces the cost of potential challengers, making it more attractive to replace. However, simultaneously it increases the loss of future savings due to forgone progress after the replacement date. From this point of view, we can argue that in some cases, technological progress may even increase the economic life beyond ΔT_2. We will demonstrate this in the following numerical example.

Example 1 *Suppose that the equipment depreciates at a constant rate (digressive depreciation):* $\frac{dV/V}{dt} = -\delta$. *In that case we can represent the value of the equipment at time t as:* $V(t) = V_0.e^{-\delta t}$, *in which* V_0 *is the value at time* 0. *A similar hypothesis will be made regarding the operating costs. Suppose that the operating costs grow at a constant rate:* $\frac{dm/m}{dt} = \lambda$. *In that case* $m(t) = m_0.e^{\lambda t}$ *(*m_0 *being the operating cost of new equipment), and* $k(t) = (i+\delta)V_0.e^{-\delta t} + m_0.e^{\lambda t}$. *The discrete time version of this equation is:* $k_n = rV_n - \Delta V_n + m_n = V_0(r-1)(1-d)^n + V_0(1-d)^{n-1} + m_0(1+l)^n$ *where* $r = e^i - 1$, $d = e^\delta - 1$, $l = e^\lambda - 1$. *To demonstrate the principle of prolonged life as a consequence of technological progress, we can insert some values in this equation and analyze the corresponding replacement decision. Let* $V_0 = 4000$, $m_0 = 100$. *Suppose also that the value of the equipment drops* 30% *a year and the operating cost increases* 10% *a year. Finally, let the opportunity cost of capital be* 2% *and assume that technology grows at a rate of* $g = 5\%$ *a year. In that case, the replacement problem can be represented as in figure 5. In the absence of technological progress, this piece of equipment would be replaced at an age of approximately 17 years. However, a myopic decision maker would keep this equipment in operation as long as over 20 years, given the present parameters. Hence, the perceived economic life of the equipment is prolonged as a consequence of technological progress.*

Another interesting aspect of this model is its second order condition of the optimal replacement date. This can be formulated as:

$$\frac{\partial k(\Delta T_1, 0)}{\partial \Delta T_1} > \frac{\partial \varphi k_e(\Delta T_2, T_1)}{\partial \Delta T_1} \qquad (11)$$

This implies that the total cost curve of the defender intersects with $\varphi k_e(\Delta T_2, t)$ from below. Notice that it is no longer required that total costs

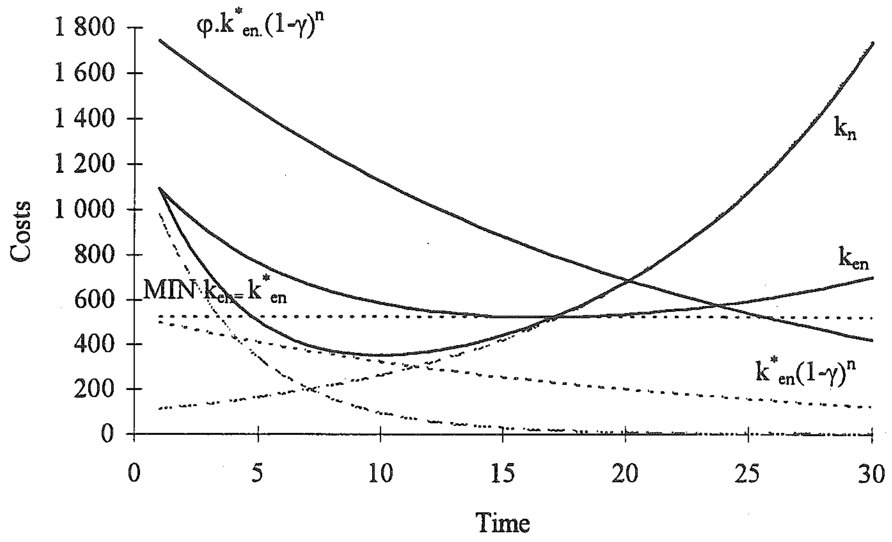

Figure 5: 2-period model with technological progress (example)

are rising to find an optimum. This is of particular interest for those cases in which the operating cost of the defender is constant over time. This case could not be optimized with the basic model, since in this case the equivalent yearly cost has no minimum. In practice this would lead to a coincidence of the economic and physical life of this kind of equipment. We will demonstrate that the previous model is able to tackle this kind of problem.

Therefore, suppose the operating cost of a piece of equipment is constant in time. Suppose also that the depreciation of the equipment goes to zero and the value of the equipment tends to a constant (the scrap value) as its age goes to infinity. In that case $\lim_{l\to\infty} k'(l,0) = 0$ and $\lim_{l\to\infty} k(l,0) = iV_s + m_0$. In the basic problem, we were unable to deal with this kind of replacement, since there is no age at which costs are rising. Since costs are always falling, so will the corresponding equivalent costflow. As age goes to infinity, we find:

$$\lim_{l\to\infty} k_e(l,0) = \lim_{l\to\infty} \frac{i}{1-e^{-il}} \int_0^l \left[iV(\theta,0) - v(\theta,0) + m_0\right].e^{-i\theta}d\theta \qquad (12)$$

The limit of the first factor is the interest rate, the second factor goes to:

$$\lim_{l\to\infty}\left[V_0 - V(l,0).e^{-il}\right] + m_0.\lim_{l\to\infty}\int_0^l e^{-i\theta}d\theta = V_0 + \frac{m_0}{i} \qquad (13)$$

Thus, when costs tend to a constant value, the corresponding equivalent costflow will be $iV_0 + m_0$.

Now, although we still have to assume the second piece of equipment will have an infinite economic life, we can find a finite economic life for the first piece. Following this model, we assume the second piece of equipment will last for ever, at a minimum equivalent costflow of $k_e(\infty, \Delta T_1) = (iV_0 + m_0).e^{-i\Delta T_1}$: Its predecessor will be replaced after ΔT_1 satisfying $k(\Delta T_1, 0) = \varphi k_e(\infty, \Delta T_1)$. All functions necessary to find the economic life of the first piece of equipment are depicted in figure 6.

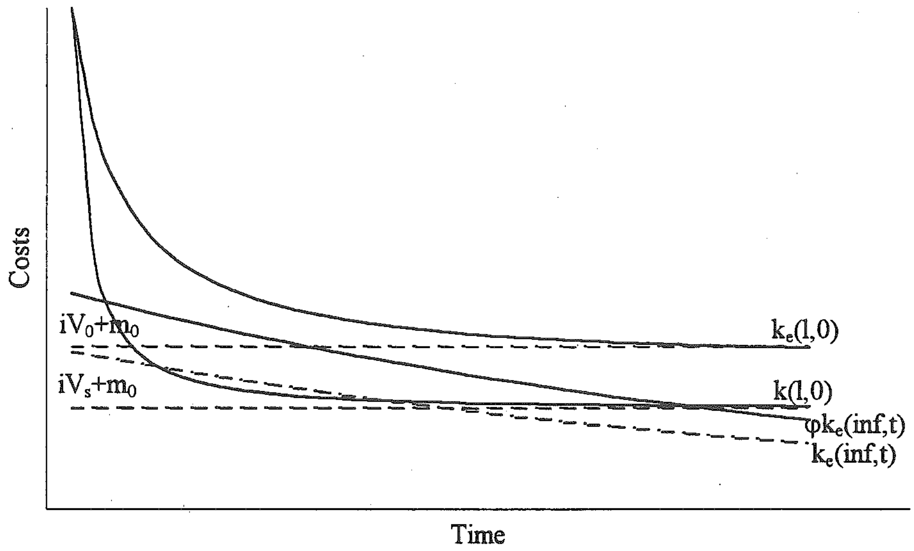

Figure 6: 2-period model with technological progress and decreasing costs

To interpret this figure, we start with the cost function $k(l,0)$, which tends to $iV_s + m_0$ and the corresponding equivalent yearly cost $k_e(l,0)$. The minimum equivalent costflow can be found at age infinity and amounts to

$k(\infty, 0) = iV_0 + m_0$. This minimum cost falls at a constant rate as the time of installation proceeds, as $k_e(\infty, t)$. Replacement will take place at time T_1 when costs of the existing equipment amount to φ times the minimum equivalent costflow of the challenger, i.e. at the intersection of $k(l, 0)$ and $\varphi k_e(\infty, t)$.

4.2 n-period model with technological progress

In the previous model, we partially limited the planning horizon to the first replacement. After this moment, we simply assumed that technological progress ceases and that the challenging equipment will eventually be replaced with identical equipment. Although this line of reasoning may be the only feasible one when there is insufficient information about technological progress in the distant future, it is not entirely correct because future technological progress after the first replacement is ignored. Since in reality, the challenger will probably not be replaced by identical equipment, the costs considered in the previous model are inaccurate. We therefore have to consider an infinite stream of challengers, each one more efficient than its predecessors.

The information necessary to find this series of replacement dates, is too vast and uncertain to be of any use in practical applications. However, we will attempt to generalize the problem from a theoretical standpoint, to assess the error made in the two-period model. We will show that if we assume that technological progress is constant, the (myopic) 2-period case can be viewed as a special case of the standard replacement-model under technological progress.

The optimization problem considered here is in essence a discrete dynamic programming problem. We will deduct the optimality conditions in a recursive way. We first optimize the final period, conditional upon the starting moment of the last period. Let T_{n-1} be the time of the last replacement, which ends technological progress (in the previous section n was 2, since technological progress ceased after the first - i.e. $(n-1)$th replacement). The economic life of all later machines will be identical, since there is no further technological progress. Then we work our way back towards the present. The optimum for the last period is easy to find, since it is not different than the basic problem. Thus we minimize a perpetuity of the equivalent costflow of technology at moment T_{n-1}.

The objective function becomes then:

$$\underset{\Delta T_n}{MIN} \ K_1(\Delta T_n) = \frac{k_e(\Delta T_n, T_{n-1})}{i} \tag{14}$$

The basic problem learned us that the optimum has to satisfy $k(\Delta T_n, 0) = k_e(\Delta T_n, 0)$ and is thus independent of T_{n-1}.

By adding one previous period, we create a variant to the 2-period problem, in which the objective is:

$$\underset{\Delta T_{n-1}}{MIN} \ K_2(\Delta T_{n-1}) = \tag{15}$$

$$\int_0^{\Delta T_{n-1}} k(l, T_{n-2}).e^{-il}dl + e^{-i.\Delta T_{n-1}} \frac{k_e(\Delta T_n, T_{n-1})}{i}$$

The objective is thus to minimize the sum of the costs of the first machine, purchased at a given time T_{n-2} and the cost of an infinite stream of machines, which starts at $T_{n-1} = T_{n-2} + \Delta T_{n-1}$.

A slightly similar formulation of the objective function shows the relation with the 2-period model from the previous section:

$$K_2(\Delta T_{n-1}) = \tag{16}$$

$$e^{-gT_{n-2}} \left[\int_0^{\Delta T_{n-1}} k(l, 0).e^{-il}dl + e^{-(i+g)\Delta T_{n-1}} \frac{k_e(\Delta T_n, 0)}{i} \right]$$

The first factor of this function is a constant, the second factor (between square brackets) is exactly the equivalent of the objective function in the 2-period model. Notice that the optimal value of ΔT_{n-1} does not depend on T_{n-2}. In fact the only difference with the 2-period model studied so far is that the starting time of technological progress is now T_{n-2} in stead of 0.

It is now possible to further expand the objective function of the model with additional previous periods. However, the deduction of the optimal economic life grows considerably more complex as new periods are added. Therefore we will first examine a 3-period model before addressing the general n-period model. The objective for period $n-2$ is then:

$$\underset{\Delta T_{n-2}}{MIN} \ K_3(\Delta T_{n-2}) = \int_0^{\Delta T_{n-2}} k(l, T_{n-3}).e^{-il}dl + \tag{17}$$

$$e^{-i\Delta T_{n-2}} \int_0^{\Delta T_{n-1}} k(l, T_{n-2}).e^{-il} dl + e^{-i(\Delta T_{n-1}+\Delta T_{n-2})} \frac{k_e(\Delta T_n, T_{n-1})}{i}$$

In this expression the first term represents the present value of all costs of the defending piece of equipment. The second term is the present value of all costs of the first challenger, and the last term the cost of the second and following challengers. Notice that, although the economic life of the first and second challenger is independent of the economic life of the defender, the costs of these challengers have to be included in the objective function, since their present value is affected by the length of the economic life of the defender.

The objective function can be transformed to:

$$K_3(\Delta T_{n-2}) = e^{-gT_{n-3}}. \left[\int_0^{\Delta T_{n-2}} k(l,0).e^{-il} dl + \right. \tag{18}$$

$$\left. e^{-(i+g)\Delta T_{n-2}} \int_0^{\Delta T_{n-1}} k(l,0).e^{-il} dl + e^{-(i+g)(\Delta T_{n-1}+\Delta T_{n-2})} \frac{k_e(\Delta T_n, 0)}{i} \right]$$

Again, this makes clear that the optimum will not depend on the time the defender is purchased (T_{n-3}). The optimum can be found by setting the first derivative of the factor between square brackets in equation 18 equal to zero. Calculations are given in detail in appendix A. The first-order condition can be written as:

$$k(\Delta T_{n-2}, 0) = \tag{19}$$

$$\varphi \left[k_e(\Delta T_{n-1}, \Delta T_{n-2}).(1 - e^{-i\Delta T_{n-1}}) + e^{-i\Delta T_{n-1}} k_e(\Delta T_n, \Delta T_{n-1} + \Delta T_{n-2}) \right]$$

The interpretation of this optimality condition is very similar to the interpretation of the optimality condition in the 2-period model. Suppose an investor where certain that the present equipment could be replaced by new (more advanced) equipment after a given ΔT_{n-2} and $\Delta T_{n-1} + \Delta T_{n-2}$ units of time. Suppose also that this replacement scheme were the only alternative to replacing the equipment with identical machines. Then, according to the basic model, it would be optimal to switch to the new replacement scheme if its equivalent costflow became smaller than the total costflow of the existing equipment. The equivalent costflow of such a scheme is exactly the factor between square brackets in equation 19. However, since technological progress is a continuous ongoing process, a replacement (temporarily) fixes the benefits of technological progress. As a result, total costs of the existing equipment have to exceed the equivalent costflow of the challenging equipment by a factor φ. Again, this reflects the opportunity cost of lost future technological progress.

In an n-period model, the objective function for the first replacement becomes:

$$\underset{\Delta T_1}{MIN} \quad K_n(\Delta T_1) = \sum_{j=1}^{n-1} \left[e^{-iT_{j-1}} \int_0^{\Delta T_j} k(l, T_{j-1}).e^{-il}\, dl \right] + e^{-iT_{n-1}} \frac{k_e(\Delta T_n, T_{n-1})}{i} \tag{20}$$

The summation represents the costs of the first $n-1$ machines. From the n-th machine on, we will switch to a system of like-for-like replacement. The cost of this perpetuity, which starts at T_{n-1},is reflected in the last term of the objective function. The optimum has to satisfy the first order condition for this problem (for the more technical aspects we refer again to appendix A):

$$k(\Delta T_1, 0).e^{-i\Delta T_1} = \tag{21}$$
$$\varphi \left\{ \sum_{j=2}^{n-1} \left[(e^{-iT_{j-1}} - e^{-iT_j}).k_e(\Delta T_j, T_{j-1}) \right] + e^{-iT_{n-1}} k_e(\Delta T_n, T_{n-1}) \right\}$$

As expected, the cost of the defender has to exceed the equivalent costflow of all challengers by a factor φ before it becomes optimal to replace.

4.3 Infinite-period model with technological progress

In all previous models, we saw that the optimal economic life of present equipment depended on the number of replacements before technological progress ceased. The time of purchase of the first defender and past replacements never influence the economic life of present equipment. Thus, we could argue that in the infinite horizon model the economic life ΔT of all successive machines will be equal, since after each replacement infinite replacements will follow. It is also obvious that in that case, the last term in the previous optimality condition vanishes. Thus, in the infinite-period case, the optimality condition is:

$$k(\Delta T, 0).e^{-i\Delta T} = \varphi \sum_{j=2}^{\infty} \left[(e^{-iT_{j-1}} - e^{-iT_j}).k_e(\Delta T, T_{j-1}) \right] \qquad (22)$$

The interpretation is again very similar to the previous models. If at time T_1 the existing equipment could be replaced by a series of new machines, one would decide to replace if the equivalent yearly cost of the challenging series (the summation at the right-hand side of equation 22) were lower than the cost of the existing equipment. However, since the challenging series grows continuously less expensive as time passes, it becomes more interesting to wait longer. The costs of the challenging equipment will have to outperform the costs of the defending equipment by a factor φ before replacement is due. The degree by which φ exceeds 1 - which could be substantial - is a measure of the opportunity cost of lost technological progress.

Perhaps a more convenient representation of the same optimality condition is (See also appendix A):

$$k(\Delta T, 0) = \frac{i+g}{1 - e^{-(i+g)\Delta T}} \int_0^{\Delta T} k(l, \Delta T).e^{-il} dl \qquad (23)$$

This rule says that replacement is due when the cost is equal to the costflow equivalent with all previous costs, provided that this equivalent is calculated with a discount rate of $(i + g)$.

5 Conclusion

The problem of optimal replacement has a long history, but still leaves opportunities for further refinement. In this paper, we tried to find answers to two important difficulties in this area. First, we focussed on the problem of the correct variable to base the replacement decision on. Although in the literature, both output and age are considered as important variables, little attention has been given to bringing these variables together in a single model. In the model presented in this paper, both variables were combined in a single model, displaying age and the intensity of utilization as variables. We showed that the economic life of equipment was dependent upon the intensity of utilization and that economic life could be prolonged by lowering this intensity. This model represented a formal explanation of the principle of functional degradation, meaning that in practice equipment often undergoes shifts in its application. It also suggested that in practice, the most demanding jobs in terms of intensity of utilization will be performed by the newest equipment.

Second, we tried to further examine the effect of technological progress on the economic life of equipment. We demonstrated that the basic model of replacement, which can be found in many textbooks on engineering economics or managerial economics, is often quite wrong when there is ongoing technological progress. We first demonstrated this by means of a myopic decision-maker, who considers the technological progress only to a limited extend. Although the objective of this decision-maker is not entirely correct, it is often the only real feasible way of analyzing the problem, because we lack the necessary data about future technological progress. Furthermore, the general idea of the two-period model can be extended to more general cases and even to the infinite-horizon case. We always found that the basic model can give quite erroneous results, because the opportunity cost of lost technological progress at the time of replacement is not taken into account.

A n-period model: mathematical derivation

In period $n-2$ the objective is given in equation 18:

$$K_3(\Delta T_{n-2}) = e^{-gT_{n-3}} \left[\int_0^{\Delta T_{n-2}} k(l,0).e^{-il}dl + \right.$$

$$\left. e^{-(i+g)\Delta T_{n-2}} \int_0^{\Delta T_{n-1}} k(l,0).e^{-il}dl + e^{-(i+g)(\Delta T_{n-1}+\Delta T_{n-2})} \frac{k_e(\Delta T_n, 0)}{i} \right]$$

The first factor can be considered as a constant. Setting the first derivative of the second factor with respect to ΔT_{n-2} equal to zero gives:

$$\begin{aligned} k(\Delta T_{n-2}, 0).e^{-i\Delta T_{n-2}} &= (i+g).e^{-(i+g)\Delta T_{n-2}} \int_0^{\Delta T_{n-1}} k(l,0).e^{-il}dl \\ &+ (i+g).e^{-(i+g)(\Delta T_{n-1}+\Delta T_{n-2})} \frac{k_e(\Delta T_n, 0)}{i} \end{aligned} \quad (24)$$

Or after simplification:

$$k(\Delta T_{n-2},0) = \tag{25}$$
$$(i+g)\int_0^{\Delta T_{n-1}} k(l,\Delta T_{n-2}).e^{-il}dl + \frac{i+g}{i}.e^{-i\Delta T_{n-1}}k_e(\Delta T_n,\Delta T_{n-1}+\Delta T_{n-2})$$

or:

$$k(\Delta T_{n-2},0) =$$
$$\varphi\left[k_e(\Delta T_{n-1},\Delta T_{n-2}).(1-e^{-i\Delta T_{n-1}}) + e^{-i\Delta T_{n-1}}k_e(\Delta T_n,\Delta T_{n-1}+\Delta T_{n-2})\right]$$

Which is the first order condition 19. It is important to notice that again, the economic life of the equipment is independent of the moment the equipment was used for the first time. If we expand the problem further, we can also derive the economic life of the first piece of equipment in a series of replacements. We assume again that the n-th machine is replaced with identical equipment. The objective 20 was then:

$$\underset{\Delta T_1}{MIN} \; K_n(\Delta T_1) =$$
$$\sum_{j=1}^{n-1}\left[e^{-iT_{j-1}}\int_0^{\Delta T_j} k(l,T_{j-1}).e^{-il}\,dl\right] + e^{-iT_{n-1}}\frac{k_e(\Delta T_n,T_{n-1})}{i}$$

Calculating the first derivative and setting equal to zero gives:

$$k(\Delta T_1,0).e^{-i\Delta T_1} = \tag{26}$$
$$\sum_{j=2}^{n-1}\left[(i+g).e^{-(i+g)T_{j-1}}\int_0^{\Delta T_j} k(l,0).e^{-il}\,dl\right] - (i+g)\,e^{-(i+g)T_{n-1}}\frac{k_e(\Delta T_n,0)}{i}$$

Rearranging and simplifying:

$$k(\Delta T_1,0).e^{-i\Delta T_1} = \tag{27}$$
$$\sum_{j=2}^{n-1}\left[(i+g).e^{-iT_{j-1}}\int_0^{\Delta T_j} k(l,T_{j-1}).e^{-il}\,dl\right] + (i+g)\,e^{-iT_{n-1}}\frac{k_e(\Delta T_n,T_{n-1})}{i}$$

Which is equivalent to equation 21:

$$k(\Delta T_1, 0).e^{-i\Delta T_1} =$$
$$\varphi.\left\{\sum_{j=2}^{n-1}\left[(e^{-iT_{j-1}} - e^{-iT_j}).k_e(\Delta T_j, T_{j-1})\right] + e^{-iT_{n-1}}k_e(\Delta T_n, T_{n-1})\right\}$$

Since economic life of present equipment depends on the number of replacements before technological progress ceased and since the time of purchase of the first defender and past replacements never influences the economic life of present equipment, we can argue that in the infinite horizon model the economic life ΔT of all successive machines will be equal. Thus, if $n \to \infty$, we find equation 22:

$$k(\Delta T, 0).e^{-i\Delta T} = \varphi \sum_{j=2}^{\infty}\left[(e^{-iT_{j-1}} - e^{-iT_j}).k_e(\Delta T, T_{j-1})\right]$$

Or:

$$k(\Delta T, 0).e^{-i\Delta T} = \varphi \sum_{j=2}^{\infty}\left[e^{-(i+g)T_{j-1}}(1 - e^{-i\Delta T}).k_e(\Delta T, 0)\right] \quad (28)$$

The two last factors do not depend on j and can be placed in front of the summation. We replace the equivalent costflow by its full notation, to find:

$$k(\Delta T, 0).e^{-i\Delta T} = (i+g).\int_0^{\Delta T} k(l, 0).e^{-il}dl\,.\sum_{j=2}^{\infty} e^{-(i+g)T_{j-1}} \quad (29)$$

Finally, replace the infinite geometric series by $\sum_{k=1}^{k=\infty} e^{-k(i+g)\Delta T} = \frac{e^{-(i+g)\Delta T}}{1-e^{-(i+g)\Delta T}}$, to find the result of equation 23:

$$k(\Delta T, 0) = \frac{(i+g)}{1 - e^{-(i+g)\Delta T}}.\int_0^{\Delta T} k(l, \Delta T).e^{-il}dl$$

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