

Sheaves and Localization

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In this note, we calculate projective limits of localization functors. We relate the results, thus obtained, to the construction of structure sheaves for noncommutative rings. © 1996 Academic Press, Inc.

INTRODUCTION

Whereas in [1, 5, 9], the authors used compatibility results between torsion and localization functors in order to construct structure sheaves for suitable topologies on the spectrum of a noncommutative ring, here we do exactly the opposite, i.e., we investigate what happens in the case of non-compatibility. One of the main results in this direction states that two idempotent kernel functors σ and τ in $R\text{-mod}$, the category of left R -modules, are compatible, i.e., $\sigma Q_\tau = Q_\tau \sigma$ (cf. [3, 4, 7] for notations and definitions) if and only if the canonical sequence of functors

$$0 \rightarrow Q_{\sigma \wedge \tau} \rightarrow Q_\sigma \oplus Q_\tau \rightarrow Q_{\sigma \vee \tau}$$

is exact. This is the torsion-theoretic analogue of the well-known fact that for any quasi-coherent sheaf \mathcal{E} on an algebraic variety X and any open subsets U, V , and W of X , there is an exact sequence

$$0 \rightarrow \mathcal{E}((U \cup V) \cap W) \rightarrow \mathcal{E}(U \cap W) \oplus \mathcal{E}(V \cap W) \rightarrow \mathcal{E}(U \cap V \cap W).$$

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In general, if σ and τ are not necessarily compatible, it appears that one may still show that the sequence (with obvious morphisms)

$$0 \rightarrow Q_{\sigma \wedge \tau} \rightarrow Q_{\sigma} \oplus Q_{\tau} \rightarrow Q_{\sigma} Q_{\tau} \oplus Q_{\tau} Q_{\sigma}$$

is exact. Actually, we will prove that for any finite family $\{\sigma_{\alpha}; \alpha \in A\}$ of idempotent kernel functors over R , the localization functor $Q_{\wedge \sigma_{\alpha}}$ is the projective limit of the family

$$\{Q_{\sigma_{\alpha}} \rightarrow Q_{\sigma_{\alpha}} Q_{\sigma_{\beta}}, Q_{\sigma_{\beta}} \rightarrow Q_{\sigma_{\alpha}} Q_{\sigma_{\beta}}; \alpha, \beta \in A\}.$$

This result is then strengthened to encompass projective limits of “localization functors” with respect to words in the free monoid over the set of all Gabriel filters for R .

1. PROJECTIVE LIMITS OF LOCALIZATION FUNCTORS

(1.1) As stated in the Introduction, two idempotent kernel functors σ and τ are mutually compatible if and only if the following sequence of functors is exact

$$0 \rightarrow Q_{\sigma \wedge \tau} \rightarrow Q_{\sigma} \oplus Q_{\tau} \rightarrow Q_{\sigma \vee \tau}.$$

Of course, the exactness of the latter sequence may also be formulated by saying that $Q_{\sigma \wedge \tau}$ is the limit (equalizer) of the projective system

$$\{Q_{\sigma} \rightarrow Q_{\sigma \vee \tau}, Q_{\tau} \rightarrow Q_{\sigma \vee \tau}\}.$$

In this section, we will take a look at this type of projective limits, when σ and τ are no longer assumed to be compatible.

Actually, one may prove:

(1.2) PROPOSITION. *Let $K = \{\sigma_{\alpha}; 1 \leq \alpha \leq n\}$ be a finite family of idempotent kernel functors over R and put $\sigma = \wedge_{\alpha} \sigma_{\alpha}$. Let M be a left R -module and denote by $j_{M, \alpha} : M \rightarrow Q_{\alpha} M$ the localization map with respect to σ_{α} . If $\mathcal{Q}_K M$ denotes the projective system*

$$\{j_{\alpha\beta}^{\alpha} : Q_{\alpha} M \rightarrow Q_{\alpha} Q_{\beta} M, j_{\alpha\beta}^{\beta} : Q_{\beta} M \rightarrow Q_{\alpha} Q_{\beta} M; 1 \leq \alpha, \beta \leq n\},$$

where $j_{\alpha\beta}^{\alpha} = Q_{\alpha} j_{M, \beta}$, and $j_{\alpha\beta}^{\beta} = j_{Q_{\beta} M, \alpha}$, then $\varprojlim \mathcal{Q}_K M = Q_{\sigma} M$.

Proof. We write $Q_{\alpha\beta} = Q_\alpha Q_\beta$ resp. $Q_{\alpha\beta\gamma} = Q_\alpha Q_\beta Q_\gamma$. One easily reduces to $n = 2$, and in this case, one has to show that the sequence

$$0 \rightarrow Q_\sigma M \xrightarrow{i} Q_1 M \oplus Q_2 M \xrightarrow{\pi} Q_{12} M \oplus Q_{21} M$$

is exact, where i is the canonical map and where for any $(m_1, m_2) \in Q_1 M \oplus Q_2 M$, we put

$$\pi(m_1, m_2) = (j_{12}^1(m_1) - j_{12}^2(m_2), j_{21}^1(m_1) - j_{21}^2(m_2)).$$

Let

$$K_{\alpha\beta} = \text{Ker}(\pi_{\alpha\beta} = j_{\alpha\beta}^1 - j_{\alpha\beta}^2 : Q_1 M \oplus Q_2 M \rightarrow Q_{\alpha\beta} M).$$

To prove the assertion, we have to verify that $K = \text{Ker}(\pi) = K_{12} \cap K_{21}$ coincides with $Q_\sigma M$ within $Q_1 M \oplus Q_2 M$. Since $Q_\sigma M \subseteq K_{12} \cap K_{21} = K$ and all of these are σ -closed, it suffices to show that $K_{12} \cap K_{21}/M$ is σ -torsion.

Localizing at σ yields exact sequences

$$0 \rightarrow Q_1 K_{12} \rightarrow Q_1 M \oplus Q_{12} M \rightarrow Q_{12} M,$$

resp.

$$0 \rightarrow Q_1 K_{21} \rightarrow Q_1 M \oplus Q_{12} M \rightarrow Q_{121} M.$$

Since (y, z) belongs to $Q_1 K_{12}$ resp. $Q_1 K_{21}$ if and only if $j_{12}^1(y) = z$ resp. $j_{12}^1(y) - z \in \text{Ker}(Q_{12} M \rightarrow Q_{121} M)$, it follows that

$$Q_1(K_{12} \cap K_{21}) = Q_1 K_{12} \cap Q_1 K_{21} = Q_1 K_{12} = Q_1 M.$$

Similarly, $Q_1(K_{12} \cap K_{21}) = Q_2 M$, so, as $\sigma = \sigma_1 \wedge \sigma_2$, we find that $K_{12} \cap K_{21}/M$ is σ -torsion, indeed. ■

(1.3) If σ and τ are mutually compatible, then $Q_\sigma Q_\tau$ is $\sigma \vee \tau$ -torsion free, so, in particular, the canonical map $Q_\sigma Q_\tau \rightarrow Q_{\sigma \vee \tau}$ is injective. Applying this to a family $\{\sigma_\alpha; 1 \leq \alpha \leq n\}$ of mutually compatible idempotent kernel functors over R , it then follows from the previous result that there is an exact sequence

$$0 \rightarrow Q_\sigma M \rightarrow \bigoplus_{\alpha} Q_\alpha M \rightarrow \bigoplus_{\alpha, \beta} Q_{\alpha \vee \beta} M,$$

where $Q_{\alpha \vee \beta}$ is the localization functor at $\sigma_\alpha \vee \sigma_\beta$.

We thus recover the results in [1, 5, 6, 10], to which we refer for the implications of the exactness of this sequence upon the construction of structure sheaves.

2. A GENERALIZATION

(2.1) Denote by \mathbb{G} the free semigroup generated by all Gabriel filters over R , i.e., elements in \mathbb{G} are words $\mathbf{L} = \mathcal{L}_1 \dots \mathcal{L}_n$, where all the \mathcal{L}_i are Gabriel filters. Recall from [2] that two Gabriel filters \mathcal{L} and \mathcal{H} yield a new filter $\mathcal{L} \circ \mathcal{H}$, which consists of all left ideals L of R , with the property that there exists some $H \in \mathcal{H}$ containing L and such that $(L : U) \in \mathcal{L}$ for some finite subset U of H . In general, $\mathcal{L} \circ \mathcal{H}$ is not a Gabriel filter anymore; however, since it is *uniform* (in the sense of [2]), torsion with respect to it may still be defined. We will denote by $\epsilon(\mathbf{L})$ the uniform filter $\mathcal{L}_1 \circ \dots \circ \mathcal{L}_n$ and by $\sigma_{\mathbf{L}}$ the associated torsion functor $\sigma_{\epsilon(\mathbf{L})}$. We write $Q_{\mathbf{L}}$ for the composition $Q_{\mathcal{L}_n} \dots Q_{\mathcal{L}_1}$.

Let us call a left R -module \mathbf{L} -torsion, if it is $\epsilon(\mathbf{L})$ -torsion. Then one may show, by induction on the length of $\mathbf{L} = \mathcal{L}_1 \dots \mathcal{L}_n \in \mathbb{G}$, that for any $M \in R\text{-mod}$ with associated canonical morphism $j_{\mathbf{L}} : M \rightarrow Q_{\mathbf{L}}(M)$, we have that $\text{Ker}(j_{\mathbf{L}}) = \sigma_{\mathbf{L}}M$ and that $\text{Coker}(j_{\mathbf{L}})$ is \mathbf{L} -torsion.

(2.2) Let $K = \{\mathbf{L}_{\alpha}; 1 \leq \alpha \leq n\}$ be a finite family of words in \mathbb{G} . Put $Q_{\mathbf{L}_{\alpha}} = Q_{\alpha}$ and consider the projective system

$$\mathcal{C}_K M = \{j_{\alpha\beta}^{\alpha} : Q_{\alpha} M \rightarrow Q_{\alpha} Q_{\beta} M, j_{\alpha\beta}^{\beta} : Q_{\beta} M \rightarrow Q_{\alpha} Q_{\beta} M; 1 \leq \alpha, \beta \leq n\},$$

where the maps are the obvious ones. Let $\mathcal{L} = \bigcap_{\alpha=1}^n \epsilon(\mathbf{L}_{\alpha})$, then the previous remarks imply that

$$\sigma_{\mathcal{L}} M \subseteq \epsilon(\mathbf{L}_{\alpha}) M = \text{Ker}(M \rightarrow Q_{\alpha} M)$$

for all $1 \leq \alpha \leq n$. The canonical map $M \rightarrow \varprojlim \mathcal{C}_K M$ thus factorizes through

$$j : M / \sigma_{\mathcal{L}} M \rightarrow \varprojlim \mathcal{C}_K M.$$

We then have:

(2.3) PROPOSITION. *With the previous notations, $\text{Ker}(j) = 0$ and the cokernel $\text{Coker}(j)$ is $\mathcal{L} \circ \mathcal{L}$ -torsion.*

Proof. Let $\sigma_{\alpha} = \sigma_{\epsilon(\mathbf{L}_{\alpha})}$ and consider the canonical map

$$j_{\alpha} : \bar{M} = M / \sigma_{\mathcal{L}} M \rightarrow M / \sigma_{\alpha} M \hookrightarrow Q_{\alpha} M.$$

Clearly, $\text{Ker}(j_{\alpha}) = \sigma_{\alpha} \bar{M}$, and from this, it easily follows that j is injective. On the other hand, assume $q = (q_{\alpha}) \in \bigoplus_{\alpha=1}^n Q_{\alpha} M$ belongs to $\varprojlim \mathcal{C}_K M$ and fix $1 \leq \beta \leq n$. Then, an easy induction argument (cf. [10] for details),

shows that for any index α , there exists some $L_\alpha \in \epsilon(\mathbf{L}_\alpha)$ such that

$$L_\alpha q_\beta \subseteq \text{Im}(M \rightarrow Q_\beta M) = \text{Im}(\bar{M} \rightarrow Q_\beta M).$$

Clearly, $L = \sum_{\alpha=1}^n L_\alpha \in \cap_{\alpha=1}^n \epsilon(\mathbf{L}_\alpha) = \mathcal{L}$ and $Lq_\beta \subseteq \text{Im}(\bar{M} \rightarrow Q_\beta M)$. Since the q_β are finite in number, we may find a single $L \in \mathcal{L}$, which works for all β , i.e., $Lq_\beta \subseteq \text{Im}(\bar{M} \rightarrow Q_\beta M)$ for all $1 \leq \beta \leq n$.

Let $l \in L$, then $lq_\alpha = j_\alpha(x_\alpha)$ for some $x_\alpha \in \bar{M}$. Fix β for a moment and, for each α , consider the commutative diagram

$$\begin{array}{ccc} \bar{M} & \xrightarrow{j_\beta} & Q_\beta M \\ j_\alpha \downarrow & & \downarrow j_{\beta\alpha}^\beta \\ Q_\alpha M & \xrightarrow{j_{\beta\alpha}^\alpha} & Q_\alpha Q_\beta M \end{array}$$

Since

$$j_{\beta\alpha}^\alpha j_\alpha(x_\beta) = j_{\beta\alpha}^\beta j_\beta(x_\beta) = l j_{\beta\alpha}^\beta(q_\beta) = l j_{\beta\alpha}^\alpha(q_\alpha) = j_{\beta\alpha}^\alpha j_\alpha(x_\alpha),$$

there exists some $J_\beta \in \epsilon(\mathbf{L}_\beta)$ with $J_\beta(j_\alpha(x_\beta) - j_\alpha(x_\alpha)) = 0$, for all $1 \leq \alpha \leq n$ and it easily follows that $J_\beta lq \subseteq j(\bar{M})$. We thus obtain that $Jlq \subseteq j(\bar{M})$, with $J = \sum_{\beta=1}^n J_\beta \in \cap_{\beta=1}^n \epsilon(\mathbf{L}_\beta) = \mathcal{L}$, so $Kq \subseteq j(\bar{M})$ for some $K \in \mathcal{L} \circ \mathcal{L}$, which proves the assertion. ■

As a consequence, let us mention:

(2.4) COROLLARY. *With the previous notations, assume that $\mathbf{L}_\alpha = \mathcal{L}_1^\alpha \dots \mathcal{L}_{n_\alpha}^\alpha$ satisfy the following properties:*

- (1) $\mathcal{L} = \cap_{\alpha=1}^n \epsilon(\mathbf{L}_\alpha)$ is a Gabriel filter;
- (2) $\mathcal{L} \subseteq \mathcal{L}_i^\alpha$ for all $1 \leq \alpha \leq n$ and $1 \leq i \leq n_\alpha$.

Then $\varprojlim \mathcal{Q}_K M = Q_{\mathcal{L}} M$.

Proof. Let $j: M/\sigma_{\mathcal{L}} M \rightarrow \varprojlim \mathcal{Q}_K M$, as before. Since \mathcal{L} is a Gabriel filter, we obtain an exact sequence

$$0 \rightarrow M \xrightarrow{j} \varprojlim \mathcal{Q}_K M \rightarrow T \rightarrow 0,$$

where T is \mathcal{L} -torsion. Now,

$$\varprojlim \mathcal{Q}_K M \subseteq \bigoplus_{\alpha=1}^n Q_\alpha M = \bigoplus_{\alpha=1}^n Q_{\mathcal{L}_{n_\alpha}^\alpha} \dots Q_{\mathcal{L}_1^\alpha} M.$$

As $\mathcal{L} \subseteq \mathcal{L}_i^\alpha$ for all $1 \leq \alpha \leq n$ and $1 \leq i \leq n_\alpha$, it follows that $Q_{\mathcal{L}}$ com-

modules with all of the $Q_{\mathcal{L}_1^\alpha}$, so $\varprojlim Q_K M$ is \mathcal{L} -closed. Applying $Q_{\mathcal{L}}$ to the above exact sequence thus easily yields the result. ■

Note that (1.2) may be recovered from (2.4) by assuming for each $1 \leq \alpha \leq n$ that \mathbf{L}_α is the Gabriel filter associated to some idempotent kernel functor σ_α over R . On the other hand, this result strengthens [8], where only Gabriel filters associated to Ore sets are considered.

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