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The Brauer Group of a Braided Monoidal Category

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INTRODUCTION

Representation theory of finite groups prompted the introduction of the Brauer group of a field and also of its Schur subgroups. The theory of quadratic forms and spaces introduced the $\mathbb{Z}/2\mathbb{Z}$-graded variant of the Brauer group together with the typical Clifford algebras and related representations; cf. the work of Wall [29]. The extension of the $\mathbb{Z}/2\mathbb{Z}$-graded theory to gradings by other cyclic groups and consequently to abelian groups, replacing the usual Clifford algebras by the so-called generalized Clifford algebras, is then natural from the algebraic point of view. However, extension to nonabelian groups did not seem to be possible, at least not by simple modifications of the abelian theory. At the same time, the generalization of the Brauer group of graded algebras obtained by Long [12, 13] made use of Hopf algebras but again (co-)commutativity conditions were present. The recent interest in quantum groups motivated the authors to introduce the Brauer group of a quantum group as the Brauer group of crossed module algebras (also called quantum Yang–Baxter module algebras) in [4, 5]. The use of the category of crossed modules originated in the case of group algebras; cf. Whitehead [30]. Module algebras (or coalgebras) of this crossed type were introduced by Radford in [24] and used by Majid in the description of modules over Drinfel'd quantum doubles.

The non-(co-)commutativity aspects of the theory concerning the Brauer group in terms of crossed module algebras do not have a counterpart in classical theory. Note that a first unifying categorical theory has been obtained by Pareigis in [23]; this theory deals with the Brauer group of a

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symmetric monoidal category. The restriction to a symmetric monoidal category, probably viewed as just a commutativity condition, does in fact restrict attention to trivial actions and coactions.

In order to formulate the construction of the Brauer group of a quantum group into a categorical framework, one has to extend the categorical construction to the situation of braided monoidal categories, recently considered by Majid [15, 18] and Joyal and Street [9]. Indeed, the fact that the category of crossed modules is a braided monoidal category is at the heart of [4, 5] so the consideration of the general braided case seems to be a natural extension. The Brauer group of a braided monoidal category generalizes precisely the notion of the Brauer−Long group to the categorical setting. This is most obvious when considering Hopf algebras over $\mathbb{C}$. It is clear that the Brauer group of such a Hopf algebra is still large and may even fail to be an abelian torsion group [28], where the usual Brauer group of $\mathbb{C}$ is trivial. The extra complexity apparent in the Brauer group of a Hopf algebra or a braided monoidal category is entirely created by the Hopf algebra action and coaction properties, resp. by the braiding properties. Concrete calculations may therefore be linked to problems of a more representation theoretic nature!

In Section 1, we recall some definitions concerning braided monoidal categories and the objects in them. The elementary algebras in a braided monoidal category $\mathscr{C}$ are studied in Section 2. Those algebras are still closely related to the Morita theory of $\mathscr{C}$ as defined by Pareigis [23]. This allows us to characterize Azumaya algebras in $\mathscr{C}$ and leads to the definition of the Brauer group of $\mathscr{C}$ which need not be an abelian group. The examples in Section 3 establish how all known Brauer groups derive from this unifying definition, for example, the Brauer group of a field, the Brauer group of a scheme, the Brauer−Wall group (see Example 3.9), the graded Brauer group (see Example 3.8), the Brauer group with respect to a bicharacter as defined by Childs, Garfinkel, and Orzech, the Brauer−Long group (see Example 3.11), and the Brauer group of a quantum group (see Example 3.12). Moreover, a general Rosenberg−Zelinsky exact sequence of an Azumaya algebra is presented (see Proposition 3.14).

In Section 4, we construct a second Brauer group $Br'(\mathscr{C})$ of separable algebras in a braided monoidal category $\mathscr{C}$. The main result of this section is Theorem 4.9. The approach to Theorem 4.9 is different from that of Pareigis even in the symmetric case where more conditions are required (see Examples 4.13). In order to stay close to the classic cases, one then has to demand that the braided monoidal category considered is cocomplete abelian and has a nice unit. There is a functorial monomorphism from the second Brauer group $Br'(\mathscr{C})$ to the (first) Brauer group $Br(\mathscr{C})$ of a braided monoidal category $\mathscr{C}$. The monomorphism becomes an isomorphism if the unit of $\mathscr{C}$ is a projective object in $\mathscr{C}$.

In order to know how far the Brauer group of a braided monoidal category is from the Brauer group of a quantum group, we have to study the behavior of the Brauer group of a braided monoidal category under base change. The Tannaka–Krein theorem will play an important role here (see Theorem 5.3).

1. PRELIMINARIES

Let us briefly recall the definition of a braided monoidal category. For full detail we refer to [9, 18]. A *braided monoidal category* is $(\mathscr{C}, \otimes, I, \phi)$ where $(\mathscr{C}, \otimes, I)$ is a monoidal category satisfying the coherence conditions [14], and ϕ, called a braiding, is a natural transformation between the two functors $\otimes$ and $\otimes^{\mathrm{op}}$ (with opposite product) from $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$. This is a collection of functorial isomorphisms $\phi_{V,W}: V \otimes W \to W \otimes V$ obeying two "hexagon" coherence identities. In our notation these are

$$\phi_{V\otimes W, Z} = \phi_{V, Z} \circ \phi_{W, Z}, \qquad \phi_{V, W\otimes Z} = \phi_{V, Z} \circ \phi_{V, W}. \tag{1}$$

The associativity morphism $V \otimes (W \otimes Z) \cong (V \otimes W) \otimes Z$ is suppressed in (1). One easily deduces identities of the following type:

$$\phi_{V, I} = \mathrm{id}_V = \phi_{I, V}. \tag{2}$$

Then morphisms associated with I are suppressed in (2). If $\phi^2 = \mathrm{id}$, then one of the hexagons is superfluous and we have an ordinary symmetric monoidal category.

Let $\mathscr{C}$ be a braided monoidal category. If $P \in \mathscr{C}$, denote by $P(X)$ the set $\mathscr{C}(X, P)$ for $X \in \mathscr{C}$. If the functor $\mathscr{C}(- \otimes P, Q)$ is representable, the representing object is denoted by $[P, Q]$. If, for any object $Q \in C$, $[P, Q]$ exists, then the tensor functor $- \otimes P$ has a right adjoint functor $[P, -]$, that is, $\forall X, P \in \mathscr{C}$,

$$\mathscr{C}(X \otimes P, Q) \cong \mathscr{C}(X, [P, Q]).$$

Similarly, if the functor $\mathscr{C}(P \otimes -, Q)$ is representable, then the representing object is denoted by $\{P, Q\}$. Now the evaluation map

$$[P, Q](X) \times P(Y) \to Q(X \otimes Y), \qquad (f, p) \mapsto f\langle p \rangle,$$

is induced by the composition of morphisms; here $[P, Q]$ operates on P from the left side. We say that P is a *finite object* of $\mathscr{C}$ if $[P, P]$ and $[P, I]$ exist and the canonical morphism

$$P \otimes [P, I] \to [P, P]$$

induced by

$$P(X) \otimes P(Y) \otimes [P, I](Z) \to P(X \otimes Y \otimes Z)$$

is an isomorphism. This is equivalent to the existence of a "dual basis" $p_0 \otimes f_0 \in P \otimes [P, I](I)$ such that $p_0 f_0 \langle p \rangle = p$ for all $p \in P(X)$ and $X \in \mathscr{C}$. Moreover, P is called *faithfully projective* if the morphism

$$[P, I] \otimes_{[P, P]} P \to I,$$

induced by the evaluation, is an isomorphism. That is to say, there exists an element $f_1 \otimes_{[P, P]} p_1 \in ([P, I] \otimes_{[P, P]} P)(I)$ with $f_1 \langle p_1 \rangle = \mathrm{id} \in I(I)$. P is said to be a progenerator if P is finite and there is an element $f_1 \otimes p_1 \in ([P, I] \otimes P)(I)$ such that $f_1 \langle p_1 \rangle = \mathrm{id} \in I(I)$. If I is projective in $\mathscr{C}$, then P is faithfully projective if and only if P is a progenerator; cf. [22].

A monoidal category $\mathscr{C}$ is *rigid* if every object $V \in \mathscr{C}$ is finite. Denote by V^* the object $[P, I]$ and by ev_V the evaluation map: $V^* \otimes V \to I$. Write coev_V for the composite map of the following canonical maps:

$$I \to [P, P] \to V \otimes V^*. \tag{3}$$

Then

$$V \xrightarrow{\mathrm{coev}} (V \otimes V^*) \otimes V \simeq V \otimes (V^* \otimes V) \xrightarrow{\mathrm{ev}} V, \tag{4}$$

$$V^* \xrightarrow{\mathrm{coev}} (V^* \otimes V) \otimes V^* \simeq V^* \otimes (V \otimes V^*) \xrightarrow{\mathrm{ev}} V^* \tag{5}$$

compose to id_V and id_{V^*}, respectively. V^* is called the left dual of V. The model is that of a finite-dimensional vector space (or a finitely generated projective module over a commutative ring).

A functor F from braided monoidal category $(\mathscr{C}, \otimes, I, \phi_C)$ to $(\mathscr{D}, \otimes, J, \phi_D)$ is called a *monoidal functor* if the natural transformations

$$\delta \colon F(X) \otimes F(Y) \to F(X \otimes Y), \qquad \xi \colon J \to FI$$

are isomorphisms, and δ, ξ should be compatible with the associativity morphisms (see [14, 23] for detail). If, in addition, F respects the braiding, then F is called a *tensor functor*.

An object A in a monoidal category $\mathscr{C}$ is called an *algebra* if there are two morphisms in $\mathscr{C}$:

$$\pi \colon A \otimes A \to A, \qquad \mu \colon I \to A$$

satisfying the associativity and unitary conditions of usual algebras but expressed in diagrams. We recall from [23] some notions concerning modules in a monoidal category. Let $\mathscr{C}$ be a monoidal category, and let A,

B be algebras in $\mathscr{C}$. An object M in $\mathscr{C}$ is called an $A-B$-bimodule if there are morphisms in $\mathscr{C}$:

$$A \otimes M \xrightarrow{m_A} M, \qquad M \otimes B \xrightarrow{m_B} M$$

satisfying the coherence conditions:

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{\mathrm{id} \otimes m_A} & A \otimes M \\ {\scriptstyle \pi \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle m_A} \\ A \otimes M & \xrightarrow{m_A} & M \end{array} \qquad \begin{array}{ccc} M \otimes B \otimes B & \xrightarrow{m_B \otimes \mathrm{id}} & M \otimes B \\ {\scriptstyle \mathrm{id} \otimes \pi} \downarrow & & \downarrow {\scriptstyle m_B} \\ M \otimes B & \xrightarrow{m_B} & M \end{array}$$

and $m_B(m_A \otimes \mathrm{id}_B) = m_A(\mathrm{id}_A \otimes m_B)$ on $A \otimes M \otimes B$. The bimodule category ${}_A\mathscr{C}_B$ consists of objects in $\mathscr{C}$ which have an $A-B$-bimodule structure and morphisms in $\mathscr{C}$ which are $A-B$-bilinear. Write ${}_A\mathscr{C}, \mathscr{C}_B$ for the categories ${}_A\mathscr{C}_I$ and ${}_I\mathscr{C}_B$, respectively. Let ${}_AP_B$ be an $A-B$-bimodule in $\mathscr{C}$. Then P induces a functor

$$F\colon \mathscr{C}_A \to \mathscr{C}_B, \qquad X \mapsto X \otimes_A P \in \mathscr{C}_B.$$

If F is representable, we denote by $[P, -]_B$ the representing object. We say that P_B is *faithfully projective* if $[P, B]$ and $[P, P]_B$ exist and

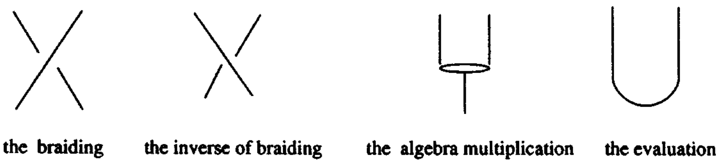
$$P \otimes_B [P, B] \to [P, P]_B, \qquad [P, B] \otimes_{[P, P]_B} P \to B$$

are isomorphisms. Then we have that F is an equivalence if and only if P_B is faithfully projective and $[P, P]_B \cong A$ in $\mathscr{C}$.

Let A be an algebra in $\mathscr{C}$. The *opposite algebra* $\overline{A}$ of A is defined as follows: $\overline{A} = A$ as an object in $\mathscr{C}$, but with multiplication $\overline{\pi}$ being $\pi \circ \phi$. One may easily prove that $\overline{A}$ is indeed an algebra in $\mathscr{C}$. If A, B are algebras in $\mathscr{C}$, then $A \otimes B$ with multiplication given by

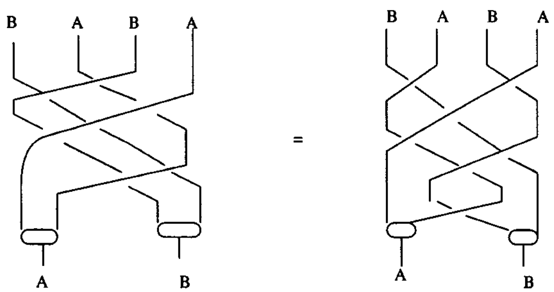
$$(A \otimes B) \otimes (A \otimes B) \xrightarrow{\mathrm{id} \otimes \phi \otimes \mathrm{id}} (A \otimes A) \otimes (B \otimes B) \to A \otimes B$$

is an algebra in $\mathscr{C}$; cf. [18]. Denote the preceding algebra by $A\#B$. In particular, we write A^e and ${}^eA$ for *$\mathscr{C}$-enveloping algebras* $A\#\overline{A}$ and $\overline{A}\#A$, respectively. For more details of algebras and Hopf algebras in a braided category, we refer to [18, 19]. In the sequel, for the sake of simplification, we will often use braiding figures in the proofs. A good and detailed explanation of the diagrammatic methods can be found in [16]. For examples:



PROPOSITION 1.1. *Let A, B be algebras in $\mathscr{C}$. $\overline{A\#B} \cong \overline{B}\#\overline{A}$.*

Proof. The morphism from $\overline{B}\#\overline{A}$ to $\overline{A\#B}$ is defined as the braiding ϕ. It is sufficient to check that ϕ is an algebra morphism. That is, to show that



But this is equivalent to showing that

$$\phi(\phi \otimes \phi)(\mathrm{id} \otimes \phi \otimes \mathrm{id}) = (\mathrm{id} \otimes \phi \otimes \mathrm{id})\phi(\phi \otimes \phi).$$

Note that

$$\phi_{12,34} = (\mathrm{id} \otimes \phi_{2,3} \otimes \mathrm{id})(\phi_{1,2} \otimes \phi_{3,4})(\mathrm{id} \otimes \phi_{2,3} \otimes \mathrm{id}).$$

Now we have

$$\begin{aligned}(\mathrm{id} \otimes \phi_{2,3} \otimes \mathrm{id})\phi_{12,34}(\phi_{1,2} \otimes \phi_{3,4}) \\ &= (\mathrm{id} \otimes \phi_{2,3} \otimes \mathrm{id})(\phi_{1,2} \otimes \phi_{3,4})\phi_{12,34} \\ &= (\mathrm{id} \otimes \phi_{2,3} \otimes \mathrm{id})(\phi_{1,2} \otimes \phi_{3,4})\end{aligned}$$

$$\times(\mathrm{id} \otimes \phi_{2,3} \otimes \mathrm{id})(\phi_{1,2} \otimes \phi_{3,4})(\mathrm{id} \otimes \phi_{2,3} \otimes \mathrm{id})$$
$$= \phi_{12,34}(\phi_{1,2} \otimes \phi_{3,4})(\mathrm{id} \otimes \phi_{2,3} \otimes \mathrm{id}),$$

where the subscriptions indicate the positions of the objects. ∎

Recall from [23] that an object M in ${}_A\mathscr{C}_B$ is called *B-coflat* if, for all algebras $C \in \mathscr{C}$ and objects $X \in {}_B\mathscr{C}_C$, $M \otimes_B X$ exists and if the natural morphism $M \otimes_B (X \otimes Y) \to (M \otimes_B X) \otimes Y$ in ${}_A\mathscr{C}_D$ is an isomorphism for $N \in \mathscr{C}_D$. M is called *bicoflat* if M is A-coflat and B-coflat.

2. THE ELEMENTARY ALGEBRAS IN $\mathscr{C}$

Let $(\mathscr{C}, \otimes, I, \phi)$ be a braided monoidal category. A Morita context in $\mathscr{C}$ is a sextuple $\{A, B, {}_AP_B, {}_BQ_A, f, g\}$ consisting of algebras $A, B \in \mathscr{C}$, an A–B-bimodule $P \in {}_A\mathscr{C}_B$, a B–A-bimodule $Q \in {}_B\mathscr{C}_A$, and bilinear morphisms

$$f\colon P \otimes_B Q \to A, \qquad g\colon Q \otimes_A P \to B$$

making commutative the following diagrams:

$$\begin{array}{ccc} P \otimes_B Q \otimes_A P & \xrightarrow{f\otimes \mathrm{id}} & A \otimes_A P \\ \downarrow{\scriptstyle \mathrm{id}\otimes g} & & \downarrow{\scriptstyle \sim} \\ P \otimes_B B & \xrightarrow{\sim} & P \end{array} \qquad \begin{array}{ccc} Q \otimes_A P \otimes_B Q & \xrightarrow{g\otimes \mathrm{id}} & B \otimes_B Q \\ \downarrow{\scriptstyle \mathrm{id}\otimes f} & & \downarrow{\scriptstyle \sim} \\ Q \otimes_A A & \xrightarrow{\sim} & Q. \end{array}$$

A morphism in $\mathscr{C}$, $h\colon M \to N$, is said to be rationally surjective if $h\colon M(I) \to N(I)$ is surjective. The following result concerning a Morita context can be found in [23, Theorems 5.1 and 5.3] and holds in general in a monoidal category.

THEOREM 2.1. *Let $\{A, B, {}_AP_B, {}_BQ_A, f, g\}$ be a Morita context in $\mathscr{C}$. If P, Q are bicoflat and f, g are rationally surjective (equivalently isomorphisms), then*

(1) $A \cong [P, P]_B \cong {}_B\{Q, Q\}$ *and* $B \cong [Q, Q]_A \cong {}_A\{P, P\}$ *as algebras in* $\mathscr{C}$.

(2) $P \simeq [Q, A]_A \simeq {}_B\{Q, B\}$ *and* $Q \simeq [P, B]_B \simeq {}_A\{P, A\}$ *as bimodules in* $\mathscr{C}$.

(3) ${}_AP$, P_B, ${}_BQ$, *and* Q_A *are faithfully projective.*

Proof. The pair of functors $(Q \otimes_A -, P \otimes_B -)$ defines an equivalence between the categories ${}_A\mathscr{C}$ and ${}_B\mathscr{C}$. So the isomorphisms concerning objects $\{-\}$ follow from [23, Theorem 5.3]. Shifting left modules to right modules, we have the equivalence between $\mathscr{C}_A$ and $\mathscr{C}_B$ defined by the pair of functors $(- \otimes_A P, - \otimes_B Q)$. An argument similar to [23, Theorem 5.3] yields the other isomorphisms concerning objects $[-]$. (3) follows from the definition of faithfully projective objects. ∎

COROLLARY 2.2. *Let P be a faithfully projective object in $\mathscr{C}$. Then*

(1) $[P, P] \cong \{P^*, P^*\}$ *as algebras in* $\mathscr{C}$.

(2) $\{P, P\} \cong [P^*, P^*]$ *as algebras in* $\mathscr{C}$.

(3) $\overline{[P, P]} \cong \{P, P\}$ *as algebras in* $\mathscr{C}$.

Proof. Let $A = [P, P]$. Then P is an A–I-bimodule, and $P^* = [P, I]$ is an I–A-bimodule in $\mathscr{C}$. Let f be the canonical map $P \otimes [P, I] \to [P, P]$ and g be the evaluation map $[P, I] \otimes_{[P,P]} P \to I$. Then $\{[P, P], I, P, P^*, f, g\}$ is a Morita context in $\mathscr{C}$. Applying Theorem 2.1, we obtain the algebra isomorphisms (1) and (2).

Now we view P as an I–$\bar{A}$-bimodule via

$$P \otimes \bar{A} \xrightarrow{\phi} A \otimes P \to P.$$

Similarly, P^* is an $\bar{A}$–I-bimodule in $\mathscr{C}$. This yields a Morita context in $\mathscr{C}$:

$$\left\{\bar{A}, I, {}_{\bar{A}}P^*_I, {}_IP_{\bar{A}}, \bar{f}, \bar{g}\right\},$$

where $\bar{f}\colon P \otimes_{\bar{A}} P^* \to I$ induced by g and $\bar{g}\colon P^* \otimes P \to \bar{A}$ induced by f are isomorphisms. It follows from Theorem 2.1 that $\bar{A} \cong \{P, P\} \cong [P^*, P^*]$. ∎

PROPOSITION 2.3. *Let M, N be faithfully projective objects in $\mathscr{C}$. Then*

(1) $[M, M] \# [N, N] \cong [M \otimes N, M \otimes N]$.

(2) $\{M, M\} \# \{N, N\} \cong \{M \otimes N, M \otimes N\}$.

Proof. (1) Let X be a finite object in $\mathscr{C}$. We may identify $[X, X]$ with $X \otimes X^*$ via the canonical map f in Corollary 2.2. In fact, $X \otimes X^*$ is an algebra and f is an algebra isomorphism. The multiplication map of $X \otimes X^*$ is

$$(X \otimes X^*) \otimes (X \otimes X^*) \xrightarrow{\mathrm{id} \otimes g \otimes \mathrm{id}} X \otimes X^*,$$

where g is the evaluation map. Note that

$$(M \otimes N)^* = [M \otimes N, I] \simeq [M, [N, I]] = [M, N^*] \simeq N^* \otimes M^*.$$

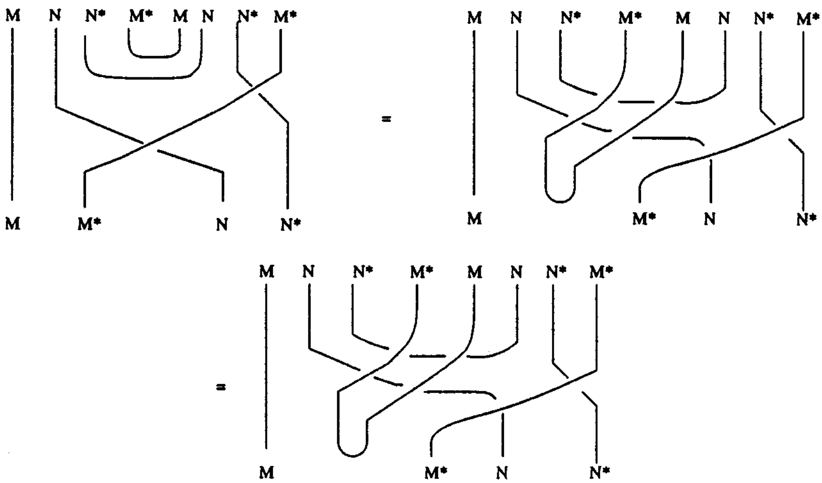
Define a morphism

$$\theta\colon M \otimes N \otimes N^* \otimes M^* \xrightarrow{\mathrm{id} \otimes \phi_{(N \otimes N^*),\, M^*}} M \otimes M^* \otimes N \otimes N^*.$$

Clearly θ is an isomorphism, and hence the composition of morphisms

$$\begin{aligned}\Theta\colon [M \otimes N, M \otimes N] &\cong (M \otimes N) \otimes (N^* \otimes M^*)\\ &\xrightarrow{\theta} (M \otimes M^*) \otimes (N \otimes N^*) \cong [M, M]\#[N, N]\end{aligned}$$

is an isomorphism, too. To show that Θ is an algebra morphism, it is sufficient to check that θ is an algebra morphism. This can be shown by the factorial property of the evaluation map ev. That is,



(2) By Corollary 2.2, we have isomorphisms $\{M, M\} \cong [M^*, M^*]$ and $\{N, N\} \cong [N^*, N^*]$. Now, applying (1), we obtain the isomorphism (2). ∎

PROPOSITION 2.4. *Let A be an algebra in $\mathscr{C}$, M a faithfully projective object in $\mathscr{C}$.*

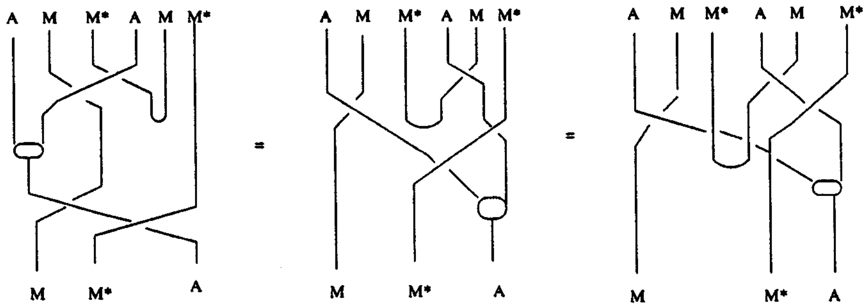
(1) $A\#[M, M] \cong [M, M]\#A$.

(2) $A\#\{M, M\} \cong \{M, M\}\#A$.

Proof. We establish statement (1). Then statement (2) follows from (1) and (2) of Corollary 2.2. Identifying $[M, M]$ with $M \otimes M^*$, we define a morphism η as the composition of the following morphisms:

$$A \otimes M \otimes M^* \xrightarrow{\phi^{-1} \otimes \mathrm{id}} M \otimes A \otimes M^* \xrightarrow{\mathrm{id} \otimes \phi} M \otimes M^* \otimes A.$$

It is clear that η is an isomorphism. We show that η is an algebra morphism diagrammatically as follows:



In the last equality, we used the functoriality of the braiding. That is, the multiplication map commutes with the braiding. ∎

3. AZUMAYA ALGEBRAS IN $\mathscr{C}$

In this section we define Azumaya algebras in a braided monoidal category $\mathscr{C}$, and construct the Brauer group of $\mathscr{C}$. Let A be an algebra in $\mathscr{C}$, and $A^e, {}^eA$ the $\mathscr{C}$-enveloping algebras of A. Then A is an object in the category ${}_{A^e}\mathscr{C}$ as well as in $\mathscr{C}_{A^e}$ via

$$A\#\overline{A} \otimes A \xrightarrow{\mathrm{id} \otimes \phi} A \otimes A \otimes A \xrightarrow{\pi} A$$

and

$$A \otimes \overline{A}\#A \xrightarrow{\phi \otimes \mathrm{id}} A \otimes A \otimes A \xrightarrow{\pi} A.$$

THEOREM 3.1. *Let $\mathscr{C}$ be a braided monoidal category, A an algebra in $\mathscr{C}$. The following statements are equivalent*:

(1) *The following two functors are equivalence functors*:

$$\begin{aligned} &\mathscr{C} \to {}_{A\#\bar{A}}\mathscr{C}, \qquad X \mapsto A \otimes X, \\ &\mathscr{C} \to \mathscr{C}_{\bar{A}\#A}, \qquad X \mapsto X \otimes A. \end{aligned} \tag{6}$$

(2) *A is faithfully projective in $\mathscr{C}$, and the following canonical morphisms are isomorphisms*:

(i) F: $A\#\bar{A}(X) \to [A, A](X)$, $F(a\#\bar{b})\langle d\rangle = a\bar{\pi}(b \otimes d)$, *where* $X, Y \in \mathscr{C}$, $d \in A(Y)$, *and* $a\bar{\pi}(b \otimes d) \in A(X \otimes Y)$.

(ii) G: $\bar{A}\#A(X) \to \{A, A\}(X)$, $\langle d\rangle G(\bar{a}\#b) = \bar{\pi}(d \otimes a)b$, *where* $X, Y \in \mathscr{C}$, $d \in A(Y)$, *and* $\bar{\pi}(d \otimes a)b \in A(X \otimes Y)$.

Proof. Follows from Morita (Theorem 2.1) and [23, Theorem 5.1]. ∎

An algebra A in $\mathscr{C}$ satisfying condition (1) or (2) is called a *$\mathscr{C}$-Azumaya algebra* (or an *Azumaya algebra in $\mathscr{C}$*).

COROLLARY 3.2. *Let A be an Azumaya algebra in $\mathscr{C}$. Then the functors $A \otimes -$ and $- \otimes A$ are monoidal functors and ${}_{A^e}\mathscr{C}, \mathscr{C}_{{}^eA}$ are monoidal categories*.

Proof. Let X be an object in ${}_{A^e}\mathscr{C}$. Then X is an object in ${}_A\mathscr{C}_A$ via the composition maps

$$A \otimes X \xrightarrow{\sim} (A\#I) \otimes X \hookrightarrow (A\#\bar{A}) \otimes X \to X, \tag{7}$$

$$X \otimes A \xrightarrow{\phi^{-1}} A \otimes X \xrightarrow{\sim} (I\#\bar{A}) \otimes X \hookrightarrow (A\#\bar{A}) \otimes X \to X. \tag{8}$$

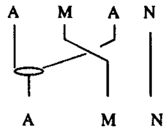
On the other hand, each object $Y \in {}_A\mathscr{C}_A$ may be viewed as an object in ${}_{A^e}\mathscr{C}$ via

$$(A\#\bar{A}) \otimes Y \xrightarrow{\mathrm{id} \otimes \phi} A \otimes Y \otimes A \to Y. \tag{9}$$

Thus one has a tensor product $\otimes_A$ in ${}_{A^e}\mathscr{C}$ and A is a unit with respect to $\otimes_A$ which makes ${}_{A^e}\mathscr{C}$ into a monoidal category. Now $A \otimes -$ is a monoidal functor between $\mathscr{C}$ and ${}_{A^e}\mathscr{C}$ because, for $M, N \in \mathscr{C}$, the isomorphism

$$(A \otimes M) \otimes_A (A \otimes N) \simeq A \otimes (M \otimes N) \in {}_{A^e}\mathscr{C}$$

is as follows:



Similarly, $-\otimes A$ is a monoidal functor and $\mathscr{C}_{e_A}$ is a monoidal category. ∎

THEOREM 3.3. *Let $\mathscr{C}$ be a braided monoidal category.*

(1) *$[P, P]$ is an Azumaya algebra in $\mathscr{C}$ if P is faithfully projective in $\mathscr{C}$.*

(2) *If A is an Azumaya algebra in $\mathscr{C}$, so is $\bar{A}$.*

(3) *$A\#B$ is an Azumaya algebra in $\mathscr{C}$ if A, B are Azumaya algebras in $\mathscr{C}$.*

Proof. (1) Let A be $[P, P]$. It is obvious that A is faithfully projective. By Theorem 2.1 we have $[P, P] \cong \{P^*, P^*\}$ and $\bar{A} \cong [P^*, P^*]$. This yields that $A\#\bar{A} \cong [P, P]\#[P^*, P^*] \cong [P \otimes P^*, P \otimes P^*] = [A, A]$, and $\bar{A}\#A \cong \{P, P\}\#\{P^*, P^*\} \cong \{P \otimes P^*, P \otimes P^*\} = \{A, A\}$. Therefore A is an Azumaya algebra in $\mathscr{C}$.

(2) Let A be an Azumaya algebra in $\mathscr{C}$. Then $\bar{A}\#\bar{\bar{A}} \cong \overline{\bar{A}\#A} \cong \overline{\{A, A\}} \cong [A, A] \cong [\bar{A}, \bar{A}]$, and $\bar{\bar{A}}\#\bar{A} \cong \overline{A\#\bar{A}} \cong \overline{[A, A]} \cong \{A, A\} \cong \{\bar{A}, \bar{A}\}$. This means that $\bar{A}$ is Azumaya in $\mathscr{C}$.

(3) Suppose that A, B are Azumaya in $\mathscr{C}$. We have

$$
\begin{aligned}
(A\#B)\#\overline{A\#B} &\simeq A\#B\#\bar{B}\#\bar{A} \\
&\simeq A\#[B, B]\#\bar{A} \\
&\simeq A\#\bar{A}\#[B, B] \\
&\simeq [A, A]\#[B, B] \\
&\simeq [A\#B, A\#B].
\end{aligned}
$$

Similarly, $\overline{A\#B}\#(A\#B) \cong \{A\#B, A\#B\}$. ∎

Now we are able to define the Brauer group of a braided monoidal category $\mathscr{C}$. Let $B(\mathscr{C})$ be the set of isomorphism classes of Azumaya algebras in $\mathscr{C}$. Then $B(\mathscr{C})$ is a semigroup with product # as before. Define a relation $\sim$ in $B(\mathscr{C})$ as follows: $A \sim B$ if and only if there exist faithfully projective objects $M, N \in \mathscr{C}$ such that

$$A\#[M, M] \cong B\#[N, N]. \tag{10}$$

It is clear that $\sim$ is an equivalence relation in $B(\mathscr{C})$. Now we have

THEOREM 3.4. *If $\mathscr{C}$ is a braided monoidal category, then the quotient set $B(\mathscr{C})/\sim$ is a group with product induced by # and an inverse operator induced by the opposite $\bar{\ }$. This group is denoted by $Br(\mathscr{C})$ and is called the Brauer group of category $\mathscr{C}$. If A is $\mathscr{C}$-Azumaya, $[A]$ indicates the Brauer class in $Br(\mathscr{C})$.*

Remark 3.5. (1) So far we do not assume the braided monoidal categories considered to be additive. The algebras in the categories may be nonadditive. Thus the (abelian) Brauer group of a symmetric category defined in [22] is the special case of the Brauer group (not necessarily abelian) of a braided category.

(2) The equivalence relation (10) turns out to be the Morita equivalence relation. One may generalize the proofs of [5, 3.8–3.10] to obtain that $\mathscr{C}$-Azumaya algebras A and B are equivalent if and only if A and B are Morita equivalent. Therefore we have that $[A] = 1 \in Br(\mathscr{C})$ if and only if $A = [P, P]$ for some faithfully projective object $P \in \mathscr{C}$.

EXAMPLE 3.6. Let $\mathscr{C}$ be the symmetric monoidal category $\mathbf{M}_k$ with the usual tensor product $\otimes_k$ where k is a field or a commutative ring. It is easy to see that $Br(\mathscr{C}) = Br(k)$, the classical Brauer group of k; cf. [2].

EXAMPLE 3.7. Let (X, O_X) be a scheme with ring structure sheaf O_X. Let $\mathscr{C}$ be the category of O_X-module sheaves. Then $\mathscr{C}$ is a symmetric category with the usual tensor product over O_X. It is a routine check that a sheaf of modules M in $\mathscr{C}$ is faithfully projective if and only if M is of finite type and locally free. Therefore a $\mathscr{C}$-Azumaya algebra is exactly a sheaf of Azumaya algebras over O_X. So we have $Br(\mathscr{C}) = Br(O_X)$; cf. [1, 20].

EXAMPLE 3.8. Let R be a commutative ring graded by an abelian group G. Let $\mathscr{C}$ be the category of graded R-modules. Then C is a symmetric category with (graded) tensor product over R. An algebra in $\mathscr{C}$ is a G-graded R-algebra. A $\mathscr{C}$-Azumaya algebra is nothing else but a G-graded R-Azumaya algebra. It follows that $Br(\mathscr{C}) = Br_G(R)$; cf. [3].

EXAMPLE 3.9. Let $\mathscr{C}$ be the graded category $Gr-R$ graded by the group $\mathbb{Z}_2$, where R is a commutative ring with trivial gradation and the tensor product is over R. Define a braiding in $\mathscr{C}$:

$$\phi: M \otimes N \to N \otimes M,$$

where $\phi(m_1 \otimes n_1) = -n_1 \otimes m_1$ if $m_1 \in M_1$, $n_1 \in N_1$ ($M = M_0 \oplus M_1$, $N = N_0 \oplus N_1$), otherwise, $\phi = tw$, the twist map. $\mathscr{C}$ together with ϕ is a braided monoidal category. The Brauer group $Br(\mathscr{C})$ is nothing else but the Brauer–Wall group $BW(R)$; cf. [29].

EXAMPLE 3.10. Let G be an abelian group, R a commutative ring with trivial gradation. Let $\chi: G \times G \to R$ be a bicharacter. Take $\mathscr{C}$ as the category of G-graded R-modules. Then $(\mathscr{C}, \otimes_R, R, \phi)$ is a braided monoidal category, where ϕ is a braiding

$$M \otimes N \to N \otimes M, \qquad m_g \otimes n_h \mapsto n_h \otimes m_g \chi(h \otimes g), \qquad g, h \in G.$$

One may easily see that the Brauer group of $\mathscr{C}$ is the Brauer group $Br_\chi(R, G)$ defined by Childs, Garfinkel, and Orzech in [6]. When χ is trivial, $\mathscr{C}$ is a symmetric category and $Br(\mathscr{C})$ is the Brauer group $Br(R, G)$ defined by Knus; cf. [10].

EXAMPLE 3.11. Let H be a commutative and cocommutative Hopf algebra over a commutative ring k. An H-dimodule M is an R-module with a left H-module structure and a right H-comodule structure which satisfy the compatibility condition:

$$\rho(h \cdot m) = \sum h \cdot m_{(0)} \otimes m_{(1)},$$

where the notation $\sum m_{(0)} \otimes m_{(1)}$ denotes the comodule structure of $\rho(m)$, $m \in M$. Let $\mathscr{C}$ be the category of H-dimodules and morphisms. Define a braiding in $\mathscr{C}$ as follows:

$$\phi: M \otimes N \to N \otimes M, \qquad m \otimes n \mapsto \sum m_{(1)} \cdot n \otimes m_{(0)}.$$

Then $(\mathscr{C}, \otimes_k, k, \phi)$ is a braided monoidal category and the Brauer group of $\mathscr{C}$ is the Brauer–Long group $BD(H, k)$; cf. [12, 13]. There are two braided monoidal subcategories of $\mathscr{C}$. One is the subcategory of left H-modules (with trivial comodule structures) ${}_H\mathbf{M}$, and another is the subcategory of right H-comodules (with the trivial left H-actions) $\mathbf{M}^H$. In both subcategories the braiding ϕ becomes the twist map. So the Brauer groups $Br({}_H\mathbf{M})$ and $Br(\mathbf{M}^H)$ are abelian subgroups of $Br(\mathscr{C})$.

The most interesting Brauer group is the Brauer group of a Hopf algebra with bijective antipode. For the details, we refer to [4, 5].

EXAMPLE 3.12. Let H be a Hopf algebra over a commutative ring k with a bijective antipode S. An H-crossed module is a k-module M on which H acts on the left side and coacts on the right side. The action and coaction satisfy the compatibility condition: $m \in M$, $h \in H$,

$$\rho(h \cdot m) = \sum h_{(2)} \cdot m_{(0)} \otimes h_{(3)} m_{(1)} S^{-1}(h_{(1)}).$$

Let $\mathscr{C}$ be the category of H-crossed modules and their morphisms. Then $(\mathscr{C}, \otimes_k, k, \phi)$ is a braided monoidal category where ϕ is defined as

$$M \otimes N \to N \otimes M, \qquad m \otimes n \mapsto \sum n_{(0)} \otimes n_{(1)} \cdot m, \qquad m \in M, n \in N.$$

The Brauer group of $\mathscr{C}$ is called the Brauer group of H-crossed module algebras, denoted by $BQ(H, k)$. This notion of Brauer group contains the Brauer–Long group as a special case. However, it is quite different from the classical Brauer groups, even the Brauer–Long group, and it essentially deals with the finite representations and structure of the Hopf algebra. For instance, let H be a finite-dimensional Hopf algebra over a field and let $\mathrm{Aut}(H)$ be the Hopf algebra automorphism group of H. We have an exact sequence of groups (cf. [28]):

$$1 \to G(D(H)^*) \to G(D(H)) \to \mathrm{Aut}(H) \to BQ(k, H),$$

where $G(-)$ is the group functor from Hopf algebras to the groups of grouplike elements of the Hopf algebras and $D(H)$ is the quantum double of H; cf. [7]. For example, if H is the Radford Hopf algebra of dimension $m2^{n+1}$, $m > 2$, then $GL_n(\mathbb{C})/U_m$ is a subgroup of $BQ(\mathbb{C}, H)$ (cf. [28]), where U_m is the subgroup of $GL_n(\mathbb{C})$ which is isomorphic to the group of mth roots of units by diagonally embedding. This example shows that the Brauer group $BQ(k, H)$ may be a nontorsion group even if the Hopf algebra is "small."

Now let H be a quasitriangular Hopf algebra. That is, H is a Hopf algebra with an element $R = \sum R^{(1)} \otimes R^{(2)}$ in $H \otimes H$ satisfying several axioms (cf. [18]). It is well known that the category ${}_H\mathbf{M}$ of left H-modules is a braided monoidal category with a braiding ϕ as follows:

$$M \otimes N \to N \otimes M, \qquad m \otimes n \mapsto \sum R^2 \cdot n \otimes R^1 \cdot m, \qquad m \in M, n \in N.$$

The Brauer group of ${}_H\mathbf{M}$ is denoted by $BM(H, k)$ which is a subgroup of $BQ(H, k)$ and contains the usual Brauer group $Br(k)$; cf. [5].

On the other hand, if H is a coquasitriangular (CQT) Hopf algebra (which is a pair $(H, \mathscr{R})$, here H is a Hopf algebra, $\mathscr{R} \in (H \otimes H)^*$ is invertible with respect to the convolution product and satisfies several axioms; cf. [11, 17]), the category $\mathbf{M}^H$ of right H-comodules is a braided monoidal category with braiding

$$M \otimes N \to N \otimes M,$$
$$m \otimes n \mapsto \sum n_{(0)} \otimes m_{(0)} \mathscr{R}\left(n_{(1)} \otimes m_{(1)}\right), \qquad m \in M, n \in N.$$

The Brauer group $Br(\mathbf{M}^H)$ is denoted by $BC(H, k)$ which also contains the usual Brauer group $Br(k)$ and is a subgroup of $BQ(H, k)$; cf. [5]. Examples 3.9 and 3.10 are special cases of $BC(k, H)$ when H is the group Hopf algebra $R\mathbb{Z}_2$ and kG, respectively.

Remark 3.13. Recently Drinfel'd and Majid generalized Hopf algebras to quasi-Hopf algebras and coquasi-Hopf algebras by deforming the comultiplication or multiplication of Hopf algebras, and then they defined the quasitriangular quasi-Hopf algebra and coquasitriangular coquasi-Hopf algebras [8, 15]. For instance, if H is a quasitriangular quasi-Hopf algebra, then the category ${}_H\mathbf{M}$ of left H-modules is a braided monoidal category. Therefore the Brauer group of ${}_H\mathbf{M}$ may be constructed in the preceding way.

To end this section, we present a generalized Rosenberg–Zelinsky sequence for an Azumaya algebra in a braided monoidal category $\mathscr{C}$. Let us first define the Picard group of $\mathscr{C}$. An object P in $\mathscr{C}$ is called an *invertible* object if $P \otimes -$ defines an equivalence between $\mathscr{C}$ and itself. That is, there exists an object $Q \in \mathscr{C}$ such that $P \otimes Q \simeq I$. The set of isomorphism classes of invertible objects in $\mathscr{C}$ becomes a group with the product $\otimes$ and unit represented by I. The Picard group of $\mathscr{C}$ is denoted by $\operatorname{Pic}(\mathscr{C})$. From the definition, one may easily see that $\operatorname{Pic}(\mathscr{C})$ contains the usual Picard group $\operatorname{Pic}(I)$. In all of the preceding examples $\operatorname{Pic}(\mathscr{C})$ is a direct product of $\operatorname{Pic}(I)$ with another group; cf. [5, Prop. 4.3].

Now let A be an algebra in $\mathscr{C}$. A morphism $f\colon A \to A$ in $\mathscr{C}$ is said to be a $\mathscr{C}$-automorphism if f is an algebra isomorphism in $\mathscr{C}$. Denote by $\operatorname{Aut}(A)$ the $\mathscr{C}$-algebra automorphism group of A. An element $\alpha \in \operatorname{Aut}(A)$ is said to be *inner* if there exists an element $a \in A(I)$ such that $\alpha(x) = axa^{-1}$ for any $x \in A(X)$, $X \in \mathscr{C}$. Note that the set $A(I)$ is a monoid with unit being the unit map μ. So a^{-1} makes sense if it exists. It is obvious that the set of all inner elements of $\operatorname{Aut}(A)$ is a subgroup of $\operatorname{Aut}(A)$, denoted by $\operatorname{INN}(A)$. Given two $\mathscr{C}$-automorphisms α, β of A, we define an object ${}_\alpha A_\beta$ in ${}_{A^e}\mathscr{C}$ as follows: ${}_\alpha A_\beta = A$ as an object in $\mathscr{C}$, but with A^e-module structure given by

$$(A\#\overline{A}) \otimes A \xrightarrow{\alpha \otimes \beta \otimes \mathrm{id}} A \otimes A \otimes A \xrightarrow{\mathrm{id} \otimes \overline{\pi}} A \otimes A \xrightarrow{\pi} A.$$

Now a straightforward verification shows that such objects as defined previously satisfy the following relations: for $\alpha, \beta, \gamma \in \mathrm{Aut}(A)$,

(i) ${}_{\alpha}A_{\beta} \simeq {}_{\gamma\alpha}A_{\gamma\beta}$ (via γ),

(ii) ${}_{\alpha}A_{\beta} \otimes_{A} {}_{\alpha}A_{\gamma} \simeq {}_{\alpha}A_{\beta\gamma}$,

(iii) ${}_{\mathrm{id}}A_{\alpha} \simeq {}_{\mathrm{id}}A_{\mathrm{id}}$ if and only if $\alpha \in \mathrm{INN}(A)$.

Given an $\alpha \in \mathrm{Aut}(A)$, we have a canonical isomorphism in ${}_{A^e}\mathscr{C}$:

$$ {}_{\mathrm{id}}A_{\alpha} \simeq A \otimes I_{\alpha}, $$

where I_{α} is an object in $\mathscr{C}$. Let $(-)^{A}$ be the inverse (equivalence) functor of $A \otimes -$ in Theorem 3.1. It is clear that $({}_{\mathrm{id}}A_{\alpha})^{A} = I_{\alpha}$. Since $A \otimes -$ is a monoidal functor, $(-)^{A}$ is a monoidal functor, too. It follows from relation (ii) that, in $\mathscr{C}$,

$$ I_{\alpha\beta} = \left({}_{\mathrm{id}}A_{\alpha\beta}\right)^{A} \simeq \left({}_{\mathrm{id}}A_{\alpha}\right)^{A} \otimes \left({}_{\mathrm{id}}A_{\beta}\right)^{A} \simeq I_{\alpha} \otimes I_{\beta}, \qquad \alpha, \beta \in \mathrm{Aut}(A). $$

It is easy to see that I_{α}, $\alpha \in \mathrm{Aut}(A)$, is an invertible object in $\mathscr{C}$ with inverse object $I_{\alpha^{-1}}$. This yields a well-defined homomorphism Φ from the group $\mathrm{Aut}(A)$ to the Picard group $\mathrm{Pic}(\mathscr{C})$, which is given by $\Phi(\alpha) = I_{\alpha}$. Finally, relation (iii) gives rise to the following:

PROPOSITION 3.14. *Let A be an Azumaya algebra in $\mathscr{C}$. The following sequence is exact*:

$$ 1 \to \mathrm{INN}(A) \to \mathrm{Aut}(A) \xrightarrow{\Phi} \mathrm{Pic}(\mathscr{C}). \tag{11} $$

4. SEPARABLE ALGEBRAS IN A BRAIDED MONOIDAL CATEGORY

In this section we study separable algebras in a braided monoidal category and construct a second Brauer group in the sense of separable algebras. In order to simplify our computations, we assume *the braided monoidal category $\mathscr{C}$ to be a k-linear cocomplete abelian category with the unit k being a commutative ring and with the ordinary tensor product over k*. Algebras in $\mathscr{C}$ are k-algebras with some extra structures; objects in $\mathscr{C}$ are k-modules with additional structures. Thus, in this section, we may use 1 to substitute the unit map $\mu: k \to A$ for an algebra A in $\mathscr{C}$ and replace $A(k)$ by A.

An algebra A in $\mathscr{C}$ is said to be *separable* if the multiplication map

$$A^e = A\#\overline{A} \xrightarrow{\pi} A$$

splits in ${}_{A^e}\mathscr{C}$.

PROPOSITION 4.1. *Let A be an algebra in $\mathscr{C}$. The following are equivalent*:

(1) *A is separable.*

(2) *There is an element $c_l \in A^e(k)$ such that $\pi(c_l) = 1$ and $(a\#\overline{1})c_l = (1\#\overline{a})c_l \in A^e(k)$.*

(3) *There is an element $c \in A\#A(k)$ such that $\pi(c) = 1$ and $(a\#1)c = c(1\#a) \in A\#A(k)$.*

(4) *There is an element $c_r \in {}^eA(k)$ such that $\pi(c_r) = 1$ and $c_r(\overline{a}\#1) = c_r(\overline{1}\#a) \in {}^eA(k)$.*

(5) *The epimorphism $A\#A \xrightarrow{\pi} A$ splits in ${}_A\mathscr{C}_A$.*

(6) *The epimorphism $\overline{A}\#A \xrightarrow{\pi} A$ splits in $\mathscr{C}_{{}^eA}$.*

Proof. One may choose c_l as the inverse image of $\pi^{-1}(1) \in A^e$ and $c = c_r = c_l$. Then a straightforward argument finishes the proof. ∎

The element $c \in A\#A(k)$ satisfying the previous conditions is called a *Casimir* element of A. Each Casimir element c of A induces a map

$$\mathrm{Tr}\colon \mathscr{C}(M, N) \to {}_A\mathscr{C}(M, N) \qquad \text{or} \qquad \mathrm{Tr}\colon \mathscr{C}(M, N) \to \mathscr{C}(M, N)_A$$

for any two objects $M, N \in \mathscr{C}_A$. Precisely, let $X \in \mathscr{C}$, $c = a\#b \in A\#A(k)$. The trace map induced by c is

$$F \mapsto \{M(X) \ni m \mapsto f(ma)b \in N(X)\} \in \mathscr{C}(M, N)_A.$$

Since the trace map is natural in X, we have

$$\mathrm{Tr}\colon \mathscr{C}(X \otimes M, N) \to \mathscr{C}(X \otimes M, N)_A.$$

This induces the trace map

$$\mathrm{Tr}\colon [M, N] \to [M, N]_A,$$

where $[M, N], [M, N]_A$ always exist because of our assumption on $\mathscr{C}$. Similarly, for $M, N \in {}_A\mathscr{C}$, we have a trace map

$$\mathrm{Tr}\colon \{M, N\} \to {}_A\{M, N\}.$$

Now since $\pi(c) = ab = 1$, we even obtain that

$$ {}_A\mathscr{C}(M, N) \to \mathscr{C}(M, N) \xrightarrow{\mathrm{Tr}} {}_A\mathscr{C}(M, N) $$

is the identity on ${}_A\mathscr{C}(M, N)$. Similarly, the composition

$$ {}_A[M, N] \to [M, N] \xrightarrow{\mathrm{Tr}} {}_A[M, N] $$

is the identity on ${}_A[M, N]$. Now we consider the opposite and the product of separable algebras in $\mathscr{C}$. To simplify the notation, we will write in the sequel, for example, $u\#v$ (instead of $\sum u_i \# v_i$) for a Casimir element c_A of a separable algebra A.

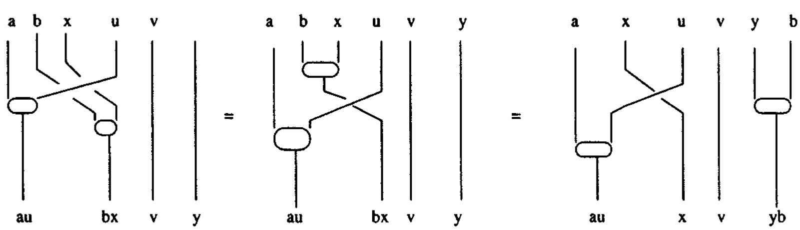
PROPOSITION 4.2. *Let A, B be separable algebras in $\mathscr{C}$. Then*

(1) *$\bar{A}$ is a separable algebra in $\mathscr{C}$.*

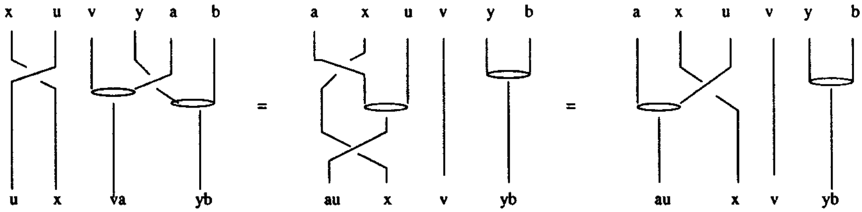
(2) *$A\#B$ is a separable algebra in $\mathscr{C}$.*

Proof. (1) Let $c_A = u\#v$ be a Casimir element of A. Then a routine computation shows that $\phi^{-1}(u \otimes v) \in \bar{A}\#\bar{A}(I)$ is a Casimir element of $\bar{A}$. So $\bar{A}$ is separable in $\mathscr{C}$.

(2) Let $c_A = u\#v$ and $c_B = x\#y$ be the Casimir elements of A and B, respectively. Write c for the element $\phi(x \otimes u)\#(v\#y) \in (A\#B)\#(A\#B)(k)$. We verify that c is a Casimir element of $A\#B$. Write $a\#b$ for an arbitrary element of $A\#B(X)$. We show that $[(a\#b)\#(1\#1)]c = c[(1\#1)\#(a\#b)]$ diagrammatically. $[(a\#b)\#(1\#1)]c =$



$= [(a\#1)\#(1\#1)]c[(1\#1)\#(1\#b)]$. On the other hand, we have $c[(1\#1)\#(a\#b)] =$



$= [(a\#1)\#(1\#1)]c[(1\#1)\#(1\#b)]$. The proof is completed. ∎

Note that in the preceding (and later) diagrams, the labeled lowercase elements are changed though we still use the same letters. We treat here the braiding of elements as we do the braiding of objects. One should understand that, for example, when element x passes a braiding it is no longer equal to x, but we still denote it by x.

DEFINITION 4.3. Let A be a separable algebra in $\mathscr{C}$. The left center and right center of A are defined respectively as follows: $Z^l(A) = \{b \in A | \forall a \in A,\ ab = \overline{\pi}(a \otimes b)\}$ and $Z^r(A) = \{b \in A | \forall a \in A,\ ba = \overline{\pi}(b \otimes a)\}$.

PROPOSITION 4.4. *Let A be a separable algebra in $\mathscr{C}$. Then*

(1) *$Z^l(A), Z^r(A)$ are objects in $\mathscr{C}$.*

(2) *${}_{A^e}\{A, A\} = Z^l(A)$ and $[A, A]_{{}^eA} = Z^r(A)$.*

Proof. (1) Let c be a Casimir element of A. Let $\rightharpoonup, \leftharpoonup$ denote the left A^e-module and right ${}^eA$-module structures of A, respectively. Then $c \rightharpoonup A = Z^l(A)$ and $A \leftharpoonup c = Z^r(A)$. In fact, $\forall x \in A$, let $b = c \rightharpoonup x$. Then, for any $a \in A$, $ab = (a\#1) \rightharpoonup b = (a\#1)c \rightharpoonup x = (1\#\bar{a})c \rightharpoonup x = (1\#\bar{a}) \rightharpoonup b = \overline{\pi}(a \otimes b)$. This implies that $c \rightharpoonup A \subseteq Z^l(A)$.

Conversely, write $c = a\#b$, $\forall x \in Z^l(A)$, $c \rightharpoonup x = a\overline{\pi}(b \otimes x) = abx = x$. $Z^l(A) \subseteq c \rightharpoonup A$ follows. The proof $Z^r(A) = A \leftharpoonup c$ is similar.

Second, we show that $c \rightharpoonup A$ and $A \leftharpoonup c$ are objects in $\mathscr{C}$. Let ξ be the composition map of the unit map followed by the splitting map of π:

$$k \to A \to A\#\overline{A}.$$

Denote by I_e the image of ξ in $A^e = A\#\overline{A}$ which is isomorphic to k. The monomorphism $I_e \to A^e$ induces a composition map in $\mathscr{C}$:

$$I_e \otimes A \to A^e \otimes A \xrightarrow{\rightharpoonup} A.$$

It is easy to see that $c \rightharpoonup A$ is exactly the image of the preceding composition map, and hence is an object in $\mathscr{C}$. Similarly, $A \leftharpoonup c$ is an object in $\mathscr{C}$, too.

(2) Now we show that ${}_{A^e}\{A, A\} = Z^l(A)$ and $[A, A]_{{}^eA} = Z^r(A)$. We establish the first identity; the second one follows in a similar way. By definition, we have to check that

$${}_{A^e}\mathscr{C}(A \otimes X, A) \simeq \mathscr{C}\big(X, Z^l(A)\big)$$

is natural in $X \in \mathscr{C}$. Define a map Ψ from ${}_{A^e}\mathscr{C}(A \otimes X, A)$ to $\mathscr{C}(X, Z^l(A))$ as follows: $\forall f \in {}_{A^e}\mathscr{C}(A \otimes X, A)$, we define $\Psi(f)$ as the following composition map:

$$X = k \otimes X \to A \otimes X \xrightarrow{f} A.$$

That Ψ is well defined follows from $c \rightharpoonup \Psi(f)(X) = \Psi(f)(c \rightharpoonup (k \otimes X)) = \Psi(f)(\pi(c)X) = \Psi(f)(X)$ for any $f \in {}_{A^e}\mathscr{C}(A \otimes X, A)$. The inverse of Ψ is given by

$$\Psi^{-1}: \mathscr{C}\big(X, Z^l(A)\big) \to {}_{A^e}\mathscr{C}(A \otimes X, A),$$
$$g \mapsto \Big(A \otimes X \xrightarrow{\mathrm{id} \otimes g} A \otimes Z^l(A) \xrightarrow{\pi} A\Big).$$

The naturality in X with respect to Ψ is clear. ∎

An algebra A in $\mathscr{C}$ is said to be *left* (or *right*) *central* if ${}_{A^e}\{A, A\}$ (or $[A, A]_{{}^eA}$) is equal to k. Let A be a separable algebra in $\mathscr{C}$ with a Casimir element $c = a\#b$. Then A is left (or right) central if and only if $a\overline{\pi}(b \otimes x) \in k$ for all $x \in A$ (or $\overline{\pi}(x \otimes a)b \in k$ for all $x \in A$).

PROPOSITION 4.5. *Let A, B be separable algebras in $\mathscr{C}$.*

(1) *${}_{A^e}\{A, A\}, [A, A]_{{}^eA}$ are direct summands of A in $\mathscr{C}$.*

(2) *If A is left (or right) central, then $\overline{A}$ is right (or left) central.*

(3) *$A\#B$ is left (or right) central if A, B are left (or right) central.*

Proof. (1) By Proposition 4.4, the subobjects ${}_{A^e}\{A, A\}, [A, A]_{^eA}$ of A exist. The trace map Tr: $A \cong_A\{A, A\} \to_{A^e}\{A, A\}$ is exactly the map induced by a Casimir element $c = u\#v \in A\#A$ as follows:

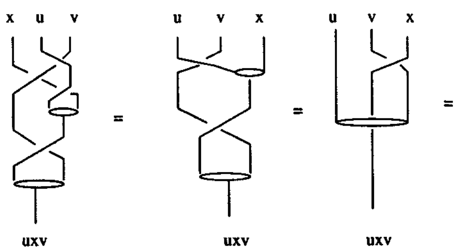
$$A \to_{A^e} \{A, A\}, \qquad x \to u\overline{\pi}(v \otimes x) = c \rightharpoonup x, \qquad x \in A.$$

This trace map splits the monomorphism ${}_{A^e}\{A, A\} \to A$. Similarly, another trace map

$$A \to [A, A]_{^eA}, \qquad x \to \overline{\pi}(x \otimes u)v = x \leftharpoonup c, \qquad x \in A,$$

splits the monomorphism $[A, A]_{^eA} \to A$. In particular, if A is left (or right) central, then k is a direct summand of A.

(2) Let $c_A = u\#v$ be a Casimir element of A. By (1) of Proposition 4.5, the element $c_{\overline{A}} = \phi^{-1}(u \otimes v) \in \overline{A}\#\overline{A}$ is a Casimir element of $\overline{A}$. If $\overline{A} \leftharpoonup c_{\overline{A}} = \overline{c_A \rightharpoonup A}$, then $\overline{A}$ is right central. Indeed, for $\bar{x} \in \overline{A}$, we have $\bar{x} \leftharpoonup c_{\overline{A}} =$

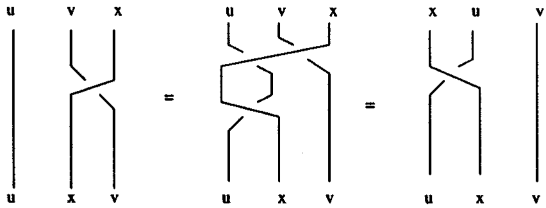


$= \overline{c_A \rightharpoonup x}$, where the first identity holds because $c_{\overline{A}} \in \overline{A}\#\overline{A}(k)$, and the second one holds because the multiplication operator commutes with the braiding.

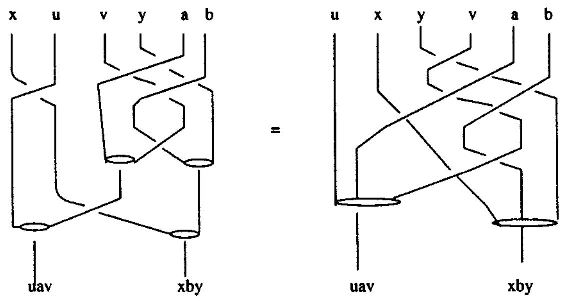
(3) It is sufficient to compute the elements of the form $c_{A\#B} \rightharpoonup d$, where $d \in A\#B$ and $c_{A\#B}$ is the Casimir element stemming from c_A and c_B, Casimir elements of A and B, respectively. Let $c_A = u\#v$, $c_B = x\#y$, and $c_{A\#B} = \phi(x \otimes u)\#(v\#y)$. Given any element $d = a\#b \in A\#B$, we have

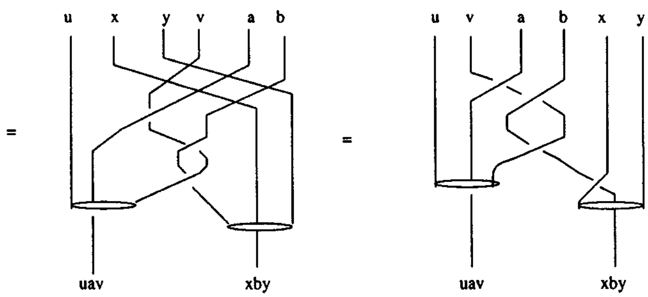
$$c_{A\#B} \rightharpoonup d = \phi(x \otimes u)\overline{\pi}((v\#y) \otimes (a\#b)).$$

In order to show that this element is in k, we use diagram chasing. First, let us point out the following equality for Casimir elements:



We have $c_{A\#B} \rightharpoonup d =$





It follows that $c_{A\#B} \rightharpoonup d \in A\#k(X)$. However, we have

$$\begin{aligned}\left[(u\#1)\#\overline{v\#1}\right]c_{A\#B} &= \left[(u\#1)\#\overline{1\#1}\right]\left[(1\#1)\#\overline{v\#1}\right]c_{A\#B} \\ &= \left[(u\#1)\#\overline{1\#1}\right]\left[(v\#1)\#\overline{1\#1}\right]c_{A\#B} \\ &= \left[(uv\#1)\#\overline{1\#1}\right]c_{A\#B} \\ &= c_{A\#B}.\end{aligned}$$

This yields

$$\begin{aligned}c_{A\#B} \rightharpoonup d &= \left[(u\#1)\#\overline{v\#1}\right]c_{A\#B} \rightharpoonup d \\ &= \left[(u\#1)\#\overline{v\#1}\right] \rightharpoonup (z\#1), \qquad z \in A \\ &= u\overline{\pi}(v \otimes z)\#1 \in Z^l(A)\#k \\ &= k\#k = k\end{aligned}$$

and the proof is completed. ∎

An algebra is *central* if both ${}_{A^e}\{A, A\}, [A, A]_{{}^eA}$ are equal to k.

Before we establish the main result of this section, we introduce a few more notions. Let A be an algebra in $\mathscr{C}$. A subobject $\mathfrak{M}$ of A is called an *ideal* or *$\mathscr{C}$-ideal* of A if $\mathfrak{M}$ is an object in ${}_A\mathscr{C}_A$ in the usual way. That is, the multiplication morphisms restrict to

$$A \otimes \mathfrak{M} \to \mathfrak{M} \qquad \text{and} \qquad \mathfrak{M} \otimes A \to \mathfrak{M}.$$

In particular, any subobject of k, unit of $\mathscr{C}$, is an ideal of k. Note that $\mathscr{C}$ is cocomplete. If $\mathfrak{M}$ is an ideal of an algebra A in $\mathscr{C}$, then there exists a maximal ideal M containing $\mathfrak{M}$. If $\mathfrak{M}$ is an ideal of algebra A, then $A/\mathfrak{M}$ is an algebra in $\mathscr{C}$. An algebra is said to be simple (or $\mathscr{C}$-simple) if A has no ideal except A or 0.

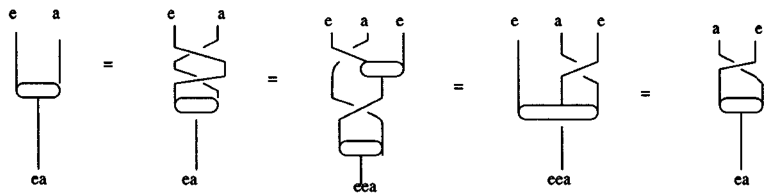
PROPOSITION 4.6. *Let A be a left (or right) central separable algebra in $\mathscr{C}$. A is simple if and only if the unit k is simple.*

Proof. Suppose that A is a simple algebra in $\mathscr{C}$ and $\mathfrak{M}$ is an ideal of k. The image of $\mathfrak{M} \otimes A \to A$, denoted by $\mathfrak{M}A$, is an ideal of A because $\mathfrak{M}A = A\mathfrak{M}$. So $\mathfrak{M}A = A$ by the assumption. Now let Tr be the trace map $A \to k$ with $\mathrm{Tr} \circ \mu = \mathrm{id}$ (assured by (1) of Proposition 4.5), where μ: $k \to A$ is the unit morphism. Then

$$k = \mathrm{Tr}(A) = \mathrm{Tr}(\mathfrak{M}A) = \mathfrak{M}\,\mathrm{Tr}(A) = \mathfrak{M}k = \mathfrak{M}.$$

It follows that k is simple.

Conversely, if k is simple, then k is a field because k is commutative. Note that A is a separable algebra over k since c_A, the Casimir element, is in $A\#A(k)$. So A is a semisimple k-algebra. Now if $\mathfrak{A}$ is a $\mathscr{C}$-ideal of A, then $\mathfrak{A}$ is a real ideal of A, and hence there is a central idempotent $e \in A$ such that $\mathfrak{A} = Ae$. We show that e is in the left (right) center of A. This follows from the diagrams:



that is, for any $a \in A$, $ea = \overline{\pi}(a \otimes e)$. But $ea = ae$, and hence e is in $Z^l(A) = k$. It follows that $\mathfrak{A} \cap k \neq 0$, which is an ideal of k, and hence $\mathfrak{A} \supseteq k$. It follows that $A = \mathfrak{A}$ and A is simple. ▮

LEMMA 4.7. *Let A be a left (or right) central separable algebra in $\mathscr{C}$, and let B be an algebra in $\mathscr{C}$.*

(1) *If $f\colon A \to B$ is an algebra epimorphism in $\mathscr{C}$, then B is a separable algebra in $\mathscr{C}$.*

(2) *$A \otimes K$ is a left (or right) central separable algebra in $\mathscr{C} \otimes K = \{X \otimes K | X \in \mathscr{C}\}$ if K is a commutative k-algebra.*

Proof. (1) One may check that the algebra morphism $f \otimes f$ maps the Casimir elements of A to the Casimir elements of B. So B is a separable algebra in $\mathscr{C}$.

(2) If c is a Casimir element of A, then $c \otimes 1$ is a Casimir element of $A \otimes K$. The left center $Z^l(A \otimes K) = (c \otimes 1) \rightharpoonup (A \otimes K) = k \otimes K = K$. ▮

LEMMA 4.8. *If A is a left (or right) central separable algebra in $\mathscr{C}$, then for any maximal $\mathscr{C}$-ideal $\mathfrak{M}$ of A, there exists a maximal ideal $\mathfrak{A}$ of k such that $\mathfrak{M} = \mathfrak{A}A$ and $\mathfrak{M} \cap k = \mathfrak{A}$.*

Proof. Let $\mathfrak{A} = \mathfrak{M} \cap A$. Denote by $\mathscr{C}'$ the category

$$\{X \in \mathscr{C} | \mathfrak{A} \otimes X = 0, X \in \mathscr{C}\}.$$

$\mathscr{C}'$ is a braided monoidal category with the tensor product over $k/\mathfrak{A}$ and unit $k/\mathfrak{A}$. It is clear that $\overline{A} = A/\mathfrak{M}$ is a simple algebra in $\mathscr{C}'$. Since $A/\mathfrak{M}$ is separable in $\mathscr{C}$, it is separable in $\mathscr{C}'$. Let c be a Casimir element of A. Then $\bar{c}$, the image of c in $A/\mathfrak{M}\#A/\mathfrak{M}$, is a Casimir element of $A/\mathfrak{M}$. The left center $Z^l(A/\mathfrak{M}) = \bar{c} \rightharpoonup A/\mathfrak{M} = (k + \mathfrak{M})/\mathfrak{M} = k/\mathfrak{A}$ is the unit of $\mathscr{C}'$. $A/\mathfrak{M}$ is left central in $\mathscr{C}'$. Now, by Proposition 4.6, $k/\mathfrak{A}$ is simple. It follows that $\mathfrak{A}$ is a maximal ideal of k. On the other hand, $A/\mathfrak{A}A$ is a separable algebra in $\mathscr{C}'$ and the left center is $\bar{c} \rightharpoonup A/\mathfrak{A}A = k/k \cap \mathfrak{A}A$

$= k/\mathfrak{A}$. Again, by Proposition 4.6 and the previous argument, $A/\mathfrak{A}A$ is simple. $\mathfrak{A}A$ is a maximal ideal contained in $\mathfrak{M}$, and hence $\mathfrak{M} = \mathfrak{A}A$. ∎

Now we are able to show the main theorem of this section.

THEOREM 4.9. *Let A be an algebra in $\mathscr{C}$. The following are equivalent*:

(1) *A is a central separable algebra in $\mathscr{C}$.*

(2) *A is a progenerator in $\mathscr{C}$, and the following canonical morphisms are isomorphisms*:

(i) F: $A\#\overline{A} \to [A, A]$, $F(a\#\overline{b})\langle c\rangle = a\overline{\pi}(b \otimes c)$,

(ii) G: $\overline{A}\#A \to \{A, A\}$, $\langle c\rangle G(\overline{a}\#b) = \overline{\pi}(c \otimes a)b$.

(3) *A is separable and Azumaya in $\mathscr{C}$.*

Proof. (1) ⇔ (3) Suppose that A is central separable in $\mathscr{C}$. By Proposition 4.4, ${}_{A^e}[A, A] = k$, and there exists a canonical Morita context in $\mathscr{C}$:

$$\{A^e, k, A, {}_{A^e}[A, A^e] = A', f, g\},$$

where f: $A \otimes A' \to A^e$, $a \otimes p \to \langle a\rangle p$ and g: $A' \otimes_{A^e} A \to k = {}_{A^e}[A, A]$, $p \otimes a \to (x \to (\langle x\rangle p) \rightharpoonup a)$.

Let ψ be the inverse map π: $A^e \to A$ such that $\pi\psi = \mathrm{id}_A$. Then $\psi \in A'(k)$. For any $x \in A$, $x = \pi\psi(x) = \psi(x) \rightharpoonup 1$. This means that $\{\psi, 1\} \in A \otimes A'(k)$ is a dual basis for ${}_{A^e}A$. It follows that g is (rationally) surjective.

Now we show that f is surjective. Let T be the image of f in A^e. T is certainly a $\mathscr{C}$-ideal of A^e. Write B for A^e, and let c be a Casimir element of A. For $a \otimes p \in A \otimes A'$,

$$\langle a\rangle p = (a\#1)\langle 1\rangle p \qquad \text{and} \qquad \langle 1\rangle p \in c \rightharpoonup B = cB.$$

So $\langle a\rangle p \in BcB$ and $T \subseteq BcB$. On the other hand, let p be defined by $\langle x\rangle p = (x\#1)c$. Then $c = \langle 1\rangle p \in T$ and $BcB \subseteq T$. Thus $T = BcB$. If $T \neq B$, then there exists a maximal ideal $\mathfrak{M}$ containing T such that $\mathfrak{M} = \mathfrak{A}B$, where $\mathfrak{A}$ is a maximal ideal of k. Now applying the morphism π, we have

$$A = \pi(BcB) \subseteq \pi(\mathfrak{M}) = \pi(\mathfrak{A}B) = \mathfrak{A}A.$$

By Lemma 4.8 we have $\mathfrak{A} = \mathfrak{A}A \cap k = A \cap k = k$, and hence $\mathfrak{M} = B$. Contradiction! f then is surjective. By the Morita theorem (Theorem 2.1) we have $A^e \cong [A, A]$ canonically and A is faithfully projective. Similarly, if one starts with ${}^eA$, then one has ${}^eA \cong \{A, A\}$ canonically. Thus we have proved that A is an Azumaya algebra in $\mathscr{C}$.

Conversely, suppose that A is Azumaya and separable in $\mathscr{C}$. Since A is faithfully projective and $A^e \cong [A, A]$ canonically, the canonical Morita context $\{A^e, k, A, [A, k], f', g'\}$ is strict; that is, f', g' are rationally surjective. This yields that ${}_{A^e}[A, A] \cong k$. This means A is left central. Similarly, ${}^{e}A \cong \{A, A\}$ and the faithful projectivity of A imply that A is right central. Therefore A is central separable in $\mathscr{C}$.

(2) $\Leftrightarrow$ (3) Suppose that (2) holds. Since a progenerator is faithfully projective, A is Azumaya in $\mathscr{C}$ by definition. We show that A has a Casimir element. Let $f_1 \otimes a_1 \in A^* \otimes A(k)$ such that $f_1\langle a_1 \rangle = 1 \in k$. Define a $g_0 \in A^*(k)$ by $g_0\langle x \rangle = f_1\langle a_1 x \rangle$. Let $a\#\bar{b}$ be the element corresponding to $1 \otimes g_0$ under the isomorphism

$$A^e \cong [A, A] \cong A \otimes A^*.$$

Then we have

$$ab = a1b = 1g_0\langle 1 \rangle = 1f_1\langle a_1 \rangle = 1.$$

Moreover, for $x, y \in A$,

$$\begin{aligned}(x\#1)(a\#\bar{b})\langle y \rangle &= xa\overline{\pi}(b \otimes y) = xg_0\langle y \rangle \\ &= (1\#\bar{x})\langle g_0\langle y \rangle 1 \rangle \\ &= (1\#\bar{x})(a\#\bar{b})\langle y \rangle.\end{aligned}$$

This implies that $(x\#\bar{1})(a\#\bar{b}) = (1\#\bar{x})(a\#\bar{b})$ for any $x \in A$. It follows from Proposition 4.1 that $a \otimes b$ is a Casimir element of A, and hence A is separable.

Finally, to show (3) $\Rightarrow$ (2), it is enough to show that A is a progenerator in $\mathscr{C}$. By definition, we have to find an element $p \otimes a \in [A, k] \otimes A$ such that $p\langle a \rangle = 1 \in k$. This immediately follows by taking $a = 1$ and $p = (\mathrm{Tr}: A \to_{A^e} [A, A] \cong k)$. ∎

Here we have to point out that (2) $\Leftrightarrow$ (3) holds in general for any braided monoidal category.

LEMMA 4.10. *If P is a progenerator in $\mathscr{C}$, then $[P, P]$ is a central separable algebra in $\mathscr{C}$.*

Proof. The proof is the same as the proof of [22, Theorem 14]. By the preceding theorem it is sufficient to show that $[P, P]$ is separable in $\mathscr{C}$. Let $p_0 \otimes f_0$ be the dual basis for P and $f_1 \otimes p_1$ be an element in $P \otimes [P, k]$ such that $f_1\langle p_1 \rangle = 1 \in k$. Identify $A = [P, P]$ with $P \otimes [P, k]$ which has the usual multiplication. Define $a\#\bar{b} =: (p_0 \otimes f_1)\#p_1 \otimes f_0 \in A^e(k)$. Then it is just routine to verify that $a\#\bar{b}$ is a Casimir element of A. ∎

Now we are able to construct the second Brauer group of the category of $\mathscr{C}$. Note that the progenerators are faithfully projective. So all the properties in Section 2 are valid for progenerators. Let $B'(\mathscr{C})$ be the set of isomorphism classes of central separable algebras in $\mathscr{C}$. Define an equivalent $\sim$ in $B'(\mathscr{C})$ as follows: $A \sim B$ if and only if there exists progenerators $P, Q \in \mathscr{C}$ such that

$$A\#[P, P] \cong B\#[Q, Q].$$

Now we have the following:

THEOREM 4.11. *The quotient set $B'(\mathscr{C})/\sim$, denoted by $Br'(\mathscr{C})$, is a group with multiplication induced by # and inverse operator* $\bar{\ }$. *There is a monomorphism from $Br'(\mathscr{C})$ into $Br(\mathscr{C})$. If k is a projective object in $\mathscr{C}$, then $Br'(\mathscr{C}) = Br(\mathscr{C})$.*

Proof. If k is projective in $\mathscr{C}$, then an object is a progenerator if and only if it is faithfully projective. $Br'(\mathscr{C}) = Br(\mathscr{C})$ follows from Theorem 4.9. ∎

Remark 4.12. (1) $Br'(\mathscr{C})$ may be defined by the separable Azumaya algebras if the braided monoidal category $\mathscr{C}$ is arbitrary. This is because conditions (2) and (3) in Theorem 4.9 are always equivalent and Proposition 4.2 holds in general. In fact, except for Proposition 4.6, the whole theory in this section can be extended to the case of an arbitrary braided monoidal category.

(2) In case $\mathscr{C}$ is a braided k-linear abelian cocomplete category, the condition (in [22, Theorem 12]) of the existence of an element $c \otimes d \otimes e \in A \otimes A \otimes A(k)$ such that $ac \otimes dbe = 1 \otimes 1$ may be dropped.

EXAMPLES 4.13. In Examples 3.6 and 3.7, the two Brauer groups $Br(\mathscr{C})$ and $Br'(\mathscr{C})$ coincide. In Examples 3.8–3.10, if the order of group G (or $\mathbb{Z}_2$) is invertible in R, then the two Brauer groups coincide. In Examples 3.11 and 3.12, if the Hopf algebras are semisimple and cosemisimple, then the two Brauer groups are equal; cf. [5, Theorem 3.25].

5. BASE CHANGE

In this section we discuss the behavior of Brauer groups under base changes. This will allow us to link the Brauer group of a braided monoidal category with the Brauer group of a Hopf algebra.

LEMMA 5.1 [22, Theorem 17 and Corollary 18]. *Let F be a monoidal functor from monoidal categories $\mathscr{C}$ to $\mathscr{D}$ (not necessarily braided). Suppose that P is finite in $\mathscr{C}$ and $[F(P), -]$ exists in $\mathscr{D}$.*

(1) *$F(P)$ is finite in $\mathscr{D}$.*

(2) *The canonical morphism Φ: $F([P, X]) \to [F(P), F(X)]$ is an isomorphism for all $X \in \mathscr{C}$, where Φ is induced by the composite morphism*

$$F[P, X] \otimes F(P) \tilde{\to} F([P, X] \otimes P) \overset{F(\mathrm{ev})}{\to} F(X).$$

(3) *If P is a progenerator, then $F(P)$ is a progenerator.*

(4) *If F preserves coequalizers, P is faithfully projective, then $F(P)$ is a faithfully projective object in $\mathscr{D}$.*

THEOREM 5.2. *If F is a tensor functor from braided monoidal categories $\mathscr{C}$ to $\mathscr{D}$ such that, for all finite objects $P \in \mathscr{C}$, $[F(P), -]$ exists, then F induces a homomorphism $\tilde{F}$: $Br'(\mathscr{C}) \to Br'(\mathscr{D})$. If, in addition, F preserves the coequalizers, then $\tilde{F}$ is a homomorphism from $Br(\mathscr{C})$ to $Br(\mathscr{D})$.*

Proof. Let A be an algebra in $\mathscr{C}$. $F(A)$ is clearly an algebra in $\mathscr{D}$, with multiplication

$$F(A) \otimes F(A) \overset{\delta}{\to} F(A \otimes A) \overset{F(\pi)}{\to} F(A)$$

and unit $J \tilde{\to} F(I) \overset{F(\mu)}{\to} F(A)$. If $[A] \in Br'(C)$, then $F(A)$ is a progenerator by Lemma 5.1. To show that $[F(A)] \in Br'(\mathscr{D})$, we have to verify that the canonical morphisms

$$F(A)\#\overline{F(A)} \to [F(A), F(A)], \qquad \overline{F(A)}\#F(A) \to \{F(A), F(A)\}$$

are in fact isomorphisms. Since F commutes with the braiding ϕ, we have $\overline{F(A)} = F(\overline{A})$. Now it follows from the following commutative diagrams that the previous canonical morphisms are isomorphisms (note that $A = \overline{A}$

as objects)

$$\begin{array}{ccc} F(A) \otimes F(\bar{A}) & \longrightarrow & [F(A), F(\bar{A})] \\ \downarrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \Phi} \\ F(A \otimes \bar{A}) & \xrightarrow{F(\sim)} & F([A, A]) \end{array}$$

$$\begin{array}{ccc} F(\bar{A}) \otimes F(A) & \longrightarrow & \{F(A), F(\bar{A})\} \\ \downarrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \Phi'} \\ F(\bar{A} \otimes A) & \xrightarrow{F(\sim)} & F(\{A, A\}). \end{array}$$

Finally, suppose that $A \sim B$ in $\mathscr{C}$. There are two progenerators $P, Q \in \mathscr{C}$ such that

$$A\#[P, P] \cong B\#[Q, Q].$$

Now we have the commutative diagram

$$\begin{array}{ccc} F(A \otimes [P, P]) & \longrightarrow & F(B \otimes [Q, Q]) \\ \downarrow{\scriptstyle \delta^{-1}} & & \downarrow{\scriptstyle \delta^{-1}} \\ F(A) \otimes F([P, P]) & \xrightarrow{\sim} & F(B) \otimes F([Q, Q]) \\ \downarrow{\scriptstyle \mathrm{id} \otimes \Phi} & & \downarrow{\scriptstyle \mathrm{id} \otimes \Phi} \\ F(A) \otimes [F(P), F(P)] & \longrightarrow & F(B) \otimes [F(Q), F(Q)], \end{array}$$

which implies that $F(A) \sim F(B)$ in $\mathscr{D}$. So F induces a well-defined homomorphism

$$\tilde{F}\colon Br'(\mathscr{C}) \to Br'(\mathscr{D}), \qquad [A] \to [F(A)].$$

Now if F preserves coequalizers, then F preserves faithfully projective objects. The same argument as before shows that the induced homomorphism $\tilde{F}$ can be extended to a homomorphism $Br(\mathscr{C}) \to Br(\mathscr{D})$. ∎

The foregoing theorem states that $Br(-)$ is a group scheme from the category of braided monoidal categories to the category of groups which restrict to an abelian group scheme on the closed subcategory of symmetric categories [22]. Let $\mathscr{C}$ be a braided monoidal category. We consider a subcategory of $\mathscr{C}$ which consists of all finite objects in $\mathscr{C}$ (it is not necessarily an abelian category), denoted by $\mathscr{C}^f$. It is clear that $\mathscr{C}^f$ is a rigid and closed (under tensor product) braided monoidal subcategory. Since all the Azumaya algebras in $\mathscr{C}$ are finite objects and fall in $\mathscr{C}^f$, it follows that the Brauer group $Br(\mathscr{C})$ equals $Br(\mathscr{C}^f)$. Let k be a commutative ring, and let $\mathbf{M}_k^f$ be the rigid category of finitely generated projective k-modules, where k is a commutative ring.

THEOREM 5.3. *Let $\mathscr{C}$ be a braided monoidal (small) category with a monoidal functor F: $\mathscr{C} \to \mathbf{M}_k$ (k a commutative ring) which sends the finite objects into $\mathbf{M}_k^f$ and preserves coequalizers. Then there exists a coquasitriangular Hopf algebra over k and a homomorphism $\tilde{F}$: $Br(\mathscr{C}) \to Br(\mathbf{M}^H)$ ($Br'(\mathscr{C}) \to Br'(\mathbf{M}^H)$) which satisfy the universal property: If H' is a coquasitriangular Hopf algebra with a tensor functor G: $\mathscr{C}^f \to \mathbf{M}^{H'}$ preserving coequalizers, through which F factors followed by the forgetful functor, then there exists a homomorphism μ: $Br(\mathbf{M}^H) \to Br(\mathbf{M}^{H'})$ such that $\tilde{G} = \mu\tilde{F}$.*

Proof. Set $H_0 = \oplus_{X \in \mathscr{F}} F(X)^* \otimes F(X)$ and $H = H_0/\sim$, where $\mathscr{F}$ is the set of all finite objects and $\sim$ is an equivalence relation such that

$$F(\xi)^*(y^*) \otimes x \sim y^* \otimes F(\xi)(x)$$

for all ξ: $X \to Y$, $x \in F(X)$, $y^* \in F(Y)^*$. H is a coquasitriangular Hopf algebra. We refer to [17, 31] for the details. For each finite object X, $F(X)$ is a right H-comodule with the comodule structure:

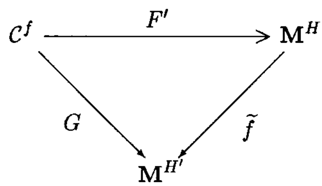
$$F(X) = I \otimes F(X) \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} F(X) \otimes F(X)^* \otimes F(X) \xrightarrow{\mathrm{id} \otimes p\iota} F(X) \otimes H,$$

where ι: $F(X)^* \otimes F(X) \to H_0$ is the usual inclusion and p: $H_0 \to H$ is the projection map. Thus the functor F induces a functor F': $\mathscr{C}^f \to \mathbf{M}^H$ sending object X to the object $F(X)$ which is now a right H-comodule. The functor F' is a tensor functor (see [17, Theorems 2.2–2.8]). Since $F'(X)$, $X \in \mathscr{C}^f$, is finitely generated projective as a k-module, $[F'(X), -]$ exists in $\mathbf{M}^H$. Moreover, F' respects coequalizers because F does. It follows from the previous theorem that the tensor functor F' induces a homomorphism $\tilde{F}$: $Br(\mathscr{C}^f) \to Br(\mathbf{M}^H)$.

Following from the Tannaka–Krein theorem [17, 18], the functor F' has the universal property. If H' is a coquasitriangular Hopf algebra and there is a tensor functor G: $\mathscr{C}^f \to \mathbf{M}^{H'}$ such that F factors through G followed by the forgetful functor from $\mathbf{M}^{H'}$ to $\mathbf{M}^k$, then there exists a coquasitriangular Hopf algebra map f: $H \to H'$ and the induced tensor functor

$$\tilde{f}\colon M^H \to \mathbf{M}^{H'}, \qquad X \mapsto X\Big(X \to X \otimes H \overset{1\otimes f}{\to} X \otimes H'\Big)$$

such that the following diagram is commutative:



Now the tensor functor $\tilde{f}$ preserves coequalizers, and hence it induces a homomorphism μ: $Br(\mathbf{M}^H) \to Br(\mathbf{M}^{H'})$ such that $\tilde{G} = \mu\tilde{F}$. ∎

Finally, there exists a Brauer group theory of coalgebras in a braided category. For instance, one may choose the braided monoidal category $\mathscr{C}$ as a k-linear abelian cocomplete category, where k is a field. Of course, k need not be the unit of $\mathscr{C}$. A basic example is $Br(\mathbf{M}^R)$, where R is a cocommutative k-coalgebra; cf. [26, 27]. A significant modification of [26] yields a theory of Azumaya coalgebras in a braided monoidal category.

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