

A Stability Result for the Modified Craig–Sneyd Scheme Applied to 2D and 3D Pure Diffusion Equations

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Abstract. The stability of ADI type schemes is considered when applied to two- and three-dimensional pure diffusion equations with mixed derivative terms. We present a first stability result that takes the actual size of the mixed derivative coefficients into account.

Keywords: initial-boundary value problems, diffusion equations, finite difference methods, ADI splitting schemes, von Neumann stability
PACS: 02.60.Lj, 02.70.Bf

INTRODUCTION

This paper deals with the stability of Alternating Direction Implicit (ADI) schemes when adapted to (semi-discretized) multi-dimensional partial differential equations (PDEs) possessing mixed spatial derivative terms. PDEs of this kind are pervasive, notably, in the field of financial mathematics, see e.g. [4, 5]. ADI schemes form a popular class of numerical solution methods. However, ADI schemes were not originally developed for PDEs with mixed derivative terms, and it is thus important to gain insight in their stability in these applications. Recently, In 't Hout & Welfert [3] proved positive stability results in the von Neumann sense for a variety of ADI schemes when applied to multi-dimensional pure diffusion equations with mixed derivative terms. These results are valid without any assumption on the size of the coefficients of the mixed derivatives, except that the diffusion equation is parabolic. In the present paper we introduce a useful refinement of the analysis from [3] that takes the actual size of the mixed derivative coefficients into account. We derive for a specific ADI scheme a first stability result in this new setting.

Consider for $k = 2$ or $k = 3$ the k -dimensional pure diffusion equation with mixed derivative terms

$$\frac{\partial u}{\partial t} = \sum_{i \neq j} d_{ij} u_{x_i x_j} + d_{11} u_{x_1 x_1} + d_{22} u_{x_2 x_2} + \cdots + d_{kk} u_{x_k x_k} \quad (1)$$

on the spatial domain $[0, 1]^k$ with given initial condition and periodic boundary condition. Here $D = [d_{ij}]_{1 \leq i, j \leq k}$ is any given constant real matrix that is always assumed to be symmetric and positive semi-definite, in particular

$$d_{jj} \geq 0 \quad \text{and} \quad |d_{ij}| \leq \sqrt{d_{ii} d_{jj}} \quad \text{for } 1 \leq i, j \leq k. \quad (2)$$

Finite difference discretization on a Cartesian grid of the spatial derivatives in equation (1) leads to initial value problems for large systems of stiff ordinary differential equations (ODEs),

$$U'(t) = AU(t) \quad (t \geq 0), \quad U(0) = U_0 \quad (3)$$

with given real matrix A and real vector U_0 . For the efficient numerical solution of semi-discrete systems (3), splitting schemes of the ADI type form an attractive class of methods. The adaptation of ADI schemes in the presence of mixed derivative terms starts with a splitting of the matrix A into

$$A = A_0 + A_1 + \cdots + A_k, \quad (4)$$

where A_0 represents all mixed derivatives and A_j (for $1 \leq j \leq k$) represents the derivative in the j -th spatial direction. We consider in this paper a specific method of the ADI type, called the Modified Craig–Sneyd (MCS) scheme. For any given time step $\Delta t > 0$, the MCS scheme defines approximations U_n to $U(n \cdot \Delta t)$ successively for $n = 1, 2, 3, \dots$ by

$$\left\{ \begin{array}{l} Y_0 = U_{n-1} + \Delta t A U_{n-1}, \\ Y_j = Y_{j-1} + \theta \Delta t A_j (Y_j - U_{n-1}) \quad (j = 1, 2, \dots, k), \\ \widehat{Y}_0 = Y_0 + \theta \Delta t A_0 (Y_k - U_{n-1}), \\ \widetilde{Y}_0 = \widehat{Y}_0 + (\frac{1}{2} - \theta) \Delta t A (Y_k - U_{n-1}), \\ \widetilde{Y}_j = \widetilde{Y}_{j-1} + \theta \Delta t A_j (\widetilde{Y}_j - U_{n-1}) \quad (j = 1, 2, \dots, k), \\ U_n = \widetilde{Y}_k. \end{array} \right. \quad (5)$$

Here $\theta > 0$ denotes any given real parameter. The MCS scheme was introduced and studied in [3] and subsequently examined in [1, 2]. Taylor expansion yields that the scheme has classical order of consistency equal to two for any θ . An inspection of (5) shows that the A_0 part is always treated in an *explicit* way, whereas the A_j parts for $1 \leq j \leq k$ are successively treated in an *implicit* way. The scheme (5) retains the key advantage of traditional ADI schemes: the linear systems are efficiently solved by LU factorization, as the pertinent matrices all possess a small bandwidth.

This paper deals with a von Neumann stability analysis of the MCS scheme. In line with the well-known von Neumann analysis we already assumed a periodic boundary condition and constant coefficients d_{ij} and we next assume also constant mesh widths in the different spatial directions. From this it follows that the A_j ($0 \leq j \leq k$) are Kronecker products of circulant (hence normal) matrices that commute with each other, and are thus simultaneously diagonalizable by a unitary matrix. This implies that stability can be rigorously analyzed by considering the scalar test equation

$$U'(t) = (\lambda_0 + \lambda_1 + \dots + \lambda_k)U(t) \quad (t \geq 0) \quad (6)$$

with λ_j corresponding eigenvalues of A_j ($0 \leq j \leq k$). In the case of (6), the MCS scheme becomes the scalar iteration

$$U_n = S_\theta(z_0, z_1, \dots, z_k)U_{n-1}, \quad (7)$$

where $z_j = \Delta t \cdot \lambda_j$ and

$$S_\theta(z_0, z_1, \dots, z_k) = 1 + \frac{z_0 + z}{p} + \theta \frac{z_0(z_0 + z)}{p^2} + (\frac{1}{2} - \theta) \frac{(z_0 + z)^2}{p^2} \quad (8)$$

with the notation

$$z = z_1 + z_2 + \dots + z_k \quad \text{and} \quad p = (1 - \theta z_1)(1 - \theta z_2) \dots (1 - \theta z_k).$$

The iteration (7) is stable if

$$|S_\theta(z_0, z_1, \dots, z_k)| \leq 1. \quad (9)$$

In [3] it was shown that for the standard central second-order FD discretization the scaled eigenvalues satisfy

$$\text{all } z_j \text{ are real,} \quad (10a)$$

$$z_j \leq 0 \quad \text{for } 1 \leq j \leq k, \quad (10b)$$

$$z + z_0 \leq 0, \quad (10c)$$

$$|z_0| \leq \sum_{i \neq j} \sqrt{z_i z_j}, \quad (10d)$$

and the positive result was proved that if $\theta \geq \frac{1}{3}$ (for $k = 2$) or $\theta \geq \frac{6}{13}$ (for $k = 3$) then the stability requirement (9) is fulfilled, and hence, the MCS scheme is unconditionally stable in the von Neumann sense. In the following we derive a useful generalization of this result, which takes the size of the mixed derivative coefficients d_{ij} ($i \neq j$) into account.

FORMULATION OF THE MAIN RESULT

Consider for $\gamma \in [0, 1]$ the condition

$$d_{jj} \geq 0 \quad \text{and} \quad |d_{ij}| \leq \gamma \sqrt{d_{ii}d_{jj}} \quad \text{for } 1 \leq i, j \leq k. \quad (11)$$

Clearly the quantity γ is a measure for the relative size of the mixed derivative coefficients in the diffusion equation (1). If $\gamma = 1$, then the condition (2) is obtained, which admits the largest possible mixed derivative coefficients. On the other hand, if $\gamma = 0$, then all mixed derivative coefficients vanish. In financial applications one usually has $0 < \gamma < 1$. We remark that, by a similar derivation as in [3], if the condition (11) holds then the scaled eigenvalues satisfy

$$|z_0| \leq \gamma \sum_{i \neq j} \sqrt{z_i z_j}. \quad (12)$$

The property (12) is obviously stronger than (10d) whenever $\gamma < 1$. The interesting question arises, to any given γ , for which range of parameter values θ is the MCS scheme unconditionally stable. We have the following neat result.

Theorem 1. *Consider equation (1) for $k = 2$ or $k = 3$ with symmetric positive semi-definite matrix D and periodic boundary condition. Assume (3), (4) is obtained by central second-order FD discretization and splitting as described in Section . Let $\gamma \in [0, 1]$ be such that (11) holds. Then the MCS scheme is unconditionally stable when applied to (3), (4) whenever $\theta \geq \max\{\frac{1}{4}, \frac{1}{6}(\gamma + 1)\}$ (for $k = 2$) or $\theta \geq \max\{\frac{1}{4}, \frac{2}{13}(2\gamma + 1)\}$ (for $k = 3$).*

The above result can be viewed as a generalization to Theorem 2.5 in [3]. The latter theorem is recovered by setting γ equal to 1.

Proof. The proof follows along the lines of [3] and uses the properties (10a)–(10c) and (12) of the scaled eigenvalues. For the sake of brevity, we consider here only the case $k = 2$ and shall incorporate the proof for $k = 3$ in a future paper.

The stability requirement (9) is equivalent to (13), where

$$2p^2 + pz + (\frac{1}{2} - \theta)z^2 + [p + (1 - \theta)z]z_0 + \frac{1}{2}z_0^2 \geq 0, \quad (13a)$$

$$p - \theta z + \frac{1}{2}(z_0 + z) \geq 0. \quad (13b)$$

The condition (13a) is fulfilled if the discriminant

$$\begin{aligned} \Delta &= [p + (1 - \theta)z]^2 - 2[2p^2 + pz + (\frac{1}{2} - \theta)z^2] \\ &= -3p^2 - 2\theta pz + \theta^2 z^2 \\ &= -(p + \theta z)(3p - \theta z) \end{aligned}$$

is non-positive, and this holds (note that $p \geq 1 - \theta z \geq 1$). Next consider the condition (13b). Write $y_j = \sqrt{-\theta z_j}$. Then

$$p = (1 + y_1^2)(1 + y_2^2) \quad \text{and} \quad z = -\frac{1}{\theta}(y_1^2 + y_2^2)$$

and

$$z + z_0 \geq z - |z_0| \geq z_1 + z_2 - 2\gamma\sqrt{z_1 z_2} = -\frac{1}{\theta}(y_1^2 + y_2^2 + 2\gamma y_1 y_2).$$

Using this one readily obtains that if

$$(1 + y_1^2)(1 + y_2^2) + y_1^2 + y_2^2 - \kappa(y_1^2 + y_2^2 + 2\gamma y_1 y_2) \geq 0 \quad (14)$$

for some $\kappa > 0$, then (13b) is fulfilled whenever $\theta \geq 1/(2\kappa)$. The inequality (14) can be rewritten as

$$(2 - \kappa)(y_1 - y_2)^2 + (y_1 y_2 + 2 - \kappa - \kappa\gamma)^2 + 1 - (2 - \kappa - \kappa\gamma)^2 \geq 0,$$

which is clearly satisfied when

$$\kappa \leq 2 \quad \text{and} \quad |2 - \kappa - \kappa\gamma| \leq 1.$$

The latter condition is equivalent to

$$\frac{1}{\gamma + 1} \leq \kappa \leq \min \left\{ 2, \frac{3}{\gamma + 1} \right\}.$$

Selecting for κ the rightmost value of this interval, it follows that (13b) holds whenever $\theta \geq \max\{\frac{1}{4}, \frac{1}{6}(\gamma + 1)\}$. \square

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