Een studie van speciale morfismen in meetkunde en analyse

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And what can I tell you my brother, my killer
What can I possibly say?
I guess that I miss you, I guess I forgive you
I'm glad you stood in my way.
- Leonard Cohen

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Chapter 2

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I wish to thank my advisor Wendy Lowen for her tireless efforts and thoughtful advice, especially in those moments when things looked rather bleak. I doubt I could have a better advisor.

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I dedicate this thesis to Houdaïfa, Kasper, Oubaïda and Milla.

Chapter 3

Introduction

In this thesis, we investigate some special morphisms in topology, geometry and analysis. In particular, we focus on noncommutative topology, algebraic geometry, differential geometry and functional analysis.

Among the most fundamental properties in general topology are the Hausdorff property and compactness of topological spaces, and their relative counterparts, separatedness and properness for continuous functions. Efforts of abstracting these notions to more general categorical contexts endowed with some notion of "closedness" or "properness" date back at least to the seventies with work of Herrlich [26], Manes [44], Penon [48], [47]. A richer theory was developed for categories endowed with a factorization system [1], see the work of Herrlich, Salicrup and Strecker [27] with applications in topology, group- and order theory. In the nineties, once the notion of a "closure operator" on a category endowed with a factorization structure was available, results quite close to the original topological situation were obtained, see [7], with applications to Top, Birkhoff closure spaces, uniform spaces, topological groups and locales. In [56] Tholen observed that both the class of separated morphisms and the class of proper

morphisms can be expressed in terms of an auxiliary class of "closed morphisms" and these ideas grew out to a theory called "functional approach to topology" developed in the context of the auxiliary class of closed morphisms linked to a given factorization system, see the work of Clementino, Giuli and Tholen [8]. Applications of this setting include approach spaces, a common generalization of topological and metric spaces [9]. Recently, in order to capture more general categories of lax algebras, Hofmann and Tholen [30] adapted the setting, replacing the class of closed morphisms by an auxiliary class of "proper maps".

The theory of functional topology introduces on a category a factorization system $(\mathbb{E}, \mathbb{M})$ and a class of morphisms $\mathbb{F}$ which represent the closed morphisms. We call this $\mathbb{F}$ an $(\mathbb{E}, \mathbb{M})$-closed class (see Definition 13.1). The main idea is that by endowing a category with this additional structure, we can interpret objects as "spaces" and study their "topological properties" like separation and compactness in a unified way. This is done not by looking at points of spaces, but by investigating how certain objects and morphisms interact with each other. Both the theory of topological spaces and the theory of locales fit nicely into this framework. Other examples are provided by taking $\mathbb{F}$ the class of open maps between topological spaces, or by taking $\mathbb{F}$ the class of torsion-preserving maps between abelian groups.

Of course, in the ideal setting of topological spaces, the class $\mathbb{F}$ can be defined containing those maps which take closed subobjects to closed subobjects. So instead of letting the class of closed maps $\mathbb{F}$ be the primary notion, we might as well take the class of closed embeddings $\mathbb{F}_0$ as primary. A closed map can then be defined as sending an element in $\mathbb{F}_0$ to an element in $\mathbb{F}_0$ (see Proposition 13.4), or more formally: a morphism f is closed if for each $m \in \mathbb{F}_0$ we have for the $(\mathbb{E}, \mathbb{M})$-factorization $fm = m'e$ that $m' \in \mathbb{F}_0$. This is the ideal situation where the theory has the best properties. But not every $(\mathbb{E}, \mathbb{M})$-closed

class arises this way.

It has been known for a long time that basic classes of morphisms in the theory of schemes in algebraic geometry satisfy many properties which also occur in the theory of functional topology. Indeed, properness and separatedness are fundamental notions in algebraic geometry and are defined in great similarity with properness and separatedness in functional topology. Furthermore, most basic properties of properness and separatedness are shared in both contexts. It was however not clear how exactly the theory of schemes would fit into the framework of $(\mathbb{E}, \mathbb{M})$-closed classes. Solving this issue is one of the topics treated in this thesis.

First of all, there is the problem on finding a suitable factorization system on schemes. The only known factorization system $(\mathbb{E}, \mathbb{M})$ makes use of the so-called scheme theoretic image of a morphism. It has the peculiarity that $\mathbb{M}$ is the class of closed immersions. So from the start, we only allow *closed* immersions as embeddings. This is somewhat unsatisfactory on a philosophical level, since we would prefer a broader class of embeddings which also encompasses the open embeddings.

But even if we would accept $\mathbb{M}$ as being the class of closed immersions, the problems do not stop there. One cannot take $\mathbb{F}$ simply as the class of closed maps, since that choice would not satisfy the definition of an $(\mathbb{E}, \mathbb{M})$-closed class (i.e. Definition 13.1), see §13.3. One could be tempted to take $\mathbb{F}_0$ the class of closed immersions and then generate $\mathbb{F}$ as those maps taking closed immersions to closed immersions. But that fails too since then *all* morphisms would be in $\mathbb{F}$. This phenomenon is an artefact from defining $\mathbb{M}$ as closed immersions and not as more general embeddings.

To solve this problem, we extend the theory of functional topology in various directions.
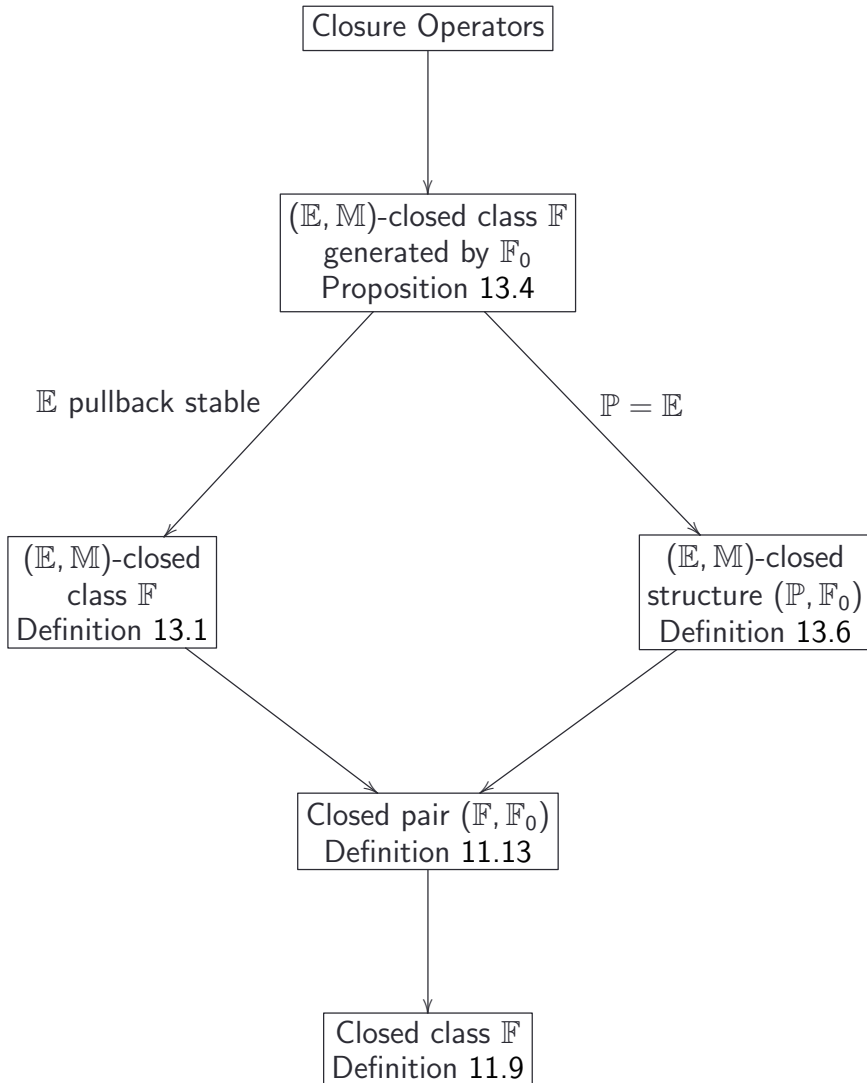
First of all, we question whether a factorization system is truly necessary for many results of functional topology. If we drop the factorization system, then we do not have natural notions of images and subobjects, but in schemes these notions did not work anyway. This leads us to defining a closed class $\mathbb{F}$ and a closed pair $(\mathbb{F}, \mathbb{F}_0)$. With the former definition, we can talk about closed maps, and in the latter definition we can additionally talk about closed embeddings. It turns out that surprisingly many results of functional topology can be extended to these two new settings. In particular, properness and separatedness make sense in these settings and behave as expected. Both the classical functional topology of Definition 13.1 and the theory of schemes fit nicely into this context.

Of course, one could ask whether we can extend the theory of functional topology while keeping the factorization system. Again, this is possible. However, instead of extending Definition 13.1, we find it more useful to extend Proposition 13.4. Thus as primary notions, we take the class $\mathbb{F}_0$ and another additional class $\mathbb{P}$ which we call "surjections." Then we can define a morphism f to be in $\mathbb{F}$ if for each subobject $m \in \mathbb{F}_0$, we have the $(\mathbb{E}, \mathbb{M})$-factorization $f m = e m'$, where $m' \in \mathbb{F}_0$ and where $e \in \mathbb{P}$. This turns out to be a convenient setting for schemes. It also generalizes the classical functional topology: indeed, by taking $\mathbb{P} = \mathbb{E}$, we recover the classical theory.

Finally, there does remain the objection that we have a factorization system $(\mathbb{E}, \mathbb{M})$ on schemes where $\mathbb{M}$ consists of closed immersions and not of more general embeddings. This can be solved if we allow images of morphisms not to be schemes, but more general objects. In particular, we can embed schemes into a suitable presheaf category. This presheaf category has a canonical factorization system and thus also a canonical notion of images. However, we lose the property that the image of a morphism is a scheme. We can however introduce a

class of closed embeddings $\mathbb{F}_0$ on the presheaf category. We then obtain notions of properness and separatedness on this presheaf category, which do restrict to the correct notions of properness and separatedness on the full subcategory of schemes.

Along with investigating how to fit schemes into a functional topology setting, we have investigated several other categories. For example, we look into the category of compactly generated spaces. These fit into Definition 13.1. We have the nice result that the Hausdorff objects in this category are the weakly Hausdorff spaces. Another category is that of Lie groups. These also fit into Definition 13.1. We get that the compact objects in this category are exactly the compact Lie groups. One is then tempted to do the same for more general manifolds, but one runs into difficulties since the category of manifolds does not behave well. In particular, the image of a smooth map does not have the natural structure of a manifold. We can extend the category of manifolds to that of diffeological spaces which are better behaved. But we can then prove the negative result that, with the natural choices of factorizations and closed embeddings, functional topology is not possible on this category (see §13.3). The issue is that the natural topology associated with a diffeological space does not behave well.

Finally, we have investigated if we could fit the theory of C^*-algebras into a functional topology setting. The theory of C^*-algebras is sometimes known as noncommutative topology and it has been known for a long time that many topological notions have a C^*-algebraic counterpart. For example, compact spaces correspond to unital C^*-algebras and compactifications correspond to unitizations. However, the natural C^*-morphisms correspond to the *proper* continuous maps and not to the general continuous maps. In order to represent all the continuous maps, we use a different notion of morphism between C^*-algebras, namely the Woronowicz morphisms [21]. The dual of the category of

C^*-algebras and its Woronowicz morphisms can then be called a category of noncommutative topology. Sadly, this category does not have very nice categorical properties. For example, the product in this category (which corresponds to the coproduct of C^*-algebras) is not so transparent. The idea is then to replace the coproduct of C^*-algebras by the tensor product. We realize this by introducing a relation R on the morphisms which tells us whether two morphisms commute or not (see [33] for the general theory of such relations, and the link with monoidal structures). In the category of "noncommutative topology", we get the tensor product. While in the standard categories of topological spaces (with all morphisms R-related by definition), we get the usual product. Apart from considering R-products with respect to such a relation R, we also introduce R-pullbacks. This allows us to generalize the various theories of functional topology that we have obtained so far. This works well and applies in particular to the category of C^*-algebras. In this category the compact objects turn out to be exactly the unital C^*-algebras, as predicted.

Most of the above paragraph on C^*-algebras works also for general associative algebras over a field. The notion of Woronowicz morphism is also available for these associative algebras. Hence, we can mimic what we did for C^*-algebras in the setting of associative algebras. This yields yet another example of functional topology. Again, we obtain that the unital associative algebras are the compact objects. These results suggest that we can generalize the situation even further to closed monoidal categories. This is a work in progress.

In noncommutative topology, an important source of examples are so-called deformations. For instance, the noncommutative torus can be viewed as a deformation of the ring of continuous functions over the torus [50]. If one attempts to understand these deformations as part of a general deformation theory of C^*-algebras similar to Ger-

stenhaber's deformation theory of associative algebras [18], one faces a number of difficulties. Whereas typically, C^*-algebras are obtained as algebras of *bounded* operators on Hilbert spaces, it turns out that natural operations fundamental to deformation theory, like the Poisson bracket, involve unbounded operators. This leads us to the theory of unbounded operators on Hilbert spaces, which works best on so-called rigged Hilbert spaces [17]. The idea would then be to construct deformations of C^*-algebras as algebras of operators on certain rigged Hilbert spaces. So far, these operators had only been studied in special cases. For example, a general spectral theorem for such operators was lacking. In the last chapter of this thesis, we present such a theorem (Theorem 23.57). Further work should determine whether rigged Hilbert spaces are part of a convenient setting for deformation theory.

Theorems in this thesis of which the proof is omitted can be found elsewhere in the literature. If a result is accompanied by a proof, then either it is a variant of a known result, or a result for which we have not been able to find a reference (and in both cases this is indicated as such in the text), or else the result is new to the best of our knowledge.

Chapter 5

Some interesting categories

In later chapters, we will extend the functional topology setting of [8] to a more general framework. We will then apply this new framework to various categories. This introductory chapter introduces the categories that we will investigate later.

First, we introduce some topological categories, namely the categories of topological spaces and compactly generated spaces. These categories are complete.

Next, we turn to some categories of algebraic geometry. We introduce the categories of ringed spaces, locally ringed spaces, schemes and complex analytic spaces. We also show how to extend the category of schemes to a category of (pre)sheaves on a Grothendieck site.

Next, we turn to some categories from differential geometry. Since manifolds do not form a very nice category (for example, categorical constructions such as pullbacks are not generally defined, and images of smooth maps are not manifolds in general), we introduce a gener-

alization of manifolds, called diffeological spaces. We also introduce
the category of Lie groups.

Finally, we introduce the notion of a C^*-algebra and we associate
with this notion two interesting categories. The first category is the
standard category of C^*-algebras with usual C^*-morphisms. The the-
orem of Gelfand–Naimark implies that C^*-algebras correspond to lo-
cally compact spaces, but the C^*-morphisms do not correspond to the
continuous maps (they only correspond to the proper maps). To rem-
edy this, we define a different kind of morphism between C^*-algebras,
called Woronowicz-morphisms. These do correspond to the continu-
ous maps. We extend the results of C^*-algebras to general associative
algebras over a more general field.

5.1   Some topological categories

Of course, the main topological category is **Top**, the category of all
topological spaces with continuous maps. The properties of this cat-
egory are well-known, in particular, it is complete and cocomplete.

Other interesting categories arise from so-called k-spaces (or com-
pactly generated spaces).

5.1 Definition. Let $(X, \mathcal{T})$ be a topological space. A subset Y of X
is called *k-closed* if for each compact Hausdorff space K and for each
continuous map $u : K \to X$ we have that $u^{-1}(Y)$ is closed.

Clearly, every closed set is k-closed, but the converse need not be true.
The collection of k-closed sets form the closed sets for a topological
space, we will denote this topological space by $k(X, \mathcal{T})$ or simply kX.
It is clear that the identity $1_X : k(X, \mathcal{T}) \to (X, \mathcal{T})$ is continuous. If

the identity is a homeomorphism (thus if $kX = X$), then we say that X is a *k-space* or *compactly generated space*.

We say that a topological space is *sequential* if for any set A the following are equivalent:

1. A is open,

2. For any convergent sequence $x_n \to x$, with $x \in A$, we have that $(x_n)_n$ is in A eventually.

We have:

5.2 Theorem. *Every sequential space (in particular, every first countable and every metric space) is a k-space. Furthermore, each locally compact Hausdorff space is a k-space.*

We let $k\mathbf{Top}$ be the category of k-spaces (as full subcategory of **Top**).

5.3 Theorem. *The full subcategory $k\mathbf{Top}$ of **Top** is coreflective. In particular, if X is a k-space and if Y is an arbitrary topological space, then a function $f : X \to Y$ is continuous iff the function $f : X \to kY$ is continuous.*

In particular, we have that $kkX = kX$ and thus kX is a k-space for each topological space X.

The category $k\mathbf{Top}$ is complete and cocomplete. The colimits in $k\mathbf{Top}$ are constructed just like in **Top**, but a limit in $k\mathbf{Top}$ is constructed as kL, where L is the limit in **Top**.

See Strickland [54] for more information and proofs.

5.2 Ringed spaces

5.4 Definition. A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and where $\mathcal{O}_X$ is a sheaf of commutative rings.

Recall that if $\varphi : X \to Y$ is a continuous map between topological spaces and if $\mathcal{F}$ is a sheaf on X and if $\varphi : X \to Y$ is a continuous map, then $\varphi_* \mathcal{F}$ is the *direct image sheaf* on Y defined by $(\varphi_* \mathcal{F})(U) = \mathcal{F}(\varphi^{-1}(U))$ for each open set U of Y. Similarly, if $\mathcal{G}$ is a sheaf on Y, then $\varphi^{-1}\mathcal{G}$ is the *inverse image sheaf*, it is defined such that the functor $\mathcal{G} \to \varphi^{-1}\mathcal{G}$ is left adjoint to the functor $\mathcal{F} \to \varphi_* \mathcal{F}$. See Tennison [55] for an explicit construction.

We can make the ringed spaces into a category:

5.5 Definition. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. A *morphism between ringed spaces* $(\varphi, \varphi^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by the following data:

- A continuous map $\varphi : X \to Y$.

- A morphism $\varphi^\flat : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ between sheaves of commutative rings.

Given morphisms $(\varphi, \varphi^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(\psi, \psi^\flat) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$. We can compose them as follows:

- The underlying continuous map is $\psi\varphi$

- The morphisms of sheaves $\varphi^\flat : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ and $\psi^\flat : \mathcal{O}_Z \to \psi_* \mathcal{O}_Y$ compose as $\psi_*(\varphi^\flat)\psi^\flat : \mathcal{O}_Z \to (\psi\varphi)_* \mathcal{O}_X$.

As such we get a category of ringed spaces. We will denote this category as **RingedSpace**. It is shown in Brandenburg [4] that this category is complete. Some interesting ringed spaces arise from geometry:

5.6 Examples. 1. Any topological space X gives rise to a ringed space $(X, \mathcal{C}(-, \mathbb{R}))$, where for any open $U \subseteq X$,

$$\mathcal{C}(U, \mathbb{R}) = \{f : U \to \mathbb{R} \mid f \text{ is continuous}\}.$$

2. Any smooth manifold M induces a ringed space

$$(M, \mathcal{C}^\infty(-, \mathbb{R})),$$

where for any open $U \subseteq M$,

$$\mathcal{C}^\infty(U, \mathbb{R}) = \{f : U \to \mathbb{R} \mid f \text{ is smooth}\}.$$

3. Any complex analytic space X gives rise to a ringed space

$$(X, \mathcal{C}^\omega(-, \mathbb{C})),$$

where for any open $U \subseteq X$,

$$\mathcal{C}^\omega(U, \mathbb{C}) = \{f : U \to \mathbb{C} \mid f \text{ is analytic}\}.$$

4. Both schemes (see §5.4) and complex analytic spaces (see §5.5) are ringed spaces.

All these examples are also locally ringed spaces.

5.3 Locally Ringed Spaces

5.7 Definition. A ringed space $(X, \mathcal{O}_X)$ is called a *locally ringed space* if the stalks $\mathcal{O}_{X,x}$ are local rings for each $x \in X$.

We do not look at the locally ringed spaces as a full subcategory of the ringed spaces. Instead, we define

5.8 Definition. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces and let $(\varphi, \varphi^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces in the sense of Definition 5.5, then we say that this is a *morphism between locally ringed spaces* if $f^\flat : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ is a local homomorphism. This means for every $x \in X$, the map induced by $f^\flat$

$$\mathcal{O}_{Y, f(x)} \xrightarrow{f_x^\flat} (\varphi_* \mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X, x}$$

sends the maximal ideal of $\mathcal{O}_{Y, f(x)}$ into the maximal ideal of $\mathcal{O}_{X, x}$.

Composition of these morphisms is defined as explained after Definition 5.5. So we have obtained a category of locally ringed spaces that we will denote by **LocRingedSpace**. It is shown in [4] that this category is complete.

More information about ringed and locally ringed spaces can be found in Tennison [55] (note however that Tennison refers to locally ringed spaces as geometric spaces).

Immersions

5.9 Definition. Consider a morphism

$$(\varphi, \varphi^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

between locally ringed spaces. We say that this morphism is an *open immersion* if φ is an open embedding in the sense of topology and if $f_x^\flat$ is an isomorphism for each $x \in X$. We say that the morphism is a *closed immersion* if φ is a closed embedding in the sense of topology and if $f_x^\flat$ is surjective for each $x \in X$.

The closed immersions arise exactly from sheaves of ideals.

5.10 Theorem.

1. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{J}$ be a sheaf of ideals of $\mathcal{O}_X$, meaning that $\mathcal{J}(U)$ is an ideal of $\mathcal{O}_X(U)$ for each open U. Define*

$$V(\mathcal{J}) = \{x \in X \mid \mathcal{J}_x \neq \mathcal{O}_{X,x}\}$$

   and let $j : V(\mathcal{J}) \to X$ denote the inclusion. Then $V(\mathcal{J})$ is a closed subset of X, $(V(\mathcal{J}), f^{-1}(\mathcal{O}_X/\mathcal{J}))$ is a locally ringed space and we have a closed immersion $(j, j^\flat)$ of this space into $(X, \mathcal{O}_X)$ where $j^\flat$ is the canonical surjection

$$\mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \to j_*(j^{-1}(\mathcal{O}_X/\mathcal{J})).$$

2. *Conversely, let $(\varphi, \varphi^\flat) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a closed immersion between locally ringed spaces. Let $\mathcal{J} = \ker \varphi^\flat$. Then f factors into an isomorphism*

$$(Y, \mathcal{O}_Y) \cong (V(\mathcal{J}), f^{-1}(\mathcal{O}_X/\mathcal{J}))$$

   followed by the canonical closed immersion defined in previous part of the theorem.

More information and the proofs can be found in Liu [40]

5.4   Schemes

Let R be a unital commutative ring. With this ring, we can associate a locally ringed space. Indeed, we define

$$\mathrm{Spec}(R) = \{\mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ is a prime ideal of } R\}.$$

Further, if $\mathfrak{a}$ is any ideal of R, then we set

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

Define further for $f \in R$, the sets $V(f) = V((f))$ and $D(f) = \mathrm{Spec}(R) \setminus D(f)$.

One can show that the sets $\{V(\mathfrak{a}) \mid \mathfrak{a} \text{ ideal of } R\}$ form the closed set of a topology on $\mathrm{Spec}(R)$. This topology is known as the Zariski topology. The collection $\{D(f) \mid f \in R\}$ forms a base for this topology. Unless mentioned otherwise, we will always look at $\mathrm{Spec}(R)$ as equipped with the Zariski topology.

We define for $f \in R$, $\mathcal{O}_{\mathrm{Spec}(R)}(D(f)) = R_f$. This definition extends to a sheaf on the topological space $\mathrm{Spec}(R)$. Then $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is a locally ringed space.

5.11 Definition. The locally ringed spaces $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ are called *affine schemes*.

The affine schemes form a full subcategory of the category of locally ringed spaces. We denote this category as **AffScheme**. This category is in fact anti-equivalent to the category of unital commutative rings, which we will denote as **ComRing**$_1$. In fact, we can define a contravariant functor

$$\mathrm{Spec} : \textbf{ComRing}_1 \rightarrow \textbf{AffScheme}$$

such that

- For each ring R, we have $\mathrm{Spec}(R) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$.

- Every morphism between rings $\varphi : R \rightarrow S$ induces a morphism between ringed spaces

$$(\varphi, \varphi^{\flat}) : (\mathrm{Spec}(S), \mathcal{O}_{\mathrm{Spec}(S)}) \rightarrow (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$$

such that $\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R) : \mathfrak{p} \to \varphi^{-1}(\mathfrak{p})$ and

$$\varphi^{\flat}_{D(f)} : A_f \to S_{\varphi(f)}$$

with $\varphi^{\flat}_{D(f)}(a/f^n) = \frac{\varphi(a)}{\varphi(f)^n}$.

The functor Spec establishes an anti-equivalence.

A scheme is a locally ringed space which is locally isomorphic to an affine scheme:

5.12 Definition. A locally ringed space $(X, \mathcal{O}_X)$ is a *scheme* if there exists an open cover $X = \bigcup_{i \in I} U_i$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic (as locally ringed spaces) to an affine scheme.

We can then look at the category **Scheme** as a full subcategory of the category of locally ringed spaces. One can show that this subcategory is finitely complete. For more information and for the proofs, we refer to Gortz, Wedhorn [19].

Subschemes

Open subschemes are easy to define:

5.13 Theorem. *Let $(X, \mathcal{O}_X)$ be a scheme and let $U \subseteq X$ be open. Then $(U, \mathcal{O}_X|_U)$ is a scheme and $(U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$ is an open immersion. We say that U is an* open subscheme *of X.*

So open immersions correspond exactly to open subsets. Closed subschemes are a bit trickier.

5.14 Definition. Let $(Z, \mathcal{O}_Z)$ and $(X, \mathcal{O}_X)$ be schemes and let $f : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a closed immersion. Then we will say that $(Z, \mathcal{O}_Z)$ is a *closed subscheme* of $(X, \mathcal{O}_X)$.

For a given closed subset Z of a scheme X, there can be many sheaves on Z that make Z into a closed subscheme. So it is necessary to specify the sheaf when looking at closed subschemes. In case of affine schemes, the closed subschemes are known precisely:

5.15 Theorem. *Let* $\mathrm{Spec}\, R$ *be an affine scheme and let* $j : Z \to \mathrm{Spec}\, R$ *be a closed immersion of schemes. Then* Z *is affine and there exists a unique ideal* J *of* R *such that* $Z \cong \mathrm{Spec}(R/J)$. *Conversely, if* J *is an ideal of* $\mathrm{Spec}\, R$, *then* $\mathrm{Spec}(R/J)$ *is a closed subscheme of* $\mathrm{Spec}\, R$, *where the closed immersion* $\mathrm{Spec}(R/J) \to \mathrm{Spec}\, R$ *is induced by the quotient morphism* $R \to R/J$.

We can define images of morphisms between schemes as follows:

5.16 Theorem. *Let* $(f, f^{\flat}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *be a morphism between schemes. Then there exists unique closed subscheme Z of Y such that*

1. *f factorizes through the inclusion $Z \to Y$.*

2. *If f factorizes through the inclusion $Z' \to Y$ of a closed subscheme Z' of Y, then Z' majorizes Z.*

We say that Z is the scheme-theoretic image *of f.*

More details can be found in Liu [40].

Functor of points

Let $\mathcal{C}$ be a small category. A sieve on X in $\mathcal{C}$ is a collection of morphisms S in $\mathcal{C}$ with common codomain X such that if $f \in S$, then $fg \in S$ for any morphism in $\mathcal{C}$ for which this composition makes sense. If S is a sieve on Y and if $h : X \to Y$ then we define the pullback sieve

$$h^*(S) = \{g : \mathrm{dom}(g) \to Y \mid fg \in S\}.$$

5.17 Definition. A *Grothendieck topology* is a collection of sieves in a category $\mathcal{C}$ (called *covering sieves*) such that the following axioms are satisfied:

1. The *maximal sieve* $\{f \mid \text{codomain of } f \text{ is } X\}$ is a covering sieve of X.

2. If $S = \{f_i : X_i \to X\}$ is a covering sieve of X and if $f : Y \to X$ is a morphism, then the pullback f^*S is a covering sieve of Y.

3. If S is a covering sieve in X and if R is any sieve on X such that $h^*(R)$ is a covering sieve of Y for any morphism $h : Y \to X$ in S, then R is a covering sieve of X.

Sometimes it is easier to describe a basis for a Grothendieck topology:

5.18 Definition. Let $\mathcal{C}$ be a finitely complete category. A *basis for a Grothendieck topology* is a collection of sets of the form $\{f_i : X_i \to X\}$ (called *covering families*) such that the following axioms are satisfied:

1. For all objects X, all morphisms $Y \to X$ and all covering families $\{X_i \to X\}_{i \in I}$, the fibered product $\{X_i \times_X Y \to Y\}_{i \in I}$ forms a covering family.

2. If $\{X_i \to X\}_{i \in I}$ is a covering family and if for each i we have a covering family $\{X_{i,j} \to X_i\}_{j \in J_i}$, then the composition $\{X_{i,j} \to X_i \to X\}_{j \in J_i,\ i \in I}$ is a covering family.

3. If $f : X \to Y$ is an isomorphism, then $\{f\}$ is a covering family.

Any basis induces a Grothendieck topology by saying that a sieve $\{f_i : X_i \to X\}$ is a covering sieve of X if and only if it contains a covering family.

Given a category $\mathcal{C}$, we let **PSh**$(\mathcal{C})$ be the category of all contravariant functors (called *presheaves*) $\mathcal{C} \to$ **Set**. If $\mathcal{C}$ carries a Grothendieck topology, then we can define special kinds of functors, which we call sheaves:

5.19 Definition. Let $\mathcal{C}$ be a category with a Grothendieck topology. Let P be a presheaf on $\mathcal{C}$.

1. If S is a covering sieve of X, then a *matching family* is a function x which associates with each element $f : Y \to X$ in S an element $x_f \in P(Y)$ such that for each morphism $g : Z \to Y$ in $\mathcal{C}$ we have that $P(g)x_f = x_{fg}$.

2. Let $\{x_f \mid f \in S\}$ be a matching family, then an *amalgamation* of this matching family is an element $x \in P(X)$ such that $P(f)x = x_f$ for each $f \in S$.

3. We say that P is a *sheaf* is every matching family has a unique amalgamation.

Let **Sh**$(\mathcal{C})$ be the full subcategory of **PSh**$(\mathcal{C})$ consisting of all sheaves on $\mathcal{C}$. See [42] for more information on Grothendieck topologies.

Now, let **AffScheme** be the category of affine schemes as a full subcategory of locally ringed spaces. This category is of course finitely complete (see Liu [40]). This category can be naturally equipped with a Grothendieck topology. Indeed, a covering family is a collection of open immersions $f_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)$ of affine schemes such that $X = \bigcup_{i \in I} f_i(X_i)$. These covering families form a basis for a Grothendieck topology (see Maclane, Moerdijk [42]), which we will call the Zariski topology.

So now we can form the presheaf category **PSh**(**AffScheme**) and the sheaf category **Sh**(**AffScheme**). Both categories are complete.

By the Yoneda lemma, there is a fully faithful embedding

$$h : \textbf{AffScheme} \to \textbf{PSh}(\textbf{AffScheme})$$

by setting $h_X = \text{Hom}_{\textbf{AffScheme}}(-, X)$. In fact, we can do more and embed schemes into **Sh**(**AffScheme**):

5.20 Theorem. *Let X be a scheme, then the functor*

$$h : \textbf{Scheme} \to \textbf{Sh}(\textbf{AffScheme})$$

defined by $h_X = \text{Hom}_{\textbf{Scheme}}(-, X)$ is a fully faithful embedding from the category of schemes into **Sh**(**AffScheme**).

We say that an $F \in \textbf{PSh}(\textbf{AffScheme})$ is *representable* if $F \cong h_X$ for a scheme X. A morphism of functors $\alpha : F \to G$ in **PSh**(**AffScheme**) is said to be a *representable morphism* if for all schemes X and all morphisms $g : h_X \to G$ we have that the pullback $F \times_G h_X$ is representable.

$$
\begin{array}{ccc}
F \times_G h_X & \xrightarrow{\;p_2\;} & h_X \\
{\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
F & \xrightarrow[\;\alpha\;]{} & G
\end{array}
$$

Let **P** be a property of morphisms of schemes such that composition from the left or right of the morphism with an isomorphism still has property **P**. We say that a representable morphism $\alpha : F \to G$ *has property* **P** if for all schemes X and all morphisms $g : h_X \to G$ we

have that the second projection $p_2 : F \times_G h_X \to h_X$ has property **P** as a morphism between schemes. In particular, we can define the notion of closed and open immersions on representable morphisms. Note however that if a scheme morphism $f : X \to Y$ has property **P** then it is not necessarily true that the associated morphism $h_X \to h_Y$ as property **P**. However, this is true if the property **P** is stable under base change.

We can now answer the question when an $F \in$ **PSh**(**AffScheme**) is representable:

5.21 Theorem. *Let $F \in$* **PSh**(**AffScheme**), *then F is representable if and only if*

1. *F is a sheaf and*

2. *There exists representable functors F_i and representable open immersions $f_i : F_i \to F$ such that $F(X) = \bigcup_i F_i(X)$ for each affine scheme X.*

More information and proofs can be found in Gortz and Wedhorn [19] and Eisenbud and Harris [13].

Quasi-coherent modules

Let $(X, \mathcal{O}_X)$ be a ringed space.

5.22 Definition. An *$\mathcal{O}_X$-module* is a sheaf $\mathcal{F}$ on X together with two morphisms of sheaves:

$$+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$$

and

$$\cdot : \mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$$

which define on each $\mathcal{F}(U)$ the structure of an $\mathcal{O}_X(U)$-module.

5.23 Definition. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi-coherent* if for each $x \in X$, there exists an open set U of x and an exact sequence

$$\mathcal{O}_X^{(J)}|_U \to \mathcal{O}_X^{(I)}|_U \to \mathcal{F}|_U \to 0.$$

where I and J are index sets (depending on x).

Let $\mathrm{Spec}(A)$ be an affine scheme and let M be a module on A. Then we can define an $\mathcal{O}_{\mathrm{Spec}(A)}$-module $\tilde{M}$ by

$$\tilde{M}(D(f)) = M_f.$$

We have the following theorem:

5.24 Theorem. *Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the following are equivalent:*

1. *$\mathcal{F}$ is quasi-coherent,*

2. *For every open affine subset $\mathrm{Spec}(A)$ of X, there exists an A-module M such that $\mathcal{F}|_U \cong \tilde{M}$.*

3. *There exists an open affine covering $(U_i)_i$ of X of X such that $U_i = \mathrm{Spec}(A_i)$ and for each i there exist A_i-modules M_i such that $\mathcal{F}|_{U_i} \cong \tilde{M}_i$.*

5.25 Theorem. *Let X be a scheme. Then there is a bijection between the set of quasi-coherent ideals of $\mathcal{O}_X$ and the closed subschemes of X. This bijection is given by attaching to each quasi-coherent ideal $\mathcal{I}$ of $\mathcal{O}_X$ the subscheme $(V(\mathcal{I}), i^{-1}(\mathcal{O}_X/\mathcal{I}))$ (see Theorem 5.10). The inverse function is given by attaching to each closed subscheme $i_Z : Z \to X$ the kernel of $\mathcal{O}_X \to (i_Z)_* \mathcal{O}_Z$.*

For more information and proofs see Gortz, Wedhorn [19].

5.5   Complex analytic spaces

Let $U \subseteq \mathbb{C}^n$ be open. If $f_1, ..., f_k : U \to \mathbb{R}$ are analytic functions, then we define

$$V(f_1, ..., f_k) = \{x \in \mathbb{C}^n \mid f_1(x) = ... = f_k(x) = 0\}.$$

Every open subset $U \subseteq \mathbb{C}^n$ has a sheaf $\mathcal{O}_U$ consisting of all holomorphic functions. Define $\mathcal{I}$ the sheaf ideal generated by $f_1, ..., f_k$. That is, it contains all holomorphic functions where we identify functions if they are equal on $V(f_1, ..., f_k)$. Then we equip $V(f_1, ..., f_k)$ with the sheaf $\mathcal{O}_U/\mathcal{I}$.

5.26 Definition. An *analytic space* is a locally ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic (as locally ringed spaces) to some $V(f_1, ..., f_n)$.

An analytic mapping of analytic spaces is a morphism of locally ringed spaces. The category of analytic spaces will be denoted as **An**.

More information can be found in Grauert, Remmert [22]

5.6   Diffeological and smooth Spaces

Diffeological spaces form a useful generalization of smooth manifolds. It gives us a notion of smoothness and differentiability for spaces which are not necessarily locally Euclidean, or more generally: locally a topological vector space. The category of diffeological spaces is very well-behaved. Unlike smooth manifolds, it is closed under quotients, exponentiation, arbitrary products, etc. The smooth manifolds form a full subcategory of the diffeological spaces. But there is more: even infinite dimensional manifolds can be seen as diffeological spaces in a canonical manner.

5.27 Definition. Let X be a set, let $n \in \mathbb{N}$ and let $U \subseteq \mathbb{R}^n$ be an open set. A function $f : U \to X$ for some $n \in \mathbb{N}$ is called a *parametrization* of X. If we want to emphasize the dimension n then we will say that f is an *n-parametrization*.

5.28 Definition. Let X be a nonempty set. A *diffeology* on X is any set $\mathcal{D}$ of parametrizations of X such that the following axioms are satisfied:

1. The covering axiom: The set $\mathcal{D}$ contains all constant functions $f : \mathbb{R}^n \to X : r \to x$ for each $n \in \mathbb{N}$ and $x \in X$.

2. The locality axiom: Let $f : U \to X$ be a parametrization. If for every point $r \in U$, there exists an open neighborhood V of r such that $f|_V$ belongs to $\mathcal{D}$, then the parametrization f belongs to $\mathcal{D}$.

3. The axiom of smooth compatibility: For every element $f : U \to X$ of $\mathcal{D}$, for every open set V of some Euclidean space $\mathbb{R}^m$ and for each smooth (= infinitely differentiable) map $F : U \to V$, we have that $F \circ f$ belongs to $\mathcal{D}$.

A *diffeological space* is a set equipped with a diffeology. The elements of the diffeology $\mathcal{D}$ are called *plots* or *n-plots* if we wish to emphasize the dimension of the parametrization.

Of course, open subsets of a Euclidean space $\mathbb{R}^n$ should form diffeological spaces:

5.29 Example. Let $n \in \mathbb{N}$ and let $U \subseteq \mathbb{R}^n$ be open. The set of all parametrizations of U which are smooth forms a diffeology on U. We call this the standard diffeology on U.

Even more generally, every smooth manifold defines a diffeological space.

5.30 Example. Let M be a smooth manifold (not necessarily Hausdorff or second countable). The collection of all parametrizations of M which are smooth in the sense of the theory of smooth manifolds is a diffeology on M.

It is not difficult to define the correct smooth maps between diffeological spaces.

5.31 Definition. Let $(X, \mathcal{D})$ and $(X', \mathcal{D}')$ be diffeological spaces. A map $f : X \to X'$ is called *smooth* if for each plot $P \in \mathcal{D}$ it we have that $f \circ P$ is a plot in $\mathcal{D}'$.

It is easy to prove that with this definition the diffeological spaces form a category, call **DiffSp**. It is also easy to see that if $(X, \mathcal{D})$ is a diffeological space, then all plots $P \in \mathcal{D}$ are smooth maps. More generally:

5.32 Example. Let M and N be smooth manifolds and let $F : M \to N$ be a function. Then F is smooth smooth map in the sense of the theory of smooth manifolds if and only if it smooth map in the sense of the above definition. Thus the smooth manifolds form a full subcategory of the category of diffeological spaces.

We can even put a diffeological structure on arbitrary Banach spaces (and thus also on Banach manifolds) which is compatible with the differentiation in Banach spaces:

5.33 Example. Let $(E, \| \ \|)$ be a Banach space. Recall that if $f : U \to E$ is a continuous n-parametrization on E, then f is said to be of class $\mathcal{C}^1$ if the quantity

$$Df(x)(u) := \lim_{h \to 0} \frac{f(x + tu) - f(x)}{t}$$

exists for each $x \in U$ and $u \in \mathbb{R}^n$ and if the map $Df(x) : \mathbb{R}^n \to E$ is a continuous linear map. The map f is said to be of class $\mathcal{C}^k$, $k > 1$

if f is of class $\mathcal{C}^1$ a,d if $D(f) : U \to \mathcal{L}(\mathbb{R}^n, E)$ is of class $\mathcal{C}^{k-1}$. The map f is said to be smooth or of class $\mathcal{C}^\infty$ if it is of class $\mathcal{C}^k$ for each $k > 0$.

The set of parametrizations of E which are smooth in this sense form a diffeology. We call it the Banach diffeology on $(E, \| \; \|)$.

A map between Banach spaces in then smooth in the Banach space sense if and only if it is smooth in the diffeology sense. Thus the Banach spaces form a full subcategory of the diffeological spaces.

Every diffeological space induces a topology in a canonical way:

5.34 Definition. Let $(X, \mathcal{D})$ be a diffeological space. The *D-topology* on X is by definition the final topology on X with respect to the plots $P \in \mathcal{D}$.

There is a faithful functor from the category of diffeological spaces to the category of topological spaces. Indeed, every smooth map between diffeological spaces is continuous.

Diffeological spaces can be seen as certain sheaves on a site. Indeed, define the category **CartSp** whose objects are open subsets of $\mathbb{R}^n$ and whose morphisms are smooth maps. This category has a Grothendieck topology in a canonical way. Indeed, we say that $\{f_i : U_i \to U\}$ cover U if $\bigcup_i f_i(U_i) = U$. This site is even a concrete site, meaning that:

5.35 Definition. A *concrete site* $\mathcal{C}$ is a site with terminal object 1 such that

1. Every representable presheaf on the site is a sheaf

2. The functor $\mathrm{Hom}_{\mathcal{C}}(1, -) : \mathcal{C} \to \mathbf{Set}$ is faithful

3. A family $(f_i : X_i \to X)_i$ is a covering family if and only if $(\mathrm{Hom}_\mathcal{C}(1, f_i) : \mathrm{Hom}_\mathcal{C}(1, X_i) \to \mathrm{Hom}_\mathcal{C}(1, X))_i$ is jointly surjective.

Intuitively, given an object D in a concrete site $\mathcal{C}$, we say that $\mathrm{Hom}_\mathcal{C}(1, D)$ are the set of ṕoints" in D. The axioms now say that two morphisms $f : D \to C$ are equal if they agree on the underlying points and that the covering condition can be verified on the underlying set of points.

Any sheaf F on a concrete site $\mathcal{C}$ has an underlying set of points, namely $F(1)$. Furthermore, an element $\varphi \in F(D)$ automatically induces a function $\widetilde{\varphi} : \mathrm{Hom}_\mathcal{C}(1, D) \to F(1)$ by sending $\widetilde{\varphi}(d) = F(d)(\varphi)$. A concrete sheaf is a sheaf where these functions determine $F(D)$ precisely:

5.36 Definition. Let $\mathcal{C}$ be a concrete site. Then a sheaf F on $\mathcal{C}$ is said to be a *concrete sheaf* if the function sending $\varphi \in F(D)$ to $\widetilde{\varphi}$ is injective.

A diffeological space now has a very simply description. Indeed, a diffeological space is exactly the category of concrete sheaves on the site **CartSp**.

One can of course also work with the category of all sheaves on **CartSp**. A sheaf F on **CartSp** is called a *smooth space*. Thus the category $\mathbf{Sh}(\mathbf{CartSp})$ is commonly called the category of smooth spaces. This category is a topos.

For more information, see Baez and Hoffnung [2].

5.7 Lie groups

5.37 Definition. A smooth manifold G is a *Lie group* if G is also a group such that the operations

$$m : G \times G \to G : (x, y) \to xy \text{ and } i : G \to G : x \to x^{-1}$$

are smooth.

We say that a group homomorphism $f : G \to H$ is a *Lie group homomorphism* if f is smooth. This makes the class of Lie groups into a category, which we call **LieGrp**. We collect some facts about Lie groups:

5.38 Theorem. *Every continuous group homomorphism between Lie groups is smooth.*

5.39 Theorem. *If $f : G \to H$ is a Lie group homomorphisms between Lie groups, then $f(G)$ is an immersed submanifold of H and thus also a Lie group.*

This defines a good notion of Lie subgroups, namely: G is a Lie subgroup of a Lie group H if G is a Lie group and if the canonical inclusion $i : G \to H$ is a Lie group homomorphism. The following important theorem states gives a useful criterion when the inclusion is an embedding:

5.40 Theorem (Closed subgroup theorem)**.** *Let H be an algebraic subgroup of a Lie group G. Then the following are equivalent:*

1. *H is closed.*

2. *H is embedded.*

3. *H is an embedded Lie subgroup of G.*

This theorem also can be used to show that the category of Lie groups admits pullbacks. Indeed, the pullback of $f : G \to H$ and $g : G' \to H$ is $\{(g, h) \in G \times H \mid f(g) = g(h)\}$ which is clearly a closed subgroup of $G \times H$.

More information and proofs can be found in Hilgert, Neeb [28].

5.8 C^*-**algebras**

5.41 Definition. A C^*-*algebra* is a (possibly noncommutative, possibly nonunital) $\mathbb{C}$-algebra equipped with a norm $\| \|$ and an involution $* : A \to A$ such that

1. $(\lambda x + \mu y)^* = \overline{\lambda} x^* + \overline{\mu} y^*$ for each $\lambda, \mu \in \mathbb{C}$ and $x, y \in A$.

2. $(xy)^* = y^* x^*$ for each $x, y \in A$.

3. $x^{**} = x$ for each $x \in A$.

4. A is complete with respect to the norm $\| \|$.

5. $\|xy\| \leq \|x\|\|y\|$ for each $x, y \in A$.

6. $\|xx^*\| = \|x\|^2$ for each $x \in A$.

A C^*-algebra is called *unital* if it is unital as an algebra, i.e. if there is a unit element $1_A \in A$ such that $1_A a = a = a 1_A$ for each $a \in A$.

Here are the main examples:

5.42 Examples.

1. Let H be a Hilbert space, then the set $\mathcal{B}(H)$ of all bounded linear operators $H \to H$ is a C^*-algebra.

2. Let $(X, \mathcal{T})$ be a compact Hausdorff space, then the space $\mathcal{C}(X, \mathbb{C})$ of continuous functions $X \to \mathbb{C}$ is a C^*-algebra.

3. Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. We say that a function $f : X \to \mathbb{C}$ vanishes at infinity if for each $\varepsilon > 0$, there exists some compact set $K_\varepsilon \subseteq X$ such that $f(x) < \varepsilon$ for each $x \notin K_\varepsilon$. Then the space $\mathcal{C}_0(X, \mathbb{C})$ of all continuous functions $f : X \to \mathbb{C}$ that vanish at infinity is a C^*-algebra.

4. Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. Then the space

$$\mathcal{C}_b(X, \mathbb{C}) = \{f : X \to \mathbb{C} \mid f \text{ is continuous and bounded}\}$$

is a C^*-algebra.

We can make the C^*-algebras into a category:

5.43 Definition. Let A and B be C^*-algebras and let $f : A \to B$ be a function. We say that f is a C^*-morphism if

1. f is linear,

2. $f(xy) = f(x)f(y)$ for each $x, y \in A$,

3. $f(x^*) = f(x)^*$ for each $x \in A$.

If A and B are unital, then we say that f is unital if $f(1_A) = 1_B$.

One can show that each C^*-morphism is continuous and has norm smaller or equal than 1. This choice of morphism makes the C^*-algebras into a category, which we will call C^*-**Alg**$_0$. The C^*-algebras can all be embedded in $\mathcal{B}(H)$ for some Hilbert space H:

5.44 Theorem (Gelfand–Naimark–Segal)**.** *Let A be a C^*-algebra, then there exists a Hilbert space H and an isometric embedding $i : A \to \mathcal{B}(H)$.*

The important theorem of Gelfand–Naimark classifies the commutative C^*-algebras.

5.45 Theorem (Gelfand–Naimark)**.** *For each commutative C^*-algebra A, there exists a locally compact Hausdorff space X such that A is isomorphic to $\mathcal{C}_0(X, \mathbb{C})$.*
A unital commutative C^*-algebra A is isomorphic to

$$\mathcal{C}(X, \mathbb{C}) = \mathcal{C}_0(X, \mathbb{C})$$

for some compact Hausdorff space X.

Sadly, the continuous functions between locally compact Hausdorff spaces do not exactly correspond to C^*-morphisms. However, we do have that the *proper* continuous functions correspond exactly to the C^*-morphisms. To be able to model all continuous functions, we need the concept of the multiplier algebra.

5.46 Definition. Let A be a C^*-algebra. We say that an ideal J of A is an *essential ideal* if every other nonzero ideal in A has nonzero intersection with A. Or equivalently, when the annihalator $J^\perp = \{a \in A \mid aJ = 0\}$ is zero.

5.47 Definition. A *unitization* of a C^*-algebra A is an embedding of A into a unital C^*-algebra B such that A is an essential ideal of B.

Since unital C^*-algebras correspond to compact Hausdorff spaces in Theorem 5.45, we see that the concept unitization corresponds to compactification. The largest compactification is of course the Čech-Stone compactification. Likewise, every C^*-algebra has a largest unitization, which we will call the multiplier algebra.

5.48 Theorem. *Let A be a C^*-algebra, then there exists a unique unitization $\mathcal{M}(A)$ of A such that if A is an ideal of a C^*-algebra B,*

then there exists a unique morphism $\mu : B \to \mathcal{M}(A)$ such that μ is the identity in A. Moreover, μ is injective iff A is essential in B.
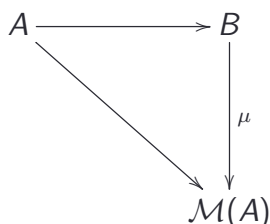
We say that $\mathcal{M}(A)$ is the multiplier algebra *of A.*

If $A = \mathcal{C}_0(X, \mathbb{C})$ is a commutative C^*-algebra, then its multiplier algebra is exactly $\mathcal{C}(\beta X, \mathcal{C}) \cong \mathcal{C}_b(X, \mathcal{C})$, where βX is the Cech-Stone compactification of X.

Sadly, the multiplier algebra is not functorial. More precisely, if $f : A \to B$ is a C^*-morphism, then this does not need to induce a morphism $\mathcal{M}(A) \to \mathcal{M}(B)$.

5.49 Definition. Let A be a C^*-algebra. Let H be a Hilbert space such that A can be seen as a subspace of $\mathcal{B}(H)$ (this exists by the theorem of Gelfand–Naimark–Segal). The *strict topology* on A is the locally convex topology generated by the seminorms $x \to \|xa\|$ and $x \to \|ax\|$ for $x \in \mathcal{B}(H)$ and $a \in A$.

One can prove that the multiplier algebra $\mathcal{M}(A)$ is the strict completion of A. Also, we are now able to state the following extension theorem:

5.50 Theorem. *Let A be a C^*-algebra and let $f : A \to \mathcal{M}(B)$ be a C^*-morphism. Then the following conditions are equivalent:*

1. *f is* nondegenerate, *meaning that $f(A)B$ is dense in B.*

2. $f(A)B = B$.

3. $Bf(A) = B$.

4. f is the restriction to A of a unique unital C^*-morphism $\mathcal{M}(A) \to \mathcal{M}(B)$ which is strictly continuous on the unit ball.

A C^*-morphism that satisfies the conditions of the previous theorem will be called a *Woronowicz-morphism* from A to B. The continuous maps between locally compact Hausdorff spaces X and Y correspond exactly to the Woronowicz-morphisms from $\mathcal{C}_0(Y, \mathbb{C})$ to $\mathcal{C}_0(X, \mathbb{C})$. With this in mind, we define the category C^*-**Alg** to be the category of all C^*-algebras with Woronowicz-morphisms (note that the above theorem tells us that we can compose such morphisms). Naturally, we are more interested in the opposite category C^*-**Alg**$^{\text{op}}$ since this category can be seen as a category of "noncommutative topological spaces."

Recall that the multiplier algebra was not functorial with respect to the C^*-morphisms. This is solved by the Woronowicz-morphisms: any Woronowicz-morphism from A to B gives rise to a unique unital C^*-morphism $A \to \mathcal{M}(B)$.

We would like to form tensor products of C^*-algebras A and B. An obvious definition is to form the algebraic tensor product $A \otimes B$ and to complete this with respect to a suitable norm. A suitable norm should satisfy $\|x \otimes y\| \leq \|x\|\|y\|$ (this is called a *subcross norm*). Sadly, the situation is complicated because there are many different subcross norms on $A \otimes B$. There are two interesting subcross norms on $A \otimes B$. The smallest possible subcross norm is called the spatial norm:

5.51 Definition. Let $\pi_A : A \to \mathcal{B}(H_1)$ and $\pi_B : B \to \mathcal{B}(H_2)$ be faithful representations of C^*-algebras. Then one can show that $\pi_A \otimes$

$\pi_B : A \otimes B \to \mathcal{B}(H_1 \otimes H_2)$ is a faithful representation. This induces a norm on $A \otimes B$ which we will call the *spatial norm*. The completion of $A \otimes B$ with respect to this norm is called the *spatial tensor product* and is denoted by $A \otimes_\sigma B$.

Another subcross seminorm is the largest possible one:

5.52 Definition. Let A and B be C^*-algebras and define for each $t \in A \otimes B$

$$\mu(t) = \sup\{\alpha(t) \mid \alpha \text{ is a } C^*\text{-seminorm on } A \otimes B\}.$$

This is called the *maximal C*-norm* on $A \otimes B$. The completion with respect to this norm is denoted by $A \otimes_\mu B$.

For some C^*-algebras, all subcross norms coincide, we call these *spatial C*-algebras*. All commutative C^*-algebras are spatial and in fact we have that $C_0(X) \otimes C_0(Y) \cong C_0(X \times Y)$.

5.53 Definition. Let A, B and C be (possibly noncommutative, possibly nonunital) rings and let $f : A \to C$ and $g : B \to C$ be ring-morphisms. We say that f and g commute if for each $a \in A$ and $b \in B$ holds that $f(a)g(b) = g(b)f(a)$.

Tensor products do not form a coproduct for C^*-algebras, but we do have the following theorem

5.54 Theorem. Let $f_k : A_k \to B_k$, $k = 1, 2$ be morphisms of C^*-algebras, then $f_1 \otimes f_2$ extends by continuity to morphisms $f_1 \otimes_\sigma f_2 : A_1 \otimes_\sigma A_2 \to B_1 \otimes_\sigma B_2$ and $f_1 \otimes_\mu f_2 : A_1 \otimes_\mu A_2 \to B_1 \otimes_\mu B_2$.
If $g_k : A_k \to C$, $k = 1, 2$ are commuting morphisms of C^*-algebras, then $g_1 \otimes g_2$ extends by continuity to a morphism $g_1 \otimes_\mu g_2 : A_1 \otimes_\mu A_2 \to C$.

We collect some facts about the category C^*-**Alg**:

5.55 Proposition. *If $f : A \to \mathcal{M}(B)$ is a Woronowicz-morphism which implements an isomorphism in C^*-**Alg**, then $f(A) = B$.*

Proof. Let $g : B \to \mathcal{M}(A)$ be the inverse of f. Take an $a \in A$, then we have by nondegeneracy of g that there exist $b_i \in B$ and $a_i \in A$ such that $a = \sum g(b_i)a_i$. But then $f(a) = \sum b_i f(a_i)$. But since B is an ideal of $\mathcal{M}(B)$, this implies that $f(a) \in B$. This shows that $f(A) \subseteq B$. Likewise, we have that $g(B) \subseteq A$. Now take $b \in B$, then $b = f(g(b))$, and thus $b \in f(A)$. ☐

5.56 Theorem. *The co-equalizer in the category C^*-**Alg** exists. In particular, the co-equalizer of two nondegenerate C^*-morphisms $f, g : A \to \mathcal{M}(B)$ is given by B/J, where J is the two-sides ideal in B generated by $f(a)b - g(a)b$ and $bf(a) - bg(a)$ for each $a \in A$ and $b \in B$.*

Proof. Let $p : B \to B/J$ be the canonical surjection. This surjection extends to $\overline{p} : \mathcal{M}(B) \to \mathcal{M}(B/J)$ (see [59]) which is clearly strictly continuous and thus a Woronowicz-morphism. We prove that $pf = pg$. Let $(b_i)_{i \in I}$ be a net in B which converges strictly to the unit in $\mathcal{M}(B)$ (this is called an approximate unit and exists in each C^*-algebra). By strict continuity, we know that $(\overline{p}(b_i))_{i \in I}$ converges to the unit in $\mathcal{M}(B/J)$ (see [59]). Thus follows that $f(a)b_i$ converges strictly to $f(a)$ and that $g(a)b_i$ converges strictly to $g(a)$. By definition of C as a quotient space, we have that $\overline{p}(f(a)b_i) = \overline{p}(g(a)b_i)$ and since $\overline{p}(f(a)b_i)$ converges to $\overline{p}(f(a))$ and $\overline{p}(g(a)b_i)$ converges to $\overline{p}(g(a))$, we have that $\overline{p}(f(a)) = \overline{p}(g(a))$.

Let $h : \mathcal{B} \to \mathcal{M}(C)$ be a strictly continuous C^*-morphism such that $hf = hg$. Then $h(J) = 0$ and thus there is a unique morphism $k : B/J \to \mathcal{M}(C)$ such that $h = kp$. This is clearly nondegenerate since h is. ☐

5.57 Proposition. *Let $f : A \to \mathcal{M}(C)$ and $g : B \to \mathcal{M}(C)$ be two Woronowicz-morphisms and let $\overline{f} : \mathcal{M}(A) \to \mathcal{M}(C)$ and $\overline{g} : \mathcal{M}(B) \to \mathcal{M}(C)$ be its unique extensions to unital C^*-morphisms. Then if f and g commute, then so do $\overline{f}$ and $\overline{g}$.*

Proof. Take $x \in \mathcal{M}(A)$ and $y \in \mathcal{M}(B)$. Then there exist nets $(a_i)_{i \in I}$ in A and $(b_j)_{j \in J}$ in B such that $a_i \to x$ strictly and $b_i \to y$ strictly. But then $f(a_i)$ converges strictly to $f(x)$ and $g(b_j)$ converges strictly to $g(y)$. Then $f(a_i)g(b_j)$ converges strictly to $\overline{f}(x)\overline{g}(y)$ and $g(b_j)f(a_i)$ converges strictly to $\overline{g}(y)\overline{f}(x)$. Since $f(a_i)g(b_j) = g(b_j)f(a_i)$, it follows that $\overline{f}(x)\overline{g}(y) = \overline{g}(y)\overline{f}(x)$. □

5.58 Proposition. *Let $m : A \to \mathcal{M}(C)$ be an injective and nondegenerate C^*-morphism. Then m extends uniquely to an isometric, unital C^*-morphism $\overline{m} : \mathcal{M}(A) \to \mathcal{M}(C)$. Furthermore, the image of $\overline{m}$ is given by the* idealizer *of $m(A)$, i.e.*

$$\overline{m}(\mathcal{M}(A)) = \{x \in \mathcal{M}(C) \mid xm(A) \subseteq m(A) \text{ and } m(A)x \subseteq m(A).$$

Proof. See Lance [37]. □

More information and proofs can be found in Wegge-Olsen [59] and Lance [37]

5.9 Associative algebras

We wish to adapt the situation of C^*-algebras to algebras over a general base field. The basic notions of the generalization can be found in De Commer, Van Daele [10]. Let k be a commutative ring with unit.

5.59 Definition. An k-algebra A is a *nondegenerate k-algebra* if for each $a \in A$ and for each k-ideal $I \subseteq A$, the following two conditions hold:

1. If $ab = 0$ for each $b \in A$, then $a = 0$.

2. If $ba = 0$ for each $b \in A$, then $a = 0$.

3. We have that $A = AA = \{\sum_i a_i a'_i \mid a_i, a'_i \in A\}$.

The category of all nondegenerate k-algebras with the usual k-algebra morphism is denotes as $k - \mathbf{Alg}_0$.

5.60 Definition. The *multiplier algebra* of a nondegenerate k-algebra A is the algebra $\mathcal{M}(A)$ consisting of couples (ρ, λ), where ρ and λ are k-linear maps $A \to A$ such that

$$a\rho(b) = \lambda(a)b \text{ for all } a, b \in A.$$

This k-algebra is a k-unital algebra equipped with the following operations:

1. For all $(\rho, \lambda), (\rho', \lambda') \in \mathcal{M}(A)$ and $r, s \in A$, we set

$$r(\rho, \lambda) + s(\rho', \lambda') = (r\rho + s\rho', r\lambda + s\lambda').$$

2. For all $(\rho, \lambda), (\rho', \lambda') \in \mathcal{M}(A)$, we set

$$(\rho, \lambda) \cdot (\rho', \lambda') = (\rho\rho', \lambda'\lambda).$$

3. The unit is given by $(\mathrm{Id}_A, \mathrm{Id}_A)$.

There is a natural k-algebra morphism $\iota : A \to \mathcal{M}(A) : a \to (\rho_a, \lambda_a)$ where

$$\rho_a(b) = ab \text{ and } \lambda_a(b) = ba.$$

5.61 Proposition. *The map ι is injective. Furthermore, we have for each $(\rho, \lambda) \in \mathcal{A}$ that*

$$(\rho, \lambda) \cdot \iota(a) = \iota(\rho(a)) \text{ and } \iota(a) \cdot (\rho, \lambda) = \iota(\lambda(a)).$$

Thus $\iota(A)$ is an ideal of $\mathcal{M}(A)$. Hence if A is unital, then ι is an isomorphism.
Furthermore, it follows that if $(\rho, \lambda) \cdot \iota(a) = 0$ for each $a \in A$, then $(\rho, \lambda) = 0$.

Proof. Assume that for an $a \in A$, we have that $\iota(a) = 0$. This means that $\rho_a = 0$ and $\lambda_a = 0$. Hence for each $b \in A$, we have that $ab = 0$ and $ba = 0$. The nondegeneracy condition then implies that $a = 0$. Thus ι is injective.
Furthermore, we have that

$$(\rho, \lambda) \cdot \iota(a) = (\rho, \lambda) \cdot (\rho_a, \lambda_a) = (\rho \rho_a, \lambda_a \lambda).$$

For $b \in A$, we have that

$$(\rho \rho_a)(b) = \rho(ab).$$

Now for each $c \in A$, we have that $c\rho(ab) = \lambda(c)ab$ and $c\rho(a) = \lambda(c)a$. Hence

$$c\rho(ab) - c\rho(a)b = \lambda(c)ab - \lambda(c)ab = 0.$$

Hence by nondegeneracy, we have that $\rho(ab) = \rho(a)b$. Thus follows that $\rho \rho_a = \rho_{\rho(a)}$.
We also have for each $b \in A$, that

$$(\lambda_a \lambda)(b) = \lambda(b)a = b\rho(a) = \lambda_{\rho(a)}(b).$$

Thus $(\rho, \lambda) \cdot \iota(a) = \iota(\rho(a))$. Analogously, it follows that $\iota(a) \cdot (\rho, \lambda) = \iota(\lambda(a))$. $\qquad \square$

In what follows, we will usually not explicitly write ι. Thus we view A as a subset of $\mathcal{M}(A)$.

5.62 Definition. Let A and B be nondegenerate k-algebras. We say that an algebra morphism $f : A \to M(B)$ is nondegenerate if $B \subseteq f(A)B$ and $B \subseteq Bf(A)$. A *Woronowicz-morphism* between A and B is then defined as a nondegenerate algebra morphism $f : A \to \mathcal{M}(B)$.

5.63 Proposition. *Let $f : A \to \mathcal{M}(B)$ be a Woronowicz-morphism, then it can be extended to a unique unital algebra morphism $\mathcal{M}(A) \to \mathcal{M}(B)$. We will denote this extension again by f.*

Proof. Let $(\rho, \lambda) \in \mathcal{M}(A)$. Every $b \in B$ can be written as $b = \sum_i f(a_i)b_i$. Now we define $\rho'(b) = \sum f(\rho(a_i))b_i$. This is well-defined because if $b = \sum f(a_j')b_j'$ and if $b''' f(a'') \in B$ is arbitrary, then

$$b''' f(a'')\left(\sum_i f(\rho(a_i))b_i - \sum_j f(\rho(a_j'))b_j'\right)$$
$$= b''' \sum_i f(a'' \rho(a_i))b_i - \sum_j b''' f(a'' \rho(a_j'))b_j'$$
$$= b''' \sum_i f(\lambda(a'')a_i)b_i - b''' \sum_j f(\lambda(a'')a_j')b_j'$$
$$= b''' f(\lambda(a''))\left(\sum_i f(a_i)b_i - \sum_j f(a_j')b_j'\right)$$
$$= b''' f(\lambda(a''))(b - b)$$
$$= 0.$$

Since B is generated by elements of the form $b''' f(a'')$, it follows from nondegeneracy of B that $\rho'(b)$ is well-defined. Similarly, we can write every $b \in B$ as $b = \sum_i b_i f(a_i)$ and it follows that $\lambda'(b) = \sum b_i f(\lambda(a_i))$ is well-defined. It is obvious that ρ' and λ' are multipliers. Thus $f' : \mathcal{M}(A) \to \mathcal{M}(B) : (\rho, \lambda) \to (\rho', \lambda')$ is well-defined. It is easy to check that this is an algebra morphism.

Assume that $f'' : \mathcal{M}(A) \to \mathcal{M}(B)$ is another extension of f. Then we have for each $x \in \mathcal{M}(A)$ and $b'f'(a') = b'f(a') \in B$ that $a'x \in A$ (since A is an ideal in $\mathcal{M}(A)$). Hence

$$b'f(a')(f(x) - f'(x)) = b'f(a'x) - b'f'(a'x) = 0$$

Thus for each $b \in B$ holds that $b(f(x) - f'(x)) = 0$. By Proposition 5.61 then follows that $f(x) = f'(x)$. $\square$

This proposition has as corollary that we can compose nondegenerate maps. Indeed, if $f : A \to \mathcal{M}(B)$ and $g : B \to \mathcal{M}(C)$ are nondegenerate, then g extends to a morphism $g : \mathcal{M}(B) \to \mathcal{M}(C)$. The composition gf then makes sense.

5.64 Lemma. *The composition gf is nondegenerate.*

Proof. Take $c \in C$, then by nondegeneracy of g, we can write $c = \sum_i g(b_i)c_i$, where $b_i \in B$ and $c_i \in C$. Now, since f is nondegenerate, we can write $b_j = \sum_j f(a_{i,j})b_{i,j}$, where $a \in A$ and $b' \in B$. Thus $c = \sum_{i,j} g(f(a_{i,j}))g(b_{i,j})c_i$ and $g(b_{i,j})c_i \in C$. Thus we have shown that $g(f(A))C = C$. Similarly, it follows that $Cg(f(A)) = C$. $\square$

We can now form a category of all k-algebras with Woronowicz morphisms. We will denote this category as $k - \mathbf{Alg}$.

5.65 Proposition. *Let $f : A \to \mathcal{M}(B)$ be an injective, nondegenerate map, then its extension $\mathcal{M}(A) \to \mathcal{M}(B)$ is injective too.*

Proof. Assume that $x \in \mathcal{M}(A)$ satisfies that $f(x) = 0$. Then for each $a \in A$, we have that $f(ax) = f(a)f(x) = 0$. Since f is injective on A and since $ax \in A$, it follows that $ax = 0$. This holds for each $a \in A$, thus follows from Proposition 5.61 that $x = 0$. $\square$

5.66 Proposition. *For every surjective algebra morphism $g : A \to B$, we have that $\iota g : A \to \mathcal{M}(B)$ is nondegenerate and that this extends to a morphism $\mathcal{M}(A) \to \mathcal{M}(B)$.*

Proof. Take $b \in B$, then there exists an $a \in A$ such that $g(a) = b$. We can write $a = \sum_i a'_i a''_i$. Hence $b = \sum_i g(a'_i) g(a''_i)$. Since $g(A) \subseteq B$, it follows that $g(a'_i), g(a''_i) \in B$. Hence $b \in g(A)B$ and $b \in Bg(A)$. $\square$

5.67 Proposition. *If $f : A \to \mathcal{M}(B)$ is a Woronowicz-morphism which implements an isomorphism in $k - \mathbf{Alg}$, then $f(A) = B$.*

Proof. Let $g : B \to \mathcal{M}(A)$ be the inverse of f. Take an $a \in A$, then we have by nondegeneracy of g that there exists $b_i \in B$ and an $a_i \in A$ such that $a = \sum g(b_i)a_i$. But then $f(a) = b_i f(a_i)$. But since B is an ideal of $\mathcal{M}(B)$, this implies that $f(a) \in B$. This shows that $f(A) \subseteq B$. Likewise, we have that $g(B) \subseteq A$. Now take $b \in B$, then $b = f(g(b))$, and thus $b \in f(A)$. $\square$

The quotients of nondegenerate algebra's do not need to be nondegenerate. But we do have the following

5.68 Proposition. *Let A be a nondegenerate algebra and let I be an ideal of A. We let $J = \{a \in A \mid ab \in I \text{ and } ba \in I \text{ for all } b \in B\}$. Then*

1. *A/J is nondegenerate.*

2. *If $f : A \to \mathcal{M}(B)$ is a nondegenerate morphism between nondegenerate algebras such that $f(I) = 0$. Then follows that $f(J) = 0$ and thus f induces a nondegenerate morphism $f : A/J \to \mathcal{M}(B)$.*

Proof.

1. Assume that $p(x)p(y) = 0$ for all $y \in A$. Then holds that $xy \in J$ for all $y \in A$. It follows that for each $y' \in A$ that $xyy' \in I$. But since A is nondegenerate, we know that every

element of A has the form $\sum_i y_i y_i'$. So for every $a \in A$, we have that $xa \in I$. Thus follows that $x \in J$ and thus $p(x) = 0$. Since $A = AA$, it follows at once that $A/J = (A/J)(A/J)$.

2. Take $x \in J$. Then holds for each $a \in A$ that $ax \in I$ and $xa \in I$. Take $b \in B$, then by nondegeneracy, we can write $b = \sum_i f(a_i)b_i$. But then

$$f(x)b = \sum_i f(x)f(a_i)b_i = \sum_i f(xa_i)b_i = 0.$$

By Proposition 5.61 follows that $f(x) = 0$. The remainder of the proof is now clear.

$\square$

In the context of nondegenerate algebras and Woronowicz-morphisms, we now redefine A/I to mean A/J (with notations as in the previous theorems).

5.69 Theorem. *The co-equalizer in the category $k - $ **Alg** exists. In particular, the co-equalizer of two nondegenerate morphisms $f, g : A \to \mathcal{M}(B)$ is given by B/J, where J is the two-sided ideal in B generated by $f(a)b - g(a)b$ and $bf(a) - bg(a)$ for each $a \in A$ and $b \in B$.*

Proof. Since B is nondegenerate, it follows that B/J is nondegenerate. Let $p : B \to B/J$ be the canonical surjection. This surjection is non-degenerate and thus extends to a surjection $p : \mathcal{M}(B) \to \mathcal{M}(B/J)$. Let $x = (\rho, \lambda) \in \mathcal{M}(B)$ and $p(b) \in B/J$. Then $p(x) = (\rho', \lambda')$, where $\rho'(p(b)) = p(\rho(b))$ and $\lambda'(p(b)) = p(\lambda(b))$.

We now prove that $pf = pg$. Indeed, take $a \in A$, then

$$p(f(a)), p(g(a)) \in \mathcal{M}(B/J).$$

Take $p(b) \in B/J$, then

$$[p(f(a)) - p(g(b))](p(b)) = p(f(a)b - g(a)b) = 0.$$

Hence $p(f(a)) = p(g(a))$ for each $a \in A$.

Let $h : B \to \mathcal{M}(C)$ be a nondegenerate morphism such that $hf = hg$. Then $h(J) = 0$ and thus there is a unique morphism $k : B/J \to \mathcal{M}(C)$ such that $h = kp$. This is clearly nondegenerate since h is. $\square$

5.70 Proposition. *Let $f : A \to \mathcal{M}(C)$ and $g : B \to \mathcal{M}(C)$ be two Woronowicz-morphisms and let $f' : \mathcal{M}(A) \to \mathcal{M}(C)$ and $g' : \mathcal{M}(B) \to \mathcal{M}(C)$ be its unique extensions to unital morphisms. Then if f and g commute, so do f' and g'.*

Proof. Take $x \in \mathcal{M}(A)$ and $y \in B$. Take $c \in C$, then we can write $c = \sum_i c_i f(a_i)$. Note that $a_i x \in A$. Then

$$
\begin{aligned}
cf(x)g(y) &= \sum_i c_i f(a_i x)g(y) \\
&= \sum_i c_i g(y)f(a_i x) \\
&= \sum_i c_i g(y)f(a_i)f(x) \\
&= \sum_i c_i f(a_i)f(y)g(x) \\
&= cg(y)f(x).
\end{aligned}
$$

By proposition 5.61 then follows that $f(x)g(y) = g(y)f(x)$. Now let $x \in \mathcal{M}(A)$ and $y \in \mathcal{M}(B)$. It follows similarly that $f(x)g(y) = g(y)f(x)$. $\square$

Let $\otimes$ be the usual tensor product of k-algebras.

5.71 Proposition. *Assume that k is a field, then if A and B are nondegenerate k-algebras, then so is $A \otimes B$.*

Proof. Assume for $a \otimes b$ we have that $aa' \otimes bb' = 0$ for all $a' \in A$ and $b' \in B$. This implies that there exists finitely supported maps $\lambda_{a',b'} : B \to R$ such that

$$bb' = \sum_{f \in B} \lambda_{a',b'}(f)f,$$

and

$$\lambda_{a',b'}(f)aa' = 0 \text{ for all } f \in B$$

If $\lambda_{a',b'}(f) = 0$ for all $f \in B$, then follows that $bb' = 0$. Otherwise, it follows by taking inverses that $aa' = 0$.
Thus if there exists some b' such that $bb' \neq 0$, then it must hold that $aa' = 0$ for all $a' \in A$. Thus by nondegeneracy of A, we have that $a = 0$ and thus $a \otimes b = 0$
Otherwise, it must hold that $bb' = 0$ for all $b' \in B$. But then $b = 0$ and hence $a \otimes b = 0$. $\square$

5.72 Proposition. *We have for nondegenerate algebra A and B that there exists commuting, nondegenerate injections*

$$j_A : A \to \mathcal{M}(A) \to \mathcal{M}(A \otimes B)$$

and

$$j_B : B \to \mathcal{M}(B) \to \mathcal{M}(A \otimes B).$$

Proof. Take $(\rho, \lambda) \in \mathcal{M}(A)$ and let $a \otimes b \in A \otimes B$, then we set $\rho'(a \otimes b) = \rho(a) \otimes b$ and $\lambda'(a \otimes b) = \lambda(a) \otimes b$. This defines an element $(\rho', \lambda') \in \mathcal{M}(A \otimes B)$. Assume that $\rho'(a \otimes b) = 0$ for each $a \otimes b \in A \otimes B$. Then for each $a \in A$ holds that $\rho(a) = \rho'(a \otimes 1) = 0$. Thus $\rho = 0$. This shows that we have an injection $\mathcal{M}(A) \to \mathcal{M}(A \otimes B)$. Similarly, we also have an injection $\mathcal{M}(B) \to \mathcal{M}(A \otimes B)$. These injections are

clearly commuting.

Note that for $(\rho_a, \lambda_a) \in \mathcal{M}(A)$, we have $\rho'_a(a' \otimes b') = aa' \otimes b'$ and $\lambda'_a = a'a \otimes b$. Thus $\rho'_a = \rho_{a \otimes 1}$ and $\lambda'_a = \lambda_{a \otimes 1}$. Thus we have that $j_A(a) = (\rho_{a \otimes 1}, \lambda_{a \otimes 1})$. We check nondegeneracy. Take $a \otimes b \in A \otimes B$. Then we can write $a = \sum_i a_i a'_i$. But then

$$a \otimes b = \sum_i (a_i \otimes 1)(a'_i \otimes b) \in j_A(A)(A \otimes B).$$

$\square$

Chapter 7

Factorization systems

In this chapter, we recall the notion of a factorization system on a category. This notion allows us to decompose a morphism $f : A \to B$ as $A \to f(A) \to B$. In particular, we get a good notion of subobjects and of images.

The standard theory of factorization systems is well known (see [1]). However, we find it useful to extend the theory of factorization systems. In particular, we define the notion of a generalized product and generalized pullback. The reason for this is that many categories (for example, the category of C^*-algebras with Woronowicz morphisms or the category of k-algebras from the previous chapter) do not have convenient pullbacks. On the other hand, these categories do have a tensor product that shares many of the convenient properties of products. We generalize this situation to arbitrary categories by introducing a relation R on the morphisms of the category. If we let all morphisms be R-related, then we get the usual product. If we only let the commuting morphisms be R-related, then we get the tensor product. Thus we see that the R-product generalizes both the standard product and the tensor product. In [33], Janelidze calls such

a relation R a *cover relation*, and he investigates cover relations obtained from factorization systems and cover relations obtained from monoidal structures. In this work, we are mainly interested in a relation R of the second type, and we go a step further by considering arbitrary R-pullbacks (apart from the usual R-products). With this notion at hand, we investigate R-pullback stability of the second class of morphisms of a factorization system (Proposition 7.14).

Finally, we apply the theory to the categories of the previous chapter.

7.1　Generalized pullbacks

Sometimes a certain category does not have useful products, or does not have products at all. For example, consider the category C^*-**Alg**$^{\text{op}}$, where it is not clear that there are products at all. This is why we will sometimes want to look at a construction that mimics the usual products.

Let $\mathcal{C}$ be a category and let C be an object in $\mathcal{C}$. The slice category $\mathcal{C}/C$ is the category of all morphisms with domain C. On every slice category, we now put a relation R with the following properties:

1. If $f : C \to A$ and $g : C \to B$ are R-related and if $h : B \to D$ is an arbitrary morphism, then f and hg are R-related.

2. If $f : C \to A$ and $g : C \to B$ are R-related and if $h : B \to C$ is an arbitrary morphism, then fh and gh are R-related.

The dual notion of such a collection of relations (which thus corresponds to a collection of relations for $\mathcal{C}^{\text{op}}$) is called a *cover relation* in [33].

Some examples are in order:

7.1 Examples.

1. If $\mathcal{C}$ is an arbitrary category, then we can say that $f : C \to A$ and $g : C \to B$ are always R-related. We will call this relation standard.

2. Let $f : C \to A$ and $g : C \to B$ be morphisms in C^*-**Alg**$^{\text{op}}$, these are represented by nondegenerate C^*-morphisms $F : A \to \mathcal{M}(C)$ and $G : B \to \mathcal{M}(C)$. We say that f and g are R-related if F and G commute.

3. Let **Rng**$_1$ be the category of all (possibly noncommutative) unital rings with as morphisms the usual ring homomorphisms. Two morphism $f : A \to B$ and $g : A \to C$ in **Rng**$_1^{\text{op}}$ are represented by ring homomorphisms $F : B \to A$ and $G : C \to A$. We say that f and g are R-related if F and G commute.

4. Two morphisms $f : C \to A$ and $g : C \to B$ in $k - $**Alg**$^{\text{op}}$ are represented by nondegenerate morphisms $F : A \to \mathcal{M}(C)$ and $G : B \to \mathcal{M}(C)$. We say that f and g are R-related if F and G commute.

With these examples in mind, we can say that an object A in $\mathcal{C}$ is a *commutative object* if the identity id_A is R-related to itself.

A relation R on the category can now be used to generalize the notion of a product and a pullback:

7.2 Definition. Let A and B be elements of a category $\mathcal{C}$. The R-*product* of A and B is an object P together with R-related morphisms $p_A : P \to A$ and $p_B : P \to B$ such that for each two R-related morphisms $q_A : Q \to A$ and $q_B : Q \to B$, there is a unique morphism $f : Q \to B$ such that $q_A = p_A f$ and $q_B = p_B f$.

Clearly, the R-product is (if it exists) unique up to isomorphism. We denote it by $A \times^R B$. We can give a similar definition for the R-pullback:

7.3 Definition. Let $f : A \to C$ and $g : B \to C$ be morphisms in a category $\mathcal{C}$. The R-*pullback* of f and g is an object P together with R-related morphisms $p_A : P \to A$ and $p_B : P \to B$ such that $f p_A = g p_B$ and such that for each two R-related morphisms $q_A : D \to A$ and $q_B : D \to B$ with $f q_A = g q_B$ we have that there is a unique morphism h such that $q_A = p_A h$ and $q_B = p_B h$.

Again, the R-pullback is (if it exists) unique up to isomorphism. We denote it by $A \times^R_C B$. It is clear that the R-pullback can be formed by taking the equalizer of an R-product.

If R is the standard relation, then the R-product and R-pullback are simply the usual product and pullback.

7.4 Remark. Note that in fact, it is possible to define arbitrary R-limits, of which the R-products and R-pullbacks are special instances.

7.5 Proposition. *Suppose $\mathcal{C}$ has R-products as well as categorical equalizers. Then $\mathcal{C}$ has R-pullbacks.*

Proof. Let $f : A \to C$ and $g : B \to C$ be morphisms. Let $A \times^R B$ be the R-product of A and B and let p_A and p_B be the morphisms associated with the product. Then the R-pullback is given by the equalizer of $f p_A$ and $g p_B$. The pullback property follows immediately from the universal property. $\qquad\square$

7.6 Theorem. *In the category C^*-$\mathbf{Alg}^{op}$, the R-products are given by the maximal tensor product.*

Proof. Let A and B be C^*-algebras. We can form the maximal tensor product $A \otimes_\mu B$. It is shown in Wegge-Olsen [59] that

$$A \otimes_\mu B \subseteq \mathcal{M}(A) \otimes_\mu \mathcal{M}(B) \subseteq \mathcal{M}(A \otimes_\mu B).$$

Thus the canonical maps $p_A : A \to \mathcal{M}(A) \otimes_\mu \mathcal{M}(B) : a \to a \otimes_\mu 1$ and $p_B : B \to \mathcal{M}(A) \otimes_\mu \mathcal{M}(B) : b \to 1 \otimes_\mu b$ extend to maps $\overline{p_A} : A \to \mathcal{M}(A \otimes_\mu B$ and $\overline{p_B} : B \to \mathcal{M}(A \otimes_\mu B)$. These maps are clearly commuting. Furthermore, they are strictly continuous since

$$\|p_A(a)(a' \otimes_\mu b')\| = \|(aa') \otimes_\mu b'\| \leq \|aa'\|\|b'\|.$$

Now let $q_A : A \to \mathcal{M}(C)$ and $q_B : B \to \mathcal{M}(C)$ be commuting nondegenerate morphisms. Then we define $h : A \otimes B \to \mathcal{M}(C)$ by $h(a \otimes b) = q_A(a)q_B(b)$. This is clearly the unique map for which $q_A = hj_A$ and $q_B = hj_B$. We must prove that h is nondegenerate. For this, take $c \in C$. Then we can write $c = \sum_i q_A(a_i)c_i$ and we can write $c_i = \sum_j q_B(b_{i,j})c_{i,j}$. Thus $c_i = \sum_{i,j} q_A(a_i)q_B(b_{i,j})c_{i,j} \in h(A \otimes B)C$. $\square$

7.7 Theorem. *In the category $k - \mathbf{Alg}^{op}$, the R-products are given by the tensor product.*

Proof. Let A and B be k-algebras. We can form the tensor product $A \otimes B$. We have shown that we have injections

$$j_A : A \to \mathcal{M}(A) \to \mathcal{M}(A \otimes B)$$

and

$$j_B : B \to \mathcal{M}(B) \to \mathcal{M}(A \otimes B)$$

which are nondegenerate and commuting.
Now let $q_A : A \to \mathcal{M}(C)$ and $q_B : B \to \mathcal{M}(C)$ be commuting nondegenerate morphisms. Then we define $h : A \otimes B \to \mathcal{M}(C)$ by $h(a \otimes b) = q_A(a)q_B(b)$. This is clearly the unique map for which $q_A = hj_A$ and $q_B = hj_B$. We must prove that h is nondegenerate. For this, take $c \in C$. Then we can write $c = \sum_i q_A(a_i)c_i$ and we can write $c_i = \sum_j q_B(b_{i,j})c_{i,j}$. Thus $c_i = \sum_{i,j} q_A(a_i)q_B(b_{i,j})c_{i,j} \in h(A \otimes B)C$. $\square$

It is not always true that the composition of R-pullback diagrams is an R-pullback diagram. To see this, take the following diagram in the category of unital algebras over $\mathbb{R}$:

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

If A is central in B and C, then it is easily checked that $D \cong B \otimes_A C$. Now take the following diagram, where $\mathbb{H}$ is the quaternion algebra:

$$
\begin{array}{ccccc}
\mathbb{R} & \xrightarrow{\ \iota\ } & \mathbb{H} & \xrightarrow{\ \mathrm{Id}\ } & \mathbb{H} \\
{\scriptstyle \mathrm{Id}}\downarrow & & {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle p} \\
\mathbb{R} & \xrightarrow{\ \iota\ } & \mathbb{H} & \xrightarrow{\ q\ } & (\mathbb{H} \otimes \mathbb{H})/I
\end{array}
$$

The left diagram is clearly an R-pullback diagram. The right diagram is by definition an R-pullback diagram, where I is the ideal generated by $h \otimes 1 - 1 \otimes h$. The composite diagram is only a pullback diagram if $(\mathbb{H} \otimes \mathbb{H})/I \cong \mathbb{H}$. This is not the case since otherwise we would have $p = q = \mathrm{Id}$, but the map $\mathrm{Id} : \mathbb{H} \to \mathbb{H}$ does not commute with itself.

Note the composition of pullback diagrams in the category of unital algebras:

$$
\begin{array}{ccccc}
C & \longrightarrow & B & \longrightarrow & D \\
\downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & A \times_C^R B & \longrightarrow & (A \times_C^R B) \times_B^R D
\end{array}
$$

This will be a pullback diagram if B is central in D. In particular, this will be the case in the category of unital commutative algebras, where the property reduces to $(A \otimes_C B) \otimes_B D \cong A \otimes_C D$.

7.2 Classes of morphisms

Let $\mathcal{C}$ be a category. In this section we introduce some notation and terminology concerning classes of morphisms in $\mathcal{C}$.

We will make use of the standard classes Mor of all morphisms, Iso of isomorphisms, Mono of monomorphisms and Epi of epimorphisms. For two classes of morphisms $\mathbb{F}$ and $\mathbb{H}$, we denote by $\mathbb{F} \subseteq \mathbb{H}$ that $f \in \mathbb{F}$ implies $f \in \mathbb{H}$.

7.8 Definition. Let $\mathbb{F}$ be a class of morphisms.

1. A morphism g is $\mathbb{F}$-*left cancellable* if $gh \in \mathbb{F}$ implies $h \in \mathbb{F}$.

2. A morphism g is $\mathbb{F}$-*right cancellable* if $hg \in \mathbb{F}$ implies $h \in \mathbb{F}$.

3. A morphism g is $\mathbb{F}$-*dense* if $g = fu$ with $f \in \mathbb{F}$ implies $f \in$ Iso.

4. A morphism g is $\mathbb{F}$-*orthogonal* if for $fu = vg$ with $f \in \mathbb{F}$, there is a unique morphism w with $fw = v$ and $wg = u$.
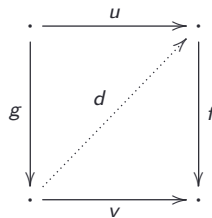
We write $g \perp f$.

We thus obtain the corresponding classes $\mathbb{F}-$LCan of $\mathbb{F}$-left cancellable morphisms, $\mathbb{F}-$RCan of $\mathbb{F}$-right cancellable morphisms, $\mathbb{F}-$Dense of $\mathbb{F}$-dense morphisms, $\mathbb{F}-$Ortho of $\mathbb{F}$-orthogonal morphisms.

We mention the following easy fact:

7.9 Lemma. *If $\mathbb{F}$ is pullback stable, then each monomorphism is $\mathbb{F}$-left cancellable.*

Proof. Let m be a monomorphism and let assume that $mh \in \mathbb{F}$. Then the following diagram is easily seen to be a pullback diagram:

$$
\begin{array}{ccc}
\cdot & \xrightarrow{\;h\;} & \cdot \\
\mathrm{Id} \downarrow & & \downarrow m \\
\cdot & \xrightarrow{\;mh\;} & \cdot
\end{array}
$$

Thus follows that $h \in \mathbb{F}$. $\square$

7.10 Remark. Note that in Lemma 7.9, we do not require that all pullbacks exist in $\mathcal{C}$, only that $\mathbb{F}$ is stable under those pullbacks that exist in $\mathcal{C}$.

7.3   Factorization systems

In this section, we recall some facts about factorization systems [1]. Let $\mathcal{C}$ be a category.

7.11 Definition. [1] A *factorization system* $(\mathbb{E}, \mathbb{M})$ consists of two classes of morphisms, a first class $\mathbb{E}$ and a second class $\mathbb{M}$, such that:

(F1) $\mathbb{E}$ and $\mathbb{M}$ are closed under composition with isomorphisms.

(F2) Every $\mathbb{E}$-morphism is $\mathbb{M}$-orthogonal.

(F3) Every morphism f decomposes as $f = me$ with $m \in \mathbb{M}$, $e \in \mathbb{E}$.

7.12 Proposition. *[1] Let $(\mathbb{E}, \mathbb{M})$ be a factorization system. We have:*

1. *$\mathbb{E} = \mathbb{M} - \mathrm{Ortho}$. Thus the class $\mathbb{E}$ is uniquely determined by the class $\mathbb{M}$.*

2. *$\mathbb{E} \cap \mathbb{M} = \mathsf{Iso}$.*

3. *The $(\mathbb{E}, \mathbb{M})$ factorization of a morphism is essentially unique, i.e. if $f = me$ and $f = m'e'$, then there is a unique isomorphism h such that $he = e'$ and $m'h = m$.*

4. *$(\mathbb{M}^{op}, \mathbb{E}^{op})$ is a factorization system on $\mathcal{C}^{op}$.*

Next we list a few properties of $\mathbb{M}$. By Proposition 7.12 (3), the dual properties hold for $\mathbb{E}$.

7.13 Proposition. *[1] Let $\mathbb{M}$ be the second class of morphisms of a factorization system. Then we have the following properties:*

1. *$\mathbb{M}$ contains all the isomorphisms.*

2. *$\mathbb{M}$ is closed under composition.*

3. *$\mathbb{M}$ is pullback stable.*

4. *If $\mathbb{M}$ consists of monomorphisms, then every morphism in $\mathbb{M}$ is $\mathbb{M}$-left cancellable.*

5. *Every monomorphism is $\mathbb{M}$-left cancellable.*

Let R be a relation as in §7.1. We can generalize the pullback stability of $\mathbb{M}$ to R-pullback stability:

7.14 Proposition. *Suppose that $\mathcal{C}$ has all R-pullbacks and suppose that $\mathbb{M}$ is the second class of morphisms of a factorization system. If for each two morphisms $f : C \to A$ and $g : C \to B$, and for every $e : D \to C$ in $\mathbb{E}$ we have that f and g are R-related whenever fe and ge are R-related, then $\mathbb{M}$ is R-pullback stable.*

Proof. Take an R-pullback diagram with $\overline{f}$ and $\overline{m}$ R-related

$$
\begin{array}{ccc}
A & \xrightarrow{\ \overline{m}\ } & B \\
{\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle f} \\
C & \xrightarrow{\ m\ } & D
\end{array}
$$

We can factorize $\overline{m} = ne$. By (F2), there exists a unique morphism d such that the following diagram commutes:

$$
\begin{array}{ccc}
\bullet & \xrightarrow{\ \overline{f}\ } & \bullet \\
{\scriptstyle e}\downarrow & {\scriptstyle d}\ \nearrow & \downarrow{\scriptstyle m} \\
\bullet & \xrightarrow{\ fn\ } & \bullet
\end{array}
$$

Since $\overline{f} = de$ is R-related to $\overline{m} = ne$, it follows that d is R-related to n. By the property of a pullback, there exists a unique morphism g such that $\overline{f}g = d$ and $\overline{m}g = n$. This implies that $\overline{f}ge = de = \overline{f}$ and $\overline{m}ge = ne = \overline{m}$. Again using the property of a pullback, we see

that $ge = \mathrm{Id}$. Thus the following diagram commutes:
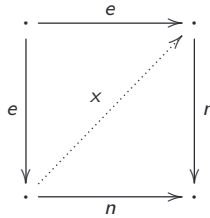
For $x = \mathrm{Id}$ and also for $x = eg$. Thus from the uniqueness requirement in $(F2)$ follows that $eg = \mathrm{Id}$. Thus e is an isomorphism and $\overline{m}$ is in $\mathbb{M}$. $\qquad\square$

We recall the following useful fact:

7.15 Proposition. *[1, Proposition 14.11] Let $(\mathbb{E}, \mathbb{M})$ be a factorization system. The following are equivalent:*

1. *All morphisms in $\mathbb{E}$ are epimorphisms.*

2. *All sections are contained in $\mathbb{M}$.*

We can now ask ourselves the question when a class of morphisms $\mathbb{M}$ is part of a factorization system.

7.16 Lemma. *Let $\mathbb{M}$ be a class of monomorphisms with $\mathbb{M}$ pullback-stable. Then $\mathbb{M} - \mathrm{Ortho} = \mathbb{M} - \mathrm{Dense}$.*

Proof. See [8, Lemma 2.2]. $\qquad\square$

7.17 Lemma. *Let $\mathbb{M} \subseteq \mathrm{Mono}$ be a class of morphisms. For a factorization $f = mu$ with $m \in \mathbb{M}$, consider the following properties:*

  (a) *The $\mathbb{M}$-subobject m is minimal among the $\mathbb{M}$-subobjects m' for which there exists a factorization $f = m'u'$, meaning that if we can write $m' = mn$, then n is an isomorphism.*

(b) *The morphism u is $\mathbb{M}$-dense.*

We have:

1. *If $\mathbb{M}$ is closed under composition, then (a) implies (b).*

2. *If each morphism in $\mathbb{M}$ is $\mathbb{M}$-left cancellable, then (b) implies (a).*

Proof. Suppose first (a). We show that u is dense. Write $u = m'u'$ with $m' \in \mathbb{M}$. Then $f = mm'u'$ with $mm' \in \mathbb{M}$ since $\mathbb{M}$ is closed under compositions. From the minimality of m we deduce that m' is an isomorphism as desired.

Suppose next (b). We show that m is minimal. Hence, consider $m' \in \mathbb{M}$ such that there are factorizations $f = m'u'$ and $m' = mn$. By $\mathbb{M}$-left cancellability of $\mathbb{M}$, $n \in \mathbb{M}$. Now $f = mnu'$ whence since m is mono, $u = nu'$. Now density of u yields that n is an isomorphism as desired.   □

We obtain the following useful criteria for $\mathbb{M} \subseteq$ Mono to be part of a factorization system:

7.18 Proposition. *Let $\mathbb{M}$ be a class of monomorphisms and let $\mathbb{E} = \mathbb{M} -$ Dense.*

1. *Suppose $\mathbb{M}$ is pullback-stable and is closed under composition with isomorphisms and $(\mathbb{E}, \mathbb{M})$ satisfies (F3). Then $(\mathbb{E}, \mathbb{M})$ is a factorization system.*

2. *Suppose $\mathbb{M}$ contains all isomorphisms and that $\mathbb{M}$ is pullback-stable and closed under composition. Furthermore, suppose for every morphism $f \in \mathcal{C}$ there is a factorization $f = mu$ such that $m \in \mathbb{M}$ is minimal among the $\mathbb{M}$-subobjects m' for which there exists a factorization $f = m'u'$. Then $(\mathbb{E}, \mathbb{M})$ is a factorization*

system and the factorization $f = mu$ as above is the $(\mathbb{E}, \mathbb{M})$-factorization of f.

Proof. This follows from the previous lemmas. □

7.4 Examples

Topological structures

In **Top**, it is well known that there is a factorization system consisting of embeddings and continuous surjections.

In k**Top**, we let

$$\mathbb{M} = \{m : kX \to Y \mid m : X \to Y \text{ is an embedding}\}$$

and

$$\mathbb{E} = \{e : X \to kY \mid e : X \to Y \text{ is a continuous surjection}\}.$$

We verify the axioms:

(F1) This is trivial.

(F2) Consider a diagram

Since e is a surjection and m is an injection, there is a unique function d which makes the above diagram commute. It remains to show that d is continuous. But we know that there exists an embedding $\overline{m} : \overline{X} \to Y$ such that $k\overline{X} = X$. Since we know that

$$ B \xrightarrow{d} k\overline{X} \xrightarrow{r} \overline{X} \xrightarrow{\overline{m}} $$

is continuous, we see that rd is continuous. Hence, since r is a coreflection arrow, it follows that d is continuous.

Schemes

Consider the category of schemes. Let $\mathbb{M}$ be the class of all closed immersions. It is well known (see Gortz, Wedhorn [19]) that $\mathbb{M}$ contains the isomorphisms, is closed under composition and is pullback stable. Furthermore, we can decompose any morphism f as $f = me$ (see Theorem 5.16). Thus it follows from Lemma 7.18, that $\mathbb{M}$ is part of a factorization system.

Toposes

Any topos has an (Epi,Mono)-factorization. In particular, we have

1. **Set** has a factorization system. Note that the epimorphisms are exactly the surjections and that the monomorphisms are exactly the injections.

2. Consider a presheaf category $\mathbf{Set}^{\mathcal{C}}$. The monomorphisms are the natural transformations $m : F \to G$ such that $m(C) : F(C) \to G(C)$ is injective for each $C \in \mathcal{C}$. The epimorphisms are the natural transformations $e : F \to G$ such that $m(C) : F(C) \to G(C)$ is surjective for each $C \in \mathcal{C}$.

3. Consider a sheaf category $\text{Sh}(\mathcal{C})$ over a site. The monomorphisms are the natural transformations $m : F \to G$ such that $m(C) : F(C) \to G(C)$ is injective for each $C \in \mathcal{C}$. The epimorphisms are the natural transformations $e : F \to G$ such that for each object C in $\mathcal{C}$ and for each $y \in G(C)$, there is a cover S of C such that for all $f : D \to C$ we have that $G(f)(y)$ is in the image of $e(D) : F(D) \to G(D)$.

C^*-**algebras**

Consider the category C^*-**Alg** of C^*-algebras with as morphisms the Woronowicz-morphisms. Let $\mathbb{E}$ be the class of nondegenerate morphisms $e : A \to \mathcal{M}(B)$ such that $e(A) = B$ and let $\mathbb{M}$ be the class of injective (and thus isometric) nondegenerate morphisms $A \to \mathcal{M}(B)$. We check the axioms:

(F1) This is obvious.

(F2) Consider a diagram

$$
\begin{array}{ccc}
\mathcal{M}(A) & \xrightarrow{\;u\;} & \mathcal{M}(C) \\
{\scriptstyle e}\big\downarrow & \nearrow{\scriptstyle d} & \big\downarrow{\scriptstyle m} \\
\mathcal{M}(B) & \xrightarrow{\;v\;} & \mathcal{M}(D)
\end{array}
$$

Since $e(A) = B$, we find for each $x \in B$ that there exists an $y \in A$ such that $e(y) = x$. Then we define $d(x) = u(y)$. This is a well-defined C^*-morphism since m is injective. It is obvious that d is nondegenerate since u is nondegenerate.

(F3) For each nondegenerate morphism $f : A \to \mathcal{M}(B)$, we can decompose it as $A \xrightarrow{e} A/\mathrm{Ker}(f) \xrightarrow{m} \mathcal{M}(B)$. This induces a suitable factorization.

Associative algebras

Consider the category $k - \mathbf{Alg}$ of k-algebras with as morphisms the Woronowicz-morphisms. Let $\mathbb{E}$ be the class of nondegenerate morphisms $e : A \to \mathcal{M}(B)$ such that $e(A) = B$ and let $\mathbb{M}$ be the class of injective nondegenerate morphisms $A \to \mathcal{M}(B)$. The verification of the axioms is along the lines of the verification of previous section on C^*-algebras.

Differential Geometry

Let $(X, \mathcal{D})$ be a diffeological space and let $f : X' \to X$ be a function. We can put on X' the pullback diffeology $f^*(\mathcal{D})$. This is defined by $P \in f^*(\mathcal{D})$ if and only if $f \circ P \in \mathcal{D}$. This makes f into a smooth map.

7.19 Definition. Let $f : (X', \mathcal{D}') \to (X, \mathcal{D})$ be a map between diffeological spaces. We say that f is an induction if

1. f is injective.

2. $\mathcal{D}' = f^*(\mathcal{D})$

It is shown in [32] that for an induction $f : X' \to X$ we have that a map $g : Y \to X'$ is smooth iff fg is smooth.

Note that it is not true that if the inclusion of a subset $i : A \to X$ is an induction, then it is a topological embedding. In other words, if $A \subseteq X$, then we can equip A with the pullback diffeology. But the

associated D-topology does not correspond with the D-topology of X. If it does, then we say that i is a smooth embedding.

We can now put a factorization system on the category of diffeological spaces by setting $\mathbb{M}$ the class of inductions and $\mathbb{E}$ the class of smooth surjections. We check the axioms:

(F1) Trivial.

(F2) Consider a diagram
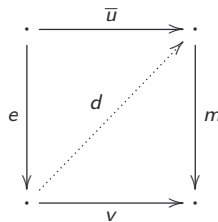
Since e is surjective and m is injective, there exists a function d such that the above diagram commutes. Furthermore, we know that $md = v$ is smooth. So since m is an induction, it follows that d is smooth.

(F3) Every smooth map $f : X \to Y$ decomposes as $X \to f(X) \to Y$. We can put on $f(X)$ the pullback topology from Y.

Lie groups

Consider the category of Lie groups with Lie homomorphisms. Let $\mathbb{M}$ be the class of homomorphisms that are injections and immersions and let $\mathbb{M}$ be the class of surjective homomorphisms. We check the axioms:

(F1) Trivial

(F2) Consider a diagram
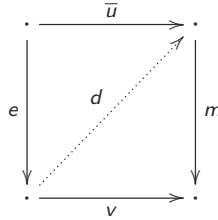
Since m is an injection and e is a surjection, we find a map d that makes the diagram commute. It is shown in Theorem 19.25 of Lee [39] that d is then smooth.

(F3) Every Lie group homomorphism $f : G \to H$ as a factorization $G \to f(G) \to H$. It is known that $f(G)$ is an immersed Lie subgroup of H, so that the factorization holds.

Chapter 11

Functional topology without factorization systems

In [8], functional topology was introduced as a categorical framework in which certain ideas from topology, revolving around closed embeddings, closed morphisms, proper morphisms and separated morphisms, can be developed. The approach in [8] makes use of a "background" factorization system for this development. In particular, one makes use of a notion of "embeddings" - the second class in the factorization system - which is such that the closed embeddings are precisely the closed morphisms that are at the same time embeddings. Finally, the approach in [8] only considers the standard pullbacks.

In this section, we present a more basic approach which only takes closed morphisms (see Definition 11.9), and in a second version closed morphisms and closed embeddings (see Definition 11.13), as an input. Furthermore, we generalize many results to R-pullbacks instead of only taking the standard pullbacks.

In this setup we introduce the notion of proper and separated morphisms. Properness still makes sense with respect to R-pullbacks in complete generality. Separatedness on the other hand turns out to be limited to so called commutative objects, for which a diagonal morphism exists

We show that many of the standard properties involving proper and separated morphisms hold true under the assumption that for the diagonal of an arbitrary morphism, it is equivalent to say that it is closed, or that it is proper, or that it is a closed embedding. This fundamental property is fulfilled in various settings where natural classes of separated and proper morphisms are present and share a number of basic properties, like pullback stability and closure under compositions of the individual classes, and the fact that separated morphisms are left cancellable with respect to proper morphisms. Thus we get the notions of closed class and closed pair. These notions encompasses both the setup of [8], the situation in various categories of locally ringed spaces, like the category of schemes and the category of complex analytic spaces, the situation of C^*-algebras and Woronowicz morphisms, and the situation of k-algebras.

On the one hand, in turns out that in the situation of C^*-algebras and k-algebras, each closed morphism is proper. On the other hand, the passage from closed to proper maps is not automatically well-behaved for a general relation R. Following [30], this leads us to taking the proper maps as fundamental instead of the closed maps in these situations. This gives us the notion of proper class and proper pair.

We characterize the compact C^*-algebras as the unital ones and similarly, we characterize the compact k-algebras as the unital ones. More generally, it seems possible to characterize monoids in a symmetric monoidal category as being precisely the compact objects in a natural

category of semigroup objects with Woronowicz morphisms. This is work in progress.

Finally, we introduce a way to compare functional topologies on different categories. We apply this to the theory of $\mathbb{C}$-schemes and topological spaces. We obtain that properness and separatedness of $\mathbb{C}$-schemes correspond to compactness and Hausdorffness of topological spaces, after applying the analytification functor. This was one of the original motivations for the correct notions of properness and separatedness in algebraic geometry.

11.1 Proper and separated morphisms

We can now finally define properness and separatedness. The results and proofs in this section are variants of results and proofs in [8].

Proper morphisms

Let $\mathcal{C}$ be a category with R-pullbacks as introduced in §7.1.

11.1 Definition. Let $\mathbb{F}$ be a class of morphisms. We say that a morphism g is $\mathbb{F}$-*proper* if every R-pullback of g is in $\mathbb{F}$.

We thus obtain the class $\mathbb{F} - \mathrm{Prop}$ of $\mathbb{F}$-proper morphisms.

11.2 Lemma. *Let $\mathbb{F}$ be a class of morphisms.*

1. *Each $\mathbb{F}$-proper morphism is in $\mathbb{F}$.*

2. *If $\mathbb{F} \subseteq \mathbb{G}$ then $\mathbb{F} - \mathrm{Prop} \subseteq \mathbb{G} - \mathrm{Prop}$.*

3. *If $\mathbb{F}_0 \subseteq \mathbb{F}$ with $\mathbb{F}_0$ R-pullback-stable, then each morphism in $\mathbb{F}_0$ is $\mathbb{F}$-proper.*

Proof.

1. Let f be an $\mathbb{F}$-proper morphism. The result follows easily by taking the R-pullback of f and the identity.

2. This is obvious.

3. This is obvious.

   $\square$

11.3 Lemma. *Let $\mathbb{F}$ be a class of morphisms and let R be a relation such that the composition of R-pullback diagrams is an R-pullback diagram.*

1. $\mathbb{F} - \mathrm{Prop}$ *is pullback-stable.*

2. *If R is the standard relation and if each isomorphism is in $\mathbb{F}$, then each isomorphism is $\mathbb{F}$-proper.*

3. *If $\mathbb{F}$ is closed under compositions, then so is $\mathbb{F} - \mathrm{Prop}$.*

Proof.   1. This is obvious.

2. This follows from the previous theorem and the fact that isomorphisms are closed under R-pullbacks.

3. This is shown by composing pullback diagrams.

   $\square$

Separated morphisms

Let $\mathcal{C}$ be a finitely complete category. For a morphism $g : X \longrightarrow Y$, the *diagonal* Δ_g of g is the unique morphism $\Delta_g = (1_X, 1_X) : X \longrightarrow$

$X \times_Y X$ to the pullback $X \times_Y X$ of g along itself.

Note that if R is not the standard relation, then 1_X does not need to be R-related to itself (this is only true for the so called commutative objects). Thus the diagonal Δ_g of a morphism $g : X \to Y$ only makes sense for commutative objects X. One can generalize many results in this section to morphisms from a commutative object to a general object.

The main concepts in this section are those of separated morphisms:

11.4 Definition. Let $\mathbb{F}$ be a class of morphisms. Let $g : X \to Y$ be a morphism such that X is a commutative object.

1. We say that g is $\mathbb{F}$-*separated* if the diagonal $\Delta_g \in \mathbb{F}$.

2. We say that g is $\mathbb{F}$-*perfect* if it is $\mathbb{F}$-proper and $\mathbb{F}$-separated.

We thus obtain the corresponding classes $\mathbb{F} - \text{Sep}$ of $\mathbb{F}$-separated morphisms and $\mathbb{F} - \text{Perf}$ of $\mathbb{F}$-perfect morphisms.

11.5 Lemma. *Let $\mathbb{F}$ be a class of morphisms.*

1. *If $\mathbb{F} \subseteq \mathbb{G}$ then $\mathbb{F} - \text{Sep} \subseteq \mathbb{G} - \text{Sep}$.*

2. *If $\mathbb{F}$ contains the isomorphisms and if X is commutative, then each monomorphism $m : X \to Y$ is $\mathbb{F}$-separated.*

Proof.

1. This is obvious.

2. This follows since if $m : X \to Y$ is a monomorphism, then the following diagram is a pullback diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \mathrm{Id}\ } & X \\
{\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle m} \\
X & \xrightarrow{\ m\ } & Y
\end{array}
$$

$\square$

11.6 Lemma. *Let R be the standard relation. The Iso-separated morphisms are exactly the monomorphisms.*

Proof. That a monomorphism is Iso-separated follows immediately from Lemma 11.5. Conversely, if f is Iso-separated, then we can take the pullback diagram:

$$
\begin{array}{ccc}
X \times_Y X & \xrightarrow{\ \bar{f}_1\ } & X \\
{\scriptstyle \bar{f}_2}\downarrow & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

If $g, h : Z \to X$ are such that $fg = fh$. Then by the property of pullbacks, there is a unique map $k : Z \to X \times_Y X$ such that $\bar{f}_1 k = g$ and $\bar{f}_2 k = h$. But since Δ_f is an isomorphism, it follows that $\bar{f}_1 = \Delta_f = \bar{f}_2$. Thus $g = h$. $\square$

11.7 Lemma. *Let R be the standard relation. Let $\mathbb{F}$ be a class of morphisms. Suppose that the diagonal Δ_f of each $\mathbb{F}$-separated morphism is actually $\mathbb{F}$-proper. Then we have:*

1. *Pullbacks of $\mathbb{F}$-separated morphisms are $\mathbb{F}$-separated.*

2. *If $\mathbb{F}$ is closed under compositions, then so is $\mathbb{F} - \mathrm{Sep}$.*

3. *If gf is $\mathbb{F}$-separated, then f is $\mathbb{F}$-separated.*

Proof.

1. Consider the following diagram

$$W \xrightarrow{\Delta_{\bar{f}}} W \times_Z W \rightrightarrows W \xrightarrow{\bar{f}} Z$$

with vertical maps $\bar{k}$, $\bar{k}$, $\bar{k}$, k and squares (1), (2), (3)

$$X \xrightarrow{\Delta_f} X \times_Y X \rightrightarrows X \xrightarrow{f} Y$$
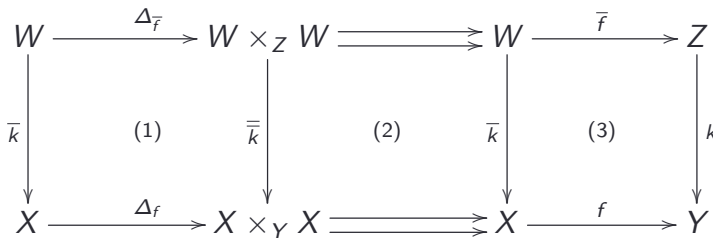
   If (3) is a pullback diagram, then $(2)\&(3)$ is also a pullback diagram. This implies that (1) is a pullback diagram. Since Δ_f is in $\mathbb{F}$, this implies by hypothesis that Δ_f is $\mathbb{F}$-proper and thus that $\Delta_{\bar{f}}$ is in $\mathbb{F}$. Thus $\bar{f}$ is $\mathcal{F}$-separated.

2&3 Let us first introduce some notation. For an arbitrary morphism u, we define u_1 and u_2 to such that the following is a pullback diagram:

$$\begin{array}{ccc} \cdot & \xrightarrow{u_1} & \cdot \\ \downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle u} \\ \cdot & \xrightarrow{u} & \cdot \end{array}$$
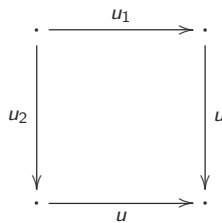
   Now consider $f : X \to Y$ and $g : Y \to Z$ and their composite $h = gf$. Let $t : X \times_Y X \to X \times_Z X$ occur in following pullback

diagram
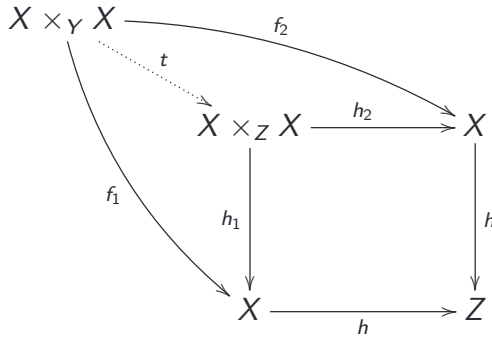
Thus we have that $h_1 t = f_1$ and $h_2 t = f_2$. It is easily checked that following diagram is commutative
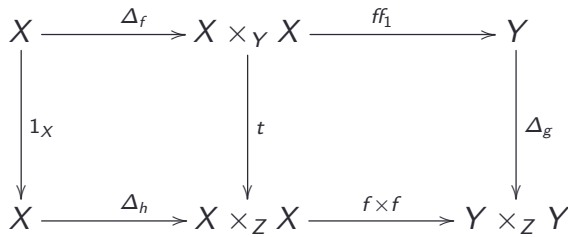
Since t is actually the equalizer of $f h_1$ and $f h_2$, this implies that the right square is a pullback diagram. Since t is a monomorphism, this implies that the left square is also a pullback square. We can now prove the assertions.

- To prove (2), note that since the right square is a pullback diagram and since Δ_g is $\mathbb{F}$-proper, it follows that t is also in $\mathbb{F}$. Since $\Delta_f \in \mathbb{F}$, it follows that $\Delta_h = t\Delta_f$ is in $\mathcal{F}$.

- To prove (3), note that the left square is a pullback diagram and that δ_h is $\mathbb{F}$-proper.

$\square$

We have the following key lemma:

11.8 Lemma. *Let R be the standard relation. If $\mathbb{F}$ contains all the isomorphism and if $\mathbb{F}$ is pullback-stable and closed under compositions, then each $\mathbb{F}$-separated map is $\mathbb{F}$-left cancellable.*

Proof. Consider $gf \in \mathbb{F}$ with g $\mathbb{F}$-separated. We obtain the following diagram of pullbacks:

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z \\
\uparrow{\scriptstyle p_1'} & & \uparrow{\scriptstyle p_1} & & \uparrow{\scriptstyle g} \\
X \times_Z Y & \xrightarrow{\ f \times 1_Y\ } & Y \times_Z Y & \xrightarrow{\ p_2\ } & Y \\
\uparrow{\scriptstyle (1,f)} & & \uparrow{\scriptstyle \Delta_g} & & \\
X & \xrightarrow{\ f\ } & Y & &
\end{array}
$$
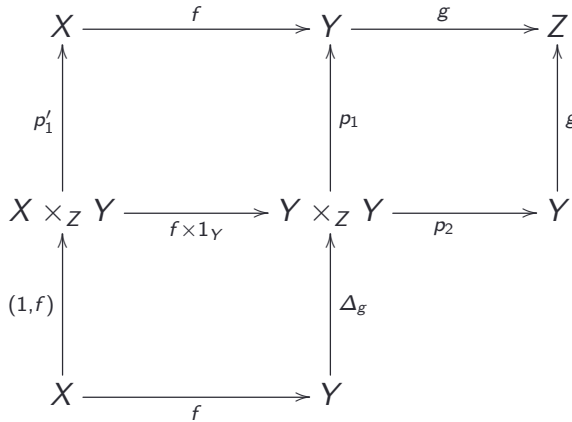
Since $gf \in \mathbb{F}$, the pullback $p_2(f \times 1_Y) \in \mathbb{F}$. Since $\Delta_g \in \mathbb{F}$, the pullback $(1, f) \in \mathbb{F}$. Thus the composition $p_2(f \times 1_Y)(1, f) = p_2 \Delta_g f = f \in \mathbb{F}$. $\qquad\square$

11.2 Closed classes and closed pairs

Let $\mathcal{C}$ be category with R-pullbacks

11.9 Definition. A *pre-closed class* on $\mathcal{C}$ is a class of morphisms $\mathbb{F}$ with which contains the isomorphisms and which is closed under composition. Morphisms in $\mathbb{F}$ are called *closed morphisms*.

If a pre-closed class $\mathbb{F}$ on $\mathcal{C}$ is chosen, we put Prop $= \mathbb{F} -$ Prop and we simply speak of *proper* morphisms.

We can define Sep $= \mathbb{F} -$ Sep and Perf $= \mathbb{F} -$ Perf and we simply speak of *proper*, *separated* and *perfect* morphisms. Furthermore, we can define that a pre-closed class $\mathbb{F}$ is a *closed class* if

(c) The diagonal Δ_f of each separated map f is proper.

A pre-closed class $\mathbb{F}$ is called a *stable class* if $\mathbb{F}$ is pullback-stable.

Clearly, a stable class is a closed class.

We obtain the following properties:

11.10 Proposition. *Let $\mathbb{F}$ be a pre-closed class on $\mathcal{C}$. And let R be a relation such that the composition of R-pullback diagrams is an R-pullback diagram*

1. *Every isomorphism is proper.*

2. *The composition and R-pullback of proper morphisms is proper.*

Furthermore, if R is the standard relation, then

1. *Every isomorphism is perfect.*

2. *Every monomorphism is separated.*

If $\mathbb{F}$ is a closed class on $\mathcal{C}$. Then

1. Sep $=$ Prop $-$ Sep.

2. Prop, Sep *and* Perf *are closed under composition and are pullback-stable.*

3. The separated morphisms are left-cancellable.

Proof. This follows immediately from the results in the previous section. □

For applications to schemes we will need the following property:

11.11 Proposition. *Let R be the standard relation and let $\mathbb{F}$ be a pre-closed class. Let $\mathbb{P}$ be closed under composition and pullback-stable such that if a diagonal Δ_f is in $\mathbb{F}$, then it is also in $\mathbb{P}$. Then the separated maps are $\mathbb{P}$-left cancellable.*

Proof. This is completely analogous to the proof of Lemma 11.7.3. □

11.12 Remarks. 1. The property (c) which makes a pre-closed class into a closed class is somehow circumvented in the approach developed in [56], where, relative to a pre-closed class $\mathbb{F}$, the relevant class of "separated" morphisms is defined to be $(\mathbb{F} - \mathrm{Prop}) - \mathrm{Sep}$.

2. What we call here a stable class is called a *topology* in Hofmann [30].

Many examples of closed classes arise in the following way:

11.13 Definition. A *pre-closed pair* $(\mathbb{F}_0, \mathbb{F})$ on $\mathcal{C}$ consists of two classes of morphisms with $\mathbb{F}_0 \subseteq \mathbb{F}$ and such that $\mathbb{F}_0$ is a class of monomorphisms which contain the isomorphisms and such that the following conditions hold:

(a) $\mathbb{F}$ is a closed class.

(b) $\mathbb{F}_0$ is R-pullback-stable and closed under composition.

Morphisms in $\mathbb{F}_0$ are called *closed immersions* and morphisms in $\mathbb{F}$ are called *closed morphisms*. We can define that a pre-closed pair is called a *closed pair* if

(c) If f is an $\mathbb{F}$-separated morphism, then the diagonal Δ_f is in $\mathbb{F}_0$.

11.14 Proposition. *Let $(\mathbb{F}_0, \mathbb{F})$ be a pre-closed pair on $\mathcal{C}$. We have:*

1. *$\mathbb{F}$ is a pre-closed class on $\mathcal{C}$.*

2. *Each morphism in $\mathbb{F}_0$ is proper.*

Furthermore, if R is the standard relation, then each morphism in $\mathbb{F}_0$ is separated.

Proof. The first statement is obvious. For the second statement: since $\mathbb{F}_0$ is R-pullback stable, it shows that each morphism in $\mathbb{F}_0$ is perfect. But every morphism in $\mathbb{F}_0$ is also $\mathbb{F}$-separated since each monomorphism is. □

11.15 Proposition. *Let $(\mathbb{F}_0, \mathbb{F})$ be a closed pair on $\mathcal{C}$. We have that $\mathbb{F}$ is a closed class on $\mathcal{C}$.*

Proof. Since $\mathbb{F}_0 \subseteq \mathbb{F}$ and $\mathbb{F}_0$ is pullback-stable, we have $\mathbb{F}_0 \subseteq \mathbb{F} - \mathrm{Prop}$ and thus $\mathbb{F} - \mathrm{Sep} \subseteq \mathbb{F}_0 - \mathrm{Sep} \subseteq (\mathbb{F} - \mathrm{Prop}) - \mathrm{Sep}$. □

11.3 Proper classes and proper pairs

Sometimes it is more convenient to describe the proper maps directly instead of the proper pairs. This is the idea in [30]. We let $\mathcal{C}$ be a category with R-pullbacks. We get the following alternative definition of Definition 11.9:

11.16 Definition. A *proper class* on $\mathcal{C}$ is a class of morphisms $\mathbb{H}$ which contain the isomorphisms, which is R-pullback stable and which is closed under compositions. Morphisms in $\mathbb{H}$ are called proper morphisms.

Closed classes and proper classes are closely related:

11.17 Proposition. *Assume that $\mathcal{C}$ is a category such that the composition of R-pullback diagrams is an R-pullback diagram and such that the pullbacks of isomorphisms are isomorphisms. If $\mathbb{F}$ is a closed class on $\mathcal{C}$, then $\mathbb{F} - \mathrm{Prop}$ is a proper class.*

11.18 Proposition. *Assume that $\mathcal{C}$ is a category with R-pullbacks. If $\mathbb{H}$ is a proper class on $\mathcal{C}$ then it also is a closed class. We also have that $\mathbb{H} - \mathrm{Prop} = \mathbb{H}$.*

Closed pairs also have a proper form:

11.19 Definition. A *proper pair* $(\mathbb{F}_0, \mathbb{H})$ on $\mathcal{C}$ consists of two classes of morphisms with $\mathbb{F}_0 \subseteq \mathbb{H}$ and such that $\mathbb{F}_0$ is a class of monomorphisms which contain the isomorphisms and such that the following conditions hold:

(a) $\mathbb{F}$ is a proper class.

(b) $\mathbb{F}_0$ is a proper class of monomorphisms.

Morphisms in $\mathbb{F}_0$ are called *closed immersions* and morphisms in $\mathbb{H}$ are called *proper morphisms*.

11.20 Proposition. *Assume that $\mathcal{C}$ is a category such that the composition of R-pullback diagrams is an R-pullback diagram and such that the pullbacks of isomorphisms are isomorphisms. If $(\mathbb{F}, \mathbb{F}_0)$ is a closed pair on $\mathcal{C}$, then $(\mathbb{F} - \mathrm{Prop}, \mathbb{F}_0)$ is a proper pair.*

11.21 Proposition. *Assume that $\mathcal{C}$ is a category with R-pullbacks. If $(\mathbb{H}, \mathbb{F}_0)$ is a proper pair on $\mathcal{C}$ then it also is a closed pair. We also have that $\mathbb{H} - \mathrm{Prop} = \mathbb{H}$.*

11.4   Examples

Most of our examples will come from a functional topology *with* a proper factorization system. So most examples will have to wait until next chapter. Nevertheless, we can indicate some interesting examples:

Topology

In the category **Top** of topological spaces, take for $\mathbb{F}$ the closed morphisms and for $\mathbb{F}_0$ the closed embeddings. Then $(\mathbb{F}_0, \mathbb{F})$ is a closed pair on **Top**, and the resulting notions of separated and proper maps are the standard ones for topological spaces [3].

Schemes

In the category **Scheme** of schemes, take for $\mathbb{F}$ the closed morphisms and for $\mathbb{F}_0$ the closed immersions. It is well known that if the diagonal is closed, then it is a closed immersion. So $(\mathbb{F}_0, \mathbb{F})$ is a closed pair. In **Scheme**, a morphism is called *separated* if it is $\mathbb{F}$-separated, it is called *universally closed* if it is $\mathbb{F}$-proper, and it is called *proper* if it is $\mathbb{F}$-perfect and of finite type.

Let $\mathbb{P}$ be the class of morphisms of finite type. Then $\mathbb{P}$ is closed under composition and is pullback-stable. Furthermore, $\mathbb{F}_0 \subseteq \mathbb{P}$. Hence, Sep $\subseteq \mathbb{P} -$ Sep and Proposition 11.10 (5) applies. Consequently, Sep $\subseteq \mathbb{P} - \mathsf{LCan}$ and Sep $\subseteq (\mathbb{P} \cap \mathsf{Perf}) - \mathsf{LCan}$, and we recover the fact that separated morphisms of schemes are left cancellable with respect to the proper maps of schemes. It is then readily seen that Proposition 11.10 and 11.14 remain valid with Perf replaced by $\mathbb{P} \cap \mathsf{Perf}$.

As another example. Consider the category **Scheme** and let $\mathbb{P}$ be the class of quasi-compact morphisms. This class contains the iso-morphisms, is pullback-stable and closed under compositions. A morphism is called *quasi-separated* if it is in $\mathbb{P} - \text{Sep}$. Thus Lemma 11.8 yields that quasi-separated morphisms are left cancellable with respect to quasi-compact morphisms.

C^*-**algebras**

For a morphism $f : A \to B$ in the category C^*-**Alg**$^{\text{op}}$, we write $F : B \to \mathcal{M}(A)$ for the associated Woronowicz-morphisms.

We define classes $f : A \to B$ to be in $\mathbb{F}_0$ if $F : B \to \mathcal{M}(A)$ is such that $F(B) = A$. We define $f \in \mathbb{H}$ if $F : B \to \mathcal{M}(A)$ is such that $F(B) \subseteq A$. Then $(\mathbb{H}, \mathbb{F}_0)$ is a proper pair. We check that $\mathbb{H}$ is closed under pullbacks: consider $f \in \mathbb{F}$ and g arbitrary, then we have the following diagram of Woronowicz-morphisms:

$$
\begin{array}{ccc}
\mathcal{M}(B) & \xrightarrow{\ F\ } & \mathcal{M}(A) \\
{\scriptstyle G}\big\downarrow & & \big\downarrow{\scriptstyle \overline{G}} \\
\mathcal{M}(C) & \xrightarrow[\ F\ ]{} & \mathcal{M}((C \otimes_\mu A)/J)
\end{array}
$$

We need to prove that for $c \in C$ we have that $\overline{F}(c) = c \otimes_\mu 1$ is in $(C \otimes_\mu A)/J$. But since $G(B)C$ is dense in C, there is a sequence $g(b_n)c_n$ that converges to c. Thus $g(b_n)c_n \otimes_\mu 1$ converges to $c \otimes_\mu 1$. But we know that $g(b_n) \otimes_\mu 1 = 1 \otimes_1 f(b_n)$ and we know that $f(b_n) \in B$. Thus $g(b_n)c_n \otimes_\mu 1 = c_n \otimes_1 f(b_n) \in (C \otimes_\mu A)/J$. Since $g(b_n)c_n \otimes_\mu 1$ converges to $c \otimes_\mu 1$ and since $(C \otimes_\mu A)/J$ is closed, we have that $c \otimes_\mu 1$ is in $(C \otimes_\mu A)/J$.

Note that in the commutative case, the morphisms in $\mathbb{F}$ reduce automatically to the continuous proper maps between topological spaces.

Associative Algebras

For a morphism $f : A \to B$ in the category $k - \mathbf{Alg}^{\mathrm{op}}$, we write $F : B \to \mathcal{M}(A)$ for the associated Woronowicz-morphism.

We define classes $f : A \to B$ to be in $\mathbb{F}_0$ if $F : B \to \mathcal{M}(A)$ is such that $F(B) = A$. We define $f \in \mathbb{H}$ if $F : B \to \mathcal{M}(A)$ is such that $F(B) \subseteq A$. Then $(\mathbb{H}, \mathbb{F}_0)$ is a proper pair. The verification of this is along the lines as the verification of this fact with C^*-algbras.

11.5   Compact and Hausdorff objects

We can now define compact and Hausdorff objects. The results and proofs in this section are variants of the results and proofs in [8].

Compactness

We will now give a nice definition of compactness.

11.22 Definition. Let $\mathbb{F}$ be a pre-closed class. Then an object X of $\mathcal{C}$ is $\mathbb{F}$-compact if for every $Y \in \mathcal{C}$ if we have that the second projection $p_2 : X \times^R Y \to Y$ is in $\mathbb{F}$.

In what follows, we will often denote the terminal object of $\mathcal{C}$ by 1. The unique morphism from an object X to 1 is denoted by $!_X : X \to 1$. Compact objects and proper morphisms are related to each other in many ways. One of these ways is the following:

11.23 Theorem. *Let $\mathbb{F}$ be a pre-closed class. An object X of $\mathcal{C}$ is $\mathbb{F}$-compact iff $!_X : X \to 1$ is $\mathbb{F}$-proper.*

Proof. The "only if" is obvious by taking $Y = 1$ in the definition. The "if" follows from the fact that the pullbacks of $!_X$ are exactly the morphisms $p_2 : X \times^R Y \to Y$. □

We give some stability properties of compact objects.

11.24 Theorem. *Let $\mathbb{F}$ be a pre-closed class. Assume that the composition of $\mathbb{F}$-proper maps is $\mathbb{F}$-proper (this is of course always true if R is the standard relation).*

1. *If $f : X \to Y$ is $\mathbb{F}$-proper and if Y is $\mathbb{F}$-compact, then X is also $\mathbb{F}$-compact.*

2. *The full subcategory $\mathbb{F}$-Comp of $\mathbb{F}$-compact objects is closed under finite R-products.*

Proof.

1. We know that f and $!_Y$ are $\mathbb{F}$-proper. Thus the composition $!_X =!_Y f$ is also $\mathbb{F}$-proper.

2. For X and Y $\mathbb{F}$-compact, we have that $p_2 : X \times^R Y \to Y$ is $\mathbb{F}$-proper. Hence, by (1), it follows that $X \times^R Y$ is $\mathbb{F}$-compact.

□

11.25 Theorem. *Let $(\mathbb{F}_0, \mathbb{F})$ be a pre-closed pair. If $m : X \to Y$ is in $\mathbb{F}_0$ and if Y is compact, then X is compact.*

Proof. This follows from previous theorem. □

Hausdorff

11.26 Definition. A commutative object X of $\mathcal{C}$ is called $\mathbb{F}$-Hausdorff if the unique morphism $!_X : X \rightarrow 1$ is $\mathbb{F}$-separated. The class of all $\mathbb{F}$-Hausdorff objects is denoted as $\mathbb{F}$-Haus.

We give some equivalent forms of $\mathbb{F}$-Hausdorffness. We assume that R is the standard relation.

11.27 Theorem. *Let $\mathbb{F}$ be a closed class. For an object X, the following are equivalent*

1. *X is $\mathbb{F}$-Hausdorff,*

2. *Every morphism $f : X \rightarrow Y$ is $\mathbb{F}$-separated,*

3. *There exists an $\mathbb{F}$-separated morphism $f : X \rightarrow Y$ with Y $\mathbb{F}$-Hausdorff,*

4. *Every projection $p_2 : X \times Y \rightarrow Y$ is $\mathbb{F}$-separation,*

5. *$X \times Y$ is $\mathbb{F}$-Hausdorff for every $\mathbb{F}$-Hausdorff object Y,*

Proof.

$(1) \Rightarrow (2)$ We have that $!_X =\, !_Y f$ is $\mathbb{F}$-separated. Theorem 11.7 implies that f is $\mathbb{F}$-separated.

$(2) \Rightarrow (3)$ Take $Y = 1$.

$(3) \Rightarrow (1)$ Since Y is $\mathbb{F}$-Hausdorff, the map $!_Y$ is $\mathbb{F}$-separated. Theorem 11.7 implies that $!_X =\, !_y f$ is $\mathbb{F}$-separated, hence X is $\mathbb{F}$-Hausdorff.

$(1) \Rightarrow (4)$ The projections $p_2 : X \times Y \rightarrow Y$ are exactly the pullbacks of $!_X : X \rightarrow 1$. Thus the result follows from the pullback stability of $\mathbb{F}$-separated morphisms.

(4) $\Rightarrow$ (5) Since Y is $\mathbb{F}$-Hausdorff, we see that the map $!_Y$ is $\mathbb{F}$-separated. Since the composition of $\mathbb{F}$-separated maps is separated, it follows that $!_{X \times Y} = !_X p_2$ is $\mathbb{F}$-separated.

(5) $\Rightarrow$ (1) Just apply (5) to $Y = 1$ (which is evidently $\mathbb{F}$-Hausdorff).

$\square$

11.28 Corollary. *Let $\mathbb{F}$ be a closed class. The full subcategory $\mathbb{F}$-Haus of $\mathbb{F}$-Hausdorff objects is closed under finite limits and under subobjects.*

Proof. It follows from (5) of previous theorems that products of $\mathbb{F}$-Hausdorff objects are $\mathbb{F}$-Hausdorff. Now take a monomorphism $m :$ $X \to Y$ with Y $\mathbb{F}$-Hausdorff. Since monomorphism are $\mathbb{F}$-separated, it follows from (3) of previous theorem that X is $\mathcal{F}$-Hausdorff. Thus $\mathbb{F}$-Hausdorffness is closed under subobjects. It is also closed under equalizers, since equalizers are a special form of subobjects. $\square$

Examples

In the category **Top**, the compact and Hausdorff objects are the usual compact and Hausdorff spaces.

The compact objects in the category C^*-**Alg**$^{\mathrm{op}}$ are now exactly the unital objects. Indeed, the final object of the category C^*-**Alg** is $\mathbb{C}$. A Woronowicz-morphism from $\mathbb{C}$ to B is then a nondegenerate morphism $f : \mathbb{C} \to \mathcal{M}(B)$. Demanding that this is proper is exactly asking that $f(\mathbb{C}) \subseteq B$. This is true if and only if B contains a unit.

We also have that every commutative object in C^*-**Alg**$^{\mathrm{op}}$ is Hausdorff. Indeed, the diagonal morphism is represented by $D : A \otimes A \to \mathcal{M}(A)$

such that $D(a \otimes b) = ab$. In C^*-algebras, every positive element has a square root and the positive elements generate the C^*-algebra. Thus $D(A \otimes A) = A$.

Similarly, an object in $k - \mathbf{Alg}^{\mathrm{op}}$ is is compact if it is unital. Every commutative object is Hausdorff.

11.6   Comparison functors

In this section, we investigate some elementary properties related to a finite limit preserving functor $\varphi : \mathcal{C} \longrightarrow \mathcal{C}'$ between finitely complete categories. We let R be the standard relation.

Let $\mathbb{F}'$ be a class of morphisms in $\mathcal{C}'$, and consider the class $\varphi^{-1}(\mathbb{F}')$ on $\mathcal{C}$.

11.29 Proposition.

 1. $\varphi^{-1}(\mathbb{F}' - \mathrm{Sep}) = \varphi^{-1}(\mathbb{F}') - \mathrm{Sep}.$

 2. $\varphi^{-1}(\mathbb{F}' - \mathrm{Prop}) \subseteq \varphi^{-1}(\mathbb{F}') - \mathrm{Prop}.$

Proof. (1) easily follows from the fact that for $f : X \longrightarrow Y$ in $\mathcal{C}$, the image under φ of the diagonal $\Delta_f : X \longrightarrow X \times_Y X$ in $\mathcal{C}$ is given by the diagonal $\varphi(\Delta_f) = \Delta_{\varphi(f)} : \varphi(X) \longrightarrow \varphi(X) \times_{\varphi(Y)} \varphi(X)$ in $\mathcal{C}'$. For (2), suppose $\varphi(f)$ is $\mathbb{F}'$-proper and consider a pullback

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\uparrow & & \uparrow \\
X' & \xrightarrow[\ f'\ ]{} & Y'
\end{array}
$$

in $\mathcal{C}$. Since φ preserves the pullback, $\varphi(f')$ is a pullback of the $\mathbb{F}'$-proper morphism $\varphi(f)$, whence $\varphi(f') \in \mathbb{F}'$. $\square$

Note that the converse inclusion in Proposition 11.29 is hard to control in general, since it requires one to consider arbitrary pullbacks of $\varphi(f)$ in $\mathcal{C}'$.

11.30 Corollary. *Let* $(\mathbb{F}_0, \mathbb{F})$ *be a closed pair on* $\mathcal{C}$ *and* $(\mathbb{F}'_0, \mathbb{F}')$ *a closed pair on* $\mathcal{C}'$. *Suppose:*

1. *One of the following holds:*

   (a) $\varphi^{-1}(\mathbb{F}') \subseteq \mathbb{F}$.
   (b) $\varphi^{-1}(\mathbb{F}' - \mathrm{Prop}) \subseteq \mathbb{F} - \mathrm{Prop}$.

2. *One of the following holds:*

   (a) $\mathbb{F}_0 \subseteq \varphi^{-1}(\mathbb{F}'_0)$
   (b) $\mathbb{F} \subseteq \varphi^{-1}(\mathbb{F}')$
   (c) $\mathbb{F} - \mathrm{Prop} \subseteq \varphi^{-1}(\mathbb{F}' - \mathrm{Prop})$

Then it follows that:

1. $\varphi^{-1}(\mathbb{F}' - \mathrm{Sep}) = \mathbb{F} - \mathrm{Sep}$.

2. $\varphi^{-1}(\mathbb{F}' - \mathrm{Prop}) \subseteq \mathbb{F} - \mathrm{Prop}$.

11.31 Example. As an application, let **FSch** be the category of finite type schemes over $\mathbb{C}$, endowed with $\mathbb{F}$ for the closed morphisms and with $\mathbb{F}_0$ for the closed immersions. Let **An** be the category of complex analytic spaces [22]. Let $(-)^{an} :$ **FSch** $\longrightarrow$ **An** be the analytification functor [52], and $|-| :$ **An** $\longrightarrow$ **Top** the forgetful functor to the underlying topological space. We consider the composed functor

$$\varphi = |(-)^{an}| : \textbf{FSch} \longrightarrow \textbf{Top}.$$

One of the fundamental motivations for the adequate nature of the notions of properness and separatedness in algebraic geometry, is the

fact that a morphism $f : X \longrightarrow Y$ of finite type schemes is separated (resp. proper) in the standard sense of Example 11.4 if and only if $|f^{an}|$ is separated (resp. perfect) in **Top** with the standard notions of Example 11.4. The entire dictionary between notions in **FSch** and **An** in [52] is based upon the subtle [52, Proposition 2.2] involving constructible sets. Although closed morphisms and universally closed morphisms are not explicitly considered in the dictionary, [52, Proposition 2.2] readily yields that closed morphisms are reflected by φ. Since by construction, closed immersions of schemes are mapped to closed analytic subspaces by analytification, and hence to closed embeddings under φ, conditions (1)(a) and (2)(a) in Corollary 11.30 are fulfilled, and thus it already follows that separated maps are preserved and reflected, and proper maps are reflected to universally closed maps. We should note however, that the direct proofs of these facts using [52, Proposition 2.2] are equally immediate. On the other hand, the fact that proper maps of schemes (with finite type included in the definition of properness) get mapped to perfect maps under φ requires an entirely different proof, based upon the familiar behaviour of projective morphisms and Chow's Lemma.

Chapter 13

Functional topology with factorization systems

In [8], a general functional topology framework was set up against a "background" factorization system on the category. This approach has some advantages. First of all, if one has natural images of morphisms, one can formulate (and prove) intuitive statements like the fact that "the image of a closed subobject under a closed morphism is again a closed subobject". In fact, such an approach can be situated half-way between the basic setup of the previous chapter and the richer setting of categories endowed with closure operators [11] [7]. Secondly, factorization systems provide a rich source of functional topology examples. Indeed, if a category has a natural class of "closed subobjects", the intuitive statement above can often be taken as the definition of a class of closed morphism.

In this chapter, we define two versions of functional topology with respect to factorization systems. Both are special cases of the functional topology of previous chapter. The first version is the classical one from [8]. We show that the theories of topological spaces, compactly gen-

erated spaces and Lie groups fit into this approach. In particular, we obtain that the Hausdorff objects in the category of compactly generated spaces are the weak Hausdorff spaces and that the compact objects in the category of Lie groups are the compact Lie groups.

A second version of functional topology introduces an auxilliary class $\mathbb{P}$ of surjections which encompasses both the situation of schemes (see next chapter), C^*-algebras and associative algebras. This version is important because it relaxes the condition in [8] which states that each morphism in $\mathbb{E}$ is $\mathbb{F}$-left cancellable. In the case that $\mathbb{P} = \mathbb{E}$ and $\mathbb{E}$ is pullback stable, we get the usual theory of [8]. This second version is in many ways more satisfactory than the first one because we obtain the geometric fact that closed morphisms send closed subobjects to closed subobjects. It is in fact a generalization of Proposition 13.4, which is not applicable in the cases of schemes, C^*-algebras and associative algebras because it does not give the right closed morphisms.

13.1 $(\mathbb{E}, \mathbb{M})$-closed classes

In this section, we discuss $(\mathbb{E}, \mathbb{M})$-closed classes as introduced by Clementino, Giuli and Tholen [8]. Let $\mathcal{C}$ be a category with R-pullbacks and endowed with a proper factorization system $(\mathbb{E}, \mathbb{M})$.

13.1 Definition. [8] An $(\mathbb{E}, \mathbb{M})$-*closed class* $\mathbb{F}$ is a class of morphisms such that the following conditions hold:

  (a) $\mathbb{F}$ contains the isomorphisms and $\mathbb{F}$ is closed under compositions.

  (b) $\mathbb{F} \cap \mathbb{M}$ is R-pullback-stable.

  (c) If $gf \in \mathbb{F}$ and $f \in \mathbb{E}$, then $g \in \mathbb{F}$.

This definition is indeed a special case of our definitions in the previous chapter, so the theorems in previous chapter do apply:

13.2 Proposition. *If the class $\mathbb{F}$ satisfies (a) and (b) in Definition 13.1, then we have that $(\mathbb{F}_0 = \mathbb{F} \cap \mathbb{M}, \mathbb{F})$ is a pre-closed pair on $\mathcal{C}$. Furthermore, if R is the standard relation, then it is a closed pair.*

Proof. Conditions (a) and (b) in Definition 11.13 are fulfilled by definition. Since $\mathbb{E}$ contains the epimorphisms, by Proposition 7.15, $\mathbb{M}$ contains the sections and hence the diagonal of a morphism is in $\mathbb{F} \cap \mathbb{M}$ provided it is in $\mathbb{F}$. □

13.3 Remark. Condition (c) in Definition 13.1 mainly ensures the intuitive fact that "under a closed map, the image of a closed subobject is a closed subobject".

Sometimes we have information about the class $\mathbb{F}_0$ instead of the full class $\mathbb{F}$. The following proposition helps to build the right class $\mathbb{F}$.

13.4 Proposition. *[8, Exercise 2.1.4] Let $\mathbb{F}_0$ be a class of monomorphisms that contains the isomorphisms. Furthermore, assume that $\mathbb{F}_0$ is R-pullback-stable and closed under composition. Define the morphism class $\mathbb{F}$ by saying that $f : X \longrightarrow Y$ belongs to $\mathbb{F}$ if and only if for every $m : X' \longrightarrow X$ in $\mathbb{F}_0$, in the $(\mathbb{E}, \mathbb{M})$-factorization $fm = \mu\varepsilon$ with $\mu \in \mathbb{M}$ and $\varepsilon \in \mathbb{E}$ we have $\mu \in \mathbb{F}_0$. Then*

1. $\mathbb{F}_0 = \mathbb{F} \cap \mathbb{M}$,

2. $\mathbb{F}$ *satisfies (a) and (b) in Definition 13.1,*

3. *If R is the standard relation, then $\mathbb{M}$ is $\mathbb{F}$-left cancellable.*

4. *If $\mathbb{E}$ is stable under R-pullbacks along morphisms in $\mathbb{F}_0$, then $\mathbb{F}$ is an $(\mathbb{E}, \mathbb{M})$-closed class.*

Proof. Most of this will follow from Proposition 13.6 below. We only need to prove (4).

Take $f : X \to Y$ and $g : Y \to Z$ morphisms. Let $gf \in \mathbb{F}$ with $f \in \mathbb{E}$. Let $m \in \mathbb{F}_0$ be a subobject of Y. We need to show that $n = \mu(gfn) \in \mathbb{F}_0$. For this, let P be the R-pullback of m along f. We have the following diagram

$$
\begin{array}{ccccc}
P & \xrightarrow{\overline{f}} & M & \xrightarrow{e} & g(M) \\
\downarrow{\overline{m}} & & \downarrow{m} & & \downarrow{n} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z
\end{array}
$$

Since $\mathbb{F}_0$ is stable under pullbacks, we get that $\overline{m} \in \mathbb{F}_0$. Since $f \in \mathbb{E}$, the assumptions of the theorem ensure that $\overline{f} \in \mathbb{E}$. Thus $n(e\overline{f})$ is the $(\mathbb{E}, \mathbb{M})$-factorization of gfm. Since $gf \in \mathbb{F}$, this is an element of $\mathbb{F}_0$. Thus $g \in \mathbb{F}$.   □

Not every class $\mathbb{F}$ from Definition 13.1 arises from the previous Proposition. In particular, this is true if $\mathbb{F}_0 = \mathbb{M}$. In the next section, rather than generalizing Definition 13.1, we will generalize Proposition 13.4.

13.2 $(\mathbb{E}, \mathbb{M})$-**closed structures**

Let $\mathcal{C}$ be a category with R-pullbacks endowed with a factorization system $(\mathbb{E}, \mathbb{M})$ with $\mathbb{M} \subseteq \mathrm{Mono}$. From now on, we denote the $(\mathbb{E}, \mathbb{M})$-factorization of a morphism $f : X \longrightarrow Y$ by

$$
X \xrightarrow{\varepsilon(f)} f(X) \xrightarrow{\mu(f)} Y.
$$

13.5 Definition. An $(\mathbb{E}, \mathbb{M})$-*pre-closed structure* $(\mathbb{P}, \mathbb{F}_0)$ (or simply *pre-closed structure* if $(\mathbb{E}, \mathbb{M})$ is understood) consists of two classes of morphisms such that:

(a) A class $\mathbb{F}_0 \subseteq \mathbb{M}$ of monomorphisms that contains all the isomorphisms and such that $\mathbb{F}_0$ closed under compositions and R-pullback-stable.

(b) A class $\mathbb{P}$ that contains the isomorphisms and that is closed under compositions.

Morphisms in $\mathbb{F}_0$ are called *closed embeddings* and morphisms in $\mathbb{P}$ are called *surjections*. With respect to a pre-closed structure, a morphism $f : X \longrightarrow Y$ is called *closed* if and only if for every $m : X' \longrightarrow X$ in $\mathbb{F}_0$, we have $\varepsilon(fm) \in \mathbb{P}$ and $\mu(fm) \in \mathbb{F}_0$. The class of closed morphisms is denoted by $\mathbb{F}$. The pre-closed structure is called a *closed structure* if the following condition is satisfied.

(c) For any morphism $f : A \to B$ with A commutative holds that if the diagonal Δ_f of f is in $\mathbb{F}$, then it is in $\mathbb{M}$.

This definition again is a particular case of the definitions of previous chapter, indeed:

13.6 Proposition. *Let* $(\mathbb{P}, \mathbb{F}_0)$ *be an* $(\mathbb{E}, \mathbb{M})$-*pre-closed structure. We have:*

1. $\mathbb{F}$ *is closed under compositions, thus* $(\mathbb{F}_0, \mathbb{F})$ *is a pre-closed pair.*

2. *If R is the standard relation, then each morphism in $\mathbb{M}$ is $\mathbb{F}$-left cancellable.*

3. $\mathbb{F}_0 = \mathbb{F} \cap \mathbb{M}$.

4. If R is the standard relation, then $\mathbb{F}$ is closed under pullback along morphisms in $\mathbb{F}_0$.

Let $(\mathbb{P}, \mathbb{F}_0)$ be an $(\mathbb{E}, \mathbb{M})$-closed structure. We have:

5. If the diagonal Δ_f of f is in $\mathbb{F}$, then it is in $\mathbb{F}_0$.

6. $(\mathbb{F}_0, \mathbb{F})$ is a closed pair.

Proof. For a composition gf and $m \in \mathbb{M}$ we consider the $(\mathbb{E}, \mathbb{M})$-factorizations:

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z \\
\uparrow{\scriptstyle m} & & \uparrow{\scriptstyle \mu} & & \uparrow{\scriptstyle \mu'} \\
X' & \xrightarrow{\ \varepsilon\ } & Y' & \xrightarrow{\ \varepsilon'\ } & Z'
\end{array}
$$

1. Suppose $f, g \in \mathbb{F}$ and $m \in \mathbb{F}_0$. Then from $f \in \mathbb{F}$ we obtain $\varepsilon \in \mathbb{P}$ and $\mu \in \mathbb{F}_0$ and from $g \in \mathbb{F}$ we obtain $\varepsilon' \in \mathbb{P}$ and $\mu' \in \mathbb{F}_0$. Now $\varepsilon'\varepsilon$ and μ' constitute the $(\mathbb{E}, \mathbb{M})$-factorization of gfm and since $\mathbb{P}$ is closed under compositions, it follows that $\varepsilon'\varepsilon \in \mathbb{P}$ and $\mu' \in \mathbb{F}_0$ as desired.

2. Take $f : X \to Y$ and $g : Y \to Z$ morphisms. Let $gf \in \mathbb{F}$ with $g \in \mathbb{M}$ and let $m : X' \to X \in \mathbb{F}_0$ From $gf \in \mathbb{F}$, it follows that $\mu(gfm) = g\mu \in \mathbb{F}_0$ and $\varepsilon(gfm) = \varepsilon \in \mathbb{P}$. Since F_0 is pullback stable and since g is a monomorphism, by Lemma 7.9 we have $\mu \in \mathbb{F}_0$. This proves that $f \in \mathbb{F}$.

3. Suppose $f, m \in \mathbb{F}_0$. Then $fm \in \mathbb{F}_0 \subseteq \mathbb{M}$ hence $\mu(fm) = fm \in \mathbb{F}_0$ and $\varepsilon(fm) = 1_{X'} \in \mathbb{P}$. It follows that $f \in \mathbb{F} \cap \mathbb{M}$. Conversely, if $f \in \mathbb{F} \cap \mathbb{M}$ we consider 1_X as $\mathbb{F}_0$-subobject. It follows that $f = \mu(f1_X) \in \mathbb{F}_0$.

4. Consider the pullback square

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle m'}\Big\uparrow & & \Big\uparrow{\scriptstyle m} \\ P & \xrightarrow[\ f'\ ]{} & Y' \end{array}$$

with $f \in \mathbb{F}$, $m \in \mathbb{F}_0$. Then by (1) $mf' = fm' \in \mathbb{F}$ whence by (2), $f' \in \mathbb{F}$.

5. Immediate from (a), (c) and (3).

6. Immediate by (1) and (5).

$\square$

13.7 Remarks.

1. Suppose in Definition 13.5 we have that $\mathbb{E}$ is a class of epimorphisms. Then by Proposition 7.15, $\mathrm{Mor} \subseteq \mathbb{M} - \mathrm{Sep}$ so condition (c) becomes automatic.

2. Suppose we take $\mathbb{P} = \mathbb{E}$. Then a morphism $f : X \longrightarrow Y$ is closed if and only if for $m : X' \longrightarrow X$ in $\mathbb{F}_0$ we have $\mu(fm) \in \mathbb{F}_0$. Thus, this is precisely the situation of Proposition 13.4.

3. Suppose we take $\mathbb{F}_0 = \mathbb{M}$. Then a morphisms $f : X \longrightarrow Y$ is closed if and only if for $m : X' \longrightarrow X$ in $\mathbb{F}_0$ we have $\varepsilon(fm) \in \mathbb{P}$.

We note that the following does not make use of the assumption that $\mathbb{E}$ consists of epimorphisms in Proposition 13.4:

13.8 Proposition. *Let* $(\mathbb{E}, \mathbb{F}_0)$ *be an* $(\mathbb{E}, \mathbb{M})$-*pre-closed structure. If* $\mathbb{E}$ *is closed under R-pullbacks along* $\mathbb{F}_0$*, then if* $gf \in \mathbb{F}$ *and* $f \in \mathbb{E}$*, then* $g \in \mathbb{F}$.

Proof. Take $f : X \to Y$ and $g : Y \to Z$ morphisms. Let $gf \in \mathbb{F}$ with $f \in \mathbb{E}$. Let $m : M \to Y$ be in F_0 arbitrary. We need to show that $\mu(gm) \in \mathbb{F}_0$. For this, we take the R-pullback P of m and f. We have the following diagram

$$
\begin{array}{ccccc}
P & \xrightarrow{\;\bar{f}\;} & M & \xrightarrow{\;\varepsilon(gm)\;} & g(M) \\
\Big\downarrow{\bar{m}} & & \Big\downarrow{m} & & \Big\downarrow{\mu(gm)} \\
X & \xrightarrow{\;f\;} & Y & \xrightarrow{\;g\;} & Z
\end{array}
$$

Since $\mathbb{F}_0$ is stable under R-pullbacks, we get that $\bar{m} \in \mathbb{F}_0$. Since $f \in \mathbb{E}$, the assumptions of the theorem ensure that $\bar{f} \in \mathcal{E}$. Thus $\mu(gf\bar{m}) = \mu(gm)$. But since gf is closed, this is an element of $\mathbb{F}_0$. $\qquad\square$

In some situations we have $\mathbb{M} = \mathbb{F}_0$. This happens for example in the settings of schemes, C^*-algebras and associative algebras. The following two propositions deal with this case:

13.9 Proposition. *Let $(\mathbb{P}, \mathbb{M})$ be an $(\mathbb{E}, \mathbb{M})$-pre-closed pair. Let $\mathbb{F}$ be the class of closed morphisms with respect to this pre-closed pair.*

1. *Each morphism in $\mathbb{M}$ is $\mathbb{F}$-left cancellable.*

2. *$\mathbb{F}$ is closed under pullback along morphisms in $\mathbb{M}$*

Proof.   1. This is proven like Proposition 13.6(2), where we now have that $\mathbb{F}_0 = \mathbb{M}$ is pullback stable (for the standard categorical pullbacks) by Proposition 7.13(3).

2. This is proven like Proposition 13.6(4), where we finish the proof by (1) of the current proposition.

$\qquad\square$

Conversely, we have:

13.10 Proposition. *Let $\mathbb{F}$ be a pre-closed class. Assume that $\mathbb{M} \subseteq \mathbb{F}$ and assume that each morphism in $\mathbb{M}$ is $\mathbb{F}$-left cancellable. Then $(\mathbb{E} \cap \mathbb{F}, \mathbb{M})$ is an $(\mathbb{E}, \mathbb{M})$-pre-closed pair, and $\mathbb{F}$ is its class of closed morphisms.*

Proof. That $(\mathbb{E} \cap \mathbb{F}, \mathbb{M})$ is an $(\mathbb{E}, \mathbb{M})$-pre-closed pair is obvious. Now assume that $f \in \mathbb{F}$ and $m \in \mathbb{M}$, then it holds obviously that $\mu(fm) \in \mathbb{M}$ and $\epsilon(fm) \in \mathbb{E}$. But since $\mu(fm)\epsilon(fm) = fm \in \mathbb{F}$, it follows by the hypothesis that $\epsilon(fm) \in \mathbb{F}$.
Conversely, assume that $f : A \to B$ is closed with respect to the $(\mathbb{E}, \mathbb{M})$-pre-closed pair $(\mathbb{E} \cap \mathbb{F}, \mathbb{M})$. Since $\mathrm{Id} \in \mathbb{M}$, it follows that $f = f1_A = \epsilon(f1_A) \in \mathbb{F}$. We also have $\mu(f1_A) \in \mathbb{M} \subseteq \mathbb{F}$. Thus it follows that $f \in \mathbb{F}$ by composition. $\qquad\square$

13.3 Examples

Topological situations

It is well-know that in **Top**, the axioms of 13.1 are satisfied with $\mathbb{F}$ the class of closed maps.
We can also consider $\mathbb{F}$ the class of open maps. In that case, the separated maps are the locally injective continuous functions and the Hausdorff spaces are the discrete spaces.

In the category k**Top**, we can take $\mathbb{F}_0$ the class of all closed embeddings. This satisfies all the axioms of Proposition 13.4, namely it is a class of monomorphisms that contains the isomorphisms, is pullback-stable and is closed under compositions. There is a nice interpretation of the Hausdorff-objects in this category as the weakly Hausdorff

spaces (See Strickland [54]). Recall that a space X is weakly Hausdorff if for every compact Hausdorff space K and every continuous map $u : K \to X$ we have that $u(K)$ is closed in X.

Noncommutative affine schemes

Let **Rng** be the category of all (possibly noncommutative) unital rings with as morphisms the ring homomorphisms. The noncommutative affine schemes are the category $\mathbf{Rng}^{\mathrm{op}}$. The tensor product gives an R-product in the category $\mathbf{Rng}^{\mathrm{op}}$, where f and g are R-related if the associated ring homomorphisms commute.

Since we have a factorization system on **Rng** consisting on injective ring homomorphisms and surjective ring homomorphisms, this induces a factorization system on $\mathbf{Rng}^{\mathrm{op}}$. Now we can put $\mathbb{F}_0 = \mathbb{M}$ and this satisfies the hypotheses of Proposition 13.4. But under this formalism, it is clear that every ring homomorphism becomes closed and thus even proper. As such, looking at its restriction to the subcategory of commutative rings, we see that this approach does not yield anything useful (see the treatment of schemes later on).

Diffeological spaces

As we have seen, the category of diffeological spaces admits a factorization system $(\mathbb{E}, \mathbb{M})$, where $\mathbb{E}$ consists of the smooth surjections and where $\mathbb{M}$ consists of the inductions. One is tempted to take as class $\mathbb{F}_0$ the class of all closed embeddings, but this does not work since the closed embeddings in the category of diffeological spaces are not pullback-stable. Indeed, consider the closed subset $S = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$ in $\mathbb{R}$ and consider a smooth map $f : \mathbb{R} \to \mathbb{R}$

that vanishes exactly on S. Then we can form the pullback

$$
\begin{array}{ccc}
S & \longrightarrow & \mathbb{R} \\
\downarrow & & \downarrow {\scriptstyle f} \\
\{0\} & \longrightarrow & \mathbb{R}
\end{array}
$$

Then one can show that the injection $S \to \mathbb{R}$ is not an embedding, and in fact one has that S has the discrete topology See [6]. The issue is of course that closed subsets of a diffeological space (even $\mathbb{R}$) are not necessarily embedded.

However, we see that open subsets are always embedded. This suggests that we take as $\mathbb{F}_0$ the class of open embeddings. Then all the hypotheses in Proposition 13.4 are fulfilled.

Lie groups

The category of Lie groups with Lie homomorphisms has a factorization system consisting of smooth, injective and immersive homomorphisms and surjective homomorphisms. We let $f : G \to H$ be in $\mathbb{F}$ if it is closed. The axioms of 13.1 are then satisfied. We check (b). Let G be a closed subgroup of H, then for an arbitrary Lie homomorphism $f : H' \to H$ then we have that $f^{-1}(G)$ is closed in H'. But a closed subgroup is always an embedded subgroup by the closed subgroup theorem. So the map $f^{-1}(G) \to H'$ is a closed embedding.

The compact objects in this category correspond exactly to the compact Lie groups (this was proven in [12]).

C^*-algebras

For a morphism $f : A \to B$ in the category C^*-**Alg**$^{\text{op}}$, we write $F : B \to \mathcal{M}(A)$ for the associated Woronowicz-morphisms. We have a factorization system on this category where $f : A \to B$ is in $\mathbb{M}$ if $F : B \to \mathcal{M}(A)$ satisfies $F(B) = A$, and f is in $\mathbb{E}$ if F is an isometry.

We define classes $\mathbb{F}_0 = \mathbb{M}$ and we let $f \in \mathbb{F}$ if $F : B \to \mathcal{M}(A)$ satisfies $F(B) \subseteq A$. We are in the situation of Proposition 13.10, and thus $(\mathbb{E} \cap \mathbb{F}, \mathbb{F}_0)$ is an $(\mathbb{E}, \mathbb{M})$-pre-closed structure and $\mathbb{F}$ is its closed maps. We are in the situation of §11.4.

Associative Algebras

For a morphism $f : A \to B$ in the category $k - $ **Alg**$^{\text{op}}$, we write $F : B \to \mathcal{M}(A)$ for the associated Woronowicz-morphism. We have a factorization system on this category where $f : A \to B$ is in $\mathbb{M}$ if $F : B \to \mathcal{M}(A)$ satisfies $F(B) = A$, and f is in $\mathbb{E}$ if F is injective.

We define classes $\mathbb{F}_0 = \mathbb{M}$ and we let $f \in \mathbb{F}$ if $F : B \to \mathcal{M}(A)$ satisfies $F(B) \subseteq A$. We are in the situation of Proposition 13.10, and thus $(\mathbb{E} \cap \mathbb{F}, \mathbb{F}_0)$ is an $(\mathbb{E}, \mathbb{M})$-pre-closed structure and $\mathbb{F}$ is its closed maps. We are in the situation of §11.4.

Schemes

Schemes do not fit into the framework of Definition 13.1. Indeed, we cannot define $\mathbb{F}$ the class of closed maps, since then $(F3)$ will not hold. To see this, consider S the discrete valuation ring $\mathbb{C}[[X]]$. Then $\text{Spec}(S)$ has two points: one closed point coming from the maximal ideal M and one point 0 coming from the zero ideal that is dense

in $\mathrm{Spec}(S)$. Now consider $X = \coprod_n \mathrm{Spec}(S/M^n)$ and consider the canonical morphism $f : X \to \mathrm{Spec}(S)$. It can be shown (see Gortz, Wedhorn [19] that $f(X)$ is the closed point $\{M\}$ and that the scheme-theoretic image of f is entire $\mathrm{Spec}(S)$ (and thus $f \in \mathbb{E}$).
Consider the canonical embedding $\varphi : \mathbb{C}[X] \to \mathbb{C}[[X]]$. Then it is easy to see that $g = \mathrm{Spec}(\varphi)$ has $g(0) = 0$ and $g(M) = (X)$. Hence fg is closed, but g is not closed (since $\{0, (X)\}$ is not closed in $\mathbb{C}[X]$).

Applying Proposition 13.4 to $\mathbb{F}_0$ the class of closed immersions does not work either, since then every morphism would be closed (and thus proper).

The theory of Definition 13.5 *does* work well, as will be seen in next chapter.

Chapter 17

Functional topology and schemes

If we want to apply the theory of the previous chapter to the category of schemes, we are faced with a number of difficulties. This category does have a natural notion of closed subobjects, the closed immersions, but there is no natural class of general (i.e. not necessarily closed) subobjects which is part of a factorization system. Indeed, it is well known that for a morphism of schemes, the image of the underlying morphism of topological spaces can obviously be endowed with the subspace topology, but it can not be endowed further with a scheme structure. In contrast, the closed immersions *are* the second class of a factorization system on schemes, the first class being given by the scheme theoretically dominant morphisms. If we compare this with the setting of a category with an "original" factorization system and a closure operator, which gives rise to a second "shadow" factorization system of dense morphisms and closed embeddings, then it is as if we find ourselves in a situation where only the shadow factorization system is present and everything else has vanished.

In this section, we show that the category of schemes is naturally in-

cluded in the theory of closed structures from §13.2. Our inspiration comes from the situation where we have two factorization systems as just described (see §17.1). If we consider for instance topological spaces, then we have the original factorization system of surjections followed by embeddings, and the shadow factorization system of dense morphisms followed by closed embeddings. A morphism of topological spaces has a closed image precisely when the two factorizations coincide. To express this, we can take either of the two factorization systems as primary: we can say that in the original factorization, the embedding has to be a closed embedding, or equivalently we can say that in the shadow factorization, the dense morphism has to be a surjection. Thus we can use the theory of closed structures of the previous chapter.

In order to understand the relation between the closed morphisms provided by this approach on the one hand, and morphisms with underlying closed morphism of topological spaces on the other hand, we are faced with the problem that the image factorization in **Scheme** is not necessarily mapped to the (dense map, closed embedding) factorization in **Top** by the forgetful functor. For morphisms where this does hold true, our approach actually yields the correct notion of closedness. In §17.3, we identify a broad class of morphisms - encompassing for instance all quasi-compact morphisms - for which this applies. Our treatment makes use of the larger category **LocRingedSpace**, and a relative "schematization" closure operator on the inclusion **Scheme** $\longrightarrow$ **LocRingedSpace**, which we introduce on the way (§17.2).

17.1   Two factorization systems

Let $\iota : \mathcal{C} \longrightarrow \mathcal{D}$ be the inclusion of a full subcategory, and suppose both categories are finitely complete. Suppose $\mathcal{D}$ is endowed with a factorization system $(\mathbb{E}, \mathbb{M})$ and $\mathcal{C}$ with a factorization system $(\mathbb{D}, \mathbb{F}_0)$

such that $\mathbb{F}_0 \subseteq \mathbb{M} \subseteq \text{Mono}_{\mathcal{D}}$. Consequently,

- $\mathbb{F}_0 \subseteq \text{Mono}_{\mathcal{C}}$,

- $\mathbb{E} = \mathbb{M} - \text{Dense in } \mathcal{D}$,

- $\mathbb{D} = \mathbb{F}_0 - \text{Dense in } \mathcal{C}$

- $\mathbb{E} \cap \mathcal{C} \subseteq \mathbb{D}$, indeed: a morphism $f \in \mathbb{E} \cap \mathcal{C}$ is orthogonal to any morphism in $\mathbb{M}$ and thus also to any morphism in $\mathbb{F}_0$. It follows that $f \in \mathbb{D}$.

For a morphism $f : X \longrightarrow Y$ in $\mathcal{D}$ we denote the $(\mathbb{E}, \mathbb{M})$-factorization as in §13.2 and for f in $\mathcal{C}$ we denote the $(\mathbb{D}, \mathbb{F}_0)$-factorization by

$$X \xrightarrow{\quad \delta(f) \quad} \overline{f(X)} \xrightarrow{\quad \varphi(f) \quad} Y.$$

17.1 Lemma. *For a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$, the following are equivalent:*

1. *The $(\mathbb{E}, \mathbb{M})$-factorization and the $(\mathbb{D}, \mathbb{F}_0)$-factorization of f are isomorphic.*

2. $\mu(f) \in \mathbb{F}_0$.

3. $\delta(f) \in \mathbb{E}$.

Proof. Assume that the $(\mathbb{E}, \mathbb{M})$-factorization and $(\mathbb{D}, \mathbb{F}_0)$-factorizations of f are isomorphic, this means that there is an isomorphism θ such that the following commutes:
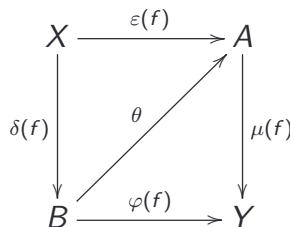
Since $\theta^{-1}\varepsilon(f) = \delta(f)$ and since $\mathbb{E}$ is closed under isomorphisms, it follows that $\delta(f) \in \mathbb{E}$. Analogously, it follows that $\mu(f) \in \mathbb{F}_0$.

Conversely, suppose that $\mu(f) : A \to Y$ is in $\mathbb{F}_0$. It then follows of course that $A \in \mathcal{C}$. Thus $\varepsilon(f) \in \mathbb{E} \cap \mathcal{C}$ and hence it follows that $\varepsilon(f) \in \mathcal{D}$. Thus $f = \mu(f)\varepsilon(f)$ and $f = \varphi(f)\delta(f)$ are two $(\mathbb{D}, \mathbb{F}_0)$ factorizations and thus are equivalent by Proposition 7.12.

Finally, assume that $\delta(f) \in \mathbb{E}$. Since $\varphi(f) \in \mathbb{F}_0 \subseteq \mathbb{M}$, it follows that $f = \mu(f)\varepsilon(f)$ and $f = \varphi(f)\delta(f)$ are two $(\mathbb{E}, \mathbb{M})$ factorizations and thus are equivalent by Proposition 7.12.   □

Put $\mathbb{E}' = \mathbb{E} \cap \mathcal{C}$.

17.2 Proposition. *We have that $(\mathbb{E}', \mathbb{F}_0)$ is a $(\mathbb{D}, \mathbb{F}_0)$-pre-closed structure, and the following are equivalent for $f : X \longrightarrow Y$ in $\mathcal{C}$:*

1. *f is closed.*

2. *For every $m : X' \longrightarrow X$ in $\mathbb{F}_0$, the $(\mathbb{E}, \mathbb{M})$-factorization and the $(\mathbb{D}, \mathbb{F}_0)$-factorization of fm are isomorphic.*

In case $\iota = 1_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$, $(\mathbb{E}, \mathbb{F}_0)$ is also an $(\mathbb{E}, \mathbb{M})$-pre-closed structure and f is closed with respect to $(\mathbb{E}, \mathbb{M})$ if and only if it is closed with respect to $(\mathbb{D}, \mathbb{F}_0)$. Further, $(\mathbb{E}, \mathbb{F}_0)$ is an $(\mathbb{E}, \mathbb{M})$-closed structure if and only if it is a $(\mathbb{D}, \mathbb{F}_0)$-closed structure.

Proof. It is clear that $(\mathbb{E}', \mathbb{F}_0)$ is a $(\mathbb{D}, \mathbb{F}_0)$-pre-closed structure. Now assume that f is closed, this means that $\delta(fm) \in \mathbb{E}'$ and $\varphi(fm) \in \mathbb{F}_0$. It follows from previous lemma that the $(\mathbb{E}, \mathbb{F}_0)$ and the $(\mathbb{D}, \mathbb{F}_0)$-factorizations are isomorphic.
Conversely, if the $(\mathbb{E}, \mathbb{M})$-factorization and $(\mathbb{D}, \mathbb{F}_0)$-factorizations are isomorphic. This implies by previous lemma that $\delta(fm) \in \mathbb{E}$. This

implies directly that f is closed.

Assume now that $\iota = 1_{\mathcal{C}}$. It is clear that $(\mathbb{E}, \mathbb{F}_0)$ is an $(\mathbb{E}, \mathbb{M})$-pre-closed structure.

Assume that f is closed with respect to $(\mathbb{E}, \mathbb{M})$, we then have that $\varepsilon(f) \in \mathbb{E}$ and $\mu(f) \in \mathbb{F}_0$. Since then follows that $\varepsilon(f) \in \mathbb{E} \cap \mathcal{C} \subseteq \mathbb{D}$, it follows that $f = \mu(f)\varepsilon(f)$ is the $(\mathbb{D}, \mathbb{F}_0)$-factorization of f, and thus follows that f is closed.

Conversely, if f is closed with respect to $(\mathbb{D}, \mathbb{F}_0)$, we have that $\delta(f) \in \mathbb{D}$ and $\varphi(f) \in \mathbb{F}_0$. From previous lemma follows that the $(\mathbb{E}, \mathbb{M})$-factorization and $(\mathbb{D}, \mathbb{F}_0)$-factorization are isomorphic, and thus f is closed with respect to $(\mathbb{E}, \mathbb{M})$.

The last statement follows at once from the previous. $\qquad\square$

Let $\mathbb{F}$ denote the class of closed morphisms as described in Proposition 17.2

17.3 Lemma. $\mathbb{F} \cap \mathbb{D} \subseteq \mathbb{E}$.

Proof. Consider $f \in \mathbb{F} \cap \mathbb{D}$. Since $f \in \mathbb{F}$, we have $\mu(f) = \varphi(f)$ and $\varepsilon(f) = \delta(f)$. But since $f \in \mathbb{D}$, $\varphi(f)$ is an isomorphism and hence $f \cong \varepsilon(f) \in \mathbb{E}$. $\qquad\square$

17.4 Example. The situation (with $\iota = 1_{\mathcal{C}}$) applies if $(\mathbb{E}, \mathbb{M})$ is a factorization system on $\mathcal{C}$ and $\mathbb{F}_0$ is the class of closed subobjects with respect to an idempotent, weakly hereditary closure operator on $\mathcal{C}$ in the sense of [11]. In §17.2, we will present a notion of relative closure operator which yields examples with $\iota \neq 1_{\mathcal{C}}$.

17.5 Example. On **Top** with its Kuratowski closure, we make the following choices:

- $\mathbb{M}$ is the class of embeddings.

- $\mathbb{E}$ is the class of surjective morphisms.

- $\mathbb{F}_0$ is the class of closed embeddings.

- $\mathbb{D}$ is the class of dense morphisms.

The $(\mathbb{E}, \mathbb{M})$-factorization of $f : X \longrightarrow Y$ arises from endowing the set theoretic image $f(X)$ with the subspace topology from Y, and the $(\mathbb{D}, \mathbb{F}_0)$-factorization of f arises from endowing the closure $\overline{f(X)}$ with the subspace topology from Y.

Then $(\mathbb{E}, \mathbb{F}_0)$ is an $(\mathbb{E}, \mathbb{M})$-closed structure by Proposition 17.2 and Remarks 13.7 (1), and consequently $(\mathbb{E}, \mathbb{F}_0)$ is a $(\mathbb{D}, \mathbb{F}_0)$-closed structure with the same class $\mathbb{F}$ of closed maps. Clearly, these are precisely the usual closed maps i.e. morphisms mapping closed subsets to closed subsets. Lemma 17.3 now becomes the familiar fact that a morphism which is closed and dense is surjective.

17.2   Relative closure operators

In this section we collect some facts concerning closure operators relative to an inclusion functor in the sense of [5]. Let $\iota : \mathcal{C} \longrightarrow \mathcal{D}$ be the inclusion of a full subcategory where $\mathcal{D}$ is endowed with a factorization system $(\mathbb{E}, \mathbb{M})$ with $\mathbb{M} \subseteq \mathrm{Mono}(\mathcal{D})$. We suppose both categories are finitely complete. We suppress ι in all notations. For $X \in \mathcal{C}$, we denote by $\mathrm{Sub}_{\mathcal{C}}(X)$ the $\mathbb{M}$-subobjects in $\mathcal{C}$ and by $\mathrm{Sub}_{\mathcal{D}}(X)$ the $\mathbb{M}$-subobjects in $\mathcal{D}$. For a morphism $f : X \longrightarrow Y$ in $\mathcal{D}$ and $m \in \mathrm{Sub}_{\mathcal{D}}(X)$, the *image* of m under f is by definition the $\mathbb{M}$-part of the $(\mathbb{E}, \mathbb{M})$-factorization of the composition fm, and is denoted by $f(m)$. The *image* of f is by definition the image of 1_X under f, and is denoted $f(X) = f(1_X)$.

17.6 Definition. Let $X \in \mathcal{C}$ and let $c = (c_X)_{X \in \mathcal{C}}$ consist of a family of maps

$$c_X : \mathrm{Sub}_{\mathcal{D}}(X) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(X).$$

1. For $X \in \mathcal{C}$, a subobject $m \in \mathrm{Sub}_{\mathcal{C}}(X)$ is *c-closed* if $c_X(m) = m$.

2. For $X \in \mathcal{C}$, a subobject $m \in \mathrm{Sub}_{\mathcal{D}}(X)$ is *c-dense* if $c_X(m) = 1_X$.

3. A morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is *c-closed* if for every c-closed $m \in \mathrm{Sub}_{\mathcal{C}}(X)$, the image $f(m)$ is c-closed as well.

4. A morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is *c-dense* if the image $f(X) = f(1_X)$ is c-dense, i.e. $c_Y(f(1_X)) = 1_Y$.

Consider the following conditions:

1. (*extension*) For $X \in \mathcal{C}$ and $m \in \mathrm{Sub}_{\mathcal{D}}(X)$, we have $m \leq c_X(m)$. In this case we denote the canonical $\mathbb{M}$-morphism from the domain of m to the domain of $c_X(m)$ by $b_X(m)$, i.e we have $m = c_X(m)b_X(m)$.

2. (*monotonicity*) For $X \in \mathcal{C}$ and $m \leq m'$ in $\mathrm{Sub}_{\mathcal{D}}(X)$, we have $c_X(m) \leq c_X(m')$ in $\mathrm{Sub}_{\mathcal{C}}(X)$.

3. (*continuity*) For $f : X \longrightarrow Y$ in $\mathcal{C}$ and $m \in \mathrm{Sub}_{\mathcal{D}}(X)$, we have $f(c_X(m)) \leq c_Y(f(m))$.

4. (*idempotency*) For $X \in \mathcal{C}$ and $m \in \mathrm{Sub}_{\mathcal{D}}(X)$, we have that $c_X(c_X(m)) = c_X(m)$.

5. (*weak heredity*) For $X \in \mathcal{C}$ and $m \in \mathrm{Sub}_{\mathcal{D}}(X)$, $b_X(m)$ is a c-dense subobject of the domain of $c_X(m)$.

If c satisfies (1) and (2), it is called a *local closure operator on ι*. If c further satisfies (3), it is called a *closure operator on ι*. A (local) closure operator is called *idempotent* if it satisfies (4) and *weakly hereditary* if it satisfies (5).

17.7 Remark. The definition of c-closed maps given in (3) is most useful for idempotent closure operators. For a nonidempotent closure operator, it is more important to understand how the image preserves closures, rather than closed subobjects.

Like for usual closure operators, we have the following familiar characterizations of continuity:

17.8 Proposition. *Let c be a local closure operator on $\iota : \mathcal{C} \longrightarrow \mathcal{D}$. Consider the following conditions:*

(3) *c satisfies (3), i.e. c is a closure operator.*

(3') *For $f : X \longrightarrow Y$ in $\mathcal{C}$ and $n \in Sub_{\mathcal{D}}(Y)$, we have $c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n))$.*

(3″) *For every c-closed $n \in Sub_{\mathcal{C}}(Y)$, we have that $f^{-1}(n) \in Sub_{\mathcal{D}}(X)$ is c-closed as well.*

Conditions (3) and (3') are equivalent. If c is idempotent, these conditions are further equivalent to (3″).

Inspired by [11], one can use certain closure operators to construct factorization systems on $\mathcal{C}$.

17.9 Proposition. *Let $\iota : \mathcal{C} \longrightarrow \mathcal{D}$ and $(\mathbb{E}, \mathbb{M})$ be as above. Let c be an idempotent, weakly hereditary closure operator on ι. Let $\mathbb{M}_c$ consist of the c-closed $\mathbb{M}$-subobjects of $\mathcal{C}$-objects and let $\mathbb{E}_c$ consist of the c-dense $\mathcal{C}$-morphisms.*

Then $(\mathbb{E}_c, \mathbb{M}_c)$ is a factorization system on $\mathcal{C}$, and for a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ with $(\mathbb{E}, \mathbb{M})$-factorization given by $f = \mu(f)\varepsilon(f)$, the $(\mathbb{E}_c, \mathbb{M}_c)$-factorization is given by $\mu_c(f) = c_Y(\mu(f))$ and $\varepsilon_c(f) = b_Y(\mu(f))\varepsilon(f)$.

Proof. In the proposed $(\mathbb{E}_c, \mathbb{M}_c)$-factorization, the map

$$\mu_c(f) = c_Y(\mu(f))$$

is c-closed by idempotency of c and $\varepsilon_c(f) = b_Y(\mu(f))\varepsilon(f)$ is c-dense since c is weakly hereditary. Using all the properties of a closure operator, it is further shown that the c-dense morphisms are $\mathbb{M}_c$-orthogonal. $\qquad\square$

Clearly, the results of §17.1 apply and we obtain:

17.10 Proposition. *Let* $\iota : \mathcal{C} \longrightarrow \mathcal{D}$, $(\mathbb{E}, \mathbb{M})$ *on* $\mathcal{D}$ *and* $(\mathbb{E}_c, \mathbb{M}_c)$ *on* $\mathcal{C}$ *be as above and put* $\mathbb{E}' = \mathbb{E} \cap \mathcal{C}$. *Then* $(\mathbb{E}', \mathbb{M}_c)$ *is an* $(\mathbb{E}_c, \mathbb{M}_c)$-*pre-closed structure on* $\mathcal{C}$ *and a* $\mathcal{C}$-*morphism is closed with respect to this structure if and only if it is c-closed.*

At the other extreme from applications with $\iota = 1_{\mathcal{C}}$, we have the following interesting situation. Let $\iota : \mathcal{C} \longrightarrow \mathcal{D}$ and $(\mathbb{E}, \mathbb{M})$ on $\mathcal{D}$ be as above. Suppose that for every $X \in \mathcal{C}$ and $m \in \mathsf{Sub}_{\mathcal{D}}(X)$, there exists a smallest $m' \in \mathsf{Sub}_{\mathcal{C}}(X)$ with $m \leq m'$. Put $c_X(m)$ equal to this smallest subobject.

17.11 Proposition. *We have that* $c = (c_X)_{X \in \mathcal{C}}$ *is an idempotent, weakly hereditary local closure operator on ι. If for every* $f : X \longrightarrow Y$ *in* $\mathcal{C}$ *and* $m \in \mathsf{Sub}_{\mathcal{C}}(Y)$, *the* $\mathcal{D}$-*pullback* $f^{-1}(m)$ *is in* $\mathcal{C}$, *c is an idempotent, weakly hereditary closure operator.*

We have $\mathbb{M}_c = \mathbb{M} \cap \mathcal{C}$, *and an arbitary $\mathcal{C}$-morphism f is c-closed if and only if for every $\mathbb{M}$-subobject in $\mathcal{C}$, the image $f(m)$ (based upon the $(\mathbb{E}, \mathbb{M})$-factorization of fm) lies in $\mathcal{C}$.*

If, for a given $\iota : \mathcal{C} \longrightarrow \mathcal{D}$ the local closure operator described in Proposition 17.11 exists, we denote it by c_{ι}. We end this section by

illustrating the special role of c_ι. Let c be an arbitrary local closure operator on ι. By restricting the domains of the maps

$$c_X : \mathsf{Sub}_{\mathcal{D}}(X) \longrightarrow \mathsf{Sub}_{\mathcal{C}}(X)$$

to $\mathsf{Sub}_{\mathcal{C}}(X)$, we obviously obtain a local closure operator $c_{\mathcal{C}}$ on $\mathcal{C}$, which inherits all the properties from c. We now have:

17.12 Proposition. *Suppose c is idempotent. Then we have*

$$c = c_{\mathcal{C}} \circ c_\iota,$$

i.e. for every $m \in \mathsf{Sub}_{\mathcal{D}}(X)$ with $X \in \mathcal{C}$, we have

$$c_X(m) = (c_{\mathcal{C}})_X (c_\iota)_X(m).$$

Proof. We omit the subscript X from the notation. Since $m \leq c_\iota(m)$, we have $c(m) \leq c(c_\iota(m)) = c_{\mathcal{C}}(c_\iota(m))$. Conversely, by definition of c_ι, we have $c_\iota(m) \leq c(m)$. Thus, it follows that $c_{\mathcal{C}}(c_\iota(m)) \leq c_{\mathcal{C}}(c(m)) = c(c(m)) = c(m)$. $\qquad\square$

17.3   Application to schemes

Let **LocRingedSpace** be the category of locally ringed spaces and **Scheme** the category of schemes. In this section, we present an application of §17.2 to the inclusion

$$\iota : \mathbf{Scheme} \longrightarrow \mathbf{LocRingedSpace}.$$

To start, we will endow the category **LocRingedSpace** with a factorization system.

17.13 Remark. The essential property of locally ringed spaces that we use is the fact that $\mathrm{supp}(\mathcal{O}_X) = \{x \in X \mid (\mathcal{O}_X)_x \neq 0\} = X$ or, equivalently, $\mathcal{O}_X(V) \neq 0$ for $V \neq \varnothing$. Everything goes through with **LocRingedSpace** replaced by the category of ringed spaces satisfying this property.

Let $(X, \mathcal{O}_X)$ be a locally ringed space. There is a corresponding Grothendieck category $\mathrm{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules. We let $\mathrm{Sub}(\mathcal{O}_X)$ be the lattice of subobjects of $\mathcal{O}_X$ in $\mathrm{Mod}(\mathcal{O}_X)$, i.e. the lattice of ideal sheaves $\mathcal{I} \subseteq \mathcal{O}_X$. Let **LocRingedSpace**$/X$ be the category of all the morphisms $f : Y \longrightarrow X$ in **LocRingedSpace**.

We consider the following map:

$$\mathcal{I} : \mathrm{Ob}(\mathbf{LocRingedSpace}/X) \longrightarrow \mathrm{Sub}(\mathcal{O}_X)$$

with

$$\mathcal{I}(f) = \mathrm{Ker}(f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y).$$

In the category **LocRingedSpace**, we recall the following special type of subobjects.

17.14 Definition. A morphism $f : Y \longrightarrow X$ of locally ringed spaces is a *closed immersion* if the following conditions hold:

1. $f : Y \longrightarrow X$ is a homeomorphism of Y onto a closed subspace $f(Y) \subseteq X$.

2. $f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is an epimorphism of sheaves.

A closed immersion of locally ringed spaces is a monomorphism in the category **LocRingedSpace**. If $f : Y \longrightarrow X$ is a closed immersion, we call Y a closed subobject (or closed subspace) of X. Let $\mathrm{Sub}_{cl}(X)$ denote the poset of (isomorphism classes of) closed subobjects of X. We obtain a restriction

$$\mathcal{I} : \mathrm{Sub}_{cl}(X) \longrightarrow \mathrm{Sub}(\mathcal{O}_X) : (i : Z \longrightarrow X) \longmapsto \mathcal{I}(Z) = \mathcal{I}(i).$$

17.15 Example. Let X be a locally ringed space and let $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal. We describe an associated subspace $Z(\mathcal{I})$ with $\mathcal{I}(Z(\mathcal{I})) = \mathcal{I}$.

The sheaf of rings $\mathcal{O}_X/\mathcal{I}$ is finitely generated and hence

$$Z = \mathrm{supp}(\mathcal{O}_X/\mathcal{I}) = \{x \in X \mid (\mathcal{O}_X)_x/\mathcal{I}_x \neq 0\}$$

is a closed subset of X. For the inclusion $i : Z \longrightarrow X$ we put $\mathcal{O}_Z = i^{-1}(\mathcal{O}_X/\mathcal{I})$. It follows that $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$ whence we can put $i^\sharp : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z$ equal to the canonical quotient map. Clearly, this is a morphism of locally ringed spaces and we have made Z into a closed subspace of X with associated ideal given by $\mathcal{I}$.

17.16 Proposition. *Let $f : Y \longrightarrow X$ be a morphism of locally ringed spaces and let $i : Z \longrightarrow X$ be a closed immersion. The following are equivalent:*

1. *There is a unique morphism $g : Y \longrightarrow Z$ with $f = ig$.*

2. $\mathcal{I}(i) \subseteq \mathcal{I}(f)$.

Proof. Since i is a monomorphism, unicity of g will be automatic.

Suppose first the existence of g. We thus obtain $f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ as a composition $f^\sharp \cong i_*(g^\sharp)i^\sharp$. Thus, $\mathcal{I}(i) = \mathrm{Ker}(i^\sharp) \subseteq \mathrm{Ker}(f^\sharp) = \mathcal{I}(f)$.

Suppose next that $\mathcal{I}(i) \subseteq \mathcal{I}(f)$. We may assume that $Z \subseteq X$ is a closed subspace. We have $f(Y) \subseteq \mathrm{supp}(f_*\mathcal{O}_Y)$ and $Z = \mathrm{supp}(i_*\mathcal{O}_Z)$. Since $i^\sharp : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z$ is surjective, the inclusion between ideals yields a factorization $f^\sharp = \varphi i^\sharp$ for some $\varphi : i_*\mathcal{O}_Z \longrightarrow f_*\mathcal{O}_Y$. Hence, if $(i_*\mathcal{O}_Z)_x = 0$ we have $f_x^\sharp = 0$ and hence $(f_*\mathcal{O}_Y)_x = 0$. This already shows that $f(Y) \subseteq Z$. Denote by $g : X \longrightarrow X$ the natural factorization such that $f = ig$. We have $f_*\mathcal{O}_Y \cong i_*g_*\mathcal{O}_Y$. On the level of sheaves of rings, we define $g^\sharp = i^{-1}(\varphi)$. Now $f = ig$ as morphisms of ringed spaces, as desired. $\square$

Recall that a *duality* between posets is an order reversing isomorphism.

17.17 Proposition. *The map $\mathcal{I}$ defines a duality*

$$\mathcal{I} : Sub_{cl}(X) \longrightarrow Sub(\mathcal{O}_X) : (i : Z \longrightarrow X) \longmapsto \mathcal{I}(\mathcal{Z}).$$

Its inverse is given by the construction in Example 17.15.

Proof. The construction in Example 17.15 shows $\mathcal{I}$ to be surjective. Furthermore, by Proposition 17.16 we have $Z_1 \leq Z_2$ if and only if $\mathcal{I}(Z_2) \subseteq \mathcal{I}(Z_1)$. □

For a morphism $f : Y \longrightarrow X$ in **LocRingedSpace**, we call the subspace $Z(\mathcal{I}(f)) \subseteq X$ the *locally ringed image of f*.

Let $\mathbb{M}$ be the class of closed immersions and $\mathbb{E}$ the class of $\mathbb{M}$-dense morphisms.

17.18 Proposition. $(\mathbb{E}, \mathbb{M})$ *constitutes a factorization system on the category* **LocRingedSpace**. *The image of a morphism with respect to this factorization system is the locally ringed image.*

Proof. This follows from Lemma 7.18 (2) by Proposition 17.16. □

Next we take a closer look at the class $\mathbb{E}$.

17.19 Definition. We say that a morphism $f : Y \longrightarrow X$ is *ringed dominant* if $f^{\sharp} : \mathcal{O}_X \longrightarrow f_* \mathcal{O}_Y$ is a monomorphism.

17.20 Proposition. *A morphism of locally ringed spaces is $\mathbb{M}$-dense if and only if it is ringed dominant.*

Proof. By definition f is $\mathbb{M}$-dense if and only if the only closed immersion through which f factors is an isomorphism. By Proposition 17.16, this corresponds to the fact that the only ideal inside $\mathcal{I}(f)$ is the zero ideal. But this means that $\mathcal{I}(f) = 0$, or, equivalently, that $f^{\sharp}$ is a monomorphism. □

17.21 Proposition. *If* $f : Y \longrightarrow X$ *is* $\mathbb{M}$*-dense, the underlying morphism of topological spaces is dense, i.e.* $\overline{f(Y)} = X$.

Proof. If f is not topologically dense, there is an open subset $\varnothing \neq U \subseteq X$ with $U \cap f(Y) = \varnothing$. Hence, $f_*\mathcal{O}_Y(U) = \mathcal{O}_Y(f^{-1}(U)) = \mathcal{O}_Y(\varnothing) = 0$. But since $\mathcal{O}_X(U) \neq 0$, $\mathcal{O}_X(U) \longrightarrow f_*\mathcal{O}_Y(U)$ cannot be a monomorphism.   □

Consider the full category **Scheme** $\subseteq$ **LocRingedSpace** of schemes, i.e. locally ringed spaces that are locally isomorphic to spectra of commutative rings. A *closed immersion* of schemes is a morphism of schemes which is a closed immersion in **LocRingedSpace**. If $f : Y \longrightarrow X$ is a closed immersion of schemes, we call Y a closed subscheme of X. We denote by $\mathrm{Sub}_{sch}(X)$ the poset of (isomorphism classes of) closed subschemes of X.

We recall the following:

17.22 Proposition. *[19, Proposition 10.30] For a quasi-compact morphism* $f : Y \longrightarrow X$ *of schemes, the sheaf* $\mathcal{I}(f) = \mathrm{Ker}(f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y)$ *is quasi-coherent.*

17.23 Lemma. *Let X be a scheme and $Z \in \mathrm{Sub}_{cl}(X)$ a locally ringed subspace. The following are equivalent:*

1. *Z is a scheme.*

2. *$\mathcal{I}(Z)$ is a quasi-coherent sheaf.*

Proof. By Proposition 17.22, (1) implies (2). The converse implication easily reduces to the case where $X = \mathrm{Spec}(A)$ is affine. In this case $\mathrm{Qch}(X) \cong \mathrm{Mod}(A)$ and in the construction of Example 17.15, an ideal $I \subseteq A$ gives rise to the corresponding scheme $Z(I) = \mathrm{Spec}(A/I)$.   □

Let $\text{Sub}_{qch}(\mathcal{O}_X)$ be the lattice of subobjects of $\mathcal{O}_X$ in the category $\text{Qch}(X)$ of quasi-coherent sheaves on X.

17.24 Proposition. *The map $\mathcal{I}$ defines a duality*

$$\mathcal{I} : \text{Sub}_{sch}(X) \longrightarrow \text{Sub}_{qch}(\mathcal{O}_X) : Z \longrightarrow \mathcal{I}(Z).$$

Now we are in the right position to define a closure operator on the inclusion $\iota : \textbf{Scheme} \longrightarrow \textbf{LocRingedSpace}$.

17.25 Proposition. *There is an idempotent, weakly hereditary closure operator* sch *on ι with, for $X \in$ **Scheme** and $Z \in \text{Sub}_{cl}(X)$,*

$$\text{sch}(Z) = \inf\{Z' \in \text{Sub}_{sch}(X) \mid Z \subseteq Z'\} \in \text{Sub}_{sch}(X).$$

Proof. For the closure operator to be well defined, we have to make sure that $\text{Sub}_{sch}(X)$ is closed under infima in $\text{Sub}_{cl}(X)$. Dually, it suffices that $\text{Sub}_{qch}(\mathcal{O}_X)$ is closed under suprema in $\text{Sub}(\mathcal{O}_X)$. But in both Grothendieck categories $\text{Qch}(X)$ and $\text{Mod}(X)$, direct sums and images coincide, and a supremum $\sup_i F_i$ of subobjects is obtained as image of the canonical morphism $\oplus_i F_i \longrightarrow \mathcal{O}_X$. Thus, the proposition is an application of Proposition 17.11 by Lemma 17.26. □

17.26 Lemma. *Consider a morphism $f : Y \longrightarrow X$ between locally ringed spaces and let $Z \in \text{Sub}_{cl}(X)$ with ideal $\mathcal{I} = \mathcal{I}(Z)$. The pullback $f^{-1}Z$ in the category **LocRingedSpace** is the closed subspace corresponding to the ideal $f^*(\mathcal{I}) \subseteq f^*\mathcal{O}_X = \mathcal{O}_Y$ for the inverse image functor $f^* : \text{Mod}(X) \longrightarrow \text{Mod}(Y)$. In particular, if f is a morphism of schemes and $Z \in \text{Sub}_{sch}(X)$, we have $f^{-1}Z \in \text{Sub}_{sch}(Y)$.*

Proof. The second claim follows from the fact that f^* preserves quasi-coherence of sheaves. □

In **Scheme**, let $\mathbb{M}_{sch} = \mathbb{M} \cap$ **Scheme** be the class of closed immersions between schemes and $\mathbb{E}_{sch}$ the class of $\mathbb{M}_{sch}$-dense (or equivalently sch-dense) morphisms. Let $\mathbb{E}' = \mathbb{E} \cap$ **Scheme** be the class of ringed dominant morphisms between schemes. For a morphism $f : Y \longrightarrow X$ in **Scheme** with locally ringed image Z, $\mathrm{sch}(Z)$ is called the *scheme theoretic image of f* (see [24], [19]).

17.27 Proposition. $(\mathbb{E}_{sch}, \mathbb{M}_{sch})$ *constitutes a factorization system on* **Scheme** *and* $(\mathbb{E}', \mathbb{M}_{sch})$ *is an* $(\mathbb{E}_{sch}, \mathbb{M}_{sch})$-*closed structure on* **Scheme**. *The following are equivalent for a morphism $f : Y \longrightarrow X$ of schemes:*

1. *f is closed with respect to $(\mathbb{E}', \mathbb{M}_{sch})$.*

2. *f is sch-closed.*

3. *For every $m \in \mathrm{Sub}_{sch}(Y)$, the locally ringed image of $f\,m$ is a scheme (and hence coincides with the scheme theoretic image).*

Proof. This follows from Propositions 17.10 and 17.11   □

17.28 Example. A quasi-compact morphism $f : Y \longrightarrow X$ is sch-closed. Indeed, for every $m \in \mathrm{Sub}_{sch}(Y)$, the composition $f\,m$ remains quasi-compact and thus by Proposition 17.22, the locally ringed image is a scheme, hence coincides with the scheme theoretic image. In particular, in this case Y is topologically dense in the scheme theoretic image. This property fails in general, as [19, Exercise 10.18] shows.

17.29 Remarks. The fact that $\mathbb{M}_{sch}$ is part of a factorization system on **Scheme** was also used in [46], without explicit reference to the larger category of locally ringed spaces.

17.4 Forgetful functors

In this section we consider a functor $|\cdot| : \mathcal{C} \longrightarrow \mathcal{A}$ between finitely complete categories. Note that we do *not* require this functor to preserve finite limits.

We suppose that $\mathcal{A}$ is endowed with a factorization system $(\mathcal{D}, \mathcal{F}_0)$ and a $(\mathcal{D}, \mathcal{F}_0)$-pre-closed structure $(\mathcal{P}, \mathcal{F}_0)$ for which $\mathcal{P} \subseteq \mathcal{D}$. Note that this applies to the situation described in §17.1. Let $\mathcal{F}$ be the class of closed morphisms.

We suppose further that $\mathcal{C}$ is endowed with a factorization system $(\mathbb{D}, \mathbb{F}_0)$ and that the following holds:

(∗) $|\mathbb{F}_0| \subseteq \mathcal{F}_0$ and every $a : A \longrightarrow |X|$ in $\mathcal{F}_0$ is isomorphic to $|m| : |X'| \longrightarrow |X|$ with $m : X' \longrightarrow X$ in $\mathbb{F}_0$.

We denote $(\mathbb{D}, \mathbb{F}_0)$-factorizations by $f = \varphi(f)\delta(f)$.

Now define:

1. $\mathbb{P} = \{f \in \mathcal{C} \mid |f| \in \mathcal{P}\}$.

2. $\mathbb{H} = \{f \in \mathcal{C} \mid |f| \in \mathcal{F}\}$.

3. $\mathbb{D}_0 = \{f \in \mathcal{C} \mid |f| \in \mathcal{D}\}$.

Both $(\mathbb{D}_0, \mathbb{F}_0)$ and $(\mathbb{P}, \mathbb{F}_0)$ are readily seen to be $(\mathbb{D}, \mathbb{F}_0)$-pre-closed structures. Denote the class of $(\mathbb{D}_0, \mathbb{F}_0)$-closed morphisms by $\mathbb{K}$ and the class of $(\mathbb{P}, \mathbb{F}_0)$-closed structures by $\mathbb{F}$.

The main example of this formalism is:

17.30 Example. Let $\mathcal{C} = $ **Scheme** be the category of schemes and consider the forgetful functor $|\cdot| : $ **Scheme** $\longrightarrow$ **Top**. On $\mathcal{A} = $ **Top**, we take:

- $\mathcal{F}_0$ the class of closed embeddings.

- $\mathcal{D}$ the class of dense morphisms.

- $\mathcal{P}$ the class of surjective morphisms.

By Example 11.4, $(\mathcal{P}, \mathcal{F}_0)$ is a $(\mathcal{D}, \mathcal{F}_0)$-closed structure and $\mathcal{F}$ is the class of standard closed morphisms. On **Scheme**, we take $\mathbb{F}_0$ to be the class of closed immersions. Then $\mathbb{F}_0 \subseteq$ Mono and according to §17.3, $(\mathbb{D} = \mathbb{F}_0 - \text{Dense}, \mathbb{F}_0)$ is a factorization system on **Scheme**. With the notations of §17.4, $\mathbb{H}$ consists of the closed morphisms of schemes (note that this class was denoted by $\mathbb{F}$ in Example 11.4), $\mathbb{P}$ consists of the surjective morphisms of schemes, $\mathbb{D}_0$ consists of the morphisms of schemes that are dense on the topological level. Further, $\mathbb{F}$ is the class of $(\mathbb{P}, \mathbb{F}_0)$-closed morphisms and $\mathbb{K}$ is the class of $(\mathbb{D}_0, \mathbb{F}_0)$-closed morphisms. Finally, let $\mathbb{E}'$ be the class of ringed dominant morphisms and $\mathbb{L}$ the class of sch-closed morphisms from §17.3.

17.31 Proposition. *We have:*

1. *$\mathbb{F} \subseteq \mathbb{H}$.*

2. *$\mathbb{F} = \mathbb{H} \cap \mathbb{K}$.*

3. *If $\mathbb{H} - \text{Prop} \subseteq \mathbb{K}$, then $\mathbb{H} - \text{Prop} = \mathbb{F} - \text{Prop}$.*

4. *If $\mathbb{H} - \text{Sep} \subseteq \mathbb{F}_0 - \text{Sep}$, then $\mathbb{H} - \text{Sep} = \mathbb{F} - \text{Sep} = \mathbb{F}_0 - \text{Sep}$ and $(\mathbb{P}, \mathbb{F}_0)$ is a $(\mathbb{D}, \mathbb{F}_0)$-closed structure.*

Proof.   1. Let $f : X \longrightarrow Y$ in $\mathbb{F}$ and consider $|f| : |X| \longrightarrow |Y|$. By the assumption $(*)$, an arbitrary $\mathcal{F}_0$-subobject of $|X|$ is given by $|m| : |X'| \longrightarrow |X|$ for some $m : X' \longrightarrow X$ in $\mathbb{F}_0$. Consider the

$(\mathbb{D}, \mathbb{F}_0)$-factorization

$$X \xrightarrow{\ f\ } Y \qquad\qquad (17.1)$$

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle m}\uparrow & & \uparrow{\scriptstyle \varphi} \\ X' & \xrightarrow{\ \delta\ } & Y'. \end{array}$$

By definition, we have $\delta \in \mathbb{P}$ and $\varphi \in \mathbb{F}_0$. Thus $|\delta| \in \mathcal{P} \subseteq \mathcal{D}$ and $|\varphi| \in \mathcal{F}_0$ is the $(\mathcal{D}, \mathcal{F}_0)$-factorization of $|f||m|$ and since $|\delta| \in \mathcal{P}$ it follows that $|f| \in \mathcal{F}$ as desired.

2. Let $f : X \longrightarrow Y$ in $\mathbb{H} \cap \mathbb{K}$ and consider $m : X' \longrightarrow X$ in $\mathbb{F}_0$. By assumption, in the factorization (17.1) we have $|\delta| \in \mathcal{D}$ and we have $|\varphi| \in \mathcal{F}_0$. Hence, $|\delta|$ and $|\varphi|$ constitute the $(\mathcal{D}, \mathcal{F}_0)$ factorization of $|f||m|$ and since $|f|$ is closed and $|m| \in \mathcal{F}_0$, it follows that $|\delta| \in \mathcal{P}$ whence $\delta \in \mathbb{P}$ as desired.

3. Since $(\mathbb{P}, \mathbb{F}_0)$ is a $(\mathbb{D}, \mathbb{F}_0)$-pre-closed structure and by (1), we have $\mathbb{F}_0 \subseteq \mathbb{F} \subseteq \mathbb{H}$ and hence $\mathbb{F}_0 \subseteq \mathbb{F} - \mathrm{Prop} \subseteq \mathbb{H} - \mathrm{Prop}$. Since $\mathbb{H}$ is closed under compositions, we thus have for every pullback of f and for every $\mathbb{F}_0$-subobject of the domain that the composition is in $\mathbb{H} - \mathrm{Prop}$. Thus, it suffices to show that $f \in \mathbb{H} - \mathrm{Prop}$ implies $f \in \mathbb{F}$. Also, for every $\mathbb{F}_0$-subobject m we have that fm satisfies the condition in (2), hence it follows that $f \in \mathbb{F}$.

4. Immediate from the inclusions $\mathbb{F}_0 \subseteq \mathbb{F} \subseteq \mathbb{H}$.

$\square$

17.5 Application to schemes

We keep the notation from Example 17.30.

17.32 Proposition. *We have:*

1. $\mathbb{F} \subseteq \mathbb{H}$.

2. $\mathbb{F} = \mathbb{H} \cap \mathbb{K}$.

3. $\mathbb{H} - \mathrm{Prop} = \mathbb{F} - \mathrm{Prop}$.

4. $\mathbb{H} - \mathrm{Sep} = \mathbb{F} - \mathrm{Sep} = \mathbb{F}_0 - \mathrm{Sep}$.

5. $(\mathbb{P}, \mathbb{F}_0)$ *is a* $(\mathbb{D}, \mathbb{F}_0)$-*closed structure.*

6. $\mathbb{L} \subseteq \mathbb{K}$.

Proof. (6) By Proposition 17.21, $\mathbb{E}' \subseteq \mathbb{D}_0$.

For the other claims, it suffices to check the conditions in Proposition 17.31.

(3) Morphisms in $\mathbb{H}$-Prop are quasi-compact, so the result follows from Example 17.28 and (6).

(4) A $\mathbb{H}$-separated morphism is well-known to be $\mathbb{F}_0$-separated. $\square$

17.33 Remark. As recalled in Example 11.4, in **Scheme**, a morphism is called *separated* if it is $\mathbb{H}$-separated. It is called *universally closed* if it is $\mathbb{H}$-proper, and it is called *proper* it it is $\mathbb{H}$-perfect and of finite type. Thus, the $(\mathbb{D}, \mathbb{F}_0)$-closed structure $(\mathbb{P}, \mathbb{F}_0)$ yields the correct class of separated morphisms and the correct class of universally closed morphisms, called proper in our terminology. When we restrict our attention to quasi-compact morphisms (or to the sch-closed or $(\mathbb{D}_0, \mathbb{F}_0)$-closed morphisms), it also yields the correct class of closed morphisms.

Comparison functors

Let $\varphi : \mathcal{C} \longrightarrow \mathcal{C}'$ be a finite limit preserving functor between finitely complete categories, and suppose $\mathcal{C}$ is endowed with an $(\mathbb{E}, \mathbb{M})$-closed structure $(\mathbb{P}, \mathbb{F}_0)$ with corresponding class $\mathbb{F}$ of closed morphisms and $\mathcal{C}'$ is endowed with an $(\mathbb{E}', \mathbb{M}')$-closed structure $(\mathbb{P}', \mathbb{F}'_0)$ with corresponding class $\mathbb{F}'$ of closed morphisms. We then have the following way of obtaining that φ reflects closed morphisms:

17.34 Proposition. *In either of the following cases, we have $\varphi^{-1}(\mathbb{F}') \subseteq \mathbb{F}$:*

1. *$\mathbb{X} \subseteq \varphi^{-1}(\mathbb{X}')$ for $\mathbb{X} = \mathbb{F}_0, \mathbb{E}, \mathbb{M}$ and $\varphi^{-1}(\mathbb{X}') \subseteq \mathbb{X}$ for $\mathbb{X} = \mathbb{F}_0, \mathbb{P}$.*

2. *$\mathbb{P} = \mathbb{E}$ and $\mathbb{X} \subseteq \varphi^{-1}(\mathbb{X}')$ for $\mathbb{X} = \mathbb{F}_0, \mathbb{E}, \mathbb{M}$ and $\varphi^{-1}(\mathbb{F}'_0) \subseteq \mathbb{F}_0$.*

3. *$\mathbb{F}_0 = \mathbb{M}$ and $\mathbb{X} \subseteq \varphi^{-1}(\mathbb{X}')$ for $\mathbb{X} = \mathbb{E}, \mathbb{M}$ and $\varphi^{-1}(\mathbb{P}') \subseteq \mathbb{P}$.*

17.35 Example. Consider the functor $\varphi : \textbf{FSch} \longrightarrow \textbf{Top}$ of Example 11.31. We take for $\mathbb{F}_0 = \mathbb{M}$ the class of closed immersions, for $\mathbb{E}$ the class of ringed dominant morphisms, and for $\mathbb{P}$ the class of surjections. We take for $\mathbb{F}'_0 = \mathbb{M}'$ the class of closed embeddings, for $\mathbb{E}'$ the class of dense morphisms and for $\mathbb{P}'$ the class of surjections. By construction, closed immersions are mapped to closed embeddings, and by the dictionary in [52] based upon [52, Proposition 2.2], ringed dominant morphisms are mapped to dense morphisms and surjectivity is reflected by φ. Thus, we are in case (3) in Proposition 17.34 whence closed morphisms are reflected. We should note however, that the fact that closed morphisms are reflected follows directly from [52, Proposition 2.2] as well.

Chapter 19

Presheaf categories

Although the extended functional topology theory developed in the previous chapter encompasses the examples of schemes, we have lost some of the simplicity by the introduction of an additional input datum of so called surjective morphisms. Thus, we still wonder whether it is possible to recover the correct classes of proper and separated morphisms in these categories using *only* the original theory from [8]. In this section, we investigate this possibility by making use of presheaf categories. The main idea is the following: if $\mathcal{C}$ is the category we are interested in, and we have natural classes of proper and separated morphisms in $\mathcal{C}$, then we try to install the original functional topology setup from [8] on the category $\mathbf{PSh}(\mathcal{C})$ of presheaves of sets on $\mathcal{C}$. The category $\mathbf{PSh}(\mathcal{C})$ is well-suited for this purpose. Indeed, it comes equipped with the $(\mathrm{Epi}, \mathrm{Mono})$-factorization system, and the class Epi is pullback stable. Thus, by Proposition 13.4, any pullback stable class $\mathbb{F}_0$ of monomorphisms which is closed under compositions is the class of closed embeddings for an $(\mathrm{Epi}, \mathrm{Mono})$-closed class $\mathbb{F}$ in the sense of [8], and $\mathbb{F}$ consists precisely of the morphisms for which the image of a closed subobject is again a closed subobject.

Since we start with the proposed classes of proper and separated mor-

phisms on $\mathcal{C}$, our aim is to define a class $\mathbb{F}_0$ on $\textbf{PSh}(\mathcal{C})$ such that a morphism in $\mathcal{C}$ is proper (resp. separated) if and only if it is proper (resp. separated) with respect to $\mathbb{F}$ when considered as a morphism in $\textbf{PSh}(\mathcal{C})$. For $\mathbb{F}_0$, we propose to take precisely the images of proper-representable $\textbf{PSh}(\mathcal{C})$-morphisms (see §19.2). After analyzing the general situation, in §19.3 we prove that taking for $\mathcal{C}$ the category of schemes, this approach works. We also identify the $\mathbb{F}_0$-morphisms that belong to $\mathcal{C}$ as being precisely the proper monomorphisms. Thus, this class contains the ordinary closed immersions, and when we restrict our attention to morphisms between schemes of finite type, the two classes coincide. However, even in this case, and even for a closed morphism, the presheaf image will in general not be a scheme, and should not be confused with the scheme-theoretic image which we used in the previous chapter.

19.1   Stable classes

Let $\mathcal{C}$ be a finitely complete category endowed with a factorization system $(\mathbb{E}, \mathbb{M})$ with $\mathbb{M} \subseteq$ Mono and $\mathbb{E}$ pullback-stable. Let $\mathbb{H}$ be a stable class of morphisms in $\mathcal{C}$ (i.e. $\mathbb{H}$ contains the isomorphisms, is closed under compositions and pullback-stable). We define the class

$$\mathbb{F}_0 = \{\mu(h) \mid h \in \mathbb{H}\}.$$

19.1 Proposition. $(\mathbb{E}, \mathbb{F}_0)$ *is an* $(\mathbb{E}, \mathbb{M})$*-pre-closed structure.*

Proof. Clearly, $\mathbb{F}_0$ contains the isomorphisms and consists of monomorphisms. Equally obvious, $\mathbb{E}$ contains the isomorphisms and is closed under compositions.

Let us show first that $\mathbb{F}_0$ is pullback-stable. For $m \in \mathbb{F}_0$, we have an $h = me$ with $h \in \mathbb{H}$ and $e \in \mathbb{E}$. Taking a pullback of this yields

$\bar{h} = \bar{m}\bar{e}$ with $\bar{h} \in \mathbb{H}$ and $\bar{e} \in \mathbb{E}$ by the assumptions. Thus, $\bar{m} \in \mathbb{F}_0$ as desired.

Let us now show that $\mathbb{F}_0$ is closed under compositions. For $m : A \longrightarrow B$ and $m' : B \longrightarrow C$ in $\mathbb{F}_0$ we write $h = me$ and $h' = m'e'$ with $h, h' \in \mathbb{H}$ and $e, e' \in \mathbb{E}$. Taking the pullback of f and e', we obtain the diagram
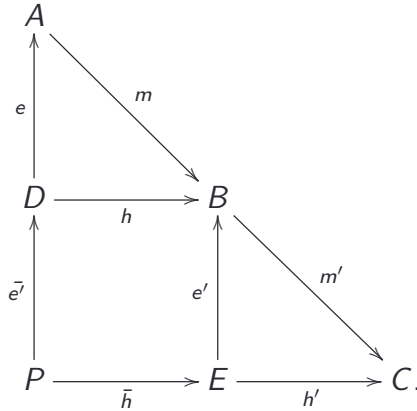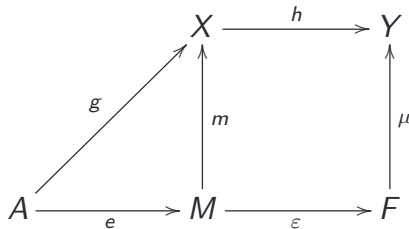

So we see that $h'\bar{h} = m'me\bar{e}'$ with $h'\bar{h} \in \mathbb{H}$. □

Let $\mathbb{F}$ be the class of closed morphisms associated to $(\mathbb{E}, \mathbb{F}_0)$.

19.2 Proposition. *Every morphism in $\mathbb{H}$ is $\mathbb{F}$-proper.*

Proof. Let $h : X \longrightarrow Y$ in $\mathbb{H}$. Since $\mathbb{H}$ is pullback-stable, it suffices to show that $h \in \mathbb{F}$. So consider $m : M \longrightarrow X$ in $\mathbb{F}_0$. We obtain a diagram

with g and hence hg in $\mathbb{H}$, e and ε and hence εe in $\mathbb{E}$, and μ in $\mathbb{M}$. Hence, $\mu \in \mathbb{F}_0$ as desired.   □

The inclusion $\mathbb{H} \subseteq \mathbb{F}$ expresses the intuitive fact that for a morphism in $\mathbb{H}$, the image of an $\mathbb{F}_0$-subobject is again an $\mathbb{F}_0$-subobject. This is comparable to the situation for $(\mathbb{E}, \mathbb{M})$-closed classes, see Remark 13.3.

19.2   Representable morphisms

In this section, starting from a closed class on a category, we define an associated stable class on an enlargement of the category.

Let $\mathcal{C} \subseteq \hat{\mathcal{C}}$ be a fully faithful left exact inclusion between finitely complete categories. Let $\mathbb{H}$ be a closed class of morphisms in $\mathcal{C}$. A morphism $f : X \longrightarrow Y$ in $\hat{\mathcal{C}}$ is called $\mathbb{H}$-*representable* if for every pullback

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;f\;\;} & Y \\
\big\uparrow & & \big\uparrow \\
P & \xrightarrow[f']{} & C
\end{array}
$$

with $C \in \mathcal{C}$, it follows that P (is isomorphic to an object) in $\mathcal{C}$ and f' is (isomorphic to a morphism) in $\mathbb{H}$. We denote the class of $\mathbb{H}$-representable morphisms by $\mathbb{H} - \mathrm{Rep}$. A morphism is called *representable* if it is $\mathrm{Mor}(\mathcal{C})$-representable and we denote $\mathrm{Rep} = \mathrm{Mor}(\mathcal{C}) - \mathrm{Rep}$.

19.3 Lemma.   *1. The isomorphisms are $\mathbb{H}$-representable.*

   2. $\mathbb{H} - \mathrm{Rep}$ is pullback-stable and closed under compositions.

> 3. The $\mathbb{H}$-*representable morphisms in* $\mathcal{C}$ *are exactly the* $\mathbb{H}$-*proper morphisms.*

Proof. This is easy to check. □

19.3 Presheaf categories

Let $\mathcal{C}$ be a finitely complete category and consider the Yoneda embedding $\mathcal{C} \subseteq \mathbf{PSh}(\mathcal{C})$. The category $\mathbf{PSh}(\mathcal{C})$ is endowed with the factorization system $(\text{Epi}, \text{Mono})$ and Epi is pullback stable. The image of a natural transformation between presheaves is computed pointwise. Let $\mathbb{H}$ be a closed class on $\mathcal{C}$ and consider the class $\mathbb{H} - \text{Rep}$ on $\mathbf{PSh}(\mathcal{C})$. We define the following class of morphisms in $\mathbf{PSh}(\mathcal{C})$:

$$\mathbb{F}_0 = \{\mu(h) \mid h \in \mathbb{H} - \text{Rep}\}.$$

19.4 Proposition. $(\text{Epi}, \mathbb{F}_0)$ *is an* $(\text{Epi}, \text{Mono})$-*closed structure on* $\mathbf{PSh}(\mathcal{C})$. *Let* $\mathbb{F}$ *denote the associated class of closed morphisms.*

1. $(\mathbb{H} - \text{Prop})_{\mathcal{C}} \subseteq \mathbb{H} - \text{Rep} \subseteq (\mathbb{F} - \text{Prop})_{\mathbf{PSh}(\mathcal{C})}$.

2. $(\mathbb{H} - \text{Prop})_{\mathcal{C}} \cap \text{Mono}_{\mathcal{C}} \subseteq \mathbb{H} - \text{Rep} \cap \text{Mono} \subseteq \mathbb{F}_0$.

Proof. According to Proposition 19.1, $(\text{Epi}, \mathbb{F}_0)$ is an $(\text{Epi}, \text{Mono})$-pre-closed structure and according to Proposition 19.2, $\mathbb{H} - \text{Rep} \subseteq (\mathbb{F} - \text{Prop})_{\mathbf{PSh}(\mathcal{C})}$. The other inclusions in (1) and (2) easily follow. Finally, it is a closed structure by Remark 13.7 (1). □

19.5 Remark. Note that the $(\text{Epi}, \text{Mono})$-closed structure $(\text{Epi}, \mathbb{F}_0)$ on $\mathbf{PSh}(\mathcal{C})$ fits entirely into Example 13.4. In particular, $\mathbb{F}$ is an $(\text{Epi}, \text{Mono})$-closed class in the sense of [8], see Definition 13.1.

Under an additional hypothesis, we can characterize the representable $\mathbb{F}_0$-morphisms as follows:

19.6 Proposition. *Suppose $\mathcal{C}$-retractions are $\mathbb{H}$-right cancellable.*

1. $\mathbb{H} - \mathsf{Rep} \cap \mathsf{Mono} = \mathbb{F}_0 \cap \mathsf{Rep}$.

2. $(\mathbb{H} - \mathsf{Prop})_\mathcal{C} \cap \mathsf{Mono}_\mathcal{C} = \mathbb{F}_0 \cap \mathsf{Mor}(\mathcal{C})$.

Proof. (2) follows from (1) by intersecting with the $\mathcal{C}$-morphisms. For (1), the inclusion left in right follows from Proposition 19.4 (2). Conversely, let $m : F \longrightarrow G$ be a representable morphism in $\mathbb{F}_0$. By taking the pullback along a $\mathcal{C}$-object, we may suppose that m is a $\mathcal{C}$-morphism in $\mathbb{F}_0$. Thus, there exists $h \in (\mathbb{H} - \mathsf{Prop})_\mathcal{C}$ with $h = me$ and e an epimorphism. Then by Lemma 11.8, e is representable and $e : H \longrightarrow F$ is a presheaf epimorphism between $\mathcal{C}$-objects, i.e. a $\mathcal{C}$-retraction. By the assumption, it follows that $m \in (\mathbb{H} - \mathsf{Prop})_\mathcal{C}$ as desired. $\qquad\square$

Next we take a closer look at separated morphisms.

19.7 Proposition. *We have*

1. $((\mathbb{H} - \mathsf{Prop})_\mathcal{C} - \mathsf{Sep})_\mathcal{C} \subseteq (\mathbb{F} - \mathsf{Sep})_{\mathsf{PSh}(\mathcal{C})} \cap \mathsf{Mor}(\mathcal{C}) = (\mathbb{F}_0 - \mathsf{Sep})_{\mathsf{PSh}(\mathcal{C})} \cap \mathsf{Mor}(\mathcal{C})$.

If $\mathcal{C}$-retractions are $\mathbb{H}$-right cancellable, then

(2) $((\mathbb{H} - \mathsf{Prop})_\mathcal{C} - \mathsf{Sep})_\mathcal{C} = (\mathbb{F} - \mathsf{Sep})_{\mathsf{PSh}(\mathcal{C})} \cap \mathsf{Mor}(\mathcal{C}) = (\mathbb{F}_0 - \mathsf{Sep})_{\mathsf{PSh}(\mathcal{C})} \cap \mathsf{Mor}(\mathcal{C})$.

If $\mathbb{H}$ is a stable class in $\mathcal{C}$, we can replace $(\mathbb{H} - \mathsf{Prop})_\mathcal{C}$ by $\mathbb{H}$ in the above.

Application to schemes

Put $\mathcal{C} = \mathbf{Scheme}$ in the previous section and take $\mathbb{H}$ to be the usual class of closed morphisms, i.e. morphisms with an underlying closed map of topological spaces. Let $\mathbf{PSh}(\mathcal{C})$, $\mathbb{F}_0$ and $\mathbb{F}$ be as above. Our aim is to prove that the $(\mathrm{Epi}, \mathrm{Mono})$-closed structure $(\mathrm{Epi}, \mathbb{F}_0)$ on $\mathbf{PSh}(\mathcal{C})$ yields the classes of $\mathbb{H}$-proper and $\mathbb{H}$-separated morphisms (after restricting to $\mathcal{C}$-morphisms). We start with properness.

19.8 Proposition.   1. $\mathbb{H} - \mathrm{Rep} = (\mathbb{F} - \mathrm{Prop})_{\mathbf{PSh}(\mathcal{C})} \cap \mathrm{Rep}$.

   2. $(\mathbb{H} - \mathrm{Prop})_{\mathcal{C}} = (\mathbb{F} - \mathrm{Prop})_{\mathbf{PSh}(\mathcal{C})} \cap \mathrm{Mor}(\mathcal{C})$.

Proof. (2) immediately follows from (1) by restricting to morphisms in $\mathcal{C}$. For (1), the inclusion from left to right follows from Proposition 19.2 (taking $\mathbb{H}$ to be $\mathbb{H} - \mathrm{Rep}$). Consider $f : F \longrightarrow G$ in $(\mathbb{F} - \mathrm{Prop})_{\mathbf{PSh}(\mathcal{C})} \cap \mathrm{Rep}$. Let $C \longrightarrow G$ be a morphism with $C \in \mathcal{C}$. We are to show that the pullback $f' : P \longrightarrow C$ is in $\mathbb{H}$. By assumption, $P \in \mathcal{C}$ and $f' \in \mathbb{F}$. For a closed subset $X \subseteq C$, let $m : M \longrightarrow C$ be a closed subscheme with $|M| = X$. By Proposition 19.4(2), $m \in \mathbb{F}_0$ and so by definition of $\mathbb{F}$ and $\mathbb{F}_0$, we obtain the following diagram
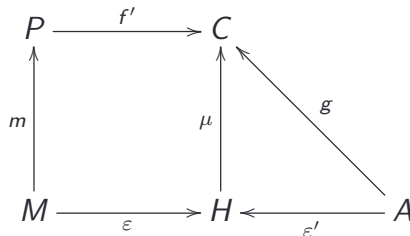
with $\mu \in \mathbb{F}_0 \cap \mathbb{M}$, $\varepsilon \in \mathbb{E}$, $g \in \mathbb{H} - \mathrm{Rep}$ and $\varepsilon' \in \mathbb{E}$. Since $C \in \mathcal{C}$, we necessarily have $g \in \mathbb{H}$. We are to show that the set theoretic image of the $\mathcal{C}$-morphism $f'm$ is a closed subset of C. But the presheaf images of $f'm$ and g coincide, whence by Lemma 19.9, so do the set theoretic images and we are ready. $\square$

19.9 Lemma. *Consider two morphism $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y$ of schemes and suppose the presheaf images $\mu = \mu(f) = \mu(f') : F \longrightarrow Y$ coincide. Then the set theoretic images $f(X) \subseteq Y$ and $f'(X') \subseteq Y$ coincide as well.*

Proof. Clearly the situation is symmetric. Consider $\varepsilon = \varepsilon(f) : X \longrightarrow F$ and $\varepsilon' = \varepsilon(f') : X' \longrightarrow F$. Take a point $x \in X$ with residue field k. This point is represented by a morphism $\xi : \mathrm{Spec}(k) \longrightarrow X$ and its image $f(x) \in Y$ is determined by the composition $f\xi : \mathrm{Spec}(k) \longrightarrow X \longrightarrow F \longrightarrow Y$. Now since $\varepsilon' : X' \longrightarrow F$ is an epimorphism, there is a morphism $\xi' : \mathrm{Spec}(k) \longrightarrow X'$ with $\mu\varepsilon\xi = \mu\varepsilon'\xi'$ and hence a point $x' \in X'$ with $f'(x') = y$. $\qquad\square$

We have a precise description of the representable $\mathbb{F}_0$-morphisms, and we obtain the correct class of separated morphisms after restricting to $\mathcal{C}$-morphisms:

19.10 Proposition.   *1.* $\mathbb{F}_0 \cap \mathrm{Rep} = \mathbb{H} - \mathrm{Rep} \cap \mathrm{Mono}$.

   2. $\mathbb{F}_0 \cap \mathrm{Mor}(\mathcal{C}) = (\mathbb{H} - \mathrm{Prop})_{\mathcal{C}} \cap \mathrm{Mono}_{\mathcal{C}}$.

   3. $(\mathbb{F}_0 - \mathrm{Sep})_{\mathbf{PSh}(\mathcal{C})} \cap \mathrm{Mor}(\mathcal{C}) = (\mathbb{F} - \mathrm{Sep})_{\mathbf{PSh}(\mathcal{C})} \cap \mathrm{Mor}(\mathcal{C}) = (\mathbb{H} - \mathrm{Sep})_{\mathcal{C}}$.

Proof. Since retractions of schemes are surjective on the level of underlying spaces, they are $\mathbb{H}$-right cancellable and the condition in Proposition 19.6 and Proposition 19.7(2) is fulfilled. $\qquad\square$

19.11 Remark. For an $\mathbb{H}$-proper morphism $f : X \longrightarrow Y$ of schemes, the presheaf image $\mu : F \longrightarrow Y$ belongs to $\mathbb{F}_0$. However, it should not be confused with the scheme theoretic image $m : Z \longrightarrow Y$ which also belongs to $\mathbb{F}_0$, where Z is a scheme rather than a presheaf, but where the corresponding $e : X \longrightarrow Z$ fails to be a presheaf epimorphism in general (the only scheme morphisms that are presheaf epimorphisms are the retractions).

19.12 Remark. Since the usual closed immersions of schemes are always proper monomorphisms, they are always contained in $\mathbb{F}_0$. For morphisms between schemes of finite type, the converse holds and the closed immersions are precisely the $\mathbb{F}_0$-morphisms between such schemes.

Chapter 23

Rigged Hilbert spaces and the spectral theorem

We wish to study the concept of a rigged Hilbert space. Informally, a rigged Hilbert space is a concept that tries to unify Hilbert space theory and the theory of distributions. This unification is important for quantum physics and really brings it on rigorous footing. Naive and formal calculations in quantum physics can be completely rigorized with the aid of rigged Hilbert spaces. Let us give a small example here. Let us think of the momentum operator $Q\varphi = -i\varphi'$, where $\varphi : \mathbb{R} \to \mathbb{C}$ is some differentiable function. To find the eigenvectors with eigenvalue λ, we need to solve the eigenvalue equation

$$-i\varphi' = \lambda\varphi.$$

The solutions to these equations are well-known to be $\varphi(x) = ce^{i\lambda x}$. However, in quantum mechanics, we only really want to deal with functions which are square-integrable, that is, functions $\varphi : \mathbb{R} \to \mathbb{C}$ such that

$$\int_{-\infty}^{+\infty} |\varphi(x)|^2 \, \mathrm{d}x < +\infty,$$

or in other words, functions in $L^2(\mathbb{R})$. The solution to our eigenvalue equation $\varphi(x) = ce^{i\lambda x}$ is not square-integrable. Thus we might say that the operator Q has no eigenvectors or eigenvalues. It is however convenient to still use the solutions $\varphi(x) = ce^{i\lambda x}$, and a way out is to regard them as distributions. So we extend the space $L^2(\mathbb{R})$ to contain distributions and then the operator Q has generalized distributional eigenvalues. The question then arises how we should extend the space $L^2(\mathbb{R})$ and which operators then have eigenvectors in this extended setting. This is solved by the theory of rigged Hilbert spaces.

A rigged Hilbert space is basically a triple $\Phi \subseteq H \subseteq \Phi'$, where H is a Hilbert space and where Φ is a nuclear space in H. The space Φ' is the space of distributions of H.

In the first few sections, we revise the theory of distributions, of nuclear spaces and of rigged Hilbert spaces. We also revise the various spectral theorems in increasing generality. We end the revision by stating the nuclear spectral theorem which gives a way to diagonalize unbounded self-adjoint operators $\Phi \to \Phi$ and to define "generalized eigenvectors." We then give several ways to construct rigged Hilbert spaces. The results in §23.5 are not new, but the proofs cannot be found in books. Then we give a generalized spectral theorem (Theorem 23.57). This spectral theorem deals with operators $\Phi' \to \Phi'$. We give a nice notion of symmetry and self-adjointness for these operators, and we show how to diagonalize them.

Finally, we give several applications of rigged Hilbert spaces. One of these applications are resolutions of the identity which is quite often used in quantum physics. Another application is to the theory of probability. This shows that many probabilistic results can be simplified and generalized considerably if we work with rigged Hilbert spaces.

23.1 Distributions

Rigged Hilbert spaces is the theory where we extend a Hilbert space with distributions. Thus first we need to review the classical theory of distributions. This is done in this section. We refer to the book [31] for more details and for the proofs.

First we need an appropriate definition of test functions:

23.1 Definition. Define for $K \subseteq \mathbb{R}$ compact, the set $\mathcal{D}_K(\mathbb{R}) = \mathcal{C}_K^\infty(\mathbb{R})$, this is the set of all smooth functions $f : \mathbb{R} \to \mathbb{C}$ such that $f(x) = 0$ for $x \notin K$.
Further, define $\mathcal{D}(\mathbb{R}) = \mathcal{C}_c^\infty(\mathbb{R}) = \bigcup_K \mathcal{D}_K(\mathbb{R})$.

For $\varphi \in \mathcal{D}(\mathbb{R})$, we define

$$\|\varphi\|_{p,K} = \max_{0 \leq r \leq p} \sup_{x \in K} |\varphi^{(r)}(x)|.$$

Then $\mathcal{D}_K(\mathbb{R})$ is a locally convex topological space with respect to the norms $\| \ \|_{p,K}$, $p \in \mathbb{N}$. And also $\mathcal{D}(\mathbb{R})$ is a locally convex topological space with respect to the collection $\| \ \|_{p,K}$, $p \in \mathbb{N}$ and K compact.

23.2 Definition. A *distribution* is a linear map $\mu : \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ which is continuous with respect to the locally convex topology of $\mathcal{D}(\mathbb{R})$ defined above. We will usually use the following notations:

$$\mu(\varphi) = \langle \mu, \varphi \rangle = \int_{-\infty}^{+\infty} \varphi(x)\mu(x)\, \mathrm{d}x.$$

The set of all distributions is denoted as $\mathcal{D}'(\mathbb{R})$.

Any locally integrable function $h : \mathbb{R} \to \mathbb{C}$ defines a distribution μ_h by

$$\mu_h(\varphi) = \int_{-\infty}^{+\infty} \varphi(x)h(x)\, \mathrm{d}x.$$

The map $L^1_{\text{loc}}(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}) : h \to \mu_h$ which associates a distribution with each locally integrable function is injective in the sense that if $\mu_h = \mu_g$, then $h = g$ a.e.

The *Dirac delta function* is defined as the distribution $\delta : \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ such that $\langle \delta, \varphi \rangle = \varphi(0)$.

23.3 Definition. For $a \in \mathbb{R}$, we define the translation operator $\tau_a : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ as $(\tau_a \varphi)(x) = \varphi(x - a)$. If μ is a distribution, then we define $\tau_a \mu$ as the distribution such that $\langle \tau_a \mu, \varphi \rangle = \langle \mu, \tau_{-a} \varphi \rangle$. We will use the notation

$$\langle \tau_a \mu, \varphi \rangle = \int_{-\infty}^{+\infty} \mu(x - a)\varphi(x)\,\mathrm{d}x.$$

23.4 Definition. If μ is any distribution, then we define the derivative of μ as the distribution μ' such that $\langle \mu', \varphi \rangle = \langle \mu, -\varphi' \rangle$.

We will often denote the kth-order derivative of a distribution μ as $D^k \mu$.

The distributions of finite order are those $\mu \in \mathcal{D}'(\mathbb{R})$ such that there exists a continuous function f such that $\mu = D^r \mu_f$ for some $r \in \mathbb{N}$. We denote the set of all distributions of finite order as $\mathcal{D}'_{\text{fin}}(\mathbb{R})$. These distributions satisfy the axioms of De Silva. That is, we have defined two linear maps $\iota : \mathcal{C}(\mathbb{R}) \to \mathcal{D}'_{\text{fin}}(\mathbb{R}) : h \to \mu_h$ and $D : \mathcal{D}'_{\text{fin}}(\mathbb{R}) \to \mathcal{D}'_{\text{fin}}(\mathbb{R})$ such that

1. ι is injective.

2. If f is continuously differentiable, then $(D\iota)(f) = \iota(f')$.

3. If $\mu \in \mathcal{D}'_{\text{fin}}$, then there is a $f \in \mathcal{C}(\mathbb{R})$ and a $r \in \mathbb{N}_0$ such that $\mu = D^r \iota(f)$.

4. For $f, g \in \mathcal{C}(\mathbb{R})$ and for $r \in \mathbb{N}$ it holds that $D^r\iota(f) = D^r\iota(g)$ if and only if $f - g$ is a polynomial of degree strictly smaller than r.

Another type of distribution are the so-called tempered distributions.

23.5 Definition. An n-dimensional multi-index is an n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of nonnegative integers. We can use multi-indices to introduce the following notations:

1. $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$

2. $D^\alpha = \dfrac{\partial^{\alpha_1 + ... + \alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}$

23.6 Definition. The *Schwartz space* or space of rapidly decreasing functions on $\mathbb{R}^n$ is the function space

$$\mathcal{S}(\mathbb{R}^n) = \{f \in \mathcal{C}^\infty(\mathbb{R}^n) \mid \|f\|_{\alpha,\beta} < +\infty \text{ for all multi-indices } \alpha, \beta\},$$

where

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|.$$

Thus $\mathcal{S}(\mathbb{R}^n)$ defines a locally convex vector space.

It is clear that $\mathcal{D}(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$ and we can prove that this inclusion is dense. Furthermore, convergence in $\mathcal{D}(\mathbb{R})$ implies convergence in $\mathcal{S}(\mathbb{R})$.

The Schwartz space is used primarily because of the Fourier transform. Indeed, for $\varphi \in \mathcal{S}(\mathbb{R})$, we define

$$\mathcal{F}(\varphi)(\omega) = \int_{-\infty}^{+\infty} \varphi(t) e^{-i\omega t} \, \mathrm{d}t.$$

We can show that $\mathcal{F}(\varphi) \in \mathcal{S}(\mathbb{R})$ and that $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is a continuous and bijective linear map with continuous inverse.

23.7 Definition. A *tempered distribution* is a linear functional μ : $\mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ which is continuous with respect to the structure of the Schwartz space. The space of tempered distributions is $\mathcal{S}'(\mathbb{R})$.

It is clear that $\mathcal{S}'(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$. Basic operations like addition, translation and differentiation of tempered distributions again yield a tempered distribution. But now we can also define the Fourier transform of a distribution. Indeed, if μ is a tempered distribution, then we define $\mathcal{F}(\mu)$ by

$$\langle \mathcal{F}(\mu), \varphi \rangle = \langle \mu, \mathcal{F}(\varphi) \rangle.$$

23.2   A collection of spectral theorems

Finite-dimensional Spectral theorem

One of the most important results in elementary linear algebra is of course the spectral theorem:

23.8 Theorem. *Let V be a finite-dimensional complex inner product space. Let $T : V \to V$ be a self-adjoint linear map, then there exists an orthonormal basis of eigenvectors for T. Every eigenvalue is real.*

This theorem can be found in any good linear algebra book, for example [51].

Spectral Theorem for compact operators

We can now ask ourselves to what extent this spectral theorem in finite dimensions extends to infinite-dimensional spaces. In order to do this, we need to extend the notion of self-adjointness to Hilbert spaces. This extension is pretty straightforward:

23.9 Definition. Let H be a complex Hilbert space. Let $T : H \to H$ be a bounded linear operator. Then there exists a unique bounded linear operator $T^* : H \to H$ such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle$$

for each $x, y \in H$. We say that T^* is the *adjoint* of T.
If $T = T^*$, then we say that T is *self-adjoint*.

For compact operators, the extension of the spectral theorem is essentially the same as the finite-dimensional case:

23.10 Theorem. *Let H be a complex Hilbert space. Let $T : H \to H$ be a self-adjoint compact operator, then there exists an orthonormal basis of eigenvectors for T. Every eigenvalue is real.*

This extends the finite-dimensional case, since any finite-dimensional complex vector space has a Hilbert space structure in a unique way and any linear operator between finite-dimensional spaces is compact.

For more details and the proof, see [49].

Spectral Theorem for bounded linear operators

The question remains in what sense this theorem still holds for arbitrary self-adjoint bounded operators. One problem is that a self-adjoint operator might not have any eigenvalues to begin with. For example, the multiplication operator

$$T : L^2[0, 1] \to L^2[0, 1]$$

defined by $T(\varphi)(t) = t\varphi(t)$ has no eigenvalues. Instead of focusing on the eigenvalues, we need to look at the spectrum of an operator

23.11 Definition. Let T be a bounded operator on a Hilbert space H. The *spectrum* of T is defined as the set of complex numbers

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not invertible}\}.$$

Clearly, in finite dimensions, the spectrum coincides with the set of eigenvalues. But in infinite dimensions, the spectrum might be much larger. We can prove that if T is a bounded operator, then the spectrum is nonempty and compact. The spectral theorem now takes the following form:

23.12 Theorem. *Let A be a bounded self-adjoint operator on a separable Hilbert space H, then there exist measures $(\mu_n)_n$ (this family can be finite or infinite) on $\sigma(A)$ and a unitary operator*

$$U : H \to \bigoplus_n L^2(\mathbb{R}, \mu_n)$$

such that

$$(UAU^{-1}\psi)_n(\lambda) = \lambda\psi_n(\lambda).$$

One easily deduces the following corollary:

23.13 Corollary. *Let A be a bounded self-adjoint operator on a separable Hilbert space H. Then there exists a finite measure space (M, μ), a bounded real-valued function F on M and a unitary map $U : H \to L^2(M, \mu)$ such that*

$$(UAU^{-1}f)(m) = F(m)f(m)$$

This form of the spectral theorem is very different from the original form in finite-dimensional vector spaces, where we explicitly found a basis. To see how at least this form extends the finite-dimensional spectral theorem, we fix a linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$. Every

orthonormal eigenbasis $(E_1, ..., E_n)$ of A determines a unitary map $U : \mathbb{C}^n \to \mathbb{C}^n$ and a function $\varphi : \{1, ..., n\} \to \mathbb{C}$. Indeed, we let $\varphi(j)$ be the eigenvalue of A corresponding with E_j. Then we have

$$UAU^{-1}\psi = \varphi\psi$$

for each $\psi : \{1, ..., n\} \to \mathbb{C}$. See [14] for more information and a proof.

But again, the general spectral theorem for bounded linear operators does not explicitly generate an eigenbasis. In fact, it might even occur that a bounded self-adjoint operator has no eigenvalues at all! Indeed, consider $A : L^2([0, 1]) \to L^2([0, 1])$ where $(A\varphi)(x) = x\varphi(x)$.

This will soon be solved with the techniques of rigged Hilbert spaces, where we will always be able to find "generalized distributional" eigenvalues. But first we need to generalize the spectral theorem even further. Indeed, in quantum mechanics, we often need unbounded operators instead of bounded ones.

Spectral Theorem for unbounded operators

The theory of unbounded operators is quite delicate. We only give the most basic results here. More details and all proofs can be found in [49].

23.14 Definition. Let H be a Hilbert space. An *unbounded operator* on H is a pair $(D(T), T)$, such that $D(T)$ is a dense subspace of H and such that $T : D(T) \to H$ is a linear map. We call $D(T)$ the domain of the operator.

With bounded operators, the domain is always the entire Hilbert space. One of the primary examples of an unbounded operator is when we

take $H = L^2(\mathbb{R})$ and $T\varphi(x) = x\varphi(x)$. A suitable domain is the set of all functions φ such that $\int x^2|\varphi(x)|^2\,\mathrm{d}x$ is finite. We could also take the Schwartz class $\mathcal{S}(\mathbb{R})$ as domain, which has the additional property that each power T^n is well-defined.

It is sometimes possible to extend the domains of operators:

23.15 Definition. Let T and S be unbounded operators on H. We say that $T \subseteq S$ if $D(T) \subseteq D(S)$ and if $T\varphi = S\varphi$ for each $\varphi \in D(T)$. We call S an *extension* of T.

An important class of operators are those of the closed and closable operators:

23.16 Definition. An unbounded operator T is called a *closed operator* if the graph
$$\{(\varphi, T\varphi) \mid \varphi \in D(T)\}$$
is a closed subset of $H \times H$.
We call T a *closable operator* if it has a closed extension. The smallest closed extension is called the closure and is denoted by $\overline{T}$.

Now we can finally introduce the notion of an adjoint:

23.17 Definition. Let T be an unbounded linear operator on a Hilbert space H. We define $D(T^*)$ the set of all $\varphi \in H$ such that there is an $\eta \in H$ with
$$\langle T\psi, \varphi \rangle = \langle \psi, \eta \rangle$$
for each $\psi \in D(T)$. For such $\varphi \in D(T^*)$, we define $T^*\varphi = \eta$. We call T^* the *adjoint* of T.

It may happen, however, that $D(T^*) = \{0\}$. The following theorem shows that this cannot happen with closable operators:

23.18 Proposition. *Let T be an unbounded operator on a Hilbert space H. Then*

1. T^* is closed.

2. T is closable if and only if $D(T^*)$ is dense in H. In this case, $\overline{T} = T^{**}$.

3. If T is closable, then $(\overline{T})^* = T^*$.

We can also generalize the notion of a spectrum:

23.19 Definition. Let T be a closed unbounded operator on a Hilbert space H. A complex number λ is in the *spectrum* $\sigma(T)$ if $T - \lambda I$ is not a bijection of $D(T)$ onto H or, in case it is a bijection, if it does not have a bounded inverse.

How can we possibly generalize the notion of self-adjointness? There are two possible definitions:

23.20 Definition. Let T be an unbounded operator on a Hilbert space H. We say that T is *symmetric* or *hermitian* if for each $\varphi, \psi \in D(T)$ it holds that
$$\langle T\varphi, \psi \rangle = \langle \varphi, T\psi \rangle$$
or equivalently, that $T \subset T^*$.

This notion will not be strong enough to allow a spectral theorem. The following notion will be:

23.21 Definition. An unbounded operator on a Hilbert space H is called *self-adjoint* if $T = T^*$. This means that $D(T) = D(T^*)$ and T is symmetric.

Here is an easy criterion to decide self-adjointness:

23.22 Proposition. *Let T be a symmetric unbounded operator on a Hilbert space H, then the following are equivalent:*

1. *T is self-adjoint,*

2. *T is closed and* $\mathrm{Ker}(T^* \pm i) = \{0\}$,

3. $\mathrm{Ran}(T \pm i) = H$.

Now we can finally state the spectral theorem:

23.23 Theorem. *Let A be a self-adjoint unbounded operator on a separable Hilbert space H. Then there is a finite measure space (M, μ), a unitary operator $U : H \to L^2(M, \mu)$ and a real valued function f on M which is finite a.e. such that*

1. $\psi \in D(A)$ *if and only if* $f(\cdot)(U\psi)(\cdot) \in L^2(M, \mu)$,

2. *If* $\varphi \in U(D(A))$, *then* $(UAU^{-1}\varphi)(m) = f(m)\varphi(m)$.

The need for generalized eigenvalues

Again, it may be possible that a self-adjoint unbounded operator might have no eigenvalues. Nevertheless, in the theory of quantum mechanics, we often act like there are eigenvalues nevertheless. Let us study two concrete examples from quantum mechanics.

For the first example, we make the following definition:

23.24 Definition. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Then we can define the unbounded operator

$$M_V : \mathrm{Dom}(M_V) \to L^2(\mathbb{R}^n)$$

defined by

$$(M_V\psi)(x) = V(x)\psi(x)$$

with domain

$$\mathrm{Dom}(M_V) = \{\psi \in L^2(\mathbb{R}^n) \mid V\psi \in L^2(\mathbb{R}^n)\}.$$

It is not difficult to prove that $\mathrm{Dom}(M_V)$ is dense in $L^2(\mathbb{R}^n)$ and that M_V is self-adjoint on this domain. The special case $V = \mathrm{pr}_j$ is called the position operator.

The spectrum of the operator M_V is given by the *essential range* of the function $V : \mathbb{R}^n \to \mathbb{R}$. This is the set of all $y \in \mathbb{R}$ such that

$$\lambda\{x \in \mathbb{R}^n \mid y - \varepsilon < V(x) < y + \varepsilon\} > 0$$

for all $\varepsilon > 0$. Note that λ indicates the Lebesgue measure.

However, it can always happen that M_V has no eigenvalues. Let us specialize to the case where $V : \mathbb{R} \to \mathbb{R} : x \to x$, this is the one-dimensional position operator. In this case, it is indeed easy to check that the operator M_V has no eigenvalues. Indeed, the spectral equation

$$x\varphi(x) = \lambda\varphi(x)$$

for $\lambda \in \mathbb{C}$ and $\varphi \in \mathrm{Dom}(M_V)$, has no solutions. Note however, that the spectrum of M_V is the entire $\mathbb{R}$.

However, it does have so-called "generalized eigenvalues". These are distributions which do solve the eigenvalue equation. The Dirac delta function δ does solve the spectral equation $x\varphi(x) = \lambda\varphi(x)$, where $\lambda = 0$. Indeed

$$\int_{-\infty}^{+\infty} x\delta(x)\,\mathrm{d}x = 0 = \int_{-\infty}^{+\infty} 0\delta(x)\,\mathrm{d}x.$$

Thus δ is an eigenvector with eigenvalue 0. Likewise, the functions $\delta(x - \lambda)$ defined by

$$\int_{-\infty}^{+\infty} \varphi(x)\delta(x - \lambda)\,\mathrm{d}x = \varphi(\lambda)$$

are eigenvectors with eigenvalue λ.

Another example is given by the momentum operator in quantum mechanics. This is defined as the operator

$$P : L^2(\mathbb{R}) \to L^2(\mathbb{R}) : \psi \to Q\psi$$

where

$$(P\psi)(x) = -i\psi'(x).$$

A suitable domain for P is given by the Schwartz class. On this domain, the operator P is *essentially self-adjoint*, meaning that there is a unique self-adjoint extension of Q on some domain. The explicit domain on which P is self-adjoint is given by the Sobolev space $H^1(\mathbb{R})$ (see Hall [23] for proofs of this):

23.25 Definition. Let $\Omega \subseteq \mathbb{R}$ be an open set, then the *Sobolev space* $W^{1,p}(\Omega)$ is defined to be the set of all $u \in L^p(\Omega)$ such that there exists some $g \in L^p(\Omega)$ such that

$$\int_\Omega u\varphi' = -\int_\Omega g\varphi$$

for each $\varphi \in C_c(\Omega)$. We set $u' = g$. We also define $H^1(\Omega) = W^{1,2}(\Omega)$.

The spectral equation for P is given by

$$-i\varphi'(x) = \lambda\varphi(x),$$

where $\lambda \in \mathbb{C}$ and $\varphi \in W^{1,p}$. The only solution to this is given by $\varphi(x) = e^{i\lambda x}$. However, this is not an element of $L^2(\mathbb{R})$ and does not count as an eigenvector. However, we can see this function as a distribution, and thus as a distributional eigenvector, because it defines a distribution

$$\mathcal{S}(\mathbb{R}) \to \mathbb{C} : \varphi \to \int \varphi(x)e^{-i\lambda x}\,\mathrm{d}x.$$

We see the necessity of expanding our Hilbert space to include distributions. The relevant construct is called a rigged Hilbert space.

23.3 Nuclear spaces

In order to define a rigged Hilbert space, we will first need to define the concept of a nuclear space. A rigged Hilbert space will then be a special case of a nuclear space. More details and proofs can be found in [58].

A nuclear space is related to a Banach space, except that there are multiple norms. One of the main examples is the Schwartz class.

23.26 Definition. A *countably Banach space* is a vector space Φ equipped with a countable number of norms $\| \|_n : \Phi \to \mathbb{R}$. These norms need to be compatible in the sense that if $x_k \to 0$ in the norm $\| \|_n$ and if $(x_k)_k$ is a Cauchy sequence in the norm $\| \|_m$, then it converges in that norm. We can put a topology on Φ by taking the following sets as a basis around 0:

$$U_{n,\varepsilon} = \{x \in \Phi \mid \|x\|_n < \varepsilon\} \text{ for all, } n \in \mathbb{N}, \ \varepsilon > 0.$$

We will denote this locally convex vector space as $(\Phi, \mathcal{T})$.
If all the norms come from an inner product, then we say that Φ is a *countably Hilbert space*.

We will not work with the original norms $\| \|_n$. Rather, we define the norms

$$\|x\|'_n = \sum_{k=1}^{n} \|x\|_n.$$

These inner products do not change the topology of Φ, but do satisfy the additional inequalities

$$\|x\|'_1 \leq \|x\|'_2 \leq \|x\|'_3 \leq \dots.$$

So in the future, we will always demand that the norms from the definition of a countably Banach space satisfy these inequalities.

Let Φ_n be the Banach space given by the completion of Φ relative to the norm $\| \ \|_n$. It follows that

$$\Phi = \bigcap_{n=1}^{+\infty} \Phi_n.$$

We can look at the situation by considering the following inclusions:

$$\Phi \subseteq \ldots \subseteq \Phi_3 \subseteq \Phi_2 \subseteq \Phi_1.$$

The dual spaces form an increasing chain

$$\Phi_1' \subseteq \Phi_2' \subseteq \Phi_3' \subseteq \ldots \subseteq \Phi'$$

with

$$\Phi' = \bigcup_{n=1}^{+\infty} \Phi_n'.$$

The spaces Φ_n' are again Banach with norm

$$\|F\|_n' = \sup_{\|x\|_n=1} |\langle F, x \rangle|.$$

These norms satisfy

$$\ldots \leq \|F\|_3' \leq \|F\|_2' \leq \|F\|_1'.$$

The sets Φ_n' form a dense subspace of Φ' under the weak topology. We have denoted $\Phi_n \subseteq \Phi_m$ as an inclusion, but it defines an operator $T_m^n : \Phi_n \to \Phi_m$. We have the identity $T_m^p = T_m^n T_n^p$ for $m \leq n \leq p$.

23.27 Definition. A countably Banach space Φ is a *nuclear space* if for each m, there is an n such that the map T_m^n is nuclear, that is, it has the form

$$T_m^n x = \sum_{k=1}^{+\infty} \lambda_k \rho_k(x) y_k,$$

with $(\rho_n)_n$ a sequence in Φ_n' with $\|\rho_n\| \leq 1$ and $(y_n)_n$ a sequence in Φ_m with $\|y_n\| \leq 1$, with $\lambda_k \geq 0$ such that $\sum \lambda_k < +\infty$.

One can prove that a nuclear space can always be generated by a system of inner products instead of just norms:

23.28 Definition. Every nuclear space Φ is a countably Hilbert space. That is: there is a countable family of inner products $(\cdot, \cdot)_n$ such that its associated norms $\| \ \|_n$ generates the countably Banach space Φ in the sense of Definition 23.26.

23.29 Examples.

1. The set of all rapidly decreasing sequences is a nuclear space. Specifically, if we define

$$s = \{(x_n)_n \in \mathbb{C}^{\mathbb{N}} \mid n^k x_n \to 0 \text{ for each } k \geq 0\}.$$

   We equip s with the norms $\|(x_n)_n\|_k = \sup_n |n^k x_n|$.

2. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is a nuclear space with the usual norms.

3. If U is an open subset of a Euclidean space $\mathbb{R}^n$, then

$$\mathcal{C}_c^{\infty}(U) = \{f \in \mathcal{C}^{\infty}(\mathbb{R}^n) \mid f \text{ has compact support}\}.$$

   is a nuclear space with respect to the norms $\|f\|_n = \|f^{(n)}\|_{\infty}$.

4. More generally, if M is a smooth manifold, then $\mathcal{C}_c^{\infty}(M)$ is a nuclear space with convergence given by $f_i \to f$ if there is a compact set K such that the support of f_i and f are contained in K and such that for each smooth vector field X on M we have that $Xf_i \to Xf$.

We have the following stability properties of nuclear spaces:

23.30 Theorem.

1. *The completion of a nuclear space is nuclear.*

2. *A linear subspace of a nuclear space is nuclear*

3. *The quotient of a nuclear space modulo a closed linear subspace is nuclear*

4. *A product of nuclear spaces is nuclear*

5. *A countable topological direct sum of nuclear spaces is nuclear*

6. *A Hausdorff projective limit of nuclear spaces is nuclear. That is: if we have a collection of nuclear spaces E_α, $\alpha \in A$, a vector space E and linear maps $\varphi_\alpha : E \to E_\alpha$ such that the linear maps separate points, then the initial topology on E gives rise to a nuclear space.*

7. *A countable inductive limit of nuclear spaces is nuclear. That is: if we have a countable collection of nuclear spaces E_α, $\alpha \in A$, a vector space E and linear maps $\varphi_\alpha : E_\alpha \to E$ such that $E = \bigcup_\alpha \varphi_\alpha(E_\alpha)$ and such that E is equipped with the finest locally convex topology such that all the φ_α are continuous, then E is a nuclear space.*

There are not many normed spaces which get to be nuclear:

23.31 Theorem. *A normed space is nuclear if and only if it is finite dimensional.*

One of the major theorems concerning nuclear spaces is the abstract kernel theorem:

23.32 Theorem. *Let Φ and Ψ be complete countably Hilbert spaces, one of which, say Φ, is nuclear. Let $B : \Phi \times \Psi \to \mathbb{C}$ be a bilinear functional, continuous in each of its arguments. Then there are values p and m such that B can be represented as*

$$B(x, y) = (Ax)y := \langle Ax, y \rangle,$$

where A is a Hilbert-Schmidt operator which maps the Hilbert space Φ_p into the Hilbert space Ψ'_m.

23.4 Rigged Hilbert spaces

We can now finally give the definition of a rigged Hilbert space and state the so-called nuclear spectral theorem. More details and proofs can be found in [17]

23.33 Definition. A *rigged Hilbert space* is a complete nuclear space Φ with an inner product $(\cdot, \cdot)_0$ (we will usually write $\langle \cdot, \cdot \rangle$) such that if $x_n \to x$ in $(\Phi, \mathcal{T})$, then $\langle x_n, y \rangle \to \langle x, y \rangle$.

We see thus that $\langle \cdot, \cdot \rangle$ is continuous with respect to one of the norms $\| \cdot \|_n$ for $n > 0$, thus we can find m and M such that

$$|\langle x, y \rangle| \leq M \|x\|_m \|y\|_m.$$

Let H be the completion of Φ under the scalar product $\langle \cdot, \cdot \rangle$, we sometimes denote H by Φ_0. Thus Φ is a dense subspace of H, and we have a continuous linear operator $T : \Phi \to H$. By transposing, we get a continuous linear embedding $T' : H' \to \Phi'$, where

$$(T'F)(x) = F(Tx),$$

for $F \in H'$ and $x \in \Phi$. If we regard the T as an inclusion, like we usually will, then the T' are of course just restriction maps, thus $T'F = F|_\Phi$. Of course, for any $F \in H'$ we can find a $y \in H$ such that $F = \langle \cdot, y \rangle$, so we can consider T' as an (antilinear) map from H to Φ'. We will often leave out the operators T and denote a rigged Hilbert space by

$$\Phi \subseteq H \subseteq \Phi'.$$

Using nuclearity of the space, we can prove the following:

23.34 Theorem. *Let $\Phi \subseteq H \subseteq \Phi'$ be a rigged Hilbert space and let T be the natural embedding operator of Φ into H. Then there is an n, an orthonormal basis $(h_k)_k$ in H and $(F_k)_k$ in Φ'_n such that*

$$Tx = \sum_{k=1}^{+\infty} \lambda_k F_k(x) h_k,$$

for each $x \in \Phi$. We also know that $\lambda_k \geq 0$ and $\sum \lambda_k$ converges.

From nuclearity of the space, we know we can associate with any rigged Hilbert space a two sided infinite decreasing chain:

$$\Phi \subseteq \ldots \subseteq \Phi_n \subseteq \ldots \subseteq \Phi_1 \subseteq H \subseteq \Phi'_1 \subseteq \ldots \subseteq \Phi'_n \subseteq \ldots \subseteq \Phi'.$$

These inclusions are actually continuous operators (some of which are antilinear).
By taking sums, one can again arrange it so that the norms satisfy

$$\|x\|_0 \leq \|x\|_1 \leq \|x\|_2 \leq \ldots.$$

and then also that

$$\ldots \leq \|F\|'_2 \leq \|F\|'_1 \leq \|F\|_0.$$

So if we identify H and H' then we get the following infinite chain

$$\ldots \leq \|x\|'_2 \leq \|x\|'_1 \leq \|x\|_0 \leq \|x\|_1 \leq \|x\|_2 \leq \ldots$$

where we set $\|x\|_n = +\infty$ for $x \notin \Phi_n$.

Let $\Phi_n^\times$ be the set of all anti-linear functionals on Φ_n. Then analogously, we can associate with any rigged Hilbert space a two-sided infinite chain

$$\Phi \subseteq \ldots \subseteq \Phi_n \subseteq \ldots \subseteq \Phi_1 \subseteq H \subseteq \Phi_1^\times \subseteq \ldots \subseteq \Phi_n^\times \subseteq \ldots \subseteq \Phi^\times.$$

These inclusions are all linear continuous operators. We again have an infinite chain of norms like before. Again, we have an infinite sequence of norms

$$... \leq \|x\|_2^\times \leq \|x\|_1^\times \leq \|x\|_0 \leq \|x\|_1 \leq \|x\|_2 \leq ...$$

where we set $\|x\|_n = +\infty$ for $x \notin \Phi_n$. So with any rigged Hilbert space, we can associate the following picture:

$$\Phi_1' \longrightarrow ... \longrightarrow \Phi'$$

$$\Phi \longrightarrow ... \longrightarrow \Phi_1 \longrightarrow H$$

$$\Phi_1^\times \longrightarrow ... \longrightarrow \Phi^\times$$
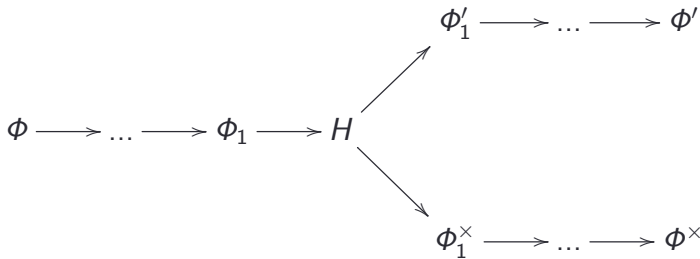
For an $F \in \Phi'$ and $\varphi \in \Phi$, we will use the notation $\langle \varphi, F \rangle := F(\varphi)$. This generalizes the inner product on H. Analogously, if $F \in \Phi^\times$, then we denote $\langle F, \varphi \rangle := F(\varphi)$.

We can extend these notions as follows. Let $F \in \Phi'$ and Let $(\varphi_i)_{i \in I}$ be a net in Φ converging weakly to a functional $G \in \Phi^\times$, meaning that for each $\varphi \in \Phi$ we have that $\langle \varphi, \varphi_i \rangle \to G(\varphi)$. Assume that $\langle \varphi_i, F \rangle$ converges, then we will define its limit as $\langle G, F \rangle$. We will not use this extension in what follows.

Another notation we will introduce is the bra-ket notation. In particular, if $\varphi \in \Phi$ then we will denote the linear operator $\langle \cdot, \varphi \rangle$ by $|\varphi\rangle$. Likewise, the anti-linear operator $\langle \varphi, \cdot \rangle$ will be denoted as $\langle \varphi|$.

We can extend this bra-ket notation by defining for each $F \in \Phi'$ that $|F\rangle := F$ and analogously for $F \in \Phi^\times$ that $\langle F| := F$. We will not use the extension of this notation here, but it is widely used in the physics literature.

It is of course well-known that any Hilbert space can be seen as a space of functions $L^2(X, \mu)$. The same holds for a rigged Hilbert space. So let $\Phi \subseteq H \subseteq \Phi'$ be a rigged Hilbert space. Consider an isometric isomorphism $f : H \to L^2(X, \mu)$. Then we can also naturally embed Φ as $f(\Phi)$. Once we have such an embedding, we can make sense of Dirac delta functions:

23.35 Proposition. *Let $\Phi \subseteq H \subseteq \Phi'$ be a rigged Hilbert space and let $f : H \to L^2(X, \mu)$ be an isometric isomorphism. Then for each $x \in X$, we can associate a linear functional F_x on Φ such that for any $\varphi \in \Phi$, we have that*

$$f(\varphi)(x_0) = F_{x_0}(\varphi)$$

holds for almost every x_0.

Let $A : \Phi \to \Phi$ be a bounded linear operator. We can associate with A the operator $A' : \Phi' \to \Phi'$ such that $A'(F)\varphi = F(A\varphi)$. A linear functional $F \in \Phi'$ is called a generalized eigenvector of A corresponding to the eigenvalue λ if $A'F = \lambda F$. To each λ, there corresponds the eigenspace Φ'_λ of all eigenvectors of eigenvalue λ. We can now formulate the spectral theorem, but first we need a generalization of the direct sum of Hilbert spaces.

For this, let X be a measure space equipped with a measure μ. For every $x \in X$, suppose there is an associated separable Hilbert space $H(x)$ of dimension m_x. Assume first that all $H(x)$ have the same dimension, then we can identify all $H(x)$ with a Hilbert space H of dimension n. Now we construct the Hilbert space K consisting of all functions $f : X \to H$ such that

1. For all φ in H, the function $x \to \langle f(x), \varphi \rangle$ is measurable with respect to μ.

2. The function $x \to \|f(x)\|$ is square-integrable with respect to μ.

Now we make K into a Hilbert space by equipping it with the pointwise operations and by defining

$$\langle f, g \rangle = \int_X \langle f(x), g(x) \rangle \, \mathrm{d}\mu(x).$$

We call K the direct integral of the Hilbert spaces $H(x)$ with respect to the measure μ and denote it by

$$K = \int_X H(x) \, \mathrm{d}\mu(x).$$

In the case that the $H(x)$ have different dimensions, we divide X into measurable subsets $X_1, X_2, ..., X_\infty$ such that on each X_k we have $m(x) = k$. We now define

$$K_k = \int_{X_k} H(x) \, \mathrm{d}\mu(x).$$

We then set

$$K = \bigoplus_{n=1}^{+\infty} K_n.$$

and we will also call K the direct integral with respect to the measure μ and denote it with the same symbol.

The nuclear spectral theorem now goes as follows:

23.36 Theorem. *Let $\Phi \subseteq H \subseteq \Phi'$ be a rigged Hilbert space and let $A : \Phi \to \Phi$ be an essentially self-adjoint operator. Then there is a direct integral*

$$K = \int_X H(x) \, d\mu(x) = \bigoplus_{k=1}^{+\infty} K_k.$$

and a unitary map (called the Fourier transform)

$$F : H \to K$$

such that for $\varphi \in \Phi$ we have $(F(\varphi))_k(x) = \langle \varphi, E_k(x) \rangle$, where $E_k(x) \in \Phi'$ are the generalized eigenvectors of A with eigenvalue $\lambda(x)$. Furthermore, we have

$$(FA\varphi)_k(x) = \lambda(x)(F\varphi)_k(x)$$

and $\sigma(A) = \{\lambda(x) \mid x \in X\}$.

It is not at all clear how this nuclear spectral theorem extends the other spectral theorems. Indeed, even for a bounded operator $A : H \to H$ between Hilbert spaces, it is not at all clear if there exists a rigged Hilbert space structure $\Phi \subseteq H \subseteq \Phi'$ such that A restrict to an operator $\Phi \to \Phi$. We can prove however that such a structure always exists. This will be clarified in the next section.

23.5   Generating rigged Hilbert spaces

The spectral theorem applies to any rigged Hilbert space $\Phi \subseteq H \subseteq \Phi'$ and any essentially self-adjoint operator $A : \Phi \to \Phi$. The spectral theorem shows that any such operator has generalized eigenvalues and that any such operator can be seen as multiplication operator. A question that remains is of course in what sense any essentially self-adjoint operator $A : dom(A) \subseteq H \to H$ can be seen as an operator $A : \Phi \to \Phi$ for some suitable rigged Hilbert space $\Phi \subseteq H \subseteq \Phi'$. The following theorem by Maurin gives a positive answer:

23.37 Theorem. *Let $(A_j : dom(A_j) \to H)_j$ be a commutative family of self-adjoint operators such that the intersection*

$$D_0 := \bigcap_{j,p} dom(A_j^p)$$

is dense in H, then there exists a nuclear space $\Phi \subseteq D_0$ such that the maps $A_j^p : \Phi \to \Phi$ are continuous with respect to the nuclear structure on Φ, such that Φ is dense in H and such that the embedding $i : \Phi \to H$ is continuous where Φ carries the nuclear structure and such that H carries its usual Hilbert space structure. Hence, $\Phi \subseteq H \subseteq \Phi'$ is a rigged Hilbert space.

See [45] for a proof.

In particular, given a self-adjoint operator $A : dom(A) \to H$ such that $\bigcap_p dom(A^p)$ is dense, we can find a rigged Hilbert space $\Phi \subseteq H \subseteq \Phi'$ such that $A : \Phi \to \Phi$ is continuous with respect to the nuclear structure. Hence the spectral theorem applies and there is a direct integral

$$K = \int_X H(x) \, d\mu(x) = \bigoplus_{k=1}^{+\infty} K_k$$

and a unitary map $F : H \to K$ such that $(F(\varphi)_k)(x) = \langle \varphi, E_k(x) \rangle$ where $E_k(x) \in \Phi'$ are the generalized eigenvalues of A and the rest of the theory applies. In particular, the entire theory applies to bounded self-adjoint operators (thus self-adjoint operators A whose domain is the entire H)

However, in quantum mechanics, we are often confronted with a family of noncommuting self-adjoint operators. For example, the position and momentum operators on $L^2(\mathbb{R})$ are defined by

$$Q : dom(P) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R}) : \varphi \to P\varphi$$

with $(P\varphi)(x) = x\varphi(x)$ and

$$P : dom(Q) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R}) : \varphi \to Q\varphi$$

with $(Q\varphi)(x) = -i\varphi'(x)$. These two operators do not commute and in fact we have that

$$PQ\varphi - QP\varphi = i\varphi$$

for those φ such that $PQ\varphi$ and $QP\varphi$ make sense (for example, for those φ in the Schwartz space $\mathcal{S}(\mathbb{R})$.). Because the operators do not commute, we cannot directly apply Maurin's theorem to obtain a suitable rigged Hilbert space. Of course, we do know that one exists (since the Schwartz space is one), but we would like a general argument.

The idea is that the position and momentum operators give rise to unitary operators on H. Indeed, define $U(\alpha) = e^{i\alpha P}$ and $V(\beta) = e^{i\beta Q}$ for $\alpha, \beta \in \mathbb{R}$. These are called the *Weyl operators*. They are easily shown to be unitary and they satisfy the following commutation relations:

$$U(\alpha)U(\beta) = U(\alpha + \beta) = U(\beta)U(\alpha),$$

$$V(\alpha)V(\beta) = V(\alpha + \beta) = V(\beta)V(\alpha),$$

and

$$U(\alpha)V(\beta) = V(\beta)U(\alpha)e^{-i\alpha\beta}.$$

The abstract algebra $\mathcal{A}_W \subseteq \mathcal{B}(H)$ generated by the $U(\alpha)$ and the $U(\beta)$ (by taking complex linear combinations and products) is called the Weyl algebra. The closure of the *Weyl algebra* in $\mathcal{B}(H)$ under the usual operator norm yields the so called Weyl C^*-algebra $\overline{\mathcal{A}_W}$.

On the other hand, we can look at the collection of all unitary operators of the form $e^{i(\alpha P + \beta Q + \lambda 1)}$ with $\alpha, \beta, \lambda \in \mathbb{R}$. This forms a topological group (with topology induced from the operator norm), which can even be shown to be a noncompact Lie group. This Lie group is called the *Heisenberg Lie group* $\mathcal{H}$. This Lie group can be alternatively described by $\mathcal{H} \cong \mathbb{R}^3$ with group law given by

$$(\alpha, \beta, \lambda) \cdot (\alpha', \beta', \lambda') = (\alpha + \alpha', \beta + \beta', (\alpha'\beta - \alpha\beta')/2).$$

So we have found a map $U : \mathcal{H} \to \mathcal{A}_W \subseteq \mathcal{B}(H)$. This is a strongly continuous unitary representation of the Heisenberg Lie group. This hints at a general situation where we have a locally compact Lie group G and a strongly continuous unitary representation $U : G \to \mathcal{B}(H)$.

Now let G be a Lie group. We know that $\mathcal{C}_c^\infty(G)$ has the structure of a complete nuclear space. Let $U : G \to H$ be a strongly continuous unitary representation of G. For $f \in \mathcal{C}_c^\infty(G)$ and $x \in H$, we define

$$T(f)x = \int_G f(g)U(g)x \, \mathrm{d}g$$

where dg is the left Haar measure on G (which exists because of local compactness). This integral is well-defined because f vanishes outside a compact set. Thus for each f, we find that $T(f)$ is a bounded linear operator on the Hilbert space. So we have a well-defined map

$$T : \mathcal{C}_c^\infty(G) \to \mathcal{B}(H).$$

This map is strongly continuous. Indeed, let $f_n \to f$ for the nuclear structure of $\mathcal{C}_c^\infty(G)$ in G, then in particular we have that there is a compact set K such that the supports of f_n and f are contained in K and we know that $f_n \to f$ uniformly. Now, if $x \in H$, then

$$
\begin{aligned}
\|T(f_n)x - T(f)x\| &= \| \int_G (f_n(g) - f(g))U(g)x \, \mathrm{d}g\| \\
&\leq \int_G |f_n(g) - f(g)| \|x\| \, \mathrm{d}g \\
&\leq \mu_g(K)\|f_n - f\|_\infty \|x\|.
\end{aligned}
$$

This proves that $T(f_n) \to T(f)$ for the strong operator topology. Thus T is strongly continuous.

Take $v_1 \in H$, we can form

$$\Phi^1 = \{T(f)v_1 \mid f \in \mathcal{C}_c^\infty(G)\}.$$

This is a nuclear space since it is the quotient of the nuclear space $\mathcal{C}_c^\infty(G)$. Let us investigate some properties about Φ^1.

23.38 Lemma. *The topology of Φ^1 is stronger than that of H.*

Proof. If $T(f_n)v_1 \to T(f)v_1$ in the nuclear structure of Φ^1, then by definition $f_n \to f$ in $\mathcal{C}_c^\infty(G)$, and thus (since T is strongly continuous), we have $T(f_n)v_1 \to T(f)v_1$ in the topology of H. $\square$

23.39 Lemma. *The space Φ^1 is invariant under the transformations $U(g)$ for each $g \in G$.*

Proof. Take $T(f)v_1 \in \Phi^1$ and $g \in G$. Then

$$
\begin{aligned}
U(g)T(f)v_1 &= U(g)\int_G f(h)U(h)v_1 \, \mathrm{d}h \\
&= \int_G f(h)U(gh)v_1 \, \mathrm{d}h \\
&= \int_G f(g^{-1}g')U(g')v_1 \, \mathrm{d}g' \\
&= T(k)v_1
\end{aligned}
$$

with $k \in \mathcal{C}_c^\infty(G)$ given by $k(g') = f(g^{-1}g')$. $\square$

23.40 Lemma. *Let $\gamma : \mathbb{R} \to G$ be a one-parameter subgroup, then we can define for $x \in H$*

$$Ax = \lim_{h \to 0} \frac{U(\gamma(h))x - U(\gamma(0))x}{h}$$

if the limit exists. This A is defined everywhere on Φ^1 and $Ax \in \Phi^1$.

Proof. Take $T(f)v_1 \in \Phi^1$. Define $k_h \in \mathcal{C}_c^\infty(G)$ by $k_h(g) = f(\gamma(h)^{-1}g)$. It is clear that

$$\lim_{h \to 0} \frac{k_h - k_0}{h}$$

exists and defines a map $k \in \mathcal{C}_c^\infty(M)$. But then

$$AT(f)v_1 = \lim_{h \to 0} \frac{T(k_h)x - T(k_0)x}{h} = T(k)x.$$

Thus $AT(f)v_1$ exists an is an element of Φ^1.

$\square$

If Φ^1 is not dense, then we can take $v_2 \notin \overline{\Phi^1}$ and form

$$\Phi^2 = \{U(f)v_2 \mid f \in \mathcal{C}_c^\infty(G)\}.$$

This is again a nuclear space and satisfies the above three results. We can iterate this process to obtain (if necessary) Φ^n for natural numbers n. We can then form

$$\Phi = \bigoplus_n \Phi^n.$$

It is easily seen that this is a nuclear space, again satisfying the above three results. Thus we have proven:

23.41 Theorem. *If $U : G \to \mathcal{B}(H)$ is a strongly continuous unitary representation of a Lie group G, then there exists a nuclear space $\Phi \subseteq H$ such that*

1. *The embedding $\Phi \to H$ is continuous.*

2. *Φ is invariant under $U(g)$ for each $g \in G$.*

3. *Define for each one-parameter subgroup $\gamma : \mathbb{R} \to G$ (if the limit exists):*
   $$Ax = \lim_{h \to 0} \frac{U(\gamma(h))x - U(\gamma(0))x}{h}.$$
   then A is a well-defined on Φ and Φ invariant under A.

The ideas behind the proof of this result appear in [45].

One possible application of this theorem is when we consider a Hilbert space H and a locally compact subgroup G of the group of unitary operators $\mathcal{U}(H)$ on H. One can prove that this is a Lie group (See [29]) and thus we can apply the theorem.

23.6 An extended spectral theorem

In the previous spectral theorem, we only discuss operators $A : \Phi \to \Phi$. The question arises if more general operators $A : \Phi' \to \Phi'$ also have a spectral theorem. We will now attempt to prove such a theorem. First we investigate different kinds of continuity for such maps.

23.42 Definition. Let $A : \Phi' \to \Phi'$ be a linear map. We say that A is *weakly continuous* if for each weakly convergent net $F_\nu \to F$ (meaning that for each $\varphi \in \Phi$, it holds that $\langle F_\nu, \varphi \rangle \to (F, \varphi)$) we have that AF_ν converges to $A(F)$, meaning that for each $\varphi \in \Phi$, we have that $\langle A(F_\nu), \varphi \rangle \to (A(F), \varphi)$.

23.43 Definition. Let $B : \Phi \to \Phi$ be a linear map. We will say that B is *bounded* with respect to the nuclear structure if for each $n \in \mathbb{N}$, there exists constant C (possibly depending on n) and an $m \in \mathbb{N}$ such that for each $\varphi \in \Phi$, we have that

$$\|B\varphi\|_n \leq C\|\varphi\|_m.$$

If we can choose $C = 1$ independently of n, then we will say that B is a *contraction*.

We note that the same definitions hold for maps $\Phi^\times \to \Phi^\times$.

Now let $A : \Phi \to \Phi$ be a bounded operator for the nuclear structure $(\Phi, \mathcal{T})$. By transposing, we can define the operator $A' : \Phi' \to \Phi'$ by $A(F)(\varphi) = F(A\varphi)$.

23.44 Proposition. *Let $A : \Phi \to \Phi$ be a bounded operator for the structure $(\Phi, \mathcal{T})$. Then $A' : \Phi' \to \Phi'$ is weakly continuous.*

Proof. That A' is weakly continuous is easy ☐

We can also extend an operator $A : \Phi \to \Phi$ to an operator on $\Phi^\times$ by the same formula $A^\times(F)(\varphi) = F(A(\varphi))$. The same result holds.

Any (densely defined) operator $T : \Phi \to \Phi$ has an adjoint T^*. We will now try to define an adjoint for operators $A : \Phi \to \Phi$ (not necessarily continuous in any topology). This adjoint must necessarily be related to the adjoints of maps $\Phi \to \Phi$.

To do this, let $A : \Phi' \to \Phi'$ be a linear operator (not necessarily continuous in any topology). For $\varphi, \psi \in \Phi$, define

$$B(\langle \varphi |)(\psi) = A(|\psi\rangle)(\varphi).$$

This is easily seen to be linear. Thus $B : dom(B) \subseteq H' \to \Phi^\times$ defines a linear operator. We use the notation $\mathcal{B}(A)$ to denote $dom(B)$.

23.45 Proposition. *If $A : \Phi' \to \Phi'$ is weakly continuous, then B is weakly continuous.*

Proof. Let F_ν be a net in $\mathcal{B}(A)$ and let F be in $\mathcal{B}(A)$. Assume that F_ν converges weakly to F, meaning that for each $\psi \in \Phi$, we have $F_\nu(\psi) \to F(\psi)$. We can write $F_\nu = \langle \varphi_\nu |$ and $F = \langle \varphi |$. We must show that

$$BF_\nu(\psi) \to BF(\psi)$$

or

$$A(|\psi\rangle)(\varphi_\nu) \to A(|\psi\rangle)\varphi$$

But since $A(|\psi\rangle)$ is continuous with respect to $(\Phi, \mathcal{T})$, we know that it takes weakly convergent sequences to weakly convergent sequences. This proves the theorem. ☐

Since $\mathcal{B}(A)$ is dense (in the weak topology) in $\Phi^\times$ and since $\Phi^\times$ is complete with respect to the weak topology, we know that we can extend B to a weakly continuous operator which we will denote as $A^\dagger : \Phi^\times \to \Phi^\times$. It is clear that on $\mathcal{B}(A)$, we have $A^\dagger(\langle\varphi|)(\psi) = A(|\psi\rangle)(\varphi)$.

23.46 Proposition. *Let $A : \Phi' \to \Phi'$ be weakly continuous. If $\varphi \in \Phi_n$ and $\psi \in \Phi$, then $A^\dagger(\langle\psi|) \in \Phi_n^\times$ and $A^\dagger(\langle\psi|)(\varphi) = A(|\varphi\rangle)(\psi)$.*

Proof. Let $(\varphi_k)_k$ be a sequence in Φ converging to φ under $\| \ \|_n$. This sequence will also converge weakly. Thus the sequence of functionals $|\varphi_k\rangle$ will converge weakly to $|\varphi\rangle$. Since A is weakly continuous, the sequence $A(|\varphi_k\rangle)(\psi)$ will converge to $A(|\varphi\rangle)(\psi)$ for every $\psi \in \Phi$. But by definition, we have that $A(|\varphi_k\rangle)(\psi) = A^\dagger(\langle\psi|)(\varphi_k)$. By boundedness of $A^\dagger(\langle\psi|)$, we see that it is well-defined on φ and that the theorem holds. $\qquad\square$

In particular, we can apply the previous theorem for $n = 0$ and we see that $A^\dagger(\langle\psi|) \in H^\times$.

23.47 Proposition. *Let $A, B : \Phi' \to \Phi'$ be weakly continuous. If $\alpha, \beta \in \mathbb{C}$, then $(\alpha A + \beta B)^\dagger = \alpha A^\dagger + \beta B^\dagger$.*

Proof. For each $\varphi, \psi \in \Phi$, we have

$$
\begin{aligned}
(\alpha A + \beta B)^\dagger(\langle\varphi|)(\psi) &= \alpha A(|\psi\rangle)(\varphi) + \beta B(|\psi\rangle)(\varphi) \\
&= (\alpha A^\dagger + \beta B^\dagger)(\langle\varphi|)(\psi)
\end{aligned}
$$

which concludes the proof. $\qquad\square$

23.48 Proposition. *Let $A, B : \Phi' \to \Phi'$ be weakly continuous. Then $(AB)^\dagger = B^\dagger A^\dagger$.*

Proof. We know that

$$(AB)^\dagger(\langle\varphi|)(\psi) = (AB)(|\psi\rangle)(\varphi) = A(B(|\psi\rangle))(\varphi).$$

$$B^\dagger(A^\dagger(\langle\varphi|))(\psi) = B^\dagger(\langle A\varphi|)(\psi) = B(|\psi\rangle)(A'\varphi).$$

We know that $A^\dagger(\langle\varphi|) \in H^\times$, thus we can put this equal to $\langle\chi|$. We get that

$$B^\dagger(A^\dagger(\langle\varphi|))(\psi) = B^\dagger(\langle\chi|)(\psi) = B(|\psi\rangle)(\chi).$$

Take a net $|\psi_\nu\rangle$ weakly converging to $B(|\psi\rangle)$. We then have

$$A(|\psi_\nu\rangle)(\varphi) = A^\dagger(\langle\varphi|)(\psi_\nu) = \langle\chi, \psi_\nu\rangle = |\psi_\nu\rangle(\chi).$$

By continuity, we get that

$$A(B(|\psi\rangle))(\varphi) = B(|\psi\rangle)(\chi).$$

$\square$

Define $S : \Phi' \to \Phi^\times$ by $SF(\varphi) = \overline{F(\varphi)}$. We can now finally define the suitable notion of adjoint.

23.49 Definition. Given a weakly continuous operator $A : \Phi^\times \to \Phi^\times$, we define $A^* : \Phi' \to \Phi'$ by $A^*F = S^{-1}(A^\dagger S(F))$. Thus for any $\varphi \in \Phi$, we have
$$A^*F(\varphi) = \overline{A^\dagger S(F)(\varphi)}.$$

If $T : \Phi \to \Phi$ is a closable operator, then we can define a canonical operator $A : \Phi' \to \Phi'$ by transposing: so we defined $AF = F \circ T$. In particular, for $|\varphi\rangle \in \mathcal{D}(A)$, we have $(A|\varphi\rangle)(\psi) = (|\varphi\rangle \circ T)(\psi) = \langle T\psi, \varphi\rangle$. Then the operator $A^\dagger$ is defined by $A^\dagger(\langle\varphi|)\psi = (A|\psi\rangle)(\varphi) = (|\psi\rangle \circ T)(\varphi) = \langle T\varphi, \psi\rangle$. So we get that $A^*(|\varphi\rangle)\psi = \overline{A^\dagger S(|\varphi\rangle)\psi} = \overline{(A^\dagger\langle\varphi|)(\psi)} = \overline{\langle T\varphi, \psi\rangle} = \langle \psi, T\varphi\rangle = \langle T^*\psi, \varphi\rangle$.

Note that since $A^\dagger(|\varphi\rangle) \in H'$, we have the same for A^*. Thus if A is weakly continuous, then $A^*(|\varphi\rangle) \in H'$.

We have the following properties:

23.50 Proposition. *Let $A, B : \Phi' \to \Phi'$ be weakly continuous and let $\alpha, \beta \in \mathbb{C}$, then $(\alpha A + \beta B)^* = \overline{\alpha}A^* + \overline{\beta}B^*$ and $(AB)^* = B^*A^*$.*

Proof. Follows immediately. □

23.51 Proposition. *Let $A : \Phi' \to \Phi'$ be weakly continuous. Then $A^{**} = A$.*

Proof. We see that

$$
\begin{aligned}
A^{**}(|\varphi\rangle)(\psi) &= \overline{(A^*)^\dagger(\langle\varphi|)(\psi)} \\
&= \overline{A^*(|\psi\rangle)(\varphi)} \\
&= \overline{A^\dagger(\langle\psi|)(\varphi)} \\
&= A(|\varphi\rangle)(\psi)
\end{aligned}
$$

□

23.52 Definition. We say that a weakly continuous operator $A : \Phi' \to \Phi'$ is symmetric if $A^* = A$. Thus if $A(|\varphi\rangle)(\psi) = \overline{A(|\psi\rangle)(\varphi)}$ for each $\varphi, \psi \in \Phi$.

As seen before, we know that for $\varphi \in \Phi$ it holds that $A(|\varphi\rangle) \in H'$ and that if $\psi \in H$ it then holds that

$$
A(|\varphi\rangle)(\psi) = \overline{A(|\psi\rangle)(\varphi)}.
$$

We cannot make the same conclusion when $\varphi \in H$.

23.53 Theorem. *Let $A : \Phi' \to \Phi'$ be symmetric. Let*

$$
\mathcal{D}(A) = \{|\varphi\rangle \in H' \mid \varphi \in \Phi \text{ and } A(|\varphi\rangle) \in H'\}.
$$

Then $A : \mathcal{D}(A) \to H'$ satisfies

1. $\langle A|\psi\rangle, |\varphi\rangle\rangle = (A|\psi\rangle)(\varphi)$ *for* $|\varphi\rangle, |\psi\rangle \in \mathcal{D}_A$.

2. $\langle A(F), G\rangle = \langle F, AG\rangle$ *for* $F, G \in \mathcal{D}_A$.

Proof. For (1), let $A(|\psi\rangle) = |\chi\rangle$ for $\chi \in H$. Then

$$\langle A|\psi\rangle, |\varphi\rangle\rangle = \langle|\chi\rangle, |\varphi\rangle\rangle = \langle \varphi, \chi\rangle.$$

But also

$$(A|\psi\rangle)(\varphi) = |\chi\rangle(\varphi) = \langle \varphi, \chi\rangle$$

. For (2), take $F = |\varphi\rangle$ and $G = |\psi\rangle$. Assume that $A(G) = |\chi\rangle$, then by (1)

$$\langle A(F), G\rangle = A(|\varphi\rangle)(\psi) = \overline{A(|\psi\rangle)(\varphi)} = \overline{\langle \varphi, \chi\rangle} = \langle \chi, \varphi\rangle.$$

On the other hand, we have

$$\langle F, A(G)\rangle = \langle|\varphi\rangle, |\chi\rangle\rangle = \langle \chi, \varphi\rangle.$$

$\square$

So we have a symmetric unbounded operator $A : \mathcal{D}(A) \subseteq H' \to H'$ on the Hilbert space H'. This operator need not be closed, but we can always extend the domain $\mathcal{D}(A)$ to a larger domain $\mathcal{D}^*(A) \subseteq H'$ such that the operator $A : \mathcal{D}^*(A) \subseteq H' \to H'$ is closed.

23.54 Theorem. *Let* $A : \Phi' \to \Phi'$ *be symmetric. Let*

$$\mathcal{D}^*(A) = \{|\varphi\rangle \in H' \mid \varphi \in H \text{ and } A(|\varphi\rangle) \in H'\}.$$

Then $A : \mathcal{D}^*(A) \to H'$ *is a closed operator extending* $A : \mathcal{D}(A) \to H'$.

Proof. Assume that $|\varphi_n\rangle \to F$ in H' with $F \in \mathcal{D}^*(A)$ and that $A(|\varphi_n\rangle) \to G$ in H'. Then convergence is also weakly and since A is weakly continuous, we have $A(F) = G$. $\square$

Let A be a symmetric operator. The previous theorem states that there exists some domain $\mathcal{D}^*(A)$ on which A is closed. We can then look at the smallest such domain, we denote this by $\overline{\mathcal{D}(A)}$. We can now introduce the notion of self-adjointness:

23.55 Definition. We say that a symmetric operator $A : \Phi' \to \Phi'$ is self-adjoint if for each $\varphi \in H$ it holds that if there exists an $\eta \in H$ such that

$$A(|\psi\rangle)(\varphi) = \langle \psi, \eta \rangle, \text{ for each } \psi \in \mathcal{D}(A)$$

then $\varphi \in \mathcal{D}(A)$.

23.56 Theorem. *Let $A : \Phi' \to \Phi'$ be self-adjoint. Then $A : \overline{\mathcal{D}(A)} \to H'$ is self-adjoint as an unbounded operator on $\overline{\mathcal{D}(A)}$.*

Proof. It suffices to prove that $A : \mathcal{D}(A) \to H'$ is essentially self-adjoint. For this, we need to find the domain of A^*, the adjoint of the unbounded operator $A : \mathcal{D}(A) \to H'$. So let $|\psi\rangle$ be in the domain of A^*, then we know that for each $|\varphi\rangle \in \mathcal{D}(A)$ it holds that

$$\langle A^*(|\psi\rangle), |\varphi\rangle \rangle = \langle |\psi\rangle, A(|\varphi\rangle) \rangle.$$

But we have

$$
\begin{aligned}
\langle |\psi\rangle, A(|\varphi\rangle) \rangle &= \overline{A(|\varphi\rangle)(\psi)} \\
&= \overline{\overline{A(|\psi\rangle)(\varphi)}} \\
&= \overline{\overline{\langle |\varphi\rangle, A(|\psi\rangle) \rangle}} \\
&= \langle A(|\psi\rangle), |\varphi\rangle \rangle
\end{aligned}
$$

and thus $A^*(|\psi\rangle) = A(|\psi\rangle)$. In particular, the domain of A^* is contained in (and thus equal to) the domain of A. $\square$

So the spectral theory of a weakly continuous self-adjoint operator $A : \Phi^* \to \Phi^*$ reduces to the spectral theory of actualy self-adjoint operators $\Phi \to H$. So to summarize, we have the main theorem:

23.57 Theorem. *Take a self-adjoint (in the sense of Definition 23.55) operator $A : \Phi' \to \Phi'$, then there exists a dense subspace $\overline{\mathcal{D}(A)}$ of H', such that the restriction of A to $\overline{\mathcal{D}(A)} \to H'$ makes sense and is a self-adjoint (in the classical sense) unbounded operator.*

We now make some final remarks on how to continue with the spectral theory of operators $A : \Phi^* \to \Phi^*$. If we apply the spectral theorem for unbounded self-adjoint operators, then there exists a unitary operator $U : H \to L^2(M)$ such that $UAU^{-1}f(m) = S(m)f(m)$. Furthermore, the structure of the rigged Hilbert space also carries over. In particular, a functional $F \in \Phi'$ induces a functional on the new rigged hilbert space by FU^{-1}. Thus we have a new rigged Hilbert space $U(\Phi) \subseteq L^2(M) \subseteq U(\Phi)'$.

Define the operator $\tilde{A}$ on the new rigged Hilbert space $U(\Phi) \subseteq L^2(M) \subseteq U(\Phi)'$ as $\tilde{A}(F)(f) = F(UAU^{-1}f)$. We can now show that A has distributional eigenvectors:

23.58 Lemma. *Let F be an element of Φ'. Then $\tilde{A}(F)(U(|\varphi\rangle)) = A(FU)(\varphi)$.*

Proof. By density, it suffices to check this for $F = |U\psi\rangle$, then

$$
\begin{aligned}
\tilde{A}(F)(U(|\varphi\rangle)) &= \langle UA|\varphi\rangle, U\psi\rangle \\
&= \langle A|\varphi\rangle, |\psi\rangle\rangle \\
&= \langle |\varphi\rangle, A|\psi\rangle\rangle \\
&= A(|\psi\rangle)(\varphi).
\end{aligned}
$$

This establishes the equality. $\qquad\square$

Now, by a previous theorem, for each $m \in M$, there exists a functional F_m on $U(\Phi)$ such that $F_m(\varphi) = \varphi(m)$. We claim that this F_m is a generalized eigenvector for $\tilde{A}$. Indeed,

$$\tilde{A}(F_m)(f) = F_m(Af) = S(m)f(m) = S(m)F_m(f).$$

Thus $F_m U$ is a generalized eigenvector for A.

23.7 Applications of rigged Hilbert spaces

Rigged Hilbert spaces and the spectral theorems have a lot of applications, we will mention a few of them

Resolution of the identity

Let A be a self-adjoint operator. In quantum mechanics we often deal with resolutions of the identity. This means that we write the identity operator as a formal integral

$$I = \int |\psi_x\rangle\langle\psi_x|\,\mathrm{d}x$$

where the $|\psi_x\rangle$ are the eigenvectors. This is a purely formal notation however, since a self-adjoint operator might not have any eigenvectors to begin with. But we can find a version of this "resolution of the identity" that works fine.

So let $\Phi \subseteq H \subseteq \Phi'$ be a rigged Hilbert space and let $A : \Phi \to \Phi$ be an essentially self-adjoint operator. By the spectral theorem, we can find a measure space (X, μ) and generalized eigenvalues $E_k(x)$ of A of eigenvalue $\lambda(x)$. We can furthermore find a direct integral

$$K = \int_X H(x)\,\mathrm{d}\mu(x) = \bigoplus_{k=1}^{+\infty} K_k$$

and a unitary map $F : H \to K$ such that $(F(\varphi))_k(x) = \langle \varphi, E_k(x) \rangle$ for $\varphi \in \Phi$. Furthermore, we can write

$$(FA\varphi)_k(x) = \lambda(x)(F\varphi)_k(x).$$

Now we can form the operator $FAF^{-1} : F(\Phi) \to F(\Phi)$. It is easy to verify that this operator has generalized eigenvectors $E_k(x)F^{-1}$ which acts on $\psi \in \Phi$ as $E_k(x)F^{-1}\psi = \psi_k(x)$. Of course, we have that in K

$$\begin{aligned} \langle \psi, \varphi \rangle_H &= \langle F\psi, F\varphi \rangle_K \\ &= \sum_{k=1}^{+\infty} \int_X (F\psi)_k(x)\overline{(F\varphi)_k(x)}\,\mathrm{d}\mu(x) \\ &= \sum_{k=1}^{+\infty} \int_X E_k(x)(\psi)S(E_k(x))(\varphi)\,\mathrm{d}\mu(x). \end{aligned}$$

We write the above formally as

$$I = \int_X \sum_{k=1}^{+\infty} E_k(x)S(E_k(x))\,\mathrm{d}\mu(x).$$

Or (by using the Gelfand-Pettis integral)

$$\varphi = \int_X \sum_{k=1}^{+\infty} E_k(x)S(E_k(x))(\varphi)\,\mathrm{d}\mu(x).$$

Applications to probability theory

Rigged Hilbert spaces can be used to simplify a lot of theorems and notations in probability theory. We give a few examples.

In this section, we will work with a the *real* Hilbert space $L^2(\mathbb{R})$. Furthermore, let $\mathcal{S}(\mathbb{R})$ be the Schwartz class. Then

$$\mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})'$$

is a rigged Hilbert space.

Let $\mathbb{P}$ be a probability measure on $\mathbb{R}$. Each such probability measure induces a functional

$$\mathcal{F}(\mathbb{P}) : \mathcal{S}(\mathbb{R}) \to \mathbb{R} : f \to \int f \, \mathrm{d}\mathbb{P}.$$

We check that this functional is well-defined and continuous on $\mathcal{S}(\mathbb{R})$ (with respect to the usual norms on the Schwartz class). Indeed,

$$|(\mathcal{F}(\mathbb{P})(f)| \leq \int |f|\mathbb{P} \leq \|f\|_{\infty}.$$

Which implies that $\mathcal{F}(\mathbb{P})$ is well-defined and continuous.

Furthermore, we can show that $\mathcal{F}$ is injective. Indeed, assume that $\mathcal{F}(\mathbb{P}) = 0)$, then for each $f \in \mathcal{S}(\mathbb{R})$ hols that $\int f \, \mathrm{d}\mathbb{P} = 0$. But since $\mathcal{S}(\mathbb{R})$ is dense in $L^1(\mathbb{R}, \mathbb{P})$ (see Lang [38]), this implies that $\int f \, \mathrm{d}\mathbb{P} = 0$ for each $f \in L^1(\mathbb{R}, \mathbb{P})$. Hence $\mathbb{P} = 0$.

Thus all probability measures on $\mathbb{R}$ can be regarded as functionals on $\mathcal{S}(\mathbb{R})$. Recall the Radon-Nikodym theorem:

23.59 Theorem (Radon-Nikodym). *Let $\mathcal{P}$ be a probability measure such that $\mathbb{P}(A) = 0$ whenever A is a Borel set for which $\lambda(A) = 0$ (where λ is the Lebesgue measure). Then there exists a integrable function f such that $\mathbb{P}(A) = \int_A f \, \mathrm{d}x$ for each Borel set A. In particular, for each integrable function g, we have $\int g \, \mathrm{d}\mathbb{P} = \int g(x)f(x) \, \mathrm{d}x$.*

But in the situation of rigged Hilbert spaces, for each continuous functional F on $\mathcal{S}(\mathbb{R})$, we use the (perhaps abusive) notation

$$F(f) = \langle F, f \rangle = \int f(x)F(x) \, \mathrm{d}x.$$

for each $f \in \mathcal{S}(\mathbb{R})$. In particular, for a probability measure $\mathbb{P}$, we have

$$\int f \, d\mathbb{P} = \int f(x)P(x) \, dx,$$

where $P = \mathcal{F}(\mathbb{P})$. This can be seen as a significant generalization of the Radon-Nikodym theorem which says that every probability measure has a density function which is perhaps a distribution. In particular, the density function of the Dirac measure for which $\mathbb{P}(A) = 1$ if $0 \in A$ and $\mathbb{P}(A) = 0$ otherwise, is given by the Dirac delta function.

Another famous result of probability theory is the following:

23.60 Theorem. *Let $\mathbb{P}$ is an absolutely continuous probability measure with respect to the Lebesgue measure, let f be its density function and let F be its cumulative distribution function, then $F'(x) = f(x)$ if f is continuous at x.*

Let us see whether we can generalize this to rigged Hilbert spaces. So fix an arbitrary measure $\mathbb{P}$ and let P be its associated functional. The cumulative density function of a probability distribution is defined as the function

$$F : \mathbb{R} \to \mathbb{R} : x \to \mathbb{P}((-\infty, x]).$$

This is of course not an element of the Schwartz-class $\mathcal{S}(\mathbb{R})$. But it does induce a distribution by

$$F : \mathcal{S}(\mathbb{R}) \to \mathbb{R} : f \to \int f(x)\mathbb{P}((-\infty, x]) \, dx.$$

The derivative of this distribution is given by G' for which

$$\begin{aligned} \langle G', f \rangle &= -\langle G, f' \rangle \\ &= -\int f'(x)\mathbb{P}((-\infty, x]) \, dx \\ &= -\int \int_0^y f'(x) \, d\mathbb{P}(y) \, dx. \end{aligned}$$

By Fubini's theorem, it follows that

$$
\begin{aligned}
\langle G', f \rangle &= -\int \int_y^{+\infty} f'(x)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(y) \\
&= -\int \left(\int_{-\infty}^{+\infty} f'(x)\,\mathrm{d}x - \int_{-\infty}^{y} f'(x)\,\mathrm{d}x \right) \mathrm{d}\mathbb{P}(y) \\
&= -\int \left(\lim_{a \to +\infty} \int_{-\infty}^{a} \int_{-\infty}^{y} f'(x)\,\mathrm{d}x - f(y) \right) \mathrm{d}\mathbb{P}(y) \\
&= -\int \left(\lim_{a \to +\infty} f(a) - f(y) \right) \mathrm{d}\mathbb{P}(y) \\
&= \int f\,\mathrm{d}\mathbb{P} \\
&= \langle P, f \rangle.
\end{aligned}
$$

So it holds in general that the derivative of the cumulative distribution function (as distribution) is the probability density function.

Chapter 29

Nederlandse samenvatting

In deze thesis onderzoeken we enkele speciale morfismen in topologie, meetkunde en analyse. In het bijzonder concentreren we ons op niet-commutatieve topologie, algebraische meetkunde, differentiaalmeetkunde en functionaalanalyse.

Enkele van fundamentele noties in topologie zijn de noties van Hausdorff and compactheid van topologische ruimtes, en hun relatieve tegenhanger van separated en proper continue functies. Al sinds de jaren zeventig zijn er pogingen gedaan om deze noties te veralgemenen naar meer algemene categorieën. Zie bijvoorbeeld het werk van Herrlich [26], Manes [44], Penon [48], [47]. Een rijke theorie werd ontwikkeld voor categorieën uitgerust met een factorizatiesysteem [1], zie bijvoorbeeld het werk van Herrlich, Salicrup en Strecker [27]. Dit had toepassingen in topologie, ordetheorie en groepentheorie. Vanaf het moment dat de notie van "sluitingsoperator" op een categorie met factorizatiesysteem beschikbaar was, werden er in de jaren negentig resultaten gevonden die heel dicht lagen bij de topologische context. Zie [7] voor toepassingen in topologie, Birkhoff sluitingsruimtes en uniforme ruimtes, topologische groepen en locales.

Tholen [56] merkte op dat zowel de klasse van separated morfismen als the klasse van proper morfismen uitgedrukt kunnen worden aan de hand van een klasse van "gesloten morfismen." Deze ideeën groeiden uit tot een theorie die "functionele topologie" genoemd werd. In Clementino, Giuli en Tholen [8], werd deze theorie ontwikkeld door een kasse van "gesloten morfismen" te nemen en deze te linken aan een factorizatiesysteem. Deze theorie kan worden toegepast in bijvoorbeeld approach ruimten, een veralgemening van topologische en metrische ruimten [9]. De algemenere theorie van lax algebras werd recent ondergebracht in deze setting door Hofmann en Tholen [30]. Zij vervangden de klasse van gesloten morfismen door een klasse van "proper afbeeldingen."

De theorie van functionele topologie introduceert een factorizatiesysteem $(\mathbb{E}, \mathbb{M})$ en een klasse van morfismen $\mathbb{F}$ op een categorie. De klasse $\mathbb{F}$ representeert de gesloten morfismen. We noemen $\mathbb{F}$ een $(\mathbb{E}, \mathbb{M})$-gesloten klasse (zie Definitie 13.1). Het idee is dat we zo objecten in een categorie kunnen bekijken als "ruimten" en dat we zo hun "topologische eigenschappen" kunnen bestuderen. We doen dit niet door te kijken naar de punten van ruimten, maar door de verbanden tussen objecten en morfismen te onderzoeken. Zowel de theorie van topologische ruimten als de theorie van locales passen in deze definities. Andere voorbeelden worden gevonden door $\mathbb{F}$ de klasse van open afbeeldingen tussen topologische ruimten te nemen, of door $\mathbb{F}$ de klasse van torsie-bewarende afbeeldingen tussen abelse groepen te nemen.

In de ideale situatie van topologische ruimten is de klasse $\mathbb{F}$ de afbeeldingen die een gesloten subobject naar een gesloten subobject sturen. Dus in plaats van de klasse van gesloten afbeeldingen $\mathbb{F}$ als primair te beschouwen, kunnen we de klasse van gesloten inbeddingen

als primair nemen. Een gesloten afbeelding kan dan gedefinieerd worden als een afbeelding die een element in $\mathbb{F}_0$ stuurt op een element in $\mathbb{F}_0$ (zie Eigenschap 13.4), of iets formeler: een morfisme f is gesloten als voor elke $m \in \mathbb{F}_0$ geldt dat de $(\mathbb{E}, \mathbb{M})$-factorizatie $fm = m'e$ voldoet aan $m' \in \mathbb{F}_0$. Dit is de ideale situatie, en de theorie heeft in deze situatie de beste eigenschappen. Maar niet elke $(\mathbb{E}, \mathbb{M})$-gesloten klasse ontstaat op deze manier.

Het is al lang bekend dat zekere klassen van morfismen tussen schema's in algebraische meetkunde aan vele eigenschappen voldoen die ook voorkomen in de theorie van functionele topologie. Inderdaad, proper en separated zijn fundamentele noties in algebraische meetkunde en worden heel analoog gedefinieerd als proper en separated in functionele topologie. Verder is het zo dat de meeste basiseigenschappen van proper en separated nog steeds gelden beide situaties. Het was echter niet duidelijk hoe de theorie van schema's juist een $(\mathbb{E}, \mathbb{M})$-gesloten klasse definieert. Het beantwoorden van deze vraag is een van de doelen van deze thesis.

Eerst en vooral moeten we een geschikt factorizatiesysteem op schema's vinden. Het enige bekende factorizatiesysteem $(\mathbb{E}, \mathbb{M})$ maakt gebruik van het zogenaamde schema-theoretisch beeld van een morfisme. Dit heeft ook als nadeel dat $\mathbb{M}$ de klasse van gesloten immersies is. Dus vanaf het begin laten we enkel *gesloten* immersies toe als inbeddingen. Dit is filosofisch ongewenst. We zouden liever een grotere klasse van inbeddingen toelaten, zoals een klasse die ook de open inbeddingen bevat.

Maar zelfs als we $\mathbb{M}$ zouden nemen als klasse van gesloten immersies, dan zijn de problemen nog niet opgelost. We kunnen immers niet zomaar $\mathbb{F}$ de klasse van gesloten afbeeldingen nemen. Deze keuze zou niet voldoen aan de definitie van een $(\mathbb{E}, \mathbb{M})$-gesloten klasse (nl. Definitie 13.1), zie §13.3. Men zou als $\mathbb{F}_0$ de klasse van gesloten

immersies kunnen nemen, en dan zou $\mathbb{F}$ gedefinieerd worden als die afbeeldingen die gesloten immersies naar gesloten immersies stuurt. Maar dan zouden *alle* morfismen in $\mathbb{F}$ zitten. Dit fenomeen ontstaat natuurlijk doordat we $\mathbb{M}$ gedefinieerd hebben als gesloten immersies en niet als algemenere inbeddingen.

Om dit probleem op te lossen, breiden we de theorie van functionele topologie uit in verschillende richtingen.

Eerst en vooral kunnen we ons afvragen of een factorizatiesysteem echt nodig is om de resultaten van functionele topologie te bekomen. Als we geen factorizatiesysteem hebben, dan zijn er natuurlijk geen goede noties van beelden en subobjecten, maar in schema's werkten deze noties toch niet goed. Dit leidt ons tot de definitie van een gesloten klasse $\mathbb{F}$ en een gesloten koppel $(\mathbb{F}, \mathbb{F}_0)$. Met de gesloten klasse $\mathbb{F}$ kunnen we praten over gesloten afbeeldingen, en met het gesloten koppel $(\mathbb{F}, \mathbb{F}_0)$, kunnen we bovendien ook praten over gesloten inbeddingen. Verrassend veel resultaten van functionele topologie zijn waar in deze setting. In het bijzonder hebben proper en separated nog steeds zin in deze context, en ze gedragen zich zoals verwacht. Zowel de klassieke functionele topologie van Definitie 13.1 als de theorie van schema's zijn voorbeelden van deze definitie.

Natuurlijk kunnen we ons ook afvragen of we de functionele topologie niet zouden kunnen uitbreiden met behoud van het factorizatiesysteem. Dit is inderdaad mogelijk. Maar in plaats van Definitie 13.1 uit te breiden, is het handiger om Eigenschap 13.4 uit te breiden. Als basisnoties nemen we dus de klassse $\mathbb{F}_0$ van gesloten inbeddingen en een andere klasse $\mathbb{P}$ die we "surjecties" zullen noemen. We kunnen dan definiëren dat $f \in \mathbb{F}$ als voor elk subobject $m \in F_0$ geldt dat de $(\mathbb{E}, \mathbb{M})$-factorizatie $fm = em'$ voldoet aan $m' \in \mathbb{F}_0$ en $e \in \mathbb{P}$. Dit generaliseert de klassieke functionele topologie: inderdaad, het volstaat om $\mathbb{P} = \mathbb{E}$ te nemen. Maar ook de theorie van schema's past

hier perfect in.

Natuurlijk blijft het spijtig dat we een factorizatiesysteem $(\mathbb{E}, \mathbb{M})$ op schema's plaatsen waarbij $\mathbb{M}$ bestaat uit gesloten immersies en niet meer algemene inbeddingen. Dit kan opgelost worden als we toelaten dat beelden van morfismen geen schema's zijn, maar algemenere objecten. In het bijzonder kunnen we de schema's inbedden in een preschoof categorie. Deze preschoof categorie heeft een canonisch factorizatiesysteem en dus een canonische notie van beelden. We verliezen echter wel de eigenschap dat het beeld van een morfisme een schema is. Maar we kunnen wel een klasse van gesloten inbeddingen $\mathbb{F}_0$ introduceren op de preschoof categorie. We verkrijgen zo noties van proper en separated op deze preschoof categorie, en deze noties restrictern zich tot de correcte noties van proper en separated op de deelcategorie van schema's.

Naast schema's hebben we ook enkele andere categorieën onderzocht. Zo hebben we gevonden dat de compact gegenereerde ruimten passen in Definitie 13.1. We hebben dan het mooie resultaat dat de Hausdorff objecten in deze categorie juist de zwakke Hausdorff ruimten zijn. Een andere categorie is deze van de Lie groepen. Deze passen ook in Definitie 13.1. De compacte objecten in deze categorie zijn juist de compacte Lie groepen. We zouden hetzelfde kunnen doen voor algemene manifolds, maar dit leidt tot problemen omdat de categorie van manifolds zich niet goed gedraagt. In het bijzonder is het beeld van een gladde afbeelding niet altijd een manifold. We kunnen de categorie van manifolds uitbreiden tot de categorie van diffeologische ruimten. Deze categorie gedraagt zich beter, maar we kunnen dan bewijzen dat de natuurlijke keuzes van factorizaties en gesloten inbeddingen geen aanleiding geven tot functionele topologie (zie §13.3). Het probleem is dat de natuurlijke topologie op diffeologische ruimten zich niet goed gedraagt.

Tenslotte hebben we onderzocht of de theorie van C^*-algebra's ook aanleiding geeft tot een functionele topologie. De theorie van C^*-algebra's is bekend als een soort niet-commutatieve topologie, en het was al lang bekend dat vele topologische noties ook gedefinieerd kunnen in C^*-algebra's. Zo komen compacte ruimten overeen met unitale C^*-algeba's en zo komen compactificaties overeen met unitizaties. Het is echter zo dat de natuurlijke C^*-morfismen overeen komen met de *proper* continue afbeeldingen en niet met de algemene continue afbeeldingen. Om alle continue afbeeldingen te kunnen beschouwen, gebruiken we een ander soort morfisme, namelijk de Woronowicz morfismen [21]. Het duaal van de categorie van C^*-algebra's en zijn Woronowicz morfismen is dan een soort niet-commutatieve topologie. Spijtig genoeg heeft deze categorie niet zo'n mooie eigenschappen. Zo is het product in deze categorie (dat overeenkomt met het coproduct van C^*-algebra's) niet zo transparant. We kunnen echter het coproduct van C^*-algebra's vervangen door het tensor product. We doen dit door een relatie R in te voeren op de morfismen die ons verteld welke twee morfismen commuteren (zie [33] voor een algemenere theorie van zulke relaties op monoidale structuren). In de categorie van "niet-commutatieve topologie" krijgen we dan het tensor product. In de standaard categorie van topologische ruimten (waar alle morfismen R-gerelateerd zijn), krijgen we het gewone product. We kunnen dan niet enkel R-producten invoeren, maar ook R-pullbacks. Dit staat ons toe om de verschillende theorieën van functionele topologie te veralgemenen. Dit werkt goed, en de C^*-algebra's passen mooi in dit verhaal. De compacte objecten zijn dan juist de unitale C^*-algebra's, zoals verwacht.

Het meeste van bovenstaande paragraaf werkt ook voor algemene associatieve algebra's over een lichaam. Zo is een notie van Woronowicz morfisme beschikbaar voor deze algebra's. Dus kunnen we alles van

C^*-algebra's nabootsen om zo resultaten over algemene associatieve algebra's te verkrijgen. Dit geeft ons een ander voorbeeld van functionele topologie. We verkrijgen zo opnieuw dat de unitale associatieve algebra's de compacte objecten zijn. Dit resultaat suggereert dat we alles zelfs nog verder zouden kunnen uitbreiden tot gesloten monoidale categorieën. Dit moet nog onderzocht worden.

Belangrijke voorbeelden in niet-commutatieve topologie bestaan uit deformaties. Zo kan de niet-commutatieve torus gezien worden als een deformatie van de C^*-algebra van continue functies over de torus [50]. Men kan proberen om zulke deformaties in te passen in een algemene deformatietheorie van C^*-algebra's, gelijkaardig aan de deformatietheorie van associatieve algebra's van Gerstenhaber [18]. Maar dan zijn er enkele moeilijkheden. Een van deze moeilijkheden is dat de C^*-algebra's kunnen worden opgevat als algebra's van *begrensde* operatoren op een Hilbertruimte, terwijl de natuurlijke operaties van deformatietheorie (zoals het Poisson haakje) aanleiding geen tot onbegrensde operatoren. Onbegrensde operatoren werken het best op de zogenaamde rigged Hilbertruimten [17]. We zouden dan deformaties van C^*-algebras kunnen construeren als operatoralgebra's over zekere rigged Hilbertruimten. Deze operatoren werden echter enkel bestudeerd in speciale gevallen. Een algemene spectraalstelling was niet beschikbaar. In het laatste hoofdstuk van deze thesis stellen we zo een stelling voor (Stelling 23.57). Verder werk zou moeten bepalen of rigged Hilbertruimten een goede setting zou kunnen zijn voor deformatietheorie.

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