Quench dynamics of an ultracold Fermi gas in the BCS regime: Spectral properties and confinement-induced breakdown of the Higgs mode

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I. INTRODUCTION

Ultracold Fermi gases have been the subject of many experimental and theoretical studies during recent years (see, e.g., [1–5]). They provide a unique system to study key concepts of condensed-matter theory. This is because in these systems many parameters such as the particle density, the Fermi energy, the confinement potential, or the interaction strength between the Fermions, which in a solid-state system are typically fixed quantities, can be externally controlled in a wide range [6]. In particular, magnetic-field Feshbach resonances provide the means for controlling the interaction strength between fermions by varying an external magnetic field. The tunability of the s-wave scattering length, which is the dominant interaction channel, makes ultracold Fermi gases ideal for exploring different regimes of interacting many-body systems in a single system. This includes the limiting regimes of weakly attracting fermions, which condense into Cooper pairs forming a Bardeen-Cooper-Schrieffer (BCS) phase below a certain temperature $T_c$, and repulsive dimers formed by two fermions, which can undergo a Bose-Einstein condensation (BEC). These two limiting regimes are separated by the strongly interacting BCS-BEC crossover regime where the scattering length diverges and the system exhibits unitary properties [7].

In addition to the variable interaction strength, ultracold atomic gases offer a unique opportunity to explore the influence of a confinement on the pairing correlations, because dimensionality and confinement can be precisely controlled by tuning external parameters [1,8–10]. Varying the confinement, which is often well approximated by a harmonic confinement potential, allows one to access new degrees of freedom. Restricting the Fermi gases to quasidimensional dimensionality, allows for the possibility of an experimental evidence of unconventional phases, like the Fulde-Ferrell-Larkin-Ovchinnikov state (11–16). Moreover it may help to study and get experimental insight into shape resonances, theoretically predicted for quasi-low-dimensional conventional superconductors [17]. In [9] the first quantitative measurements of the transition from two-dimensional (2D) to quasi-2D and three-dimensional behavior in a weakly interacting Fermi gas have been reported.

At low atom numbers, the shell structure associated with the filling of individual transverse oscillator states has been observed. On the theoretical side the ground-state properties of a $^6$Li gas confined in a cigar-shaped laser trap have been investigated predicting size-dependent resonances of the superfluid gap [10], similar to the case of superconducting nanowires [17], yielding an atypical BCS-BEC crossover.

An important effort is now devoted to the exploration of the out-of-equilibrium behavior of trapped ultracold atomic Fermi gases and, in particular, to the determination of their dynamical properties. The dynamics has been studied in the normal as well as in the condensed phase, observing second sound [5] and soliton trains [3], and showing a low-frequency oscillation of the cloud after a change of the system confinement or optical excitation [18–22]. Furthermore, state-of-the-art technology allows one to change the coupling constant on such short time scales that it is possible to explore the regime where the many-body system is governed by a unitary evolution with nonequilibrium initial conditions. In ultracold atomic Fermi gases the dynamics may be initiated by readjusting the pairing interaction through switching an external magnetic field in the region of a Feshbach resonance (i.e., a quantum quench) or by a rapid change of the confinement potential of the trap [19]. Due to the small energies in the trapping potential the dynamics in the Fermi gases take place on a millisecond time scale. Therefore, in contrast to metallic superconductors, where subpicosecond excitations are required to achieve nonadiabatic dynamics [23], in atomic gases the nonadiabatic regime can be reached already by excitations in the submillisecond range.

Spontaneous symmetry breaking gives rise to collective modes of the order parameter which are classified into the Higgs amplitude mode, and the Goldstone mode, the latter corresponding to a phase oscillation of the gap [24–26]. The Higgs mode has been the subject of intensive theoretical and experimental [27–29] research efforts in the past. On the theoretical side the nonadiabatic temporal response of the order parameter to (quasi-) instantaneous perturbations has been studied. Different regimes of an oscillatory temporal
behavior of the pairing potential were theoretically predicted in homogeneous fermionic condensates \cite{30–33}. It was shown that the amplitude of the order parameter oscillates without damping when the coupling constant is increased above a certain critical value \cite{31}. On the other hand, the gap vanishes when the coupling constant is decreased below another critical value. In between these two limiting regimes the amplitude exhibits damped dephased oscillations and the system goes to a stationary steady state with a finite gap \cite{31}. In extended systems the approach to a stationary state occurs in an oscillatory way with an inverse square-root decay in time of the amplitude of the oscillations \cite{32}. This inverse square-root decay after a quench was also found for an inhomogeneous Fermi gas confined in a box with periodic boundary conditions in two dimensions and a harmonic confinement in the third dimension \cite{34}. Using the time-dependent Bogoliubov-de Gennes equation a reduced mean value of the order parameter oscillation compared to the obtained equilibrium value of the order parameter has been found. Furthermore, in a two-dimensional pancake shaped Fermi gas a long-lived Higgs mode has been predicted assuming particle-hole symmetry around the Fermi surface \cite{35}. For conventional bulk superconductors an evolution similar to the homogeneous case was predicted \cite{23,36} where the nonadiabatic regime is reached by excitation with short, intense terahertz pulses. An experimental realization was reported in \cite{37}. In contrast, in finite length superconducting nanowires a breakdown of the damped oscillation and a subsequently rather irregular dynamics have been predicted \cite{38}.

In this paper we present a theoretical analysis of the short-time BCS dynamics of a $^6$Li gas confined in a cigar-shaped laser trap. The excitation is modeled by a sudden change of the interaction strength which can be achieved through a Feshbach resonance by an abrupt change of the external magnetic field \cite{6}. Applying the well-known BCS theory in mean field approximation to ultracold Fermi gases we show that the change of the coupling strength induces a collective oscillation of the Bogoliubov quasiparticles close to the Fermi level. This results in a damped amplitude oscillation of the BCS gap, which corresponds to the Higgs mode \cite{39}. Like in the case of confined BCS superconductors this oscillation breaks down after a certain time revealing rather chaotic dynamics afterwards. We explain these dynamics in terms of coupled harmonic oscillators which can be derived by linearizing the quasiparticle dynamics obtained from the Heisenberg equation of motion.

In doing so we first derive the quasiparticle equations of motion from the inhomogeneous Bogoliubov-de Gennes Hamiltonian (Sec. II) which, because of using the standard contact-type interaction, requires a proper regularization of the gap equation. Starting from the ground state calculated according to \cite{10} we then calculate the dynamics of the superfluid gap after an instantaneous change of the coupling constant. The numerical results as well as their explanation follow in Sec. III, where we first discuss a rather small system and then proceed to a larger, experimentally more easily accessible system. Finally, in Sec. IV we summarize our results and give some concluding remarks.

II. THEORETICAL APPROACH

Our approach aims at describing the dynamics of the superfluid order parameter $\Delta(r,t)$ of an ultracold $^6$Li Fermi gas, confined in a cigar-shaped, axial symmetric harmonic trapping potential:

$$V_{\text{conf}}(x,y,z) = \frac{1}{2} m \omega_x^2 x^2 + \frac{1}{2} m \omega_y^2 y^2 + \frac{1}{2} m \omega_z^2 z^2.$$  \hspace{1cm} (1)

Here, $m$ is the mass of the $^6$Li atoms and $\omega_{\perp}$ ($\omega_{\parallel}$) is the confinement frequency in the $x$-$y$ plane ($z$ direction), respectively. Choosing $\omega_{\perp} \gg \omega_{\parallel}$ yields an elongated cigar-shaped trap where the oscillator length $l_g = \sqrt{\hbar/(m \omega_{\perp})}$ provides a measure of the system length. The eigenvalues

$$\epsilon_m = \hbar \omega_{\perp} (m_x + m_y + 1) + \hbar \omega_{\parallel} (m_z + \frac{1}{2}) - E_F$$  \hspace{1cm} (2)

are measured with respect to the Fermi energy $E_F$. The index $m$ refers to the combination of quantum numbers $m_x, m_y, m_z$. For this geometry the one-particle states form one-dimensional sub-bands, labeled by $(m_x, m_y)$ [see Fig. 1(a)], while the states within each sub-band are labeled by $m_z$. Each sub-band has a constant one-particle density of states and thus the overall density of states exhibits finite jumps whenever a new sub-band appears.

We consider the gas to be composed of two spin states, $\uparrow$ and $\downarrow$, and start from the inhomogeneous BCS Hamiltonian at $T = 0$ K. Within the Anderson approximation we then derive equations of motion for the corresponding Bogoliubov quasiparticle excitations.

A. Hamiltonian

The usual inhomogeneous BCS Hamiltonian for an effective BCS-type contact interaction reads \cite{40}

$$H_{\text{BCS}} = \int \left[ \Psi_1^\dagger(r) H \Psi_1(r) + \Psi_1^\dagger(r) H \Psi_1(r) \right] d^3r$$

$$- g \int \Psi_1^\dagger(r) \Psi_1^\dagger(r) \Psi_1(r) \Psi_1(r) d^3r,$$  \hspace{1cm} (3)

where $\Psi_1(r)$ and $\Psi_1^\dagger(r)$ are the field operators for up and down spin, respectively; $H = p^2/2m + V_{\text{conf}} - E_F$ is the one-particle Hamiltonian; and $g$ is the coupling constant of the contact interaction $V(r) = -g \delta(r)$. In the limit of low temperatures the main contribution to the interaction between two fermionic atoms in different internal spin states is given by scattering processes at low momentum. The description of those can be replaced by the widely known pseudopotential only depending on the scattering length $a$ \cite{1], which yields $g = -\frac{4\pi\hbar^2}{m} a$.

A BCS-like mean field expansion in terms of anomalous expectation values and a particle-hole transformation, leaving spin-up operators unchanged, $\Psi_1^\dagger = \Phi_1^\dagger$, while interchanging spin-down ones, $\Psi_1^\dagger = \Phi_1$, leads to the Bogoliubov-de Gennes (BdG) Hamiltonian \cite{41}:

$$H_{\text{BdG}} = \int \Phi_1^\dagger(r) H \Phi_1(r) d^3r - \int \Phi_1^\dagger(r) H \Phi_1(r) d^3r$$

$$+ \int \left( \Delta(r) \Phi_1^\dagger(r) \Phi_1(r) + \Delta^*(r) \Phi_1^\dagger(r) \Phi_1(r) \right) d^3r,$$  \hspace{1cm} (4)
where

\[ \Delta(r) = -g \langle \Psi_{\uparrow}(r) | \Psi_{\downarrow}(r) \rangle = -g \langle \Phi_{\uparrow}(r) | \Phi_{\downarrow}(r) \rangle \]

\[(5)\]

is the BCS order parameter. From Eq. (4) it becomes apparent that the corresponding eigenvalue equation can be written as the one-particle Bogoliubov-de Gennes equation:

\[ \begin{pmatrix} H & \Delta(r) \\ \Delta^*(r) & -H^* \end{pmatrix} \begin{pmatrix} u_M(r) \\ v_M(r) \end{pmatrix} = E_M \begin{pmatrix} u_M(r) \\ v_M(r) \end{pmatrix}. \]

\[(6)\]

\(H_{\text{BdG}}\) can thus be diagonalized by Bogoliubov’s transformation, using the eigenfunctions \(u_M(r)\) and \(v_M(r)\). This introduces noninteracting quasiparticles with energy \(E_M\) obeying fermionic commutation relations with the corresponding creation operator:

\[ \gamma_M^\dagger = \int [u_M(r) \phi_{\uparrow}(r) + v_M(r) \phi_{\downarrow}(r)] d^3r. \]

\[(7)\]

The spectrum of the BdG equation is symmetric with respect to the Fermi energy and thus the eigenstates of the BdG equation occur in pairs. Labeling \(M \rightarrow m, \alpha\) for states \(E_M > 0\) and \(M \rightarrow m, \beta\) for \(E_M < 0\), respectively, one finds the relations \(u_{m, \beta} = -v_{m, \alpha}^*\) and \(v_{m, \beta} = u_{m, \alpha}^*\) for the eigenstates. Therefore, all quantities can be expressed solely using the \(\alpha\) state wave functions [41]. In the following we omit the \(\alpha, \beta\) index of the eigenfunctions while they are still necessary for the quasiparticle operators. Hereafter—in the case of the eigenfunctions—the index \(m\) refers to the \(\alpha\) states. For our further calculations it is convenient to transform to the excitation picture \((\alpha \rightarrow a, \beta \rightarrow b)\) with \(\gamma_{ma} = y_{ma}^\dagger\) and \(\gamma_{m\beta} = y_{m\beta}^\dagger\), where all quasiparticle excitations vanish in the ground state. We can rewrite the order parameter in the basis given by the eigenfunctions, where it reads

\[ \Delta(r, t) = -g \sum_{m,n} u_m^*(r) u_n(r) (\gamma_{ma}^\dagger \gamma_{na} + u_m(r) u_n(r) (\gamma_{mb}^\dagger \gamma_{na}) \]

\[ - u_m^*(r) v_n^*(r) (\gamma_{ma}^\dagger \gamma_{nb}^\dagger) + u_m(r) v_n(r) (\gamma_{mb}^\dagger \gamma_{na}^\dagger) - \delta_{mn}. \]

\[\text{(8)}\]

This yields the well-known result for the ground-state order parameter:

\[ \Delta_{GS}(r) = g \sum_m u_m(r) v_m^*(r), \]

\[\text{(9)}\]

which has to be solved self-consistently with Eq. (6) [40]. Focusing on the underlying physics, we exploit the Anderson approximation (AA) [42], choosing the BdG wave functions \(u_m(r)\) and \(v_m(r)\) proportional to the one-particle wave functions of the confinement potential \(\phi_m(r)\), i.e.,

\[ u_m(r) = u_m \phi_m(r) \quad \text{and} \quad v_m(r) = v_m \phi_m(r). \]

\[\text{(10)}\]

Here the amplitudes of the BdG wave functions \(u_m, v_m\) are obtained from the BdG Eq. (6) and read

\[ u_m = \sqrt{\frac{1}{2} \left( 1 + \frac{\varepsilon_m}{E_m} \right)}, \quad v_m = \sqrt{\frac{1}{2} \left( 1 - \frac{\varepsilon_m}{E_m} \right)} \]

\[\text{(11)}\]

with the quasiparticle energies given by

\[ E_m = \sqrt{\varepsilon_m^2 + \Delta_m^2} \]

\[\text{(12)}\]

and the one-particle energies \(\varepsilon_m\) given by Eq. (2). Applying the Anderson approximation to Eq. (6) additionally yields \(\Delta_m = \langle m | \Delta(r) | n \rangle = \Delta_m \delta_{m,n}\), where \(m\) are the one-particle eigenfunctions. The Anderson approximation has been tested in several nanostructured geometries and no qualitative deviations have been found [43]. It is applied to all our calculations.

From Eqs. (6) and (9) we obtain the well-known BCS-like self-consistency equation, also referred to as the gap equation. The ground-state order parameter in the state \(| m \rangle\) is given by

\[ \Delta_m = - \sum_{m'} V_{mm'} \frac{\Delta_{m'}}{2E_{m'}} \]

\[\text{(13)}\]

Here \(V_{mm'}\) is the interaction matrix element:

\[ V_{mm'} = -g \int |\phi_m(r)|^2 |\phi_{m'}(r)|^2 d^3r, \]

\[\text{(14)}\]

which exhibits maxima for states at the sub-band minimum (i.e., states with low \(m_z\)).
The contact interaction used here leads to a well-known ultraviolet divergence in the summation over all states, i.e., in Eq. (13), which can be regularized by applying a scattering length regularization [1]. This has been established for the homogeneous gap equation and subsequently extended to the inhomogeneous gap equation (13), where [44] gives a careful derivation for confined systems. However, [10,45] state that a much simpler regularization is sufficient since the results are not sensitive to the details of the method used. The corresponding regularized gap equation reads

$$\Delta_m = -\frac{1}{2} \sum_{m'} V_{mm'} \Delta_m \left( \frac{1}{E_m} - \frac{1}{E_m' + E_F} \right),$$

which can be rewritten as a multiplication by a factor

$$\chi_{m'} = \left( 1 - \frac{E_m}{E_m' + E_F} \right).$$

Thus, each quasiparticle state $m'$ is weighted by a factor $\chi_{m'}$. In order to extend this in a consistent manner to the nonequilibrium case, in which $\Delta$ deviates from its ground-state value $\Delta_{GS}$ and is determined by the full Eq. (8), the same procedure has to be applied to the nonequilibrium version of Eq. (15) as will be discussed below [see Eq. (27)] [46].

Figure 1(b) shows the dependence of the quasiparticle energies $E_m$ on the one-particle energies $\epsilon_m$. For all sub-bands crossing the Fermi energy, finite minima of the quasiparticle density of states [see Fig. 1(c)]. Sub-bands showing this feature will be called resonant in the following and systems dependent according to $\Delta_m - \Delta_{GS} \approx h/\Delta_{GS} \sim 1$ ms [30]. The excitations considered in this paper require a shift in the magnetic field of a few gauss, which indeed can be assumed to be instantaneous on the typical ms time scale in ultracold Fermi gases. In this case during the switching of the magnetic field the state of the system remains unchanged.

As usual, the dynamics of a quantum-mechanical system can be described in different basis systems, which from a mathematical point of view are all equivalent. In our case, to calculate the dynamics of the order parameter after a quench from the initial system $[u_m(r), \tilde{v}_m(r), g]$ to the final system $[u_m(r), v_m(r), g]$ we choose a time-independent basis rather than remaining in the diagonal basis. For convenience we take the basis corresponding to the eigenstates of the system after the switching, i.e., to the coupling constant $g$. All our calculations are thus carried out in the basis $u_m(r), v_m(r)$.

The initial state, which is characterized by the ground-state order parameter $\Delta_{GS}(r)$, corresponding to the coupling constant $\tilde{g}$ and the basis functions $u_m(r), \tilde{v}_m(r)$, has to be expressed in terms of the basis $u_m(r), v_m(r)$ which in particular gives rise to nonvanishing quasiparticle excitations in this basis. Since the confinement potential is unchanged, also the corresponding one-particle wave functions $\psi_n(r)$ remain unchanged. According to Eq. (10), in the present case only the BdG amplitudes $\tilde{u}_m$ and $\tilde{v}_m$ change while the spatial shapes of $u_m(r)$ and $v_m(r)$ remain unchanged. Therefore, all orthogonality relations are preserved and only diagonal quasiparticles are populated. For the initial values of the normal and anomalous expectation values, respectively, one finds

$$\langle \gamma^+_{ma} \gamma_{ma} \rangle_{t=0} = (\tilde{v}_m \tilde{u}_m - u_m \tilde{v}_m)^2,$$

$$\langle \gamma^+_{ma} \gamma^+_{mb} \rangle_{t=0} = (\tilde{v}_m \tilde{u}_m - u_m \tilde{v}_m)(\tilde{v}_m \tilde{u}_m + u_m \tilde{v}_m).$$

However, the field operators and the particle density of the cloud remain unchanged during the quench. According to Eq. (5) this yields for the gap in the excited system at $t = 0$

$$\Delta(r)_{t=0} = \frac{\tilde{g}}{g} \Delta_{GS}(r).$$

In addition

$$\langle \gamma^+_{ma} \gamma_{ma} \rangle = \langle \gamma^+_{mb} \gamma_{mb} \rangle, \langle \gamma^+_{mb} \gamma_{ma} \rangle = \langle \gamma^+_{ma} \gamma^+_{mb} \rangle^*$$

holds for all times $t \geq 0$.

Since the instantaneous order parameter $\Delta(t)$ deviates from the ground-state value of the final system $\Delta_{GS}$, the Hamiltonian in the basis $u_m(r), v_m(r)$ becomes nondiagonal depending on the difference $[\Delta(t) - \Delta_{GS}]$. It thus becomes explicitly time dependent according to

$$H_{BdG} = \sum_m E_{ma} \gamma^+_{ma} \gamma_{ma} - E_{mb}(1 - \gamma^+_{mb} \gamma_{mb})$$

$$+ \sum_{m,n} \left[ (\Delta - \Delta_{GS}) u^*_n \psi_m + (\Delta^* - \Delta_{GS}^*) \psi^*_m u_n \right] \gamma^+_{ma} \gamma_{na}$$

$$+ \sum_{m,n} \left[ (\Delta - \Delta_{GS}) \psi^*_n u_m - (\Delta^* - \Delta_{GS}^*) \psi^*_m u_n \right] \gamma^+_{ma} \gamma_{na}.$$
oscillators via the factor (of Eqs. (24) and (25) contain nonlinear couplings to all other

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where

\[ (\Delta - \Delta_{\text{GS}})_{mn} = \int u_m^* (r) \big[ \Delta (r,t) - \Delta_{\text{GS}}(r) \big] v_n (r) d^3 r \]

with

\[ (\Delta - \Delta_{\text{GS}})_{mn} = \int u_m^* (r) \big[ \Delta (r,t) - \Delta_{\text{GS}}(r) \big] v_n (r) d^3 r \]

Here, the Anderson approximation has been applied to the dynamical equations since the requirements hold for all time during dynamical calculations as proposed in [38]. This yields \((\Delta - \Delta_{\text{GS}})_{mn} \approx (\Delta - \Delta_{\text{GS}})_{m} \delta_{mn}\). According to Eq. (8) the time evolution of the system is thus described by the time-dependent quasiparticle expectation values. The corresponding equations of motion can be obtained via Heisenberg’s equation of motion.

For the considered instantaneous change of the coupling constant only diagonal expectation values are excited. The required equations of motion read

\[ i \hbar \frac{d}{dt} \langle \gamma_m \gamma_m \rangle = a_m \langle \gamma_m \gamma_m \rangle - a_m^* \langle \gamma_m \gamma_m \rangle, \]

\[ i \hbar \frac{d}{dt} \langle \gamma_m^\dagger \gamma_m \rangle = -2 E_m^{\text{(ren)}} \langle \gamma_m^\dagger \gamma_m \rangle + a_m (1 - 2 \langle \gamma_m \gamma_m \rangle), \]

where

\[ E_m^{\text{(ren)}} = E_m + 2 u_m v_m \Re \left[ (\Delta - \Delta_{\text{GS}})_m \right], \]

\[ a_m = v_m^2 (\Delta - \Delta_{\text{GS}})_m - u_m^2 (\Delta - \Delta_{\text{GS}})_m^*, \]

and

\[ (\Delta - \Delta_{\text{GS}})_m = - \sum_k \left[ 2 u_k u_k \langle \gamma_k \gamma_k \rangle + u_k^2 \langle \gamma_k \gamma_k \rangle \right] V_{mk} \chi_k. \]

In Eq. (27) again the regularization factor \(\chi_k\) has been introduced. Equations (23)–(27) represent a finite set of coupled ordinary differential equations that we solve numerically.

It is interesting to note that the evolution of the anomalous expectation values [Eq. (24)] corresponds to a set of harmonic oscillators with energies approximately given by 2\(E_m\) [first term in Eq. (24)] while Eq. (23) as well as the second terms of Eqs. (24) and (25) contain nonlinear couplings to all other oscillators via the factor \((\Delta - \Delta_{\text{GS}})_m\), which vanishes when the order parameter agrees with its ground-state value. We will come back to this separation into linear and nonlinear terms in Sec. III B.

III. RESULTS

In the following the temporal evolution of the amplitude of the spatially averaged gap

\[ \Delta = \frac{1}{V} \int d^3 r \Delta (r) \]

will be shown and analyzed for different system parameters. Here, the normalization volume \(V\) is set to \(V = l_x l_y l_z\) with \(l_a\) being the oscillator length in the \(a\) direction [49].

In order to concentrate on the physics we will start our analysis by investigating a very small system: All the main features occurring in the dynamics of larger, experimentally accessible confinements arise in small systems, too, but with a strongly reduced degree of numerical complexity. Thus, the dynamics of the superfluid gap will at first be explained on the basis of small systems. The results for a larger system will be shown afterwards.

A. Full model, small system

An exemplary result for the gap dynamics after a quantum quench, obtained by changing the scattering length from \(a = -140\) nm to \(a = -135\) nm for a system with the confinement frequencies \(f_\parallel = \omega_\parallel / 2\pi = 11.2\) kHz and \(f_\perp = \omega_\perp / 2\pi = 240\) Hz, is shown in Fig. 2. The Fermi energy has been set to \(E_F = 100\) h\(w_0\) yielding \(1/(k_F a) \approx -1\) according to [10]. As can be seen the amplitude of the gap shows an initial drop corresponding to the decreased coupling and thus decreased ground-state gap. Afterwards a smoothly damped oscillation of the gap occurs, which after a certain transition time \(t_c\) turns into an irregular, rather chaotic oscillation. Here \(t_c\) is defined as the time of the first deviation [50] from a smooth oscillation.

The inset of Fig. 2 suggests that this irregular oscillation after \(t_c\) is the result of a superposition of several frequencies.

FIG. 2. (Color online) Dynamics of the spatially averaged gap after a sudden change of the scattering length from \(-140\) to \(-135\) nm (normalized to the spatially averaged ground-state gap of the system before the quench \(\Delta_{\text{GS}}\)). Inset: Fourier transform of the gap dynamics. The confinement frequencies are \(f_\parallel = 11.2\) kHz and \(f_\perp = 240\) Hz.
Here a segment of the Fourier spectrum of the gap dynamics is shown. The spectrum is composed of a series of sharp peaks in the range of 8.9 peV to about 30 peV, which can each be assigned to a corresponding quasiparticle state, i.e., $\hbar \omega_m \approx 2E_m$.

The main peaks at the lower end of this series correspond to the frequency of the initial damped oscillation and to the dominant frequencies of the irregular dynamics afterwards. Their values are given by the quasiparticle energies closest to the Fermi level. These lie in the vicinity of a quasiparticle sub-band minimum. The corresponding frequencies are thus given by $\hbar \omega_m \approx 2\Delta_{[m_x,m_y,m_z]}$, where $m_{\text{min}}$ is the $z$ quantum number referring to the state with minimal quasiparticle energy, i.e., the state at the Fermi energy. The other peaks belong to higher quasiparticle states and decrease continuously with increasing energy.

While the qualitative picture of the gap dynamics is the same for all investigated systems, the quantitative values of the features mentioned above crucially depend on the system parameters. On the one hand the ground-state gap and thus the mean value of the oscillation and its frequency contributions strongly depend on the perpendicular confinement $f_\perp$ (due to the size-dependent superfluid resonances [10]) and on the scattering length $a$. The transition time, on the other hand, increases with decreasing parallel confinement $f_\parallel$—i.e., with increasing system length—as can be seen in Fig. 3.

Here the gap dynamics is shown for the same perpendicular confinement and excitation as in Fig. 2 but for increasing system length, i.e., decreasing $f_\parallel$ (from bottom to top). Figure 3 shows that the transition time $t_c$ moves to larger times as the length of the system increases. In addition a revival of the oscillation can be seen for the two largest systems with $f_\parallel = 96$ and 80 Hz, which for smaller systems would occur after the breakdown.

A quantitative analysis of the transition time for different perpendicular confinements $f_\perp$ over a wide range of parallel confinements is shown in Fig. 4. Here the transition time is plotted against the inverse parallel confinement frequency $1/f_\parallel$ compared to the prediction from the linearized theory (dashed curve).

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A quantitative analysis of the transition time for different perpendicular confinements $f_\perp$ over a wide range of parallel confinements is shown in Fig. 4. Here the transition time is plotted against the inverse parallel confinement frequency $1/f_\parallel$. One can see that $t_c$ is independent of the size of the gas in the $x$-$y$ plane, since the values for every $f_\perp$ lie on the same curve. It is only influenced by the confinement in the $z$ direction, where a linear dependence on $1/f_\parallel$ can be observed. This is in full agreement with the behavior found for superconducting quantum wires [38]. Even a beatinglike pattern was found for thin quantum wires which corresponds to the revivals seen in Fig. 3.

To investigate the smooth regime of the gap dynamics Fig. 5 shows calculations for a rather large system length and two different perpendicular confinements. For this case of large lengths in [38] it was found that thick quantum wires exhibit a damping of the gap oscillation given by a power law $\sim t^{-\alpha}$ with $\alpha = 3/4$ for resonant systems and $\alpha = 1/2$ for off-resonant ones. However, thin quantum wires were found to differ from this power law, showing a more irregular oscillation but still a rather fast decay of the gap oscillation when resonant sub-bands are present.

Figure 5 shows that this situation applies to ultracold Fermi gases as well. Here a calculation of the gap dynamics is shown for a resonant system (upper, red curve), which is again characterized by the same perpendicular confinement as in Figs. 2 and 3, as well as for a system far away from resonance (lower, blue curve; $f_\perp = 12.9$ kHz). The excitation is the same as before and the parallel confinement frequency is chosen as $f_\parallel = 48$ Hz, which corresponds to a rather long cloud. It can be seen that both systems show an initial decay.
of the gap oscillation until a minimal amplitude is reached. In the resonant case this initial decay is rather strong and fast. Here, the amplitude of the oscillation falls close to zero. Afterwards it exhibits revivals until the smooth oscillation breaks down. In contrast, the off-resonant system shows an only moderate, comparatively slow decay of the oscillation, which after a short time exhibits a nearly constant amplitude. (The larger amplitude compared to the resonant system is due to the weak excitation investigated in this paper, which leads to \(|\langle \gamma_{mb}^0 \rangle \| \| \langle \gamma_{ma}^0 \rangle \| \approx 1\). In doing so, Eq. (23) can be neglected since by inserting Eqs. (26) and (27) into this equation only terms of at least second order in the normal excitations or products of anomalous and normal excitations contribute, which are to be neglected in the linearized case.

Equation (24) reduces to a closed set of equations for the anomalous excitations which, by performing the derivative in time of one equation and inserting the other, can be separated into its real and imaginary parts leading to

\[
\frac{d^2}{dt^2} \langle \gamma_{ma}^+ \gamma_{mb}^+ \rangle + \omega_m^2 \langle \gamma_{ma} \gamma_{mb} \rangle = \sum_{k \neq m} [A_{km} \text{Re}(\langle \gamma_{ka}^+ \gamma_{kb}^+ \rangle) + i A_{mk} \text{Im}(\langle \gamma_{ka}^+ \gamma_{kb}^+ \rangle)] \chi_k, \tag{29}
\]

where again all terms nonlinear in the quasiparticle expectation values have been neglected. This equation describes a set of linearly coupled harmonic oscillators with the uncoupled frequencies

\[
\omega_m = \sqrt{\left( \frac{2E_m}{\hbar} \right)^2 - A_{mm} \chi_m} \tag{30}
\]

and

\[
A_{km} = \frac{2}{\hbar^2} V_{km} \left( \frac{E_k + \epsilon_m \epsilon_k}{E_k} - \frac{1}{\hbar^2} \sum_l \frac{\epsilon_k \epsilon_l V_{ml} V_{lk}}{E_k E_l} \right). \tag{31}
\]

The coupling strength of the oscillators \(A_{km}\) with \(k \neq m\) therein is weak due to the— in this case—mostly small matrix elements \(V_{km}\). The shift of the eigenfrequencies of the coupled system [Eq. (29)] with respect to the uncoupled frequencies [Eq. (30)] should therefore be small, as should be the shift of the uncoupled frequencies compared to the bare ones \(2E_m/\hbar\).

Figure 6 shows the dynamics of the BCS gap obtained from Eq. (29) compared to the full dynamics [51]. The parameters correspond to the system shown in Fig. 2 and are exemplary for all investigated systems. Apart from a small offset—in addition to the artificial one of +0.02 in Fig. 6—the linearized equations clearly reproduce the full dynamics and all its features in very good agreement. The inset shows that the positions as

![Figure 5](image-url)
oscillator frequencies corresponding frequencies are determined by the uncoupled of a linear superposition of simple (co)sine oscillators. The analytical solution can be expressed in terms of a spectrum with rather dense, strong frequency contributions near $\hbar \omega = 2\Delta (m_x, m_y, m_z)$ and relatively widespread suppressed frequency contributions at higher energies. The explanations given above have shown that the gap dynamics can be understood as a linear superposition of quasiparticle oscillations which themselves are given by a sum of simple oscillators. To finally explain the main features of the gap dynamics in the time domain—the damping, the transition, and the irregular oscillation—one can therefore use an even simpler picture of a set of independent cosine oscillators with the frequencies $\omega_m$. Due to the abrupt excitation all these oscillators start in phase at maximum deflection. Starting to oscillate they will soon dephase. A sum of all oscillators (i.e., the gap) thus performs a damped oscillation (see [32]). Here systems with several strong frequency contributions—i.e., resonant systems—show a rather fast and persistent damping since a large part of the cumulative amplitude is able to dephase. Off-resonant systems in contrast exhibit only a slight decay of the oscillation since the main part of the cumulative oscillation is carried by one single mode.

Proceeding in time the damped oscillation continues until all oscillators are completely dephased and the amplitude of the oscillation is minimal. Then the oscillators start to rephase, the amplitude grows, and the oscillation reappears (see Fig. 3; for off-resonant systems this effect is strongly suppressed since one single frequency dominates the spectrum). As soon as the first adjacent pair of oscillators goes back in phase again this beatinglike pattern is interrupted. At this moment a spike in the cumulative oscillation indicates the breakdown of the regular oscillation and thus the transition time $t_c$. Afterwards all other frequencies rephase successively and create a rapid sequence of spikes which leaves a picture of an irregular oscillation. The preceding argumentation suggests that the time of this breakdown should be inversely proportional to the maximum spacing of adjacent quasiparticle energies. Strictly speaking the transition times are found to be determined by adjacent quasiparticles from the same sub-band [38], i.e.,

$$t_c \approx \frac{2\pi \hbar}{2\delta E_{max}} \approx \frac{\pi \hbar}{(\varepsilon_{m_x,m_y,m_z+1} - \varepsilon_{m_x,m_y,m_z})} = \frac{1}{2f_{||}}.$$  \hspace{1cm} (32)

It thus increases $\sim 1/f_{||}$ with decreasing parallel confinement frequency since the energy spacing of the atomic spectrum then decreases. This is in full agreement with the relation found before. In fact, Eq. (32) gives exactly the linear curve shown in Fig. 4. This indicates that the breakdown of the smooth...
initial oscillation of the BCS gap indeed is due to adjacent frequencies rephasing in time. However, an experimental observation of the transition time may be obscured by thermalization of the system which can be a competing source of damping by creating excitations which are not covered in the presented formalism. Indeed the spacing between adjacent quasiparticle energies defines a temperature of the order of $T \sim 10 \mu K$. This gives a rough estimate for a temperature scale above which thermalization might interfere with the observed rephasing. Nevertheless, the relevant time scale of thermalization in a nonequilibrium system can be much larger than the one obtained from this estimate depending on the underlying mechanism. In the case of s-wave scattering thermalization times of the order of 100 ms have been found at $T = 5 \mu K$. Hence, we do not expect thermalization to obscure an observation of the transition time $t_c$. In addition a thermal excitation during the quench should not affect the qualitative dynamics for the same reason since the induced broadening of single-particle states is connected with a finite lifetime which is determined by relaxation processes present in the system.

### C. Full model, large system

As a final example the gap dynamics of a rather large system with $f_l = 50 \text{ Hz}$ and $f_\perp \approx 1.03 \text{ kHz}$ again after a sudden change of the scattering length from $-140$ to $-135 \text{ nm}$ is shown in Fig. 7. This corresponds to a gas with $l_\parallel \approx 5.4 \mu \text{ m}$ and $l_\perp \approx 1.2 \mu \text{ m}$ and is thus on an experimentally accessible length scale. The Fermi energy is chosen as $E_F = 250 \hbar \omega_\parallel$, which corresponds to $N = 12,258$ atoms in the trap.

Figure 7 shows that the qualitative behavior of the gap dynamics is the same as for the smaller systems: A slowly decaying oscillation of the gap occurs. The transition time $t_c = 10 \text{ ms}$ calculated from Eq. (32) is in good agreement with a small bump in the curve at $t \approx 9.7 \text{ ms}$, the first deviation from a smooth oscillation. Afterwards more and more deviations occur and the gap dynamics becomes successively irregular.

When looking at the damping of the gap oscillation one can see that although the system is resonant—one sub-band is close to the Fermi energy—the strength of the damping is situated somewhere between the resonant and the off-resonant case of Fig. 5. This is due to the weak perpendicular confinement and therefore large number of states contributing to the condensate: Compared to the overall number of relevant states the resonant ones give only a small contribution to the gap. For larger systems the resonances are thus less pronounced.

The Fourier transform in the inset of Fig. 7 shows a familiar picture, too, with strong contributions at the lower end of the spectrum and successively decaying peaks towards higher energies. The difference with respect to the spectra shown before is on the one hand the high density of peaks for the large system. This is due to the weaker confinement and thus higher density of bare and quasiparticle states. On the other hand the gap in the Fourier spectrum and thus the main frequency of the oscillation is comparatively small. This is because of the larger ratio of the perpendicular to the parallel length $l_\perp/l_\parallel$. With the Fermi energy fixed at $E_F = 250 \hbar \omega_\parallel$ this leads to a comparatively low particle density of the trapped gas and thus a weaker condensate and a smaller gap.

### IV. CONCLUSION

In conclusion, we have calculated the Higgs amplitude dynamics in the BCS phase of an ultracold $^6$Li gas confined in a cigar-shaped trap. The dynamics is induced by a quantum quench resulting from a sudden change of an external magnetic field. We have shown that the amplitude of the spatially averaged gap performs a damped oscillation breaking down after a certain time $t_c$, which is determined by the parallel confinement frequency $f_\parallel$, i.e., by the length of the cloud. Afterwards a rather irregular oscillation involving many different frequencies occurs.

We have investigated the influence of the confinement on the gap dynamics and the impact of the size-dependent superfluid resonances on its qualitative behavior. It turned out that in the case of a resonant system, i.e., a system where the Fermi energy is close to a sub-band minimum, the dynamics of the order parameter exhibits a strong damping and, for sufficiently long systems, a revival before eventually the irregular regime is reached. In contrast, in an off-resonant system the damping is much less pronounced and the oscillation before the transition to the irregular regime is mainly determined by a single frequency.

By analyzing the linearized version of the equations of motion for the quasiparticle excitations we were able to interpret the observed features of the dynamics. It turned out that for the excitations studied in this paper the linearized equations well reproduce the dynamical behavior of the gap, except for some slight details resulting from the nonlinearities in the full equations of motion. From the linearized model it becomes evident that the system approximately behaves like a set of weakly coupled harmonic oscillators. The frequencies
as well as the couplings of these oscillators are completely determined by the system parameters after the excitation while the amplitudes of the different eigenmodes depend on the details of the excitation. The analysis revealed in particular that the transition time to the irregular dynamics is directly related to the energy separation of the one-particle energies while the differences between resonant and nonresonant systems are caused by the different densities of states and coupling efficiencies close to the sub-band minima.

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[39] In the case of the nonlinear system given here the Higgs and the Goldstone mode are in general coupled. However, due to the considered weak excitations and thus weak dynamical coupling the system behaves approximately linearly and the Higgs and the Goldstone mode decouple.
[46] This regularization introduces an additional dependency on the quasiparticle energy $E_m$ and thus the ground-state gap $\Delta_m$ in the dynamical equations. Since—in the dynamical case—the time-dependent gap $\Delta_m(t)$ differs from the ground-state value this may seem unusual. However, we have checked that the method is not sensitive on the detailed parameters used, e.g., on using one constant value for all $\Delta_m$ in order to omit the newly introduced dependence. Thus the proposed regularization scheme is a suitable extension since the physical behavior of the system remains unchanged.

[49] We want to remark that this definition is only for illustrative purposes. The normalization only scales the gap dynamics but does not enter the dynamical equations.

[50] In the numerical analysis the transition time $t_c$ is assumed to be reached if the distance between two adjacent maxima shows a deviation of more than 50% compared to the averaged distance of all preceding maxima.

[51] In case of possible divergences of the linearized solution one can artificially lower the coupling strength in the dynamical calculations in order to prevent them. This means that, while the ground state is still calculated with the coupling $g$, in the dynamical calculations [i.e., in Eqs. (29)–(31)] the coupling has to be slightly lowered by a small amount, i.e., $g \rightarrow g' \lesssim g$. It should be mentioned that the full (nonlinear) equations never lead to divergences.

[52] The main features can actually be reproduced by plotting a weighted sum of such cosine functions.