What do normative indices of multidimensional inequality really measure?

Reference:
Bosmans Kristof, Decancq Koen, Ooghe Erwin.- What do normative indices of multidimensional inequality really measure?
Journal of public economics - ISSN 0047-2727 - 130(2015), p. 94-104
Full text (Publishers DOI): http://dx.doi.org/doi:10.1016/j.jpubeco.2015.03.003

What do normative indices of multidimensional inequality really measure?*

Kristof Bosmans[a] · Koen Decancq[b] · Erwin Ooghe[c]

[a] *Department of Economics, Maastricht University, Tongersestraat 53, 6211 LM Maastricht, The Netherlands*

[b] *Herman Deleeck Centre for Social Policy, University of Antwerp, Sint-Jacobstraat 2, 2000 Antwerp, Belgium*

[c] *Center for Economic Studies, Katholieke Universiteit Leuven, Naamsestraat 69, 3000 Leuven, Belgium*

November 28, 2014

Abstract

We argue that normative indices of multidimensional inequality do not only measure a distribution's extent of inequity (i.e., the gaps between the better-off and the worse-off), but also its extent of inefficiency (i.e., the non-realized mutually beneficial exchanges of goods). We provide a decomposition that allows us to quantify these two parts of inequality. Strikingly, the inequity component turns out to be a two-stage measure, that is, a measure that applies a unidimensional inequality measure to the vector of individual well-being levels. The decomposition also clarifies existing controversies surrounding two prominent transfer axioms, viz., uniform majorization and correlation increasing majorization. An application to inequality in human development illustrates the analysis.

Keywords. Multidimensional inequality · Equity · Efficiency · Uniform majorization · Correlation increasing majorization

JEL classification. D31 · D63 · I31

*The paper has benefited from discussions with Rolf Aaberge, Matthew Adler, Francisco Ferreira, Marc Fleurbaey, François Maniquet, Pauline Mornet, Erik Schokkaert, Alain Trannoy, Dirk Van de gaer and Claudio Zoli. We thank audiences in Ghent, Wavres, New Delhi and Marseille for valuable comments. The authors are responsible for remaining shortcomings.

E-mail addresses: k.bosmans@maastrichtuniversity.nl (K. Bosmans), koen.decancq@uantwerpen.be (K. Decancq), erwin.ooghe@kuleuven.be (E. Ooghe).

1 Introduction

Individual well-being is now widely accepted to be multidimensional. Academic research in various disciplines, synthesized recently by the Stiglitz-Sen-Fitoussi Commission (2009), stresses the need to go beyond income by adding material and nonmaterial dimensions. Also in policy circles multidimensional measures of well-being gain prominence. An established example is the Human Development Index of the United Nations, which measures the performance of countries in terms of material living standards, health and education.[1]

A rapidly growing literature explicitly takes into account the multidimensionality of well-being in measuring inequality.[2] Two alternative approaches to measure multidimensional inequality may be distinguished: the normative approach and the two-stage approach.

The normative approach was originally developed in the unidimensional setting by Dalton (1920), Kolm (1969), Atkinson (1970) and Sen (1973), and was extended to the multidimensional setting by Kolm (1977), Atkinson and Bourguignon (1982) and Tsui (1995). In the normative approach, inequality measures are derived from social welfare functions. Inequality is defined as the social welfare gain that could be obtained by optimally redistributing the available goods: the greater is this potential social welfare gain, the greater is inequality. Since social welfare functions subsume values regarding distributive justice, made explicit by axioms, the derived inequality measures are given a normative basis.

The two-stage approach was pioneered by Maasoumi (1986, 1999). The first stage associates a well-being level to the bundle of goods of each individual. The second stage applies a unidimensional inequality measure to the obtained vector of individual well-being levels. The two-stage approach offers no guidelines for the choice of the appropriate functional form of the well-being measure in the first stage. In practice this choice is made on an ad hoc basis.[3]

Presumably because of its intuitive attraction, the two-stage approach dominates the empirical literature.[4] However, because of its dependence

[1] Yang (2014) lists 101 multidimensional measures of well-being and progress.

[2] For surveys, see Weymark (2006), Lugo (2007), Chakravarty (2009) and Aaberge and Brandolini (2014).

[3] Maasoumi (1986) proposes a specific class of two-stage measures based on information-theoretic considerations. Many others have employed two-stage measures without stating theoretic foundations.

[4] Empirical applications have studied inequality in individual well-being defined as a function of income and, among others, life expectancy (Bourguignon and Morrisson,

on the arbitrary choice of a well-being measure, the two-stage approach is generally regarded as less theoretically appealing than the normative approach.[5] In the latter, the axioms imposed on the social welfare function determine the properties of the underlying measure of individual well-being. Nevertheless, our examination will reveal a surprising connection between the two approaches. In particular, we show that two-stage measures receive a theoretical foundation from within the normative approach.

The starting point of our examination is the simple, yet novel, observation that normative indices of multidimensional inequality do not measure only inequity, but also inefficiency. Recall that the normative approach defines inequality as the social welfare gain that could be obtained by optimally redistributing the available goods. If there is a single good, then social welfare can be increased only by redistributing from better off to worse off individuals, i.e., by improving equity. Hence, in a unidimensional setting, inequality as defined in the normative approach coincides with inequity. The identity of inequality and inequity vanishes once we move to the multidimensional setting. In addition to equity improvements, now also efficiency improvements become possible: by exchanging goods between individuals with unequal marginal rates of substitution, the well-being levels of all individuals involved can be increased. Because normative indices of multidimensional inequality gauge the total potential social welfare gain, they capture both inequity and inefficiency.

Let us consider a simple example, with two individuals and two goods, to illustrate the importance of distinguishing between inequity and inefficiency in multidimensional inequality judgments. Consider the left-hand panel of Figure 1. Depicted is distribution $X = (x_1, x_2)$, where $x_1 = (70, 10)$ is the bundle of individual 1 and $x_2 = (10, 70)$ is the bundle of individual 2. Also depicted is distribution $Z = (z_1, z_2)$, where $z_1 = z_2 = (40, 40)$, which is under standard assumptions the welfare maximizing distribution given the available goods in X. In distribution X, the two individuals have equal well-being levels since their bundles are on the

2002, and Becker, Philipson and Soares, 2005), life expectancy and education (Noorbakhsh, 2007, and Decancq, Decoster and Schokkaert, 2009), leisure and intra-household public goods (Lise and Seitz, 2011), family composition (Slesnick, 2001), and price information (Pendakur, 2002, Deaton, 2010, Almås, 2012, and Moretti, 2013). Also inequality in lifetime utility, defined as a function of the entire life-time income profile, is essentially two-stage (Flinn, 2002, Bowlus and Robin, 2004, and Huggett, Ventura and Yaron, 2011).

[5] As Duclos, Sahn and Younger (2011, pp. 226-227) put it, the two-stage approach "... reduces the multivariate problem to a more familiar univariate one, but at a significant cost: it requires specifying a particular definition of [individual well-being], something that is necessarily arbitrary."

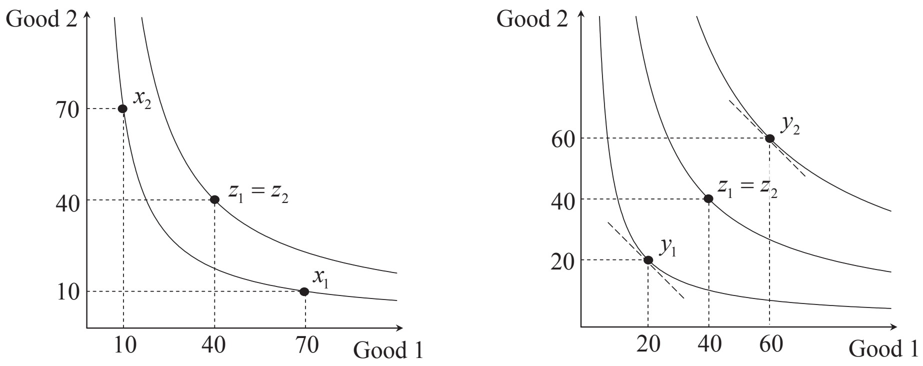

Figure 1. Distribution X exhibits inefficiency but is perfectly equitable (left), whereas distribution Y exhibits inequity but is perfectly efficient (right)

same iso-well-being curve, and hence X is perfectly equitable. The entire social welfare gain in moving to the ideal distribution Z is due to efficiency improving exchanges of goods between the individuals. Consider now the right-hand panel of Figure 1, which depicts distribution $Y = (y_1, y_2)$, where $y_1 = (20, 20)$ and $y_2 = (60, 60)$. In distribution Y, there are no possibilities for efficiency improving exchanges of goods, and hence Y is perfectly efficient.[6] Now the entire social welfare gain in moving to the ideal distribution Z is due to equity improving transfers of goods from the better off individual 2 to the worse off individual 1.

The sources of potential social welfare gain, and hence the meaning of inequality in the normative approach, can differ dramatically depending on the distribution under consideration. The distributions in the example are extreme: inequality coincides with inefficiency in the case of distribution X, whereas it coincides with inequity in the case of distribution Y.[7] For a real-world distribution, inequality typically consists of both inequity and inefficiency.

The observation that normative indices of multidimensional inequality measure a combination of inequity and inefficiency is the impetus of

[6]Indeed, the marginal rates of substitution of the two individuals are equal.

[7]Note that normative inequality measures may indicate higher inequality for X than for Y. For example, the inequality measure obtained by setting $\alpha = 1$, $\beta = 1$ and $r_1 = r_2$ in equations (2), (5) and (6) below, takes the values 0.34 for X and 0.13 for Y. This illustrates that it is highly misleading to identify inequality with inequity in the normative approach: although measured inequality is greater in X than in Y, inequity is smaller in X than in Y.

our examination of the normative approach. Using a suggestion by Graaff (1977), we show that normative measures can be neatly decomposed into an inequity and an inefficiency part.[8] Each of the two components receives an exact formula for a generic class of social welfare functions. Moreover, we use the decomposition to shed light on the prominent, but controversial, uniform majorization and correlation increasing majorization axioms. An empirical application to between-country inequality in the period 1980 to 2011, focusing on the three well-being dimensions of the Human Development Index, illustrates the analysis.

Two contributions of the paper deserve special emphasis. First, as mentioned before, the decomposition reveals a connection between the normative approach and the two-stage approach. We show that the inequity component associated with a normative measure is itself a two-stage measure. This means that two-stage measures, generally regarded as theoretically flawed, receive a strong theoretical justification as measures of multidimensional inequity from within the normative approach. If one would insist that inequality measures should be concerned with inequity alone, and not with inefficiency, then we arrive at the striking conclusion that the normative approach itself pushes two-stage measures to the forefront. Second, we disentangle the inequity and inefficiency effects of the transfers recommended by the uniform majorization and correlation increasing majorization axioms. These two multidimensional transfer axioms are key in the normative approach, but have nevertheless each received considerable criticism. Our decomposition illuminates these controversies.

The next section introduces notation and basic assumptions. Section 3 presents the decomposition of multidimensional inequality into inequity and inefficiency parts. This section also discusses the connection with the two-stage approach. In Section 4, we critically discuss uniform majorization and correlation increasing majorization in the light of the decomposition. Section 5 provides the illustrative empirical application. Section 6 concludes.

2 Notation and assumptions

There are n individuals and m goods. The quantity of good k owned by individual i is a positive real number x_{ik}. The bundle of individual i is a vector $x_i = (x_{i1}, x_{i2}, \ldots, x_{im})$. A distribution is an $n \times m$ matrix X

[8] Graaff (1977) did not consider multidimensional inequality measurement, but was instead interested in establishing the conditions under which aggregate income growth leads to an improvement in efficiency.

with bundle x_i at the ith row. The set of all bundles is $B = \mathbb{R}^m_{++}$ and the set of all distributions is $D = B^n$. We use μ_X to denote the bundle $(\mu_1, \mu_2, \ldots, \mu_m)$ with $\mu_k = (x_{1k} + x_{2k} + \cdots + x_{nk})/n$ the average quantity of good k in distribution X. We use X_μ to denote the distribution in which each individual has a bundle equal to μ_X. We write 1_ℓ for the ℓ-vector with a one at each entry.

A social welfare function is a function $W : D \to \mathbb{R}$. The value $W(X)$ is to be interpreted as the social welfare level associated with distribution X in D. We focus on the class of social welfare functions $\mathcal{W}$. A social welfare function W is in $\mathcal{W}$ if and only if there exist a continuous, concave, linearly homogenous[9] and strictly increasing function $U : B \to \mathbb{R}$ with $U(1_m) = 1$ and a continuous, Schur-concave,[10] linearly homogenous and strictly increasing function $V : \mathbb{R}^n \to \mathbb{R}$ with $V(1_n) = 1$ such that, for each distribution X in D, we have

$$W(X) \;=\; V(U(x_1), U(x_2), \ldots, U(x_n)). \tag{1}$$

As we prove in Appendix A, the class $\mathcal{W}$ is characterized by the following seven axioms: (i) anonymity (social welfare is invariant to rearranging bundles of individuals), (ii) monotonicity (increasing the amount of a good owned by an individual increases social welfare), (iii) continuity (small changes in distributions do not cause large changes in their social welfare ranking), (iv) weak uniform majorization (progressive transfers uniformly applied to each good do not decrease social welfare), (v) normalization (if each individual has an amount λ of each good, then the social welfare level is λ), (vi) individualism (social welfare is measured in two steps, a first step to aggregate across dimensions for each individual, and a second step to aggregate the obtained values across individuals) and (vii) homotheticity (the social welfare ranking of two distributions is preserved if each good of each individual is multiplied by the same factor in both distributions). These seven axioms are standard in the literature.[11] Nevertheless, the use

[9]A function $\psi : \mathbb{R}^\ell \to \mathbb{R}$ is linearly homogenous, or homogenous of degree one, if $\psi(\lambda t) = \lambda\psi(t)$ for each t in $\mathbb{R}^\ell$ and each positive real number λ.

[10]A function $\psi : \mathbb{R}^\ell \to \mathbb{R}$ is Schur-concave if $\psi(t) \le \psi(Qt)$ for each t in $\mathbb{R}^\ell$ and each bistochastic matrix Q. A bistochastic matrix is a nonnegative square matrix of which each row sum and each column sum is equal to 1. A permutation matrix is a bistochastic matrix of which each component is either 0 or 1.

[11]With the exception of normalization and individualism, all these axioms are discussed in the survey by Weymark (2006). Normalization is an innocent axiom that ensures a convenient cardinalization of the social welfare function. Individualism is very common in the literature: some studies explicitly impose individualism (e.g., Kolm, 1977, and Seth, 2013), whereas others assume stronger requirements of additive separa-

of weak uniform majorization is not without controversy, an issue to which we return in Section 4.

We will interpret the function U in equation (1) as a well-being measure common to all individuals. This interpretation is in line with the literature. Some studies explicitly treat U (or a similar function) as a measure of individual well-being and discuss it using the language of utility theory (e.g., Atkinson and Bourguignon, 1982, Atkinson, 2003, and Bourguignon and Chakravarty, 2003). Other studies are less explicit, but admit the possibility of this interpretation. For example, Tsui (1995, p. 264) notes that axioms such as (i) to (vii) above do not only impose requirements on a social welfare function, but also on an underlying concept of individual well-being.[12] Hence, these axioms can be interpreted as describing an ethical observer's value judgments concerning individual well-being, an interpretation already noted by Kolm (1977, pp. 2-3).

Two final comments are in order. First, the function U is a least concave representation of the common individual well-being ranking (see Proposition 1 in Kihlstrom and Mirman, 1981).[13] Therefore, the social welfare function in equation (1) is in a form that allows a clear separation between, on the one hand, the ordinal properties of the individual well-being ranking, such as the degree of complementarity of the goods and, on the other hand, the cardinal properties of the social aggregation, such as the degree of inequity aversion. With the cardinal representation of the individual well-being ranking fixed, the former category of properties is in U, whereas the latter category is in V. This convenient form was advocated by Atkinson and Bourguignon (1982, p. 191).

Second, the value $W(X)$ in equation (1) is the equally distributed equivalent level of individual well-being associated with X. That is, $W(X)$ is the level of individual well-being that, if equally attained by all individuals, yields the same level of social welfare as X.

---

bility that imply individualism (e.g., Tsui, 1995). A noteworthy exception is the study by Gajdos and Weymark (2005). Gajdos and Weymark propose a social welfare function that is not individualistic, but rather first aggregates across individuals for each dimension, and second aggregates the obtained values across dimensions. Their approach can be seen as a way to make the best of a situation in which the joint distribution of the dimensions is unknown.

[12] For example, anonymity makes the individual well-being measure common to all individuals, monotonicity makes it increasing in the amounts of goods, and weak uniform majorization makes it quasi-concave.

[13] A function U is a least concave representation of a well-being relation if, for each concave representation $\hat{U}$ of the same well-being relation, we have $\hat{U} = \psi \circ U$, where ψ is a concave and increasing function (Debreu, 1976).

3 Equity and efficiency as components of equality

3.1 Measuring inequality

The normative approach to inequality measurement identifies inequality with the potential social welfare gain that could be obtained by distributing the available goods optimally. In order to avoid that the values of the inequality measure depend on an arbitrary choice of cardinalization of the social welfare function, the literature has opted to measure this potential social welfare gain in terms of goods rather than directly in terms of social welfare. Kolm (1977, footnote 3) and Tsui (1995) proposed what is now the standard procedure to derive a relative multidimensional inequality measure from a social welfare function in $\mathcal{W}$. A relative inequality measure is invariant to a multiplication of each good of each individual by the same factor.[14]

The Kolm-Tsui procedure is as follows: equality in distribution X is measured by the smallest fraction of the total good amounts in X required to maintain the level of social welfare of X.

Definition 1. Let X be a distribution in D. Let $f(X)$ be the smallest real number for which there exists a distribution X^* in D such that $W(X^*) = W(X)$ and $\mu_{X^*} = f(X)\times\mu_X$. Then, $f(X)$ is the equality level and $1-f(X)$ is the inequality level of distribution X.

Because a social welfare function W in $\mathcal{W}$ satisfies weak uniform majorization, the available goods in distribution X are optimally distributed if each individual is given the average bundle μ_X, yielding distribution X_μ. Since X^* in Definition 1 must be optimal, we have that $X^* = (\mu_{X^*}, \mu_{X^*}, \ldots, \mu_{X^*})$ and thus $X^* = (f(X)\times\mu_X, f(X)\times\mu_X, \ldots, f(X)\times\mu_X)$. Hence, equality $f(X)$ is defined by $W(X) = W(X^*) = W(f(X)X_\mu)$. Since W is linearly homogenous, we obtain

$$f(X) = \frac{W(X)}{W(X_\mu)}. \tag{2}$$

The relative inequality measure $1 - f$ is standard in the normative approach.[15] The inequality level $1 - f(X)$ is the percentage social welfare loss incurred by moving from the optimal distribution X_μ to the actual distribution X.

[14] Appendix B provides the analogous analysis for absolute inequality measures, which are invariant to an addition of the same amount to each good of each individual.

[15] The linear homogeneity of W indeed ensures that $1 - f$ is relative, i.e., we have $1 - f(X) = 1 - f(\lambda X)$ for each distribution X in D and each positive real number λ.

The example in the introduction revealed that the potential social welfare gain measured by $1 - f$ reflects both efficiency and equity improvements. This makes the meaning and interpretation of variations in inequality ambiguous, thus calling for a decomposition of equality into efficiency and equity parts.

3.2 A decomposition

To decompose equality into efficiency and equity parts, we use a procedure suggested by Graaff (1977). Graaff's procedure distinguishes two steps involved in the social welfare optimization underlying Definition 1. The first step consists of efficiency improving exchanges of goods, which increase the well-being levels of all involved individuals. The second step consists of equity improving redistributions of goods, which decrease the individual well-being gaps. Graaff proposes an efficiency measure and an equity measure corresponding, respectively, to each of these two steps.

The efficiency measure is related closely to Debreu's (1951) coefficient of resource utilization: efficiency in distribution X is measured by the smallest fraction of the total good amounts in X required to maintain the individual well-being levels associated with X.

Definition 2. Let X be a distribution in D. Let $d(X)$ be the smallest real number for which there exists a distribution X' in D such that $U(x_i') = U(x_i)$ for each individual i and $\mu_{X'} = d(X) \times \mu_X$. Then, $d(X)$ is the efficiency level and $1 - d(X)$ is the inefficiency level of distribution X.

Distribution X' in Definition 2 is perfectly efficient, but it maintains the level of inequity of distribution X because all individual well-being levels are the same in the two distributions. Therefore, the level of equity of distribution X can be measured by applying Definition 1 to distribution X': equity in distribution X is the smallest fraction of the good amounts in distribution X' required to maintain the level of social welfare of X'.

Definition 3. Let X be a distribution in D. Let X' be a distribution in D obtained from distribution X as in Definition 2. Let $e(X)$ be the smallest real number for which there exists a distribution X'' such that $W(X'') = W(X')$ and $\mu_{X''} = e(X) \times \mu_{X'}$. Then, $e(X)$ is the equity level and $1 - e(X)$ is the inequity level of distribution X.

We obtain a multiplicative decomposition of equality into its efficiency and equity parts. To see this, note first that X^* in Definition 1 must coincide with X'' in Definition 3. Combining $\mu_{X''} = e(X) \times \mu_{X'}$ and $\mu_{X'} = d(X) \times \mu_X$ from Definitions 2 and 3 yields $\mu_{X''} = e(X) \times d(X) \times \mu_X$.

Since also $\mu_{X''} = \mu_{X^*} = f(X) \times \mu_X$ from Definition 1, we obtain[16]

$$f(X) \;=\; d(X) \;\times\; e(X).$$

We will now derive formulas for the efficiency measure d and the equity measure e. We start with the efficiency measure. Figure 2 gives an illustration for the two-individual distribution $X = (x_1, x_2)$. Because the individual well-being function U is homothetic, each individual must receive in the efficient distribution X' a bundle proportional to the average bundle μ_X.[17] That is, for each individual i, we have $x'_i = s_i\mu_X$ with $s_i > 0$. Distribution X' must moreover maintain the individual well-being levels of distribution X. That is, for each individual i, we have $U(x_i) = U(s_i\mu_X)$. Definition 2 implies that $\mu_{X'} = d(X) \times \mu_X$. Since $x'_i = s_i\mu_X$ for each individual i, we have also that $\mu_{X'} = \sum_{i=1}^n s_i\mu_X/n$. It follows that $d(X) = \sum_{i=1}^n s_i/n$. Because $U(x_i) = U(s_i\mu_X)$ for each individual i and because U is linearly homogenous, we moreover have $s_i = U(x_i)/U(\mu_X)$ for each individual i. We obtain

$$d(X) \;=\; \frac{\frac{1}{n}\sum_{i=1}^n U(x_i)}{U(\mu_X)}. \tag{3}$$

Equation (3) has a straightforward interpretation. It reveals that the efficiency measure d is in fact the equality measure f in equation (2) applied to the inequity neutral version of the social welfare function W, i.e., the social welfare function obtained by replacing in equation (1) the function V by the mean operator.[18]

To provide a formula for e, we use that $e = f/d$. Substituting equations (2) and (3) and using that $U(\mu_X) = W(X_\mu)$ for each social welfare function W in $\mathcal{W}$, we obtain

$$e(X) \;=\; \frac{W(X)}{\frac{1}{n}\sum_{i=1}^n U(x_i)}. \tag{4}$$

Importantly, the inequity measure $1-e$ is a two-stage measure. The first stage associates to distribution X a vector of individual well-being levels $(U(x_1), U(x_2), \ldots, U(x_n))$. The second stage computes inequity by applying to this vector of well-being levels a generic unidimensional inequality

[16] Note that social welfare can be decomposed as $W(X) = W(X_\mu) \times d(X) \times e(X)$.

[17] To see this, note that the marginal rates of substitution (assuming these are well-defined) of the individuals are equal if all bundles are on the same ray through the origin.

[18] Inequity aversion is defined here in analogy with risk aversion in the case of many commodities (Kihlstrom and Mirman, 1974). The fact that the mean of least concave representations can be interpreted as the risk neutral, or inequity neutral, case is discussed by Kihlstrom and Mirman (1981).

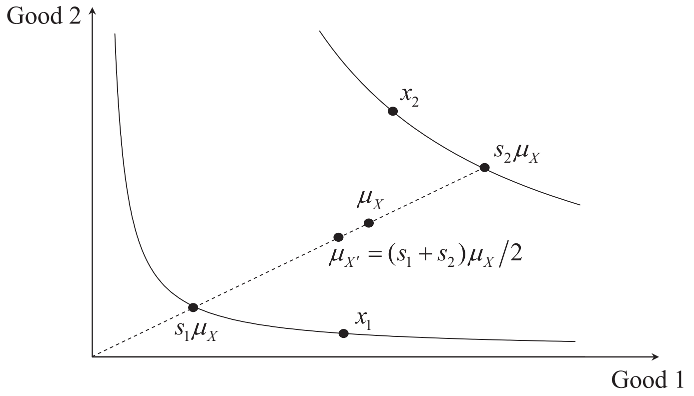

Figure 2. The efficiency level of distribution X is $d(X) = (s_1 + s_2)/2$

measure as defined by Sen (1973, p. 42, equation (2.17)): one minus the ratio of the equally distributed equivalent well-being level $V(U(x_1), U(x_2), \ldots, U(x_n)) = W(X)$ and the average well-being level $\sum_{i=1}^{n} U(x_i)/n$.[19]

It is striking that the two-stage approach turns out to deliver the inequity component of the normative inequality measure $1 - f$. Indeed, the literature has interpreted the two-stage approach as inherently distinct from the normative approach as well as theoretically less appealing (see, e.g., the discussions by Bourguignon, 1999, and Weymark, 2006). Our examination reveals that the two approaches are nevertheless intimately connected. The normative approach moreover provides an axiomatic underpinning of the two-stage inequity measure $1 - e$: the axioms determine the properties both of the well-being measure U used in the first stage and of the function V that fixes the unidimensional measure used in the second stage.

One plausible view may be that the true interest lies in the measurement of inequity, not of inefficiency. On this view, our examination surprisingly reveals that the normative approach itself insists on the use of two-stage measures.

[19] This class of generic unidimensional inequality measures includes the well-known Atkinson (1970) and S-Gini (Donaldson and Weymark, 1980, 1983) classes. Note that Maasoumi (1986) proposed to apply the generalized entropy class of unidimensional inequality measures to the vector of levels of a CES individual well-being measure. The generalized entropy class includes a subclass that is ordinally equivalent to the Atkinson class.

Note, for later use, that the inefficiency measure and the inequity measure have a straightforward graphical interpretation in terms of Figure 2. The inefficiency measure summarizes the distance from the actual bundles to the constructed efficient bundles on the dashed line. The inequity measure measures the inequality between these constructed efficient bundles on the dashed line.[20]

*3.3 The double-*CES *class*

We now apply the decomposition to the class of double-CES social welfare functions, a popular subclass of $\mathcal{W}$ of which the members satisfy additive separability and replication invariance.[21] The double-CES class is used in the discussion of Section 4 and the empirical illustration of Section 5.

A double-CES social welfare function uses a CES function to aggregate at the social level and a second CES function to aggregate at the individual level. A social welfare function W is a member of the double-CES class if, for each distribution X in D, we have

$$W(X) = \begin{cases} \left(\dfrac{1}{n} \displaystyle\sum_{i=1}^{n} U(x_i)^{1-\alpha} \right)^{\frac{1}{1-\alpha}} & \text{for } \alpha \geq 0 \text{ and } \alpha \neq 1, \\ \displaystyle\prod_{i=1}^{n} U(x_i)^{\frac{1}{n}} & \text{for } \alpha = 1, \end{cases} \tag{5}$$

and

$$U(x_i) = \begin{cases} \left(\displaystyle\sum_{k=1}^{m} r_k x_{ik}^{1-\beta} \right)^{\frac{1}{1-\beta}} & \text{for } \beta \geq 0 \text{ and } \beta \neq 1, \\ \displaystyle\prod_{k=1}^{m} x_{ik}^{r_k} & \text{for } \beta = 1, \end{cases} \tag{6}$$

where $r_k > 0$ for each good k and $r_1 + r_2 + \cdots + r_m = 1$. The shape of the iso-well-being curves is determined by the weights r_k on the different dimensions and by the parameter β. The parameter β determines the degree of complementarity, with $\beta = 0$ corresponding to the case of perfect

[20] Because $1 - e$ is a relative two-stage measure, it is simply a unidimensional measure applied to the vector $(s_1, s_2, \ldots, s_n)$.

[21] For axiomatic characterizations of the double-CES class (or subclasses thereof), see Decancq and Ooghe (2010), Lasso de la Vega and Urrutia (2011), Seth (2013) and Tsui (1995). For discussions of the class, see Atkinson (2003) and Bourguignon (1999).

substitutes and $\beta \to \infty$ to the case of perfect complements. The parameter α determines the degree of inequity aversion. The case $\alpha = 0$ corresponds to inequity neutrality with social welfare equal to the mean of least concave representations, and the case $\alpha \to \infty$ corresponds to maximin with social welfare measured by the well-being level of the individual in the worst position.

We apply equations (2), (3) and (4) to the double-CES class (omitting the cases $\alpha = 1$ and $\beta = 1$). We have

$$f(X) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\sum_{k=1}^{m} r_k x_{ik}^{1-\beta}}{\sum_{k=1}^{m} r_k \mu_k^{1-\beta}}\right)^{\frac{1-\alpha}{1-\beta}}\right]^{\frac{1}{1-\alpha}}, \qquad (7)$$

$$d(X) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\sum_{k=1}^{m} r_k x_{ik}^{1-\beta}}{\sum_{k=1}^{m} r_k \mu_k^{1-\beta}}\right)^{\frac{1}{1-\beta}}, \qquad (8)$$

and

$$e(X) = \frac{\left(\frac{1}{n}\sum_{i=1}^{n}(\sum_{k=1}^{m} r_k x_{ik}^{1-\beta})^{\frac{1-\alpha}{1-\beta}}\right)^{\frac{1}{1-\alpha}}}{\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{m} r_k x_{ik}^{1-\beta}\right)^{\frac{1}{1-\beta}}}. \qquad (9)$$

The parameters α and β of the double-CES class play different roles for the inefficiency and inequity components.[22] First, the interpretation of α as a parameter of inequity aversion is clear: efficiency d is not influenced by α, whereas inequity $1 - e$ increases if inequity aversion α increases. Hence, also the relative importance of the inequity component increases with inequity aversion. Inequality $1 - f$ is a pure measure of inefficiency if $\alpha = 0$, i.e., in that case $1 - f = 1 - d$. Second, as the parameter β decreases, the different dimensions become more substitutable. If the goods are perfect substitutes, i.e., if $\beta = 0$, then inequality $1 - f$ reduces to a pure measure of inequity. In this case we have $1 - f = 1 - e$.[23] In general, the effects of a change in β on efficiency d depends on the distribution and the chosen weights on the different dimensions. Typically, also the value of e will change as β varies. The reason is that $1-e$ is equal to the unidimensional Atkinson (1970) inequality measure applied to the vector of individual well-being levels, and these well-being levels typically change if β changes.

[22] The values of the weights r_k for the different goods have a complex impact on both d and e.

[23] For a characterization of a multidimensional social welfare function with a linear individual well-being function, see Bosmans, Lauwers and Ooghe (2009).

4 How do transfers affect equity and efficiency?

Multidimensional transfer axioms endow the social welfare function with a concern for distribution. They therefore play a key role in the normative approach to inequality measurement. Two prominent transfer axioms are uniform majorization (Kolm, 1977, and Tsui, 1995) and correlation increasing majorization (Atkinson and Bourguignon, 1982, and Tsui, 1999). Although these two axioms are widely used in the literature, both have received considerable criticism. Uniform majorization has been shown to recommend in some cases transfers that are unambiguously unfavorable in terms of equity (Dardanoni, 1995). Correlation increasing majorization has been claimed to be inappropriate if the goods are complements (Bourguignon and Chakravarty, 2003). We will disentangle the efficiency and equity effects of the transfers recommended by each of the two axioms and thus provide new insights into these two critiques.

First, we consider uniform majorization, which advocates progressive transfers in each dimension with a uniform structure across dimensions. The principle is a strict version of the weak uniform majorization axiom of Section 2.

Uniform majorization. For all distributions X and Y in D such that $X \neq Y$, if $X = QY$ with Q a bistochastic matrix that is not a permutation matrix, then $W(X) > W(Y)$.

Dardanoni (1995) provides an example in which uniform majorization recommends transfers that actually widen the gaps between individual well-being levels. He argues that this puts into question the appropriateness of uniform majorization in the context of inequality measurement. Figure 3 presents an example qualitatively similar to Dardanoni's. The figure extends the example in the left-hand panel of Figure 1 by adding in distributions X and Z a third individual with bundle $x_3 = z_3 = (10, 10)$.[24] Equity clearly worsens in going from X to Z because the gap between the equally well off individuals 1 and 2, on the one hand, and the worst off individual 3, on the other hand, further widens. Nevertheless, uniform majorization demands that equality increases as we move from X to Z, i.e., $f(X) < f(Z)$.[25] Our decomposition helps to understand this coun-

[24] The iso-well-being curves in the figure are for the symmetric Cobb-Douglas case. This is not essential: examples with the same conclusion can easily be constructed if U is not in the Cobb-Douglas form or is not symmetric.

[25] Because $Z = QX$ and $X \neq Y$ with $Q = ((0.5, 0.5, 0), (0.5, 0.5, 0), (0, 0, 1))$, we have $W(X) < W(Z)$. Since, in addition, $W(X_\mu) = W(Z_\mu)$, we obtain $f(X) < f(Z)$ using equation (2).

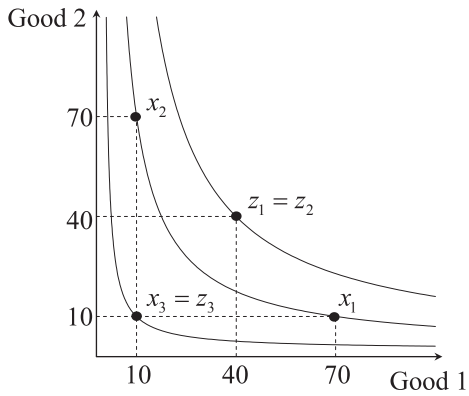

Figure 3. Uniform majorization implies $W(X) < W(Z)$

terintuitive implication. It is straightforward to show that, for all social welfare functions in $\mathcal{W}$ satisfying uniform majorization, the move from X to Z is a worsening in terms of equity, but an improvement in terms of efficiency, i.e., we have $e(X) > e(Z)$ and $d(X) < d(Z) = 1$. That is, uniform majorization imposes that the efficiency improvement in going from X to Z must outweigh the equity worsening, so that equality as measured by f goes up.[26]

The conclusion that the transfers recommended by uniform majorization increase efficiency extends beyond the above example. It holds in general: we have $d(QX) \geq d(X)$ for each distribution X in D and each non-permutation bistochastic matrix Q such that $X \neq QX$.[27] In terms of Figure 2, the intuition is that multiplication by a bistochastic matrix brings the bundles closer to the dashed line. To sum up, the effect on efficiency is positive in general, whereas, as the above example shows, the effect on equity is not always positive. We can conclude, therefore, that uniform majorization is more successful at capturing the efficiency aspect

[26]Dardanoni (1995) uses a two-stage measure to argue against this conclusion. Weymark (2006, pp. 310-311) claims that Dardanoni's example shows that it is the two-stage approach that is problematical, rather than the uniform majorization axiom. But, we saw in the previous section that the inequity measure in the normative approach is a two-stage measure. Hence, if we interpret Dardanoni's critique as arguing that uniform majorization is not a satisfactory principle of equity, then his critique remains valid also within the normative approach.

[27]Let $Y = QX$. We have $\sum_{i=1}^{n} U(x_i)/n \leq \sum_{i=1}^{n} U(y_i)/n$ (Kolm, 1977, Theorem 3), while $U(\mu_X) = U(\mu_Y)$. Using equation (3), we obtain $d(QX) \geq d(X)$.

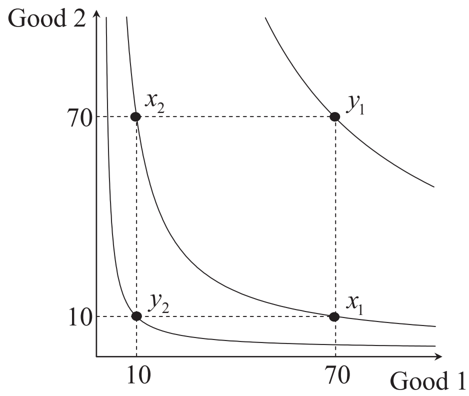

Figure 4. Correlation increasing majorization implies $W(X) > W(Y)$

of multidimensional inequality than at capturing the equity aspect.[28]

Next, we discuss correlation increasing majorization, which advocates exchanges of goods that decrease the correlation between dimensions. Let X and Y be distributions in D that coincide except for individuals i and j, let $x_i \neq x_j$, and let y_i be the componentwise minimum and y_j the componentwise maximum of x_i and x_j.[29] Then, Y is said to be obtained from X by a correlation increasing switch.

Correlation increasing majorization. For all distributions X and Y in D, if Y is obtained from X by a correlation increasing switch, then $W(X) > W(Y)$.

Figure 4 gives an example where distribution $X = (x_1, x_2)$ with $x_1 = (70, 10)$ and $x_2 = (10, 70)$ and distribution $Y = (y_1, y_2)$ with $y_1 = (70, 70)$ and $y_2 = (10, 10)$. For the double-CES class, correlation increasing majorization is satisfied if $\alpha > \beta$.

Bourguignon and Chakravarty (2003, pp. 35-36) claim that correlation increasing majorization is inappropriate if the goods are complements. We use our decomposition to examine this claim. To define complementarity

[28]It is by now fair to say that the uniform majorization axiom is controversial as a principle of equity. Trannoy (2006) and Fleurbaey (2006) argue that it is inappropriate that the axiom imposes quasi-concavity on individual well-being functions. Duclos, Sahn and Younger (2011) show that an example where the principle recommends transfers that reduce equity can also be found in the two-individual case if the assumption of homothetic individual well-being rankings is dropped.

[29]That is, $y_i = (\min\{x_{ik}, x_{jk}\})_{k=1,\ldots,m}$ and $y_j = (\max\{x_{ik}, x_{jk}\})_{k=1,\ldots,m}$.

we rely on the Auspitz-Lieben-Edgeworth-Pareto (ALEP) notion as refined by Kannai (1980): goods are complements if U is strictly supermodular.[30] A general conclusion can be drawn about the effect of a correlation increasing switch on efficiency: if U is strictly supermodular, then we have $d(X) < d(Y)$ for all distributions X and Y in D such that Y is obtained from X by a correlation increasing switch.[31] In other words, the transfers advocated by correlation increasing majorization always worsen efficiency in the case of complements. In terms of Figure 2, the intuition is roughly that these transfers move the bundles away from the dashed line. But the transfers also unambiguously reduce the well-being gap between the two individuals involved, irrespective of whether the goods are complements or substitutes. Summing up, correlation increasing majorization demands that we accept an efficiency leak in a transfer from a better off individual to a worse off individual. Hence, it does not follow that correlation increasing majorization is a priori inappropriate if the goods are complements: the principle just becomes more demanding, i.e., imposes a greater tolerance of efficiency leaks, as the degree of complementary increases.

We consider the distributions X and Y in Figure 4 to make the above point more concrete. For a strictly supermodular U, we indeed have $d(X) < d(Y) = 1$, whereas the gap between the well-being levels of individuals 1 and 2 has widened in going from X to Y. The efficiency loss that has to be tolerated in return for the reduction in the well-being gap can be made precise by considering the well-being vectors corresponding to distributions X and Y. In Figure 4 we have $\beta = 1$ and hence U is strictly supermodular. We have $(U(x_1), U(x_2)) = (27, 27)$ and $(U(y_1), U(y_2)) = (70, 10)$. The efficiency loss in moving from Y to X translates in terms of well-being as a leaky-bucket transfer: individual 1 gave up 43 units, but individual 2

[30]A function F is supermodular if $F(x \vee y) + F(x \wedge y) \geq F(x) + F(y)$ for all x and y in B, where $x \vee y$ is the componentwise maximum and $x \wedge y$ is the componentwise minimum of x and y. If, in addition, the inequality holds strictly for all unordered pairs x and y, then F is strictly supermodular. For a twice-differentiable F, supermodularity is equivalent to non-negative cross-derivatives. According to the ALEP notion, complementarity is equivalent to strict supermodularity of some functional representation of the well-being ranking. The refinement of Kannai (1980) consists in applying the ALEP notion specifically to the least concave representation of the well-being ranking, U in our case, and thus removes all arbitrariness concerning the choice of the representation. Unfortunately, Bourguignon and Chakravarty (2003) rely on the original ALEP definition, which renders their treatment of complements versus substitutes unsatisfactory. We refer to Atkinson (2003, pp. 57-60) for a thorough treatment of this point.

[31]Let Y be obtained from X by a correlation increasing switch. We have $\sum_{i=1}^{n} U(x_i)/n < \sum_{i=1}^{n} U(y_i)/n$ by strict supermodularity of U, while $U(\mu_X) = U(\mu_Y)$. Using equation (3), we obtain $d(X) < d(Y)$.

received only 17. The leak that the principle asks us to tolerate increases as the degree of complementarity β increases. For example, this leak is zero units if the goods are substitutes ($\beta = 0$), it is 12 units if $\beta = 0.5$, it is 26 units if $\beta = 1$ as we saw above, and it approaches the full 60 unit well-being difference between individuals 1 and 2 as $\beta \to \infty$. This also helps to understand the requirement $\alpha > \beta$ for a double-CES social welfare function: as the degree of complementarity β increases, the efficiency leak becomes larger, and hence the tolerance of leaks has to increase accordingly by increasing inequity aversion α.

The analysis of this section shows that neither uniform majorization nor correlation increasing majorization captures the simplicity of the unidimensional Pigou-Dalton principle. The Pigou-Dalton principle expresses a clear and elementary idea: an improvement in equity (in the unidimensional case, a progressive transfer), while keeping efficiency at the same level (which in the unidimensional case amounts to preserving the mean), increases social welfare. On the other hand, we have seen that the two multidimensional transfer axioms have implications in cases where equity and efficiency change simultaneously. In the case of uniform majorization the recommendation sometimes even is unequivocally in the direction of worse equity. It would be interesting to examine, in future research, a new multidimensional transfer principle that, like the unidimensional Pigou-Dalton principle, recommends transfers that improve equity but preserve efficiency.

5 Empirical illustration

We illustrate the proposed decomposition with an empirical application to population-weighted between-country inequality in the period 1980 to 2011. Each individual in a country is assigned the same bundle of goods, the average bundle of his country. Following the well-known Human Development Index (HDI), we consider the following three 'goods': living standards, health and education.[32] We only retain the countries for which

[32]We define the three goods as in the HDI-methodology of the United Nations Development Programme (UNDP). Living standards are measured using the logarithm of GNI per capita (in 2005 US$ PPP), health is measured by life expectancy at birth, and education by a geometric average of normalized mean years of schooling and normalized expected years of schooling. Each dimension is normalized between 0 and 1. See UNDP (2010, pp. 216-217) for details. The data have been downloaded from the UNDP website in December 2012. Note that the use of a logarithmic transformation in the definition of living standards is controversial. See Decancq, Decoster and Schokkaert (2009) and Ravallion (2012) for critical discussions.

the data between 1980 and 2011 are available, leaving us with 105 countries covering 84% of the population in 2011. Appendix C lists the countries. We measure inequality and its inefficiency and inequity components using the double-CES class in equations (7), (8) and (9).

The HDI aggregates the goods in a way that is consistent with our analysis. It also uses the CES well-being function in equation (6), with equal weighting of the three goods. The value of β has recently been changed: β was equal to 0 prior to 2010 and equal to 1 since (UNDP, 2010, p. 15). This change in the value of β in the definition of the HDI is interesting, as it constitutes a move away from perfect substitutability ($\beta = 0$) to a case of complementarity ($\beta = 1$), and hence introduces the efficiency aspect. Note that the inequity component in our analysis is the unidimensional Atkinson inequality measure applied to the HDI well-being measure (for β equal to 0 or 1) or to similar well-being measures (for other values of β).[33]

Figure 5 presents an overview of the data. The two left-hand panels are for 1980, the two right-hand panels for 2011. The two top panels show each country's achievements in living standards and health, the two bottom panels show achievements in living standards and education. Larger points correspond to more highly populated countries. The dashed line in each panel, which connects the origin with the average bundle, has the same interpretation as the dashed line in Figure 2. Efficiency appears to have improved between 1980 and 2011, as points have moved closer to the dashed line. Equity also seems to have improved, as highly populated countries that traditionally were poorer, such as China and India, have moved more rapidly upward along the dashed line.

Let us first consider the results for a specific choice of the parameters. In line with the formulation chosen for the HDI, we consider first the benchmark case with equal weights for the three goods, i.e., $r_1 = r_2 = r_3 = 1/3$. We let $\beta = 1$, as in the new formulation of the HDI, and $\alpha = 0.5$, reflecting a moderate degree of inequity aversion. The top left-hand panel of Figure 6 shows the evolution of inequality, inefficiency and inequity for these parameter values.[34] There is a clear decrease of inequality, with the loss in social welfare dropping from 4.8% in 1980 to 1.6% in 2011. Note that

[33]Noorbakhsh (2007) applies the unidimensional Gini index to the HDI values, clearly a two-stage approach. Decancq, Decoster and Schokkaert (2009) contrast the normative and two-stage approaches in the context of global well-being inequality.

[34]The decomposition presented in the figure is approximately additive. Recall that the decomposition is multiplicative, i.e., $f(X) = d(X) \times e(X)$. Hence, the logarithm can be decomposed additively, i.e., $\ln f(X) = \ln d(X) + \ln e(X)$. For values of t close to 1, we indeed have $\ln t \approx 1 - t$, so that $1 - f(X) \approx 1 - d(X) + 1 - e(X)$.

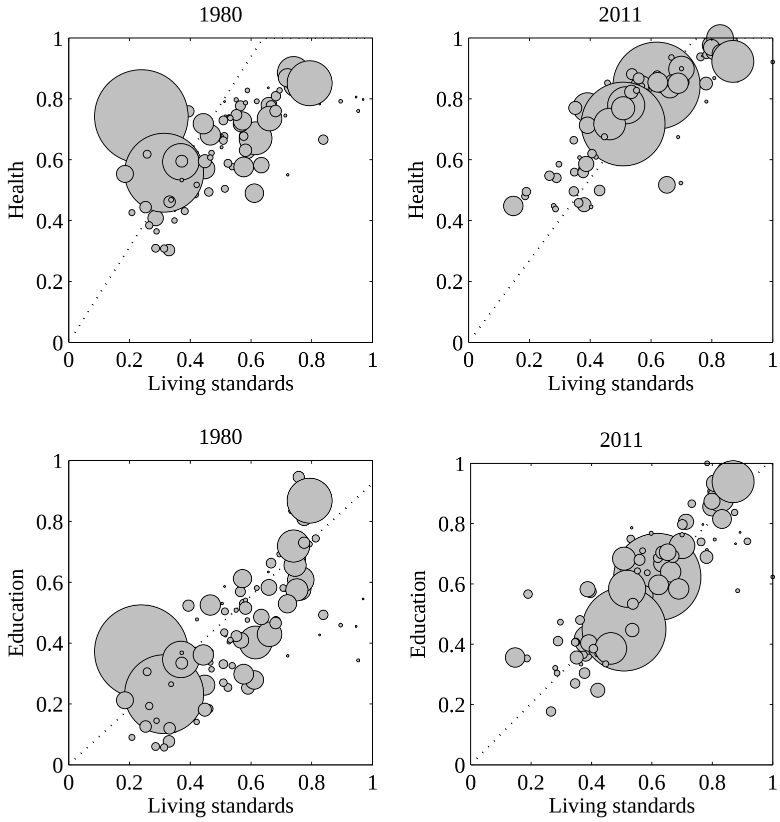

Figure 5. Living standards and health (top) and living standards and education (bottom) in 1980 (left) and 2011 (right)

a value of 1.6% is only seemingly low because of the way the three goods are constructed. This percentage corresponds to a considerable social welfare waste: for a country with the average bundle in 2011, it is equivalent to a loss in living standards of $1,335 out of $7,922 (keeping the health and education levels fixed). The decreasing trends for the inefficiency and inequity components are in line with the first impression from Figure 5.

The remaining panels of Figure 6 test the sensitivity of this finding to the use of three alternative weighting schemes. In each panel we drop one dimension and weight the other two dimensions equally. The trend in inequality remains similar. But the composition of inequality, and hence its interpretation, does depend on the weighting scheme. In the top-right panel, we consider only the goods living standards and health. In 1980

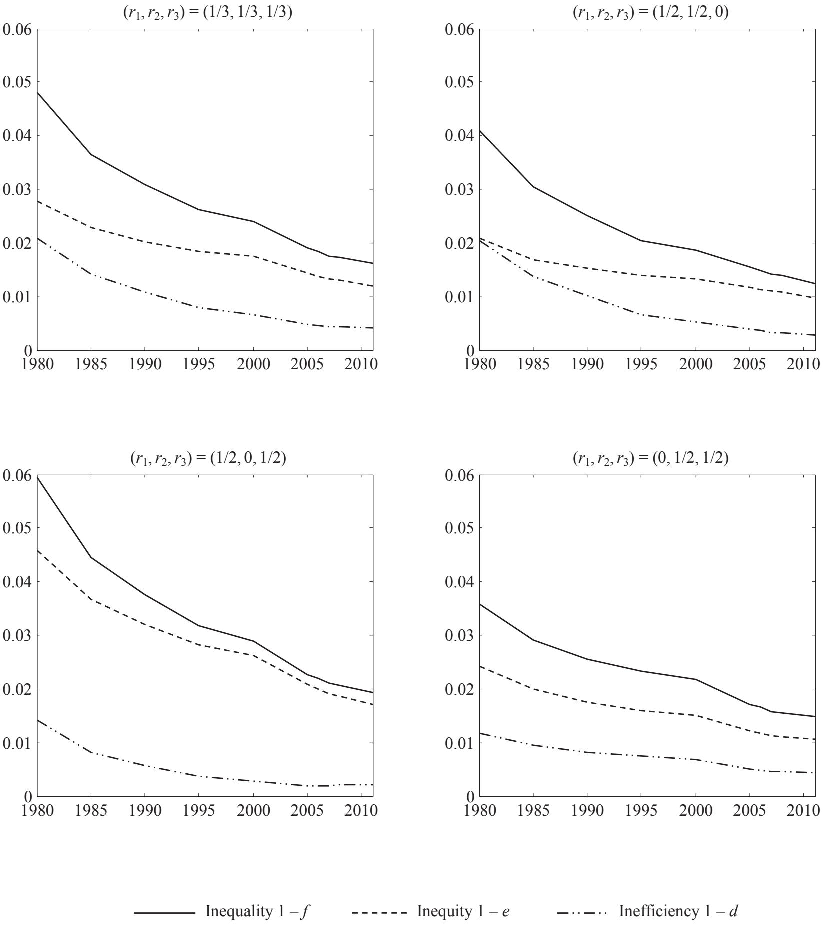

Figure 6. Inequality, inequity and inefficiency between 1980 and 2011 ($\alpha = 0.5$, $\beta = 1$)

the inequity and inefficiency components are roughly equal. Afterwards, inefficiency falls at a considerably faster rate than inequity. In the bottom left-hand panel, we leave out health. In this case inequality mainly consists of inequity. Finally, if living standards are left out, the inequity and inefficiency components both decrease mildly, with their relative importance remaining stable.

Table 1 presents the results for equal weights ($r_1 = r_2 = r_3 = 1/3$) and for a variety of combinations of the parameters α and β for the years

Table 1. Inequality, inefficiency and inequity in 1980 and 2011

		1980			2011		
α	β	Inequality	Inefficiency	Inequity	Inequality	Inefficiency	Inequity
0	0	0.00	0.00	0.00	0.00	0.00	0.00
	0.5	1.03	1.03	0.00	0.21	0.21	0.00
	1	2.09	2.09	0.00	0.42	0.42	0.00
	2	4.02	4.02	0.00	0.80	0.80	0.00
	5	7.26	7.26	0.00	1.59	1.59	0.00
0.5	0	2.22	0.00	2.22	1.08	0.00	1.08
	0.5	3.50	1.03	2.49	1.34	0.21	1.14
	1	4.81	2.09	2.78	1.61	0.42	1.20
	2	7.28	4.02	3.40	2.12	0.80	1.33
	5	11.66	7.26	4.74	3.20	1.59	1.63
1	0	4.34	0.00	4.34	2.18	0.00	2.18
	0.5	5.83	1.03	4.84	2.51	0.21	2.30
	1	7.38	2.09	5.40	2.84	0.42	2.44
	2	10.35	4.02	6.60	3.50	0.80	2.71
	5	15.76	7.26	9.16	4.92	1.59	3.38
2	0	8.21	0.00	8.21	4.48	0.00	4.48
	0.5	10.05	1.03	9.11	4.96	0.21	4.76
	1	12.00	2.09	10.12	5.46	0.42	5.06
	2	15.85	4.02	12.32	6.47	0.80	5.71
	5	23.04	7.26	17.01	8.86	1.59	7.39
5	0	17.31	0.00	17.31	12.48	0.00	12.48
	0.5	19.82	1.03	18.98	13.65	0.21	13.47
	1	22.78	2.09	21.13	15.01	0.42	14.65
	2	30.10	4.02	27.18	18.24	0.80	17.58
	5	45.23	7.26	40.94	27.32	1.59	26.15

1980 and 2011. For all considered values of the parameters, inequality, inefficiency and inequity decrease. The relative drop in efficiency is more pronounced than the relative drop in inequity across different parameter values (except of course for $\beta = 0$, in which case efficiency is zero). As β increases, the relative decrease in inefficiency is stronger. As α increases, the share of inequity in inequality increases, a consequence of the fact that increasing α increases inequity, while leaving inefficiency unaffected. The relative decrease in inequity falls as α increases. While cases where inequality and inequity move in opposite directions do not occur in the table, such cases do occur in the data. For example, given $\beta = 0.33$ and a high degree of inequality aversion $\alpha = 10$, inequality falls between 1980 and 2005, while inequity rises. This serves as an illustration that one should be cautious to identify inequality with inequity in empirical research.

6 Conclusion

Normative indices of multidimensional inequality capture both the inequity and inefficiency exhibited by a distribution. Using a suggestion by Graaff (1977), we have provided a decomposition of normative inequality into its inequity and inefficiency parts for a generic class of social welfare functions.

Our analysis has revealed an intimate link between the normative approach and the two-stage approach. This is striking, as the literature regards these two approaches as inherently different. While the two-stage approach in its original conception by Maasoumi (1986, 1999) does not rest on axiomatic foundations, we have shown that it nevertheless has a solid justification within the axiomatic normative approach. A plausible view on inequality measurement may be that it should not be concerned with inefficiency at all, making the measurement of inequity the only objective. If this view is taken, then, as our analysis has shown, two-stage inequality measures are the appropriate measures to use.

The decomposition has also yielded new insights into the two main multidimensional transfer axioms, viz., uniform majorization and correlation increasing majorization. These axioms recommend transfers that have both equity and efficiency effects, making the overall implication of these transfers difficult to evaluate a priori. In particular, if again the view is taken that the objective is to measure inequity and not inefficiency, then uniform majorization is clearly an unappealing requirement, as it in some cases recommends transfers that unambiguously increase inequity, an intuition already apparent in Dardanoni (1995).

We end with two ways forward. First, the decomposition is useful to study the impact of real-world phenomena, such as globalization and redistributive policies, on inequality and welfare. Indeed, such phenomena are likely to have a different effect on the inequity and inefficiency components. Second, our analysis has started from the restrictive, but standard, assumption of a common individual well-being function. A challenging question is how to proceed if the dimensions of well-being are weighted differently across individuals. As recently stressed by Fleurbaey and Maniquet (2011), multidimensional transfer axioms in this setting severely restrict the possibilities for social welfare comparisons. Again, these transfer axioms mix equity and an efficiency. An axiom that focuses on transfers that improve equity, but preserve efficiency, may offer new possibilities (Bosmans, Decancq and Ooghe, 2014). We leave both issues for further research.

References

Aaberge R, Brandolini A, 2014. Multidimensional poverty and inequality. *Temi di discussione* (*Economic working papers*) 976, Bank of Italy, Economic Research and International Relations Area. Forthcoming in: AB Atkinson, F Bourguignon (eds) *Handbook of Income Distribution*, vol. 2A. Amsterdam: Elsevier.

Almås, 2012. International income inequality: measuring PPP bias by estimating Engel curves for food. *American Economic Review* **102**, 1093-1117.

Atkinson AB, 1970. On the measurement of inequality. *Journal of Economic Theory* **2**, 244-263.

Atkinson AB, 2003. Multidimensional deprivation: contrasting social welfare and counting approaches. *Journal of Economic Inequality* **1**, 51-65.

Atkinson AB, Bourguignon F, 1982. The comparison of multidimensioned distributions of economic status. *Review of Economic Studies* **49**, 183-201.

Becker G, Philipson T, Soares R, 2005. The quantity and quality of life and the evolution of world inequality. *American Economic Review* **95**, 277-291.

Bosmans K, Decancq K, Ooghe E, 2014. Who's afraid of aggregating money metrics? *Mimeo.*

Bosmans K, Lauwers L, Ooghe E, 2009. A consistent multidimensional Pigou-Dalton transfer principle. *Journal of Economic Theory* **144**, 1358-1371.

Bourguignon F, 1999. Comment on 'Multidimensioned approaches to welfare analysis' by E. Maasoumi. In: Silber J (ed) *Handbook of Income Inequality Measurement.* London: Kluwer Academic.

Bourguignon F, Chakravarty SR, 2003. The measurement of multidimensional poverty *Journal of Economic Inequality* **1**, 25-49.

Bourguignon F, Morrisson C, 2002. Inequality among world citizens: 1820-1992. *American Economic Review* **92**, 727-744.

Bowlus A, Robin J-M, 2004. Twenty years of rising inequality in U.S. lifetime labour income values. *Review of Economic Studies* **71**, 709-742.

Chakravarty SR, 2009. *Inequality, Polarization and Poverty.* Heidelberg: Springer.

Dalton H, 1920. The measurement of the inequality of incomes. *Economic Journal* **30**, 348-361.

Dardanoni V, 1995. On multidimensional inequality measurement. In: Dagum C, Lemmi A (eds) *Income Distribution, Social Welfare, Inequality and Poverty* (*Research on Economic Inequality*, vol. 6). Stamford, CT: JAI Press.

Deaton A, 2010. Price Indexes, inequality, and the measurement of world poverty. *American Economic Review* **100**, 5-34.

Debreu G, 1951. The coefficient of resource utilization. *Econometrica* **19**, 273-292.

Debreu G, 1976. Least concave utility functions. *Journal of Mathematical Economics* **3**, 121-129.

Decancq K, Decoster A, Schokkaert E, 2009. The evolution of world inequality in well-being. *World Development* **37**, 11-25.

Decancq K, Ooghe E, 2010. Has the world moved forward? A robust multidimensional evaluation. *Economics Letters* **107**, 266-269.

Donaldson D, Weymark JA, 1980. A single-parameter generalization of the Gini indices of inequality. *Journal of Economic Theory* **22**, 67-86.

Donaldson D, Weymark JA, 1983. Ethically flexible Gini indices for income distributions in the continuum. *Journal of Economic Theory* **29**, 353-358.

Duclos J-Y, Sahn DE, Younger SD, 2011. Partial multidimensional inequality orderings. *Journal of Public Economics* **95**, 225-238.

Fleurbaey M, 2006. Social welfare, priority to the worst-off and the dimensions of individual well-being. In: Farina F, Savaglio E (eds) *Inequality and Economic Integration*. London: Routledge.

Fleurbaey M, Maniquet F, 2011. *A Theory of Fairness and Social Welfare*. Cambridge: Cambridge University Press.

Flinn C, 2002. Labour market structure and inequality: a comparison of Italy and the U.S. *Review of Economic Studies* **69**, 611-645.

Gajdos T, Weymark JA, 2005. Multidimensional generalized Gini indices. *Economic Theory* **26**, 471-496.

Graaff J de V, 1977. Equity and efficiency as components of the general welfare. *South African Journal of Economics* **45**, 362-375.

Huggett M, Ventura G, Yaron A, 2011. Sources of lifetime inequality. *American Economic Review* **101**, 2923-2954.

Jehle A, Reny PJ, 2011. Advanced Microeconomic Theory, third edition. Harlow: Pearson Education Limited.

Kannai Y, 1980. The ALEP definition of complementarity and least concave utility functions. *Journal of Economic Theory* **22**, 115-117.

Kihlstrom RE, Mirman LJ, 1974. Risk aversion with many commodities. *Journal of Economic Theory* **8**, 361-368.

Kihlstrom RE, Mirman LJ, 1981. Constant, increasing and decreasing risk aversion with many commodities *Review of Economic Studies* **48**, 271-280.

Kolm S-Ch, 1969. The optimal production of social justice. In: Margolis J, Guitton H (eds) *Public Economics*. London: Macmillan.

Kolm S-Ch, 1977. Multidimensional egalitarianisms. *Quarterly Journal of Economics* **91**, 1-13.

Lasso de la Vega MC, Urrutia A, 2011. Characterizing how to aggregate the individuals deprivations in a multidimensional framework. *Journal of Economic Inequality* **9**, 183-194.

Lise J, Seitz S, 2011. Consumption inequality and intra-household allocations. *Review of Economic Studies* 78, 328-355.

Lugo MA, 2007. Comparing multidimensional indices of inequality: methods and application, In: Bishop J, Amiel Y (eds) *Inequality and Poverty* (*Research on Economic Inequality*, vol. 14). Emerald Group Publishing Limited.

Maasoumi E, 1986. The measurement and decomposition of multi-dimensional inequality. *Econometrica* **54**, 991-998.

Maasoumi E, 1999. Multidimensioned approaches to welfare analysis. In: Silber J (ed) *Handbook of Income Inequality Measurement*. London: Kluwer Academic.

Moretti E, 2013. Real wage inequality. *American Economic Journal: Applied Economics* **5**, 65-103.

Noorbakhsh F, 2007. International convergence or higher inequality in human development? Evidence for 1975-2002. In: Mavrotas G, Shorrocks T (eds) *Advancing Development: Core Themes in Global Economics*. Basingstoke, New York: Palgrave Macmillan.

Pendakur K, 2002. Taking prices seriously in the measurement of inequality. *Journal of Public Economics* **86**, 47-69.

Ravallion M, 2012. Troubling tradeoffs in the Human Development Index. *Journal of Development Economics* **99**, 201-209.

Sen AK, 1973. *On Economic Inequality*. Oxford: Clarendon Press.

Seth S, 2013. A class of distribution and association sensitive multidimensional welfare indices. *Journal of Economic Inequality* **11**, 133-162.

Slesnick D, 2001. *Consumption and Social Welfare: Living Standards and their Distribution in the United States*. Cambridge: Cambridge University Press.

Stiglitz JE, Sen AK, Fitoussi J-P, 2009. *Report by the Commission on the Measurement of Economic Performance and Social Progress*. Mimeo.

Trannoy A, 2006. Multidimensional egalitarianism and the dominance approach: a lost paradise? In: Farina F, Savaglio E (eds) *Inequality and Economic Integration*. London: Routledge.

Tsui K-Y, 1995. Multidimensional generalizations of the relative and absolute inequality indices: the Atkinson-Kolm-Sen approach. *Journal of Economic Theory* **67**, 251-265.

Tsui K-Y, 1999. Multidimensional inequality and multidimensional generalized entropy measures: an axiomatic derivation. *Social Choice and Welfare* **16**, 145-157.

United Nations Development Programme, 2010. *Human Development Report 2010*. New York: Oxford University Press.

Weymark JA, 2006. The normative approach to the measurement of multidimensional inequality. In: Farina F, Savaglio E (eds) *Inequality and Economic Integration*. London: Routledge.

Yang L, 2014. An inventory of composite measures of human progress. *Occasional Paper on Methodology*, UNDP Human Development Report Office.

Appendix A

We provide a characterization of the class of social welfare functions $\mathcal{W}$. First, we formally define the seven axioms presented in Section 2. We write $X > Y$ if $x_{ij} \geq y_{ij}$ for all individuals i and all goods k with at least one inequality holding strictly. We use $1_{n\times m}$ to denote the $n \times m$ matrix with a one at each entry.

(i) *Anonymity.* For all distributions X and Y in D, if X and Y are equal up to a rearrangement of rows, then $W(X) = W(Y)$.

(ii) *Monotonicity.* For all distributions X and Y in D, if $X > Y$, then $W(X) > W(Y)$.

(iii) *Continuity.* The function W is continuous.

(iv) *Weak uniform majorization.* For all distributions X and Y in D, if $X = QY$ with Q a bistochastic matrix that is not a permutation matrix, then $W(X) \geq W(Y)$.

(v) *Normalization.* For each real number λ, we have $W(\lambda 1_{n\times m}) = \lambda$.

(vi) *Individualism.* There exist a function $u_i : B \to \mathbb{R}$ for each individual i and a function $v : \mathbb{R}^n \to \mathbb{R}$ such that, for each distribution X in D, we have $W(X) = v(u_1(x_1), u_2(x_2), \ldots, u_n(x_n))$.

(vii) *Homotheticity.* For all distributions X and Y in D and each positive real number λ, we have $W(X) \geq W(Y)$ if and only if $W(\lambda X) \geq W(\lambda Y)$.

We now prove the following result.

A social welfare function W satisfies axioms (i) *to* (vii) *if and only if there exists a continuous, concave, linearly homogenous and strictly increasing function $U : B \to \mathbb{R}$ with $U(1_m) = 1$ and a continuous, Schur-concave, linearly homogenous and strictly increasing function $V : \mathbb{R}^n \to \mathbb{R}$ with $V(1_n) = 1$ such that, for each distribution X in D, we have $W(X) = V(U(x_1), U(x_2), \ldots, U(x_n))$.*

Proof. It is easy to verify that the specified social welfare function satisfies axioms (i) to (vii). We therefore focus on the reverse implication.

Let W be a social welfare function that satisfies axioms (i) to (vii). By anonymity, monotonicity, continuity and individualism, there exist a continuous and strictly increasing function $\hat{U} : B \to \mathbb{R}$ and a continuous and strictly increasing function $\hat{V} : \mathbb{R}^n \to \mathbb{R}$ such that, for each distribution X in D, we have $W(X) = \hat{V}(\hat{U}(x_1), \hat{U}(x_2), \ldots, \hat{U}(x_n))$.

Homotheticity implies that, for all bundles x and y in B and each positive real number λ, we have $W((x, x, \ldots, x)') \geq W((y, y, \ldots, y)')$ if and only if $W((\lambda x, \lambda x, \ldots, \lambda x)') \geq W((\lambda y, \lambda y, \ldots, \lambda y)')$. It follows that, for all bundles x and y in B and each positive real number λ, we have $\hat{U}(x) \geq \hat{U}(y)$ if and only if $\hat{U}(\lambda x) \geq \hat{U}(\lambda y)$, i.e., $\hat{U}$ is a homothetic function. Hence, there exists a linearly homogenous function $U : \mathbb{R}^n \to \mathbb{R}$ with $U(1_m) = 1$ and a strictly increasing function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\hat{U} = \psi \circ U$. Let the function $V : \mathbb{R}^n \to \mathbb{R}$ be such that, for each $(t_1, t_2, \ldots, t_n)$ in $\mathbb{R}^n$, we have $V(t_1, t_2, \ldots, t_n) = \hat{V}(\psi(t_1), \psi(t_2), \ldots, \psi(t_n))$. It follows that, for each distribution X in D, we have $W(X) = V(U(x_1), U(x_2), \ldots, U(x_n))$. The functions U and V inherit continuity and strict increasingness from the functions $\hat{U}$ and $\hat{V}$. Furthermore, by normalization, V is linearly homogenous and $V(1_n) = 1$. What remains to be shown is that U is concave and that V is Schur-concave.

We show that U is concave. We first show that U is quasi-concave. Seeking a contradiction, suppose that U is not quasi-concave, i.e., there exist two bundles a and b in B such that $U(0.5a + 0.5b) < U(a) = U(b)$. Let Q be an $n \times n$ matrix, where each row $k \neq i, j$ has 1 at the kth entry and 0 at each other entry, and where rows i and j have 0.5 at the ith entry, 0.5 at the jth entry and 0 at each other entry. Let X be a distribution in D with bundle a at the ith row and bundle b at the jth row. Distribution QX has bundle $0.5a+0.5b$ at both the ith and the jth rows and coincides with X on all other rows. Since $U(0.5a+0.5b) < U(a) = U(b)$, we have $W(X) > W(QX)$. But weak uniform majorization implies $W(X) \leq W(QX)$ because Q is a bistochastic matrix that is not a permutation matrix. We have a contradiction and hence U is quasi-concave. Because U is continuous, strictly increasing, linearly homogeneous, and quasi-concave, U must also be concave (see, e.g., Jehle and Reny, 2011, Theorem 3.1).

We show that V is Schur-concave. Jointly, anonymity and weak uniform majorization imply that, for each $(t_1, t_2, \ldots, t_n)$ in $\mathbb{R}^n$ and each bistochastic matrix Q, we have $W((t_1 1_m, t_2 1_m, \ldots, t_n 1_m)') \leq W(Q(t_1 1_m, t_2 1_m, \ldots, t_n 1_m)')$. Using that U is linearly homogenous function and $u(1_m) = 1$, it follows that, for each $(t_1, t_2, \ldots, t_n)$ in $\mathbb{R}^n$ and each bistochastic matrix Q, we have $V(t_1, t_2, \ldots, t_n) \leq V(Q(t_1, t_2, \ldots, t_n)')$. Hence, V is Schur-concave. $\square$

Appendix B

We provide a concise treatment of the analysis of Section 3 for the absolute case. We omit all explanations that are similar to those in Section 3. The only change we make to the basic assumptions in Section 2 is that good amounts can be zero or negative in addition to positive. That is, the set of all bundles is $B = \mathbb{R}^m$.

We focus on the class of social welfare functions $\bar{\mathcal{W}}$. A social welfare function W is in $\bar{\mathcal{W}}$ if and only if there exist a continuous, concave, unit-translatable[35] and strictly increasing function $U : B \to \mathbb{R}$ with $U(1_m) = 1$ and a continuous, Schur-concave, unit-translatable and strictly increasing function $V : \mathbb{R}^n \to \mathbb{R}$ with $V(1_n) = 1$ such that, for each distribution X in D, we have $W(X) = V(U(x_1), U(x_2), \dots, U(x_n))$.

The class $\bar{\mathcal{W}}$ is characterized by the combination of axioms (i) to (vi) in Appendix A and the following axiom of translatability (we omit the proof).

Translatability. For all distributions X and Y in D and each positive real number λ, we have $W(X) \geq W(Y)$ if and only if $W(X + \lambda 1_{n\times m}) \geq W(Y + \lambda 1_{n\times m})$.

The definitions of inequality and the inefficiency and inequity components for the absolute case are as follows.

*Definition B.*1. Let X be a distribution in D. Let $\bar{f}(X)$ be the largest real number for which there exists a distribution X^* in D such that $W(X^*) = W(X)$ and $\mu_{X^*} = \mu_X - \bar{f}(X)1_m$. Then, $\bar{f}(X)$ is the inequality level of distribution X.

*Definition B.*2. Let X be a distribution in D. Let $\bar{d}(X)$ be the largest real number for which there exists a distribution X' in D such that $U(x'_i) = U(x_i)$ for each individual i and $\mu_{X'} = \mu_X - \bar{d}(X)1_m$. Then, $\bar{d}(X)$ is the inefficiency level of distribution X.

*Definition B.*3. Let X be a distribution in D. Let X' be a distribution in D obtained from distribution X as in Definition $B.2$. Let $e(X)$ be the largest real number for which there exists a distribution X'' such that $W(X'') = W(X')$ and $\mu_{X''} = \mu_{X'} - \bar{e}(X)1_m$. Then, $\bar{e}(X)$ is the inequity level of distribution X.

We obtain an additive decomposition of inequality into inefficiency and inequity parts. That is, for each distribution X in D, we have

$$\bar{f}(X) \;=\; \bar{d}(X) \;+\; \bar{e}(X).$$

[35] A function $\psi : \mathbb{R}^\ell \to \mathbb{R}$ is unit translatable if $\psi(t + \lambda 1_\ell) = \lambda + \psi(t)$ for each t in $\mathbb{R}^\ell$ and each real number λ.

For a distribution X in D, inequality $\bar{f}(X)$ is defined by $W(X) = W(X_\mu - \bar{f}(X)1_{n\times m})$. Using unit translatability of W, we obtain

$$\bar{f}(X) \;=\; W(X_\mu) - W(X).$$

For each individual i, define t_i by $U(x_i) = U(\mu_X - t_i 1_m)$. The efficient distribution X' equals $(\mu_X - t_1 1_m, \mu_X - t_2 1_m, \ldots, \mu_X - t_n 1_m)'$ and therefore $\mu_{X'} = \mu_X - (\sum_{i=1}^n t_i/n)1_m$. By Definition B.2, we have $\bar{d}(X)1_m = \mu_X - \mu_{X'} = (\sum_{i=1}^n t_i/n)1_m$. Hence, $\bar{d}(X) = \sum_{i=1}^n t_i/n$. Moreover, $t_i = U(\mu_X) - U(x_i)$ because U is unit translatable. We obtain

$$\bar{d}(X) \;=\; U(\mu_X) - \frac{1}{n}\sum_{i=1}^n U(x_i).$$

Finally, using $\bar{e} = \bar{f} - \bar{d}$ and $U(\mu_X) = W(X_\mu)$, we obtain

$$\bar{e}(X) \;=\; \frac{1}{n}\sum_{i=1}^n U(x_i) - W(X).$$

Appendix C

Table 2. Countries in the data set

Afghanistan	Germany	Nicaragua
Algeria	Ghana	Niger
Argentina	Greece	Norway
Australia	Guatemala	Pakistan
Austria	Guyana	Panamá
Bahrain	Haiti	Papua New Guinea
Bangladesh	Honduras	Paraguay
Belgium	Hong Kong	Peru
Belize	Hungary	Philippines
Benin	Iceland	Portugal
Bolivia	India	Qatar
Botswana	Indonesia	Republic of Korea
Brazil	Iran	Rwanda
Brunei Darussalam	Ireland	Saudi Arabia
Burundi	Israel	Senegal
Cameroon	Italy	Sierra Leone
Canada	Jamaica	South Africa
Central African Republic	Japan	Spain
Chile	Jordan	Sri Lanka
China	Kenya	Sudan
Colombia	Kuwait	Sweden
Congo	Lesotho	Switzerland
Costa Rica	Luxembourg	Syria
Côte d'Ivoire	Malawi	Thailand
Democratic Republic of the Congo	Malaysia	Togo
Denmark	Mali	Trinidad and Tobago
Dominican Republic	Malta	Tunisia
Ecuador	Mauritania	Turkey
Egypt	Mauritius	United Arab Emirates
El Salvador	Mexico	United Kingdom
Fiji	Morocco	United States
Finland	Myanmar	Uruguay
France	Nepal	Venezuela
Gabon	Netherlands	Zambia
Gambia	New Zealand	Zimbabwe