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Stability of Central Finite Difference Schemes on Non–Uniform Grids for 1D Partial Differential Equations with Variable Coefficients

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Abstract. This paper deals with stability in the numerical solution of general one-dimensional partial differential equations with variable coefficients. We will generalize stability results for central finite difference schemes on non-uniform grids that were obtained by In 't Hout & Volders (2009) for the Black–Scholes equation. Subsequently we will apply our stability results to the CEV model.

Keywords: partial differential equations, finite difference discretization, stability
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INTRODUCTION

In [3] stability results were obtained for central finite difference schemes on non-uniform spatial grids for the Black–Scholes equation. In this paper some of these stability estimates will be extended for general one-dimensional partial differential equations (PDEs) with variable coefficients.

Consider the one-dimensional partial differential equation

$$\frac{\partial u}{\partial t}(s,t) = d(s)\frac{\partial^2 u}{\partial s^2}(s,t) + a(s)\frac{\partial u}{\partial s}(s,t) + r(s)u(s,t) \qquad (L < s < S, 0 < t \leq T). \tag{1}$$

This PDE is a time-dependent advection-diffusion-reaction equation with variable coefficients $d(s)$, $a(s)$ and $r(s)$, such that $d(s) > 0$. The interval (L,S) is a given spatial domain. We assume that an initial condition is provided and that the boundary conditions are of Dirichlet type.

For the numerical solution of (1) we consider finite difference (FD) schemes on non-uniform grids, which we denote by $L = s_0 < s_1 < \ldots < s_m < s_{m+1} = S$ and set $h_i = s_i - s_{i-1}$.

We next describe the finite difference schemes that will be used for the semi-discretization of (1). Let $f : [L,S] \to \mathbb{R}$ be any given function and $1 \leq i \leq m$. Let $H_i = h_i + h_{i+1}$. Consider the following two central FD schemes for approximating the first derivative $f'(s_i)$:

$$f'(s_i) \approx \frac{f(s_{i+1}) - f(s_{i-1})}{H_i}, \tag{2a}$$

$$f'(s_i) \approx -\frac{h_{i+1}}{h_i H_i} f(s_{i-1}) + \frac{h_{i+1} - h_i}{h_i h_{i+1}} f(s_i) + \frac{h_i}{h_{i+1} H_i} f(s_{i+1}). \tag{2b}$$

We refer to (2a) as *method A* and to (2b) as *method B*, cf. Veldman & Rinzema [5]. For approximating the second derivative $f''(s_i)$ the central FD scheme

$$f''(s_i) \approx \frac{2}{h_i H_i} f(s_{i-1}) - \frac{2}{h_i h_{i+1}} f(s_i) + \frac{2}{h_{i+1} H_i} f(s_{i+1}). \tag{3}$$

is used. The FD formulas (2a) and (3) have a second-order truncation error whenever the grid is smooth. The FD formula (2b) has a second-order truncation error for arbitrary grids. If the grid is uniform, then (2a) and (2b) both reduce to the standard second-order central scheme for advection, and (3) becomes the standard second-order central scheme for diffusion.

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Semi-discretization of (1) leads to initial value problems for systems of ordinary differential equations (ODEs) of the form

$$U'(t) = AU(t) + b(t) \quad (0 \le t \le T), \quad U(0) = U_0. \tag{4}$$

Here A is a given $m \times m$–matrix and $b(t)$ $(0 \le t \le T)$, U_0 are given m–vectors. The vector U_0 is given by the initial condition and the vector function $b(t)$ is determined by the boundary conditions. The entries of the solution vector $U(t)$ to (4) form approximations to the exact solution values $u(s,t)$ at the spatial grid points $s = s_1, s_2, \ldots, s_m$.

This paper deals with stability of the semi-discretization (4) of (1) on non-uniform grids, in the sense that we are interested in upper bounds

$$||e^{tA}|| \le Ke^{t\omega} \quad (t \ge 0), \tag{5}$$

where $||\cdot||$ denotes a given induced matrix norm and $\omega \in \mathbb{R}$, $K \ge 1$ are constants. If ω and K are of moderate size, then (5) implies that any errors, e.g. rounding errors or spatial truncation errors, can only grow in a moderate fashion.

In this paper we make use of the *logarithmic norm* as an important tool in our stability analysis. The logarithmic norm of a matrix A is defined as

$$\mu[A] = \lim_{t \downarrow 0} \frac{||I + tA|| - 1}{t},$$

where I is the $m \times m$ identity matrix. The following theorem provides a key property of the logarithmic norm; see e.g. [2].

Theorem 1 *Let $A \in \mathbb{R}^{m\times m}$ and $\omega \in \mathbb{R}$. Then*

$$\mu[A] \le \omega \quad \Longleftrightarrow \quad ||e^{tA}|| \le e^{t\omega} \ \textit{for all} \ \ t \ge 0.$$

This paper gives practical upper bounds on $\mu[A]$. By virtue of Theorem 1, this provides stability estimates of the type (5) with $K = 1, \omega = \mu[A]$.

Let $|\cdot|_2$ be the Euclidean vector norm and define the diagonal matrix H by

$$H = \frac{1}{2}\mathrm{diag}(H_1, H_2, \ldots, H_m).$$

For vectors $x \in \mathbb{R}^m$ we define the scaled Euclidean norm

$$|x|_H = |H^{1/2}x|_2,$$

and denote the induced matrix norm by $||\cdot||_H$. Note that this scaling is natural when considering non-uniform spatial grids.

STABILITY ESTIMATES IN THE SCALED EUCLIDEAN NORM

FD discretization of (1) on a general grid $L = s_0 < s_1 < \ldots < s_m < s_{m+1} = S$ and with Dirichlet boundary conditions, leads to a system of ODEs (4) with certain $m \times m$–matrix A. If the advection term $a(s)u_s$ is discretized using the FD scheme (2a), then the corresponding part of A is given by the tridiagonal matrix

$$A_1 = \mathrm{tridiag}\left(-\frac{a(s_i)}{H_i}\,,\, 0\,,\, \frac{a(s_i)}{H_i}\right). \tag{6}$$

If the advection term $a(s)u_s$ is discretized according to the FD scheme (2b), then the corresponding part of A is given by

$$A_2 = \mathrm{tridiag}\left(-\frac{a(s_i)h_{i+1}}{h_iH_i}\,,\, \frac{a(s_i)(h_{i+1}-h_i)}{h_ih_{i+1}}\,,\, \frac{a(s_i)h_i}{h_{i+1}H_i}\right). \tag{7}$$

For the discretization of the diffusion term $d(s)u_{ss}$ with the FD scheme (3), the corresponding part of A is

$$A_3 = 2\cdot\mathrm{tridiag}\left(\frac{d(s_i)}{h_iH_i}\,,\, -\frac{d(s_i)}{h_ih_{i+1}}\,,\, \frac{d(s_i)}{h_{i+1}H_i}\right). \tag{8}$$

Theorems 2, 3 and 4 below provide useful upper bounds for $||e^{tA_1}||_H, ||e^{tA_2}||_H$ and $||e^{tA_3}||_H$. For the sake of brevity all proofs in this paper will be omitted. They are fully analogous to those in [3].

Theorem 2 *Consider the matrix A_1 given by (6). Then*

$$||e^{tA_1}||_H \le e^{t\omega_1}$$

with

$$\omega_1 = \max_{2\le i\le m} \frac{|a(s_{i-1}) - a(s_i)|}{\sqrt{H_{i-1}H_i}}. \tag{9}$$

If the function $a(s)$ satisfies a Lipschitz condition, i.e.

$$|a(\tilde{s}) - a(s)| \le \Lambda_1 \cdot |\tilde{s} - s| \qquad (\textit{whenever } L \le s, \tilde{s} \le S)$$

then

$$\omega_1 \le \frac{\Lambda_1}{1+\sqrt{\varepsilon_0}}$$

with

$$\varepsilon_0 = \min_{2\le i\le m} \left\{ \frac{h_{i-1}h_{i+1}}{h_i^2} \right\}. \tag{10}$$

We note that a stability estimate of this type has been presented in [2] for first order upwind schemes on uniform grids for the variable-coefficient advection equation.

Theorem 3 *Consider the matrix A_2 given by (7). Let*

$$\kappa = \max_{1\le i\le m} \frac{|a(s_i)(h_{i+1} - h_i)|}{h_i h_{i+1}}, \tag{11}$$

$$\gamma = \max \left\{ \frac{x + \frac{y}{x}}{\sqrt{(x+1)(1+\frac{y}{x})}} \; : \; x \in \{\delta_0, \delta_1\}, y \in \{\varepsilon_0, \varepsilon_1\} \right\}, \tag{12}$$

where

$$\delta_0 = \min_{2\le i\le m} \frac{h_{i-1}}{h_i}, \;\; \varepsilon_0 = \min_{2\le i\le m} \frac{h_{i-1}h_{i+1}}{h_i^2}, \tag{13}$$

$$\delta_1 = \max_{2\le i\le m} \frac{h_{i-1}}{h_i}, \;\; \varepsilon_1 = \max_{2\le i\le m} \frac{h_{i-1}h_{i+1}}{h_i^2}. \tag{14}$$

Then

$$||e^{tA_2}||_H \le e^{t\omega_2} \quad (t \ge 0) \quad \textit{with} \;\; \omega_2 = \kappa + \gamma\kappa + \omega_1$$

where ω_1 is given by (9).

We remark that κ is of moderate size for smooth grids [3]. Clearly the surprising result holds that $\omega_1 \le \omega_2$, which implies that the stability bound in the case of the straightforward advection method A is more favorable than that in the case of the more sophisticated advection method B. Numerical experiments for the Black–Scholes equation are in agreement with this observation, see [3].

Theorem 4 *Consider the matrix A_3 given by (8). Then*

$$||e^{tA_3}||_H \le e^{t\omega_3} \quad (t \ge 0) \quad \textit{with} \;\; \omega_3 = \max_{1\le i\le m} \left\{ \frac{d(s_{i-1})}{h_i H_i} - \frac{d(s_i)}{h_i h_{i+1}} + \frac{d(s_{i+1})}{h_{i+1} H_i} \right\}.$$

If the function $d(s)$ is twice continuously differentiable and

$$\max_{L\leq s\leq S} d''(s) \leq \Lambda_2$$

then

$$\omega_3 \leq \frac{\Lambda_2}{2}.$$

Note that for the reaction term $r(s)u$ the trivial stability result

$$||e^{tR}||_H \leq e^{t\omega_4} \quad \text{with} \quad \omega_4 = \max\{r(s_1), r(s_2), \dots, r(s_m)\}$$

where $R = \mathrm{diag}(r(s_i))$ holds.

Theorems 2, 3 and 4 are generalizations of Theorems 2.2, 2.4 and 2.6 in [3] respectively.

APPLICATION TO THE CEV MODEL

As an example, we apply the obtained results to the Constant Elasticity of Variance (CEV) model [1, 4]

$$\frac{\partial u}{\partial t}(s,t) = \frac{1}{2}\sigma^2 s^{2\alpha}\frac{\partial^2 u}{\partial s^2}(s,t) + (r-q)s\frac{\partial u}{\partial s}(s,t) - ru(s,t) \qquad (L < s < S, 0 < t \leq T).$$

We assume here that $L > 0$ is a strictly positive lower barrier, r is the risk-free interest rate, q is the dividend yield, σ is a volatility parameter and $0 < \alpha \leq 1$. Note that the case $\alpha = 1$ corresponds to the Black-Scholes equation and $\alpha = 1/2$ represents the Cox-Ingersoll-Ross model. By using Theorem 2, we find that

$$||e^{tA_1}||_H \leq e^{t\omega_1} \quad \text{with} \quad \omega_1 = \frac{|r-q|}{1+\sqrt{\varepsilon_0}} \tag{15}$$

where ε_0 is given by (10). By applying Theorem 3, we obtain that

$$||e^{tA_2}||_H \leq e^{t\omega_2} \quad \text{with} \quad \omega_2 = \kappa + \gamma\kappa + \omega_1$$

where ω_1 is given by (15) and

$$\kappa = |r-q| \cdot \max_{1\leq i\leq m} \frac{s_i|h_{i+1}-h_i|}{h_i h_{i+1}}$$

and γ is given by (12). Finally we use Theorem 4 to establish the stability result

$$||e^{tA_3}||_H \leq e^{t\omega_3} \quad \text{with} \quad \omega_3 = \max\left\{\frac{\sigma^2}{2}\alpha(2\alpha-1)L^{2\alpha-2}, 0\right\}.$$

Note that for $\alpha \leq 1/2$ the strong result holds that $\omega_3 = 0$.

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