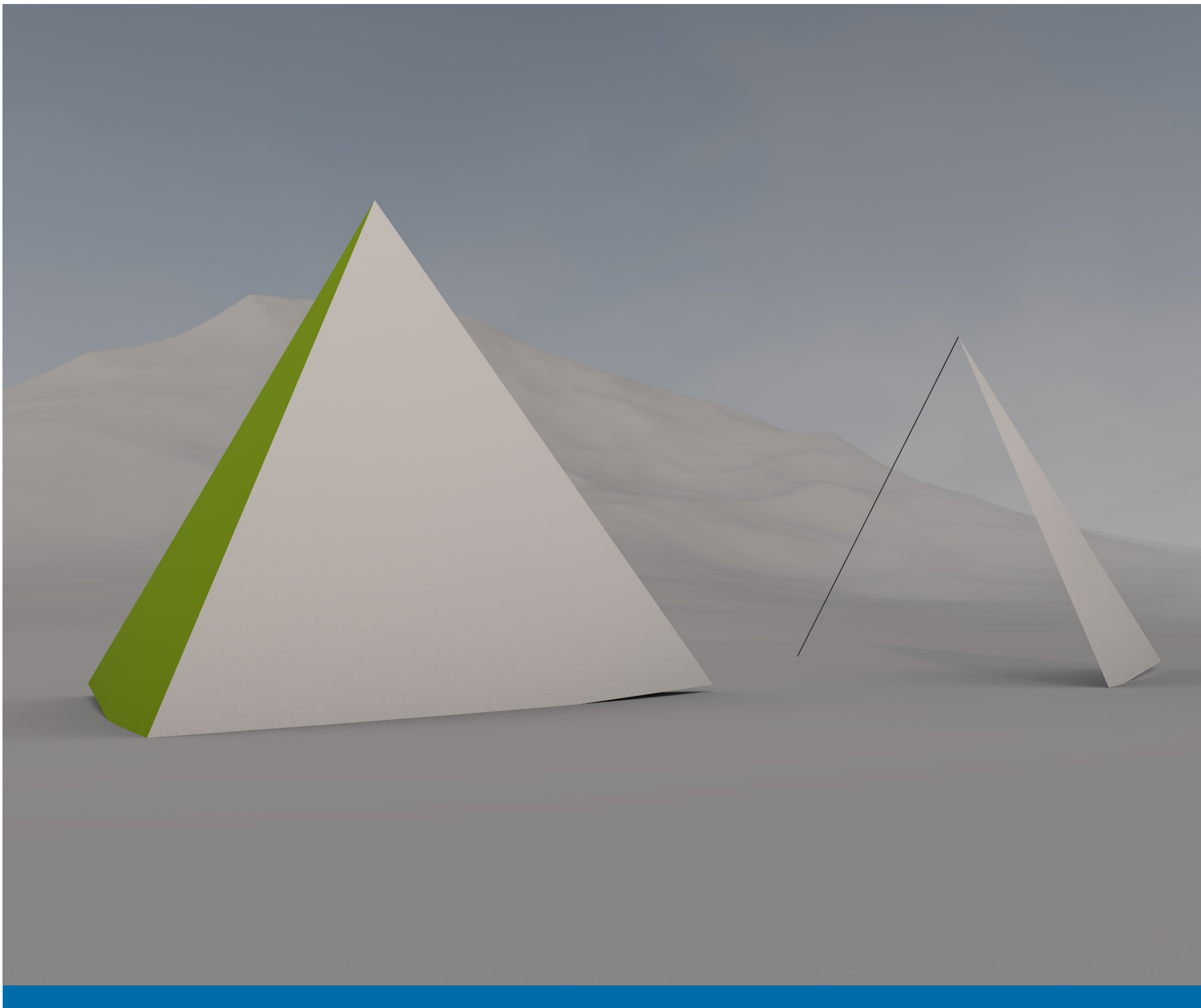
# Templicial objects: Simplicial objects in a monoidal category

## Templiciale objecten: simpliciale objecten in een monoïdale categorie


**Arne Mertens**


Supervisor **prof. dr. Wendy Lowen**



Thesis submitted in fulfilment of the requirements for the degree of Doctor of Science: Mathematics

**Faculty of Science | Antwerp, 2022**

University of Antwerp

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# Contents

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*"This song is new to me, but I am honored to be a part of it."*

- Solanum (Outer Wilds)

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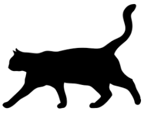


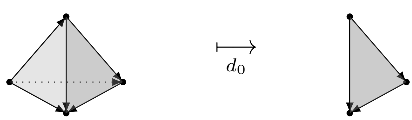
# Introduction

Simplicial sets are collections of simplices (vertices, edges, triangles, tetrahedra, ...) that are glued together along common faces. They are foundational objects in algebraic topology and higher category theory, appearing as combinatorial variants of topological spaces and in many different models of $(\infty, 1)$-categories like quasi-categories, Segal categories and simplicial categories.

Formally, a *simplicial set* is a functor $X : \mathbf{\Delta}^{op} \to \mathrm{Set}$ where $\mathbf{\Delta}$ is known as the simplex category. Explicitly, $X$ is given by a certain commutative diagram of maps of sets

$$\cdots \; \substack{\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow} \; X_2 \; \substack{\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow} \; X_1 \; \substack{\longrightarrow\\\longleftarrow\\\longrightarrow} \; X_0 \tag{1}$$

So in particular we are given maps $d_0, ..., d_n : X_n \to X_{n-1}$ for all $n \geq 1$. These are called *face maps*. The elements of each set $X_n$ should be interpreted as simplices of dimension $n$, and the maps express how these simplices are connected. For instance, the face map $d_i : X_n \to X_{n-1}$ takes an $n$-simplex and sends it to its $i$th face, which is a simplex of one dimension lower.




A classical construction associates to every (small) category $\mathcal{C}$ a simplicial set $N(\mathcal{C})$. This is known as the *nerve functor* $N : \mathrm{Cat} \to \mathrm{SSet}$. For every $n \geq 0$, the $n$-simplices of $N(\mathcal{C})$ are given by sequences $(f_1, ..., f_n)$ of composable morphisms in $\mathcal{C}$. In other words,

$$N(\mathcal{C})_n = \coprod_{A_0,...,A_n \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(A_0, A_1) \times ... \times \mathcal{C}(A_{n-1}, A_n)$$

Its face maps $d_j$ for $0 < j < n$ are defined by composing morphisms in the sequence: $(f_1, ..., f_j, f_{j+1}, ..., f_n) \mapsto (f_1, ..., f_{j+1} \circ f_j, ..., f_n)$. The face maps $d_0$ and $d_n$ on the other hand are defined by projection, e.g. $d_0$ uses the projection map

$$\mathcal{C}(A_0, A_1) \times \mathcal{C}(A_1, A_2) \times ... \times \mathcal{C}(A_{n-1}, A_n) \to \mathcal{C}(A_1, A_2) \times ... \times \mathcal{C}(A_{n-1}, A_n)$$

Now suppose we have a linear category $\mathcal{C}$ over a unital commutative ring $k$ (that is, each $\mathcal{C}(A, B)$ has a $k$-module structure and the composition is bilinear). What is the appropriate definition of a nerve for $\mathcal{C}$?

Analogous to the classical situation, we can try setting

$$N_k(\mathcal{C})_n = \coprod_{A_0,...,A_n \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(A_0, A_1) \otimes ... \otimes \mathcal{C}(A_{n-1}, A_n)$$

Then the face maps $d_j$ for $0 < j < n$ can still be defined through composition, but the face maps $d_0$ and $d_n$ cannot be defined in the same way because we lack projection morphisms

$$\mathcal{C}(A_0, A_1) \otimes \mathcal{C}(A_1, A_2) \otimes ... \otimes \mathcal{C}(A_{n-1}, A_n) \to \mathcal{C}(A_1, A_2) \otimes ... \otimes \mathcal{C}(A_{n-1}, A_n)$$

In other words, we do not obtain a simplicial $k$-module $N_k(\mathcal{C})$. But what do we get?

To answer this question, this thesis aims to develop a generalization of simplicial sets which may be interpreted as "simplicial objects in a monoidal category", and study what properties they possess. In particular, they allow to define a nerve for general enriched categories (e.g. $k$-linear categories). We call them *tensor-simplicial* or *templicial objects*.

Below we go into a little more detail to outline the ideas appearing in the thesis. We will formally define everything in the main text, but for now we just give a rough sketch of the objects involved.

## Simplicial objects in a monoidal category

Let us illustrate the main philosophy with a simple example. Consider a directed graph $G$, that is, a collection of vertices and edges between them. Formally, $G$ is given by a pair of sets $(G_1, G_0)$ where $G_0$ contains the vertices and $G_1$ the edges. Moreover, $G$ comes equipped with maps

$$G_1 \overset{s}{\underset{t}{\rightrightarrows}} G_0 \tag{2}$$

which send every edge to its source and target vertex respectively. We may denote an edge of $G$ as $e : a \to b$ to indicate that $s(e) = a$ and $t(e) = b$. A (small) category can then be equivalently described as a graph $G = (G_1, G_0)$ along with maps $m : G_1 \times_{G_0} G_1 \to G_1$ and $u : G_0 \to G_1$ specifying for each pair of edges $f : a \to b$ and $g : b \to c$ their composition $g \circ f = m(f, g)$, and for each vertex $a$ the identity $\mathrm{id}_a = u(a)$ on $a$. Of course $m$ and $u$ have to satisfy the appropriate associativity and unitality conditions, as well as compatibility conditions with respect to $s$ and $t$.

In their PhD thesis [Agu97], Aguiar defined graphs and categories internal to a monoidal category $(\mathcal{V}, \otimes, I)$ (e.g. $\mathcal{V} = \mathrm{Mod}(k)$ is the category of modules over a unital commutative ring $k$ with $\otimes$ the tensor product). For instance, the former is a pair $(G_1, G_0)$ with $G_0$ a comonoid in $\mathcal{V}$ and $G_1$ a bicomodule over $G_0$. That is, we have morphisms

$$\begin{gathered}\mu_{0,0} : G_0 \to G_0 \otimes G_0, \quad \epsilon : G_0 \to I \\ \mu_{0,1} : G_1 \to G_0 \otimes G_1 \quad \text{and} \quad \mu_{1,0} : G_1 \to G_1 \otimes G_0\end{gathered} \tag{3}$$

which satisfy some appropriate coherence diagrams. When $\mathcal{V} = \mathrm{Set}$ with the cartesian monoidal structure (i.e. $\otimes = \times$ is the cartesian product), then graphs and categories internal to $\mathcal{V}$ recover graphs and categories in the usual sense. For example, $s : G_1 \to G_0$ can be recovered by composing $\mu_{0,1}$ with the projection $G_0 \times G_1 \to G_0$. The morphisms $\mu_{0,1}$ and $\mu_{1,0}$ may thus be regarded as a replacement for the morphisms $s$ and $t$ when $\mathcal{V}$ is a general (non-cartesian) monoidal category.

Moreover, under some hypotheses on $\mathcal{V}$, $\mathcal{V}$-enriched categories can be recovered from categories internal to $\mathcal{V}$ as well. Indeed, let $F : \mathrm{Set} \to \mathcal{V}$ denote the left-adjoint of the forgetful functor $U = \mathcal{V}(I, -) : \mathcal{V} \to \mathrm{Set}$. Then a category $(G_1, G_0)$ internal to $\mathcal{V}$ recovers a $\mathcal{V}$-enriched category with object set $S$ if $G_0 \simeq F(S)$, such that the comultiplication and counit of $G_0$ are induced by the diagonal $S \to S \times S$ and the terminal map $S \to 1$ respectively.

Note how the diagram of a simplicial set (1) extends that of a directed graph (2). We can do the same for graphs internal to $\mathcal{V}$. Following an idea of Leinster in [Lei00], we can define a "simplicial object internal to $\mathcal{V}$" as a colax monoidal functor $X : \mathbf{\Delta}_f^{op} \to \mathcal{V}$. Here, $\mathbf{\Delta}_f$ is the category of *finite intervals*, a subcategory of the usual simplex category $\mathbf{\Delta}$. The colax structure provides morphisms in $\mathcal{V}$

$$\mu_{k,l} : X_{k+l} \to X_k \otimes X_l \quad \text{for all } k, l \geq 0$$

extending the bicomodule structure (3). Leinster showed in particular that if $\mathcal{V} = \mathrm{Set}$, then this indeed recovers simplicial sets. As for directed graphs, we can also restrict to the case where $X_0 \simeq F(S)$ in the appropriate way. The latter is essentially the definition of a templicial object, although we formalize them slightly differently, using $\mathcal{V}$-enriched quivers instead.

By a $\mathcal{V}$-enriched quiver, we mean a pair $(Q, S)$ with $S$ a set and $Q$ a collection $(Q(a,b))_{a,b \in S}$ of objects $Q(a,b) \in \mathcal{V}$. Quivers with a fixed set $S$ can be organized into a monoidal category $(\mathcal{V}\,\mathrm{Quiv}_S, \otimes_S, I_S)$. A *templicial object* (Definition 2.1.9) of $\mathcal{V}$ is then defined as a pair $(X, S)$ with $S$ a set and

$$X : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$$

a strongly unital, colax monoidal functor. To emphasize the monoidal structure involved, we will denote the category of templicial objects (with varying sets $S$) by

$$S_{\otimes}\mathcal{V}$$

When $\mathcal{V} = \mathrm{Set}$, this again recovers the category $\mathrm{SSet}$ of simplicial sets (Proposition 2.1.15).

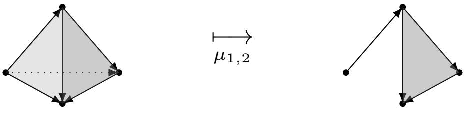
Let us unpack this definition a bit. The elements of $S$ should be considered as vertices of the templicial object $(X, S)$. Then for every $a, b \in S$ we have a diagram in $\mathcal{V}$:

$$\cdots \;\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow}\; X_2(a,b) \;\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow}\; X_1(a,b) \longleftarrow X_0(a,b)$$

Note that the outer face maps of (1) have disappeared. They have been replaced by the colax monoidal structure, which provides morphisms in $\mathcal{V}$:

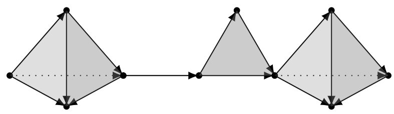
$$(\mu_{k,l})_{a,b} : X_{k+l}(a,b) \to \coprod_{c \in S} X_k(a,c) \otimes X_l(c,b) \quad \text{for all } k, l \geq 0$$

Even though they are not sets, we can intuitively think about the objects $X_n(a,b)$ as containing the "$n$-simplices with first vertex $a$ and last vertex $b$". The morphisms $\mu_{k,l}$ may then be interpreted as "pulling apart" a $(k+l)$-simplex into a $k$-simplex and an $l$-simplex which are joint at a vertex.



Thus from a simplex of $X$, we can no longer access its outer faces directly. But we can recover faces which are joint at a vertex. This naturally leads us to considering necklaces.

A *necklace* (Definition 2.2.3) is a simplicial set composed of a finite sequence of simplices (called *beads*) that are glued along vertices (as opposed to higher dimensional faces). They were introduced by Dugger and Spivak in [DS11b] to demystify the categorification functor $\mathfrak{C} : \mathrm{SSet} \to \mathrm{Cat}_\Delta$ relating simplicial sets to simplicial categories.



Given a simplicial set $X$ and a necklace $T$ with beads of dimensions $n_1, ..., n_k$, a map $T \to X$ corresponds to an element of the set $X_{n_1} \times_{X_0} ... \times_{X_0} X_{n_k}$. We can consider such an element as being a necklace of shape $T$ *in* $X$. If $(X, S)$ is a templicial object in $\mathcal{V}$, then we can similarly consider the quiver $X_T = X_{n_1} \otimes_S ... \otimes_S X_{n_k}$. This construction extends to a coreflective embedding $(-)^{nec} : S_\otimes \mathcal{V} \hookrightarrow \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ which associates to $X$ a certain enriched category $X^{nec}$ which we call a *necklace category* (Definition 3.2.3). Many proofs can be simplified by passing to necklace categories, for instance that $S_\otimes \mathcal{V}$ is complete (Proposition 3.2.33) and locally presentable (Theorem 3.2.29) whenever $\mathcal{V}$ is.

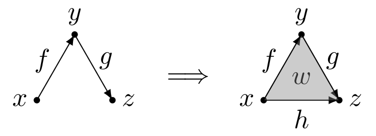
# Quasi-categories

Quasi-categories are one of the many models of $(\infty, 1)$-categories. Their theory was developed by Joyal [Joy02] and extensively expounded upon by Lurie [Lur09a]. Since then they have been studied by many others.

Given $n \geq 0$, we denote $\Delta^n$ for the simplicial set consisting of a single simplex in dimension $n$ and all its faces. For $0 \leq j \leq n$, the *$j$th horn* $\Lambda^n_j$ is obtained from $\Delta^n$ by removing the interior and the $j$th face. A *quasi-category* (or $\infty$*-category*) is then defined as a simplicial set $X$ which satisfies the *weak Kan condition*. This means that for all $0 < j < n$, every map of simplicial sets $\Lambda^n_j \to X$ can be extended to a map $\Delta^n \to X$.

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

A quasi-category exhibits behaviour that resembles that of a category. For example, consider the weak Kan condition where $n = 2$ and $j = 1$. This tells us that for any two edges $f : a \to b$ and $g : b \to c$, there exists a 2-simplex $w \in X_2$ filling up the horn formed by $f$ and $g$:




We can then consider the edge $h = d_1(w)$ as a composition of $f$ and $g$. Note that $w$ is only assumed to exist however, not that it is unique. Indeed, usually $w$ will not be uniquely determined by $f$ and $g$, but it will be unique up to a notion of *homotopy* that can be defined inside of $X$. In other words, composition is no longer given by a map $m : (f, g) \mapsto g \circ f$, but is witnessed by a piece of data (in this case $w$) in a higher dimension. Similarly, the identities and associativity for the composition are not given by equations but by higher dimensional simplices witnessing them.

To any templicial object $(X, S)$ in a monoidal category $\mathcal{V}$, with $a, b \in S$, the assignment $T \mapsto X_T(a, b) \in \mathcal{V}$ defines a functor $X_\bullet(a, b) : \mathcal{N}ec^{op} \to \mathcal{V}$ where $\mathcal{N}ec$ denotes the category of necklaces. We will call $(X, S)$ a *quasi-category in* $\mathcal{V}$ (Definition 2.2.26) if for all $a, b \in S$ and $0 < j < n$, we have the following lifting property:

$$\begin{tikzcd}
F\left((\Lambda^n_j)_\bullet(0,n)\right) \arrow[r] \arrow[d, hook] & X_\bullet(a,b) \\
F\left(\Delta^n_\bullet(0,n)\right) \arrow[ur, dashed] &
\end{tikzcd}$$

in $\mathcal{V}^{\mathcal{N}ec^{op}}$. Similarly, for $n = 2$ and $j = 1$, this lifting property expresses that for any element $\alpha \in U\left(\coprod_{c\in S} X_1(a, c) \otimes X_1(c, b)\right)$ we can find some $w \in U(X_2(a, b))$ such that $\mu_{1,1}(w) = \alpha$. We can then consider $d_1(w) \in U(X_1(a, b))$ as a composition of $\alpha$ in $X$.

The better part of this thesis is devoted to constructing templicial analogues of classical examples of quasi-categories:

- The classical nerve functor $N : \mathrm{Cat} \to \mathrm{SSet}$ generalizes to the *templicial nerve functor* $N_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat} \to S_\otimes\mathcal{V}$ which associates to every small $\mathcal{V}$-enriched category a quasi-category in $\mathcal{V}$ (Construction 2.3.4).

- There is the homotopy coherent nerve functor $N^{hc} : \mathrm{Cat}_\Delta \to \mathrm{SSet}$ of Cordier [Cor82], associating to every simplicial category $\mathcal{C}$ (that is, a category enriched in simplicial sets), a simplicial set $N^{hc}(\mathcal{C})$. Cordier and Porter showed in [CP86] that $N^{hc}(\mathcal{C})$ is a quasi-category when every hom-object $\mathcal{C}(A, B)$ is a Kan complex. We generalize this to the *templicial homotopy coherent nerve functor* $N^{hc}_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat}_\Delta \to S_\otimes\mathcal{V}$ (Definition 4.1.13) where $\mathcal{V}\,\mathrm{Cat}_\Delta$ denotes the category of small categories enriched in simplicial objects $S\mathcal{V}$. Further, $N^{hc}_{\mathcal{V}}(\mathcal{C})$ will be a quasi-category in $\mathcal{V}$ if every hom-object $\mathcal{C}(A, B) \in S\mathcal{V}$ has an underlying Kan complex.

- There is the differential graded (dg) nerve $N^{dg} : k\,\mathrm{Cat}_{dg} \to \mathrm{SSet}$ [Lur16] associating to every small dg-category $\mathcal{C}_\bullet$ over a ring $k$ a quasi-category $N^{dg}(\mathcal{C})$. We will lift this to the *linear dg-nerve* $N^{dg}_k : k\,\mathrm{Cat}_{dg} \to S_\otimes\,\mathrm{Mod}(k)$ (Definition 4.2.46) which associates to every dg-category over $k$, a quasi-category in $\mathrm{Mod}(k)$.

## Frobenius structures

Further, we will introduce *Frobenius structures* (Defintion 2.2.34) on templicial objects $X$. To motivate them, let us again consider an example. Consider a (small) simplicial category $\mathcal{C}$. Thus $\mathcal{C}$ has a set of objects $\mathrm{Ob}(\mathcal{C})$ and for every $A, B \in \mathrm{Ob}(\mathcal{C})$, we have a simplicial set $\mathcal{C}(A, B)$. In Cordier's homotopy coherent nerve $N^{hc}(\mathcal{C})$, vertices are given by $\mathrm{Ob}(\mathcal{C})$ and edges are given by the 0-simplices $f \in \mathcal{C}_0(A, B)$. A 2-simplex in $N^{hc}(\mathcal{C})$ is given by a diagram

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow\sigma"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

with $f \in \mathcal{C}_0(A, B)$, $g \in \mathcal{C}_0(B, C)$, $h \in \mathcal{C}_0(A, C)$ and $\sigma \in \mathcal{C}_1(A, C)$ a 1-simplex from $h$ to the composition $g \circ f \in \mathcal{C}_0(A, C)$.

For general $\mathcal{C}$, $N^{hc}(\mathcal{C})$ is not a quasi-category. Nonetheless, some lifing properties are still satisfied for $N^{hc}(\mathcal{C})$. For example, a map $\Lambda^2_1 \to N^{hc}(\mathcal{C})$ corresponds to a pair of 0-simplices $f \in \mathcal{C}_0(A, B)$ and $g \in \mathcal{C}_0(B, C)$ for some $A, B, C \in \mathrm{Ob}(\mathcal{C})$. Extending this map to $\Delta^2 \to N^{hc}(\mathcal{C})$ is then equivalent to finding some $h \in \mathcal{C}_0(A, C)$ and a 1-simplex $\sigma \in \mathcal{C}_1(A, C)$ from $h$ to $g \circ f$. But this is trivial. Just choose $h = g \circ f$ and let $\sigma$ be the degenerate 1-simplex on $g \circ f$:

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow ="] & \\ A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & C \end{tikzcd}$$

The horn $\Lambda^2_1$ is in particular also a necklace and this procedure generalizes to arbitrary necklaces. In fact, we can define maps

$$Z^{k,l} : N^{hc}(\mathcal{C})_k \times_{N^{hc}(\mathcal{C})_0} N^{hc}(\mathcal{C})_l \to N^{hc}(\mathcal{C})_{k+l}$$

assigning to every necklace in $N^{hc}(\mathcal{C})$ with two beads a simplex filling up the necklace.

Similarly, we will define a Frobenius structure on a templicial object $(X, S)$ as a collection of quiver morphisms

$$Z^{k,l} : X_k \otimes_S X_l \to X_{k+l} \quad \text{for all } k, l \geq 0$$

satisfying associativity and certain compatibility conditions with the morphisms $\mu_{k,l}$ of $X$. A templicial object $X$ with a Frobenius structure is in particular a Frobenius monoidal functor in the sense of Day and Pastro [DP08]. We will see that the templicial nerve, templicial homotopy coherent nerve and linear dg-nerve all naturally come equipped with Frobenius structures. In fact, $N^{dg}_k$ is defined through an equivalence of categories

$$k\,\mathrm{Cat}_{dg,\geq 0} \simeq S^{Frob}_{\otimes}\,\mathrm{Mod}(k)$$

between non-negatively graded dg-categories over $k$ and templicial $k$-modules with a Frobenius structure (Corollary 4.2.45).

More generally, we will consider *non-associative Frobenius (naF)* structures, for which the associativity condition above is dropped. As (non-associative) Frobenius structures also represent a filling condition, one might expect them to be related to quasi-categories in $\mathcal{V}$, and indeed they are. We will show:

- (Proposition 3.1.32) If $X$ is a projective quasi-category in $\mathcal{V}$, then $X$ has a naF-structure.
- (Theorem 4.2.62) If $\mathcal{V} = \mathrm{Mod}(k)$, then every templicial $k$-module with a naF-structure is a quasi-category in $\mathrm{Mod}(k)$.

# Towards homotopy theory

At present, quasi-categories in $\mathcal{V}$ merely exist in analogy to classical quasi-categories and a thorough study of their homotopical properties is still lacking.

In [Joy08], Joyal completely formalized the homotopy theory of quasi-categories by equipping the category $\mathrm{SSet}$ of simplicial sets with a model structure. The fibrant objects are precisely the quasi-categories and the cofibrations are the monomorphisms. So far we have not been able to build a similar model structure for quasi-categories in $\mathcal{V}$ and we leave this to future research. As a small step in that direction, we introduce *projective templicial morphisms* (Definition 3.1.24) which are the left lifting class in a weak factorization system on the category of templicial objects $S_{\otimes}\mathcal{V}$ (Theorem 3.1.28). If $\mathcal{V} = \mathrm{Set}$, they recover the monomorphisms of simplicial sets. As such, the author believes projective templicial morphisms to be the appropriate cofibrations in $S_{\otimes}\mathcal{V}$. With fibrant objects given by the quasi-categories in $\mathcal{V}$, this would completely determine the model structure.

Quasi-categories (or $(\infty, 1)$-categories in general) are often viewed as "categories weakly enriched in spaces". This idea was made formal by Gepner and Haugseng [GH15] who defined categories weakly enriched in a general monoidal $\infty$-category $\mathcal{M}$. It would be a mistake to view quasi-categories in $\mathcal{V}$ as being "weakly enriched in $\mathcal{V}$". With $\mathcal{V}$ being a plain monoidal category, there is no weak structure to exploit. Instead, it is probably more accurate to view them as being weakly enriched in simplicial objects $S\mathcal{V}$. Through the templicial homotopy coherent nerve $N^{hc}_{\mathcal{V}}$, the author believes a Quillen equivalence between $S_{\otimes}\mathcal{V}$ and $\mathcal{V}\,\mathrm{Cat}_{\Delta}$ should exist. Following [Hau15], this would relate templicial objects with categories weakly enriched in the monoidal $\infty$-category associated to $S\mathcal{V}$.

Leinster's idea of using colax monoidal functors instead of simplicial objects has also been applied by Bacard in [Bac10] to define a notion of Segal categories enriched in a monoidal model category $\mathcal{M}$ called *Segal $\mathcal{M}$-categories*. If $\mathcal{M}$ is the Quillen model category on $\mathrm{SSet}$, then this recovers classical Segal categories. As such, Segal $\mathcal{M}$-categories can also be considered as categories "weakly enriched in $\mathcal{M}$". This thesis represents the first steps in applying the same philosophy to the theory of quasi-categories as opposed to Segal categories. Though currently still conjectural, the author believes quasi-categories in $\mathcal{V}$ to be related to Segal $S\mathcal{V}$-categories in the same way that ordinary quasi-categories are related to Segal categories. We also leave this to future research.

## Notations and assumptions

- Integers are always denoted by lowercase letters, usually $i, j, k, l, m, n, ...$
- Sets, posets and necklaces are denoted by capital letters $S, T, U, ...$
- Both simplicial sets and general templicial objects are also denoted by capital letters. We'll distinguish between them by choosing different letters. Usually, that is $K, L, ...$ for simplicial sets and $X, Y, ...$ for templicial objects.
- Generic (enriched) categories are usually denoted by calligraphic letters $\mathcal{C}, \mathcal{D}, ...$, while objects of a category are usually denoted by capital letters $A, B, C, ...$ Given objects $A$ and $B$ of an enriched category $\mathcal{C}$, we denote its hom-object by $\mathcal{C}(A, B)$, rather than $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
- In a given category, the initial and terminal objects (if they exist) are denoted by $0$ and $1$ respectively.
- The symbols $\pi$ and $\iota$ usually refer to the canonical morphisms out of a limit and into a colimit respectively. More precisely, if $F : \mathcal{J} \to \mathcal{C}$ is a functor such that its colimit exists in $\mathcal{C}$, then we'll always denote the canonical morphism $F(j) \to \operatorname{colim} F$ by $\iota_j$ for all $j \in \mathcal{J}$. Similarly, if the limit of $F$ exists in $\mathcal{C}$, we'll denote the canonical morphism $\pi_j : \lim F \to F(j)$ for all $j \in \mathcal{J}$.

We assume the axiom of choice. Further, we work with three nested Grothendieck universes (see [AGV71, Exposé I] for details). The sets in each of these universes are called *small*, *large* and *very large* respectively. Without adjective, a set is assumed to be small, and a large set is also called a *class*. A category is assumed to be large and locally small unless stated otherwise. We'll underline categories to indicate that they are very large. For example, $\mathrm{Cat}$ denotes the large category of small categories while $\underline{\mathrm{Cat}}$ is the very large category of large categories.

**Standing hypotheses:** From Chapter 2 onwards, we let $(\mathcal{V}, \otimes, I)$ be a fixed monoidal category that is cocomplete and finitely complete such that the monoidal product $- \otimes -$ preserves colimits in each variable. Further conditions on $\mathcal{V}$ may be imposed at the beginning of some (sub)sections. Other monoidal categories which are introduced in the text are not assumed to have these properties unless stated otherwise.

## Overview of the thesis

About half of the thesis, spread out over the text, is based on the preprint [LM20] by the author and their supervisor. It should be noted that some definitions, including that of a quasi-category in $\mathcal{V}$, have changed since then. The preprint is therefore somewhat outdated, both in philosophy and terminology.

In Chapter 1 we present some well-known concepts for later reference. No new results will appear in this chapter.

We proceed in Chapter 2 by introducing the major players of this thesis, that is templicial objects, necklaces, quasi-categories in $\mathcal{V}$ and (non-associative) Frobenius structures. Further, we show some elementary results. Amongst other things, we show that the category $S_{\otimes}\mathcal{V}$ of templicial objects is cocomplete and construct an adjunction $\tilde{F} : \mathrm{SSet} \leftrightarrows S_{\otimes}\mathcal{V} : \tilde{U}$. We show that templicial objects in $\mathcal{V}$ and quasi-categories in $\mathcal{V}$ recover simplicial sets and ordinary quasi-categories when $\mathcal{V} = \mathrm{Set}$. Moreover, $\tilde{U}$ preserves quasi-categories. Finally, we construct the templicial nerve $N_{\mathcal{V}}$ and its left-adjoint $h_{\mathcal{V}} : S_{\otimes}\mathcal{V} \to \mathcal{V}\,\mathrm{Cat}$ by analogy with the classical situation.

Chapter 3 discusses some properties of $S_{\otimes}\mathcal{V}$ as a category. In the first section we introduce free and projective templicial morphisms. Projective templicial morphisms are precisely retracts of free morphisms. They also form the left lifting class of a weak factorization system on $S_{\otimes}\mathcal{V}$ which reduces to the weak factorization system on $\mathrm{SSet}$ of monomorphisms and trivial fibrations, if $\mathcal{V} = \mathrm{Set}$. Finally, we explain how free templicial objects have a well-behaved notion of non-degenerate simplices, and we show an analogue of the Eilenberg-Zilber lemma.
In the second section, we embed $S_{\otimes}\mathcal{V}$ into the larger category of necklace categories, which allows us to prove some properties of $S_{\otimes}\mathcal{V}$ like local presentability and completeness. Necklace categories will reappear as a useful tool in Chapter 4 as well.

In Chapter 4, we introduce two major sources of examples of quasi-categories in $\mathcal{V}$: the templicial homotopy coherent nerve of an $S\mathcal{V}$-enriched category and the linear dg-nerve of a dg-category. This is by far the longest and most technical chapter.
To construct the templicial homotopy coherent nerve functor $N^{hc}_{\mathcal{V}}$ and its left-adjoint $\mathfrak{C}_{\mathcal{V}} : S_{\otimes}\mathcal{V} \to \mathcal{V}\,\mathrm{Cat}_{\Delta}$, we adapt Dugger and Spivak's [DS11b] description of the classical categorification functor $\mathfrak{C}$. We make essential use of necklace categories as an intermediate step in this construction.
Next, we restrict to the case where $\mathcal{V}$ is the category $\mathrm{Mod}(k)$ of modules over a commutative ring $k$, and construct the linear dg-nerve $N^{dg}_{k} : k\,\mathrm{Cat}_{dg} \to S_{\otimes}\,\mathrm{Mod}(k)$. We show that non-negatively graded dg-categories over $k$ are equivalent to templicial $k$-modules with a Frobenius structure via an augmented version of the Dold-Kan correspondence. The fact that $N^{dg}_{k}(\mathcal{C})$ is always a quasi-category in $\mathrm{Mod}(k)$ will then follow from the more general result that every templicial $k$-module with a naF-structure is a quasi-category in $\mathrm{Mod}(k)$. Finally, we will show that $\tilde{U} \circ N^{dg}_{k}$ coincides with the classical dg-nerve functor and we will compare $N^{dg}_{k}$ to the templicial homotopy coherent nerve functor $N^{hc}_{k}$.

Chapter 5 discusses some open questions and possible avenues for answering them, in particular concerning the model structure alluded to above.

Finally, the appendix provides an alternative definition of templicial objects as colax monoidal functors $X : \mathbf{\Delta}^{op}_{f} \to \mathcal{V}$ (instead of $\mathcal{V}\,\mathrm{Quiv}_{S}$). The discrete set of vertices is then obtained by imposing that $X_0$ is a free object of $\mathcal{V}$ in a compatible way. We will identify some conditions on $\mathcal{V}$ for which both definitions coincide. None of the results in the main chapters depend on the appendix.

# Nederlandse samenvatting

Simpliciale verzamelingen zijn fundamentele objecten in de algebraïsche topologie en homotopietheorie. Ze verschijnen als combinatorische varianten voor topologische ruimten en worden gebruikt om de meeste modellen voor $(\infty, 1)$-categorieën te definiëren, zoals quasi-categorieën, Segal categorieën en simpliciale categorieën. Het doel van deze thesis is het ontwikkelen en bestuderen van een veralgemening van simpliciale verzamelingen die kunnen geïnterpreteerd worden als "simpliciale objecten in een monoïdale categorie". We noemen deze *tensor-simpliciale* of *templiciale objecten*.

In hun doctoraatsthesis [Agu97] introduceert Aguiar grafen en categorieën *inwendig* tot een monoïdale categorie $\mathcal{V}$. Simpliciale verzamelingen kunnen gezien worden als grafen in hogere dimensies. Gebaseerd op een observatie van Leinster [Lei00] kunnen we Aguiar's aanpak uitbreiden om zo templiciale objecten te definiëren. Noteren we $\mathbf{\Delta}_f$ voor de categorie van eindige intervallen (dit is een deelcategorie van de simplex categorie $\mathbf{\Delta}$), dan is een templiciaal object een koppel $(X, S)$ met $S$ een verzameling en

$$X : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$$

een sterk unitale, colax monoïdale functor. Hier stelt $\mathcal{V}\,\mathrm{Quiv}_S$ de categorie voor van $\mathcal{V}$-verrijkte grafen met $S$ als puntenverzameling. De elementen van $S$ moeten we interpreteren als de punten van $(X, S)$. Voor alle $a, b \in S$ en $n \geq 0$ hebben we een object $X_n(a, b) \in \mathcal{V}$ dat we kunnen beschouwen als een abstractie van de verzameling van $n$-simplexen met als eerste punt $a$ en laatste punt $b$. De colaxstructuur van $X$ bestaat dan uit morfismen

$$(\mu_{k,l})_{a,b} : X_{k+l}(a, b) \to \coprod_{c \in S} X_k(a, c) \otimes X_l(c, b) \quad \text{voor alle } k, l \geq 0$$

Deze morfismen vervangen de uiterste zijvlakafbeeldingen van een simpliciale verzameling en we kunnen ze interpreteren als het "uiteen trekken" van een $(k + l)$-simplex tot een $k$-simplex en een $l$-simplex die verbonden zijn aan een punt. Gegeven een $n$-simplex van $X$ hebben we dus geen rechtstreekse toegang meer tot het 0de of $n$de zijvlak. Maar we kunnen wel zijvlakken bereiken die verbonden zijn aan een punt.

Dit leidt ons tot Dugger en Spivak's [DS11b] kettingen. Een *ketting* is een simpliciale verzameling bestaande uit een eindige keten van simplexen die met elkaar verbonden zijn aan een punt. Hierop gebaseerd introduceren we zekere verrijkte categorieën die we *kettingcategorieën* noemen. We kunnen de categorie $S_{\otimes}\mathcal{V}$ van templiciale objecten inbedden in de categorie $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ van kettingcategorieën. Dit maakt heel wat eigenschappen van $S_{\otimes}\mathcal{V}$ makkelijker te bewijzen.

Zo bewijzen we dat $S_{\otimes}\mathcal{V}$ cocompleet, compleet of lokaal presenteerbaar is van zodra $\mathcal{V}$ dit is. Er bestaat een vergelijkende adjunctie $\tilde{F} : \mathrm{SSet} \leftrightarrows S_{\otimes}\mathcal{V} : \tilde{U}$ die een equivalentie wordt als $\mathcal{V} = \mathrm{Set}$. Verder introduceren we *vrije* en *projectieve* templiciale morfismen. Projectieve morfismen zijn precies de retracties van vrije morfismen en ze vormen de linker liftklasse in een zwak factorisatiesysteem op $S_\otimes\mathcal{V}$. Als $\mathcal{V} = \mathrm{Set}$, vinden we het klassieke zwakke factorisatiesysteem van monomorfismen en triviale fibraties op simpliciale verzamelingen $\mathrm{SSet}$ terug.

Geïnspireerd door Day en Pastro's [DP08] Frobenius-monoïdale functoren, voeren we *Frobeniusstructuren* in, alsook een niet-associatieve variant genaamd *naF-structuren*. Een naF-structuur op een templiciaal object $(X, S)$ bestaat uit een collectie morfismen

$$Z^{k,l}_{a,b} : \coprod_{c\in S} X_k(a,c) \otimes X_l(c,b) \to X_{k+l}(a,b) \quad \text{voor alle } k,l \geq 0$$

die aan zekere compatibiliteitsvoorwaarden met de morfismen $\mu_{k,l}$ voldoen. Intuïtief laten naF-structuren dus toe om "kettingen op te vullen tot een simplex".

Joyal ontwikkelde in [Joy02] de theorie van quasi-categorieën als model voor $(\infty,1)$-categorieën. Deze theorie werd sterk uitgebreid door Lurie [Lur09a] en ondertussen vele anderen. Formeel is een *quasi-categorie* een simpliciale verzameling $X$ die voldoet aan zekere liftconditie, de *zwakke Kan conditie* genaamd.

We zijn vooral geinteresseerd in de eigenschappen van templiciale objecten naar analogie met quasi-categorieën. Daarom introduceren we *quasi-categorieën in* $\mathcal{V}$ als een templiciaal object dat aan een analoge liftconditie voldoet. Als $\mathcal{V} = \mathrm{Set}$, dan vinden we klassieke quasi-categorieën terug. Bovendien worden quasi-categorieën bewaard door $\tilde{U}$. Quasi-categorieën in $\mathcal{V}$ zijn gerelateerd aan naF-structuren in de zin dat elke projectieve quasi-categorie in $\mathcal{V}$ steeds een naF-structuur heeft.

Het grootste deel van deze thesis bespreekt hoe verschillende klassieke voorbeelden van quasi-categorieën kunnen veralgemeend worden voor templiciale objecten. Al deze voorbeelden kunnen worden uitgerust met natuurlijke Frobeniusstructuren.

- De nerffunctor $N : \mathrm{Cat} \to \mathrm{SSet}$, van categorieën naar simpliciale verzamelingen, wordt veralgemeend tot de *templiciale nerffunctor* $N_\mathcal{V} : \mathcal{V}\,\mathrm{Cat} \to S_\otimes\mathcal{V}$, van $\mathcal{V}$-verrijkte categorieën naar templiciale objecten. Bovendien tonen we dat $N_\mathcal{V}(\mathcal{C})$ een quasi-categorie in $\mathcal{V}$ is voor elke $\mathcal{V}$-verrijkte categorie $\mathcal{C}$.
- Cordier's [Cor82] homotopiecoherente nerffunctor $N^{hc} : \mathrm{Cat}_\Delta \to \mathrm{SSet}$, van simpliciale categorieën naar simpliciale verzamelingen, wordt veralgemeend tot de *templiciale homotopiecoherente nerffunctor* $N^{hc}_\mathcal{V} : \mathcal{V}\,\mathrm{Cat}_\Delta \to S_\otimes\mathcal{V}$, van categorieën verrijkt in simpliciale objecten $S\mathcal{V}$ naar templiciale objecten. Dit is gebaseerd op het werk van Dugger en Spivak [DS11b] en maakt gebruik van kettingcategorieën. Als $\mathcal{C}$ een $S\mathcal{V}$-verrijkte categorie is zodat elk hom-object $\mathcal{C}(A,B)$ een onderliggend Kan-complex heeft, dan is $N^{hc}_\mathcal{V}(\mathcal{C})$ bovendien een quasi-categorie in $\mathcal{V}$.
- De differentiaal gegradeerde (dg) nerffunctor $N^{dg} : k\,\mathrm{Cat}_{dg} \to \mathrm{SSet}$ [Lur16], van dg-categorieën over een ring $k$ naar simpliciale verzamelingen, wordt gelift tot de *lineaire dg-nerffunctor* $N^{dg}_k : k\,\mathrm{Cat}_{dg} \to S_\otimes\,\mathrm{Mod}(k)$, van dg-categorieën over $k$ naar templiciale $k$-modulen. We tonen dat elk templiciaal $k$-moduul met een naF-structuur ook een quasi-categorie in $\mathrm{Mod}(k)$ is. Het volgt dan dat $N^{dg}_k(\mathcal{C})$ een quasi-categorie in $\mathrm{Mod}(k)$ is voor elke dg-categorie $\mathcal{C}_\bullet$.

# Chapter 1

# Categorical preliminaries

> *"Wait, this next test does require some explanation. Let me give you the fast version. [fast gibberish]. There. If you have any questions, just remember what I said, in slow motion."*
>
> — GLaDOS (Portal 2)

The first chapter is devoted to establishing some preliminaries that we will use in the rest of the thesis. We assume that the reader is familiar with the basics of category theory. For background on category theory we refer to the many books on the subject, for instance [Mac71], [Bor94a], [Lei14], [Rie17].

Below are three sections, in each of which we outline some preparatory notions. The reader familiar with any of these is free to skip the corresponding section. We'd like to highlight Definition 1.1.20 and Remark 1.1.22 however, since we adopt a slightly different definition of enriched categories than usual.

## 1.1 Monoidal and enriched categories

Recall that a *monoidal category* is a triple $(\mathcal{V}, \otimes, I)$ with $\mathcal{V}$ a category, $- \otimes - : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ a functor called the *monoidal product* and $I \in \mathcal{V}$ an object called the *monoidal unit*. Moreover, it comes equipped with specified isomorphisms

$$\lambda_A : I \otimes A \xrightarrow{\sim} A \quad \text{and} \quad \rho_A : A \otimes I \xrightarrow{\sim} A$$

called the *left* and *right unit isomorphisms*, and

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$$

called the *associator*, such that $\lambda_A$, $\rho_A$ and $\alpha_{A,B,C}$ are natural in $A, B, C \in \mathcal{V}$. These have to satisfy the triangle and pentagon identities, which require that certain diagrams involving $\lambda_A$, $\rho_A$ and $\alpha_{A,B,C}$ commute (see [Bor94b, §6.1] for details). We call the monoidal category $(\mathcal{V}, \otimes, I)$ *strict* if $\lambda_A$, $\rho_A$ and $\alpha_{A,B,C}$ are all identities.

Further, $(\mathcal{V}, \otimes, I)$ is called *symmetric* if it comes equipped with an isomorphism

$$\sigma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$$

which is natural in $A, B \in \mathrm{Ob}(\mathcal{V})$ and fits into certain commutative diagrams (again, see [Bor94b, §6.1]).

The monoidal category $(\mathcal{V}, \otimes, I)$ is called *closed* if for all objects $A \in \mathcal{V}$, the functors $A \otimes -$ and $- \otimes A$ from $\mathcal{V}$ to itself have right-adjoints $\underline{\mathcal{V}}(A, -)_l$ and $\underline{\mathcal{V}}(A, -)_r$ respectively. Letting $A$ vary, we obtain functors $\underline{\mathcal{V}}(-,-)_l, \underline{\mathcal{V}}(-,-)_r : \mathcal{V}^{op} \times \mathcal{V} \to \mathcal{V}$ which we call the *internal hom-objects* of $\mathcal{V}$. If $\mathcal{V}$ is symmetric, then $\underline{\mathcal{V}}(-,-)_l$ and $\underline{\mathcal{V}}(-,-)_r$ are naturally isomorphic and we denote both by $\underline{\mathcal{V}}(-,-)$.

We call $(\mathcal{V}, \otimes, I)$ a *Bénabou cosmos* if it is bicomplete and symmetric monoidal closed.

Consider the corepresentable functor $\mathcal{V}(I, -) : \mathcal{V} \to \mathrm{Set}$. For every object $A \in \mathcal{V}$, we call $\mathcal{V}(I, A)$ the *underlying set* of $A$.

Usually, we will denote the monoidal category $(\mathcal{V}, \otimes, I)$ simply by $\mathcal{V}$.

The main monoidal categories we will be interested in are the following:

- The category of sets $\mathrm{Set}$ with the monoidal product given by the cartesian product $- \times -$ and the monoidal unit given by the singleton $\{*\}$. This monoidal category is symmetric and closed.
- More generally, if $\mathcal{C}$ is a category with finite products, then it carries a monoidal structure $(\times, 1)$ where $- \times -$ is the cartesian product and $1$ the terminal object. We refer to this as the *cartesian* monoidal structure. This monoidal category is always symmetric but not necessarily closed.
- The category of $k$-modules $\mathrm{Mod}(k)$ for a fixed unital commutative ring $k$. The monoidal product is the tensor product $- \otimes_k -$ over $k$ and the monoidal unit is the free $k$-module on one generator, $k$ itself. This monoidal category is symmetric and closed.

### 1.1.1 Monoidal functors and natural transformations

Details for this subsection can be found in [AM10, Chapter 3].

**Definition 1.1.1.** Let $(\mathcal{V}, \otimes, I)$ and $(\mathcal{W}, \boxtimes, J)$ be monoidal categories. A *colax monoidal functor* $\mathcal{V} \to \mathcal{W}$ is a triple $(F, \mu, \epsilon)$ where $F : \mathcal{V} \to \mathcal{W}$ is a functor between the underlying categories, $\mu$ is a natural transformation between functors $\mathcal{V} \times \mathcal{V} \to \mathcal{W}$:

$$\mu : F(- \otimes -) \to F(-) \boxtimes F(-)$$

called the *comultiplication* and $\epsilon$ is a morphism $F(I) \to J$ in $\mathcal{W}$ called the *counit*. This data must moreover satisfy the following conditions:

(a) (Coassociativity) For all $A, B, C \in \mathcal{V}$, the following diagram commutes:

$$\begin{tikzcd}
F((A\otimes B)\otimes C) \arrow[r, "\mu_{A\otimes B,C}"'] \arrow[d, "F(\alpha_{A,B,C})"', "\sim"] & F(A\otimes B)\boxtimes F(C) \arrow[r, "\mu_{A,B}\boxtimes \mathrm{id}_{F(C)}"'] & (F(A)\boxtimes F(B))\boxtimes F(C) \arrow[d, "\sim"', "\alpha_{F(A),F(B),F(C)}"] \\
F(A\otimes(B\otimes C)) \arrow[r, "\mu_{A,B\otimes C}"'] & F(A)\boxtimes F(B\otimes C) \arrow[r, "\mathrm{id}_{F(A)}\boxtimes \mu_{B,C}"'] & F(A)\boxtimes(F(B)\boxtimes F(C))
\end{tikzcd}$$

(b) (Counitality) For all $A \in \mathcal{V}$, the following diagrams commute:

$$\begin{tikzcd}
F(I\otimes A) \arrow[r, "\mu_{I,A}"] \arrow[d, "F(\lambda_A)"', "\sim"] & F(I)\boxtimes F(A) \arrow[d, "\epsilon\boxtimes \mathrm{id}_{F(A)}"] \\
F(A) & J\boxtimes F(A) \arrow[l, "\sim", "\lambda_{F(A)}"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F(A\otimes I) \arrow[r, "\mu_{A,I}"] \arrow[d, "F(\rho_A)"', "\sim"] & F(A)\boxtimes F(I) \arrow[d, "\mathrm{id}_{F(A)}\boxtimes \epsilon"] \\
F(A) & F(A)\boxtimes J \arrow[l, "\sim", "\rho_{F(A)}"']
\end{tikzcd}$$

A *lax monoidal functor* $\mathcal{V} \to \mathcal{W}$ is defined as a colax monoidal functor $\mathcal{V}^{op} \to \mathcal{W}^{op}$. More explicitly, it is a triple $(F, m, u)$ with $m : F(-) \boxtimes F(-) \to F(- \otimes -)$ a natural transformation between functors $\mathcal{V} \times \mathcal{V} \to \mathcal{W}$ called the *multiplication* and $u : J \to F(I)$ a morphism in $\mathcal{W}$ called the *unit*, such that the duals of the above diagrams commute.

Usually, we will abuse notation and denote a lax or colax monoidal functor simply by its underlying functor $F$.

Further, we call a colax monoidal functor $(F, \mu, \epsilon) : \mathcal{V} \to \mathcal{W}$ *strongly unital* if the counit $\epsilon : F(I) \to J$ is an isomorphism in $\mathcal{W}$. We call $(F, \mu, \epsilon)$ *strong monoidal* if it is strongly unital and for all $A, B \in \mathcal{V}$, the comultiplication

$$\mu_{A,B} : F(A \otimes B) \xrightarrow{\sim} F(A) \boxtimes F(B)$$

is an isomorphism in $\mathcal{W}$. Note that a strong monoidal functor is also lax monoidal where the multiplication and unit are given by the inverses of the comultiplication and counit.

**Definition 1.1.2.** Let $(F, \mu^F, \epsilon^F), (G, \mu^G, \epsilon^G) : (\mathcal{V}, \otimes, I) \to (\mathcal{W}, \boxtimes, J)$ be colax monoidal functors between monoidal categories. A natural transformation $\alpha : F \to G$ is called *monoidal* if the following diagrams commute for all $A, B \in \mathcal{V}$:

$$\begin{tikzcd}
F(A\otimes B) \arrow[r, "\alpha_{A\otimes B}"] \arrow[d, "\mu^F_{A,B}"'] & G(A\otimes B) \arrow[d, "\mu^G_{A,B}"] \\
F(A)\boxtimes F(B) \arrow[r, "\alpha_A\boxtimes\alpha_B"'] & G(A)\boxtimes G(B)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F(I) \arrow[rr, "\alpha_I"] \arrow[dr, "\epsilon^F"'] & & G(I) \arrow[dl, "\epsilon^G"] \\
& J &
\end{tikzcd}$$

Dually, a natural transformation $\alpha : F \to G$ between lax monoidal functors $F, G : \mathcal{V} \to \mathcal{W}$ is called *monoidal* if it is a monoidal natural transformation between the corresponding colax monoidal functors $F, G : \mathcal{V}^{op} \to \mathcal{W}^{op}$.

**Notation 1.1.3.** Let $\mathcal{V}$ and $\mathcal{W}$ be monoidal categories. Then the colax monoidal functors $\mathcal{V} \to \mathcal{W}$ and monoidal natural transformations form a (non-locally small) category which we denote by

$$\mathrm{Colax}(\mathcal{V}, \mathcal{W})$$

Dually, we denote by

$$\mathrm{Lax}(\mathcal{V}, \mathcal{W})$$

the (non-locally small) category of all lax monoidal functors $\mathcal{V} \to \mathcal{W}$ and monoidal natural transformations between them.

**Lemma 1.1.4.** *Let $(F, \mu, \epsilon) : (\mathcal{V}, \otimes, I) \to (\mathcal{W}, \boxtimes, J)$ be a colax monoidal functor. Suppose the underlying functor $F : \mathcal{V} \to \mathcal{W}$ has a right-adjoint $G : \mathcal{W} \to \mathcal{V}$. Then $G$ is lax monoidal with multiplication $G(A) \otimes G(B) \to G(A \boxtimes B)$ adjoint to*

$$F(G(A) \otimes G(B)) \xrightarrow{\mu^F_{G(A),G(B)}} FG(A) \boxtimes FG(B) \to A \boxtimes B$$

*for all $A, B \in \mathcal{V}$, and with unit $I \to G(J)$ adjoint to $\epsilon : F(I) \to J$.*

**Definition 1.1.5.** Let $(\mathcal{V}, \otimes, I)$ and $(\mathcal{W}, \boxtimes, J)$ be monoidal categories and $F : \mathcal{V} \to \mathcal{W}$ a functor with a right-adjoint $G : \mathcal{W} \to \mathcal{V}$. We call the adjunction $F \dashv G$ *monoidal* if $F$ comes equipped with a strong monoidal structure. Then by Lemma 1.1.4, $G$ has an induced lax monoidal structure.

**Definition 1.1.6.** A strong monoidal functor $F : \mathcal{V} \to \mathcal{W}$ between monoidal categories is called a *monoidal equivalence* if there exists a strong monoidal functor $G : \mathcal{W} \to \mathcal{V}$ along with monoidal natural isomorphisms

$$G \circ F \simeq \mathrm{id}_{\mathcal{V}} \quad \text{and} \quad F \circ G \simeq \mathrm{id}_{\mathcal{V}'}$$

**Theorem 1.1.7** ([Mac63]). *For every monoidal category $\mathcal{V}$, there exists a strict monoidal category $\mathcal{V}'$ which is monoidally equivalent to $\mathcal{V}$.*

*Remark* 1.1.8. By Theorem 1.1.7, we may always replace a monoidal category by an equivalent strict monoidal category. We will use this as justification to abuse notation and treat the associator, and left and right unit isomorphisms as though they were identities. Given an arbitrary monoidal category $(\mathcal{V}, \otimes, I)$, and objects $A_1, ..., A_n \in \mathcal{V}$, we will therefore simply write

$$A_1 \otimes ... \otimes A_n$$

for any possible bracketting of this expression. Similarly, we may sometimes identify the expressions $I \otimes A$ ans $A \otimes I$ with simply the object $A$.

**Proposition 1.1.9.** *Let $(F, \mu^F, \epsilon^F) : \mathcal{U} \to \mathcal{V}$ and $(G, \mu^G, \epsilon^G) : \mathcal{V} \to \mathcal{W}$ be colax monoidal functors. Then the composite $G \circ F : \mathcal{U} \to \mathcal{W}$ has the structure of a colax monoidal functor with comultiplication $\mu^G_{F(-),F(-)} \circ G(\mu^F_{-,-})$ and counit $\epsilon^G \circ G(\epsilon^F)$.*

*Further, for any monoidal category $\mathcal{V}$, the identity functor $\mathrm{id}_{\mathcal{V}}$ is strong monoidal.*

**Notation 1.1.10.** We denote by

$$\underline{\mathrm{MonCat}}$$

the (very large) 2-category whose objects are all monoidal categories, whose morphisms are the colax monoidal functors between them, and whose 2-morphisms are the monoidal natural transformations between those. So for any two fixed monoidal categories $\mathcal{V}$ and $\mathcal{W}$, we have

$$\underline{\mathrm{MonCat}}(\mathcal{V}, \mathcal{W}) = \mathrm{Colax}(\mathcal{V}, \mathcal{W})$$

### 1.1.2 Quivers and enriched categories

For more details about the basic concepts of enriched category theory, we refer to [Kel05].

For this subsection, we fix a monoidal category $(\mathcal{V}, \otimes, I)$ with coproducts such that the functor $- \otimes -$ preserves coproducts in each variable.

**Definition 1.1.11.** A *$\mathcal{V}$-enriched quiver* or *$\mathcal{V}$-quiver* is a pair $(Q, S)$ with $S$ a set whose elements are called *vertices*, and $Q = (Q(a, b))_{a,b\in S}$ a collection of objects $Q(a, b) \in \mathcal{V}$. A *quiver morphism* $(Q, S) \to (P, T)$ is a pair $(f, f_0)$ with $f_0 : S \to T$ a map of sets and $f = (f_{a,b})_{a,b\in S}$ a collection of morphisms $f_{a,b} : Q(a, b) \to P(f_0(a), f_0(b))$ in $\mathcal{V}$.

Given quiver morphisms $(f, f_0) : (Q, S) \to (P, T)$ and $(g, g_0) : (P, T) \to (R, U)$, the *composition* $(g, g_0) \circ (f, f_0)$ is the quiver morphism $(Q, S) \to (R, U)$ given by the pair $(gf, g_0 f_0)$ where $gf = (g_{f_0(a),f_0(b)} \circ f_{a,b})_{a,b\in S}$. The *identity* on a $\mathcal{V}$-quiver $(Q, S)$ is the quiver morphism $(Q, S) \to (Q, S)$ given by the pair $(\mathrm{id}_Q, \mathrm{id}_S)$ with $\mathrm{id}_Q = (\mathrm{id}_{Q(a,b)})_{a,b\in S}$. This data defines a category which we denote by

$$\mathcal{V}\,\mathrm{Quiv}$$

**Notation 1.1.12.** Given a set $S$, we denote by

$$\mathcal{V}\,\mathrm{Quiv}_S$$

the subcategory of $\mathcal{V}\,\mathrm{Quiv}$ consisting of all $\mathcal{V}$-quivers $(Q, S)$ and all quiver morphisms $(f, \mathrm{id}_S)$. Note that $\mathcal{V}\,\mathrm{Quiv}_S$ is canonically isomorphic to the category $\mathcal{V}^{S\times S}$ of functors $S \times S \to \mathcal{V}$ where we consider the set $S \times S$ as a discrete category.

**Notation 1.1.13.** If $\mathcal{V} = \mathrm{Set}$ is the cartesian monoidal category of sets, then we denote $\mathcal{V}\,\mathrm{Quiv} = \mathrm{Quiv}$ and $\mathcal{V}\,\mathrm{Quiv}_S = \mathrm{Quiv}_S$ for any set $S$.

**Construction 1.1.14.** Let $S$ be a set. For any two $\mathcal{V}$-quivers $Q$ and $P$, we define a $\mathcal{V}$-quiver $Q \otimes_S P$ as follows. For all $a, b \in S$, set

$$(Q \otimes_S P)(a, b) = \coprod_{c\in S} Q(a, c) \otimes P(c, b)$$

Similarly, for quiver morphisms $f : Q \to Q'$ and $g : P \to P'$ in $\mathcal{V}\,\mathrm{Quiv}_S$, we define $f \otimes_S g : Q \otimes_S P \to Q' \otimes_S P'$ as follows. For all $a, b \in S$:

$$(f \otimes_S g)_{a,b} = \coprod_{c\in S} f_{a,c} \otimes g_{c,b}$$

This clearly defines a functor $- \otimes_S - : \mathcal{V}\,\mathrm{Quiv}_S \times \mathcal{V}\,\mathrm{Quiv}_S \to \mathcal{V}\,\mathrm{Quiv}_S$.

Further, we define a $\mathcal{V}$-quiver $I_S$ by setting for all $a, b \in S$:

$$I_S(a, b) = \begin{cases} I & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

It easily follows from the hypotheses on $\mathcal{V}$ that $\otimes_S$ and $I_S$ define a monoidal category $(\mathcal{V}\,\mathrm{Quiv}_S, \otimes_S, I_S)$. We will sometimes drop the subscript $S$ from $\otimes_S$ and $I_S$ when it is clear from context.

Note that the monoidal category $\mathcal{V}\,\mathrm{Quiv}_S$ is generally not symmetric.

**Example 1.1.15.** Let $S = \{*\}$ be a singleton set. Then we have an isomorphism of monoidal categories $\mathcal{V}\,\mathrm{Quiv}_S \simeq \mathcal{V}$.

**Construction 1.1.16.** Let $f : S \to T$ be a map between sets. We define a functor

$$f^* : \mathcal{V}\,\mathrm{Quiv}_T \to \mathcal{V}\,\mathrm{Quiv}_S$$

as follows. For all $\mathcal{V}$-quivers $(Q, T)$, and all $a, b \in S$, set:

$$f^*(Q)(a, b) = Q(f(a), f(b))$$

and for any morphism $g : Q \to P$ in $\mathcal{V}\,\mathrm{Quiv}_T$, and all $a, b \in S$:

$$f^*(g)_{a,b} = g_{f(a),f(b)}$$

Note that by identifying $\mathcal{V}\,\mathrm{Quiv}_S \simeq \mathcal{V}^{S\times S}$, $f^*$ is in fact the precomposition functor $- \circ (f \times f)$. Consequently, it has a left-adjoint given by the left Kan extension

$$f_! = \mathrm{Lan}_{f\times f}(-) : \mathcal{V}\,\mathrm{Quiv}_S \to \mathcal{V}\,\mathrm{Quiv}_T$$

In this case, $f_!$ is easily seen to be given by,

$$f_!(Q)(x, y) = \coprod_{\substack{a\in f^{-1}(x)\\ b\in f^{-1}(y)}} Q(a, b)$$

for all $Q \in \mathcal{V}\,\mathrm{Quiv}_S$ and $x, y \in T$.

*Remark* 1.1.17. It is easy to see that if $\mathcal{V}$ is complete or cocomplete, then so is $\mathcal{V}\,\mathrm{Quiv}$. For a fixed set $S$, the limits and colimits of $\mathcal{V}\,\mathrm{Quiv}_S \simeq \mathcal{V}^{S\times S}$ are given pointwise. Then given a diagram $F : \mathcal{J} \to \mathcal{V}\,\mathrm{Quiv} : j \mapsto (Q^j, S^j)$, the limit $(Q, S)$ of $F$ is given by

$$S = \lim_{j\in\mathcal{J}} S^j \quad \text{in Set} \quad \text{and} \quad Q = \lim_{j\in\mathcal{J}} \pi_j^*(Q^j) \quad \text{in } \mathcal{V}\,\mathrm{Quiv}_S$$

where $\pi_j$ denotes the canonical map $S \to S^j$ for all $j \in \mathcal{J}$. Similarly, the colimit $(Q, S)$ of $F$ is given by

$$S = \operatorname*{colim}_{j\in\mathcal{J}} S^j \quad \text{in Set} \quad \text{and} \quad Q = \operatorname*{colim}_{j\in\mathcal{J}} (\iota_j)_!(Q^j) \quad \text{in } \mathcal{V}\,\mathrm{Quiv}_S$$

where $\iota_j$ denotes the canonical map $S^j \to S$ for all $j \in \mathcal{J}$.

**Lemma 1.1.18.** *For any function $f : S \to T$, $f^*$ is a lax monoidal functor and $f_!$ is a colax monoidal functor.*

*Proof.* By the dual of Lemma 1.1.4, it suffices to show that $f^*$ is lax monoidal. Define the unit $u : I_S \to f^*(I_T)$ of $f^*$ by

$$u_{a,b} = \begin{cases} I \xrightarrow{\mathrm{id}} I & \text{if } a = b \\ 0 \to I & \text{if } a \neq b, f(a) = f(b) \\ 0 \to 0 & \text{if } f(a) \neq f(b) \end{cases}$$

for all $a, b \in S$. Further, we have for any $Q, P \in \mathcal{V}\,\mathrm{Quiv}_T$ that

$$f^*(Q \otimes_T P)(a,b) = \coprod_{x \in T} Q(f(a), x) \otimes Q(x, f(b))$$
$$(f^*(Q) \otimes_S f^*(P))(a,b) = \coprod_{c \in S} Q(f(a), f(c)) \otimes P(f(c), f(b))$$

which gives a canonical morphism of quivers

$$m_{Q,P} : f^*(Q) \otimes_S f^*(P) \to f^*(Q \otimes_T P)$$

It is readily verified that $m_{Q,P}$ is natural in $Q$ and $P$, and that it is associative and unital with respect to $u$. □

In the next proposition, we consider Set as a 2-category with discrete hom-categories. $\underline{\mathrm{MonCat}}$ is the 2-category of monoidal categories from Notation 1.1.10.

**Proposition 1.1.19.** *The assignments $S \mapsto \mathcal{V}\,\mathrm{Quiv}_S$ and $f \mapsto f_!$ define a pseudofunctor*

$$(-)_! : \mathrm{Set} \to \underline{\mathrm{MonCat}}$$

*Proof.* For any $\mathcal{V}$-quiver $Q$ with vertex set $S$, we obviously have that

$$(\mathrm{id}_S)_!(Q)(x,y) = \coprod_{\substack{a \in \mathrm{id}_S^{-1}(x) \\ b \in \mathrm{id}_S^{-1}(y)}} Q(a,b) \simeq Q(x,y)$$

for all $x, y \in S$ and thus $(\mathrm{id}_S)_!(Q) \simeq Q$. Further, given maps of sets $f : R \to S$ and $g : S \to T$, we have for all $Q \in \mathcal{V}\,\mathrm{Quiv}_S$ and $x, y \in T$:

$$(g \circ f)_!(Q)(x,y) = \coprod_{\substack{r \in (g\circ f)^{-1}(x) \\ s \in (g \circ f)^{-1}(y)}} Q(r,s)$$
$$g_!(f_!(Q))(x,y) = \coprod_{\substack{a \in g^{-1}(x) \\ b \in g^{-1}(y)}} \coprod_{\substack{r \in f^{-1}(a) \\ s \in f^{-1}(b)}} Q(r,s)$$

So we have an isomorphism $(g \circ f)_!(Q) \simeq g_!(f_!(Q))$.

It follows from a direct verification that these isomorphisms make $(-)_!$ into a well-defined pseudofunctor. □

**Definition 1.1.20.** A *$\mathcal{V}$-enriched category* or *$\mathcal{V}$-category* $\mathcal{C}$ is a pair $(\mathcal{C}, \mathrm{Ob}(\mathcal{C}))$ with $\mathrm{Ob}(\mathcal{C})$ a class and $\mathcal{C}$ a monoid in $(\mathcal{V}\,\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}, \otimes_{\mathrm{Ob}(\mathcal{C})}, I_{\mathrm{Ob}(\mathcal{C})})$. We say $(\mathcal{C}, \mathrm{Ob}(\mathcal{C}))$ is *small* if $\mathrm{Ob}(\mathcal{C})$ is a set.

A *$\mathcal{V}$-enriched functor* or *$\mathcal{V}$-functor* $(\mathcal{C}, \mathrm{Ob}(\mathcal{C})) \to (\mathcal{D}, \mathrm{Ob}(\mathcal{D}))$ is a pair $(F, f)$ with $f : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ a map of sets and $F : \mathcal{C} \to f^*(\mathcal{D})$ a morphism of monoids in $\mathcal{V}\,\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{D})}$ (where we used the lax structure of $f^*$, see Lemma 1.1.18).

The composition and identities of $\mathcal{V}$-functors are defined as in $\mathcal{V}\,\mathrm{Quiv}$. This data defines a (large) category of (small) $\mathcal{V}$-categories which we denote by

$$\mathcal{V}\,\mathrm{Cat}$$

**Examples 1.1.21.**
1. For $\mathcal{V} = \mathrm{Set}$ the cartesian monoidal category of sets, the category $\mathrm{Set}\text{-}\mathrm{Cat}$ is isomorphic to the category of small categories $\mathrm{Cat}$.
2. For $\mathcal{V} = \mathrm{Mod}(k)$ the monoidal category of modules over a fixed unital commutative ring $k$, we refer to $\mathrm{Mod}(k)$-categories as *$k$-linear categories* and we write $k\,\mathrm{Cat}$ for $\mathrm{Mod}(k)\text{-}\mathrm{Cat}$.
3. For $\mathcal{V} = \mathrm{Cat}$ the cartesian category of small categories, the category $\mathrm{Cat}\text{-}\mathrm{Cat}$ is isomorphic to the category of small strict 2-categories $2\text{-}\mathrm{Cat}$.

The following remark is subtle but it bears mentioning.

*Remark* 1.1.22. The monoidal product $\otimes_S$ in $\mathcal{V}\,\mathrm{Quiv}_S$ from Construction 1.1.14 is chosen so that it is more convenient when defining templicial objects in Chapter 2. However, this introduces a discrepancy between Definition 1.1.20 and how enriched categories are usually defined.

Traditionally, the composition law of a $\mathcal{V}$-category $\mathcal{C}$ is defined as a collection of morphisms, for all $A, B, C \in \mathrm{Ob}(\mathcal{C})$:

$$m_{A,B,C} : \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \to \mathcal{C}(A, C) \tag{1.1}$$

However, following Definition 1.1.20, a monoid $\mathcal{C}$ in $(\mathcal{V}\,\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}, \otimes_{\mathrm{Ob}(\mathcal{C})}, I_{\mathrm{Ob}(\mathcal{C})})$ comes equipped with morphisms, for all $A, B, C \in \mathrm{Ob}(\mathcal{C})$:

$$\tilde{m}_{A,B,C} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C) \tag{1.2}$$

So this in fact a category enriched in the *reverse* monoidal category $\mathcal{V}^{rev}$ whose monoidal product $- \otimes^{rev} -$ is defined by $A \otimes^{rev} B = B \otimes A$.

If $\mathcal{V}$ is symmetric, then $\mathcal{V}^{rev}$ and $\mathcal{V}$ are monoidally equivalent via the symmetry $\sigma$ of $\mathcal{V}$. So we can safely pass between the two composition laws (1.1) and (1.2):

$$\tilde{m}_{A,B,C} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \xrightarrow[\sim]{\sigma_{\mathcal{C}(A,B),\mathcal{C}(B,C)}} \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \xrightarrow{m_{A,B,C}} \mathcal{C}(A, C)$$

In most cases, there is thus no risk of confusion. Beware however that when $\mathcal{V}$ is the symmetric monoidal category of chain complexes for example, the symmetry $\sigma$ introduces a sign change. We will return to this point in Remark 4.2.42.

Nonetheless, even when $\mathcal{V}$ is not symmetric, we will still adopt the convention (1.2) as dictated by Definition 1.1.20. To make the distinction with (1.1) absolutely clear, we will always denote the composition in a $\mathcal{V}$-category by $\tilde{m}$ instead of $m$ and refer to it as its *reverse composition law*.

**Proposition 1.1.23.** *Let $F : \mathcal{V} \to \mathcal{W}$ be a lax monoidal functor between monoidal categories. Then there is an induced functor*

$$\mathcal{F} : \mathcal{V}\,\mathrm{Cat} \to \mathcal{W}\,\mathrm{Cat}$$

*which is given as follows. For each $\mathcal{V}$-category $\mathcal{C}$, the $\mathcal{W}$-category $\mathcal{F}(\mathcal{C})$ has the same set of objects as $\mathcal{C}$ and for all $A, B \in \mathcal{C}$, $\mathcal{F}(\mathcal{C})(A, B) = F(\mathcal{C}(A, B))$.*

*Moreover, if $F$ has a lax monoidal right-adjoint $G : \mathcal{W} \to \mathcal{V}$, then the induced functors form an adjunction*

$$\mathcal{F} : \mathcal{V}\,\mathrm{Cat} \leftrightarrows \mathcal{W}\,\mathrm{Cat} : \mathcal{G}$$

## 1.2 Lifting properties

Fix a cocomplete category $\mathcal{C}$ for this section. We briefly discuss weak factorization systems on $\mathcal{C}$. Then we highlight the particular weak factorization system of projecive morphisms and regular epimorphisms.

### 1.2.1 Weak factorization systems

Proofs for this subsection can be found in [Hov99] for example.

**Definition 1.2.1.** A morphism $f : A \to B$ is a *retract* of a morphism $g : C \to D$ if $f$ is a retract of $g$ as objects in the category of morphisms $\mathrm{Mor}(\mathcal{C}) = \mathrm{Fun}([1], \mathcal{C})$. That is, there exist morphisms $a$, $b$, $c$ and $d$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "a"'] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_A"] & C \arrow[r, "c"'] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "b"] \arrow[rr, bend right, "\mathrm{id}_B"'] & D \arrow[r, "d"] & B \end{tikzcd}$$

We call $f$ a *strong retract* of $g$ if $A = C$ and there exists a commutative diagram as above with $a = c = \mathrm{id}_A$.

**Definition 1.2.2.** Let $\lambda > 0$ be an ordinal. A *$\lambda$-sequence* is a colimit-preserving functor $X : \lambda \to \mathcal{C}$. The canonical morphism

$$\iota_0 : X(0) \to \operatorname*{colim}_{\alpha<\lambda} X(\alpha)$$

is called the *transfinite composition* of $X$.

Let $\mathcal{A}$ be a class of morphisms in $\mathcal{C}$. If for all ordinals $\beta$ with $\beta + 1 < \lambda$ we have that the morphism $\iota_{\beta,\beta+1} : X(\beta) \to X(\beta + 1)$ belongs to $\mathcal{A}$, then we call $X$ a *$\lambda$-sequence in $\mathcal{A}$*.

**Definition 1.2.3.** Let $\mathcal{A}$ be a class of morphisms in $\mathcal{C}$. We call $\mathcal{A}$ *weakly saturated* if

(a) $\mathcal{A}$ is closed under pushouts, that is, for all $f \in \mathcal{A}$, the pushout of $f$ along any morphism in $\mathcal{C}$ belongs to $\mathcal{A}$.

(b) $\mathcal{A}$ is closed under transfinite compositions, that is, for every ordinal $\lambda$ and every $\lambda$-sequence $X : \lambda \to \mathcal{C}$ in $\mathcal{A}$, the transfinite composition $\iota_0 : X(0) \to \operatorname{colim} X$ belongs to $\mathcal{A}$.

(c) $\mathcal{A}$ is closed under retracts, that is, if $f \in \mathcal{A}$ and $g$ is a retract of $f$ in $\mathcal{C}$, then $g \in \mathcal{A}$.

It is clear that arbitrary intersections of weakly saturated classes are again weakly saturated. If $\mathcal{S}$ is a class of morphisms in $\mathcal{C}$, we write $\overline{\mathcal{S}}$ for the smallest weakly saturated class of morphisms in $\mathcal{C}$ that contains $\mathcal{S}$. In other words,

$$\overline{\mathcal{S}} = \bigcap_{\substack{\mathcal{S}\subseteq\mathcal{A} \\ \mathcal{A} \text{ weakly saturated}}} \mathcal{A}$$

We call $\overline{\mathcal{S}}$ the *weak saturated closure* of $\mathcal{S}$.

**Definition 1.2.4.** A *lifting problem* in $\mathcal{C}$ is any commutative square of solid arrows:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed] & D \end{tikzcd}$$

A morphism $h : B \to C$ is called a *lift* or a *solution* of the lifting problem if $hi = f$ and $ph = g$.

Given morphisms $i : A \to B$ and $p : X \to Y$ in $\mathcal{C}$, we say that $i$ has the *left lifting property* with respect to $p$ and that $p$ has the *right lifting property* with respect to $i$ if every lifting problem as above has a solution. In this case, we write

$$i \boxslash p$$

**Definition 1.2.5.** If $\mathcal{A}$ is a class of morphisms in $\mathcal{C}$, then we define the *left lifting class* and *right lifting class* of $\mathcal{A}$ respectively by

$$\mathcal{A}^{\boxslash} = \{p \in \mathrm{Mor}(\mathcal{C}) \mid \forall i \in \mathcal{A} : i \boxslash p\} \quad \text{and} \quad {}^{\boxslash}\mathcal{A} = \{i \in \mathrm{Mor}(\mathcal{C}) \mid \forall p \in \mathcal{A} : i \boxslash p\}$$

**Proposition 1.2.6.** *For every class of morphisms $\mathcal{B}$ in $\mathcal{C}$, ${}^{\boxslash}\mathcal{B}$ is weakly saturated.*

**Lemma 1.2.7** (Retract argument)**.** *Let $i : X \to A$ and $p : A \to Y$ be morphisms in $\mathcal{C}$ and set $f = pi$, then*

- *If $f$ has the left lifting property with respect to $p$, then $f$ is a (strong) retract of $i$.*
- *If $f$ has the right lifting property with respect to $i$, then $f$ is a retract of $p$.*

**Definition 1.2.8.** A pair $(\mathcal{L}, \mathcal{R})$ of classes of morphims of $\mathcal{C}$ is called a *weak factorization system* on $\mathcal{C}$ if

(a) $\mathcal{L}^{\boxslash} = \mathcal{R}$ and $\mathcal{L} = {}^{\boxslash}\mathcal{R}$, and

(b) every morphism $f$ of $\mathcal{C}$ can be factored as $f = pi$ with $i \in \mathcal{L}$ and $p \in \mathcal{R}$.

**Definition 1.2.9.** Let $\kappa$ be a regular cardinal. An ordinal $\lambda$ is called *$\kappa$-directed* if for every collection of ordinals $(\alpha_i)_{i\in I}$ with $|I| < \kappa$ and $\alpha_i < \lambda$ for all $i \in I$, there exists an ordinal $\alpha < \lambda$ such that $\alpha_i \leq \alpha$ for all $i \in I$.

An object $A$ of $\mathcal{C}$ is called *$\kappa$-small* if the corepresentable functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathrm{Set}$ preserves transfinite compositions of $\lambda$-sequences for all $\kappa$-directed ordinals $\lambda > 0$.

We say $A$ is *small* if it is $\kappa$-small for some regular cardinal $\kappa$.

**Example 1.2.10.** Let $\mathcal{J}$ be a small category. Then every object in the functor category $\mathrm{Set}^{\mathcal{J}}$ is small.

**Proposition 1.2.11** (Small object argument)**.** *Let $S$ be a set of morphisms in $\mathcal{C}$. Assume that the domains of all morphisms in $S$ are small. Then $(\overline{S}, S^{\boxslash})$ is a weak factorization system on $\mathcal{C}$.*

### 1.2.2 Free and projective morphisms

Further assume that $\mathcal{C}$ has pullbacks.

**Definition 1.2.12.** A morphism $g : X \to Y$ in $\mathcal{C}$ is called a *regular epimorphism* if it is the coequalizer of its kernel pair:

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \arrow[r, two heads, "g"] & Y \end{tikzcd}$$

*Remark* 1.2.13. What we defined here as a regular epimorphism is usually called an *effective epimorphism*, while a regular epimorphism is defined as a coequalizer of any pair of morphisms. For categories with pullbacks, the two notions are equivalent so there is no risk of confusion.

**Definition 1.2.14.** An object $P$ of $\mathcal{C}$ is called *projective* if the initial morphism $0 \to P$ has the left lifting property with respect to all regular epimorphisms $X \twoheadrightarrow Y$:

$$\begin{tikzcd} & X \arrow[d, two heads] \\ P \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

We call a morphism $f : A \to B$ in $\mathcal{C}$ *projective* if it is projective as an object of the under category $\mathcal{C}_{A/}$. Equivalently, $f$ has the left lifting property with respect to all regular epimorphisms $X \twoheadrightarrow Y$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

**Examples 1.2.15.** Let us describe the projective morphisms in our main categories of interest.

1. In the category of sets $\mathrm{Set}$ the projective morphisms are precisely the injective maps of sets.

2. Fix a unital commutative ring $k$. In the category of $k$-modules $\mathrm{Mod}(k)$, a morphism $f : A \to B$ is projective if and only if there is a projective $k$-module $P$ (in the usual sense) and an isomorphism $B \simeq A \oplus P$ such that $f$ corresponds to the coprojection $A \to A \oplus P$.

   In particular, if $k$ is a field then every $k$-vectorspace is projective and every injective $k$-linear map splits. It follows that the projective morphisms in $\mathrm{Mod}(k)$ are precisely the injective $k$-linear maps.

**Notation 1.2.16.** For any object $P \in \mathcal{C}$, we denote $F_P$ for the functor

$$F_P : \mathrm{Set} \to \mathcal{C} : S \mapsto \coprod_{a \in S} P$$

It is uniquely determined (up to natural isomorphism) by the conditions that $F_P(\{*\}) \simeq P$ and $F_P$ preserves colimits. Then $F_P$ is left-adjoint to the corepresentable functor

$$U_P = \mathcal{C}(P, -) : \mathcal{C} \to \mathrm{Set}$$

*Remark* 1.2.17. Note that $P$ is a projective object of $\mathcal{C}$ if and only if the functor $U_P : \mathcal{C} \to \mathrm{Set}$ preserves regular epimorphisms.

**Definition 1.2.18.** Fix an object $P \in \mathcal{C}$. An object $A$ of $\mathcal{C}$ is called *free* if it is isomorphic to $F_P(S)$ for some set $S$.

We call a morphism $f : A \to B$ in $\mathcal{C}$ *free* if it is free as an object of the under category $\mathcal{C}_{A/}$ (with respect to the composite left-adjoint $A \amalg F_P(-) : \mathrm{Set} \to \mathcal{C}_{A/}$). Equivalently, there exists a set $S$ and an isomorphism $B \simeq A \amalg F_P(S)$ in $\mathcal{C}$ such that $f$ corresponds to the coprojection $A \to A \amalg F_P(S)$.

**Examples 1.2.19.** Let us describe the free morphisms in our main categories of interest.

1. Choose $P = \{*\}$ in the category of sets $\mathrm{Set}$. Then the free morphisms in $\mathrm{Set}$ coincide with the projective morphisms and thus with the injective maps of sets.
2. Fix a unital commutative ring $k$. Choose $P = k$ in the category of $k$-modules $\mathrm{Mod}(k)$. A morphism $f : A \to B$ is free in $\mathrm{Mod}(k)$ if and only if there is a free $k$-module $F$ (in the usual sense) and an isomorphism $B \simeq A \oplus F$ such that $f$ corresponds to the coprojection $A \to A \oplus F$.

   In particular, if $k$ is a field then every $k$-vectorspace is free and every injective $k$-linear map splits. It follows that the free morphisms of $\mathrm{Mod}(k)$ coincide with the projective morphisms and thus with the injective $k$-linear maps.

The following properties are easy to show.

**Proposition 1.2.20.** *Let $P$ be a projective object of $\mathcal{C}$ such that $U_P$ also reflects epimorphisms. Then the following statements are true.*

1. *The functor $F_P : \mathrm{Set} \to \mathcal{C}$ sends monomorphisms to free morphisms.*
2. *A morphism in $\mathcal{C}$ is projective if and only if it is a (strong) retract of a free morphism.*
3. *Every morphism in $\mathcal{C}$ can be factored as a free morphism followed by a regular epimorphism.*
4. *The classes of projective morphisms and regular epimorphisms form a weak factorization system on $\mathcal{C}$.*

*Remark* 1.2.21. Let $S$ be a set. As $\mathcal{C}\,\mathrm{Quiv}_S \simeq \mathcal{C}^{S \times S}$, we have for every object $P \in \mathcal{C}$ an induced adjunction $F_P : \mathrm{Quiv}_S \leftrightarrows \mathcal{C}\,\mathrm{Quiv}_S : U_P$. Moreover, a quiver morphism $f : Q_1 \to Q_2$ is a regular epimorphism if and only if the morphism $f_{a,b} : Q_1(a,b) \to Q_2(a,b)$ is a regular epimorphism in $\mathcal{V}$ for all $a, b \in S$. Let us call $f$ *free* (resp. *projective*) if $f_{a,b}$ is free (resp. projective) in $\mathcal{C}$ for all $a, b \in S$. Then the properties of Proposition 1.2.20 hold for $\mathcal{C}\,\mathrm{Quiv}_S$ as well.

## 1.3 Quasi-categories

Quasi-categories are one of the many models of $(\infty, 1)$-categories. They were originally considered by Boardman and Vogt under the name *weak Kan complexes* [BV73]. Later Joyal introduced the term *quasi-category* and studied them extensively [Joy02]. Lurie greatly expounded on their theory in [Lur09a]. Some modern resources include [Rez17] and [Lur18]. Quasi-categories are often referred to as simply $\infty$-*categories*. To make the distinction with other models, we will adopt Joyal's terminology.

In this section we give the very basic first definitions in the study of quasi-categories.

### 1.3.1 Simplicial sets and quasi-categories

A simplicial set can be interpreted geometrically as a collection of simplices of varying dimensions, which are glued together along common faces. They are formalized combinatorially as presheaves on the simplex category $\mathbf{\Delta}$. For more details on simplicial sets, see [May67] for example.

**Definition 1.3.1.** Let $\mathbf{\Delta}$ be the category of all posets $[n] = \{0, ..., n\}$ with $n \geq 0$ an integer and order morphisms $f : [n] \to [m]$ between them. We call $\mathbf{\Delta}$ the *simplex category*.

**Definition 1.3.2.** Let $n > 0$ be an integer and $i \in [n]$. We call the order morphism

$$\delta_i : [n-1] \to [n] : j \mapsto \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

a *coface map*. Let $n \geq 0$ be an integer and $i \in [n]$. We call the order morphism

$$\sigma_i : [n+1] \to [n] : j \mapsto \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

a *codegeneracy maps*.

We call a coface map $\delta_i : [n-1] \to [n]$ *inner* if $0 < i < n$ and *outer* if $i = 0$ or $i = n$.

**Lemma 1.3.3** ([Mac71], VII.5). *Every morphism $f : [n] \to [m]$ of $\mathbf{\Delta}$ has a unique representation*

$$f = \delta_{i_1} ... \delta_{i_s} \sigma_{j_1} ... \sigma_{j_t}$$

*with $0 \leq i_s < ... < i_1 \leq m$, $0 \leq j_1 < ... < j_t < n$ and $s, t \geq 0$ such that $n - t + s = m$.*

**Definition 1.3.4.** Let $\mathcal{C}$ be a category. A *simplicial object* of $\mathcal{C}$ is a functor $Y : \mathbf{\Delta}^{op} \to \mathcal{C}$. For all $n \geq 0$ we denote $Y_n = Y([n])$, Further, for all integers $n > 0$ and $i \in [n]$ we denote

$$d_i = Y(\delta_i) : Y_n \to Y_{n-1}$$

and call these the *face morphisms* of $Y$. Similarly, for all integers $n \geq 0$ and $i \in [n]$, we denote

$$s_i = Y(\sigma_i) : Y_n \to Y_{n+1}$$

and call these the *degeneracy morphisms* of $Y$. We denote

$$S\mathcal{C} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{C})$$

for the category of simplicial objects and call its morphisms *simplicial morphisms*.

**Proposition 1.3.5.** *Let $\mathcal{C}$ be a category. A simplicial object $Y$ of $\mathcal{C}$ is equivalent to a collection $(Y_n)_{n\geq 0}$ of objects of $\mathcal{C}$ with morphisms $d_i : Y_n \to Y_{n-1}$ for all integers $n > 0$ and $i \in [n]$, and $s_i : Y_n \to Y_{n+1}$ for all integers $n \geq 0$ and $i \in [n]$, satisfying the following identities:*

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases} \tag{1.3}$$
$$d_i d_j = d_{j-1} d_i \text{ if } i < j \qquad s_i s_j = s_j s_{i-1} \text{ if } i > j$$

**Definition 1.3.6.** A *simplicial set* is a simplicial object of the category of sets $\mathrm{Set}$. In this case, we denote
$$\mathrm{SSet} = S\,\mathrm{Set}$$
We call the morphisms of $\mathrm{SSet}$ *simplicial maps*.

Given a simplicial set $K$ and an integer $n \geq 0$, we call the elements of the set $K_n$ the *$n$-simplices* of $K$. We shall also refer to $0$-simplices as *vertices*, and to $1$-simplices as *edges*. We denote $f : a \to b$ to indicate that $f \in K_1$ with vertices $d_0(f) = b$ and $d_1(f) = a$.

An $n$-simplex $x \in K_n$ with $n > 0$ is called *degenerate* if there exists a $y \in K_{n-1}$ and $0 \leq i \leq n-1$ such that $x = s_i(x)$, and *non-degenerate* otherwise.

The following result is known as the Eilenberg-Zilber lemma.

**Lemma 1.3.7** ([EZ50], (8.3))**.** *Let $K$ be a simplicial set and $n \geq 0$. For any $n$-simplex $x$ of $K$, there is a unique surjective morphism $\sigma : [n] \twoheadrightarrow [k]$ in $\mathbf{\Delta}$ and a unique non-degenerate $k$-simplex $y$ of $K$ such that $x = K(\sigma)(y)$.*

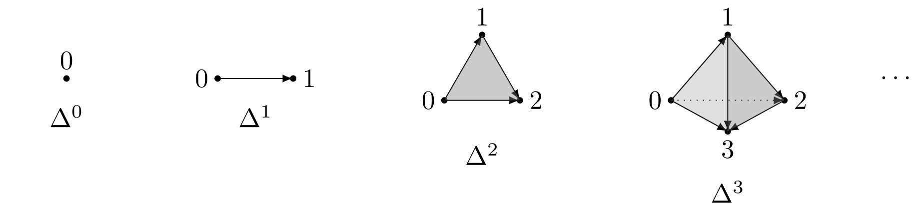
**Definition 1.3.8.** Let $n \geq 0$ be an integer.

- The *standard $n$-simplex* $\Delta^n$ is the simplicial set
$$\Delta^n = \mathbf{\Delta}(-, [n]) : \mathbf{\Delta}^{op} \to \mathrm{Set}$$

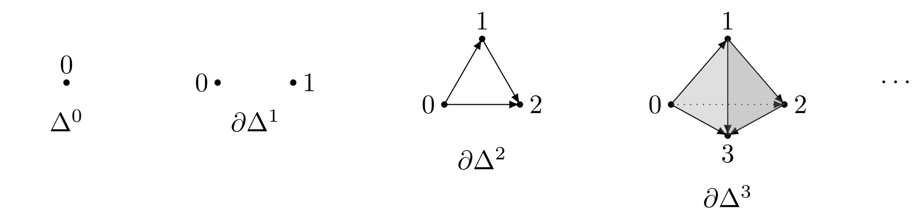
- The *$n$th boundary* $\partial\Delta^n$ is the simplicial subset of $\Delta^n$ defined by setting, for all integers $m \geq 0$:
$$(\partial\Delta^n)_m = \{f : [m] \to [n] \mid f([m]) \neq [n]\} \subseteq (\Delta^n)_m$$

- For $0 \leq j \leq n$, the *$j$th horn* $\Lambda^n_j$ is the simplicial subset of $\Delta^n$ defined by setting, for all integers $m \geq 0$:
$$(\Lambda^n_j)_m = \{f : [m] \to [n] \mid f([m]) \not\supseteq [n] \setminus \{j\}\} \subseteq (\Delta^n)_m$$
We call $\Lambda^n_j$ an *inner* horn if $0 < j < n$ and an *outer* horn if $j = 0$ or $j = n$.

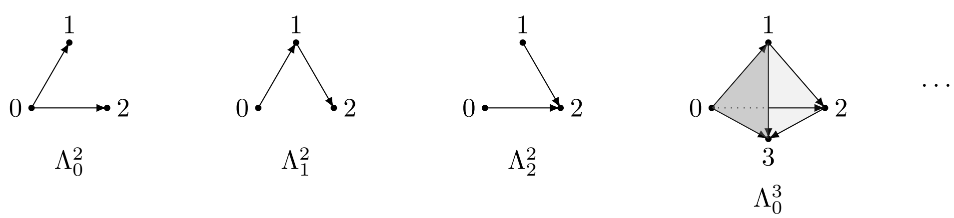
*Remark* 1.3.9. Let $n \geq 0$ be an integer. The standard $n$-simplex may be visualized geometrically by a simplex of dimension $n$.




The $n$th boundary is obtained from the standard $n$-simplex by removing the interior. Alternatively, it can be described by the union of all of the faces of the standard $n$-simplex.




Given an integer $0 \leq j \leq n$, the $j$th horn can be described as the union of all the faces of the standard simplex, except the $j$th face.




The next result is an immediate consequence of the Yoneda lemma.

**Proposition 1.3.10.** *Let $K$ be a simplicial set. There is a bijection*

$$K_n \simeq \mathrm{SSet}(\Delta^n, K)$$

*between the sets of $n$-simplices of $K$ and of simplicial maps $\Delta^n \to K$, which is natural in $n \geq 0$.*

**Proposition 1.3.11.** *Let $\mathcal{D}$ be a cocomplete category and $C : \mathbf{\Delta} \to \mathcal{D}$ a functor. Then there is an adjunction*

$$\begin{tikzcd} \mathrm{SSet} \arrow[r, shift left, "\overline{C}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "N^C"] \end{tikzcd}$$

*where $\overline{C}$ is the left Kan extension of $C$ along the Yoneda embedding $\mathcal{Y} : \mathbf{\Delta} \to \mathrm{SSet}$ and $N^C$ is defined by*

$$N^C(D)_n = \mathcal{D}(C([n]), D) \tag{1.4}$$

*for all $D \in \mathcal{D}$ and $n \geq 0$.*

*Moreover, if $L : \mathrm{SSet} \to \mathcal{D}$ is a functor preserving colimits, then $L \simeq \overline{C}$ where $C = L \circ \mathcal{Y}$.*

**Definition 1.3.12.** A simplicial set $K$ is called a *Kan complex* if it satisfies the *Kan condition*, that is for all $0 \leq j \leq n$, every diagram in $\mathrm{SSet}$

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a lift.

**Definition 1.3.13.** A simplicial set $\mathcal{C}$ is called a *weak Kan complex* or *quasi-category* if it satisfies the *weak Kan condition*, that is for all $0 < j < n$, every diagram in $\mathrm{SSet}$

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a lift.

In this case we refer to the $0$-simplices of $K$ as the *objects* of $K$ and to the $1$-simplices of $K$ as the *morphisms* of $K$.

**Proposition 1.3.14.** *A simplicial set $\mathcal{C}$ is a quasi-category if and only if for all $0 < j < n$ and every collection of $(n-1)$-simplices $(x_0, x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)$ of $\mathcal{C}$ satisfying, for all $0 \leq i < i' \leq n$ with $i \neq j \neq i'$:*

$$d_i(x_{i'}) = d_{i'-1}(x_i)$$

*there exists an $n$-simplex $x$ of $\mathcal{C}$ such that $d_i(x) = x_i$ for all $0 \leq i \leq n$ with $i \neq j$.*

**Proposition 1.3.15.** *The class of monomorphisms in $\mathrm{SSet}$ is equal to the weak saturated closure of the set of boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for all $n \geq 0$.*

**Definition 1.3.16.** Let $f : K \to L$ be a simplicial map. We call $f$

- *anodyne* if $f$ belongs to the weak saturated closure of the set of all horn inclusions $\Lambda^n_j \hookrightarrow \Delta^n$ with $0 \leq j \leq n$,
- *inner anodyne* if $f$ belongs to the weak saturated closure of the set of all inner horn inclusions $\Lambda^n_j \hookrightarrow \Delta^n$ with $0 < j < n$,
- a *Kan fibration* if $f$ has the right lifting property with respect to all horn inclusions,
- an *inner fibration* if $f$ has the right lifting property with respect to all inner horn inclusions,
- a *trivial fibration* if $f$ has the right lifting property with respect to all boundary inclusions.

*Remark* 1.3.17. Note that a simplicial set $\mathcal{C}$ is a quasi-category if and only if the terminal map $\mathcal{C} \to 1$ is an inner fibration.

*Remark* 1.3.18. In view of Example 1.2.10, all simplicial sets are small in the sense of Definition 1.2.9. Thus it follows from the Small object argument (Proposition 1.2.11) that we have weak factorization systems on $\mathrm{SSet}$ given by (anodyne, Kan fibration), (inner anodyne, inner fibration) and (monomorphism, trivial fibration).

**Definition 1.3.19.** Let $K$ be a simplicial set. We define the set $\pi_0(K)$ of *connected components* of $K$ as the colimit of $K$ as a functor $\mathbf{\Delta}^{op} \to \mathrm{Set}$. Equivalently, it is given by the reflexive coequalizer

$$\begin{tikzcd} K_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & K_0 \arrow[l, "s_0" description] \arrow[r, two heads] & \pi_0(K) \end{tikzcd}$$

In other words, $\pi_0(K)$ is the quotient set $K_0/\sim$ where the equivalence relation is generated by letting $a \sim b$ if there exists an edge $f : a \to b$ in $K$ (for all $a, b \in K_0$). We call $K$ *connected* if $\pi_0(K)$ is a singleton.

This construction clearly extends to a functor

$$\pi_0 : \mathrm{SSet} \to \mathrm{Set}$$

**Construction 1.3.20.** Let $\mathrm{Top}$ denote the category of topological spaces. Given $n \geq 0$, consider the topological $n$-simplex $|\Delta^n| \in \mathrm{Top}$. This defines a functor $|\Delta^{(-)}| : \mathbf{\Delta} \to \mathrm{Top}$. By Proposition 1.3.11, we obtain an adjunction

$$\begin{tikzcd} \mathrm{SSet} \arrow[r, shift left=2, "|-|"] \arrow[r, phantom, "\perp"] & \mathrm{Top} \arrow[l, shift left=2, "\mathrm{Sing}"] \end{tikzcd}$$

The left-adjoint $|-|$ is called the *geometric realization functor*.

Given a simplicial set $K$, the set of path components of $|K|$ is bijective to $\pi_0(K)$.

**Definition 1.3.21.** A simplicial map $f : K \to L$ is called a *weak homotopy equivalence* if the induced continuous map $|f| : |K| \to |L|$ is a weak homotopy equivalence of topological spaces.

It is clear that a weak homotopy equivalence $f : K \to L$ of simplicial sets induces a bijection on the sets of connected components $\pi_0(f) : \pi_0(K) \xrightarrow{\sim} \pi_0(L)$.

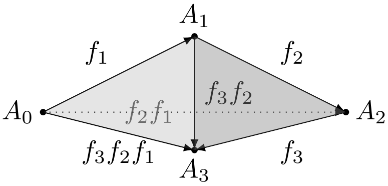
### 1.3.2 The nerve and the homotopy category

**Definition 1.3.22.** Let $\mathcal{C}$ be a small category. The *nerve* of $\mathcal{C}$ is the simplicial set

$$N(\mathcal{C}) = \mathrm{Cat}(-, \mathcal{C}) : \mathbf{\Delta}^{op} \to \mathrm{Set}$$

where we consider $\mathbf{\Delta}$ as a full subcategory of $\mathrm{Cat}$.

Explicitly, for all integers $n \geq 0$, the set $N(\mathcal{C})_n$ consists of all sequences $(f_1, ..., f_n)$ of composable morphisms in $\mathcal{C}$. That is, we have objects $A_0, ..., A_n \in \mathrm{Ob}(\mathcal{C})$ such that $f_i : A_{i-1} \to A_i$ for all $i \in \{1, ..., n\}$. In particular, we can identify $N(\mathcal{C})_0$ with the set $\mathrm{Ob}(\mathcal{C})$ and $N(\mathcal{C})_1$ with the set $\mathrm{Mor}(\mathcal{C})$ of morphisms of $\mathcal{C}$.



For any integer $n > 0$ and $i \in [n]$ the face map $d_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n-1}$ is given by

$$d_i(f_1, ..., f_n) = \begin{cases} (f_2, ..., f_n) & \text{if } i = 0 \\ (f_1, ..., f_{i-1}, f_{i+1} \circ f_i, f_{i+2}, ..., f_n) & \text{if } 0 < i < n \\ (f_1, ..., f_{n-1}) & \text{if } i = n \end{cases}$$

For any integer $n \geq 0$ and $i \in [n]$ the degeneracy map $s_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n+1}$ is given by

$$s_i(f_1, ..., f_n) = (f_1, ..., f_i, \mathrm{id}_{A_i}, f_{i+1}, ..., f_n)$$

Moreover, this defines a functor

$$N : \mathrm{Cat} \to \mathrm{SSet}$$

called the *nerve functor*.

**Example 1.3.23.** For all $n \geq 0$, we have

$$N([n]) = \mathrm{Cat}(-, [n]) \simeq \mathbf{\Delta}(-, [n]) = \Delta^n$$

**Proposition 1.3.24.** *The nerve functor* $N : \mathrm{Cat} \to \mathrm{SSet}$ *is fully faithful and a simplicial set* $K$ *is isomorphic to the nerve of a category if and only if for all* $0 < j < n$*, every diagram in* SSet

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

*has a unique lift. In particular every nerve of a small category is a quasi-category.*

**Construction 1.3.25.** Let $K$ be a simplicial set and let $\mathbb{F}(K)$ be the free category on the graph whose vertices and edges are the 0-simplices and 1-simplices of $K$ respectively.

Thus $\mathrm{Ob}(\mathbb{F}(K)) = K_0$ and for all $a, b \in K_0$ a morphism $a \to b$ in $\mathbb{F}(K)$ is given by a sequence

$$(f_1, ..., f_n)$$

with $n \geq 0$ and for all $i \in \{1, ..., n\}$, $f_i$ is an edge $a_{i-1} \to a_i$ of $K$, for some vertices $a = a_0, a_1, ..., a_{n-1}, a_n = b$ of $K$. Composition is given by concatenation of sequences and the identity is given by the empty sequence $()$.

**Proposition 1.3.26.** *The nerve functor* $N : \mathrm{Cat} \to \mathrm{SSet}$ *has a left-adjoint* $h : \mathrm{SSet} \to \mathrm{Cat}$ *which is given on objects as follows.*

*For any simplicial set* $K$*,* $hK$ *is the quotient category* $\mathbb{F}(K)/\sim$ *where* $\sim$ *is the equivalence relation generated by*

- $(f, g) \sim h$ *for all 2-simplices* $w \in K_2$ *with* $d_0(w) = g, d_1(w) = h, d_2(w) = f$*,*
- $() \sim s_0(a)$ *for all* $a \in K_0$*.*

**Lemma 1.3.27.** *Let $\mathcal{C}$ be a quasi-category and $f, g : a \to b$ morphisms in $\mathcal{C}$. The following statements are equivalent.*

*(1) There exists a* 2*-simplex $w \in \mathcal{C}_2$ such that $d_0(w) = s_0(b)$, $d_1(w) = g$ and $d_2(w) = f$.*

*(2) There exists a* 2*-simplex $w \in \mathcal{C}_2$ such that $d_0(w) = f$, $d_1(w) = g$ and $d_2(w) = s_0(a)$.*

$$
\begin{tikzcd}
 & b \arrow[dr, "s_0(b)"] \arrow[d, phantom, "w"] & \\
a \arrow[ur, "f"] \arrow[rr, "g"'] & & b
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & a \arrow[dr, "f"] \arrow[d, phantom, "w"] & \\
a \arrow[ur, "s_0(a)"] \arrow[rr, "g"'] & & b
\end{tikzcd}
$$

*In this case, we denote $f \sim g$ and say that $f$ is* homotopic *to $g$. Moreover, $\sim$ defines an equivalence relation on the set $\mathcal{C}_1$ of morphisms in $\mathcal{C}$.*

**Proposition 1.3.28.** *Let $\mathcal{C}$ be a quasi-category. Then for all objects $a$ and $b$ of $\mathcal{C}$, there is a bijection*

$$h\mathcal{C}(a, b) \simeq \{[f] \mid f : a \to b \text{ in } \mathcal{C}\}$$

*where $[f]$ denotes the equivalence class of $f$ under $\sim$. Under these bijections, we have that*

- *the identity in $h\mathcal{C}$ on an object $a \in \mathcal{C}_0$ is given by $[s_0(a)]$, and*
- *for any two morphisms $f : a \to b$ and $g : b \to c$ in $\mathcal{C}$, we have $[g] \circ [f] = [d_1(w)]$ for any* 2*-simplex $w \in \mathcal{C}_2$ with $d_0(w) = g$ and $d_2(w) = f$.*

# Chapter 2

# Templicial objects

*"I don't know where I am, but there is something beautiful about this place. I will explore and see what I can discover."*

— @ v17.1.0054 (The Talos Principle)

In this first non-preparatory chapter we introduce our main objects of study, *templicial objects* (Definition 2.1.9). These should be viewed as simplicial objects in a sense internal to a suitable monoidal category $\mathcal{V}$. As mentioned in the introduction, this philosophy is based on the work of Aguiar [Agu97] and Leinster [Lei00]. We will always require templicial objects to have a set of vertices. This will be achieved by means of $\mathcal{V}$-enriched quivers (see §1.1.2). Concretely, a templicial object is a pair $(X, S)$ with $S$ a set and

$$X : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$$

a strongly unital, colax monoidal functor. We will see that these indeed recover ordinary simplicial sets when $\mathcal{V} = \mathrm{Set}$. Moreover, we will establish some first easy properties of templicial objects and introduce the tools that will turn up in the later chapters.

The chapter is divided into three sections as follows. Section 2.1 covers the very basics. We define templicial objects and their category $S_\otimes\mathcal{V}$, and show it is cocomplete (Corollary 2.1.23). It will follow that we have an adjunction $\tilde{F} : \mathrm{SSet} \leftrightarrows S_\otimes\mathcal{V} : \tilde{U}$ (Proposition 2.1.25). We end the section by generalizing the classical simplicial skeleton construction to the context of templicial objects in §2.1.5.

Next, we introduce our proposed analogue of quasi-categories in the templicial context, the *quasi-categories in a monoidal category* $\mathcal{V}$ (Definition 2.2.26). Crucial for this, and for the rest of the thesis, are *necklaces*. These were first introduced by Dugger and Spivak in [DS11b]. We open Section 2.2 by introducing these necklaces and then defining quasi-categories in $\mathcal{V}$. Then we show that for any quasi-category $X$ in $\mathcal{V}$, the simplicial set $\tilde{U}(X)$ is always an ordinary quasi-category and if $\mathcal{V} = \mathrm{Set}$ the two notions coincide (see Propositions 2.2.30 and 2.2.31). Finally, we introduce *(non-associative) Frobenius structures*, which are based on Day and Pastro's Frobenius monoidal functors [DP08]. These are related to quasi-categories in $\mathcal{V}$ and will prove to be a useful tool when we consider templicial $k$-modules (i.e. when $\mathcal{V} = \mathrm{Mod}(k)$) in Chapter 4.

In the final section of this chapter, Section 2.3, we generalize the classical nerve functor $N : \mathrm{Cat} \to \mathrm{SSet}$ and its left-adjoint, which takes the homotopy category of a simplicial set.

These templicial versions will satisfy analogous properties to their classical counterparts. For instance, the description of the homotopy category of a templicial object becomes significantly easier if it is a quasi-category in $\mathcal{V}$ (Construction 2.3.19).

Recall the standing hypotheses. We fix a cocomplete and finitely complete monoidal category $(\mathcal{V}, \otimes, I)$ such that the monoidal product $- \otimes -$ preserves coproducts in each variable.

## 2.1 Basic definitions

### 2.1.1 Simplicial objects and colax monoidal functors

An observation originally due to Leinster (see Proposition 2.1.6) states that simplicial objects in a cartesian monoidal category can be equivalently described as certain colax monoidal functors. We argue that these colax monoidal functors are a good replacement for simplicial objects even when the monoidal category is not cartesian. As such, they will be essential when we define templicial objects in the next subsection.

Let us start by introducing some variants of the simplex category $\mathbf{\Delta}$ (Definition 1.3.1).

**Definition 2.1.1.** We define the following simplex categories:

- $\mathbf{\Delta}_+$ is the *augmented simplex category*. Its objects are the posets $[n] = \{0, ..., n\}$ with $n \geq -1$ (where $[-1] = \emptyset$), and its morphisms are the order morphisms $[m] \to [n]$. We denote the unique morphism $[-1] \to [0]$ by $\delta_0$ and call it a *coface* map as well.
- $\mathbf{\Delta}_f$ is the category of *finite intervals*, which is the subcategory of $\mathbf{\Delta}$ consisting of all morphisms $f : [m] \to [n]$ that preserve the endpoints, that is $f(0) = 0$ and $f(m) = n$.
- $\mathbf{\Delta}_{surj}$ is the subcategory of $\mathbf{\Delta}_f$ consisting of all surjective morphisms $f : [m] \twoheadrightarrow [n]$.

Note that we have inclusions of subcategories $\mathbf{\Delta}_{surj} \subseteq \mathbf{\Delta}_f \subseteq \mathbf{\Delta} \subseteq \mathbf{\Delta}_+$.

*Remark* 2.1.2. Note that $\mathbf{\Delta}_+$ contains all coface and codegeneracy maps of $\mathbf{\Delta}$, as well as the coface map $\delta_0 : [-1] \to [0]$. On the other hand, $\mathbf{\Delta}_f$ only contains the inner coface maps but still all codegeneracy maps. Finally, $\mathbf{\Delta}_{surj}$ only contains the codegeneracy maps but no coface maps.

It follows from Lemma 1.3.3 that the categories $\mathbf{\Delta}_+$, $\mathbf{\Delta}_f$ and $\mathbf{\Delta}_{surj}$ are also generated by the coface and codegeneracy maps that they contain. So for example, every morphism $f : [m] \to [n]$ in $\mathbf{\Delta}_f$ has a unique representation

$$f = \delta_{j_1} ... \delta_{j_s} \sigma_{i_1} ... \sigma_{i_t}$$

with $0 < j_s < ... < j_1 < n$, $0 \leq i_1 < ... < i_t < m$ and $s, t \geq 0$ such that $m - t + s = n$.

*Remark* 2.1.3. In contrast to the category $\mathbf{\Delta}$, both the categories $\mathbf{\Delta}_+$ and $\mathbf{\Delta}_f$ are naturally endowed with monoidal structures.

The monoidal structure $(\star, [-1])$ on $\mathbf{\Delta}_+$ is given by juxtaposition of posets and morphisms, as follows. For $m, n \geq -1$:

$$[m] \star [n] = [m+n+1]$$

For morphisms $f : [m] \to [m']$ and $g : [n] \to [n']$ in $\mathbf{\Delta}_+$:

$$(f \star g)(i) = \begin{cases} f(i) & \text{if } i \leq m \\ m' + 1 + g(i - m - 1) & \text{if } i > m \end{cases}$$

Similarly, the monoidal structure $(+, [0])$ on $\mathbf{\Delta}_f$ is given by identifying respective top and bottom endpoints, as follows. For all $m, n \geq 0$:

$$[m] + [n] = [m+n]$$

For morphisms $f : [m] \to [m']$ and $[n] \to [n']$ in $\mathbf{\Delta}_f$:

$$(f + g)(i) = \begin{cases} f(i) & \text{if } i \leq m \\ m' + g(i - m) & \text{if } i \geq m \end{cases}$$

Of course, $(\mathbf{\Delta}_{surj}, +, [0])$ also becomes a monoidal category with the inherited monoidal structure from $\mathbf{\Delta}_f$.

There is a well-known monoidal equivalence $\mathbf{\Delta}_+ \simeq \mathbf{\Delta}_f^{op}$, the relevant functor in each direction being obtained by considering posets of morphisms into $[1]$ (see [Joy97]).

*Remark* 2.1.4. Let $f : [m] \to [n]$ be a morphism in $\mathbf{\Delta}_f$ and let $0 \leq k \leq m$. Then there exist unique morphisms $f_1 : [k] \to [p]$ and $f_2 : [m-k] \to [n-p]$ in $\mathbf{\Delta}_f$ such that

$$f_1 + f_2 = f$$

They are defined by setting $p = f(k)$, $f_1(i) = f(i)$ for all $i \in [k]$ and $f_2(i) = f(i+k) - p$ for all $i \in [n-k]$.

*Remark* 2.1.5. Let $(\mathcal{W}, \otimes, I)$ be an arbitrary monoidal category. We will be particularly interested in colax monoidal functors $X : \mathbf{\Delta}_f^{op} \to \mathcal{W}$. Similarly to Proposition 1.3.5, it follows from Remark 2.1.2 that $X$ is equivalent to a collection $(X_n)_{n \geq 0}$ of objects in $\mathcal{W}$ endowed with the following data:

- morphisms $d_j^X : X_n \to X_{n-1}$ for all $0 < j < n$ which we call the *inner face morphisms*,
- morphisms $s_i^X : X_n \to X_{n+1}$ for all $0 \leq i \leq n$ which we call the *degeneracy morphisms*,
- morphisms $\mu_{k,l}^X : X_{k+l} \to X_k \otimes X_l$ for all $k, l \geq 0$ which we call the *comultiplication morphisms*,
- a morphism $\epsilon^X : X_0 \to I$ which we call the *counit morphism*.

This data moreover has to satisfy the simplicial identities (1.3) as well as:

- (Naturality of $\mu^X$) For all $k, l \geq 0$ and $0 < j < k+l+1$, $0 \leq i \leq k+l-1$, we have

$$\begin{aligned} \mu^X_{k,l} d^X_j &= \begin{cases} (d^X_j \otimes \mathrm{id}_{X_l})\mu^X_{k+1,l} & \text{if } j \leq k \\ (\mathrm{id}_{X_k} \otimes d^X_{j-k})\mu^X_{k,l+1} & \text{if } j > k \end{cases} \\ \mu^X_{k,l} s^X_i &= \begin{cases} (s^X_i \otimes \mathrm{id}_{X_l})\mu^X_{k-1,l} & \text{if } i < k \\ (\mathrm{id}_{X_k} \otimes s^X_{i-k})\mu^X_{k,l-1} & \text{if } i \geq k \end{cases} \end{aligned} \tag{2.1}$$

- (Coassociativity of $\mu^X$) For all $r, s, t \geq 0$, we have

$$(\mathrm{id}_{X_r} \otimes \mu^X_{s,t})\mu^X_{r,s+t} = (\mu^X_{r,s} \otimes \mathrm{id}_{X_t})\mu^X_{r+s,t} \tag{2.2}$$

- (Counitality of $\mu^X$ with $\epsilon^X$) For all $n \geq 0$, we have

$$(\mathrm{id}_{X_n} \otimes \epsilon^X)\mu^X_{n,0} = \mathrm{id}_{X_n} = (\epsilon^X \otimes \mathrm{id}_{X_n})\mu^X_{0,n} \tag{2.3}$$

Note that by the coassociativity, we have a well-defined morphism

$$\mu^X_{k_1,\dots,k_n} : X_{k_1+\dots+k_n} \to X_{k_1} \otimes \dots \otimes X_{k_n}$$

for all $n \geq 2$ and $k_1, \dots, k_n \geq 0$. Further, we will set $\mu^X_{k_1,\dots,k_n}$ to be the identity on $X_{k_1}$ if $n = 1$, and to be the counit $\epsilon^X$ if $n = 0$.

Moreover, under these identifications a monoidal natural transformation $\alpha : X \to Y$ between colax monoidal functors $X, Y : \mathbf{\Delta}^{op}_f \to \mathcal{W}$ is equivalent to a collection of morphisms $(\alpha_n : X_n \to Y_n)_{n \geq 0}$ which satisfy:

- (Naturality of $\alpha$) For all $0 < j < n$ and $0 \leq i \leq n$, we have

$$\alpha_{n-1} d^X_j = d^Y_j \alpha_n \quad \text{and} \quad \alpha_{n+1} s^X_i = s^Y_i \alpha_n$$

- (Monoidality of $\alpha$) For all $k, l \geq 0$, we have

$$\mu^Y_{k,l}\alpha_{k+l} = (\alpha_k \otimes \alpha_l)\mu^X_{k,l} \quad \text{and} \quad \epsilon^Y \alpha_0 = \epsilon^X$$

Finally, we will often drop the superscript $X$ from the notation when it is clear from the context which colax monoidal functor these morphisms belong to.

Recall that a monoidal category $\mathcal{W}$ is called *cartesian* if the monoidal product is given by the categorical product (also see Section 1.1).

**Proposition 2.1.6** ([Lei00], Proposition 3.1.7)**.** *Let $\mathcal{W}$ be a cartesian monoidal category. There is an isomorphism of categories*

$$\mathrm{Colax}(\mathbf{\Delta}^{op}_f, \mathcal{W}) \simeq S\mathcal{W}.$$

*Proof.* Let $X : \mathbf{\Delta}^{op} \to \mathcal{W}$ be a functor. We can equip the restriction $X|_{\mathbf{\Delta}^{op}_f}$ with a colax monoidal structure as follows. Define for all $k, l \geq 0$:

$$\mu^X_{k,l} = (d_{k+1} \dots d_{k+l}, \underbrace{d_0 \dots d_0}_{k \text{ times}}) : X_{k+l} \to X_k \times X_l$$

Given morphisms $f : [k] \to [k']$ and $g : [l] \to [l']$ in $\mathbf{\Delta}_f$, we have

$$(f+g)\delta_{k+l}...\delta_{k+1} = \delta_{k'+l'}...\delta_{k'+1}f \quad \text{and} \quad (f+g)\underbrace{\delta_0...\delta_0}_{k \text{ times}} = \underbrace{\delta_0...\delta_0}_{k' \text{ times}} g$$

which shows that $\mu^X$ is a natural in the sense of (2.1). Further, let $\epsilon^X : X_0 \to 1$ be the terminal morphism. Then $\mu^X$ and $\epsilon^X$ trivially satisfy counitality (2.3) and $\mu^X$ is also coassociative (2.2) by the following observation:

$$d_{s+1}...d_{s+t}\underbrace{d_0...d_0}_{r \text{ times}} = \underbrace{d_0...d_0}_{r \text{ times}} d_{r+s+1}...d_{r+s+t} \quad (\forall r,s,t \geq 0)$$

Take $X, Y \in S\mathcal{W}$ and let $\alpha : X|_{\mathbf{\Delta}_f^{op}} \to Y|_{\mathbf{\Delta}_f^{op}}$ be a natural transformation between their restrictions $\mathbf{\Delta}_f^{op} \to \mathcal{W}$. Then $\alpha$ extends to a morphism $X \to Y$ in $S\mathcal{W}$ if and only if $\alpha$ is monoidal. Indeed, this follows from the observation that for all $k, l \geq 0$:

$$(\alpha_k \times \alpha_l)\mu^X_{k,l} = (\alpha_k d_{k+1}...d_{k+l}, \alpha_l d_0...d_0), \quad \mu^Y_{k,l}\alpha_{k+l} = (d_{k+1}...d_{k+l}\alpha_{k+l}, d_0...d_0\alpha_{k+l})$$

and $\epsilon^Y\alpha_0 = \epsilon^X$ always holds. We thus obtain a fully faithful functor

$$\varphi : S\mathcal{W} \to \mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathcal{W})$$

It remains to show that $\varphi$ is also bijective on objects.

Given a colax monoidal functor $X : \mathbf{\Delta}_f^{op} \to \mathcal{W}$ with comultiplication $\mu$ and counit $\epsilon$, we can extend $X$ to a functor $\overline{X} : \mathbf{\Delta}^{op} \to \mathcal{W}$ as follows. It suffices to define the outer face morphisms of $\overline{X}$. For $n > 0$, set

$$d_0 : X_n \xrightarrow{\mu_{1,n-1}} X_1 \times X_{n-1} \xrightarrow{\pi_2} X_{n-1} \quad \text{and} \quad d_n : X_n \xrightarrow{\mu_{n-1,1}} X_{n-1} \times X_1 \xrightarrow{\pi_1} X_{n-1}$$

Then the coassociativity and naturality of $\mu$ implies that for all $n \geq 2$:

$$\begin{aligned} d_0 d_n &= \pi_2\mu_{1,n-2}\pi_1\mu_{n-1,1} = \pi_2(\mu_{1,n-2} \times \mathrm{id}_{X_1})\mu_{n-1,1} = \pi_2(\mathrm{id}_{X_1} \times \mu_{n-2,1})\mu_{1,n-1} \\ &= \pi_1\mu_{n-2,1}\pi_2\mu_{1,n-1} = d_{n-1}d_0 \\ d_0 d_0 &= \pi_2\mu_{1,n-2}\pi_2\mu_{1,n-1} = \pi_3(\mathrm{id}_{X_1} \times \mu_{1,n-2})\mu_{1,n-1} = \pi_3(\mu_{1,1} \times \mathrm{id}_{X_{n-2}})\mu_{2,n-2} \\ &= \pi_2\mu_{2,n-2} = \pi_2(d_1 \times \mathrm{id}_{X_{n-2}})\mu_{2,n-2} = \pi_2\mu_{1,n-2}d_1 = d_0 d_1 \end{aligned}$$

and similarly, $d_{n-1}d_n = d_{n-1}d_{n-1}$. The other simplicial identities involving the outer face morphisms follow from the naturality of $\mu$. Thus $\overline{X}$ is a well-defined simplicial object in $\mathcal{V}$. Now, it follows from the coassociativity of $\mu$ that for all $k, l \geq 0$:

$$\pi_1\mu_{k,l} = (\pi_1\mu_{k,1})(\pi_1\mu_{k+1,1})\cdots(\pi_1\mu_{k+l-1,1}) = d_{k+1}...d_{k+l}$$

and similarly $\pi_2\mu_{k,l} = d_0...d_0$. Hence, we find that $\varphi(\overline{X}) = (\overline{X}|_{\mathbf{\Delta}_f^{op}}, \mu^{\overline{X}}, \epsilon^{\overline{X}}) = (X, \mu, \epsilon)$. Finally, if $Y$ is a simplicial object in $\mathcal{V}$ such that $\varphi(Y) = (X, \mu, \epsilon)$, then it follows that $Y|_{\mathbf{\Delta}_f^{op}} = X$ and $d_0 = \pi_2\mu_{1,n-1}$, $d_n = \pi_1\mu_{n-1,1}$ and thus $Y = \overline{X}$. □

**Example 2.1.7.** If $\mathcal{W} = \mathrm{Set}$, Proposition 2.1.6 supplies an isomorphism of categories:

$$\mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathrm{Set}) \simeq \mathrm{SSet}$$

*Remark* 2.1.8. Suppose $\mathcal{W}$ is cartesian and let $X : \mathbf{\Delta}_f^{op} \to \mathcal{W}$ be a colax monoidal functor. It follows from the proof of Proposition 2.1.6 that the outer face morphisms $d_0$ and $d_n$ are obtained as

$$d_0 : X_n \xrightarrow{\mu_{1,n-1}} X_1 \times X_{n-1} \xrightarrow{\pi_2} X_{n-1}$$

and

$$d_n : X_n \xrightarrow{\mu_{n-1,1}} X_{n-1} \times X_1 \xrightarrow{\pi_1} X_{n-1}$$

where we have made use of the projections $\pi_1$ and $\pi_2$ from the product to its factors.

If $\mathcal{W}$ is not necessarily cartesian, these projections are not available in general and the comultiplication $\mu$ of a colax monoidal functor can be considered as a replacement for the outer face morphisms in the monoidal context.

### 2.1.2 Templicial objects

We are now ready to define our main object of study, the templicial objects in $\mathcal{V}$. Further, we give some examples and show that when $\mathcal{V} = \mathrm{Set}$ they recover simplicial sets (Proposition 2.1.15). Finally, we show how the category of templicial objects $S_{\otimes}\mathcal{V}$ can be constructed by means of a Grothendieck construction.

Recall the monoidal category $(\mathcal{V}\,\mathrm{Quiv}_S, \otimes_S, I_S)$ of Construction 1.1.14, and the base change functors $f_! : \mathcal{V}\,\mathrm{Quiv}_S \leftrightarrows \mathcal{V}\,\mathrm{Quiv}_T : f^*$ for a given map of sets $f : S \to T$ (Construction 1.1.16).

**Definition 2.1.9.** A *tensor-simplicial* or *templicial object* in $\mathcal{V}$ is a pair $(X, S)$ with $S$ a set and

$$X : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$$

a colax monoidal functor which is strongly unital, i.e. its counit $\epsilon : X_0 \to I_S$ is an isomorphism. We call the elements of $S$ the *vertices* of $X$. For $n > 0$, an *$n$-simplex* of $X$ is an element of the underlying set of $X_n(a, b) \in \mathcal{V}$ for some $a, b \in S$.

Let $(X, S)$ and $(Y, T)$ be templicial objects. A *templicial morphism* $(X, S) \to (Y, T)$ is a pair $(\alpha, f)$ with $f : S \to T$ a map of sets and $\alpha : f_!X \to Y$ a monoidal natural transformation between colax monoidal functors $\mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_T$. Here, we used the colax monoidal structure of $f_!$ (see Lemma 1.1.18).

Sometimes we will denote a templicial object $(X, S)$ or a templicial morphism $(\alpha, f)$ simply by $X$ or $\alpha$ respectively.

*Remark* 2.1.10. An alternative way to realize a set of vertices $S$ consists in turning the monoidal category $\mathbf{\Delta}_f$ (which is a one object bicategory) into a bicategory with object set $S$. This approach goes back to [Lur09a] and was used in [Sim12], [Bac10].

Before discussing the category of templicial objects, let us first give some examples in the case where $\mathcal{V} = \mathrm{Mod}(k)$ is the category of $k$-modules for some unital commutative ring $k$. The monoidal product is the tensor product $\otimes_k$ over $k$ and the monoidal unit is $k$ itself.

**Example 2.1.11.** Let $M$ be a $k$-module. Consider the $\mathrm{Mod}(k)$-enriched quiver $Q$ with only two distinct vertices $a$ and $b$, and

$$Q(a,b) = M, \quad Q(b,a) = 0 \quad \text{and} \quad Q(a,a) = Q(b,b) = k$$

Then there exists a templicial object $(X, \{a,b\})$ such that the quiver $X_1$ is equal to $Q$ (also see Example 2.1.38.2).

We consider the elements of $M$ as edges $a \to b$ of $Q$. As $M$ is an abelian group, we can take the sum of two edges $f, g : a \to b$ to get another edge $f + g : a \to b$. Note that this is reflected in the comultiplication map $\mu_{1,0} : X_1 \to X_1 \otimes_S X_0$ of $X$. Indeed, for $f, g \in M$ we have the equations

$$\mu_{1,0}(f) = f \otimes b \quad \text{and} \quad \mu_{1,0}(g) = g \otimes b$$

which express that $f$ and $g$ are edges of $X$ with common target $b$. Now because $\mu_{1,0}$ is assumed to be a linear map, we have

$$\mu_{1,0}(f+g) = \mu_{1,0}(f) + \mu_{1,0}(g) = f \otimes b + g \otimes b = (f+g) \otimes b$$

which expresses that $f + g$ is also an edge of $X$ with target $b$.

This may all seem a bit tautological, because it is. But notice that a simplicial $k$-module cannot capture the same behavior. The analogue of the map $\mu_{1,0}$ for a simplicial $k$-module $X$ would be the face map $d_0 : X_1 \to X_0$. Now for $f, g \in X_1$, the linearity of $d_0$ implies that $d_0(f+g) = d_0(f) + d_0(g)$. So the targets of the edges in $X_1$ are not invariant under addition. In other words, the set $\{f \in X_1 \mid d_1(f) = a, d_0(f) = b\}$ is not a $k$-module in any canonical way.

**Example 2.1.12.** Let $A$ be a $k$-algebra. We can make $A$ into a templicial $k$-module resembling the bar construction. For all $n \geq 0$, define

$$N_k(A)_n = A^{\otimes n} \in \mathrm{Mod}(k)$$

We define the inner face maps and degeneracy maps of $N_k(A)$ as follows. For $0 < j < n$, $0 \leq i \leq n$ and $a_1, ..., a_n \in A$, set

$$\begin{aligned} d_j(a_1 \otimes ... \otimes a_n) &= a_1 \otimes ... \otimes a_{j-1} \otimes a_{j+1}a_j \otimes a_{j+2} \otimes ... \otimes a_n \\ s_i(a_1 \otimes ... \otimes a_n) &= a_1 \otimes ... \otimes a_i \otimes 1_A \otimes a_{i+1} \otimes ... \otimes a_n \end{aligned}$$

It follows from the associativity of the multiplication in $A$ and the unit property of $1_A$ that the simplicial identities (1.3) are satisfied. Finally, the counit and comultiplication maps of $N_k(A)$ are given by the isomorphisms $\epsilon : A^{\otimes 0} \xrightarrow{\sim} k$ and $\mu_{p,q} : A^{\otimes p+q} \xrightarrow{\sim} A^{\otimes p} \otimes A^{\otimes q}$. We thus obtain a strong (and thus colax) monoidal functor

$$N_k(A) : \mathbf{\Delta}_f^{op} \to \mathrm{Mod}(k)$$

So by Example 1.1.15, we can view $N_k(A)$ as a templicial object with one vertex. It was noted by Maclane that this completely determines the algebra $A$ up to isomorphism, as far back as [Mac71, Proposition VII.5.1].

In §2.3.1, we will extend this construction to general $\mathcal{V}$-enriched categories.

**Example 2.1.13.** Templicial objects behave fundamentally differently to simplicial objects when the enriching category $\mathcal{V}$ is not cartesian. We already illustrated this in Example 2.1.11. Let us give another example. Choose $k = \mathbb{Z}$ so that $\mathcal{V} = \mathrm{Ab}$ is the category of abelian groups. Let $S = \{*\}$ and define

$$X_n = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Q}/\mathbb{Z} & \text{if } n > 0 \end{cases} \in \mathrm{Ab}$$

where $s_0 : X_0 \to X_1$ is the zero map and the other face and degeneracy maps are the identity on $\mathbb{Q}/\mathbb{Z}$. As $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} \simeq 0$, we can define $\mu_{k,l} : X_{k+l} \to X_k \otimes X_l$ as the zero map for all $k, l > 0$, and as the left or right unit isomorphism if $l = 0$ or $k = 0$. We thus obtain a templicial abelian group $(X, S)$.

On the other hand, if $A$ is a simplicial abelian group, note that if any face map $d_i : A_n \to A_{n-1}$ is the zero map, then necessarily $A_{n-1} = 0$.

**Definition 2.1.14.** Given templicial morphisms $(\alpha, f) : (X, S) \to (Y, T)$ and $(\beta, g) : (Y, T) \to (Z, U)$, the *composition* $(\beta, g) \circ (\alpha, f)$ is defined to be the templicial morphism $(\gamma, g \circ f) : (X, S) \to (Y, T)$ with

$$\gamma : (gf)_! X \simeq g_! f_! X \xrightarrow{g_! \alpha} g_! Y \xrightarrow{\beta} Z$$

where the isomorphism is given by Proposition 1.1.19.

Further, for any templicial object $(X, S)$ we define the *identity* on $(X, S)$ as the templicial morphism $(\varphi, \mathrm{id}_S) : (X, S) \to (X, S)$ where $\varphi : (\mathrm{id}_S)_! X \xrightarrow{\sim} X$ is the isomorphism given by Proposition 1.1.19.

It then follows that templicial objects and templicial morphisms define a category which we denote by

$$S_{\otimes} \mathcal{V}$$

**Proposition 2.1.15.** *There is an equivalence of categories:*

$$S_{\times}\,\mathrm{Set} \simeq \mathrm{SSet}$$

*Proof.* Let $K$ be a simplicial set. By Proposition 2.1.6, we may consider $K$ as a colax monoidal functor $\mathbf{\Delta}_f^{op} \to \mathrm{Set}$ with comultiplication $\mu$ and counit $\epsilon$. Then define for all $n \geq 0$ and $a, b \in K_0$:

$$\begin{aligned} K_n(a, b) &= \{x \in K_n \mid \mu_{0,n,0}(x) = (a, x, b)\} \\ &= \{x \in K_n \mid d_1 ... d_n(x) = a, d_0 ... d_0(x) = b\} \end{aligned}$$

Given $f : [m] \to [n]$ in $\mathbf{\Delta}_f$, it follows from the simplicial identities that $K(f) : K_n \to K_m$ restricts to a map $K(f)_{a,b} : K_n(a, b) \to K_m(a, b)$. Moreover, it is clear that for all $k, l \geq 0$ and $a, b \in K_0$, $\mu_{k,l}$ restricts to

$$\mu_{k,l}|_{K_{k+l}(a,b)} : K_{k+l}(a, b) \to \coprod_{c \in K_0} K_k(a, c) \times K_l(c, b)$$

and $K_0(a, a) = \{a\}$ if $a = b$, and $K_0(a, b) = \emptyset$ if $a \neq b$. Consequently, the functor

$$\varphi(K) : \mathbf{\Delta}_f^{op} \to \mathrm{Quiv}_{K_0} : [n] \mapsto (K_n(a, b))_{a,b \in K_0}$$

is strongly unital and colax monoidal, whereby $(\varphi(K), K_0)$ is a templicial object.

Conversely, if $(X, S)$ is a templicial object in Set, then we can define a simplicial set $\mathfrak{c}(X)$ by setting for all $n \geq 0$:

$$\mathfrak{c}(X)_n = \coprod_{a,b\in S} X_n(a, b)$$

It is readily verified that the assignments $K \mapsto \varphi(K)$ and $X \mapsto \mathfrak{c}(X)$ can be extended to mutually inverse equivalences between SSet and $S_\times$ Set. □

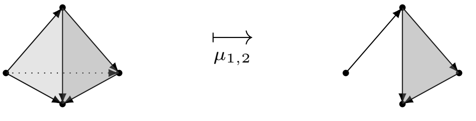
*Remark* 2.1.16. More generally, we can use the same method as in the proof of Proposition 2.1.15 to "pull apart" the objects $X_n \in \mathcal{V}$ of a colax monoidal functor $X : \mathbf{\Delta}_f^{op} \to \mathcal{V}$ into objects $X_n(a, b)$ indexed over a set, as long as $X_0$ is free in the appropriate sense. From this we can obtain an alternative definition of templicial objects. In Appendix A, we will present this comparison for suitable monoidal categories $\mathcal{V}$.

*Remark* 2.1.17. Let $(X, S)$ be a templicial object in $\mathcal{V}$ and consider $a, b \in S$. Then the proof of Proposition 2.1.15 suggests that $X_n(a, b) \in \mathcal{V}$ should be interpreted as the *object of $n$-simplices of $X$ with first vertex $a$ and last vertex $b$.*

Moreover, for all $k, l \geq 0$ and $a, b \in S$, the comultiplication morphism

$$(\mu^X_{k,l})_{a,b} : X_{k+l}(a, b) \to \coprod_{c\in S} X_k(a, c) \otimes X_l(c, b)$$

should be interpreted as taking a $(k+l)$-simplex from $a$ to $b$ and sending it to a $k$-simplex from $a$ to some $c \in S$, along with an $l$-simplex from $c$ to $b$, which are outer faces of the original $(k + l)$-simplex.




We can recover the category $S_\otimes\mathcal{V}$ as a subcategory of a Grothendieck construction. This will be useful later. Let <u>Cat</u> denote the (very large) strict 2-category of (large) categories, functors and natural transformations

**Proposition 2.1.18.** *Consider the pseudofunctor*

$$\Phi_{\mathcal{V}} = \mathrm{Colax}(\mathbf{\Delta}_f^{op}, (-)_!) : \mathrm{Set} \to \underline{\mathrm{Cat}}$$

*where* $(-)_! : \mathrm{Set} \to \underline{\mathrm{MonCat}}$ *is the pseudofunctor of Proposition 1.1.19. Then there is fully faithful functor*

$$S_\otimes\mathcal{V} \hookrightarrow \int \Phi_{\mathcal{V}}$$

*embedding the category of templicial objects into the Grothendieck construction of* $\Phi_{\mathcal{V}}$.

*Proof.* The pseudofunctor $\Phi_{\mathcal{V}}$ sends a set $S$ to the category $\mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathcal{V}\,\mathrm{Quiv}_S)$. A map of sets $f : S \to T$ is sent to the post-composition functor $f_! \circ -$. Thus the Grothendieck construction $\int \Phi_{\mathcal{V}}$ has as objects all pairs. $(X, S)$ with $S$ a set and $X : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$ a colax monoidal functor. A morphism from $(X, S)$ to $(Y, T)$ is given by a pair $(\alpha, f)$ with $f : S \to T$ a map of sets and $\alpha : f_! X \to Y$ a monoidal natural transformation in $\Phi_{\mathcal{V}}(T)$.

It is thus clear that $S_{\otimes}\mathcal{V}$ may be identified with the full subcategory of $\int \Phi_{\mathcal{V}}$ spanned by all pairs $(X, S)$ for which the colax monoidal functor $X : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$ is strongly unital. □

**Construction 2.1.19.** Consider a monoidal category $(\mathcal{W}, \otimes, I)$ with coproducts, such that $- \otimes -$ preserves coproducts in each variable. Let $H : \mathcal{W} \to \mathcal{V}$ be a strongly unital colax monoidal functor that preserves coproducts. Then for any set $S$, $H$ induces a colax monoidal functor

$$H_S : \mathcal{W}\,\mathrm{Quiv}_S \to \mathcal{V}\,\mathrm{Quiv}_S : Q \mapsto \big(H(Q(a, b))\big)_{a,b\in S}$$

If $f : S \to T$ is a map of sets, then because $H$ preserves coproducts, we have a monoidal natural isomorphism

$$f_! \circ H_S \simeq H_T \circ f_!$$

and one can check that the functors $(H_S)_S$ form a pseudonatural transformation $H_*$. Thus we have a pseudonatural transformation

$$\mathrm{Colax}(\mathbf{\Delta}_f^{op}, H_*) : \Phi_{\mathcal{U}} \to \Phi_{\mathcal{V}}$$

Then the Grothendieck construction supplies us with a functor

$$\tilde{H} : \int \Phi_{\mathcal{W}} \to \int \Phi_{\mathcal{V}}$$

Explicitely, a pair $(X, S)$ in $\int \Phi_{\mathcal{W}}$ is sent to the pair $(H_S \circ X, S)$ in $\int \Phi_{\mathcal{V}}$

Finally, as $H$ is assumed to be strongly unital, each $H_S$ is strongly unital as well and thus $\tilde{H}$ restricts to a functor

$$\tilde{H} : S_{\otimes}\mathcal{W} \to S_{\otimes}\mathcal{V}$$

### 2.1.3 Colimits of templicial objects

In this subsection we show that the category $S_{\otimes}\mathcal{V}$ of templicial objects is cocomplete and explicitly describe its colimits. We make use of the following result from the literature.

**Proposition 2.1.20** ([Her93], Corollary 3.3.7)**.** *Let $\mathcal{C}$ be a category and $\Psi : \mathcal{C} \to \underline{\mathrm{Cat}}$ a pseudofunctor. Assume that*

*(a) $\mathcal{C}$ is cocomplete,*

*(b) for every object $C$ of $\mathcal{C}$, the category $\Psi(C)$ is cocomplete,*

*(c) for every morphism $f$ in $\mathcal{C}$, the functor $\Psi(f)$ preserves colimits.*

*Then the Grothendieck construction $\int \Psi$ is cocomplete and a colimit of objects $(X_i, C_i)$ with $C_i \in \mathcal{C}$ and $X_i \in \Psi(C_i)$ is obtained as*

$$\operatorname*{colim}_i(X_i, C_i) = (\operatorname*{colim}_i \Psi(\iota^i)(X_i), \operatorname*{colim}_i C_i)$$

*for the canonical morphisms $\iota^i : C_i \to \mathrm{colim}_i\, C_i$ in $\mathcal{C}$.*

**Corollary 2.1.21.** *The category $\int \Phi_{\mathcal{V}}$ is cocomplete.*

*Proof.* Recall the pseudofunctor $\Phi_{\mathcal{V}} : \mathrm{Set} \to \underline{\mathrm{Cat}}$ from Proposition 2.1.18. Since $\mathcal{V}$ is cocomplete, so is $\mathcal{V}\,\mathrm{Quiv}_S \simeq \mathcal{V}^{S\times S}$ for every set $S$. It is not difficult to show that then also $\Phi_{\mathcal{V}}(S) = \mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathcal{V}\,\mathrm{Quiv}_S)$ is cocomplete, with colimits given pointwise. Moreover, if $f$ is a map of sets, then $f_!$ is left adjoint to $f^*$ and thus preserves colimits. It follows that $\Phi_{\mathcal{V}}(f)$ preserves colimits as well. Thus by Proposition 2.1.20, the category $\int \Phi_{\mathcal{V}}$ is cocomplete. □

*Remark* 2.1.22. Let us explicitly describe the colimits of $\int \Phi_{\mathcal{V}}$. Consider a diagram

$$D : \mathcal{J} \to \int \Phi_{\mathcal{V}}$$

with $\mathcal{J}$ a small category. Write $D(j) = (X^j, S^j)$ for every $j \in \mathcal{J}$ and $D(t) = (\alpha^t, f^t) : D(i) \to D(j)$ for every $t : i \to j$ in $\mathcal{J}$. Then the colimit of $D$ is given by

$$(\mathrm{colim}\, \tilde{D}, S)$$

where $S = \mathrm{colim}_{j\in\mathcal{J}} S^j$ in $\mathrm{Set}$ with canonical maps $\iota_j : S^j \to S$, and

$$\tilde{D} : \mathcal{J} \to \mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathcal{V}\,\mathrm{Quiv}_S)$$

is defined by for all $i, j \in \mathcal{J}$ and $t : i \to j$ in $\mathcal{J}$:

$$\tilde{D}(j) = (\iota_j)_! X^j \quad \text{and} \quad \tilde{D}(t) : (\iota_i)_! X^i \simeq (\iota_j)_! f^t_! X^i \xrightarrow{(\iota_j)_!\alpha^t} (\iota_j)_! X^j$$

where the isomorphism $(\iota_i)_! X^i \simeq (\iota_j)_! f^t_! X^i$ is given by the fact that $\iota_j f^t = \iota_j$.

Moreover, the colimit of $\tilde{D}$ is given pointwise. So for all $n \geq 0$ we have

$$\left(\mathrm{colim}\, \tilde{D}\right)_n \simeq \mathop{\mathrm{colim}}_{j\in\mathcal{J}} (\iota_j)_! X^j_n$$

**Proposition 2.1.23.** *The category $S_\otimes\mathcal{V}$ is cocomplete and the embedding $S_\otimes\mathcal{V} \hookrightarrow \int \Phi_{\mathcal{V}}$ preserves colimits.*

*Proof.* We check that the subcategory $S_\otimes\mathcal{V}$ is closed under colimits in $\int \Phi_{\mathcal{V}}$. So let $\mathcal{J}$ be a small category and $D : \mathcal{J} \to S_\otimes\mathcal{V} \subseteq \int \Phi_{\mathcal{V}}$ a diagram. With notations as in Remark 2.1.22, the colimit of $D$ in $\int \Phi_{\mathcal{V}}$ is the pair $(\mathrm{colim}\, \tilde{D}, S)$. For every $j \in \mathcal{J}$, write $\epsilon^j$ and $\xi^j$ for the counits of $X^j$ and $(\iota_j)_!$ respectively.

Boiling down the definitions, we see that the counit $(\mathrm{colim}\, \tilde{D})_0 \to I_S$ of $\mathrm{colim}\, \tilde{D}$ is the composition

$$\mathop{\mathrm{colim}}_{j\in\mathcal{J}} (\iota_j)_!(X^j_0) \xrightarrow{\mathrm{colim}_{j\in\mathcal{J}}(\iota_j)_!(\epsilon^j)} \mathop{\mathrm{colim}}_{j\in\mathcal{J}} (\iota_j)_!(I_{S^j}) \xrightarrow{\mathrm{colim}_{j\in\mathcal{J}}\xi^j} \mathop{\mathrm{colim}}_{j\in\mathcal{J}} I_S \xrightarrow{\nabla} I_S$$

in $\mathcal{V}\,\mathrm{Quiv}_S$, where $\nabla$ is the codiagonal. Now the composite $\nabla \circ \mathrm{colim}_{j\in\mathcal{J}}\xi^j$ is an isomorphism because for any $x, y \in S$, we have

$$(\mathop{\mathrm{colim}}_{j\in\mathcal{J}} (\iota_j)_!(I_{S^j}))(x,y) \simeq \begin{cases} \mathrm{colim}_{j\in\mathcal{J}} \coprod_{a\in(\iota_j)^{-1}(x)} I \simeq I & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \simeq I_S(x,y)$$

Since each $\epsilon^j$ is assumed to be an isomorphism as well, we conclude that $\operatorname{colim}\tilde{D}$ is strongly counital and thus that $\operatorname{colim} D$ is a templicial object. $\square$

**Proposition 2.1.24.** *Let $(\mathcal{W}, \otimes, I)$ be a cocomplete monoidal category such that $- \otimes -$ preserves colimits in each variable. Let $H : \mathcal{W} \to \mathcal{V}$ be a strongly unital colax monoidal functor. Assume that $H$ preserves colimits. Then the induced functor of Construction 2.1.19*

$$\tilde{H} : S_{\otimes}\mathcal{W} \to S_{\otimes}\mathcal{V}$$

*preserves colimits.*

*Proof.* Let $\mathcal{J}$ be a small category and $D : \mathcal{J} \to S_{\otimes}\mathcal{W}$ a diagram. With notations as in Remark 2.1.22, we have a monoidal natural isomorphism

$$H_S \circ \operatorname{colim}\tilde{D} = H_S \circ \operatorname*{colim}_{j\in\mathcal{J}} (\iota_j)_! X^j \simeq \operatorname*{colim}_{j\in\mathcal{J}} (\iota_j)_! H_{S^j} X^j$$

because $H$ preserves colimits and $H_T f_! \simeq f_! H_S$ for every map of sets $f : S \to T$. It follows that $\tilde{H}$ preserves colimits. $\square$

### 2.1.4 Comparison with simplicial sets

Consider the colimit-preserving, strong monoidal functor

$$F : \mathrm{Set} \to \mathcal{V} : S \mapsto \coprod_{a\in S} I$$

(this is the functor $F_P$ of Notation 1.2.16 with $P = I$). Then its right-adjoint

$$U = \mathcal{V}(I, -) : \mathcal{V} \to \mathrm{Set}$$

is lax monoidal by Lemma 1.1.4. So $F \dashv U$ is a monoidal adjunction in the sense of Definition 1.1.5.

Combining Construction 2.1.19 and Proposition 2.1.15, we obtain a functor

$$\tilde{F} : \mathrm{SSet} \simeq S_{\times}\,\mathrm{Set} \to S_{\otimes}\mathcal{V}$$

**Proposition 2.1.25.** *The functor $\tilde{F} : \mathrm{SSet} \to S_{\otimes}\mathcal{V}$ has a right adjoint*

$$\tilde{U} : S_{\otimes}\mathcal{V} \to \mathrm{SSet}$$

*that is given by, for all templicial objects $X$ and $n \geq 0$,*

$$\tilde{U}(X)_n = S_{\otimes}\mathcal{V}(\tilde{F}(\Delta^n), X)$$

*Proof.* By Proposition 1.3.11, it suffices to note that $\tilde{F}$ preserves colimits, which follows by Proposition 2.1.24. $\square$

**Proposition 2.1.26.** *Let $K$ be a simplicial set and $(X, S)$ a templicial object in $\mathcal{V}$. Then a simplicial map $K \to \tilde{U}(X)$ is equivalent to the following data:*

- *a map of sets* $f : K_0 \to S$*,*
- *an element* $\alpha_\sigma \in U\left(X_n(f(a), f(b))\right)$ *for all* $n > 0$*,* $a, b \in S$ *and all non-degenerate* $\sigma \in K_n(a, b)$*,*

*which must satisfy:*

$$d_j(\alpha_\sigma) = \alpha_{d_j(\sigma)} \quad \text{and} \quad \mu_{j,n-j}(\alpha_\sigma) = \alpha_{d_{j+1}\ldots d_n(\sigma)} \otimes \alpha_{d_0\ldots d_0(\sigma)}$$

*for all* $0 < j < n$*.*

*Proof.* By adjunction, the map $K \to \tilde{U}(X)$ corresponds to a templicial morphism $(\alpha, f) : \tilde{F}(K) \to X$, where $f : K_0 \to S$ is a map of sets and $\alpha : f_!\tilde{F}(K) \to X$ a monoidal natural transformation. Now for all $n \geq 0$ and $a, b \in K_0$,

$$\tilde{F}(K)_n(a, b) = F(K_n(a, b)) \simeq \coprod_{\sigma \in K_n(a,b)} I,$$

and thus $\alpha$ is determined by a collection of morphisms $\alpha_\sigma : I \to X_n(f(a), f(b))$ for all $\sigma \in K_n(a, b)$. As $\alpha$ is natural with respect to the degeneracy maps in $\tilde{F}(K)$, it follows that $s_i(\alpha_\sigma) = \alpha_{s_i(\sigma)}$ for all $0 \leq i \leq n$. Thus we may restrict to non-degenerate simplices $\sigma \in K_n(a, b)$. The naturality of $\alpha$ with respect to the inner face maps in $\tilde{F}(K)$ and the monoidality of $\alpha$ now translate to the conditions in the statement. $\square$

**Corollary 2.1.27.** *Consider a templicial object* $(X, S)$ *in* $\mathcal{V}$ *and* $n \geq 0$*. An* $n$*-simplex of* $\tilde{U}(X)$ *is equivalent to a choice of vertices* $a_0, ..., a_n \in S$ *along with a collection of elements*

$$\left(\alpha_{i,j} \in U(X_{j-i}(a_i, a_j))\right)_{0 \leq i < j \leq n} \tag{2.4}$$

*such that for all* $0 \leq i < k < j \leq n$*, we have*

$$\mu_{k-i,j-k}(\alpha_{i,j}) = \alpha_{i,k} \otimes \alpha_{k,j} \tag{2.5}$$

*Proof.* Apply Proposition 2.1.26 to the case $K = \Delta^n$. The map $f : [n] \to S$ is equivalent to a choice of vertices $a_0, ..., a_n \in S$. Further, we can identify every non-degenerate $m$-simplex of $\Delta^n$ by its sequence of vertices $[i_0, ..., i_m]$ with $0 \leq i_0 < ... < i_m \leq n$. Note that each $[i_0, i_1, ..., i_m]$ can be obtained from the simplex $[i_0, i_0 + 1, ..., i_m]$ by iteratively applying face maps in $\Delta^n$. Thus the collection $\left(\alpha_{[i_0,...,i_m]}\right)_{0 \leq i_0 < ... < i_m \leq n}$ is completely determined by the elements $\alpha_{i,j} = \alpha_{[i,i+1,...,j]} \in U(X_{j-i}(a_i, a_j))$ with $0 \leq i < j \leq n$. $\square$

**Notation 2.1.28.** Let $(X, S)$ be a templicial object and $\alpha = (\alpha_{i,j})_{0 \leq i < j \leq n}$ an $n$-simplex of $\tilde{U}(X)$ with vertices $a_0, ..., a_n \in S$. We will sometimes write $\alpha$ more compactly as $(\alpha_{i,j})_{0 \leq i \leq j \leq n}$ where $\alpha_{i,i} = a_i$ for all $0 \leq i \leq n$.

*Remark* 2.1.29. Take a templicial object $(X, S)$. Let us describe the face and degeneracy maps of $\tilde{U}(X)$. Given an $n$-simplex $\alpha = (\alpha_{i,j})_{0 \leq i \leq j \leq n}$ of $\tilde{U}(X)$, we have

$$d_k(\alpha) = (\beta_{i,j})_{0 \leq i \leq j \leq n-1} \quad \text{with } \beta_{ij} = \begin{cases} \alpha_{i+1,j+1} & \text{if } k \leq i \\ d^X_{k-i}(\alpha_{i,j+1}) & \text{if } i < k \leq j \\ \alpha_{i,j} & \text{if } j < k \end{cases}$$

and

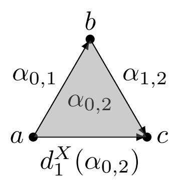
$$s_k(\alpha) = (\beta_{i,j})_{0\leq i\leq j\leq n+1} \quad \text{with } \beta_{ij} = \begin{cases} \alpha_{i-1,j-1} & \text{if } k < i \\ s^X_{k-i}(\alpha_{i,j-1}) & \text{if } i \leq k < j \\ \alpha_{i,j} & \text{if } j \leq k \end{cases}$$

for all $0 \leq k \leq n$.

**Example 2.1.30.** Given a templicial object $(X, S)$, let us discuss the $n$-simplices of $\tilde{U}(X)$ in low dimensions. By Corollary 2.1.27 we have bijections $\tilde{U}(X)_0 \simeq S$, and

$$\tilde{U}(X)_1 \simeq \coprod_{a,b\in S} U(X_1(a, b))$$

Further, a 2-simplex of $\tilde{U}(X)$ is a tuple $(a, b, c, \alpha_{0,1}, \alpha_{1,2}, \alpha_{0,2})$ with $a, b, c \in S$ and $\alpha_{0,1} \in U(X_1(a, b))$, $\alpha_{1,2} \in U(X_1(b, c))$ and $\alpha_{0,2} \in U(X_2(a, c))$ such that $\mu_{1,1}(\alpha_{0,2}) = \alpha_{0,1} \otimes \alpha_{1,2}$. The edges of this 2-simplex are given by $\alpha_{0,1}$, $\alpha_{1,2}$ and $d^X_1(\alpha_{0,2})$. Visually, we might represent this as




### 2.1.5 Skeleta

We now introduce the templicial analogue of the classical skeleton construction for simplicial sets. This requires introducing a truncated version of templicial objects. Some of the proofs are analogous to those in §2.1.2 and thus we will not always give them in full detail.

Throughout this subsection, we fix a positive integer $n \geq 0$.

**Notation 2.1.31.** We define $\mathbf{\Delta}^{\leq n}_f$ as the full subcategory of $\mathbf{\Delta}_f$ spanned by the objects $[0], ..., [n]$.

**Construction 2.1.32.** Let $S$ be a set. We define a category $\Phi^{\leq n}_{\mathcal{V}}(S)$ whose objects are functors

$$X : (\mathbf{\Delta}^{\leq n}_f)^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$$

with a morphism $\epsilon : X_0 \to I_S$ and for all pairs $k, l \geq 0$ with $k + l \leq n$, a quiver morphism

$$\mu_{k,l} : X_{k+l} \to X_k \otimes X_l$$

which satisfy the naturality, coassociativity and counitality conditions (2.1), (2.2) and (2.3) whenever they are well-defined. A morphism $X \to Y$ in $\Phi^{\leq n}_{\mathcal{V}}(S)$ is given by a natural transformation $\alpha : X \to Y$ such that for all $k, l \geq 0$ with $k + l \leq n$, we have

$$\mu^Y_{k,l}\alpha_{k+l} = (\alpha_k \otimes \alpha_l)\mu^X_{k,l} \quad \text{and} \quad \epsilon^Y\alpha_0 = \epsilon^X$$

Analogously to Proposition 2.1.18, we obtain a pseudofunctor

$$\Phi^{\leq n}_{\mathcal{V}} : \mathrm{Set} \to \underline{\mathrm{Cat}} : S \mapsto \Phi^{\leq n}_{\mathcal{V}}(S), f \mapsto f_! \circ -$$

Then consider the Grothendieck construction $\int \Phi^{\leq n}_{\mathcal{V}}$ of $\Phi^{\leq n}_{\mathcal{V}}$. Explicitly, $\int \Phi^{\leq n}_{\mathcal{V}}$ is the category whose objects are all pairs $(X, S)$ with $S$ a set and $X : (\mathbf{\Delta}^{\leq n}_{f})^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$ a functor with the extra structure described above. A morphism from $(X, S)$ to $(Y, T)$ is given by a pair $(\alpha, f)$ with $f : S \to T$ a map of sets and $\alpha : f_! X \to Y$ a morphism in $\Phi^{\leq n}_{\mathcal{V}}(T)$.

**Definition 2.1.33.** We define $S^{\leq n}_{\otimes}\mathcal{V}$ as the full subcategory of $\int \Phi^{\leq n}_{\mathcal{V}}$ spanned by all pairs $(X, S)$ such that the counit $\epsilon : X_0 \to I_S$ is an isomorphism. Its objects are called *$n$-truncated templicial objects*.

There is an obvious restriction functor

$$\tau_{\leq n} : S_{\otimes}\mathcal{V} \to S^{\leq n}_{\otimes}\mathcal{V} : (X, S) \mapsto (X|_{(\mathbf{\Delta}^{\leq n}_{f})^{op}}, S)$$

which we call the *nth truncation functor*.

**Proposition 2.1.34.** *The category $S^{\leq n}_{\otimes}\mathcal{V}$ is cocomplete and $\tau_{\leq n} : S_{\otimes}\mathcal{V} \to S^{\leq n}_{\otimes}\mathcal{V}$ preserves colimits.*

*Proof.* Let $\mathcal{J}$ be a small category and $D : \mathcal{J} \to S^{\leq n}_{\otimes}\mathcal{V} : j \mapsto (X^j, S^j)$ a diagram. It follows completely analogously to the proof of Corollary 2.1.23 that the colimit of $D$ is given by the pair

$$(\operatorname*{colim}_{j \in \mathcal{J}} (\iota_j)_! X^j, \operatorname*{colim}_{j \in \mathcal{J}} S^j)$$

where $\iota_j : S^j \to \mathrm{colim}_{j \in \mathcal{J}} S^j$ denotes the canonical map and the colimit $\mathrm{colim}_{j \in \mathcal{J}} (\iota_j)_! X^j$ is taken in $\Phi^{\leq n}_{\mathcal{V}}(\mathrm{colim}_{j \in \mathcal{J}} S^j)$. It is then clear that $\tau_{\leq n}$ preserves colimits. □

**Examples 2.1.35.** Let us describe the category $S^{\leq n}_{\otimes}\mathcal{V}$ for low values of $n$.

1. If $n = 0$, then $\mathbf{\Delta}^{\leq 0}_{f}$ is the discrete category with one object and thus $S^{\leq 0}_{\otimes}\mathcal{V}$ is equivalent to the category $\mathrm{Set}$ of sets.

2. If $n = 1$, then $\mathbf{\Delta}^{\leq 1}_{f}$ consists of a single non-identity morphism $[1] \to [0]$. It follows that $S^{\leq 1}_{\otimes}\mathcal{V}$ is equivalent to the category of *unital $\mathcal{V}$-enriched quivers* $\mathcal{V}\,\mathrm{Quiv}_u$. Its objects are triples $(Q, S, u)$ with $S$ a set, $Q \in \mathcal{V}\,\mathrm{Quiv}_S$ a quiver and $u : I_S \to Q$ a quiver morphism. A morphism $(Q, S, u) \to (P, T, v)$ is a pair $(\alpha, f)$ with $f : S \to T$ a map of sets and $\alpha : f_!(Q) \to P$ a quiver morphism such that the following diagram commutes:

$$\begin{tikzcd} f_!(I_S) \arrow[r] \arrow[d, "f_!(u)"'] & I_T \arrow[d, "v"] \\ f_!(Q) \arrow[r, "\alpha"'] & P \end{tikzcd}$$

**Example 2.1.36.** Let $\mathbf{\Delta}^{\leq n}$ denote the full subcategory of $\mathbf{\Delta}$ spanned by the objects $[0], ..., [n]$. Analogously to Proposition 2.1.15, we obtain an equivalence of categories:

$$S^{\leq n}_{\times}\,\mathrm{Set} \simeq \mathrm{Set}^{(\mathbf{\Delta}^{\leq n})^{op}}$$

Clearly the truncation functor $\tau_{\leq n} : S_\times \operatorname{Set} \to S_\times^{\leq n} \operatorname{Set}$ then corresponds to the classical truncation $\tau_{\leq n} : \operatorname{SSet} \to \operatorname{Set}^{(\mathbf{\Delta}^{\leq n})^{op}}$ under this equivalence.

**Construction 2.1.37.** Let $\iota_{\leq n} : \mathbf{\Delta}_f^{\leq n} \to \mathbf{\Delta}_f$ denote the inclusion functor and let $S$ be a set. Consider the restriction functor

$$- \circ \iota_{\leq n} : \operatorname{Fun}(\mathbf{\Delta}_f^{op}, \mathcal{V}\operatorname{Quiv}_S) \to \operatorname{Fun}((\mathbf{\Delta}_f^{\leq n})^{op}, \mathcal{V}\operatorname{Quiv}_S)$$

As $\mathcal{V}$ and thus $\mathcal{V}\operatorname{Quiv}_S$ is cocomplete, $- \circ \iota_{\leq n}$ has a left-adjoint given by the left Kan extension along the inclusion $\iota_{\leq n}$, which we denote by $\operatorname{sk}_n = \operatorname{Lan}_{\iota_{\leq n}}$. Explicitly, given a functor $X : (\mathbf{\Delta}_f^{\leq n})^{op} \to \mathcal{V}\operatorname{Quiv}_S$ and $k \geq 0$, we have

$$\operatorname{sk}_n(X)_k = \operatorname*{colim}_{\substack{[k]\twoheadrightarrow[p] \\ 0\leq p\leq n}} X_p$$

The colimit is taken over the opposite of the under category $(\mathbf{\Delta}_{surj}^{\leq n})_{[k]/}$, where $\mathbf{\Delta}_{surj}^{\leq n}$ denotes the full subcategory of $\mathbf{\Delta}_{surj}$ spanned by the objects $[0], ..., [n]$.

Further, for $f : [m] \to [k]$ in $\mathbf{\Delta}_f$, the quiver morphism $\operatorname{sk}_n(X)(f) : \operatorname{sk}_n(X)_k \to \operatorname{sk}_n(X)_m$ is given as follows. Let $\sigma : [k] \twoheadrightarrow [p]$ be a surjective morphism in $\mathbf{\Delta}_f$. By Remark 2.1.2, we can factor $\sigma f$ uniquely as a surjective morphism $\sigma' : [m] \twoheadrightarrow [q]$ followed by an injective morphism $f' : [q] \to [p]$ in $\mathbf{\Delta}_f$. In particular, $q \leq p \leq n$ and thus $f'$ belongs to $\mathbf{\Delta}_f^{\leq n}$. Then

$$\operatorname{sk}_n(X)(f)\iota_\sigma = \iota_{\sigma'} X(f')$$

Now suppose $(X, S)$ is an $n$-truncated templicial object. We can equip $\operatorname{sk}_n(X)$ with a strongly unital colax monoidal structure as follows. Note that $\operatorname{sk}_n(X)_0 \simeq X_0 \simeq I_S$. Take $k, l \geq 0$. For all surjective $\sigma : [k+l] \twoheadrightarrow [p]$ with $0 \leq p \leq n$, we can write $\sigma = \sigma_1 + \sigma_2$ for some unique $\sigma_1 : [k] \twoheadrightarrow [p_1]$ and $\sigma_2 : [l] \twoheadrightarrow [p_2]$ (Remark 2.1.4). Then consider the morphism

$$(\iota_{\sigma_1} \otimes \iota_{\sigma_2})\mu^X_{p_1,p_2} : X_p \to \operatorname{sk}_n(X)_k \otimes \operatorname{sk}_n(X)_l$$

These morphisms form a cocone since for any other surjective morphism $\sigma' : [k+l] \twoheadrightarrow [q]$ with $h : [q] \twoheadrightarrow [p]$ such that $h\sigma' = \sigma$, we can also write $\sigma' = \sigma'_1 + \sigma'_2$ and $h = h_1 + h_2$ so that $h_1\sigma'_1 = \sigma_1$ and $h_2\sigma'_2 = \sigma_2$. Thus

$$\begin{aligned}(\iota_{\sigma_1} \otimes \iota_{\sigma_2})\mu^X_{p_1,p_2} &= (\iota_{h_1\sigma'_1} \otimes \iota_{h_2\sigma'_2})\mu^X_{p_1,p_2} \\ &= (\iota_{\sigma'_1} \otimes \iota_{\sigma'_2})(X(h_1) \otimes X(h_2))\mu^X_{p_1,p_2} = (\iota_{\sigma'_1} \otimes \iota_{\sigma'_2})\mu^X_{q_1,q_2}X(h)\end{aligned}$$

We thus obtain a canonical morphism

$$\mu_{k,l} : \operatorname{sk}_n(X)_{k+l} \to \operatorname{sk}_n(X)_k \otimes \operatorname{sk}_n(X)_l$$

such that $\mu_{k,l}\iota_\sigma = (\iota_{\sigma_1} \otimes \iota_{\sigma_2})\mu^X_{p_1,p_2}$ for all surjective $\sigma : [k+l] \twoheadrightarrow [p]$ with $0 \leq p \leq n$.

It now follows easily from the definitions that $\mu_{k,l}$ is natural in $[k], [l] \in \mathbf{\Delta}_f$ and that it is coassociative and counital with respect to $\operatorname{sk}_n(X)_0 \simeq I_S$. Hence, $(\operatorname{sk}_n(X), S)$ is a templicial object.

**Examples 2.1.38.** Let us consider the functor $\operatorname{sk}_n$ for low values of $n$.

1. If $n = 0$, let $S$ be a set. Then $\mathrm{sk}_0(S)$ is the templicial object $(\underline{I_S}, S)$ where $\underline{I_S}$ is the constant functor $\mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S : [n] \mapsto I_S$. Equivalently, as $\tilde{F}$ preserves colimits, $\mathrm{sk}_0(S)$ is isomorphic to the coproduct $\coprod_{a\in S} \tilde{F}(\Delta^0)$ in $S_\otimes\mathcal{V}$.
2. If $n = 1$, let $Q \in \mathcal{V}\,\mathrm{Quiv}_S$ be a quiver with unit morphism $u : I_S \to Q$. Then $\mathrm{sk}_1(Q)$ is given as follows. For all $n \geq 0$,
$$\mathrm{sk}_1(Q)_n = \underbrace{Q \amalg_{I_S} Q \amalg_{I_S} \ldots \amalg_{I_S} Q}_{n \text{ terms}} \in \mathcal{V}\,\mathrm{Quiv}_S$$
That is, we have one copy of $Q$ for each morphism $h : [n] \to [1]$ in $\mathbf{\Delta}_f$. Let $\iota_h$ denote the coprojection $Q \to \mathrm{sk}_1(Q)_n$ corresponding to $h$. Then for any morphism $f : [m] \to [n]$ in $\mathbf{\Delta}_f$, $\mathrm{sk}_1(Q)(f) : \mathrm{sk}_1(Q)_n \to \mathrm{sk}_1(Q)_m$ is given by
$$\mathrm{sk}_1(Q)(f)\iota_h = \iota_{hf} \quad \text{for all } h : [n] \to [1] \text{ in } \mathbf{\Delta}_f$$
Finally, for all $k, l \geq 0$ the comultiplication morphism $\mu_{k,l} : X_{k+l} \to X_k \otimes X_l$ is given as follows. For any $h : [n] \to [1]$ in $\mathbf{\Delta}_f$, we have $h = h_1 + h_2$ for some unique $h_1 : [k] \to [h(k)]$ and $h_2 : [l] \to [1 - h(k)]$. Then
$$\mu_{k,l}\iota_h = \begin{cases} X(h_1) \otimes \iota_{h_2} & \text{if } h(k) = 0 \\ \iota_{h_1} \otimes X(h_2) & \text{if } h(k) = 1 \end{cases}$$

**Proposition 2.1.39.** *The assignment $X \mapsto \mathrm{sk}_n(X)$ of Construction 2.1.37 extends to a fully faithful functor $\mathrm{sk}_n : S_\otimes^{\leq n}\mathcal{V} \to S_\otimes\mathcal{V}$ which is left-adjoint to the nth truncation functor $\tau_{\leq n} : S_\otimes\mathcal{V} \to S_\otimes^{\leq n}\mathcal{V}$.*

*Proof.* Let $(X, S)$ be an $n$-truncated templicial object. Consider the inclusion functor $\iota_{\leq n} : \mathbf{\Delta}_f^{\leq n} \to \mathbf{\Delta}_f$. The unit of the adjunction $\mathrm{Lan}_{\iota_{\leq n}} \dashv - \circ \iota_{\leq n}$ supplies a natural transformation $\eta_X : X \to \tau_{\leq n}(\mathrm{sk}_n(X))$. For all $0 \leq k \leq n$, $\eta_{X_k}$ is just the canonical morphism $\iota_\sigma : X_k \to \mathrm{colim}_{[k]\twoheadrightarrow[p], 0\leq p\leq n} X_p$ where $\sigma = \mathrm{id}_{[k]}$. Moreover, $\eta_{X_k}$ is an isomorphism as $\mathrm{id}_{[k]}$ is the terminal object in the opposite category of $(\mathbf{\Delta}_{surj}^{\leq n})_{[k]/}$. It follows easily from the definitions that $\eta_X$ is in fact an isomorphism in $S_\otimes^{\leq n}\mathcal{V}$.

Now let $(Y, T)$ be a templicial morphism and $f : S \to T$ a map of sets. There is a bijection between natural transformations $\alpha : f_! X \to Y|_{(\mathbf{\Delta}_f^{\leq n})^{op}}$ and natural transformations $\alpha' : f_!(\mathrm{Lan}_{\iota_{\leq n}}(X)) \simeq \mathrm{Lan}_{\iota_{\leq n}}(f_!(X)) \to Y$ where $\alpha'|_{(\mathbf{\Delta}_f^{\leq n})^{op}} \circ f_!(\eta_X) = \alpha$ ($f_!$ preserves colimits by Lemma 1.1.18). It easily follows from the definitions that $\alpha : f_! X \to \tau_{\leq n}(Y)$ is a morphism in $\Phi_\mathcal{V}^{\leq n}(T)$ if and only if $\alpha' : f_!(\mathrm{sk}_n(X)) \to Y$ is a monoidal natural transformation. We conclude that $\eta_X$ is the unit of an adjunction $\mathrm{sk}_n \dashv \tau_{\leq n}$. Finally, as $\eta_X$ is an isomorphism, $\mathrm{sk}_n : S_\otimes^{\leq n}\mathcal{V} \to S_\otimes\mathcal{V}$ is fully faithful. $\square$

**Definition 2.1.40.** Given an $n$-truncated templicial object $X$, we call $\mathrm{sk}_n(X)$ the *nth skeleton* of $X$. We call a templicial object $X$ *$n$-skeletal* or *$n$-truncated* if the counit $\mathrm{sk}_n(\tau_{\leq n}X) \to X$ is an isomorphism.

Note that this terminology is compatible with Definition 2.1.33 since by Proposition 2.1.39, $\mathrm{sk}_n$ identifies $S_\otimes^{\leq n}\mathcal{V}$ with the full subcategory of $S_\otimes\mathcal{V}$ spanned by all $n$-truncated templicial objects.

From now on we will abuse notation and write the composite $\mathrm{sk}_n \circ \tau_{\leq n}$ as just $\mathrm{sk}_n$. So we have an endofunctor $\mathrm{sk}_n : S_\otimes\mathcal{V} \to S_\otimes\mathcal{V}$ whose essential image consists of the $n$-skeletal templicial objects.

*Remark* 2.1.41. Let $n \geq m \geq 0$. It is clear from Construction 2.1.37 that we have canonical natural isomorphisms of endofunctors on $S_\otimes\mathcal{V}$:

$$\mathrm{sk}_m \mathrm{sk}_n \simeq \mathrm{sk}_m \simeq \mathrm{sk}_n \mathrm{sk}_m$$

In particular, every $m$-truncated templicial object is also $n$-truncated.

Moreover, the canonical natural transformation $\mathrm{sk}_n \to \mathrm{id}_{S_\otimes\mathcal{V}}$ induces an infinite sequence of natural transformations

$$\mathrm{sk}_0 \to \mathrm{sk}_1 \to \cdots \to \mathrm{sk}_n \to \cdots \to X$$

**Proposition 2.1.42.** *The canonical templicial morphism*

$$\operatorname*{colim}_{n\geq 0} \mathrm{sk}_n(X) \to X$$

*is an isomorphism which is natural in all templicial objects $X$.*

*Proof.* For any $k \geq 0$, the induced quiver morphism

$$\operatorname*{colim}_{n\geq 0} \mathrm{sk}_n(X)_k \simeq \operatorname*{colim}_{\substack{[k]\twoheadrightarrow[p]\\ p\geq 0}} X_p \to X_k$$

is an isomorphism because $\mathrm{id}_{[k]} : [k] \to [k]$ is the terminal object in the opposite category of $(\mathbf{\Delta}_{surj})_{[k]/}$ with $p \geq 0$. $\square$

*Remark* 2.1.43. Let $K$ be a simplicial set and $n > 0$. There is a well-known result that the canonical map $\mathrm{sk}_{n-1}(K) \to \mathrm{sk}_n(K)$ fits in a pushout diagram

$$\begin{tikzcd} \coprod\limits_{\substack{\sigma\in K_n\\ \text{non deg.}}} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}(K) \arrow[d] \\ \coprod\limits_{\substack{\sigma\in K_n\\ \text{non deg.}}} \Delta^n \arrow[r] & \mathrm{sk}_n(K) \end{tikzcd}$$

Beware that this is no longer the case for templicial objects. The first obstacle is that there doesn't seem to be a good notion of non-degenerate simplices of a general templicial object $X$. This can be resolved by restricting to what we call *free* templicial objects (see §3.1.3). The most straightforward way to construct an analogous pushout in $S_\otimes\mathcal{V}$ would be to apply $\tilde{F}$ to the simplicial sets $\partial\Delta^n$ and $\Delta^n$. But Example 2.1.44 shows that even when $X$ is free, this diagram need not exist.

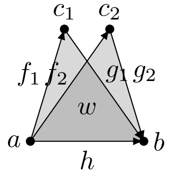
**Example 2.1.44.** Let $\mathcal{V} = \mathrm{Mod}(\mathbb{Z}) = \mathrm{Ab}$ be the monoidal category of abelian groups with the tensor product as monoidal product and $\mathbb{Z}$ as monoidal unit. We define a 2-truncated templicial abelian group $X$ as follows. Let $S = \{a, c_1, c_2, b\}$ be a set with four elements and consider for all $x, y \in S$ the following free abelian groups:

$$X_0(x,y) = \begin{cases} \mathbb{Z}x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}, \quad X_1(x,y) = \begin{cases} \mathbb{Z}f_i & \text{if } x = a, y = c_i \\ \mathbb{Z}g_i & \text{if } x = c_i, y = b \\ \mathbb{Z}h & \text{if } x = a, y = b \\ \mathbb{Z}s_0(x) & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and} \quad X_2(x,y) = \begin{cases} \mathbb{Z}s_0(f_i) \oplus s_1(f_i) & \text{if } x = a, y = c_i \\ \mathbb{Z}s_0(g_i) \oplus s_1(g_i) & \text{if } x = c_i, y = b \\ \mathbb{Z}w \oplus s_0(h) \oplus s_1(h) & \text{if } x = a, y = b \\ \mathbb{Z}s_0s_0(x) & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

The degeneracy maps are defined on the generators as shown, while the face map $d_1 : X_2 \to X_1$ and the comultiplication map $\mu_{1,1} : X_2 \to X_1 \otimes X_1$ are uniquely determined by the conditions

$$d_1(w) = h \quad \text{and} \quad \mu_{1,1}(w) = f_1 \otimes g_1 + f_2 \otimes g_2$$




The generator $w$ of $X_2(a,b)$ not in the image of the joint degeneracy map $(s_0, s_1) : X_1(a,b) \oplus X_1(a,b) \to X_2(a,b)$. So we would expect to have a commutative square

$$\begin{array}{ccc} \tilde{F}(\partial\Delta^2) & \longrightarrow & \mathrm{sk}_1(X) \\ \downarrow & & \downarrow \\ \tilde{F}(\Delta^2) & \longrightarrow & \mathrm{sk}_2(X) \end{array}$$

such that the bottom map sends the unique non-degenerate 2-simplex of $\Delta^2$ to $w$. However, such a map does not exist. Indeed, this would be equivalent to a 2-simplex $(\alpha_{i,j})_{0 \leq i \leq j \leq 2}$ of $\tilde{U}(X)$ with $\alpha_{0,2} = w$. But $\mu_{1,1}(w)$ is not a pure tensor while $\mu_{1,1}(\alpha_{0,2}) = \alpha_{0,1} \otimes \alpha_{1,2}$.

## 2.2 Quasi-categories and Frobenius structures

### 2.2.1 Necklaces

Necklaces were first introduced in [DS11b] by Dugger and Spivak. As necklaces will also play a crucial role in this thesis, we will devote this subsection to recalling them. We also give a combinatorial description of the category of necklaces (Proposition 2.2.5). Finally, we introduce some new terminology like *active* and *inert* necklace maps (Definition 2.2.7), and the *splitting* of a necklace over another. These will pop up from time to time throughout the thesis.

**Definition 2.2.1.** We denote by $\mathrm{SSet}_{*,*} = (\partial\Delta^1 \downarrow \mathrm{SSet})$ the category of *bipointed simplicial sets*. Its objects can be identified with tuples $(K, a, b)$ where $K$ is a simplicial set and $a, b \in K_0$ are called the *distinguished points* of $K$. We will also denote $K_{a,b} = (K, a, b)$. A morphism $K_{a,b} \to L_{c,d}$ in $\mathrm{SSet}_{*,*}$ is a simplicial map $f : K \to L$ such that $f(a) = c$ and $f(b) = d$.

Let $K_{a,b}$ and $L_{c,d}$ be bipointed simplicial sets. The *wedge sum* $K \vee L$ of $K$ and $L$ is constructed by glueing $K$ and $L$ at the distinguished points $b$ and $c$. More precisely, $K \vee L$ is the coequalizer

$$\Delta^0 \overset{b}{\underset{c}{\rightrightarrows}} K \amalg L \twoheadrightarrow K \vee L$$

We consider $K \vee L$ again as bipointed with distinguished points $(a, d)$.

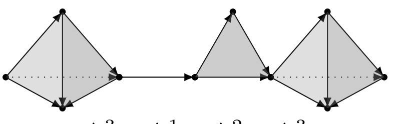
*Remark* 2.2.2. It is not difficult to verify that the wedge $\vee$ is a monoidal product on the category of bipointed simplicial sets $\mathrm{SSet}_{*,*}$ whose unit is given by $\Delta^0$.

**Definition 2.2.3.** For any $n \geq 0$, we consider the standard simplex $\Delta^n$ as bipointed with distinguished points $0$ and $n$. A *necklace* $T$ is an iterated wedge of standard simplices. That is,

$$T = \Delta^{n_1} \vee ... \vee \Delta^{n_k} \in \mathrm{SSet}_{*,*}$$

for some $k \geq 0$ and $n_1, ..., n_k > 0$ (if $k = 0$, then $T = \Delta^0$). We refer to the standard simplices $\Delta^{n_1}, ..., \Delta^{n_k}$ as the *beads* of $T$. The number of beads $k$ is called the *length* of $T$ and is denoted by $\ell(T)$. The distinguished points in every bead are called the *joints* of $T$.

We let $\mathcal{N}ec$ denote the full subcategory of $\mathrm{SSet}_{*,*}$ spanned by all necklaces. By construction, $(\mathcal{N}ec, \vee, \Delta^0)$ is again a monoidal category.



$\Delta^3 \vee \Delta^1 \vee \Delta^2 \vee \Delta^3$

*Remark* 2.2.4. Note that for any two necklaces $T$ and $U$, we have that $\ell(T \vee U) = \ell(T) + \ell(U)$.

**Proposition 2.2.5.** *The category of necklaces $\mathcal{N}ec$ is equivalent to the category defined as follows:*

*The objects are pairs $(T, p)$ with $p \geq 0$ and $\{0, p\} \subseteq T \subseteq [p]$. The morphisms $(T, p) \to (U, q)$ are morphisms $f : [p] \to [q]$ in $\mathbf{\Delta}_f$ such that $U \subseteq f(T)$, with compositions and identities defined as in $\mathbf{\Delta}_f$.*

*Moreover, under this equivalence, the wedge* $\vee$ *corresponds to*

$$(T, p) \vee (U, q) = (T \cup (p + U), p + q)$$

*where* $p + U = \{p + u \mid u \in U\}$. *Further,* $\ell(T) = |T| - 1$.

*Proof.* Clearly, a necklace $T = \Delta^{n_1} \vee ... \vee \Delta^{n_k}$ is determined up to isomorphism by the sequence $(n_1, ..., n_k)$. Setting $p = n_1 + ... + n_k$, this sequence is in turn determined by $I = \{0 < n_1 < n_1 + n_2 < ... < p\}$ as a subset of $[p]$. Note that under these identifications, $[p]$ and $I$ correspond to the sets of vertices and joints of $T$ respectively.

Further, let $T \to T'$ be a map of necklaces. As above, we may identify $T$ and $T'$ with pairs $(I, p)$ and $(J, q)$ respectively. Note that the map $T \to T'$ is completely determined on vertices and must preserve the order of these vertices. Hence, under these identifications, this map corresponds to an order morphism $[p] \to [q]$ which preserves the endpoints.

Suppose now that $J \not\subseteq f(I)$. We can write $I = \{0 = i_0 < i_1 < ... < i_k = p\}$ and $J = \{0 = j_0 < j_1 < ... < j_l = q\}$. Then we can choose $\beta \in \{1, ..., l\}$ and $\alpha \in \{1, ..., k\}$ such that $f(i_{\alpha-1}) = j_{\beta-1}$ but $f(i_\alpha) > j_\beta$. Now the unique edge of $T$ between the joints $i_{\alpha-1}$ and $i_\alpha$ must be sent to an edge of $T'$ between the vertices $j_{\beta-1}$ and $f(i_\alpha)$. But there is no such edge. Hence, we must have $J \subseteq f(I)$.

Conversely, consider a morphism $f : (I, p) \to (J, q)$. Let $T$ and $T'$ be the necklaces corresponding to $(I, p)$ and $(J, q)$ respectively. With the same notations for $I$ and $J$ as above, Remark 2.1.4 allows us to write

$$f = f_1 + ... + f_k$$

with $f_i : [i_\alpha - i_{\alpha-1}] \to [f(i_\alpha) - f(i_{\alpha-1})]$ in $\mathbf{\Delta}_f$ for each $\alpha \in \{1, ..., k\}$. Since $J \subseteq f(I)$, there is an $\alpha_\beta \in \{1, ..., k\}$ such that $j_\beta = f(i_{\alpha_\beta})$ for any $\beta \in \{1, ..., l\}$. Now there is a unique $\beta$ such that $\alpha_{\beta-1} \leq \alpha - 1 < \alpha \leq \alpha_\beta$ and thus we have $j_{\beta-1} \leq f(i_{\alpha-1}) \leq f(i_\alpha) \leq j_\beta$. So we can extend $f_i$ to an order morphism $[i_\alpha - i_{\alpha-1}] \to [j_\beta - j_{\beta-1}]$, which induces a simplicial map $\Delta^{i_\alpha - i_{\alpha-1}} \to \Delta^{j_\beta - j_{\beta-1}} \to T'$. These maps combine to give a map of necklaces $T \to T'$.

Clearly, this correspondence is functorial and preserves the wedge. □

**Notation 2.2.6.** Henceforth, we will identify $\mathcal{N}ec$ with the category described in Proposition 2.2.5. So we will also use the notation

$$T = \{0 = t_0 < t_1 < t_2 < ... < t_k = p\}$$

to refer to the necklace $\Delta^{t_1} \vee \Delta^{t_2 - t_1} \vee ... \vee \Delta^{p - t_{k-1}}$. We will often refer to a necklace $(T, p)$ just by its underlying set of joints $T$.

**Definition 2.2.7.** Let $f : (T, p) \to (U, q)$ be a map of necklaces. We say $f$ is *inert* if $p = q$ and $f = \mathrm{id}_{[p]}$. We say $f$ is *active* if $f(T) = U$.

*Remark* 2.2.8. A simplex $\Delta^n$, considered as a necklace with a single bead, is represented in $\mathcal{N}ec$ by the pair $(\{0 < n\}, n)$. On the other hand, the necklace $([n], n)$ represents the *spine* of $\Delta^n$, that is the union of the edges $0 \to 1 \to ... \to n$ in $\Delta^n$.

More generally for any necklace $(T, p)$ we can consider $([p], p)$, which is the spine passing through all the vertices of $T$. Note that there is a unique inert map $([p], p) \to (T, p)$ which represents the inclusion of the spine into $T$. Further let $k = \ell(T)$, then there is a unique order isomorphism $[k] \simeq T$. Thus there is a unique active map $([k], k) \to (T, p)$, which is the inclusion of the spine passing through all the joints of $T$.

**Notation 2.2.9.** Let $(T, p)$ be a necklace. We denote the poset

$$\mathcal{P}_T = \{U \subseteq [p] \mid T \subseteq U\}$$

ordered by inclusion. Equivalently, it is the poset of inert necklace maps $U \hookrightarrow T$.

If $T = \{0 < p\}$ is a simplex, we also write $\mathcal{P}_T = \mathcal{P}_p$.

*Remark* 2.2.10. It is easy to see that the assignment $T \mapsto \mathcal{P}_T$ extends to a strong monoidal functor

$$\mathcal{P} : \mathcal{N}ec \to \mathrm{Cat}$$

where for every necklace map $f : T \to U$, $\mathcal{P}(f)$ sends $V \in \mathcal{P}_T$ to $f(V) \in \mathcal{P}_U$. For necklaces $T$ and $U$, the lax monoidal structure is given by

$$\mathcal{P}_T \times \mathcal{P}_U \to \mathcal{P}_{T \vee U} : (V, W) \mapsto (V \vee W)$$

which is clearly an order isomorphism.

**Definition 2.2.11.** Let $T, U \in \mathcal{P}_p$ with $p \geq 0$. Suppose $U = \{0 = u_0 < ... < u_l = p\}$. Then there exist unique necklaces $T_1, ..., T_l$ such that

$$T \cup U = T_1 \vee ... \vee T_l$$

where for every $i \in \{1, ..., l\}$, we have $T_i \in \mathcal{P}_{u_i - u_{i-1}}$. More precisely,

$$T_i = \{0\} \cup \{t - u_{i-1} \mid t \in T, u_{i-1} \leq t \leq u_i\} \cup \{u_i - u_{i-1}\}$$

We call the sequence

$$(T_1, ..., T_l)$$

the *splitting* of $T$ over $U$.

**Proposition 2.2.12.** *Let $p \geq 0$ and $T, U \in \mathcal{P}_p$. The following statements are true.*

1. *For any $V \in \mathcal{P}_p$ with $T \cup U = V \cup U$, the splitting of $T$ over $U$ is equal to the splitting of $V$ over $U$.*

2. *If $T \subseteq U$ with $(T_1, ..., T_l)$ the splitting of $T$ over $U$ and $(U_1, ..., U_k)$ the splitting of $U$ over $T$, then $\ell(T_i) = 1$ for all $i \in \{1, ..., p\}$ and $U = U_1 \vee ... \vee U_k$.*

*Proof.*

1. This follows from the uniqueness of the expression $T \cup U = V \cup U = X_1 \vee ... \vee X_l$ with $X_i \in \mathcal{P}_{u_i - u_{i-1}}$ where $U = \{0 = u_0 < ... < u_l = p\}$.

2. As $U = T \cup U$, it is obvious that $U = U_1 \vee ... \vee U_k$. Further, $U = T \cup U = T_1 \vee ... \vee T_l$ and thus $l = \ell(T_1) + ... + \ell(T_l)$. But the length of every $T_i$ is at least $1$, so we must have $\ell(T_i) = 1$ for all $i \in \{1, ..., l\}$.

□

**Notation 2.2.13.** As in $\mathbf{\Delta}$, we distinguish some special maps in $\mathcal{N}ec$.

- For any $0 < j < n$, we write
$$\delta_j : \{0 < n-1\} \to \{0 < n\}$$
for the active necklace map whose underlying morphism is the inner coface map $\delta_j$ in $\mathbf{\Delta}_f$ of Definition 1.3.2.
- For any $0 \leq i \leq n$, we write
$$\sigma_i : \{0 < n+1\} \to \{0 < n\}$$
for the active necklace map whose underlying morphism is the codegeneracy map $\sigma_i$ in $\mathbf{\Delta}_f$ of Definition 1.3.2.
- For any $k, l > 0$, we write
$$\nu_{k,l} : \{0 < k < k+l\} \to \{0 < k+l\}$$
for the inert necklace map. More generally, for any necklace $(T, p)$, we write
$$\nu_T : T \to \{0 < p\}$$
for the inert necklace map.

*Remark* 2.2.14. The necklace maps of Notation 2.2.13 generate $\mathcal{N}ec$ as a monoidal category in the following sense. A necklace map $f : (T, p) \to (U, q)$ can be uniquely factorized as an active map followed by an inert map:
$$(T, p) \xrightarrow{f'} (f(T), q) \xrightarrow{\iota} (U, q)$$
Now suppose $T = \{0 = t_0 < t_1 < ... < t_{k-1} < t_k = p\} \subseteq [p]$. By Remark 2.1.4, the underlying morphism $f : [p] \to [q]$ in $\mathbf{\Delta}_f$ can be written as $f = f_1 + ... + f_k$ for some unique $f_i : [t_i - t_{i-1}] \to [f(t_i) - f(t_{i-1})]$ with $j \in \{1, ..., k\}$. By Remark 2.1.2, each $f_i$ has a unique representation
$$f_i = \delta_{j_1^i} ... \delta_{j_{r_i}^i} \sigma_{l_1^i} ... \sigma_{l_{s_i}^i}$$
It follows that
$$f' = f_1 \vee ... \vee f_k = (\delta_{j_1^1} ... \delta_{j_{r_1}^1} \sigma_{l_1^1} ... \sigma_{l_{s_1}^1}) \vee ... \vee (\delta_{j_1^k} ... \delta_{j_{r_k}^k} \sigma_{l_1^k} ... \sigma_{l_{s_k}^k})$$
Further, write $U = \{0 = u_0 < u_1 < ... < u_{l-1} < u_l = q\} \subseteq f(T)$. Let $(V_1, ..., V_l)$ be the splitting of $f(T)$ over $U$. It follows from Proposition 2.2.12 that
$$\iota = \nu_{V_1} \vee ... \vee \nu_{V_l}$$
Note that each $\nu_{V_i}$ can be further written as a composition of wedges of some maps $\nu_{r,s}$ with $r, s > 0$. But this decomposition will no longer be unique. For example, the unique inert map $\nu_{r,s,t} : \{0 < r < r+s < r+s+t\} \to \{0 < r+s+t\}$ has the representations
$$\nu_{r,s,t} = \nu_{r,s+t}(\mathrm{id}_{\{0<r\}} \vee \nu_{s,t}) = \nu_{r+s,t}(\nu_{r,s} \vee \mathrm{id}_{\{0<t\}})$$

### 2.2.2 Quasi-categories in a monoidal category

We now describe a generalization of quasi-categories in the context of templicial objects, which we call *quasi-categories in $\mathcal{V}$*. They are similarly defined by means of a lifting property along horn inclusions (Definition 2.2.26).

As a naive attempt, one might want to consider templicial objects $X$ with the right lifting property with respect to all inner horn inclusions $\tilde{F}(\Lambda^n_j) \to \tilde{F}(\Delta^n)$ in $S_\otimes\mathcal{V}$. However, in Example 2.1.44 we have seen that the templicial morphisms $\tilde{F}(\Delta^n) \to X$ are rather bad behaved. More precisely, not every $n$-simplex $x \in U(X_n(a,b))$ is represented by such a morphism $\tilde{F}(\Delta^n) \to X$, unlike the classical situation of Proposition 1.3.10. To resolve this issue, we can pass to the category $\mathcal{V}^{\mathcal{N}ec^{op}}$ of functors $\mathcal{N}ec^{op} \to \mathcal{V}$. Let us start by explaining how to obtain a functor $X_\bullet(a,b) : \mathcal{N}ec^{op} \to \mathcal{V}$ from a given templicial object $X$ with vertices $a$ and $b$.

**Notation 2.2.15.** Let $(X,S)$ be a templicial object with comultiplication $\mu$. For any necklace $T = \{0 = t_0 < t_1 < ... < t_{k-1} < t_k = p\}$, we write

$$X_T = X_{t_1} \otimes_S X_{t_2-t_1} \otimes_S ... \otimes_S X_{p-t_{k-1}} \in \mathcal{V}\,\mathrm{Quiv}_S$$

and

$$\mu_T = \mu_{t_1,t_2-t_1,...,p-t_{k-1}} : X_p \to X_T$$

**Construction 2.2.16.** Let $(X,S)$ be a templicial object of $\mathcal{V}$ with vertices $a,b \in S$. We can extend the assignment $T \mapsto X_T$ to a strong monoidal functor

$$X_\bullet : \mathcal{N}ec^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$$

as follows. In view of Remark 2.2.14, it suffices to define $X_\bullet$ on inert and active necklace maps. Let $f : (T,p) \to (U,q)$ be a map of necklaces.

- If $f$ is inert, then $p = q$ and $U \subseteq T$. Let $(T_1,...,T_l)$ be the splitting of $T$ over $U$ so that $T = T_1 \vee ... \vee T_l$ by Proposition 2.2.12. Then set

$$X(f) : X_U \xrightarrow{\mu_{T_1}\otimes...\otimes\mu_{T_l}} X_T$$

- If $f$ is active, write the necklace $T$ as $\{0 = t_0 < t_1 < ... < t_k = p\}$. Then there exist unique $f_i : [t_i - t_{i-1}] \to [f(t_i) - f(t_{i-1})]$ in $\mathbf{\Delta}_f$ such that $f = f_1 + ... + f_k$. Now set

$$X(f) : X_U \simeq X_{f(t_1)} \otimes ... \otimes X_{q-f(t_{k-1})} \xrightarrow{X(f_1)\otimes...\otimes X(f_k)} X_T$$

  where the isomorphism is induced by the strong unitality of $X$ and the fact that $U = f(T)$.

It follows from the coassociativity of $\mu$ that $X_\bullet$ is functorial on inert morphisms, and from the functoriality of $X$ that $X_\bullet$ is functorial on active morphisms. Then it follows from the naturality of $\mu$ that $X_\bullet$ is functorial on all morphisms.

If we fix vertices $a,b \in S$, then we obtain a functor

$$X_\bullet(a,b) : \mathcal{N}ec^{op} \to \mathcal{V} : T \mapsto X_T(a,b)$$

**Example 2.2.17.** Assume $\mathcal{V} = \mathrm{Set}$. Let $K$ be a simplicial set with vertices $a$ and $b$ and $T = \{0 = t_0 < t_1 < ... < t_k = p\}$ a necklace. It follows from Proposition 1.3.10 that

$$K_T(a,b) = \coprod_{a_1,\ldots,a_{k-1}\in K_0} K_{t_1}(a,a_1)\times \ldots \times K_{p-t_{k-1}}(a_{k-1},b)$$

is in bijection with the set of maps $T \to K_{a,b}$ in $\mathrm{SSet}_{*,*}$. Clearly, this bijection is natural in $T$ so that we have an isomorphism of functors $\mathcal{N}ec^{op} \to \mathrm{Set}$:

$$K_\bullet(a,b) \simeq \mathrm{SSet}_{*,*}(-,K_{a,b})$$

Recall the inner horns and boundaries in SSet (Definition 1.3.8). Let us investigate how they behave under the construction $K_{a,b} \mapsto K_\bullet(a,b)$. It turns out the resulting objects in $\mathrm{Set}^{\mathcal{N}ec^{op}}$ can be described very similarly.

**Proposition 2.2.18.** *For any $n > 0$,*

$$\partial\Delta^n_\bullet(0,n) = \bigcup_{i=1}^{n-1} \delta_i(\Delta^{n-1})_\bullet(0,n) \cup \bigcup_{k=1}^{n-1} (\Delta^k \vee \Delta^{n-k})_\bullet(0,n)$$

*and for every $0 < j < n$,*

$$(\Lambda^n_j)_\bullet(0,n) = \bigcup_{\substack{i=1\\ i\neq j}}^{n-1} \delta_i(\Delta^{n-1})_\bullet(0,n) \cup \bigcup_{k=1}^{n-1} (\Delta^k \vee \Delta^{n-k})_\bullet(0,n)$$

*as subfunctors of $\Delta^n_\bullet(0,n)$ in $\mathrm{Set}^{\mathcal{N}ec^{op}}$.*

*Proof.* We prove the statement for $\Lambda^n_j$. The case for $\partial\Delta^n$ is proven similarly. Let $0 < j < n$. For all $0 < k, i < n$ with $i \neq j$, we have inclusions $\Delta^k \vee \Delta^{n-k} \subseteq \Lambda^n_j$ and $\delta_i(\Delta^{n-1}) \subseteq \partial\Delta^n$ in SSet. It follows that

$$\bigcup_{\substack{i=1\\ i\neq j}}^{n-1} \delta_i(\Delta^{n-1})_\bullet(0,n) \cup \bigcup_{k=1}^{n-1} (\Delta^k \vee \Delta^{n-k})_\bullet(0,n) \subseteq (\Lambda^n_j)_\bullet(0,n)$$

Conversely, let $f : T \to (\Lambda^n_j)_{0,n}$ be a map in $\mathrm{SSet}_{*,*}$ with $(T,p)$ a necklace. Suppose first that $f$ is surjective on vertices. As the unique non-degenerate $n$-simplex of $\Delta^n$ is not contained in $\Lambda^n_j$, there must be some $k \in T$ such that $0 < f(k) < n$. Therefore, $f$ factors through $\Delta^l \vee \Delta^{n-l}$ with $l = f(k)$. Now suppose that $f$ is not surjective on vertices. Then $f$ must factor through $\delta_i(\Delta^{n-1})$ for some $i \in [n] \setminus \{j\}$. As a map in $\mathrm{SSet}_{*,*}$, $f$ always reaches the vertices $0$ and $n$ of $\Delta^n$ and thus $0 < i < n$. $\square$

**Example 2.2.19.** The outer horns aren't as well-behaved in $\mathrm{Set}^{\mathcal{N}ec^{op}}$ as the inner horns. For example, $\Lambda^2_0$ is the pushout $\Delta^1 \amalg_{\{0\}} \Delta^1$ in SSet, but $(\Lambda^2_0)_\bullet(0,2)$ is isomorphic to just $\Delta^1_\bullet(0,1)$ as all maps $T \to (\Lambda^2_0)_{0,2}$ in $\mathrm{SSet}_{*,*}$ must factor through the edge $0 \to 2$ of $\Lambda^2_0$.

**Corollary 2.2.20.** *Let $(T,p)$ be a necklace. For all $n > 0$ we have a bijection*

$$\partial\Delta^n_T(0,n) \simeq \{f : T \to \Delta^n \text{ in } \mathcal{N}ec \mid f([p]) \neq [n] \text{ or } \{0 < n\} \subsetneq f(T)\}$$

*and for all $0 < j < n$ we have a bijection*

$$(\Lambda^n_j)_T(0,n) \simeq \{f : T \to \Delta^n \text{ in } \mathcal{N}ec \mid f([p]) \not\supseteq [n]\setminus\{j\} \text{ or } \{0 < n\} \subsetneq f(T)\}$$

*Proof.* This follows from Proposition 2.2.18 by observing that a map $f : T \to \Delta^n$ in $\mathcal{N}ec$ factors through $\delta_i(\Delta^{n-1})$ with $0 < i < n$ if and only if $f([p]) \subseteq [n] \setminus \{i\}$ and $f$ factors through $\Delta^k \vee \Delta^{n-k}$ with $0 < k < n$ if and only if $k \in f(T)$. □

**Proposition 2.2.21.** *Let $K$ be a simplicial set with $a, b \in K_0$. Then there is a canonical isomorphism*

$$\tilde{F}(K)_\bullet(a,b) \simeq F(K_\bullet(a,b))$$

*where $F : \mathrm{Set}^{\mathcal{N}ec^{op}} \to \mathcal{V}^{\mathcal{N}ec^{op}}$ is the functor given by post-composition with $F : \mathrm{Set} \to \mathcal{V}$.*

*Proof.* This follows immediately from the definitions since $F$ is strong monoidal and preserves coproducts. □

**Corollary 2.2.22.** *Let $(X, S)$ be a templicial object with $a, b \in S$.*

1. *Let $T$ be a necklace. There is a bijective correspondence between morphisms $\tilde{F}(T)_\bullet(0,n) \to X_\bullet(a,b)$ in $\mathcal{V}^{\mathcal{N}ec^{op}}$ and elements $\sigma \in U(X_T(a,b))$.*

2. *Let $n > 0$ be an integer. There is a bijective correspondence between morphisms $\tilde{F}(\partial\Delta^n)_\bullet(0,n) \to X_\bullet(a,b)$ in $\mathcal{V}^{\mathcal{N}ec^{op}}$ and elements*

$$x_k \in U((X_k \otimes_S X_{n-k})(a,b)) \quad \text{and} \quad y_i \in U(X_{n-1}(a,b))$$

   *for all $0 < k, i < n$, which satisfy:*

   - *for all $0 < i < i' < n$,*
   $$d_{i'-1}(y_i) = d_i(y_{i'}),$$
   - *for all $0 < k < l < n$,*
   $$(\mathrm{id}_{X_k} \otimes \mu_{l-k,n-l})(x_k) = (\mu_{k,l-k} \otimes \mathrm{id}_{X_{n-l}})(x_l)$$
   - *for all $0 < k < n-1$ and $0 < i < n$,*
   $$\mu_{k,n-k-1}(y_i) = \begin{cases} (d_i \otimes \mathrm{id}_{X_{n-k-1}})(x_{k+1}) & \text{if } i \leq k \\ (\mathrm{id}_{X_k} \otimes d_{i-k})(x_k) & \text{if } i > k \end{cases}$$

3. *Let $0 < j < n$ be integers. There is a bijective correspondence between morphisms $\tilde{F}(\Lambda^n_j)_\bullet(0,n) \to X_\bullet(a,b)$ in $\mathcal{V}^{\mathcal{N}ec^{op}}$ and elements*

$$x_k \in U((X_k \otimes_S X_{n-k})(a,b)) \quad \text{and} \quad y_i \in U(X_{n-1}(a,b))$$

   *for all $0 < k, i < n$ with $i \neq j$, which satisfy:*

   - *for all $0 < i < i' < n$ with $i \neq j \neq i'$,*
   $$d_{i'-1}(y_i) = d_i(y_{i'}),$$
   - *for all $0 < k < l < n$,*
   $$(\mathrm{id}_{X_k} \otimes \mu_{l-k,n-l})(x_k) = (\mu_{k,l-k} \otimes \mathrm{id}_{X_{n-l}})(x_l)$$

- *for all* $0 < k < n-1$ *and* $0 < i < n$ *with* $i \neq j$,

$$\mu_{k,n-k-1}(y_i) = \begin{cases} (d_i \otimes \mathrm{id}_{X_{n-k-1}})(x_{k+1}) & \text{if } i \leq k \\ (\mathrm{id}_{X_k} \otimes d_{i-k})(x_k) & \text{if } i > k \end{cases}$$

*Proof.* By Proposition 2.2.21, a morphism $\tilde{F}(T)_\bullet(0,n) \to X_\bullet(a,b)$ corresponds to a morphism $T_\bullet(0,n) \to U(X_\bullet(a,b))$ in $\mathrm{Set}^{\mathcal{N}ec^{op}}$, which corresponds to an element $\sigma \in U(X_T(a,b))$ by applying the Yoneda lemma to the necklace $T$. This shows 1. Statements 2 and 3 follow from Construction 2.2.16 and Proposition 2.2.18. □

**Notation 2.2.23.** We denote

$$\mathrm{Horn} = \left\{ \tilde{F}(\Lambda^n_j)_\bullet(0,n) \to \tilde{F}(\Delta^n)_\bullet(0,n) \text{ in } \mathcal{V}^{\mathcal{N}ec^{op}} \;\middle|\; 0 < j < n \right\}$$
$$\mathrm{Cell} = \left\{ \tilde{F}(\partial\Delta^n)_\bullet(0,n) \to \tilde{F}(\Delta^n)_\bullet(0,n) \text{ in } \mathcal{V}^{\mathcal{N}ec^{op}} \;\middle|\; n > 0 \right\}$$

where the morphisms are induced by the inclusions $\Lambda^n_j \subseteq \Delta^n$ and $\partial\Delta^n \subseteq \Delta^n$ respectively.

*Remark* 2.2.24. Assume the monoidal unit $I$ of $\mathcal{V}$ is small in the sense of Definition 1.2.9. Then the forgetful functor $U = \mathcal{V}(I,-) : \mathcal{V} \to \mathrm{Set}$ preserves $\lambda$-sequences with $\lambda > 0$ a $\kappa$-directed ordinal for sufficiently large regular cardinals $\kappa$. Since every object of $\mathrm{Set}^{\mathcal{N}ec^{op}}$ is small by Example 1.2.10, it follows from Proposition 2.2.21 and the adjunction $F \dashv U$ that $\tilde{F}(K)_\bullet(a,b)$ is small in $\mathcal{V}^{\mathcal{N}ec^{op}}$ for every simplicial set $K$ with $a, b \in K_0$.

Consequently, by the Small object argument (Proposition 1.2.11), we obtain weak factorization systems $(\overline{\mathrm{Horn}}, \mathrm{Horn}^{\boxslash})$ and $(\overline{\mathrm{Cell}}, \mathrm{Cell}^{\boxslash})$ for $\mathcal{V}^{\mathcal{N}ec^{op}}$.

**Lemma 2.2.25.** *We have an inclusion of weakly saturated classes in* $\mathcal{V}^{\mathcal{N}ec^{op}}$*:*

$$\overline{\mathrm{Horn}} \subseteq \overline{\mathrm{Cell}}$$

*Proof.* It suffices to show that the morphism $\tilde{F}(\Lambda^n_j)_\bullet(0,n) \to \tilde{F}(\Delta^n)_\bullet(0,n)$ belongs to $\overline{\mathrm{Cell}}$ for all $0 < j < n$. From Proposition 2.2.18 we have that

$$(\partial\Delta^n)_\bullet(0,n) = (\Lambda^n_j)_\bullet(0,n) \cup (\delta_j(\Delta^{n-1}))_\bullet(0,n)$$

in $\mathrm{Set}^{\mathcal{N}ec^{op}}$. Further, we have that $\delta_j(\partial\Delta^{n-1})_\bullet(0,n) = (\Lambda^n_j)_\bullet(0,n) \cap (\delta_j(\Delta^{n-1}))_\bullet(0,n)$. Thus by Proposition 2.2.21 and the fact that $F$ preserves colimits, we obtain a pushout in $\mathcal{V}^{nec^{op}}$:

$$\begin{array}{ccc} \tilde{F}(\partial\Delta^{n-1})_\bullet(0,n) & \xrightarrow{\tilde{F}(\delta_j)} & \tilde{F}(\Lambda^n_j)_\bullet(0,n) \\ \downarrow & & \downarrow \\ \tilde{F}(\Delta^{n-1})_\bullet(0,n) & \xrightarrow[\tilde{F}(\delta_j)]{} & \tilde{F}(\partial\Delta^n)_\bullet(0,n) \end{array}$$

It now suffices to note that the morphism $\tilde{F}(\Lambda^n_j)_\bullet(0,n) \to \tilde{F}(\Delta^n)_\bullet(0,n)$ is the composition of the right vertical morphism with $\tilde{F}(\partial\Delta^n)_\bullet(0,n) \to \tilde{F}(\Delta^n)_\bullet(0,n)$. □

**Definition 2.2.26.** Let $Y : \mathcal{N}ec^{op} \to \mathcal{V}$ be a functor. We say $Y$ *lifts inner horns* if it has the right lifting property with respect to Horn in $\mathcal{V}^{\mathcal{N}ec^{op}}$. That is, for all $0 < j < n$ any lifting problem

$$\begin{tikzcd} \tilde{F}(\Lambda^n_j)_\bullet(0,n) \arrow[r] \arrow[d] & Y \\ \tilde{F}(\Delta^n)_\bullet(0,n) \arrow[ur, dashed] & \end{tikzcd}$$

has a solution in $\mathcal{V}^{\mathcal{N}ec^{op}}$. We say $Y$ *lifts inner horns uniquely* if every such lifting problem has a unique solution in $\mathcal{V}^{nec^{op}}$.

A templicial object $(X, S)$ in $\mathcal{V}$ is called a *quasi-category in* $\mathcal{V}$ if the functor $X_\bullet(a, b)$ lifts inner horns for all $a, b \in S$. In this case, we will refer to the elements of $S$ as the *objects* of $X$ and to elements of $U(X_1(a, b))$ as the *morphisms* $a \to b$ in $X$.

*Remark* 2.2.27. Let $Y : \mathcal{N}ec^{op} \to \mathcal{V}$ be a functor. Note that by Proposition 2.2.21 and the adjunction $F \dashv U$, $Y$ lifts inner horns in $\mathcal{V}^{\mathcal{N}ec^{op}}$ if and only if the composite $UY : \mathcal{N}ec^{op} \to \mathrm{Set}$ lifts inner horns in $\mathrm{Set}^{\mathcal{N}ec^{op}}$.

As for ordinary quasi-categories, there is an elementwise characterization of quasi-categories in $\mathcal{V}$, although it is bit more cumbersome to describe.

**Proposition 2.2.28.** *Let $(X, S)$ be a templicial object. The following statements are equivalent.*

*(1) $X$ is a quasi-category in $\mathcal{V}$.*

*(2) Let $a, b \in S$ and $0 < j < n$. For all collections of elements $(x_k)_{k=1}^{n-1}$, $(y_i)_{i=1,i\neq j}^{n-1}$ satisfying the conditions of Corollary 2.2.22.3, there exists an element $z \in U(X_n(a, b))$ such that*

$$\mu_{k,n-k}(z) = x_k \quad \textit{and} \quad d_i(z) = y_i$$

*for all $0 < k, i < n$ with $i \neq j$.*

*Proof.* This immediately follows from Corollary 2.2.22. □

*Remark* 2.2.29. Note the similarities with the classical elementwise characterization (see Proposition 1.3.14). The elements $y_i$ with $0 < i < n$, $i \neq j$ represent all inner faces of the horn $\Lambda^n_j$. They still have to satisfy the same conditions as before. However, the two outer faces of the horn are replaced by the elements $x_k$ with $0 < k < n$. The two new conditions of Corollary 2.2.22.3 merely express that these outer faces are glued to each other and to the inner faces in the appropriate way.

Indeed, in case $\mathcal{V} = \mathrm{Set}$ we recover the classical notion of a quasi-category.

**Proposition 2.2.30.** *A simplicial set is a quasi-category if and only if it is a quasi-category in* Set *(in the sense of Definition 2.2.26) .*

*Proof.* Let $X$ be a simplicial set, considered as a templicial set with $X_0$ its set of vertices. Then the assignment $(x_k)_{k=1}^{n-1} \mapsto (x^1_{n-1}, x^2_1)$ defines a bijection between the set of all collections of elements

$$\left(x_k = (x^1_k, x^2_k) \in X_k \times X_{n-k}\right)_{k=1}^{n-1}$$

satisfying $(x_k^1, \mu_{l-k,n-l}(x_k^2)) = (\mu_{k,l-k}(x_k^1), x_l^2)$ for all $0 < k < l < n$, and the set of all pairs $(y_n, y_0) \in X_{n-1} \times X_{n-1}$ satisfying $d_{n-1}(y_0) = d_0(y_n)$. It follows that condition (2) of Proposition 2.2.28 is equivalent to

(2′) Let $0 < j < n$. Consider elements $y_i \in X_{n-1}$ for all $0 \leq i \leq n$ with $i \neq j$, which satisfy for all $0 \leq i < i' \leq n$ with $i \neq j \neq i'$:

$$d_{i'-1}(y_i) = d_i(y_{i'})$$

Then there is an element $z \in X_n$ such that $d_i(z) = y_i$ for all $0 \leq i \leq n$ with $i \neq j$.

But this precisely expresses that $X$ is a quasi-category by Proposition 1.3.14. □

**Proposition 2.2.31.** *Let $X$ be a quasi-category in $\mathcal{V}$. Then $\tilde{U}(X)$ is a quasi-category.*

*Proof.* Suppose $(X, S)$ is a quasi-category in $\mathcal{V}$. Consider a simplicial map $\alpha : \Lambda_j^n \to \tilde{U}(X)$ with $0 < j < n$. It follows from Proposition 2.1.26 this is equivalent to a choice of vertices $a_0, ..., a_n \in S$ along with elements

$$\alpha_{k,l} \in U(X_{l-k}(a_k, a_l)) \quad \text{and} \quad \beta_i \in U(X_{n-1}(a_0, a_n))$$

for all $0 \leq k < l \leq n$ with $(k, l) \neq (0, n)$ and $0 < i < n$ with $i \neq j$, which satisfy

- for all $0 < i < i' < n$ with $i \neq j \neq i'$,

$$d_{i'-1}(\beta_i) = d_i(\beta_{i'})$$

- for all $0 \leq k < i < l \leq n$ with $(k, l) \neq (0, n)$,

$$\mu_{i-k,l-i}(\alpha_{k,l}) = \alpha_{k,i} \otimes \alpha_{i,l}$$

- for all $0 < k < n - 1$ and $0 < i < n$ with $i \neq j$,

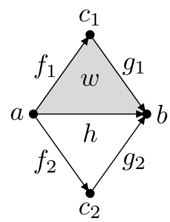
$$\mu_{k,n-k-1}(\beta_i) = \begin{cases} d_i(\alpha_{0,k}) \otimes \alpha_{k+1,n} & \text{if } i \leq k \\ \alpha_{0,k} \otimes d_{i-k}(\alpha_{k,n}) & \text{if } i > k \end{cases}$$

Now set $x_k = \alpha_{0,k} \otimes \alpha_{k,n}$ and $y_i = \beta_i$ for all $0 < k, i < n$ with $i \neq j$. Then by Proposition 2.2.28, there exists an element $\alpha_{0,n} \in U(X_n(a_0, a_n))$ such that $\mu_{k,n-k}(\alpha_{0,n}) = \alpha_{0,k} \otimes \alpha_{k,n}$ and $d_i(\alpha_{0,n}) = \beta_i$. Now the vertices $a_0, ..., a_n$ and the collection $(\alpha_{k,l})_{0 \leq k < l \leq n}$ define a map $\Delta^n \to \tilde{U}(X)$ which extends $\alpha$ by Corollary 2.1.27. □

The converse to Proposition 2.2.31 does not hold.

**Example 2.2.32.** Consider the over category $\mathcal{V} = \mathrm{Ab}/\mathbb{Z}$ of abelian groups $A$ with a $\mathbb{Z}$-linear map $p : A \to \mathbb{Z}$. Then $\mathcal{V}$ is bicomplete and symmetric monoidal closed with monoidal unit given by $\mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$. The forgetful functor $U : \mathcal{V} \to \mathrm{Set}$ associates to every map $p : A \to \mathbb{Z}$ the set $\{a \in A \mid p(a) = 1\}$.

Now consider the simplicial set $\Delta^2 \amalg_{\{0,2\}} \Lambda_1^2$:




Set $X = \tilde{F}(\Delta^2 \amalg_{\{0,2\}} \Lambda^2_1) \in S_\otimes \mathrm{Ab}$. We can promote $X$ to a templicial object in $\mathcal{V}$ by equipping it with $\mathbb{Z}$-linear maps $p : X_n(x,y) \to \mathbb{Z}$ defined as follows:

$$p(f_1) = p(f_2) = p(g_2) = p(h) = 1, \quad p(g_2) = 2 \quad \text{and} \quad p(w) = 1$$

Then for example $U(X_1(a,c_2)) = \{f_2\}$ but $U(X_1(c_2,b)) = \emptyset$. Consider $\tilde{U} : S_\otimes\mathcal{V} \to \mathrm{SSet}$ as induced by the forgetful functor $U$ above (not by $\mathrm{Ab} \to \mathrm{Set}$). Then it follows that $\tilde{U}(X) \simeq \Delta^2 \amalg_{\{0\}} \Delta^1$, which is clearly a quasi-category.

However, $X$ is not a quasi-category in $\mathcal{V}$. To see this, consider the element

$$\alpha = f_2 \otimes g_2 - f_1 \otimes g_1 \in U((X_1 \otimes X_1)(a,b))$$

(note that indeed, $(p \otimes p)(\alpha) = p(f_2)p(g_2) - p(f_1)p(g_1) = 1$). But there exists no element $\xi \in U(X_2(a,b))$ such that $\mu_{1,1}(\xi) = \alpha$.

### 2.2.3 Frobenius structures

We introduce *Frobenius structures* on a templicial object, which are based on the Frobenius monoidal functors of Day and Pastro [DP08]. These will mostly come into play in Section 4.2 when we restrict to $\mathcal{V} = \mathrm{Mod}(k)$ for some unital commutative ring $k$. But even for general $\mathcal{V}$, Frobenius structures turn up naturally. Many examples of templicial objects that we will encounter, like the templicial variants of the nerve (§2.3.1), homotopy coherent nerve (§4.1.2) and dg-nerve (§4.2.3) all carry canonical Frobenius structures.

First we introduce the more general notion of a *non-associative Frobenius structure* on an arbitrary colax monoidal functor. Then we discuss how this applies to templicial objects, and how a naF-structure interacts with the comultiplication morphisms using splittings of necklaces (see Proposition 2.2.40).

**Definition 2.2.33.** Let $H : \mathcal{U} \to \mathcal{V}$ be a functor between monoidal categories with a colax monoidal structure $(\mu, \epsilon)$. A *nonassociative Frobenius (naF) structure* on $H$ is a pair $(Z, \eta)$ with $\eta : I \to H(I)$ a morphism in $\mathcal{V}$, called the *unit*, and

$$Z : H(-) \otimes H(-) \to H(- \otimes -)$$

a natural transformation, called the *multiplication*, such that the following diagrams commute for all $A, B, C \in \mathcal{U}$:

$$\begin{tikzcd} H(A \otimes B) \otimes H(C) \arrow[r, "\mu_{A,B} \otimes \mathrm{id}"] \arrow[d, "Z_{A\otimes B, C}"'] & H(A) \otimes H(B) \otimes H(C) \arrow[d, "\mathrm{id} \otimes Z_{B,C}"] \\ H(A \otimes B \otimes C) \arrow[r, "\mu_{A, B\otimes C}"'] & H(A) \otimes H(B \otimes C) \end{tikzcd} \tag{2.6}$$

$$\begin{tikzcd} H(A)\otimes H(B\otimes C) \arrow[r, "\mathrm{id}\otimes\mu_{B,C}"] \arrow[d, "Z_{A,B\otimes C}"'] & H(A)\otimes H(B)\otimes H(C) \arrow[d, "Z_{A,B}\otimes\mathrm{id}"] \\ H(A\otimes B\otimes C) \arrow[r, "\mu_{A\otimes B,C}"'] & H(A\otimes B)\otimes H(C) \end{tikzcd} \tag{2.7}$$

and

$$\begin{tikzcd} H(A)\otimes H(I) \arrow[r, "Z_{A,I}"] & H(A\otimes I) \arrow[d, "H(\rho_A)", "\wr"'] & H(I)\otimes H(A) \arrow[r, "Z_{I,A}"] & H(I\otimes A) \arrow[d, "H(\lambda_A)", "\wr"'] \\ H(A)\otimes I \arrow[u, "H(A)\otimes\eta"] \arrow[r, "\sim", "\rho_{H(A)}"'] & H(A) & I\otimes H(A) \arrow[u, "\eta\otimes H(A)"] \arrow[r, "\sim", "\lambda_{(A)}"'] & H(A) \end{tikzcd}$$

where $\lambda$ and $\rho$ denote the left and right unit isomorphisms respectively.

For the purposes of this thesis, we will always assume that a naF-structure is *strongly unital*. That is, $\epsilon$ is invertible and

$$\eta = \epsilon^{-1}$$

Then the naF-structure $(Z, \eta)$ is completely determined by $Z$.

**Definition 2.2.34.** Let $H : \mathcal{U} \to \mathcal{V}$ be a colax monoidal functor with a naF-structure. In the special case where the multiplication $Z$ is associative, that is

$$Z_{A\otimes B,C}(Z_{A,B}\otimes \mathrm{id}_C) = Z_{A,B\otimes C}(\mathrm{id}_A \otimes Z_{B,C}) \tag{2.8}$$

for all $A, B, C \in \mathcal{U}$, we refer to the naF-structure $(Z, \eta)$ as a *Frobenius structure* and we call $H$ a *Frobenius monoidal functor*. Note that in this case, $H$ is both a lax and colax monoidal functor.

Given Frobenius monoidal functors $H, H' : \mathcal{U} \to \mathcal{V}$, we call a natural transformation $H \to H'$ *bimonoidal* if it is monoidal with respect to both the lax and colax structures of $H$ and $H'$.

*Remark* 2.2.35. A Frobenius monoidal functor as defined above is precisely a Frobenius monoidal functor of [DP08] for which the unit and counit are each others inverses.

**Example 2.2.36.** A strong monoidal functor is exactly a Frobenius monoidal functor whose multiplication and comultiplication are each others inverses. In particular, the Frobenius structure is uniquely determined.

Let $(X, S)$ be a templicial object. Then in particular we have a colax monoidal functor $X : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$. So it makes sense to consider naF-structures on $X$. Suppose $X$ has a naF-structure whose multiplication we denote by $Z$. Then $Z$ consists of quiver morphisms

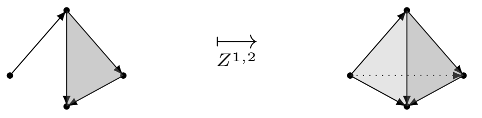
$$(Z^{p,q} : X_p \otimes_S X_q \to X_{p+q})_{p,q\geq 0}$$

which are natural in $p$ and $q$. The diagrams (2.6) and (2.7) then come down to

$$\mu_{k,l}Z^{p,q} = \begin{cases} (Z^{p,k-p}\otimes \mathrm{id}_{X_l})(\mathrm{id}_{X_p}\otimes\mu_{k-p,l}) & \text{if } p \leq k \\ (\mathrm{id}_{X_k}\otimes Z^{p-k,q})(\mu_{k,p-k}\otimes \mathrm{id}_{X_q}) & \text{if } p \geq k \end{cases} \tag{2.9}$$

for all $k, l, p, q \geq 0$ such that $k + l = p + q$. Note that in particular $\mu_{k,l}Z^{k,l} = \mathrm{id}_{X_k\otimes X_l}$ for all $k, l \geq 0$ by the strong unitality.

In §2.1.2 we discussed how the comultiplication morphisms $\mu_{k,l} : X_{k+l} \to X_k \otimes_S X_l$ should be interpreted as "pulling a $(k+l)$-simplex apart into outer faces". Similarly, the morphisms $Z^{k,l} : X_k \otimes_S X_l \to X_{k+l}$ should be interpreted as "filling up necklaces to an entire simplex". In that respect, a templicial object with a naF-structure is reminiscent of a quasi-category in $\mathcal{V}$. But there are two crucial differences. First, naF-structures only allow to fill up necklaces and not inner horns. Second, Frobenius structures give a specified choice of fillers while quasi-categories only requires that they exist. Nonetheless, both are related. In Proposition 3.1.32 we'll see that a quasi-category in $\mathcal{V}$ can be equipped with a naF-structure if it satifies an additional projectivity hypothesis. Moreover, if we restrict to $\mathcal{V} = \mathrm{Mod}(k)$, then the converse holds as well (see Theorem 4.2.62).




**Proposition 2.2.37.** *Let $(\mathcal{W}, \otimes, I)$ be a monoidal category with coproducts such that $- \otimes -$ preserves coproducts in each variable. Let $H : \mathcal{W} \to \mathcal{V}$ be a strong monoidal functor. Assume $H$ preserves coproducts. If $(X, S)$ is a templicial object of $\mathcal{W}$ with naF-structure $Z$, then the quiver morphisms*

$$\left( Z^{p,q}_{\tilde{H}(X)} : H_S(X_p) \otimes H_S(X_q) \xrightarrow{\sim} H_S(X_p \otimes X_q) \xrightarrow{H_S(Z^{p,q})} H_S(X_{p+q}) \right)_{p,q \geq 0}$$

*define a naF-structure on $\tilde{H}(X) \in S_{\otimes}\mathcal{V}$, with $\tilde{H}$ as in Construction 2.1.19.*

*Proof.* Write $\mu$ and $\epsilon$ for the comultiplication and counit of $X$ respectively. Given $p, q \geq 0$, denote by $\varphi_{p,q}$ the isomorphism of $k$-quivers

$$H_S(X_p \otimes X_q) \xrightarrow{\sim} H_S(X_p) \otimes H_S(X_q)$$

Then by definition, we have for all $p, q \geq 0$:

$$Z^{p,q}_{\tilde{H}(X)} = H_S(Z^{p,q}) \circ \varphi^{-1}_{p,q}$$

while the comultiplication of $\tilde{H}(X)$ is given by, for all $k, l \geq 0$:

$$\mu^{\tilde{H}(X)}_{k,l} = \varphi_{k,l} \circ H_S(\mu_{k,l})$$

It easily follows that $(Z^{p,q}_{\tilde{H}(X)})_{p,q\geq 0}$ is a naF-structure on $\tilde{H}(X)$. $\square$

**Notation 2.2.38.** Let $(X, S)$ be a templicial object with naF-structure $Z$. Given a necklace $T = \{0 = t_0 < t_1 < ... < t_k = p\}$, recall the quiver morphism $\mu_T : X_p \to X_T$ of Notation 2.2.15. We'd like to similarly define a quiver morphism

$$Z^T : X_T \to X_p$$

However, since $Z$ is not assumed to be associative, this will depend on how we compose the two-variable morphisms $Z^{k,l}$. Nevertheless, making an arbitrary choice, we can define

$$Z^{p_1,\dots,p_k} = Z^{p_1,p_2+\dots+p_k}(\mathrm{id}_{X_{p_1}} \otimes Z^{p_2,\dots,p_k})$$

inductively on $k \geq 2$, for all $p_1, ..., p_k \geq 0$, and subsequently set

$$Z^T = \begin{cases} \epsilon^{-1} & \text{if } k = 0 \\ \mathrm{id}_{X_p} & \text{if } k = 1 \\ Z^{t_1, t_2 - t_1, ..., p - t_{k-1}} & \text{if } k \geq 2 \end{cases}$$

*Remark* 2.2.39. Let $(X, S)$ be a templicial object with naF-structure $Z$. Consider a necklace $T = \{0 = t_0 < ... < t_k = p\}$. It follows from the naturality of $Z$ that for all $i, j \in [p] \setminus T$:

$$\begin{aligned} d_j Z^T &= Z^{\delta_j^{-1}(T)}(\mathrm{id} \otimes ... \otimes \mathrm{id} \otimes d_{j-i_{m-1}} \otimes \mathrm{id} \otimes ... \otimes \mathrm{id}) \\ s_i Z^T &= Z^{\sigma_i^{-1}(T)}(\mathrm{id} \otimes ... \otimes \mathrm{id} \otimes s_{i-i_{m-1}} \otimes \mathrm{id} \otimes ... \otimes \mathrm{id}) \end{aligned}$$

where $m \in \{1, ..., k\}$ is minimal such that $i < i_m$ or $j < i_m$ respectively. On the other hand, if $i \in T$, then

$$s_i Z^T = Z^{\sigma_i^{-1}(T)}(\mathrm{id} \otimes ... \otimes \mathrm{id} \otimes s_0 \epsilon^{-1} \otimes \mathrm{id} \otimes ... \otimes \mathrm{id})$$

However, if $0 < j < n$ and $j \in T$ the naturality of $Z$ doesn't supply us with a formula to pass the face map $d_j$ through $Z$.

**Proposition 2.2.40.** *Let $(X, S)$ be a templicial object with naF-structure $Z$. Let $p \geq 0$ and $T, U \in \mathcal{P}_p$. Then*

$$\mu_T Z^U = (Z^{U_1} \otimes ... \otimes Z^{U_k})(\mu_{T_1} \otimes ... \otimes \mu_{T_l}) \tag{2.10}$$

*where $(U_1, ..., U_k)$ is the splitting of $U$ over $T$ and $(T_1, ..., T_l)$ is the splitting of $T$ over $U$.*

*Proof.* We use induction on $k = \ell(T)$ and $l = \ell(U)$. If either $k = 0$ or $l = 0$, then both are zero and (2.10) is trivially true. For $k = 1$, both sides of (2.10) reduce to $Z^U$. Similarly, if $l = 1$ both sides reduce to $\mu_T$.

Assume further that $k, l \geq 2$. Let $t \in T$ and $u \in U$ be minimal such that $0 < t$ and $0 < u$. We can write $T = \{0 < t\} \vee T'$ and $U = \{0 < u\} \vee U'$ for some unique $T' \in \mathcal{P}_{p-t}$ and $U' \in \mathcal{P}_{p-u}$. Then:

$$\mu_T Z^U = (\mathrm{id}_{X_t} \otimes \mu_{T'}) \mu_{t,p-t} Z^{u,p-u} (\mathrm{id}_{X_u} \otimes Z^{U'})$$

If $t \leq u$, then $\mu_{t,p-t} Z^{u,p-u} = (\mathrm{id}_{X_t} \otimes Z^{u-t,p-u})(\mu_{t,u-t} \otimes \mathrm{id}_{X_{p-u}})$ by (2.9), and we can write $T_1 = \{0 < t\} \vee T'_1$ for some unique $T'_1 \in \mathcal{P}_{u-t}$. So, by the induction hypothesis, we have

$$\begin{aligned} \mu_T Z^U &= (\mathrm{id}_{X_t} \otimes \mu_{T'} Z^{u-t,p-u})(\mu_{t,u-t} \otimes Z^{U'}) \\ &= (\mathrm{id}_{X_t} \otimes \mu_{T'} Z^{\{0<u-t\} \vee U'})(\mu_{t,u-t} \otimes \mathrm{id}_{X_{U'}}) \\ &= (\mathrm{id}_{X_t} \otimes Z^{U_2} \otimes ... \otimes Z^{U_k})((\mathrm{id}_{X_t} \otimes \mu_{T'_1}) \mu_{t,u-t} \otimes \mu_{T_2} \otimes ... \otimes \mu_{T_l}) \\ &= (Z^{U_1} \otimes Z^{U_2} \otimes ... \otimes Z^{U_k})(\mu_{T_1} \otimes \mu_{T_2} \otimes ... \otimes \mu_{T_l}) \end{aligned}$$

where we used that $U_1 = \{0 < t\}$ since $t \leq u$. A similar argument shows the case for $t \geq u$. $\square$

**Corollary 2.2.41.** *Let $p \geq 0$ and $T, U \in \mathcal{P}_p$. The following statements are true.*

1. *If $T \subseteq U$, and $(U_1, ..., U_k)$ is the splitting of $U$ over $T$, then*
$$\mu_T Z^U = Z^{U_1} \otimes ... \otimes Z^{U_k}$$
2. *If $U \subseteq T$, and $(T_1, ..., T_l)$ is the splitting of $T$ over $U$, then*
$$\mu_T Z^U = \mu_{T_1} \otimes ... \otimes \mu_{T_l}$$
3. *We have $\mu_T Z^U \mu_U = \mu_T Z^{T \cup U} \mu_{T \cup U}$.*

*Proof.* Statements 1 and 2 follow from Propositions 2.2.12.2 and 2.2.40.

To prove 3, consider the splittings $(U_1, ..., U_k)$ and $(T_1, ..., T_l)$ of $U$ over $T$ and $T$ over $U$ respectively. By Proposition 2.2.12.1, $(U_1, ..., U_k)$ is also the splitting of $T \cup U$ over $T$. Thus as $T \subseteq T \cup U$, it follows from 1 that

$$\begin{aligned} \mu_T Z^{T \cup U} \mu_{T \cup U} &= (Z^{U_1} \otimes ... \otimes Z^{U_k}) \mu_{T \cup U} = (Z^{U_1} \otimes ... \otimes Z^{U_k}) \mu_{T_1 \vee ... \vee T_l} \\ &= (Z^{U_1} \otimes ... \otimes Z^{U_k})(\mu_{T_1} \otimes ... \otimes \mu_{T_l}) \mu_U = \mu_T Z^U \mu_U \end{aligned}$$

where we used the coassociativity of $\mu$. $\square$

## 2.3 Enriched categories as templicial objects

### 2.3.1 The templicial nerve

Recall the classical nerve functor $N : \mathrm{Cat} \to \mathrm{SSet}$ (Definition 1.3.22). Given a small $\mathcal{V}$-enriched category $\mathcal{C}$, one might want to similarly define a simplicial object $N(\mathcal{C}) \in S\mathcal{V}$. By analogy with the classical case, we can set for $n \geq 0$:

$$N(\mathcal{C})_n = \coprod_{A_0,\dots,A_n \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(A_0, A_1) \otimes \dots \otimes \mathcal{C}(A_{n-1}, A_n)$$

Then the degeneracy maps and inner face maps can be defined in completely the same way by using the reverse composition law and the identities of $\mathcal{C}$. However to define the outer face map $d_0$ (and similarly $d_n$), we run into a problem because there exists no projection morphism

$$\mathcal{C}(A_0, A_1) \otimes \mathcal{C}(A_1, A_2) \otimes \dots \otimes \mathcal{C}(A_{n-1}, A_n) \to \mathcal{C}(A_1, A_2) \otimes \dots \otimes \mathcal{C}(A_{n-1}, A_n)$$

in general (non-cartesian) monoidal categories $\mathcal{V}$.

This issue can be resolved by considering templicial objects instead. We will construct a fully faithful functor $N_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat} \to S_{\otimes}\mathcal{V}$ which recovers the classical nerve functor when $\mathcal{V} = \mathrm{Set}$. Moreover, just like Proposition 1.3.24, the nerves of $\mathcal{V}$-categories are characterized by a unique horn lifting property (see Proposition 2.3.8).

**Construction 2.3.1.** Let $\mathcal{C}$ be a small $\mathcal{V}$-enriched category. We denote its underlying quiver in $\mathcal{V}\,\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$ by $\mathcal{C}$ as well. Let $u_{\mathcal{C}} : I_S \to \mathcal{C}$ denote the unit of $\mathcal{C}$. Consider the reverse composition law $\tilde{m}_{\mathcal{C}} : \mathcal{C} \otimes_{\mathrm{Ob}(\mathcal{C})} \mathcal{C} \to \mathcal{C}$ of Remark 1.1.22:

$$(\tilde{m}_{\mathcal{C}})_{A,C} : \coprod_{B \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C)$$

for all $A, C \in \mathrm{Ob}(\mathcal{C})$.

For all $n \geq 0$, define the $\mathcal{V}$-quiver

$$N_{\mathcal{V}}(\mathcal{C})_n = \mathcal{C}^{\otimes n}$$

and for all $0 \leq i \leq n$ and $0 < j < n$, define

$$\begin{aligned} d_j &= \mathrm{id}_{\mathcal{C}}^{\otimes j-1} \otimes \tilde{m}_{\mathcal{C}} \otimes \mathrm{id}_{\mathcal{C}}^{\otimes n-j-1} : \mathcal{C}^{\otimes n} \to \mathcal{C}^{\otimes n-1} \\ s_i &= \mathrm{id}_{\mathcal{C}}^{\otimes i} \otimes u_{\mathcal{C}} \otimes \mathcal{C}^{\otimes n-i} : \mathcal{C}^{\otimes n} \to \mathcal{C}^{\otimes n+1} \end{aligned}$$

By the associativity and unitality conditions on $\mathcal{C}$, this defines a functor

$$N_{\mathcal{V}}(\mathcal{C}) : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$$

Further, for any $k, l \geq 0$ we let

$$\mu_{k,l} : \mathcal{C}^{\otimes k+l} \to \mathcal{C}^{\otimes k} \otimes_{\mathrm{Ob}(\mathcal{C})} \mathcal{C}^{\otimes l} \quad \text{and} \quad \epsilon : \mathcal{C}^{\otimes 0} \to I_{\mathrm{Ob}(\mathcal{C})}$$

be the canonical isomorphisms in $\mathcal{V}\,\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$. Thus this defines a strong monoidal structure on $N_{\mathcal{V}}(\mathcal{C})$. In particular, we obtain a templicial object

$$(N_{\mathcal{V}}(\mathcal{C}), \mathrm{Ob}(\mathcal{C}))$$

which we call the *templicial nerve* of $\mathcal{C}$.

Recall the base change functors $f_! : \mathcal{V}\,\mathrm{Quiv}_S \to \mathcal{V}\,\mathrm{Quiv}_T$ and its right-adjoint $f^* : \mathcal{V}\,\mathrm{Quiv}_T \to \mathcal{V}\,\mathrm{Quiv}_S$ for a given map of sets $f : S \to T$ (see Construction 1.1.16).

**Lemma 2.3.2.** *Let $(X, S)$ be a templicial object, $\mathcal{C}$ a small $\mathcal{V}$-enriched category and $f : S \to \mathrm{Ob}(\mathcal{C})$ a map of sets. Then we have a bijection between monoidal natural transformations $f_!X \to N_{\mathcal{V}}(\mathcal{C})$ and quiver morphisms $H : X_1 \to f^*(\mathcal{C})$ such that the diagrams*

$$\begin{tikzcd} X_1^{\otimes 2} \arrow[r, "H^{\otimes 2}"] & f^*(\mathcal{C})^{\otimes 2} \arrow[r] & f^*(\mathcal{C}^{\otimes 2}) \arrow[d, "f^*(\tilde{m}_{\mathcal{C}})"] \\ X_2 \arrow[u, "\mu_{1,1}"] \arrow[r, "d_1"'] & X_1 \arrow[r, "H"'] & f^*(\mathcal{C}) \end{tikzcd} \qquad \begin{tikzcd} & I_S \arrow[r] & f^*(I_{\mathrm{Ob}(\mathcal{C})}) \arrow[d, "f^*(u_{\mathcal{C}})"] \\ X_0 \arrow[ur, "\epsilon", "\sim"'] \arrow[r, "s_0"'] & X_1 \arrow[r, "H"'] & f^*(\mathcal{C}) \end{tikzcd} \tag{2.11}$$

*commute.*

*Proof.* For a monoidal natural transformation $\alpha : f_!X \to N_{\mathcal{V}}(\mathcal{C})$, define $H_\alpha : X_1 \to f^*(\mathcal{C})$ to be the adjoint of $\alpha_1 : f_!(X_1) \to \mathcal{C}$. It follows from the monoidality of $\alpha$ that for all $n \geq 0$, $\alpha_n$ is the composite

$$f_!(X_n) \xrightarrow{f_!(\mu_{1,\ldots,1})} f_!(X_1^{\otimes n}) \to f_!(X_1)^{\otimes n} \xrightarrow{\alpha_1^{\otimes n}} \mathcal{C}^{\otimes n}$$

where we used the colax monoidal structure of $f_!$ (see Lemma 1.1.18). So the assignment $\alpha \mapsto H_\alpha$ is injective. Moreover, it then follows from the naturality of $\alpha$ that $H_\alpha$ satisfies (2.11).

Conversely, if $H : X_1 \to f^*(\mathcal{C})$ satisfies (2.11), then defining $\alpha_1$ as adjoint to $H$ and $\alpha_n$ as above, it follows that $\alpha : f_!X \to N_{\mathcal{V}}(\mathcal{C})$ is a natural transformation. It is immediate that $\alpha$ is also monoidal. $\square$

*Remark* 2.3.3. Let $\mathcal{C}$ and $\mathcal{D}$ be small $\mathcal{V}$-enriched categories, $f : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ a map of sets and $H : \mathcal{C} \to f^*(\mathcal{D})$ a morphism in $\mathcal{V}\,\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$. Then the diagrams (2.11) with $X = N_{\mathcal{V}}(\mathcal{C})$ precisely express that $(H, f)$ is a $\mathcal{V}$-enriched functor $\mathcal{C} \to \mathcal{D}$.

**Construction 2.3.4.** Let $(H, f) : \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-enriched functor between small $\mathcal{V}$-enriched categories. By Lemma 2.3.2, there exists a unique templicial morphism

$$N_{\mathcal{V}}(H) : N_{\mathcal{V}}(\mathcal{C}) \to N_{\mathcal{V}}(\mathcal{D})$$

such that the quiver morphism $N_{\mathcal{V}}(H)_1 : f_!(\mathcal{C}) \to \mathcal{D}$ corresponds to $H : \mathcal{C} \to f^*(\mathcal{D})$ by adjunction. Explicitly,

$$N_{\mathcal{V}}(H)_n : f_!(\mathcal{C}^{\otimes n}) \to f_!(\mathcal{C})^{\otimes n} \xrightarrow{N_{\mathcal{V}}(H)_1^{\otimes n}} \mathcal{D}^{\otimes n}$$

for all $n \geq 0$. It follows that we obtain a functor

$$N_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat} \to S_{\otimes}\mathcal{V},$$

which we call the *templicial nerve functor*.

*Remark* 2.3.5. It is clear from the construction that in case $\mathcal{V} = \mathrm{Set}$, the templicial nerve functor $N_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat} \to S_{\otimes}\mathcal{V}$ reduces to the classical nerve functor $N : \mathrm{Cat} \to \mathrm{SSet}$.

**Proposition 2.3.6.** *The templicial nerve functor $N_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat} \to S_{\otimes}\mathcal{V}$ is fully faithful.*

*Proof.* This follows from Lemma 2.3.2 and Remark 2.3.3. □

**Lemma 2.3.7.** *Let $(X, S)$ be a templicial object such that for all $a, b \in S$, $X_\bullet(a, b)$ lifts inner horns uniquely. Suppose $U : \mathcal{V} \to \mathrm{Set}$ is conservative. Then for any inert necklace map $(T, p) \to (T', p)$, the induced quiver morphism*

$$X(f) : X_{T'} \to X_T$$

*is an isomorphism.*

*Proof.* As $T \to T'$ is inert, we have that $T' \subseteq T$. Let $(T_1, ..., T_l)$ be the splitting of $T$ over $T'$. Then $X(f) = \mu_{T_1} \otimes ... \otimes \mu_{T_l}$. Thus we are reduced to showing that $\mu_T : X_p \to X_T$ is an isomorphism for all necklaces $T$. Writing $T = \{0 = t_1 < ... < t_k = p\}$, we have

$$\mu_T = (\mathrm{id}_{X_{t_1}} \otimes ... \otimes \mathrm{id}_{X_{t_{k-2}-t_{k-3}}} \otimes \mu_{t_{k-1}-t_{k-2}, p-t_{k-1}}) \cdots (\mathrm{id}_{X_{t_1}} \otimes \mu_{t_2-t_1, p-t_2}) \mu_{t_1, p-t_1}$$

and thus it suffices to show that each comultiplication morphism $\mu_{k,n-k}$ with $0 < k < n$ is an isomorphism. We proceed by induction on $n$.

If $n = 1$, there is nothing prove. So let $n \geq 2$ and $0 < k < n$. Take $a, b \in S$ and $x_k \in U((X_k \otimes X_{n-k})(a, b))$. For any $0 < l < n$ with $l \neq k$, define

$$x_l = \begin{cases} (\mathrm{id}_{X_l} \otimes \mu^{-1}_{k-l,n-k})(\mu_{l,k-l} \otimes \mathrm{id}_{X_{n-k}})(x_k) & \text{if } l < k \\ (\mu^{-1}_{k,l-k} \otimes \mathrm{id}_{X_{n-l}})(\mathrm{id}_{X_k} \otimes \mu_{l-k,n-l})(x_k) & \text{if } l > k \end{cases}$$

Further set, for all $0 < i < n$ with $i \neq k$:

$$y_i = \begin{cases} \mu^{-1}_{k-1,n-k}(d_i \otimes \mathrm{id}_{X_{n-k}})(x_k) & \text{if } i < k \\ \mu^{-1}_{k,n-k-1}(\mathrm{id}_{X_k} \otimes d_{i-k})(x_k) & \text{if } i > k \end{cases}$$

It follows that the elements $(x_l)_{l=1}^{n-1}$ and $(y_i)_{i=1, i\neq k}^{n-1}$ satisfy the conditions of Corollary 2.2.22.3 and thus there is a unique element $z \in U(X_n(a, b))$ such that $\mu_{l,n-l}(z) = x_l$ and $d_i(z) = x_i$ for all $0 < l, i < n$ with $i \neq k$. In particular $\mu_{k,n-k}(z) = x_k$. For any other $z' \in U(X_n(a, b))$ with $\mu_{k,n-k}(z') = x_k$, it follows from the definitions of the $x_l$ and $y_i$ that also $\mu_{l,n-l}(z') = x_l$ and $d_i(z') = y_i$ for all $0 < l, i < n$ with $i \neq k$. Thus $z' = z$ and hence the map

$$U(\mu_{k,n-k}) : U(X_n(a, b)) \to U((X_k \otimes X_{n-k})(a, b))$$

is a bijection. As $U$ is conservative, $\mu_{k,n-k} : X_n \to X_k \otimes X_{n-k}$ is an isomorphism of $\mathcal{V}$-enriched quivers. □

**Proposition 2.3.8.** *Let $(X, S) \in S_\otimes\mathcal{V}$. Consider the following statements.*

*(1) The functor $X : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$ is strong monoidal.*

*(2) $(X, S)$ is isomorphic to the templicial nerve of a small $\mathcal{V}$-category.*

*(3) For all $a, b \in S$, $X_\bullet(a, b)$ lifts inner horns uniquely.*

*Then (1) and (2) are equivalent and they imply (3). Moreover, if the functor $U : \mathcal{V} \to \mathrm{Set}$ is conservative, then (1), (2) and (3) are all equivalent.*

*Proof.* The implication (2) $\Rightarrow$ (1) is clear by definition of the templicial nerve. Conversely, suppose $X$ is strong monoidal, i.e. its comultiplication $\mu$ is an isomorphism. Then we have isomorphisms for all $n \geq 0$:

$$\mu_{1,...,1} : X_n \xrightarrow{\sim} X_1 \otimes_S ... \otimes_S X_1$$

in $\mathcal{V}\,\mathrm{Quiv}_S$. Through these isomorphisms, the face $d_1 : X_2 \to X_1$ and degeneracy $s_0 : X_0 \to X_1$, give us quiver morphisms

$$\tilde{m} : X_1 \otimes_S X_1 \to X_1 \quad \text{and} \quad u : I_S \to X_1$$

It follows by the simplicial identities and the naturality, coassociativity and counitality of $\mu$ that these morphisms define the structure of a $\mathcal{V}$-enriched category on $X_1$ with set of objects given by $S$. Again by the naturality of $\mu$, the morphisms $\mu_{1,...,1}$ combine to give an isomorphism $X \simeq N_{\mathcal{V}}(X_1)$ between functors $\mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$. This natural isomorphism is monoidal by the coassociativity of $\mu$, showing that $(X, S)$ is isomorphic to $(N_{\mathcal{V}}(X_1), S)$ in $S_{\otimes}\mathcal{V}$.

Assume that (1) holds and let $0 < j < n$ and $a, b \in S$. Take $x_k \in U((X_k \otimes X_{n-k})(a, b))$ and $y_i \in U(X_{n-1}(a, b))$ for all $0 < k, i < n$ with $i \neq j$ which satisfy the conditions of Corollary 2.2.22.3. We wish to show that there is a unique $z \in X_n(a, b)$ such that $\mu_{k,n-k}(z) = x_k$ and $d_i(x) = y_i$ for all $0 < k, i < n$ with $i \neq j$. As the $\mu_{k,n-k}$ are isomorphisms, we have by the hypotheses on the $x_k$ that

$$\mu^{-1}_{1,n-1}(x_1) = \mu^{-1}_{2,n-2}(x_2) = ... = \mu^{-1}_{n-1,1}(x_{n-1})$$

Setting $z$ to be equal to these elements, it follows from the hypotheses on the $x_k$ and $y_i$ that for all $0 < i < n$ with $i \neq j$, we have $d_i(z) = y_i$. This shows (3).

Assume that (3) holds and that $U$ is conservative. Then by Lemma 2.3.7, the comultiplication morphism $\mu_{k,l}$ is an isomorphism for all $k, l > 0$. As $\mu_{n,0}$ and $\mu_{0,n}$ are always isomorphisms for $n \geq 0$, this shows (1). $\square$

**Corollary 2.3.9.** *For any small $\mathcal{V}$-enriched category $\mathcal{C}$, the nerve $N_{\mathcal{V}}(\mathcal{C})$ is a quasi-category in $\mathcal{V}$ with a unique Frobenius structure.*

*Proof.* This immediately follows from Proposition 2.3.8 and Example 2.2.36. $\square$

We end this subsection with some compatibility results.

**Notation 2.3.10.** The adjunction $F \dashv U$ between $\mathrm{Set}$ and $\mathcal{V}$ also induces an adjunction between small categories and small $\mathcal{V}$-enriched categories by Proposition 1.1.23. We will denote this adjunction by

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, shift left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\,\mathrm{Cat} \arrow[l, shift left, "\mathcal{U}"] \end{tikzcd}$$

**Proposition 2.3.11.** *We have a natural isomorphism*

$$N_{\mathcal{V}} \circ \mathcal{F} \simeq \tilde{F} \circ N$$

*Proof.* Let $\mathcal{C}$ be an ordinary small category. Consider its nerve $N(\mathcal{C})$ as a templicial set (Proposition 2.1.15). As $F$ is strong monoidal and preserves colimits, we obtain a canonical isomorphism

$$\alpha_n : F(\mathcal{C}^{\times n}) \xrightarrow{\sim} \mathcal{F}(\mathcal{C})^{\otimes n}$$

where $\mathcal{C}^{\times n}$ denotes the $n$-fold monoidal product of $\mathcal{C}$ as a quiver in $\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$. It is easy to see that these isomorphisms combine to give an isomorphism of templicial objects:

$$\alpha : \tilde{F}(N(\mathcal{C})) \xrightarrow{\sim} N_{\mathcal{V}}(\mathcal{F}(\mathcal{C}))$$

which is clearly natural in $\mathcal{C}$. □

**Proposition 2.3.12.** *We have a natural isomorphism*

$$\tilde{U} \circ N_{\mathcal{V}} \simeq N \circ \mathcal{U}$$

*Proof.* Let $\mathcal{C}$ be a small $\mathcal{V}$-category and $n \geq 0$. Then we have the following isomorphisms, natural in $n$ and $\mathcal{C}$:

$$\begin{aligned}\tilde{U}(N_{\mathcal{V}}(\mathcal{C}))_n &= S_{\otimes}\mathcal{V}(\tilde{F}(\Delta^n), N_{\mathcal{V}}(\mathcal{C})) \simeq S_{\otimes}\mathcal{V}(N_{\mathcal{V}}(\mathcal{F}([n])), N_{\mathcal{V}}(\mathcal{C}))\\ &\simeq \mathcal{V}\,\mathrm{Cat}(\mathcal{F}([n]), \mathcal{C}) \simeq \mathrm{Cat}([n], \mathcal{U}(\mathcal{C})) \simeq N(\mathcal{U}(\mathcal{C}))_n\end{aligned}$$

where we subsequently used the isomorphism $\Delta^n \simeq N([n])$, Proposition 2.3.11, Proposition 2.3.6, and the adjunction $\mathcal{F} \dashv \mathcal{U}$. □

### 2.3.2 The homotopy category of a templicial object

Just like the classical nerve functor, the templicial nerve functor $N_{\mathcal{V}}$ of Construction 2.3.4 has a left-adjoint $h_{\mathcal{V}} : S_{\otimes}\mathcal{V} \to \mathcal{V}\,\mathrm{Cat}$ which associates to every templicial object its *homotopy category* (Proposition 2.3.14). Moreover, in Proposition 2.3.25 we'll show that the homotopy category $h_{\mathcal{V}}X$ is significantly easier to describe when the templicial object $X$ is a quasi-category in $\mathcal{V}$. This generalizes the classical Proposition 1.3.28.

**Construction 2.3.13.** Let $(X, S)$ be a templicial object and $a, b \in S$. We construct an object $h_{\mathcal{V}}X(a, b) \in \mathcal{V}$ by the following coequalizer:

$$\coprod_{\substack{T \in \mathcal{N}ec\\ T \neq \{0\}}} X_T(a, b) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{p>0} X_1^{\otimes p}(a, b) \xrightarrow{q} h_{\mathcal{V}}X(a, b) \tag{2.12}$$

where $\alpha$, $\beta$ are defined as follows. For $(T, p) \in \mathcal{N}ec$ with $p > 0$, let $k = \ell(T)$. Then set

$$\alpha\iota_T = \iota_p X(([p], p) \xrightarrow{\text{inert}} (T, p))$$
$$\beta\iota_T = \iota_k X(([k], k) \xrightarrow{\text{active}} (T, p))$$

where we used the unique inert and active maps of Remark 2.2.8. Note that this coequalizer is reflexive where the common section $\gamma$ of $\alpha$ and $\beta$ is given by $\gamma\iota_p = \iota_{([p],p)}$.

We construct a $\mathcal{V}$-category $h_{\mathcal{V}}X$ with object set $S$, whose hom-objects are given by $h_{\mathcal{V}}X(a,b)$ for all $a,b \in S$. Consider $h_{\mathcal{V}}X$ as a quiver. Then define

$$u : I_S \xrightarrow{s_0\epsilon^{-1}} X_1 \xrightarrow{\iota_1} \coprod_{p>0} X_1^{\otimes p} \overset{q}{\twoheadrightarrow} h_{\mathcal{V}}X$$

Further, as the coequalizer (2.12) is reflexive, it is preserved by $- \otimes -$ in both variables simultanously, so that we obtain a reflexive coequalizer of quivers

$$\coprod_{\substack{T,U \in \mathcal{N}ec\\ T,U \neq \{0\}}} X_{T\vee U} \overset{\alpha\otimes\alpha}{\underset{\beta\otimes\beta}{\rightrightarrows}} \coprod_{r,s>0} X_1^{\otimes r+s} \xrightarrow{q\otimes q} h_{\mathcal{V}}X \otimes_S h_{\mathcal{V}}X$$

It follows that there is a unique quiver morphism

$$\tilde{m} : h_{\mathcal{V}}X \otimes_S h_{\mathcal{V}}X \to h_{\mathcal{V}}X$$

such that $\tilde{m}(q\iota_r \otimes q\iota_s) = q\iota_{r+s}$ for all $r,s > 0$. It easily follows that $\tilde{m}$ is associative. Moreover, it is unital with respect to $u$. Indeed, for $p > 0$, set $T = [p-1] \vee \{0 < 2\}$. Then we have

$$\begin{aligned}
\tilde{m}(\mathrm{id}_{h_{\mathcal{V}}X} \otimes u)q\iota_p &= \tilde{m}(q \otimes q)(\iota_p \otimes \iota_1 s_0\epsilon^{-1}) = q\iota_{p+1}(\mathrm{id}_{X_1}^{\otimes p} \otimes s_0\epsilon^{-1}) \\
&= q\iota_{p+1}(\mathrm{id}_{X_1}^{\otimes p-1} \otimes \mu_{1,1}s_1) = q\alpha\iota_T(\mathrm{id}_{X_1}^{\otimes p-1} \otimes s_1) \\
&= q\beta\iota_T(\mathrm{id}^{\otimes p-1} \otimes s_1) = q\iota_p(\mathrm{id}_{X_1}^{\otimes p-1} \otimes d_1 s_1) = q\iota_p
\end{aligned}$$

and thus $\tilde{m}(\mathrm{id}_{h_{\mathcal{V}}X} \otimes u) = \mathrm{id}_{h_{\mathcal{V}}X}$. Similarly, $\tilde{m}(u \otimes \mathrm{id}_{h_{\mathcal{V}}X}) = \mathrm{id}_{h_{\mathcal{V}}X}$. Thus $\tilde{m}$ and $u$ define the structure of a $\mathcal{V}$-category on the quiver $h_{\mathcal{V}}X$.

**Proposition 2.3.14.** *The assignment $X \mapsto h_{\mathcal{V}}X$ of Construction 2.3.13 extends to a functor $h_{\mathcal{V}} : S_{\otimes}\mathcal{V} \to \mathcal{V}\,\mathrm{Cat}$ which is left adjoint to the templicial nerve functor $N_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat} \to S_{\otimes}\mathcal{V}$.*

*Proof.* We use the same notation as in Construction 2.3.13. Consider the composite quiver morphism $q\iota_1 : X_1 \to h_{\mathcal{V}}X$. Then the following diagrams commute

$$\begin{tikzcd} X_2 \arrow[r, "\mu_{1,1}"] \arrow[dr, "d_1"'] & X_1^{\otimes 2} \arrow[r, "(q\iota_1)^{\otimes 2}"] & h_{\mathcal{V}}X^{\otimes 2} \arrow[d, "\tilde{m}"] \\ & X_1 \arrow[r, "q\iota_1"'] & h_{\mathcal{V}}X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I_S \arrow[d, "s_0\epsilon^{-1}"'] \arrow[dr, "u"] & \\ X_1 \arrow[r, "q\iota_1"'] & h_{\mathcal{V}}X \end{tikzcd}$$

Indeed, the right hand diagram commutes by definition and the left hand diagram commutes because

$$\tilde{m}(q\iota_1 \otimes q\iota_1)\mu_{1,1} = q\iota_2\mu_{1,1} = q\alpha\iota_{\{0<2\}} = q\beta\iota_{\{0<2\}} = q\iota_1 d_1$$

Thus by Lemma 2.3.2, there is a unique templicial morphism $\eta_X : X \to N_{\mathcal{V}}(h_{\mathcal{V}}X)$ such that $\eta_{X_1} : X_1 \to h_{\mathcal{V}}X$ is $q\iota_1$. We claim that $\eta_X$ is the unit of an adjunction $h_{\mathcal{V}} \dashv N_{\mathcal{V}}$.

Now let $\mathcal{C}$ be an arbitrary small $\mathcal{V}$-category and $(\zeta, f) : X \to N_{\mathcal{V}}(\mathcal{C})$ a templicial morphism. By Lemma 2.3.2, $\zeta$ corresponds to a quiver morphism $H : X_1 \to f^*(\mathcal{C})$ such that the diagrams (2.11) commute. Then it follows from the associativity of $\tilde{m}_{\mathcal{C}}$, and the coassociativity of the comultiplication $\mu$ that the diagram

$$\begin{tikzcd} X_T \arrow[r, "X([p]\to T)"] \arrow[d, "X([k]\to T)"'] & X_1^{\otimes p} \arrow[r, "H^{\otimes p}"] & f^*(\mathcal{C})^{\otimes p} \arrow[r] & f^*(\mathcal{C}^{\otimes p}) \arrow[d, "f^*(\tilde{m}_{\mathcal{C}}^{(p)})"] \\ X_1^{\otimes k} \arrow[r, "H^{\otimes k}"'] & f^*(\mathcal{C})^{\otimes k} \arrow[r] & f^*(\mathcal{C}^{\otimes k}) \arrow[r, "f^*(\tilde{m}_{\mathcal{C}}^{(k)})"'] & f^*(\mathcal{C}) \end{tikzcd}$$

commutes as well for all necklaces $(T, p)$ with $p > 0$ and $k = \ell(T)$. Thus by (2.12), we get a unique quiver morphism $\overline{H} : h_{\mathcal{V}}X \to f^*(\mathcal{C})$ such that $\overline{H}q$ is the composite

$$\coprod_{p>0} X_1^{\otimes p} \xrightarrow{\coprod_{p>0} H^{\otimes p}} \coprod_{p>0} f^*(\mathcal{C})^{\otimes p} \to \coprod_{p>0} f^*(\mathcal{C}^{\otimes p}) \xrightarrow{(f^*(\tilde{m}_{\mathcal{C}}^{(p)}))_{p>0}} f^*(\mathcal{C})$$

It follows from the definition of the composition in $h_{\mathcal{V}}X$ that $\overline{H}$ defines a $\mathcal{V}$-functor $h_{\mathcal{V}}X \to \mathcal{C}$ and it is clearly unique such that $N_{\mathcal{V}}(\overline{H}) \circ \eta_X = (\zeta, f)$. □

*Remark* 2.3.15. By comparing left-adjoints, Remark 2.3.5 shows that $h_{\mathcal{V}} : S_{\otimes}\mathcal{V} \to \mathcal{V}\,\mathrm{Cat}$ reduces to the classical homotopy functor $\mathrm{SSet} \to \mathrm{Cat}$ when $\mathcal{V} = \mathrm{Set}$.

**Corollary 2.3.16.** *We have a natural isomorphism*

$$h_{\mathcal{V}} \circ \tilde{F} \simeq \mathcal{F} \circ h$$

*Proof.* This follows by comparing left-adjoints using Proposition 2.3.12. □

We collect some of the previous results in the following theorem.

**Theorem 2.3.17.** *There is a diagram of adjunctions*

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, shift left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "N"] \arrow[d, phantom, "\dashv"] & \mathcal{V}\,\mathrm{Cat} \arrow[l, shift left, "\mathcal{U}"] \arrow[d, shift left, "N_{\mathcal{V}}"] \arrow[d, phantom, "\dashv"] \\ \mathrm{SSet} \arrow[u, shift left, "h"] \arrow[r, shift left, "\tilde{F}"] \arrow[r, phantom, "\perp"] & S_{\otimes}\mathcal{V} \arrow[l, shift left, "\tilde{U}"] \arrow[u, shift left, "h_{\mathcal{V}}"] \end{tikzcd}$$

*which commutes in the sense that we have natural isomorphisms:*

$$N_{\mathcal{V}} \circ \mathcal{F} \simeq \tilde{F} \circ N, \quad \tilde{U} \circ N_{\mathcal{V}} \simeq N \circ \mathcal{U} \quad \textit{and} \quad \mathcal{F} \circ h \simeq h_{\mathcal{V}} \circ \tilde{F}$$

*Proof.* Combine Propositions 2.3.11 and 2.3.12, and Corollary 2.3.16. □

Consider the unit $\mathrm{id}_{S_{\otimes}\mathcal{V}} \to N_{\mathcal{V}}h_{\mathcal{V}}$ of the adjunction $h_{\mathcal{V}} \dashv N_{\mathcal{V}}$. Applying $\tilde{U}$, Proposition 2.3.12 provides a natural transformation

$$\tilde{U} \to \tilde{U}N_{\mathcal{V}}h_{\mathcal{V}} \simeq N\mathcal{U}h_{\mathcal{V}}$$

which by the adjunction $h \dashv N$ corresponds to a natural transformation

$$h\tilde{U} \to \mathcal{U}h_{\mathcal{V}}$$

In general this will not be an isomorphism, as is shown in the following example. However, assuming some conditions on the forgetful functor $U : \mathcal{V} \to \mathrm{Set}$, we do find an isomorphism if we restrict to quasi-categories in $\mathcal{V}$ (see Corollary 2.3.26).

**Example 2.3.18.** Let $\mathcal{V} = \mathrm{Mod}(k)$ with $k$ an arbitrary unital commutative ring. In this case we denote $h_k = h_{\mathrm{Mod}(k)}$. Consider the templicial $k$-module $X = \tilde{F}(\partial\Delta^2)$. Then by Corollary 2.3.16, the hom-object $(h_k X)(0,2)$ is isomorphic to

$$F(h(\partial\Delta^2)(0,2)) = F(\{\ 0\bullet \nearrow \overset{1}{\bullet} \searrow \bullet 2\ ,\ 0\bullet\!\longrightarrow\!\bullet 2\ \}) \simeq k \oplus k$$

On the other hand, the set $h\tilde{U}(X)(0,2)$ consists of equivalence classes of sequences of edges $(a_1, ..., a_n)$ from $0$ to $2$ in $\tilde{U}(X)$. Note that each edge in $\tilde{U}(X)$ between two given vertices is uniquely determined by an element $a_i \in k$. One can check that there is a bijection

$$h\tilde{U}\tilde{F}(\partial\Delta^2)(0,2) \xrightarrow{\sim} U(k) \amalg_{U(0)} U(k)$$

which sends a sequence $(a_1, ..., a_n)$ to its product $a_n \cdots a_1$ in $k$. The two terms $U(k)$ correspond to paths either passing through the vertex 1 or not. Now the induced map $h\tilde{U}\tilde{F}(\partial\Delta^2)(0,2) \to U((h_k X)(0,2))$ on hom-sets corresponds to the canonical map

$$U(k) \amalg_{U(0)} U(k) \to U(k \oplus k)$$

which is certainly not a bijection if $k$ is not the zero ring. Hence, the canonical functor

$$h\tilde{U}(X) \to \mathcal{U}(h_k X)$$

is not an equivalence of categories.

We now turn our attention to the description of the homotopy category $h_{\mathcal{V}}X$ when $X$ is a quasi-category in $\mathcal{V}$.

**Construction 2.3.19.** Let $(X, S)$ be a templicial object and $a, b \in S$. We define an object $\mathrm{Hom}^L_X(a,b)_1 \in \mathcal{V}$ by the following pullback:

$$\begin{tikzcd}
\mathrm{Hom}^L_X(a,b)_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X_2(a,b) \arrow[d, "\mu^X_{1,1}"] \\
X_1(a,b) \arrow[r, "-\otimes s^X_0"'] & (X_1 \otimes_S X_1)(a,b)
\end{tikzcd}$$

Further, we denote $d_1 = \pi_1$, $d_0 = d^X_1 \pi_2$ and we let $s_0 : X_1(a,b) \to \mathrm{Hom}^L_X(a,b)_1$ be the unique morphism such that $\pi_1 s_0 = \mathrm{id}_{X_1(a,b)}$ and $\pi_2 s_0 = s^X_1$. We obtain a reflexive pair:

$$\begin{tikzcd}
\mathrm{Hom}^L_X(a,b)_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_1(a,b) \arrow[l, "s_0" description]
\end{tikzcd}$$

Finally, we define an object $h'_{\mathcal{V}}X(a,b)$ by the following coequalizer:

$$\begin{tikzcd}
\mathrm{Hom}^L_X(a,b)_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_1(a,b) \arrow[r, two heads, "q"] & h'_{\mathcal{V}}X(a,b)
\end{tikzcd} \tag{2.13}$$

*Remark* 2.3.20. It is possible to extend Construction 2.3.19 to obtain a simplicial object $\mathrm{Hom}^L_X(a,b) : \mathbf{\Delta}^{op} \to \mathcal{V}$ which generalizes the *left-pinched morphism space* of a simplicial set (as defined in [Lur18, Tag 01KX]). In particular, $\mathrm{Hom}^L_X(a,b)_0 = X_1(a,b)$. The morphisms $d_0, d_1 : \mathrm{Hom}^L_X(a,b)_1 \rightrightarrows X_1(a,b)$ and $s_0 : X_1(a,b) \to \mathrm{Hom}^L_X(a,b)_1$ then constitute the lowest dimensional face and degeneracy morphisms of $\mathrm{Hom}^L_X(a,b)$. We will not go into them here however and leave their investigation to future research (also see Chapter 5).

*Remark* 2.3.21. Note that as $U$ preserves pullbacks, we find that $U(\mathrm{Hom}^L_X(a,b)_1)$ is the set of all 2-simplices $(\alpha_{0,2}, \alpha_{0,1}, \alpha_{1,2})$ of $\tilde{U}(X)$ (see Corollary 2.1.27) with vertices $a_0 = a, a_1 = b, a_2 = b$ and $\alpha_{1,2} = s_0(b)$.

Assuming that $\tilde{U}(X)$ is a quasi-category and that $U$ preserves reflexive coequalizers, it follows from Lemma 1.3.27 and Proposition 1.3.28 that we have an isomorphism:

$$U(h'_{\mathcal{V}}X(a,b)) \simeq h\tilde{U}(X)(a,b)$$

and the canonical morphism $X_1(a,b) \twoheadrightarrow h'_{\mathcal{V}}X(a,b)$ precisely takes the homotopy class $[f]$ in $h\tilde{U}(X)$ of any $f \in U(X_1(a,b))$.

**Lemma 2.3.22.** *Assume that $U : \mathcal{V} \to \mathrm{Set}$ preserves reflexive coequalizers. Let $X$ be a quasi-category in $\mathcal{V}$ with objects $a$ and $b$. For any $w, w' \in U(X_2(a,b))$ such that $(q \otimes q)\mu_{1,1}(w) = (q \otimes q)\mu_{1,1}(w')$, we have that $q(d^X_1(w)) = q(d^X_1(w'))$ in $h'_{\mathcal{V}}X(a,b)$.*

*Proof.* Let $Q$ denote the quiver given by $\mathrm{Hom}^L_X(a,b)_1$ for all objects $a$ and $b$ of $X$. Let $\sigma \in U((Q \otimes Q)(a,b))$ and $w, w' \in U(X_2(a,b))$ be such that $\mu_{1,1}(w) = (d_0 \otimes d_0)(\sigma)$ and $\mu_{1,1}(w') = (d_1 \otimes d_1)(\sigma)$. Then:

- The elements $x_1 = (d_1 \otimes s^X_0 d_1)(\sigma) \in U((X_1 \otimes X_2)(a,b))$, $x_2 = (\pi_2 \otimes d_1)(\sigma) \in U((X_2 \otimes X_1)(a,b))$ and $y_2 = w \in U(X_2(a,b))$ define a horn $\tilde{F}(\Lambda^3_1)_\bullet(0,3) \to X_\bullet(a,b)$ which extends to an element $z \in U(X_3(a,b))$. Then set $w'' = d^X_1(z) \in U(X_2(a,b))$. Note that $d^X_1(w'') = d^X_1(w)$.
- Likewise, the elements $x_1 = (d_0 \otimes \pi_2)(\sigma) \in U((X_1 \otimes X_2)(a,b))$, $x_2 = w'' \otimes s^X_0(b) \in U((X_2 \otimes X_1)(a,b))$ and $y_2 = w'$ define a horn $\tilde{F}(\Lambda^3_1)_\bullet(0,3) \to X_\bullet(a,b)$ which extends to an element $z \in U(X_3(a,b))$. Then set $\tau = d^X_1(z) \in U(X_2(a,b))$.

It follows that $\mu_{1,1}(\tau) = d^X_1(w) \otimes s^X_0(b)$ and $d^X_1(\tau) = d^X_1(w')$. Hence, $qd^X_1(w) = qd^X_1(w')$.

As the diagram (2.13) is a reflexive coequalizer, it is preserved by $- \otimes -$ in both variables simultaneously so that we again have a reflexive coequalizer

$$(Q \otimes Q)(a,b) \underset{d_1 \otimes d_1}{\overset{d_0 \otimes d_0}{\rightrightarrows}} (X_1 \otimes X_1)(a,b) \xrightarrow{q \otimes q} (h'_{\mathcal{V}}X \otimes h'_{\mathcal{V}}X)(a,b)$$

Now assume that $(q \otimes q)\mu_{1,1}(w) = (q \otimes q)\mu_{1,1}(w')$. As $U$ preserves reflexive coequalizers, there exist $\alpha_0, \ldots, \alpha_n \in U((X_1 \otimes X_1)(a,b))$ such that $\mu_{1,1}(w) = \alpha_0$, $\alpha_n = \mu_{1,1}(w)$ and for all $i \in \{1, \ldots, n\}$ there exists a $\sigma \in U((Q \otimes Q)(a,b))$ such that

$$\begin{aligned} & \alpha_{i-1} = (d_0 \otimes d_0)(\sigma) \text{ and } (d_1 \otimes d_1)(\sigma) = \alpha_i \\ \text{or} \qquad & \alpha_{i-1} = (d_1 \otimes d_1)(\sigma) \text{ and } (d_0 \otimes d_0)(\sigma) = \alpha_i \end{aligned}$$

For every $0 < i < n$, $\alpha_i$ defines a horn $\tilde{F}(\Lambda_1^2)_\bullet(0,2) \to X_\bullet(a,b)$ which we can extend to an element $w_i \in U(X_2(a,b))$ so that $\mu_{1,1}(w_i) = \alpha_i$. Thus it follows by the previous that

$$qd_1(w) = qd_1(w_1) = ... = qd_1(w_{n-1}) = qd_1(w')$$

□

**Lemma 2.3.23.** *Assume that $U : \mathcal{V} \to \mathrm{Set}$ is faithful. Let $g : X \to Y$ and $f : X \to Z$ be morphisms in $\mathcal{V}$ such that $g$ is a regular epimorphism. Suppose that for all $x, y \in U(X)$, we have*

$$g(x) = g(y) \quad \Rightarrow \quad f(x) = f(y)$$

*Then there exists a unique morphism $h : Y \to Z$ such that $hg = f$.*

*Proof.* Denote the kernel pair $X \times_Y X \rightrightarrows X$ of $g$ by $\pi_1$ and $\pi_2$. In view of Definition 1.2.12, it suffices to show that $f\pi_1 = f\pi_2$. As $U$ is faithful, this is equivant to showing that for all $(x,y) \in U(X) \times_{U(Y)} U(X)$, we have $f(x) = f(y)$. But this is equivalent to the hypothesis on $f$ and $g$. □

**Construction 2.3.24.** Assume that $U : \mathcal{V} \to \mathrm{Set}$ is faithful and preserves and reflects reflexive coequalizers. Let $(X, S)$ be a quasi-category in $\mathcal{V}$. We construct a $\mathcal{V}$-enriched category $h'_{\mathcal{V}}X$ whose hom-objects are given by $h'_{\mathcal{V}}X(a,b)$ of Construction 2.3.19. Let $h'_{\mathcal{V}}X$ denote the quiver given by $h'_{\mathcal{V}}X(a,b)$ for all $a, b \in S$, and let $q : X_1 \to h'_{\mathcal{V}}X$ denote the canonical quiver morphism.

First define $u : I_S \xrightarrow{s_0} X_1 \xrightarrow{q} h'_{\mathcal{V}}X$. Note that $U$ also reflects regular epimorphisms (as they are the coequalizer of their kernel pair). Thus as $X$ is a quasi-category in $\mathcal{V}$, the comultiplication $\mu_{1,1} : X_2 \to X_1 \otimes_S X_1$ is a regular epimorphism. Further, $q$ is a regular epimorphism by definition (also see Remark 1.2.13). Now $- \otimes -$ preserves reflexive coequalizers in each variable and thus also regular epimorphisms. It follows that $q^{\otimes 2} \circ \mu_{1,1}$ is a regular epimorphism as well. Using Lemmas 2.3.22 and 2.3.23 , we have a unique quiver morphism $\tilde{m} : h'_{\mathcal{V}}X \otimes_S h'_{\mathcal{V}}X \to h'_{\mathcal{V}}X$ such that the following diagram commutes:

$$\begin{tikzcd} X_2 \arrow[r, "\mu_{1,1}"] \arrow[dr, "d_1"'] & X_1^{\otimes 2} \arrow[r, "q^{\otimes 2}"] & (h'_{\mathcal{V}}X)^{\otimes 2} \arrow[d, "\tilde{m}"] \\ & X_1 \arrow[r, "q"'] & h'_{\mathcal{V}}X \end{tikzcd}$$

Given a 2-simplex $(\alpha_{02}, \alpha_{01}, \alpha_{12})$ of $\tilde{U}(X)$ with vertices $a$, $b$ and $c$, we have $\mu_{1,1}(\alpha_{02}) = \alpha_{01} \otimes \alpha_{12}$ and thus $\tilde{m}(q(\alpha_{01}) \otimes q(\alpha_{02})) = q(d_1(\alpha_{02}))$. Therefore, the induced map

$$U(h'_{\mathcal{V}}X(a,b)) \times U(h'_{\mathcal{V}}X(b,c)) \to U(h'_{\mathcal{V}}X(a,b) \otimes h'_{\mathcal{V}}X(b,c)) \xrightarrow{U(\tilde{m}_{a,b,c})} U(h'_{\mathcal{V}}X(a,c))$$

coincides with the reverse composition law of $h\tilde{U}(X)$ (see Remark 1.1.22) under the isomorphisms supplied by Remark 2.3.21. The element $u_a = q(s_0(a)) : I \to h'_{\mathcal{V}}X(a,a)$ is clearly the identity at $a$ in $h\tilde{U}(X)$. It then follows from the faithfulness of $U$ that $\tilde{m}$ is associative and unital with respect to $u$. So we obtain a $\mathcal{V}$-category $h'_{\mathcal{V}}X$.

Note that by construction we have an isomorphism of categories

$$\mathcal{U}(h'_{\mathcal{V}}X) \simeq h\tilde{U}(X)$$

**Proposition 2.3.25.** *Assume that $U : \mathcal{V} \to \mathrm{Set}$ is faithful and preserves and reflects reflexive coequalizers. The assignment $X \mapsto h'_{\mathcal{V}}X$ of Construction 2.3.24 extends to a functor $h'_{\mathcal{V}}$ from the full subcategory of $S_{\otimes}\mathcal{V}$ spanned by all quasi-categories in $\mathcal{V}$ to $\mathcal{V}\,\mathrm{Cat}$, which is left-adjoint to the templicial nerve functor $N_{\mathcal{V}}$.*

*In particular, there exists a canonical isomorphism of $\mathcal{V}$-enriched categories:*

$$h_{\mathcal{V}}X \simeq h'_{\mathcal{V}}X$$

*for every quasi-category $X$ in $\mathcal{V}$.*

*Proof.* We'll show this similarly to the proof of Proposition 2.3.14, using Lemma 2.3.2. By Construction 2.3.24, the appropriate diagrams commute so that we have a unique templicial morphism $\eta_X : X_1 \to N_{\mathcal{V}}(h'_{\mathcal{V}}X)$ such that $\eta_{X_1} : X_1 \to h'_{\mathcal{V}}X$ is precisely $q$. We claim that $\eta_X$ is the unit of an adjunction $h'_{\mathcal{V}} \dashv N_{\mathcal{V}}$.

Now let $\mathcal{C}$ be an arbitrary small $\mathcal{V}$-category and $(\zeta, f) : X \to N_{\mathcal{V}}(\mathcal{C})$ a templicial morphism. By Lemma 2.3.2, $\zeta$ corresponds to a quiver morphism $H : X_1 \to f^*(\mathcal{C})$ such that the diagrams (2.11) commute. Letting $Q$ denote the quiver given by $\mathrm{Hom}^L_X(a,b)_1$ for all objects $a$ and $b$ of $X$, we have a commutative diagram

$$\begin{tikzcd}
Q \arrow[r, "\pi_2"'] \arrow[rr, bend left, "d_0"] \arrow[d, "d_1=\pi_1"'] & X_2 \arrow[r, "d_1^X"'] \arrow[d, "\mu^X_{1,1}"] & X_1 \arrow[r, "H"] & f^*(\mathcal{C}) \\
X_1 \arrow[r, "-\otimes s_0^X"] \arrow[rr, bend right, "H\otimes u"'] & X_1^{\otimes 2} \arrow[r, "H^{\otimes 2}"] & f^*(\mathcal{C})^{\otimes 2} \arrow[r] & f^*(\mathcal{C}^{\otimes 2}) \arrow[u, "f^*(\tilde{m}_{\mathcal{C}})"']
\end{tikzcd}$$

It follows that $Hd_0 = Hd_1 : Q \to f^*(\mathcal{C})$ and thus there exists a unique quiver morphism $H' : h'_{\mathcal{V}}X \to f^*(\mathcal{C})$ such that $H'q = H$. By construction, $H'$ defines a $\mathcal{V}$-functor $h'_{\mathcal{V}}X \to \mathcal{C}$ which is clearly unique such that $N_{\mathcal{V}}(H) \circ \eta_X = (\zeta, f)$. $\square$

**Corollary 2.3.26.** *Assume that $U : \mathcal{V} \to \mathrm{Set}$ is faithful and preserves and reflects reflexive coequalizers. Let $X$ be a quasi-category in $\mathcal{V}$. The canonical functor*

$$h\tilde{U}(X) \to \mathcal{U}(h_{\mathcal{V}}X)$$

*is an isomorphism of categories.*

*Proof.* This is now an immediate consequence of Proposition 2.3.25 and the fact that $\mathcal{U}(h'_{\mathcal{V}}X) \simeq h\tilde{U}(X)$. $\square$

# Chapter 3

# Categorical properties of templicial objects

*"What the hell is going on?! What do you know about these things?"*

— Captain Robert Witterel (Return of the Obra Dinn)

Now that the basic definitions are covered, we are ready to discuss some properties of $S_{\otimes}\mathcal{V}$ as a category. This chapter is divided into two main sections. In both sections, but more prominently in the second, Dugger and Spivak's necklaces (see §2.2.1) will play an essential role.

In Section 3.1 we define and study *free* and *projective* templicial objects and morphisms (Definitions 3.1.6 and 3.1.24). They are based on, and behave similarly to, the free and projective morphisms discussed in §1.2.2. As such, the projective templicial morphisms appear as the left lifting class in a weak factorization system on $S_{\otimes}\mathcal{V}$ (Theorem 3.1.28), where we call the morphisms in the right lifting class *contractible* (Definition 3.1.18). In case $\mathcal{V} = \mathrm{Set}$, then the classes of projective and free templicial morphisms both coincide with the class of monomorphisms of simplicial sets, while the contractible templicial morphisms coincide with the trivial fibrations. As such, every simplicial set is free (Corollary 3.1.9). The concept of free templicial objects thus only becomes meaningful for other choices of $\mathcal{V}$. Classically, trivial fibrations are characterized as those simplicial maps having the right lifting property with respect to all boundary inclusions. A similar characterization can be shown for contractible templicial morphisms, but this requires the use of necklaces. More precisely, a templicial morphism is contractible if and only if the induced morphisms in $\mathcal{V}^{\mathcal{N}ec^{op}}$ under Construction 2.2.16 have the right lifting property with respect to all boundary inclusions (Proposition 3.1.19). Finally, we will explain how free templicial objects allow for a notion of non-degenerate simplices, which general templicial objects lack.

Next, Section 3.2 introduces *necklace categories*, which are categories enriched in the category $\mathcal{V}^{\mathcal{N}ec^{op}}$, considered as a monoidal category with the Day convolution (Construction 3.2.1). We then extend Construction 2.2.16 to a fully faithful left-adjoint $(-)^{nec} : S_{\otimes}\mathcal{V} \hookrightarrow \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ (Construction 3.2.5), where $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ denotes the category of small necklace categories. The rest of the section is devoted to showing how passing to necklace categories can simplify proofs for templicial objects.

## 3.1 Freeness and projectivity

This section is heavily inspired by the theory of Reedy categories. For more details on this, we refer to the literature (e.g. [RV14] or [Hir03, Chapter 15]).

Recall that the simplex category $\mathbf{\Delta}$ is a Reedy category which allows to inductively build up a simplicial set $X$ through its skeleta $\mathrm{sk}_n(X)$. We adapted this approach to define skeleta for templicial objects in §2.1.5. Further, one can use the Reedy structure to lift the weak factorization system (injective, surjective) on $\mathrm{Set}$ to a weak factorization system on $\mathrm{SSet}$ where the left lifting class consists of the monomorphisms and the right lifting class consists of the trivial fibrations (Definition 1.3.16). We can adapt this approach as well, starting from the weak factorization system (projective, regular epimorphic) on $\mathcal{V}$ (see Proposition 1.2.20.4). This will result in a weak factorization system on $S_{\otimes}\mathcal{V}$ where the templicial morphisms in the left lifting class will be called *projective* and those in the right lifting class *contractible*. Beware that we are not constructing a Reedy model structure on $S_{\otimes}\mathcal{V}$. We are simply using Reedy techniques to lift a single weak factorization system from $\mathcal{V}$ to $S_{\otimes}\mathcal{V}$.

In [Bac12, Definition 6.1], Bacard introduced *locally Reedy* 2*-categories*, that is, a category enriched in Reedy categories. Further, in [Bac13] they defined latching and matching objects for colax functors $\mathcal{R} \to \mathcal{M}$ where $\mathcal{R}$ is a locally Reedy 2-category and $\mathcal{M}$ is a 2-category. The category of finite intervals $\mathbf{\Delta}_f$ is well known to be Reedy, but it is also a monoidal Reedy category (i.e. a locally Reedy 2-category with one object). In Definitions 3.1.1 and 3.1.14 we define latching and matching objects for a given templicial object $(X, S)$. Although defined slightly differently, they ultimately coincide with those of Bacard, applied to the colax monoidal functor $X : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$. Constructing the weak factorization system on $S_{\otimes}\mathcal{V}$ now follows completely analogously to the case for classical Reedy categories. Because we still have to deal with the base change of the sets $S$, and to make this section more self-contained, we'll still provide all proofs in full. But it is important to note that all Reedy-type proofs (that is Remark 3.1.21, Proposition 3.1.22 and Theorem 3.1.28) were essentially already shown in [Bac12] and [Bac13].

We also introduce *free* templicial morphisms which occupy a slightly smaller class than the projective ones. In fact, every templicial morphism can be factored as $\beta\alpha$ where $\beta$ is contractible and $\alpha$ is free, not just projective (Proposition 3.1.22). Free templicial objects also provide the right context to talk about non-degenerate simplices, which is impossible for general templicial objects. We then also obtain an analogue of the classical Eilenber-Zilber lemma (Lemma 3.1.39).

For this section we impose the additional standing hypotheses that the forgetful functor $U = \mathcal{V}(I, -) : \mathcal{V} \to \mathrm{Set}$ preserves and reflects regular epimorphisms. Note that then in particular the monoidal unit $I$ of $\mathcal{V}$ is a projective object in the sense of Definition 1.2.14. Thus we may apply Proposition 1.2.20.

### 3.1.1 Free templicial morphisms

We will first focus on free and contractible templicial morphisms (see Definitions 3.1.6 and 3.1.18). For both, we give equivalent characterizations in Propositions 3.1.7 and

3.1.19. Further, Proposition 3.1.22 shows that every templicial morphism may be factored as a free templicial morphism followed by a contractible one. Free and contractible morphisms do not form a weak factorization system on $S_{\otimes}\mathcal{V}$ however as the free templicial morphisms are not closed under retracts. Taking their closure under retracts yields the projective templicial morphisms, whose discussion we postpone to the next subsection.

Recall the category $\mathbf{\Delta}_{surj} \subseteq \mathbf{\Delta}_f$ consisting of all surjective morphisms in $\mathbf{\Delta}_f$ (see Definition 2.1.1).

**Definition 3.1.1.** Let $(X, S)$ be a templicial object. For every $n > 0$, we define the *nth latching object* of $X$ as the following colimit in $\mathcal{V}\,\mathrm{Quiv}_S$:

$$X_n^{deg} = L_n X = \operatorname*{colim}_{\substack{\sigma:[n]\twoheadrightarrow[k]\\ 0\leq k<n}} X_k$$

where the colimit is taken over the full subcategory of $\big((\mathbf{\Delta}_{surj})_{[n]/}\big)^{op}$ spanned by all morphisms $\sigma : [n] \twoheadrightarrow [k]$ in $\mathbf{\Delta}_{surj}$ with $0 \leq k < n$. Note that we have a canonical quiver morphism

$$X_n^{deg} \to X_n$$

For any $a, b \in S$, we may also refer to $X_n^{deg}(a, b) \in \mathcal{V}$ as the *object of degenerate $n$-simplices* of $X$ with first vertex $a$ and last vertex $b$.

*Remark* 3.1.2. Note that in view of Construction 2.1.37, we have

$$X_n^{deg} \simeq \mathrm{sk}_{n-1}(X)_n$$

and thus the $n$th latching object of $X$ only depends on the skeleton $\mathrm{sk}_{n-1}(X)$. In fact, $X_n^{deg}$ only depends on the functor $X|_{(\mathbf{\Delta}_{surj}^{<n})^{op}} : (\mathbf{\Delta}_{surj}^{<n})^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$ where $\mathbf{\Delta}_{surj}^{<n}$ is the full subcategory of $\mathbf{\Delta}_{surj}$ spanned by all the objects $[k]$ with $k < n$.

The following is well-known.

**Example 3.1.3.** In case $\mathcal{V} = \mathrm{Set}$, let $K$ be a simplicial set and $n > 0$. Take $a, b \in K_0$ and $x \in K_k(a, b)$, $y \in K_l(a, b)$ along with morphisms $\sigma : [n] \twoheadrightarrow [k]$ and $\tau : [n] \twoheadrightarrow [l]$ in $\mathbf{\Delta}_{surj}$ such that $K(\sigma)(x) = K(\tau)(y)$. By the Eilenberg-Zilber lemma (Lemma 1.3.7), there exist morphisms $\sigma' : [k] \twoheadrightarrow [m]$ and $\tau' : [l] \twoheadrightarrow [m]$ in $\mathbf{\Delta}_{surj}$ and a non-degenerate $m$-simplex $z$ of $K$ such that $x = K(\sigma')(z)$, $y = K(\tau')(z)$ and $\sigma'\sigma = \tau'\tau$. Therefore $x$ and $y$ represent the same element in the colimit $K_n^{deg}(a, b) = \mathrm{colim}_{\sigma:[n]\twoheadrightarrow[k],k<n} K_k(a, b)$. Hence, the quiver map

$$K_n^{deg} \to K_n$$

is a monomorphism and thus projective in $\mathrm{Quiv}_S$. In fact, the set $\coprod_{a,b\in K_0} K_n^{deg}(a, b)$ can be identified with the set of degenerate $n$-simplices of $K$.

An important distinction between simplicial sets and templicial objects is that the canonical quiver morphism

$$X_n^{deg} \to X_n$$

need not be projective in general as Example 3.1.4 shows.

Intuitively, we might interpret the quiver $X_n^{deg}$ only as a prototype expressing how the degenerate simplices of $X$ "should" behave. For example:

- If $n = 1$, then $X_1^{deg} \simeq X_0$. Intuitively, this means that $X$ should have a unique degenerate 1-simplex for each 0-simplex.
- If $n = 2$, then $X_2^{deg}$ is given by the cokernel pair $X_1 \amalg_{X_0} X_1$ of $s_0 : X_0 \to X_1$. Intuitively, this means that $X$ should have two degenerate 2-simplices for each 1-simplex, which coincide precisely if the 1-simplex was already degenerate.

**Example 3.1.4.** Consider the ring $\mathbb{Z}/2\mathbb{Z}$ as a one object Ab-enriched category and take its templicial nerve $X = N_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) \in S_{\otimes}\,\mathrm{Ab}$ (Construction 2.3.1). Then the canonical map $X_1^{deg} \simeq X_0 \to X_1$ is given by the quotient map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$, which is clearly not projective.

We will isolate the templicial objects for which the degenerate simplices are well-behaved as those for which the canonical quiver morphism $X_n^{deg} \to X_n$ is free (Definition 3.1.6) or more broadly, projective (Definition 3.1.24).

**Definition 3.1.5.** Let $(\alpha, f) : (X, S) \to (Y, T)$ be a templicial morphism. As $f_!$ is a left-adjoint and thus preserves colimits (Construction 1.1.16), we have a canonical quiver morphism for every $n > 0$:

$$f_!(X_n^{deg}) \to Y_n^{deg}$$

We define the *nth relative latching morphism* of $(\alpha, f)$ as the induced quiver morphism

$$Y_n^{deg} \amalg_{f_!(X_n^{deg})} f_!(X_n) \to Y_n$$

**Definition 3.1.6.** We call a templicial morphism $(\alpha, f) : (X, S) \to (Y, T)$ *free* if

(a) the map $f : S \to T$ is injective, and

(b) the $n$th relative latching morphism $Y_n^{deg} \amalg_{f_!(X_n^{deg})} f_!(X_n) \to Y_n$ is free in $\mathcal{V}\,\mathrm{Quiv}_T$ (in the sense of Remark 1.2.21) for all $n > 0$.

In particular, we call a templicial object *free* if the initial morphism $0 \to X$ in $S_{\otimes}\mathcal{V}$ is free. Equivalently, the quiver morphism $X_n^{deg} \to X_n$ is free for all $n > 0$.

**Proposition 3.1.7.** *Let $(\alpha, f) : (X, S) \to (Y, T)$ be a templicial morphism. Then $(\alpha, f)$ is free if and only if*

*(a) the map $f : S \to T$ is injective, and*

*(b) there exists a functor $Z : \mathbf{\Delta}_{surj}^{op} \to \mathrm{Quiv}_T$ such that for all $n > 0$ and $a, b \in T$ the canonical map $Z_n^{deg}(a, b) \to Z_n(a, b)$ is injective, along with an isomorphism $Y|_{\mathbf{\Delta}_{surj}^{op}} \simeq f_!X|_{\mathbf{\Delta}_{surj}^{op}} \amalg FZ$ in $\mathrm{Fun}(\mathbf{\Delta}_{surj}^{op}, \mathcal{V}\,\mathrm{Quiv}_T)$ such that the natural transformation $\alpha : f_!X \to Y$ corresponds to the coprojection $f_!X \to f_!X \amalg FZ$.*

*In particular, the induced morphism $X_n(a, b) \to Y_n(f(a), f(b))$ is free in $\mathcal{V}$ for any $n > 0$ and $a, b \in S$.*

*Proof.* Suppose $(\alpha, f)$ satisfies conditions $(a)$ and $(b)$. As $f_!$ and $F$ preserve colimits, we have for all $n > 0$ that

$$Y_n^{deg} = \operatorname*{colim}_{\substack{\sigma:[n]\twoheadrightarrow[k]\\ 0\leq k<n}} Y_k \simeq \operatorname*{colim}_{\substack{\sigma:[n]\twoheadrightarrow[k]\\ 0\leq k<n}} (f_!X_k \amalg F(Z_k)) \simeq f_!(X_n^{deg}) \amalg F(Z_n^{deg})$$

and therefore

$$Y_n^{deg} \amalg_{f_!(X_n^{deg})} f_!(X_n) \simeq f_!(X_n) \amalg F(Z_n^{deg})$$

Under these isomorphisms, the $n$th relative latching morphism of $(\alpha, f)$ becomes the following quiver morphism induced by the monomorphism $Z_n^{deg} \to Z_n$:

$$f_!(X_n) \amalg F(Z_n^{deg}) \to f_!(X_n) \amalg F(Z_n)$$

which is clearly free.

Conversely suppose that $(\alpha, f)$ is free. Then condition $(a)$ holds by assumption. We construct a functor $Z : \mathbf{\Delta}_{surj}^{op} \to \mathrm{Quiv}_T$ as in condition $(b)$ by induction. First define

$$Z_0(x,y) = \begin{cases} \{*\} & \text{if } x = y \in T \setminus f(S) \\ \emptyset & \text{otherwise} \end{cases}$$

As $f$ is injective, we have a bijection of sets: $T \simeq (T \setminus f(S)) \amalg S$. It follows that $\alpha_0 : f_! X_0 \to Y_0$ is isomorphic to the coprojection $f_!(I_S) \to f_!(I_S) \amalg F(Z_0)$.

Now let $n > 0$ and let $\mathbf{\Delta}_{surj}^{<n}$ denote the full subcategory of $\mathbf{\Delta}_{surj}$ spanned by all objects $[k]$ with $k < n$. Assume we have already defined a functor $Z_{<n} : (\mathbf{\Delta}_{surj}^{<n})^{op} \to \mathrm{Quiv}_T$ such that $Z_m^{deg} \to Z_m$ is a monomorphism for all $0 \leq m < n$, and an isomorphism $Y|_{(\mathbf{\Delta}_{surj}^{<n})^{op}} \simeq f_! X|_{(\mathbf{\Delta}_{surj}^{<n})^{op}} \amalg F Z_{<n}$ in $\mathrm{Fun}((\mathbf{\Delta}_{surj}^{<n})^{op}, \mathrm{Quiv}_T)$ such that the natural transformation $\alpha_{<n} : f_! X \to Y$ corresponds to the coprojection $f_! X \to f_! X \amalg F Z_{<n}$. As $Y_n^{deg}$ is depends only on $Y|_{(\mathbf{\Delta}_{surj}^{<n})^{op}}$ (Remark 3.1.2), we have by the same argument as above that

$$Y_n^{deg} \amalg_{f_!(X_n^{deg})} f_!(X_n) \simeq f_!(X_n) \amalg F(Z_n^{deg})$$

Since the $n$th relative latching morphism of $(\alpha, f)$ is free, it is isomorphic to the coprojection $f_!(X_n) \amalg F(Z_n^{deg}) \to \big(f_!(X_n) \amalg F(Z_n^{deg})\big) \amalg F(Z'_n)$ for some $Z'_n \in \mathrm{Quiv}_S$. Thus setting $Z_n = Z_n^{deg} \amalg Z'_n$, we can extend $Z_{<n}$ to a functor

$$Z_{<n+1} : (\mathbf{\Delta}_{surj}^{<n+1})^{op} \to \mathrm{Quiv}_T$$

such that also $Z_n^{deg} \to Z_n$ is a monomorphism and $\alpha_{<n}$ is isomorphic to the coprojection $f_! X \to f_! X \amalg F Z_{<n+1}$ in $\mathrm{Fun}((\mathbf{\Delta}_{surj}^{<n+1})^{op}, \mathrm{Quiv}_T)$. Finally, the functors $(Z_{<n})_{n>0}$ combine to define a functor $Z : \mathbf{\Delta}_{surj}^{op} \to \mathrm{Quiv}_T$ as in condition $(b)$.

Finally, take $n > 0$ and $a, b \in S$. As $f$ is injective, we have $f_!(X_n)(f(a), f(b)) \simeq X_n(a,b)$. Thus the induced morphism $X_n(a,b) \to Y_n(f(a), f(b))$ is obtained by evaluating $\alpha_n : f_!(X_n) \to f_!(X_n) \amalg F(Z_n)$ in the pair $(f(a), f(b))$ and is therefore free. $\square$

**Corollary 3.1.8.** *A templicial object $(X, S)$ is free if and only if there exists a functor $Z : \mathbf{\Delta}_{surj}^{op} \to \mathrm{Quiv}_S$ such that for all $n > 0$ and $a, b \in S$, the canonical map $Z_n^{deg}(a,b) \to Z_n(a,b)$ is injective, along with an isomorphism $X|_{\mathbf{\Delta}_{surj}^{op}} \simeq FZ$ in $\mathrm{Fun}(\mathbf{\Delta}_{surj}^{op}, \mathcal{V}\,\mathrm{Quiv}_S)$.*

*Proof.* This is an immediate consequence of Proposition 3.1.7. $\square$

**Corollary 3.1.9.** *If $\mathcal{V} = \mathrm{Set}$, then a simplicial map is free if and only if it is a monomorphism. In particular, every simplicial set is free.*

*Proof.* Let $A \to B$ be a simplicial map. If it is either free or a monomorphism, then the map $f : A_0 \to B_0$ is injective. Recall by Example 1.2.19.1 that a morphism in Set is free if and only if it is an injective map of sets.

So if $A \to B$ is free we have by Proposition 3.1.7 that for all $n > 0$ and $a, b \in A_0$ that the map $A_n(a,b) \to B_n(f(a), f(b))$ is injective. Taking the coproduct over all $a, b \in A_0$, we see that $A_n \to B_n$ is an injective map as well.

Conversely, if $A \subseteq B$ is a simplicial subset, we can define a functor

$$B \setminus A : \mathbf{\Delta}^{op}_{surj} \to \mathrm{Quiv}_{B_0} : [n] \mapsto B_n \setminus f_! A_n$$

Then $B|_{\mathbf{\Delta}^{op}_{surj}} \simeq f_! A \amalg (B \setminus A)$ in $\mathrm{Fun}(\mathbf{\Delta}^{op}_{surj}, \mathrm{Quiv}_{B_0})$ and the natural transformation $f_! A \to B$ corresponds to the coprojection $f_! A \to f_! A \amalg (B \setminus A)$. Finally, for all $n > 0$ and $a, b \in B_0$, the map $(B \setminus A)^{deg}_n(a,b) \to (B \setminus A)_n(a,b)$ is a restriction of $B^{deg}_n(a,b) \to B_n(a,b)$ and is therefore injective by Example 3.1.3. $\square$

**Corollary 3.1.10.** *For any monomorphism $A \hookrightarrow B$ of simplicial sets, the induced templicial morphism $\tilde{F}(A) \to \tilde{F}(B)$ is free. In particular, $\tilde{F}(K)$ is a free templicial object for every simplicial set $K$.*

*Proof.* This follows from Corollary 3.1.9 as $F : \mathrm{Set} \to \mathcal{V}$ preserves colimits and free morphisms (see Proposition 1.2.20.1). $\square$

**Example 3.1.11.** Let $X$ be the templicial abelian group of Example 2.1.44. Then $X$ is free but it is not isomorphic to $\tilde{F}(K)$ for any simplicial set $K$. Indeed, like any free templicial object, the degeneracy maps of $X$ preserve the basis elements by Corollary 3.1.8. But the inner face maps of $X$ don't.

We now turn our attention to contractible morphisms of templicial objects. Note that while the latching objects are defined completely analogously as for classical Reedy categories, the matching objects require passing to necklaces via Construction 2.2.16.

**Notation 3.1.12.** Let $(T, n)$ be a necklace. We denote $d(T)$ for the maximal dimension of all beads of $T$. More precisely, if we write $T = \{0 = t_0 < t_1 < ... < t_k = p\}$, then

$$d(T) = \max\{t_i - t_{i-1} \mid 1 \leq i \leq k\}$$

**Notation 3.1.13.** We denote

$$\mathcal{N}ec_{inj}$$

for the subcategory of $\mathcal{N}ec$ consisting of all necklace maps $(T, p) \to (U, q)$ for which the underlying morphism $[p] \to [q]$ in $\mathbf{\Delta}_f$ is injective. Note that $\mathcal{N}ec_{inj}$ contains both the active injective maps as well as the inert maps.

**Definition 3.1.14.** Let $(X, S)$ be a templicial object. For every $n > 0$, we define the *nth matching object* of $X$ as the following limit in $\mathcal{V}\,\mathrm{Quiv}_S$:

$$M_n X = \lim_{\substack{T \hookrightarrow \{0 < n\} \\ d(T) < n}} X_T$$

where the limit is taken over the full subcategory of $((\mathcal{N}ec_{inj})_{/\{0<n\}})^{op}$ spanned by all necklaces maps $T \hookrightarrow \{0 < n\}$ in $\mathcal{N}ec_{inj}$ with $d(T) < n$. Note that we have a canonical quiver morphism

$$X_n \to M_n X$$

*Remark* 3.1.15. Note that the $n$th matching object $M_nX$ only depends on the functors $X_\bullet|_{(\mathcal{N}ec_{inj}^{<n})^{op}} : (\mathcal{N}ec_{inj}^{<n})^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$ (see Construction 2.2.16) where $\mathcal{N}ec_{inj}^{<n}$ denotes the full subcategory of $\mathcal{N}ec_{inj}$ spanned by all necklaces $T$ with $d(T) < n$.

**Example 3.1.16.** Given a templicial object $(X, S)$, let us analyze the matching objects $M_nX$ for low values of $n$.

- If $n = 1$, then $M_1X$ is the terminal object of $\mathcal{V}\,\mathrm{Quiv}_S$.
- If $n = 2$, then $M_2X \simeq X_1 \times (X_1 \otimes_S X_1)$.
- If $n = 3$, then $M_3X$ is the limit of the following diagram of quivers:

$$\begin{tikzcd}
X_1 \otimes_S X_2 \arrow[d, "\mathrm{id} \otimes \mu_{1,1}"'] \arrow[dr, "\mathrm{id} \otimes d_1" near start] & X_2 \otimes_S X_1 \arrow[dl, "\mu_{1,1} \otimes \mathrm{id}"' near start] \arrow[dr, "d_1 \otimes \mathrm{id}" near start] & X_2 \arrow[dl, "\mu_{1,1}"' near start] \arrow[dr, "d_1" near start] & X_2 \arrow[dl, "\mu_{1,1}"' near start] \arrow[d, "d_1"] \\
X_1 \otimes_S X_1 \otimes_S X_1 & X_1 \otimes_S X_1 & X_1 \otimes_S X_1 & X_1
\end{tikzcd}$$

**Definition 3.1.17.** Let $(\alpha, f) : (X, S) \to (Y, T)$ be a templicial morphism. As $f^*$ is a right-adjoint and thus preserves limits (Construction 1.1.16), we have a canonical quiver morphism for every $n > 0$:

$$M_nX \to f^*M_nY$$

We define the *$n$th relative matching morphism* of $(\alpha, f)$ as the induced quiver morphism

$$X_n \to f^*Y_n \times_{f^*M_nY} M_nX$$

**Definition 3.1.18.** We call a templicial morphism $(\alpha, f) : (X, S) \to (Y, T)$ *contractible* if

(a) the map $f : S \to T$ is surjective, and

(b) the $n$th relative matching morphism $X_n \to f^*Y_n \times_{f^*M_nY} M_nX$ is a regular epimorphism in $\mathcal{V}\,\mathrm{Quiv}_S$ for all $n > 0$.

**Proposition 3.1.19.** *Let $(\alpha, f) : (X, S) \to (Y, T)$ be a templicial morphism. Then $(\alpha, f)$ is contractible if and only if*

*(a) the map $f : S \to T$ is surjective, and*

*(b) for all $a, b \in S$, the induced morphism $X_\bullet(a, b) \to Y_\bullet(f(a), f(b))$ has the right lifting property with respect to* Cell *in $\mathcal{V}^{\mathcal{N}ec^{op}}$.*

*Proof.* Take $a, b \in S$ and $n > 0$. As $U : \mathcal{V} \to \mathrm{Set}$ preserves limits, we see that an element of $U(M_nX(a, b))$ is equivalent to a choice of collections $(x_k)_{k=1}^{n-1}$ and $(y_i)_{i=1}^{n-1}$ with $x_k \in U((X_k \otimes_S X_{n-k})(a, b))$ and $y_i \in U(X_{n-1}(a, b))$ satisfying the conditions of Corollary 2.2.22.2 which thus determine a morphism $\tilde{F}(\partial\Delta^n)_\bullet(0, n) \to X_\bullet(a, b)$ in $\mathcal{V}^{\mathcal{N}ec^{op}}$. On the other hand, an element of $U(X_n(a, b))$ is equivalent to a morphism $\tilde{F}(\Delta^n)_\bullet(0, n) \to X_\bullet(a, b)$ by Corollary 2.2.22.1.

Since $U$ preserves and reflects regular epimorphisms, we conclude that the $n$th relative latching morphism $X_n(a,b) \to Y_n(f(a),f(b)) \times_{M_nY(f(a),f(b))} M_nX(a,b)$ is a regular epimorphism in $\mathcal{V}$ if and only if every lifting problem

$$\begin{tikzcd}
\tilde{F}(\partial\Delta^n)_\bullet(0,n) \arrow[r] \arrow[d] & X_\bullet(a,b) \arrow[d] \\
\tilde{F}(\Delta^n)_\bullet(0,n) \arrow[r] \arrow[ur, dashed] & Y_\bullet(f(a),f(b))
\end{tikzcd}$$

in $\mathcal{V}^{\mathcal{N}ec^{op}}$ has a solution. □

**Corollary 3.1.20.** *Let $(\alpha, f) : (X,S) \to (Y,T)$ be a contractible templicial morphism. Then $\tilde{U}(\alpha) : \tilde{U}(X) \to \tilde{U}(Y)$ is a trivial fibration of simplicial sets.*

*Proof.* By hypothesis, $f = \tilde{U}(\alpha)_0 : \tilde{U}(X)_0 \to \tilde{U}(Y)_0$ is surjective. So $\tilde{U}(\alpha)$ has the right lifting property with respect to the simplicial map $\emptyset = \partial\Delta^0 \to \Delta^0$.

Take $n > 0$ and consider a lifting problem in SSet:

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] & \tilde{U}(X) \arrow[d, "\tilde{U}(\alpha)"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \tilde{U}(Y)
\end{tikzcd}$$

Then by Propostion 2.1.26, the top horizontal map is equivalent to a choice of vertices $a_0, ..., a_n \in S$ along with elements

$$\beta_{k,l} \in U(X_{l-k}(a_k, a_l)) \quad \text{and} \quad \gamma_i \in U(X_{n-1}(a_0, a_n))$$

for all $0 \le k < l \le n$ with $(k,l) \neq (0,n)$ and $0 < i < n$, which satisfy

- for all $0 < i < i' < n$,
$$d_i(\gamma_{i'}) = d_{i'-1}(\gamma_i)$$
- for all $0 \le k < i < l \le n$ with $(k,l) \neq (0,n)$,
$$\mu_{k-i,l-k}(\beta_{k,l}) = \beta_{k,i} \otimes \beta_{i,l}$$
- for all $0 < k < n-1$ and $0 < i < n$,
$$\mu_{k,n-k-1}(\gamma_i) = \begin{cases} d_i(\beta_{0,k}) \otimes \beta_{k+1,n} & \text{if } i \le k \\ \beta_{0,k} \otimes d_{i-k}(\beta_{k,n}) & \text{if } i > k \end{cases}$$

Further, the bottom horizontal map making the diagram commute is equivalent to an element $\beta'_{0,n} \in U(Y_n(f(a_0), f(a_n)))$ such that $\mu_{k,n-k}(\beta'_{0,n}) = \alpha_k(\beta_{0,k}) \otimes \alpha_{n-k}(\beta_{k,n})$ and $d_i(\beta'_{0,n}) = \alpha_{n-1}(\gamma_i)$ for all $0 < k, i < n$. Solving the lifting problem then comes down to finding an element $\beta_{0,n} \in U(X_n(a_0, a_n))$ such that $\alpha_n(\beta_{0,n}) = \beta'_{0,n}$ and $\mu_{k,n-k}(\beta_{0,n}) = \beta_{0,k} \otimes \beta_{k,n}$, $d_i(\beta_{0,n}) = \gamma_i$ for all $0 < i < n$.

Now setting $x_k = \beta_{0,k} \otimes \beta_{k,n}$ and $y_i = \gamma_i$ we obtain collections satisfying the conditions Corollary 2.2.22.2, so that the collections $(x_k)_{k=1}^{n-1}$ and $(y_i)_{i=1}^{n-1}$ and the element $\beta'_{0,n}$ define a commutative diagram in $\mathcal{V}^{\mathcal{N}ec^{op}}$:

$$\begin{tikzcd}
\tilde{F}(\partial\Delta^n)_\bullet(0,n) \arrow[r] \arrow[d] & X_\bullet(a,b) \arrow[d] \\
\tilde{F}(\Delta^n)_\bullet(0,n) \arrow[r] \arrow[ur, dashed] & Y_\bullet(f(a),f(b))
\end{tikzcd}$$

which has a lift by Proposition 3.1.19. This lift provides the element $\beta_{0,n}$ as desired. $\square$

*Remark* 3.1.21. Let $X$ be an $(n-1)$-skeletal templicial object for some $n > 0$. Then we have a canonical quiver morphism

$$X_n^{deg} = \mathrm{sk}_{n-1}(X)_n \simeq X_n \to M_n X$$

It follows from the definitions of the latching and matching objects that there is a bijective correspondence between (isomorphism classes of) $n$-skeletal templicial objects $\overline{X}$ with $\mathrm{sk}_{n-1}(\overline{X}) \simeq X$, and (isomorphism classes of) factorizations

$$X_n^{deg} \to \overline{X}_n \to M_n X$$

of the canonical quiver morphism $X_n^{deg} \to M_n X$.

Let $(\alpha, f) : (X,S) \to (Y,T)$ be a morphism between $(n-1)$-skelatal templicial objects and let $\overline{X}$ and $\overline{Y}$ be $n$-skeletal templicial objects such that $\mathrm{sk}_{n-1}(\overline{X}) \simeq X$ and $\mathrm{sk}_n(\overline{Y}) \simeq Y$. Then there is a bijective correspondence between templicial morphisms $\overline{\alpha} : \overline{X} \to \overline{Y}$ and commutative diagrams in $\mathcal{V}\,\mathrm{Quiv}_T$:

$$\begin{tikzcd}
f_! X_n^{deg} \arrow[r] \arrow[d] & f_! \overline{X}_n \arrow[r] \arrow[d, dashed] & f_! M_n X \arrow[d] \\
Y_n^{deg} \arrow[r] & \overline{Y}_{n+1} \arrow[r] & M_n Y
\end{tikzcd}$$

where the top and bottom horizontal rows are the factorizations corresponding to $\overline{X}$ and $\overline{Y}$, and the left and right vertical morphisms are induced by $\alpha$.

**Proposition 3.1.22.** *Every templicial morphism can be factored as a free templicial morphism followed by a contractible templicial morphism.*

*Proof.* Take a templicial morphism $(\alpha, f) : (X,S) \to (Y,T)$. We use induction on $n \geq 0$ to construct factorizations

$$\mathrm{sk}_n(\alpha) : \mathrm{sk}_n(X) \xrightarrow{(\beta_{\leq n}, g)} Z_{\leq n} \xrightarrow{(\gamma_{\leq n}, h)} \mathrm{sk}_n(Y)$$

in $S_\otimes^{\leq n}\mathcal{V}$ such that $g$ is injective, $h$ is surjective and for all $0 < m \leq n$, the $m$th relative latching morphism of $\beta_{\leq n}$ is free and the $m$th relative matching morphism of $\gamma_{\leq n}$ is a regular epimorphism.

If $n = 0$, then $S_\otimes^{\leq 0}\mathcal{V} \simeq \mathrm{Set}$ and we can use Proposition 1.2.20.4 to factor $f$ as an injective map $g : S \hookrightarrow U$ followed by a surjective map $h : U \twoheadrightarrow T$. If $n > 0$, assume that we have already constructed a factorization of $\mathrm{sk}_{n-1}(\alpha)$ as above. Consider the following morphism in $\mathcal{V}\,\mathrm{Quiv}_U$:

$$\psi : Z_n^{deg} \amalg_{g_!(X_n^{deg})} g_!(X_n) \to h^*(Y_n) \times_{h^*(M_n Y)} M_n Z$$

determined by the following quiver morphisms:

- $Z_n^{deg} \to h^*(Y_n)$ is adjoint to the morphism $h_!(Z_n^{deg}) \to Y_n^{deg} \to Y_n$ induced by $\gamma_{\leq n-1}$.
- $g_!(X_n) \to h^*(Y_n)$ is adjoint to the morphism $h_! g_!(X_n) \simeq f_!(X_n) \xrightarrow{\alpha_n} Y_n$.
- $Z_n^{deg} \to M_n Z$ is the canonical morphism of Remark 3.1.21.
- $g_!(X_n) \to M_n Z$ is adjoint to the morphism $X_n \to M_n X \to g^*(M_n Z)$ induced by $\beta_{\leq n-1}$.

Then by Proposition 1.2.20.3 that we can factor $\psi$ as

$$Z_n^{deg} \amalg_{g_!(X_n^{deg})} g_!(X_n) \xrightarrow{\psi_1} Z_n \xrightarrow{\psi_2} h^*(Y_n) \times_{h^*(M_n Y)} M_n Z$$

in $\mathcal{V}\,\mathrm{Quiv}_U$ where $\psi_1$ is free and $\psi_2$ is a regular epimorphism. Consequently, by Remark 3.1.21 we have an object $Z_{\leq n} \in S_{\otimes}^{\leq n}\mathcal{V}$ and morphisms

$$(\beta_{\leq n}, g) : \mathrm{sk}_n(X) \to Z_{\leq n} \quad \text{and} \quad (\gamma_{\leq n}, h) : Z_{\leq n} \to \mathrm{sk}_n(Y)$$

which factor $\mathrm{sk}_n(\alpha)$ as above, such that $\mathrm{sk}_{n-1}(Z_{\leq n}) = Z_{\leq n-1}$, $\mathrm{sk}_{n-1}(\beta_{\leq n}) = \beta_{\leq n-1}$ and $\mathrm{sk}_{n-1}(\gamma_{\leq n}) = \gamma_{\leq n-1}$.

Finally, taking the colimit over $n \geq 0$, we obtain a factorization of $(\alpha, f)$ by Proposition 2.1.42:

$$(\alpha, f) : (X, S) \xrightarrow{(\beta,g)} (Z, U) \xrightarrow{(\gamma,h)} (Y, T)$$

where $Z = \mathrm{colim}_{n\geq 0} Z_{\leq n}$. Moreover, by construction $(\beta, g)$ is free and $(\gamma, h)$ is contractible. $\square$

### 3.1.2 Projective templicial morphisms

Despite Proposition 3.1.22, the classes of free and contractible templicial morphisms do not form a weak factorization system on $S_{\otimes}\mathcal{V}$. Indeed, as one might expect, Example 3.1.23 shows that the class of free templicial morphisms is not closed under retracts and thus it cannot be a left lifting class. Taking the closure under retracts, we obtain the projective templicial morphisms (Definition 3.1.24), and this class does yield a weak factorization system with the contractible templicial morphisms (see Theorem 3.1.28).

Moreover, in Proposition 3.1.32 we show that a projective quasi-category in $\mathcal{V}$ can always be equipped with a naF-structure.

**Example 3.1.23.** The class of free templicial objects (and more generally morphisms) is not closed under retracts. Indeed, consider a unital commutative ring $k$ and any non-free $k$-module $P$ that is a retract of $k$ in $\mathrm{Mod}(k)$ (e.g. $k = \mathbb{Z}/6\mathbb{Z}$ and $P = \mathbb{Z}/2\mathbb{Z}$). Then we have a map $k \twoheadrightarrow P$ so that we can consider $P$ as a unital $\mathrm{Mod}(k)$-enriched quiver with vertex set $S = \{*\}$ (see Example 2.1.35.2). Similarly, we can consider $k$ as a unital quiver with the identity on $k$ as unit. Then $P$ is also a retract of $k$ in $k\,\mathrm{Quiv}_u \simeq S_{\otimes}^{\leq 1}\,\mathrm{Mod}(k)$. Under the embedding $S_{\otimes}^{\leq 1}\,\mathrm{Mod}(k) \hookrightarrow S_{\otimes}\,\mathrm{Mod}(k)$, we can consider $P$ and $k$ as templicial $k$-modules. It is now easy to see that $k$ is free (in fact, it is isomorphic to $\tilde{F}(\Delta^0)$), but $P$ is not.

**Definition 3.1.24.** We call a templicial morphism $(\alpha, f) : (X, S) \to (Y, T)$ *projective* if

(a) the map $f : S \to T$ is injective, and

(b) the $n$th relative latching morphism $Y_n^{deg} \amalg_{f_!(X_n^{deg})} f_!(X_n) \to Y_n$ is projective in $\mathcal{V}\,\mathrm{Quiv}_T$ (in the sense of Remark 1.2.21) for all $n > 0$.

In particular, we call a templicial object $X$ *projective* if the initial morphism $0 \to X$ is projective.

**Example 3.1.25.** If $\mathcal{V} = \mathrm{Set}$, it follows from Example 1.2.19.1 and Corollary 3.1.9 that a simplicial map is projective if and only if it is free, if and only if it is a monomorphism.

**Lemma 3.1.26.** *Every free templicial morphism is projective.*

*Proof.* This is clear from the definitions and Proposition 1.2.20.2. □

**Lemma 3.1.27.** *The classes of projective and contractible templicial morphisms are closed under retracts.*

*Proof.* Let $(\alpha, f)$ be a projective templicial morphism and let $(\beta, g)$ be a retract of $(\alpha, f)$. Then in particular $g$ is a retract of $f$ in $\mathrm{Set}$ and is therefore again injective. Moreover, each relative latching morphism of $(\beta, g)$ is projective as it can be written as a retract of (a base change of) a relative latching morphism of $(\alpha, f)$ (projective morphisms in $\mathcal{V}$ are closed under retracts since they form a left lifting class by Proposition 1.2.20.4). Thus $(\beta, g)$ is also projective.

The proof for contractible templicial morphisms is similar. □

**Theorem 3.1.28.** *The classes of projective and contractible templicial morphisms form a weak factorization system on $S_{\otimes}\mathcal{V}$.*

*Proof.* First note that by Propositions 3.1.22 and 3.1.26, every templicial morphism may be factored as a projective morphism followed by a contractible one.

Let $(\alpha, f) : (A, S) \to (B, T)$ be projective and $(\beta, g) : (X, U) \to (Y, V)$ contractible in $S_\otimes \mathcal{V}$. We wish to solve the following lifting problem in $S_\otimes \mathcal{V}$:

$$\begin{tikzcd}
(A,S) \arrow[r, "(\gamma,h)"] \arrow[d, "(\alpha,f)"'] & (X,U) \arrow[d, "(\beta,g)"] \\
(B,T) \arrow[r, "(\overline{\gamma},\overline{h})"'] \arrow[ur, dashed] & (Y,V)
\end{tikzcd} \tag{3.1}$$

We show by induction that each induced lifting problem in $S_\otimes^{\leq n}\mathcal{V}$

$$\begin{tikzcd}
\mathrm{sk}_n(A) \arrow[r, "\mathrm{sk}_n(\gamma)"] \arrow[d, "\mathrm{sk}_n(\alpha)"'] & \mathrm{sk}_n(X) \arrow[d, "\mathrm{sk}_n(\beta)"] \\
\mathrm{sk}_n(B) \arrow[r, "\mathrm{sk}_n(\overline{\gamma})"'] \arrow[ur, dashed, "\gamma'_{\leq n}"'] & \mathrm{sk}_n(Y)
\end{tikzcd}$$

has a solution for all $n \geq 0$. If $n = 0$, then $S_\otimes^{\leq 0}\mathcal{V} \simeq \mathrm{Set}$ and it follows from Proposition 1.2.20.4 that we have a map of sets $h' : T \to U$ such that $h'f = h$ and $gh' = \overline{h}$. If $n > 0$, assume we have already defined a lift $\gamma'_{\leq n-1} : \mathrm{sk}_{n-1}(B) \to \mathrm{sk}_{n-1}(X)$ as above. Then consider the following lifting problem of quivers:

$$\begin{tikzcd}
h'_!(B_n^{deg}) \amalg_{h_!(A_n^{deg})} h_!(A_n) \arrow[r] \arrow[d] & X_n \arrow[d] \\
h'_!(B_n) \arrow[r] \arrow[ur, dashed] & g^*(Y_n) \times_{g^*(M_nY)} M_nX
\end{tikzcd}$$

where the left vertical morphism is given by applying $h'_!$ to the $n$th relative latching morphism of $(\alpha, f)$ and the right vertical morphism is the $n$th relative matching morphism of $(\beta, g)$. The top and bottom horizontal morphisms are induced by $\gamma$, $\overline{\gamma}$ and $\gamma'_{\leq n-1}$. By hypothesis, the left vertical morphism is projective and the right vertical morphism is a regular epimorphism, so that we have a lift $\gamma'_n : h'_!B_n \to X_n$. Then it follows by Remark 3.1.21 that we have a lift $\gamma'_{\leq n} : \mathrm{sk}_n(B) \to \mathrm{sk}_n(X)$ of the diagram above such that $\mathrm{sk}_{n-1}(\gamma'_{\leq n}) = \gamma'_{\leq n-1}$. Finally, it follows by Proposition 2.1.42 that the original diagram (3.1) has a lift.

Now take a templicial morphism $\alpha$ having the left lifting property with respect to all contractible templicial morphisms. We can factor $\alpha = \gamma\beta$ with $\beta$ projective and $\gamma$ contractible. By the Retract argument (Lemma 1.2.7), $\alpha$ is a retract of $\beta$ and is therefore itself projective by Lemma 3.1.27. A similar argument shows that every templicial morphism having the right lifting property with respect to all projective templicial morphisms is contractible. ☐

**Corollary 3.1.29.** *A templicial morphism is projective if and only if it is a (strong) retract of a free templicial morphism.*

*Proof.* Suppose $\alpha$ is a projective templicial morphism. By Proposition 3.1.22, we can factor $\alpha = \gamma\beta$ in $S_\otimes\mathcal{V}$ with $\beta$ free and $\gamma$ contractible. As $\alpha$ has the left lifting property with respect to $\gamma$ by Theorem 3.1.28, it follows by the Retract argument (Lemma 1.2.7) that $\alpha$ is a retract of $\beta$. Moreover, this retract may always be chosen to be strong.

Conversely, it suffices to note that every free templicial morphism is projective by Proposition 3.1.26 and so are their retracts by Lemma 3.1.27. $\square$

**Corollary 3.1.30.** *If $\mathcal{V} = \mathrm{Set}$, then a simplicial map is contractible if and only if it is a trivial fibration.*

*Proof.* This follows from Remark 1.3.18, Example 3.1.25 and Theorem 3.1.28. $\square$

**Lemma 3.1.31.** *Let $f : A \to B$ and $g : C \to D$ be free (resp. projective) morphisms in $\mathcal{V}$. Then the canonical morphism*

$$f \boxtimes g : (A \otimes D) \amalg_{A\otimes C} (B \otimes C) \to B \otimes D$$

*is free (resp. projective).*

*Proof.* Suppose first that $f$ and $g$ are free. So we have sets $S$ and $T$ such that $f$ is the coprojection $A \to A \amalg F(S)$ and $g$ is the coprojection $C \to C \amalg F(T)$. Then the domain and codomain of $f \boxtimes g$ are respectively

$$(A \otimes (C \amalg F(T))) \amalg_{A\otimes C} ((A \amalg F(S)) \otimes C) \simeq (A \otimes F(T)) \amalg (A \otimes C) \amalg (F(S) \otimes C)$$
$$(A \amalg F(S)) \otimes (B \amalg F(T)) \simeq (A \otimes F(T)) \amalg (A \otimes C) \amalg (F(S) \otimes C) \amalg F(S \times T)$$

and $f \boxtimes g$ is given by the coprojection. Thus $f \boxtimes g$ is free.

Now suppose $f$ and $g$ are projective. Then by Proposition 1.2.20.2, $f$ and $g$ are strong retracts of some free morphisms $f'$ and $g'$ respectively. It follows that $f \boxtimes g$ is a retract of $f' \boxtimes g'$. So by the previous, $f \boxtimes g$ is projective. $\square$

**Proposition 3.1.32.** *Let $(X, S)$ be a projective templicial object. If $X$ is a quasi-category in $\mathcal{V}$, then $X$ has a naF-structure.*

*Proof.* Given $0 < j < n$ let us define

$$M_{j,n}X = \lim_{\substack{f:T\hookrightarrow\{0<n\}\\ d(T)<n\\ f\neq\delta_j}} X_T \in \mathcal{V}\,\mathrm{Quiv}_S$$

where the limit is taken over the full subcategory of $\left((\mathcal{N}ec_{inj})_{/\{0<n\}}\right)^{op}$ spanned by all necklace maps $T \hookrightarrow \{0 < n\}$ in $\mathcal{N}ec_{inj}$ with $d(T) < n$, except the necklace map $\delta_j : \{0 < n-1\} \hookrightarrow \{0 < n\}$. As $U : \mathcal{V} \to \mathrm{Set}$ preserves limits, we see that an element of $U(M_{j,n}X(a,b))$ with $a, b \in S$ may be identified with collections $(x_k)_{k=1}^{n-1}$ and $(y_i)_{l=1,i\neq j}^{n-1}$ satisfying the conditions of Corollary 2.2.22.3. Hence, an element of $U(M_{j,n}X(a,b))$ is equivelant to a morphism $\tilde{F}(\Lambda^n_j)_\bullet(0,n) \to X_\bullet(a,b)$ in $\mathcal{V}^{\mathcal{N}ec^{op}}$. Consequently, the condition that $X$ is a quasi-category in $\mathcal{V}$ is equivalent to the condition that the canonical quiver morphism $X_n \to M_{j,n}X$ is a regular epimorphism (as $U$ preserves and reflects regular epimorphisms).

We define quiver morphisms $Z^{p,q} : X_p \otimes X_q \to X_{p+q}$ by induction on $n = p + q$, for all $p, q \geq 0$. Define $Z^{p,0}$ and $Z^{0,q}$ to be the left and right unit isomorphisms. Now take

$n > 0$ and let $p, q > 0$ be such that $p + q = n$. Then consider the following commutative diagram:

$$\begin{tikzcd}
(X_p^{deg} \otimes X_q) \amalg_{(X_p^{deg} \otimes X_q^{deg})} (X_p \otimes X_q^{deg}) \arrow[r] \arrow[d] & X_n \arrow[d] \\
X_p \otimes X_q \arrow[r] \arrow[ur, dashed] & M_{p,n}X
\end{tikzcd}$$

As $X$ is a projective templicial object, the left vertical morphism is projective by Lemma 3.1.31. The top horizontal morphism is induced by the already defined morphisms $Z^{k,l}$ with $k + l < n$, which is well-defined by the fact that the $Z^{k,l}$ are natural with respect to the degeneracy maps of $X$. The bottom horizontal morphism is determined by the morphisms

$$X_p \otimes X_q \xrightarrow{\xi_k} X_k \otimes X_{n-k} \quad \text{and} \quad X_p \otimes X_q \xrightarrow{\zeta_i} X_{n-1}$$

for all $0 < k, i < n$ with $i \neq p$, where

$$\xi_k = \begin{cases} (Z^{p,k-p} \otimes \mathrm{id}_{X_l})(\mathrm{id}_{X_p} \otimes \mu_{k-p,l}) & \text{if } p \leq k \\ (\mathrm{id}_{X_k} \otimes Z^{p-k,q})(\mu_{k,p-k} \otimes \mathrm{id}_{X_q}) & \text{if } p \geq k \end{cases}, \quad \zeta_i = \begin{cases} Z^{p-1,q}(d_i \otimes \mathrm{id}_{X_q}) & \text{if } i < p \\ Z^{p,q-1}(\mathrm{id}_{X_p} \otimes d_{i-p}) & \text{if } i > p \end{cases}$$

Hence, there exists a lift $Z^{p,q} : X_p \otimes X_q \to X_n$ which by construction is natural with respect to the degeneracy and inner face morphisms of $X$, and satisfies the Frobenius equations (2.9). $\square$

**Example 3.1.33.** Since every simplicial set is projective by Example 3.1.25, it follows from Propositions 2.2.30 and 3.1.32 that every quasi-category has a naF-structure.

The converse to Proposition 3.1.32 is false in general, as Example 3.1.34 shows. However, in Chapter 4 we will see that the converse does hold in case $\mathcal{V} = \mathrm{Mod}(k)$ for a unital commutative ring $k$ (see Theorem 4.2.62).

**Example 3.1.34.** Let $X$ be the simplicial set defined as the colimit of

$$\begin{tikzcd}
 & \Lambda^3_3 \arrow[dl] \arrow[dr] & & \Lambda^3_0 \arrow[dl] \arrow[dr] & \\
\Delta^3 & & \Delta^3 & & \Delta^3
\end{tikzcd}$$

It is the standard 3-simplex $\Delta^3$, whose simplices we will represent by their vertices $[i_0, ..., i_m]$, with two non-degenerate 3-simplices $x$ and $y$ glued on. We have

$$\forall i \in \{0,1,2\} : d_i(x) = [0, ..., \cancel{i}, ..., 3] \quad \text{but } d_3(x) \neq [0,1,2]$$
$$\forall j \in \{1,2,3\} : d_j(y) = [0, ..., \cancel{j}, ..., 3] \quad \text{but } d_0(y) \neq [1,2,3]$$

In $X$, not all horns can be filled. Indeed, since

$$d_0 d_3(x) = d_2([1,2,3]) = [1,2] = d_0([0,1,2]) = d_2 d_0(y),$$
$$d_2 d_3(x) = [0,1] = d_2([0,1,3]) \quad \text{and} \quad d_1 d_0(y) = [1,3] = d_0([0,1,3])$$

the faces $d_3(x)$, $d_0(y)$ and $[0,1,3]$ form a horn $\Lambda^3_1$ in $X$. But there is no 3-simplex in $X$ with these faces.

However, $X$ does have a naF-structure. It suffices to define $Z^{p,q}(a,b)$ on non-degenerate simplices $a$ and $b$. For those contained in $\Delta^3$, define

$$Z^{p,q}([i_0,...,i_p],[i_p,...,i_{p+q}]) = [i_0,...,i_{p+q}]$$

note that this includes all edges of $X$. Further, set

$$Z^{2,1}(d_3(x),[2,3]) = x, \quad \text{and} \quad Z^{1,2}([0,1],d_0(y)) = y$$

It is easy to check that this satisfies the Frobenius equations (2.9).

Proposition 3.1.32 does not hold without assuming projectivity.

**Example 3.1.35.** Let $\mathcal{V} = \mathrm{Mod}(\mathbb{Z}) = \mathrm{Ab}$ and consider the unital $\mathrm{Ab}$-enriched quiver $Q$ with vertex set $S = \{a,b\}$ and

$$Q(x,y) = \begin{cases} \mathbb{Z} & \text{if } x = a, y = b \\ \mathbb{Z}/2\mathbb{Z} & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

The unit of $Q$ is given by the quotient map $q : \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$. Consider $Q$ as a templicial abelian group $X$ via the embedding $S^{\leq 1}_{\otimes}\,\mathrm{Ab} \hookrightarrow S_{\otimes}\,\mathrm{Ab}$. Then $X$ is easily seen to be a quasi-category in $\mathrm{Ab}$, but the canonical map $X^{deg}_1(a,a) \to X_1(a,a)$ is given by $q$ which is not projective. Note that $X$ does not have a naF-structure, because this would require the existence of a map

$$Z^{1,1} : (X_1 \otimes X_1)(a,b) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to X_2(a,b) \simeq \mathbb{Z} \oplus \mathbb{Z}$$

which is a section of $\mu_{1,1} = q \oplus q : \mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

### 3.1.3 Non-degenerate simplices

Unlike the case for simplicial sets, a general templicial object does not have a well-defined notion of non-degenerate simplices. Given a templicial object $X$ and $n \geq 0$, one might expect to have a quiver $N \in \mathcal{V}\,\mathrm{Quiv}_S$ such that the canonical morphism $X^{deg}_n \to X_n$ is isomorphic to the coprojection

$$X^{deg}_n \to X^{deg}_n \amalg N$$

We might then consider $N(a,b)$ as the "object of non-degenerate $n$-simplices of $X$ from $a$ to $b$" where $a$ and $b$ are vertices of $X$. Such a quiver $N$ need not exist however, as Example 3.1.36 shows. But if we restrict to free templicial objects, a choice for $N$ can always be made (Definition 3.1.38).

**Example 3.1.36.** Consider the monoidal category $\mathcal{V} = \mathrm{Mod}(\mathbb{Z}) = \mathrm{Ab}$ of abelian groups. Let $S = \{*\}$ be a singleton and define a functor $X : \mathbf{\Delta}^{op}_f \to \mathrm{Ab}$ by setting $X_n = \mathbb{Z}$ for all $n \geq 0$ with $s_0 : X_0 = \mathbb{Z} \xrightarrow{2\cdot} X_1 = \mathbb{Z}$ and all other face and degeneracy maps given by the identity on $\mathbb{Z}$. Then $X$ is a strongly unital, colax monoidal functor with comultiplication map $\mu_{k,l}$ for $k, l \geq 0$ given by

$$\mu_{k,l} : X_{k+l} = \mathbb{Z} \to X_k \otimes X_l \simeq \mathbb{Z} : z \mapsto \begin{cases} 2z & \text{if } k, l > 0 \\ z & \text{if } k = 0 \text{ or } l = 0 \end{cases}$$

We thus find a templicial abelian group $(X, S)$. Now note that the image of $X_1^{deg} \to X_1$ is given by the submodule $2\mathbb{Z} \subseteq \mathbb{Z}$ which doesn't have a direct complement.

*Remark* 3.1.37. Note that actually we do not use the hypothesis that $U : \mathcal{V} \to \mathrm{Set}$ preserves or reflects regular epimorphisms to obtain the result of Propositoin 3.1.7. Thus the following definition and lemma apply even without this assumption.

**Definition 3.1.38.** Let $(X, S)$ be a free templicial object and let $Z : \mathbf{\Delta}_{surj}^{op} \to \mathrm{Quiv}_S$ be a functor such that $X|_{\mathbf{\Delta}_{surj}^{op}} \simeq FZ$ in $\mathrm{Fun}(\mathbf{\Delta}_{surj}^{op}, \mathcal{V}\,\mathrm{Quiv}_S)$ and $Z_n^{deg}(a, b) \to Z_n(a, b)$ is injective for all $n > 0$ and $a, b \in S$. Then in particular $X_n^{deg} \simeq F(Z_n^{deg})$ for all $n > 0$. We define the *object of non-degenerate $n$-simplices* of $X$ from $a$ to $b$ as

$$X_n^{nd}(a, b) = F(Z_n(a, b) \setminus Z_n^{deg}(a, b)) \in \mathcal{V}$$

This yields a quiver $X_n^{nd} \in \mathcal{V}\,\mathrm{Quiv}_S$ along with an isomorphism

$$X_n \simeq X_n^{deg} \amalg X_n^{nd}$$

which identifies the morphism $X_n^{deg} \to X_n$ with the coprojection $X_n^{deg} \to X_n^{deg} \amalg X_n^{nd}$. Further, we set $X_0^{nd} = X_0$.

Whenever writing $X_n^{nd}$ for a free templicial object $X$, we implicitly assume a functor $Z : \mathbf{\Delta}_{surj}^{op} \to \mathrm{Quiv}_S$ as above has been chosen.

Recall the Eilenberg-Zilber lemma for simplicial sets (Lemma 1.3.7). Equivalently, this lemma states that for every simplicial set $K$ and $n \geq 0$, there exists a bijection

$$K_n \simeq \coprod_{\substack{\sigma : [n] \twoheadrightarrow [k] \\ \text{in } \mathbf{\Delta}_{surj}}} K_k^{nd}$$

where $K_k^{nd} \subseteq K_k$ denotes the subset of non-degenerate $k$-simplices of $K$. We can prove the analogue for templicial objects if we assume that they are free.

**Lemma 3.1.39.** *Let $X$ be a free templicial object and $n \geq 0$. We have an isomorphism of quivers:*

$$X_n \simeq \coprod_{\substack{\sigma : [n] \twoheadrightarrow [k] \\ in\ \mathbf{\Delta}_{surj}}} X_k^{nd}$$

*Proof.* By definition, $X_0 = X_0^{nd}$. Take $n > 0$, then it follows by induction that

$$X_n \simeq X_n^{nd} \amalg X_n^{deg} = X_n^{nd} \amalg \operatorname*{colim}_{\substack{[n] \twoheadrightarrow [k] \\ 0 \leq k < n}} X_k \simeq X_n^{nd} \amalg \operatorname*{colim}_{\substack{[n] \twoheadrightarrow [k] \\ 0 \leq k < n}} \coprod_{\sigma : [k] \twoheadrightarrow [l]} X_l^{nd}$$

$$\simeq X_n^{nd} \amalg \coprod_{\substack{\sigma : [n] \twoheadrightarrow [l] \\ 0 \leq l < n}} \operatorname*{colim}_{\substack{[n] \overset{\sigma_1}{\twoheadrightarrow} [k] \overset{\sigma_2}{\twoheadrightarrow} [l] \\ \sigma = \sigma_2 \sigma_1}} X_l^{nd} \simeq X_n^{nd} \amalg \coprod_{\substack{\sigma : [n] \twoheadrightarrow [l] \\ 0 \leq l < n}} X_l^{nd}$$

The last isomorphism is obtained by noting that the colimit on the left hand side is taken over a category which has a terminal object given by the factorization $[n] \overset{=}{\twoheadrightarrow} [n] \overset{\sigma}{\twoheadrightarrow} [l]$. $\square$

## 3.2 Necklace categories

There is a classical adjunction $\mathfrak{C} : \mathrm{SSet} \leftrightarrows \mathrm{Cat}_{\Delta} : N^{hc}$ between simplicial sets and simplicial categories, originally constructed by Cordier and Porter [CP86]. The composition in a quasi-category is only well-defined, associative and unital up to homotopy, while the composition in a simplicial category is by definition associative and unital "on the nose". One can thus see the functor $\mathfrak{C}$ as a "rigidification" of quasi-categories. We will consider the adjunction $\mathfrak{C} \dashv N^{hc}$ in more detail in Chapter 4 when we generalize it to an adjunction $\mathfrak{C}_{\mathcal{V}} \dashv N^{hc}_{\mathcal{V}}$ between templicial objects and categories enriched in simplicial objects $S\mathcal{V}$ (see §4.1.2).

This section introduces a functor $(-)^{nec} : S_{\otimes}\mathcal{V} \hookrightarrow \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ from templicial objects (and thus simplicial sets if $\mathcal{V} = \mathrm{Set}$) to a category of small enriched categories $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ which we call *necklace categories*. Like $\mathfrak{C}$, the functor $(-)^{nec}$ can thus also be seen as a type of rigidification. The functor $\mathfrak{C}$ is not a full embedding but we will see that $(-)^{nec}$ is fully faithful in Proposition 3.2.6. So we can interpret $(-)^{nec}$ as a rigidification which does not lose any information. Moreover, the functor $\mathfrak{C}_{\mathcal{V}}$ above will factor through $(-)^{nec}$.

Passing to necklace categories makes a lot of facts about templicial objects easier to prove. Many constructions, like the templicial nerve $N_{\mathcal{V}}$ and the functor $\tilde{U}$ actually factor through $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ (Propositions 3.2.11 and 3.2.14). Moreover, we will identify conditions on a necklace category $\mathcal{C}$ so that its associated templicial object $\mathcal{C}^{temp}$ is a quasi-category in $\mathcal{V}$ or has a Frobenius structure (Proposition 3.2.20 and Corollary 3.2.22). Finally, we use $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ to show that $S_{\otimes}\mathcal{V}$ is locally presentable (Theorem 3.2.29) and we briefly discuss its limits.

### 3.2.1 Coreflective embedding

We open the section by equipping the functor category $\mathcal{V}^{\mathcal{N}ec^{op}}$ with the (non-symmetric) monoidal structure of the Day convolution (Construction 3.2.1). Necklace categories are then defined as categories enriched in $\mathcal{V}^{\mathcal{N}ec^{op}}$. We continue by constructing the full embedding $(-)^{nec}$ (Construction 3.2.5) and showing that it is coreflective in the sense that it has a right-adjoint $(-)^{temp} : \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec} \to S_{\otimes}\mathcal{V}$. The functor $(-)^{temp}$ can be described relatively explicitly by induction on the dimension (Construction 3.2.8).

**Construction 3.2.1.** Consider the category $\mathcal{V}^{\mathcal{N}ec^{op}}$ of functors $\mathcal{N}ec^{op} \to \mathcal{V}$. As $\mathcal{N}ec^{op}$ and $\mathcal{V}$ are both monoidal categories, we can endow $\mathcal{V}^{\mathcal{N}ec^{op}}$ with the monoidal structure given by Day convolution (see [Day70]). We denote the resulting monoidal category by $(\mathcal{V}^{\mathcal{N}ec^{op}}, \otimes_{Day}, \underline{I})$.

Given two functors $X, Y : \mathcal{N}ec^{op} \to \mathcal{V}$, their Day convolution $X \otimes_{Day} Y$ is obtained by the left Kan extension of the composite

$$\mathcal{N}ec^{op} \times \mathcal{N}ec^{op} \xrightarrow{X \times Y} \mathcal{V} \times \mathcal{V} \xrightarrow{-\otimes-} \mathcal{V}$$

along $\vee : \mathcal{N}ec^{op} \times \mathcal{N}ec^{op} \to \mathcal{N}ec^{op}$:

$$X \otimes_{Day} Y = \mathrm{Lan}_{\vee}(X(-) \otimes Y(-))$$

Further, the monoidal unit of $\mathcal{V}^{\mathcal{N}ec^{op}}$ is given by the representable functor on the monoidal unit $\{0\}$ of $\mathcal{N}ec$. As $\{0\}$ is also the terminal object of $\mathcal{N}ec$, we find that $F(\mathcal{N}ec(-,\{0\})) \simeq \underline{I}$ is the constant functor on $I$, the monoidal unit of $\mathcal{V}$.

Beware that the monoidal category $(\mathcal{V}^{\mathcal{N}ec^{op}}, \otimes_{Day}, \underline{I})$ is not symmetric.

**Proposition 3.2.2.** *Let $X$ and $Y$ be functors $\mathcal{N}ec^{op} \to \mathcal{V}$ and $T$ a necklace. Then there is a reflexive coequalizer*

$$\coprod_{\substack{U,N,V \in \mathcal{N}ec \\ U \vee N \vee V = T}} X_U \otimes Y_V \overset{\alpha}{\underset{\beta}{\leftleftarrows\!\!\!\rightrightarrows}} \coprod_{\substack{U,V \in \mathcal{N}ec \\ U \vee V = T}} X_U \otimes Y_V \twoheadrightarrow (X \otimes_{Day} Y)_T \tag{3.2}$$

*where $\alpha$ and $\beta$ are given by, for all $U, N, V \in \mathcal{N}ec$ with $U \vee N \vee V = T$:*

$$\begin{aligned} \alpha\iota_{U,N,V} &= \iota_{U\vee N,V}(X(\mathrm{id}_U \vee \sigma_N) \otimes \mathrm{id}_{Y_V}) \\ \beta\iota_{U,N,V} &= \iota_{U,N\vee V}(\mathrm{id}_{X_U} \otimes Y(\sigma_N \vee \mathrm{id}_V)) \end{aligned}$$

*with $\sigma_N : N \to \{0\}$ the terminal necklace map.*

*Proof.* By Construction 3.2.1, we have a (reflexive) coequalizer diagram:

$$\coprod_{\substack{h:T\to U\vee V \\ f:U\to U' \\ g:V\to V'}} X_{U'} \otimes Y_{V'} \overset{\alpha'}{\underset{\beta'}{\rightrightarrows}} \coprod_{\substack{T \xrightarrow{h} U\vee V \\ \text{in } \mathcal{N}ec}} X_U \otimes Y_V \overset{q'}{\twoheadrightarrow} (X \otimes_{Day} Y)_T$$

where $\alpha'\iota_{h,f,g} = \iota_h(X(f) \otimes Y(g))$ and $\beta'\iota_{h,f,g} = \iota_{(f\vee g)h}$ for all $h : T \to U \vee V$ and $f : U \to U'$, $g : V \to V'$ in $\mathcal{N}ec$.

On the other hand, let $q : \coprod_{U\vee V=T} X_U \otimes Y_V \twoheadrightarrow C$ denote the coequalizer of $\alpha$ and $\beta$ in (3.2). Consider the morphism $\varphi : \coprod_{U\vee V=T} X_U \otimes Y_V \to \coprod_{T\to U\vee V} X_U \otimes Y_V$ defined by sending a pair of necklaces $(U, V)$ with $U \vee V = T$ to the identity on $T$. Then it easy to see that $\varphi$ decends to a morphism

$$\overline{\varphi} : C \to (X \otimes_{Day} Y)_T$$

Conversely take necklaces $(U, p)$, $(V, q)$ and $(T, n)$, and a necklace map $h : T \to U \vee V$. Let $k \in T$ be minimal such that $h(k) = p$. This minimum exists as $U \vee V \subseteq h(T)$. As a morphism in $\mathbf{\Delta}_f$, $h : [n] \to [p + q]$ has a unique representation as $h = h_U + h_V$ with $h_U : [k] \to [p]$ and $h_V : [n - k] \to [q]$ in $\mathbf{\Delta}_f$. Now set $U' = \{t \mid t \in T, t \leq k\}$ and $V' = \{t - k \mid t \in T, k \leq t\}$. Then $(U', k)$ and $(V', n - k)$ are necklaces such that $T = U' \vee V'$. Moreover, we have induced necklace maps $h_U : U' \to U$ and $h_V : V' \to V$ such that $h = h_U \vee h_V$. Define a morphism

$$\psi : \coprod_{\substack{T \xrightarrow{h} U\vee V \\ \text{in } \mathcal{N}ec}} X_U \otimes Y_V \to \coprod_{\substack{U,V \in \mathcal{N}ec \\ U\vee V=T}} X_U \otimes Y_V$$

by setting $\psi\iota_h = \iota_{U',V'}(X(h_U) \otimes Y(h_V))$. It follows that $\psi$ descends to a morphism

$$\overline{\psi} : (X \otimes_{Day} Y)_T \to C$$

Indeed, take necklace maps $h_1 : T \to U_1 \vee V_1$ and $f : U_1 \to U_2$, $g : V_1 \to V_2$, and set $h_2 = (f \vee g)h_1$. As above, we have $(T, n) = (U_i', k_i) \vee (V_i', n - k_i)$ and $h_i = h_{U_i} \vee h_{V_i}$ with $h_{U_i} : U_i' \to U_i$ and $h_{V_i} : V_i' \to V_i$, for $i \in \{1, 2\}$. Observe that

$$q\psi\alpha'\iota_{h_1,f,g} = q\psi\iota_{h_1}(X(f) \otimes Y(g)) = q\iota_{U_1',V_1'}(X(fh_{U_1}) \otimes Y(gh_{V_1}))$$

By the minimality of $k_2 \in T$, it follows that $U_1' = U_2' \vee N$, $V_2' = N \vee V_1'$ and $fh_{U_1} = h_{U_2} \vee \sigma_N$, $h_{V_2} = \sigma_N \vee gh_{V_1}$ for some necklace $N$ and $\sigma_N : N \to \{0\}$. Thus as $q\alpha = q\beta$, we have

$$q\iota_{U_1',V_1'}(X(fh_{U_1}) \otimes Y(gh_{V_1})) = q\iota_{U_2',V_2'}(X(h_{U_2}) \otimes Y(h_{V_2})) = q\psi\beta'\iota_{h_1,f,g}$$

Hence, $q\psi\alpha' = q\psi\beta'$ and thus there exists a unique morphism $\overline{\psi}$ such that $q\psi = \overline{\psi}q'$. It easily follows that $\overline{\psi}$ is inverse to $\overline{\varphi}$.

Finally, we can define a morphism $\gamma : \coprod_{U\vee V=T} X_U \otimes Y_V \to \coprod_{U\vee N\vee V=T} X_U \otimes Y_V$ by setting $\gamma\iota_{U,V} = \iota_{U,\{0\},V}$. Then clearly $\alpha\gamma = \mathrm{id} = \beta\gamma$ so that the coequalizer (3.2) is reflexive. □

**Definition 3.2.3.** Consider the category

$$\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec} = \mathcal{V}^{\mathcal{N}ec^{op}}\text{-}\,\mathrm{Cat}$$

of small categories enriched in the monoidal category $(\mathcal{V}^{\mathcal{N}ec^{op}}, \otimes_{Day}, \underline{I})$ of Construction 3.2.1. We call the objects of $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ *necklace categories* and its morphisms *necklace functors*.

If $\mathcal{V} = \mathrm{Set}$, we simply write $\mathrm{Cat}_{\mathcal{N}ec}$ for $\mathrm{Set}\,\mathrm{Cat}_{\mathcal{N}ec}$.

**Construction 3.2.4.** Let $(X, S)$ be a templicial object and consider the strong monoidal functor of Construction 2.2.16:

$$X_\bullet : \mathcal{N}ec^{op} \to \mathrm{Quiv}_S(\mathcal{V})$$

We construct a necklace category $X^{nec}$ with object set $S$ and hom-objects given by the functors $X_\bullet(a, b) : \mathcal{N}ec^{op} \to \mathcal{V}$ for all $a, b \in S$.

Take $a, b, c \in S$, then for any necklaces $U$ and $V$, we have a canonical morphism

$$\tilde{m}_{U,V} : X_U(a, b) \otimes X_V(b, c) \to (X_U \otimes_S X_V)(a, c) \simeq X_{U\vee V}(a, c)$$

By the coequalizer diagram (3.2), it follows that we have an induced morphism in $\mathcal{V}^{nec^{op}}$:

$$\tilde{m}_{a,b,c} : X_\bullet(a, b) \otimes_{Day} X_\bullet(b, c) \to X_\bullet(a, c)$$

Further, note that $X_{\{0\}}(a, a) = X_0(a, a) \simeq I$ for all $a \in S$ and thus we have an induced morphism

$$u : \underline{I} \to X_\bullet(a, a)$$

It is easy to check that $\tilde{m}$ is associative and unital with respect to $u$ so that we obtain a $\mathcal{V}^{\mathcal{N}ec^{op}}$-enriched category $X^{nec}$.

**Construction 3.2.5.** Let $(\alpha, f) : (X, S) \to (Y, T)$ be a templicial morphism, we define a necklace functor $\alpha^{nec} : X^{nec} \to Y^{nec}$ as follows. On objects it is given by the map $f : S \to T$. Further, for any $a, b \in S$, we have a morphism in $\mathcal{V}^{\mathcal{N}ec^{op}}$:

$$\alpha_{a,b} : X_\bullet(a, b) \to Y_\bullet(f(a), f(b))$$

which is given by, for all necklaces $U = \{0 = u_0 < u_1 < ... < u_k = p\}$:

$$(\alpha_{a,b})_U : X_U(a,b) \xrightarrow{\alpha_{u_1}\otimes ...\otimes \alpha_{p-u_{k-1}}} Y_U(f(a), f(b))$$

where we identified $\alpha_n : f_! X_n \to Y_n$ with its adjoint $X_n \to f^* Y_n$ for all $n \geq 0$ (see Construction 1.1.16). The compatibility of $\alpha_{a,b}$ with inert morphisms in $\mathcal{N}ec$ follows from the monoidality of $\alpha$ and the compatibility with active morphisms follows from the naturality of $\alpha$. It is easy to verify that this defines a $\mathcal{V}^{\mathcal{N}ec^{op}}$-enriched functor $\alpha^{nec}$ : $X^{nec} \to Y^{nec}$.

Finally, we clearly obtain a functor

$$(-)^{nec} : S_\otimes \mathcal{V} \to \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$$

**Proposition 3.2.6.** *The functor $(-)^{nec} : S_\otimes \mathcal{V} \to \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ is fully faithful.*

*Proof.* Take templicial objects $(X, S)$ and $(Y, T)$ and a necklace functor $H : X^{nec} \to Y^{nec}$. Let $f : S \to T$ be the object map of the functor $H$. For $n \geq 0$ we define

$$\alpha_n : f_!(X_n) = f_!(X_{\{0<n\}}) \to Y_{\{0<n\}} = Y_n$$

to be adjoint to the quiver morphism $H_{\{0<n\}} : X_{\{0<n\}} \to f^*(Y_{\{0<n\}})$. As $H$ is compatible with active morphisms, $\alpha_n$ is natural in $n$. Let $U = \{0 = u_0 < u_1 < ... < u_k = p\}$ be a necklace. It follows from Construction 3.2.4 that the quiver morphism $H_U : X_U \to Y_U$ is equal to $H_{\{0<u_1\}} \otimes ... \otimes H_{\{0<p-u_{k-1}\}}$ and thus $\alpha$ is monoidal and $H = \alpha^{nec}$. Clearly, $(\alpha, f)$ is also unique with this property. $\square$

*Remark* 3.2.7. Let $S$ be a set. As $\mathcal{V}\,\mathrm{Quiv}_S$ is isomorphic to the category $\mathcal{V}^{S\times S}$, we have a canonical equivalence of categories $\mathcal{V}^{\mathcal{N}ec^{op}}\,\mathrm{Quiv}_S \simeq \mathcal{V}\,\mathrm{Quiv}_S^{\mathcal{N}ec^{op}}$. Equipping $\mathcal{V}\,\mathrm{Quiv}_S^{\mathcal{N}ec^{op}}$ with the Day convolution as well, this equivalence extends to an equivalence of monoidal categories.

It is well known (see for example [MMSS01, Proposition 22.1]) that monoids in a category of functors equipped with the Day convolution are equivalent to lax monoidal functors. In our situation, this amounts to an equivalence of categories:

$$\mathrm{Mon}(\mathcal{V}^{\mathcal{N}ec^{op}}\,\mathrm{Quiv}_S) \simeq \mathrm{Mon}(\mathcal{V}\,\mathrm{Quiv}_S^{\mathcal{N}ec^{op}}) \simeq \mathrm{Lax}(\mathcal{N}ec^{op}, \mathcal{V}\,\mathrm{Quiv}_S)$$

letting $S \in \mathrm{Set}$ vary and applying the Grothendieck construction on both sides, we obtain an equivalence of categories

$$\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec} \simeq \int_{S\in\mathrm{Set}} \mathrm{Lax}(\mathcal{N}ec^{op}, \mathcal{V}\,\mathrm{Quiv}_S)$$

Thus we may identify necklace categories with lax monoidal functors $\mathcal{N}ec^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$ with $S$ a set. Under this identification, the essential image of the functor $(-)^{nec}$ consists of the strong monoidal functors $\mathcal{N}ec^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$ with $S$ a set.

We will now construct a right-adjoint to the inclusion $(-)^{nec} : S_\otimes \mathcal{V} \hookrightarrow \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$.

**Construction 3.2.8.** Fix a necklace category $\mathcal{C}$ with set of objects $S$. For every neckace $T$, we have a $\mathcal{V}$-enriched quiver $\mathcal{C}_T = (\mathcal{C}_T(a,b))_{a,b\in S}$ Then the composition and identities of $\mathcal{C}$ induce quiver morphisms

$$\tilde{m}_{U,V} : \mathcal{C}_U \otimes_S \mathcal{C}_V \to \mathcal{C}_{U\vee V} \quad \text{and} \quad u : I_S \to \mathcal{C}_{\{0\}}$$

for all necklaces $U$ and $V$. We construct a templicial object $(\mathcal{C}^{temp}, S)$ as follows. Set $\mathcal{C}_0^{temp} = I_S$ and $p_0 = u : \mathcal{C}_0^{temp} \to \mathcal{C}_{\{0\}}$. Now let $n > 0$. We inductively define an object $\mathcal{C}_n^{temp} \in \mathcal{V}\,\mathrm{Quiv}_S$ along with morphisms $p_n$ and $\mu_{k,l}$ as the limit of the following diagram of solid arrows in $\mathcal{V}\,\mathrm{Quiv}_S$:

$$\begin{tikzcd}
\mathcal{C}_n^{temp} \arrow[r, dashed, "(\mu_{k,l})_{k,l}"] \arrow[d, dashed, "p_n"'] & \prod\limits_{\substack{k,l>0\\k+l=n}} \mathcal{C}_k^{temp} \otimes_S \mathcal{C}_l^{temp} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "\prod_{k,l} p_k \otimes p_l"] & \prod\limits_{\substack{r,s,t>0\\r+s+t=n}} \mathcal{C}_r^{temp} \otimes_S \mathcal{C}_s^{temp} \otimes_S \mathcal{C}_t^{temp} \\
\mathcal{C}_{\{0<n\}} \arrow[dr, "(\mathcal{C}(\nu_{k,l}))_{k,l}"'] & \prod\limits_{\substack{k,l>0\\k+l=n}} \mathcal{C}_{\{0<k\}} \otimes_S \mathcal{C}_{\{0<l\}} \arrow[d, "\prod_{k,l} \tilde{m}_{\{0<k\},\{0<l\}}"] & \\
& \prod\limits_{\substack{k,l>0\\k+l=n}} \mathcal{C}_{\{0<k<k+l\}} &
\end{tikzcd} \tag{3.3}$$

where $\alpha$ and $\beta$ are defined by

$$\pi_{r,s,t}\alpha = (\mathrm{id}_r \otimes \mu_{s,t})\pi_{r,s+t} \quad \text{and} \quad \pi_{r,s,t}\beta = (\mu_{r,s} \otimes \mathrm{id}_t)\pi_{r+s,t}$$

For example, $\mathcal{C}_1^{temp} = \mathcal{C}_{\{0,1\}}$ with $p_1 = \mathrm{id}_{\mathcal{C}_{\{0,1\}}}$, and $\mathcal{C}_2^{temp}$ is the pullback of $\tilde{m}_{\{0<1\},\{0<1\}}$ and $\mathcal{C}(\nu_{1,1})$. We further set $\mu_{0,n}$ and $\mu_{n,0}$ to be the left and right unit isomorphisms respectively:

$$\mathcal{C}_n^{temp} \xrightarrow{\sim} \mathcal{C}_0^{temp} \otimes_S \mathcal{C}_n^{temp}, \quad \mathcal{C}_n^{temp} \xrightarrow{\sim} \mathcal{C}_n^{temp} \otimes_S \mathcal{C}_0^{temp}$$

Further, let $f : [m] \to [n]$ be a morphism in $\mathbf{\Delta}_f$. We define a quiver morphism $\mathcal{C}^{temp}(f) : \mathcal{C}_n^{temp} \to \mathcal{C}_m^{temp}$ by induction on $m$. Set $\mathcal{C}^{temp}(\mathrm{id}_{[0]})$ to be the identity on $I_S$. If $m > 0$, we let $\mathcal{C}^{temp}(f)$ be the unique morphism satisfying, for all $k, l > 0$ with $k + l = m$:

$$\mu_{k,l}\mathcal{C}^{temp}(f) = (\mathcal{C}^{temp}(f_1) \otimes_S \mathcal{C}^{temp}(f_2))\mu_{p,q}$$

and

$$p_m\mathcal{C}^{temp}(f) = \mathcal{C}(f)p_n$$

where $f_1 : [k] \to [p]$ and $f_2 : [l] \to [q]$ are unique in $\mathbf{\Delta}_f$ such that $f_1 + f_2 = f$. (Note that in case $m = 1$, the first condition is empty and $\mathcal{C}^{temp}(f)$ is just $\mathcal{C}(f)p_n$.)

We have thus constructed a well-defined functor

$$\mathcal{C}^{temp} : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$$

By construction, $\mathcal{C}^{temp}$ is strongly unital and colax monoidal with comultiplication given by the morphisms $(\mu_{k,l})_{k,l\geq 0}$.

**Theorem 3.2.9.** *The assignment $\mathcal{C} \mapsto \mathcal{C}^{temp}$ of Construction 3.2.8 extends to a functor*

$$(-)^{temp} : \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec} \to S_{\otimes}\mathcal{V}$$

*that is right adjoint to the functor $(-)^{nec}$.*

*Proof.* Take a necklace category $\mathcal{C}$. For every necklace $T = \{0 = t_0 < t_1 < ... < t_k = p\}$, consider the quiver morphism

$$\varepsilon_{\mathcal{C}_T} : (\mathcal{C}^{temp})_T \xrightarrow{p_{t_1} \otimes ... \otimes p_{p-t_{k-1}}} \mathcal{C}_{\{0,t_1\}} \otimes ... \otimes \mathcal{C}_{\{0,p-t_{k-1}\}} \xrightarrow{\tilde{m}} \mathcal{C}_T$$

By Construction 3.2.8 this morphism is natural in $T$ and it follows immediately that we have a necklace functor

$$\varepsilon_{\mathcal{C}} : (\mathcal{C}^{temp})^{nec} \to \mathcal{C}$$

For a necklace functor $H : X^{nec} \to \mathcal{C}$ with $(X, S)$ a templicial object, define $\alpha_0$ as the canonical quiver morphism $X_0 \simeq I_S \to f^*(I_{\mathrm{Ob}(\mathcal{C})}) = f^*(\mathcal{C}_0^{temp})$ where $f : S \to \mathrm{Ob}(\mathcal{C})$ is the object map of $H$. Then we define a morphism $\alpha_n : X_n \to f^*(\mathcal{C}_n^{temp})$ by induction on $n > 0$. Let $\beta_n : f_!(X_n) = f_!(X_{\{0,n\}}) \to \mathcal{C}_{\{0,n\}}$ be the adjoint to $H_{\{0<n\}}$. By the construction of $\mathcal{C}^{temp}$, we have a unique morphism $\alpha_n : f_!(X_n) \to \mathcal{C}_n^{temp}$ such that

$$p_n \alpha_n = \beta_n$$

and for all $k, l > 0$ with $k + l = n$, $\mu_{k,l} \circ \alpha_n$ is equal to the composite

$$f_!(X_n) \xrightarrow{f_!(\mu^X_{k,l})} f_!(X_k \otimes X_l) \to f_!(X_k) \otimes f_!(X_l) \xrightarrow{\alpha_k \otimes \alpha_l} \mathcal{C}_k^{temp} \otimes \mathcal{C}_l^{temp}$$

where we used to colax monoidal structure of $f_!$ (Lemma 1.1.18). Hence, we obtain a templicial morphism $(\alpha, f) : (X, S) \to (\mathcal{C}^{temp}, \mathrm{Ob}(\mathcal{C}))$. Moreover, by the compatibility of $H$ with the compositions of $X^{nec}$ and $\mathcal{C}$, we have that

$$\varepsilon_{\mathcal{C}} \circ \alpha^{nec} = H$$

It is clear by construction that $\alpha$ is the unique templicial morphism with this property. $\square$

### 3.2.2 Some past constructions revisited

We show that the functor $\tilde{U} : S_{\otimes}\mathcal{V} \to \mathrm{SSet}$ (Proposition 2.1.25) and the templicial nerve $N_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat} \to S_{\otimes}\mathcal{V}$ (Construction 2.3.4) factor through the category $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ of necklace categories.

**Notation 3.2.10.** By post-composition, the adjunction $F : \mathrm{Set} \leftrightarrows \mathcal{V} : U$ induces an adjunction $F : \mathrm{Set}^{\mathcal{N}ec^{op}} \leftrightarrows \mathcal{V}^{\mathcal{N}ec^{op}} : U$. Note that as $F$ is strong monoidal and preserves colimits, the induced functor $F : \mathrm{Set}^{\mathcal{N}ec^{op}} \to \mathcal{V}^{\mathcal{N}ec^{op}}$ is strong monoidal as well. Therefore, Proposition 1.1.23 provides an adjunction which we denote by

$$\begin{tikzcd} \mathrm{Cat}_{\mathcal{N}ec} \arrow[r, shift left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec} \arrow[l, shift left, "\mathcal{U}"] \end{tikzcd}$$

**Proposition 3.2.11.** *There is diagram of adjunctions*

$$\begin{tikzcd} \mathrm{SSet} \arrow[r, shift left, "\tilde{F}"] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "(-)^{nec}"'] \arrow[d, phantom, "\dashv"] & S_{\otimes}\mathcal{V} \arrow[l, shift left, "\tilde{U}"] \arrow[d, shift right, "(-)^{nec}"'] \arrow[d, phantom, "\dashv"] \\ \mathrm{Cat}_{\mathcal{N}ec} \arrow[u, shift right, "(-)^{temp}"'] \arrow[r, shift left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec} \arrow[l, shift left, "\mathcal{U}"] \arrow[u, shift right, "(-)^{temp}"'] \end{tikzcd}$$

*which commutes in the sense that we have natural isomorphisms*

$$(-)^{nec} \circ \tilde{F} \simeq \mathcal{F} \circ (-)^{nec} \quad \text{and} \quad \tilde{U} \circ (-)^{temp} \simeq (-)^{temp} \circ \mathcal{U}$$

*In particular, we have a natural isomorphism*

$$\tilde{U} \simeq (-)^{temp} \circ \mathcal{U} \circ (-)^{nec}$$

*Proof.* It suffices to show the commutativity of the left-adjoints. But this immediately follows from the fact that $F : \mathrm{Set} \to \mathcal{V}$ is strong monoidal and preserves colimits. The final isomorphism $\tilde{U} \simeq (-)^{temp} \circ \mathcal{U} \circ (-)^{nec}$ follows from the fact that $(-)^{nec}$ is fully faithful. □

**Lemma 3.2.12.** *Let $\mathcal{C}$ be a finitely complete category. Let $f : A \to B$ be a morphism in $\mathcal{C}$ and $n \geq 2$. Then $A$ is the limit of the following diagram of solid arrows:*

$$\begin{tikzcd} A \arrow[r, dashed, "\Delta"] \arrow[d, dashed, "f"'] & \prod_{\substack{k,l>0 \\ k+l=n}} A \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "\prod_{k,l} f"] & \prod_{\substack{r,s,t>0 \\ r+s+t=n}} A \\ B \arrow[r, "\Delta"'] & \prod_{\substack{k,l>0 \\ k+l=n}} B & \end{tikzcd}$$

*where $\Delta$ is the diagonal morphism and $\alpha$ and $\beta$ are defined by*

$$\pi_{r,s,t} \circ \alpha = \pi_{r+s,t} \quad \text{and} \quad \pi_{r,s,t} \circ \beta = \pi_{r,s+t}$$

*for all $r, s, t > 0$ with $r + s + t = n$.*

*Proof.* Suppose $Z \in \mathcal{C}$ with $u : Z \to \prod_{k,l} A$ and $v : Z \to B$ such that $(\prod_{k,l} f)u = \Delta v$ and $\alpha u = \beta u$. Then note that for all $k, l, k', l' > 0$ with $k + l = k' + l' = n$, the projections $\pi_{k,l}u$ and $\pi_{k',l'}u$ coincide. Indeed, we may assume that $k < k'$ (i.e. $l > l'$) and thus

$$\pi_{k,l}u = \pi_{k,l-l',l'}\beta u = \pi_{k,l-l',l'}\alpha u = \pi_{k',l'}u$$

We set $h : Z \to A$ to be equal to these morphisms $\pi_{k,l}u$ (note that there exists at least one such morphism since $n \geq 2$). Then by construction, $\Delta h = u$. Moreover,

$$fh = \pi_{k,l}\Delta fh = \pi_{k,l}\left(\prod_{k,l} f\right)\Delta h = \pi_{k,l}\left(\prod_{k,l} f\right)u = \pi_{k,l}\Delta v = v$$

It is clear that $h$ is unique such that $\Delta h = u$ and $fh = v$. □

**Construction 3.2.13.** Let $\underline{(-)} : \mathcal{V} \to \mathcal{V}^{\mathcal{N}ec^{op}}$ denote the diagonal functor associating to every object $A \in \mathcal{V}$ the constant functor on $A$.

It is easy to see that for all $A, B \in \mathcal{V}$ and $T \in \mathcal{N}ec$, the following diagram is a coequalizer:

$$\begin{tikzcd} \coprod\limits_{\substack{U,N,V \in \mathcal{N}ec \\ U \vee N \vee V = T}} A \otimes B \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \coprod\limits_{\substack{U,V \in \mathcal{N}ec \\ U \vee V = T}} A \otimes B \arrow[r, two heads, "\nabla"] & A \otimes B \end{tikzcd}$$

where $\nabla$ denotes the codiagonal and $\alpha\iota_{U,N,V} = \iota_{U\vee N,V}$, $\beta\iota_{U,N,V} = \iota_{U,N\vee V}$ for all $U, N, V \in \mathcal{N}ec$ with $U \vee N \vee V = T$. It then follows from Proposition 3.2.2 that $\underline{(-)}$ is strong monoidal and thus we have an induced functor

$$\underline{(-)} : \mathcal{V}\,\mathrm{Cat} \to \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$$

**Proposition 3.2.14.** *There is a diagram of functors*

$$\begin{tikzcd} \mathcal{V}\,\mathrm{Cat} \arrow[rr, "N_{\mathcal{V}}"] \arrow[dr, "\underline{(-)}"'] & & S_{\otimes}\mathcal{V} \\ & \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec} \arrow[ur, "(-)^{temp}"'] & \end{tikzcd}$$

*which commutes in the sense that we have a natural isomorphism*

$$N_{\mathcal{V}} \simeq (-)^{temp} \circ \underline{(-)}$$

*Proof.* Given a small $\mathcal{V}$-enriched category $\mathcal{C}$, denote its underlying $\mathcal{V}$-enriched quiver also by $\mathcal{C}$. By definition, $\underline{\mathcal{C}}_0^{temp} = I_{\mathrm{Ob}(\mathcal{C})}$. We proceed by induction on $n > 0$ to show that $\underline{\mathcal{C}}_n^{temp} \simeq \mathcal{C}^{\otimes n}$ and $p_n : \underline{\mathcal{C}}_n^{temp} \to \mathcal{C}$ is the reverse composition $\tilde{m}^{(n)} : \mathcal{C}^{\otimes n} \to \mathcal{C}$ (see Remark 1.1.22). Indeed, by Construction 3.2.8, $\underline{\mathcal{C}}_n^{temp}$ is the limit of the following diagram of solid arrows:

$$\begin{tikzcd} \underline{\mathcal{C}}_n^{temp} \arrow[r, dashed, "\Delta"] \arrow[d, dashed, "p_n"'] & \prod\limits_{\substack{k,l>0 \\ k+l=n}} \mathcal{C}^{\otimes k} \otimes \mathcal{C}^{\otimes l} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "\prod_{k,l} \tilde{m}^{(k)} \otimes \tilde{m}^{(l)}"] & \prod\limits_{\substack{r,s,t>0 \\ r+s+t=n}} \mathcal{C}^{\otimes r} \otimes \mathcal{C}^{\otimes s} \otimes \mathcal{C}^{\otimes t} \\ \mathcal{C} \arrow[dr, "\Delta"'] & \prod\limits_{\substack{k,l>0 \\ k+l=n}} \mathcal{C} \otimes \mathcal{C} \arrow[d, "\prod_{k,l} \tilde{m}"] & \\ & \prod\limits_{\substack{k,l>0 \\ k+l=n}} \mathcal{C} & \end{tikzcd}$$

So as $\tilde{m}(\tilde{m}^{(k)} \otimes \tilde{m}^{(l)}) = \tilde{m}^{(k+l)}$, it follows from Lemma 3.2.12 that we may identify $\underline{\mathcal{C}}_n^{temp}$ with $\mathcal{C}^{\otimes n}$ and $p_n$ with $\tilde{m}^{(n)}$.

It quickly follows that this identification induces an isomorphism of templicial objects $\underline{\mathcal{C}}^{temp} \simeq N_{\mathcal{V}}(\mathcal{C})$, which is clearly natural in $\mathcal{C}$. $\square$

### 3.2.3 Quasi-categories and Frobenius structures

In this subsection we identify sufficient conditions on a necklace category $\mathcal{C}$ such that its associated templicial object $\mathcal{C}^{temp}$ is a quasi-category in $\mathcal{V}$ (Corollary 3.2.22) or has a Frobenius structure (Proposition 3.2.20). To achieve the latter, we embed the category of necklaces $\mathcal{N}ec$ into a larger category $\overline{\mathcal{N}ec}$. This category also contains what we call *coinert* maps which parametrize the multiplication morphisms of a Frobenius structure. For example, there is a coinert map $\{0 < p+q\} \to \{0 < p < p+q\}$ in $\overline{\mathcal{N}ec}$ which should be interpreted as parametrizing the morphism

$$Z^{p,q} : X_p \otimes_S X_q \to X_{p+q}$$

of a Frobenius structure $Z$ on a templicial object $(X, S)$. Therefore, if a necklace category $\mathcal{C}$ can be extended to a $\mathcal{V}^{\overline{\mathcal{N}ec}^{op}}$-enriched category, then the associated templicial object $\mathcal{C}^{temp}$ has a Frobenius structure.

**Definition 3.2.15.** We define a monoidal category $\overline{\mathcal{N}ec}$ as follows:

The objects of $\overline{\mathcal{N}ec}$ are the same as those of $\mathcal{N}ec$. Given two necklaces $(T, p)$ and $(U, q)$, a morphism $(T, p) \to (U, q)$ in $\overline{\mathcal{N}ec}$ is a pair $(f, U')$ with $f : [p] \to [q]$ in $\mathbf{\Delta}_f$ and $f(T) \cup U \subseteq U' \subseteq [q]$.

The composition of two morphisms $(f, U') : T \to U$ and $(g, V') : U \to V$ is given by the pair $(gf, V' \cup g(U'))$ and the identity on a necklace $T$ is given by the pair $(\mathrm{id}_{[p]}, T)$.

The category $\overline{\mathcal{N}ec}$ has a monoidal structure given on morphisms by

$$(f, U') \vee (g, V') = (f \vee g, U' \vee T')$$

with monoidal unit given by the necklace $(\{0\}, 0)$.

Finally, note that we can identify $\mathcal{N}ec$ with the non-full monoidal subcategory of $\overline{\mathcal{N}ec}$ that consists of all morphisms $(f, U') : T \to U$ such that $U' = f(T)$.

*Remark* 3.2.16. Let $\mathcal{N}ec^{inert}$ denote the subcategory of $\mathcal{N}ec$ consisting of all inert necklace maps. Note that we can also embed the opposite category $(\mathcal{N}ec^{inert})^{op}$ into $\overline{\mathcal{N}ec}$ by sending an inert map $f : (T, q) \to (U, q)$ to the pair $f^{co} = (\mathrm{id}_{[q]}, T) : U \to T$ (this is well-defined as $U \subseteq T$). Let us call such a morphism a *coinert map*. Then we can uniquely decompose every morphism $f : T \to U$ in $\overline{\mathcal{N}ec}$ as

$$T \xrightarrow{f_1} T_1 \xrightarrow{f_2^{co}} T_2 \xrightarrow{f_3} U$$

where $f_1$ is an active necklace map, $f_2^{co}$ is a coinert map and $f_3$ is an inert necklace map. It follows that $\overline{\mathcal{N}ec}$ is characterized by the following universal property:

Let $\mathcal{C}$ be a category and $F : \mathcal{N}ec \to \mathcal{C}$ and $G : (\mathcal{N}ec^{inert})^{op} \to \mathcal{C}$ functors such that for every commutative diagram of necklace maps

$$\begin{array}{ccc} T' & \xrightarrow{g'} & T \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U' & \xrightarrow[g]{} & U \end{array}$$

with $g$ and $g'$ inert and $f'(T') = U' \cup f(T)$, we have that

$$G(g)F(f) = F(f')G(g')$$

Then there is a unique functor

$$\overline{F} : \overline{\mathcal{N}ec} \to \mathcal{C}$$

such that $\overline{F}|_{\mathcal{N}ec} = F$ and $\overline{F}|_{(\mathcal{N}ec^{inert})^{op}} = G$.

*Remark* 3.2.17. Consider the functor category $\mathcal{V}^{\overline{\mathcal{N}ec}^{op}}$. As for $\mathcal{V}^{\mathcal{N}ec^{op}}$, we can equip $\mathcal{V}^{\overline{\mathcal{N}ec}^{op}}$ with the monoidal closed structure given by Day convolution (see [Day70]).

**Proposition 3.2.18.** *The restriction functor $\mathcal{V}^{\overline{\mathcal{N}ec}^{op}} \to \mathcal{V}^{\mathcal{N}ec^{op}}$ is lax monoidal.*

*Proof.* First note that $\{0\}$ is still the terminal object of $\overline{\mathcal{N}ec}$ and thus as in Construction 3.2.1, the monoidal unit of $\mathcal{V}^{\overline{\mathcal{N}ec}^{op}}$ is given by the constant functor $\underline{I}$. Thus the restriction functor $\mathcal{V}^{\overline{\mathcal{N}ec}^{op}} \to \mathcal{V}^{\mathcal{N}ec^{op}}$ is at least strongly unital.

Now take functors $X, Y : \overline{\mathcal{N}ec}^{op} \to \mathcal{V}$ and let $X \overline{\otimes}_{Day} Y$ denote their monoidal product in $\mathcal{V}^{\overline{\mathcal{N}ec}^{op}}$. Then the strong monoidal inclusion $\iota : \mathcal{N}ec \hookrightarrow \overline{\mathcal{N}ec}$ induces a canonical morphism in $\mathcal{V}^{\mathcal{N}ec^{op}}$

$$X\iota \otimes_{Day} Y\iota = \mathrm{Lan}_{\vee}(X\iota(-) \otimes Y\iota(-)) \to \mathrm{Lan}_{\vee}(X(-) \otimes Y(-))\iota = (X \overline{\otimes}_{Day} Y)\iota$$

It is now easy to see that this morphism is natural in $X$ and $Y$ and that it equips the forgetful functor $\mathcal{V}^{\overline{\mathcal{N}ec}^{op}} \to \mathcal{V}^{\mathcal{N}ec^{op}}$ with a lax monoidal structure. □

**Notation 3.2.19.** We denote

$$\mathcal{V}\,\mathrm{Cat}_{\overline{\mathcal{N}ec}} = \mathcal{V}^{\overline{\mathcal{N}ec}^{op}}\text{-}\,\mathrm{Cat}$$

Note that by Proposition 3.2.18 there is an induced forgetful functor

$$\mathcal{V}\,\mathrm{Cat}_{\overline{\mathcal{N}ec}} \to \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$$

**Proposition 3.2.20.** *Let $\mathcal{C}$ be a necklace category. Assume that there is an object $\overline{\mathcal{C}}$ in $\mathcal{V}\,\mathrm{Cat}_{\overline{\mathcal{N}ec}}$ that restricts to $\mathcal{C}$ when applying the functor $\mathcal{V}\,\mathrm{Cat}_{\overline{\mathcal{N}ec}} \to \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$. Then $\mathcal{C}^{temp}$ has a Frobenius structure.*

*Proof.* Denote the comultiplication of $\mathcal{C}^{temp}$ by $\mu$. We define a Frobenius structure

$$\left(Z^{k,l} : \mathcal{C}^{temp}_k \otimes \mathcal{C}^{temp}_l \to \mathcal{C}^{temp}_{k+l}\right)_{k,l\geq 0}$$

on $\mathcal{C}^{temp}$ by induction on the pairs $(k, l)$. If $k = 0$, we set $Z^{0,l}$ to be the left unit isomorphism $Z^{0,l} : I_S \otimes \mathcal{C}^{temp}_l \xrightarrow{\sim} \mathcal{C}^{temp}_l$. Similarly, if $l = 0$, we set $Z^{k,0}$ to be the right unit isomorphism. This forces that condition (2.8) of Definition 2.2.34 holds.

Assume further that $k, l > 0$ and set $n = k + l$. For all $p, q > 0$ with $p + q = n$, define a morphism $\xi_{p,q} : \mathcal{C}^{temp}_k \otimes \mathcal{C}^{temp}_l \to \mathcal{C}^{temp}_p \otimes \mathcal{C}^{temp}_q$ by

$$\xi_{p,q} = \begin{cases} (Z^{k,l-q} \otimes \mathrm{id}_{\mathcal{C}^{temp}_q})(\mathrm{id}_{\mathcal{C}^{temp}_k} \otimes \mu_{p-k,q}) & \text{if } k < p \\ \mathrm{id}_{\mathcal{C}^{temp}_k \otimes \mathcal{C}^{temp}_l} & \text{if } k = p \\ (\mathrm{id}_{\mathcal{C}^{temp}_k} \otimes Z^{k-p,l})(\mu_{p,q-l} \otimes \mathrm{id}_{\mathcal{C}^{temp}_l}) & \text{if } k > p \end{cases} \tag{3.4}$$

If $k < p$, we have a commutative diagram in $\overline{\mathcal{N}ec}$:

$$\begin{tikzcd}
\{0<k<p<n\} \arrow[r, "\mathrm{id}_{\{0<k\}} \vee \nu_{p-k,q}"] & \{0<k<n\} \\
\{0<p<n\} \arrow[u, "\nu^{co}_{k,p-k} \vee \mathrm{id}_{\{0<q\}}"] \arrow[r, "\nu_{p,q}"'] & \{0<n\} \arrow[u, "\nu^{co}_{k,n-k}"']
\end{tikzcd}$$

Moreover, we have that

$$\nu_{k,p-k} \circ \nu^{co}_{k,p-k} = \mathrm{id}_{\{0,p\}}$$

It now follows from the construction of $\mathcal{C}^{temp}$ (diagram (3.3)) and the induction hypothesis that

$$\begin{aligned}
&\mathcal{C}(\nu_{p,q})\overline{\mathcal{C}}(\nu^{co}_{k,n-k})\tilde{m}_{\{0<k\},\{0<l\}}(p_k \otimes p_l) \\
&= \overline{\mathcal{C}}(\nu^{co}_{k,p-k} \vee \mathrm{id}_{\{0<q\}})\mathcal{C}(\mathrm{id}_{\{0<k\}} \vee \nu_{p-k,n-p})\tilde{m}_{\{0<k\},\{0<l\}}(p_k \otimes p_l) \\
&= \overline{\mathcal{C}}(\nu^{co}_{k,p-k} \vee \mathrm{id}_{\{0<q\}})\tilde{m}_{\{0<k\},\{0<p-k<l\}}(\mathrm{id}_{\mathcal{C}_{\{0<k\}}} \otimes \mathcal{C}(\nu_{p-k,n-p}))(p_k \otimes p_l) \\
&= \overline{\mathcal{C}}(\nu^{co}_{k,p-k} \vee \mathrm{id}_{\{0<q\}})\tilde{m}_{\{0<k\},\{0<p-k<l\}}(p_k \otimes \tilde{m}_{\{0<p-k\},\{0<q\}}(p_{p-k} \otimes p_q)\mu_{p-k,q}) \\
&= \overline{\mathcal{C}}(\nu^{co}_{k,p-k} \vee \mathrm{id}_{\{0<q\}})\tilde{m}_{\{0<k<p\},\{0<q\}}(\tilde{m}_{\{0<k\},\{0<p-k\}}(p_k \otimes p_{p-k})\mu_{k,p-k} \otimes p_q)\xi_{p,q} \\
&= \overline{\mathcal{C}}(\nu^{co}_{k,p-k} \vee \mathrm{id}_{\{0<q\}})\tilde{m}_{\{0<k<p\},\{0<q\}}(\mathcal{C}(\nu_{k,p-k}) \otimes \mathrm{id}_{\mathcal{C}_{\{0<q\}}})(p_p \otimes p_q)\xi_{p,q} \\
&= \overline{\mathcal{C}}(\nu^{co}_{k,p-k} \vee \mathrm{id}_{\{0<q\}})\mathcal{C}(\nu_{k,p-k} \vee \mathrm{id}_{\mathcal{C}_{\{0<q\}}})\tilde{m}_{\{0<p\},\{0<q\}}(p_p \otimes p_q)\xi_{p,q} \\
&= \tilde{m}_{\{0<p\},\{0<q\}}(p_p \otimes p_q)\xi_{p,q}
\end{aligned}$$

Similarly, $\mathcal{C}(\nu_{p,q})\overline{\mathcal{C}}(\nu^{co}_{k,n-k})\tilde{m}_{\{0<k\},\{0<l\}}(p_k \otimes p_l) = \tilde{m}_{\{0<p\},\{0<q\}}(p_p \otimes p_q)\xi_{p,q}$ also holds when $k > p$ or $k = p$. Hence, by the construction of $\mathcal{C}^{temp}_n$ as the limit (3.3), there is a unique morphism

$$Z^{k,l} : \mathcal{C}^{temp}_k \otimes \mathcal{C}^{temp}_l \to \mathcal{C}^{temp}_n$$

such that $p_n Z^{k,l} = \overline{\mathcal{C}}(\nu^{co}_{k,n-k})\tilde{m}_{\{0<k\},\{0<l\}}(p_k \otimes p_l)$ and $\mu_{p,q}Z^{k,l} = \xi_{p,q}$ for all $p, q > 0$ with $p + q = n$. In particular, the Frobenius condition (2.9) is satisfied.

A similar argument from induction shows that the morphisms $Z^{k,l}$ are natural in $k, l \geq 0$ and satisfy associativity (2.8). □

Let us now investigate when the templicial object $\mathcal{C}^{temp}$ is a quasi-category in $\mathcal{V}$.

**Proposition 3.2.21.** *Let $\mathcal{C}$ be a necklace category with object set $S$. For any $a, b \in S$, the canonical morphism $\varepsilon_{\mathcal{C}} : (\mathcal{C}^{temp})_\bullet(a,b) \to \mathcal{C}_\bullet(a,b)$ has the right lifting property with respect to* Cell *in $\mathcal{V}^{nec^{op}}$.*

*Proof.* Let $a, b \in S$ and $n > 0$, and consider a commutative diagram

$$\begin{tikzcd}
\tilde{F}(\partial\Delta^n)_\bullet(0,n) \arrow[r] \arrow[d] & \mathcal{C}^{temp}_\bullet(a,b) \arrow[d, "\epsilon"] \\
\tilde{F}(\Delta^n)_\bullet(0,n) \arrow[r] & \mathcal{C}_\bullet(a,b)
\end{tikzcd}$$

in $\mathcal{V}^{\mathcal{N}ec^{op}}$. The top horizontal morphism corresponds to some collections of elements $(x_k)_{k=1}^{n-1}$ and $(y_i)_{i=1}^{n-1}$ with $x_k \in U((\mathcal{C}^{temp}_k \otimes \mathcal{C}^{temp}_{n-k})(a,b))$ and $y_i \in U(\mathcal{C}^{temp}_{n-1}(a,b))$, satisfying the conditions of Corollary 2.2.22.2. Moreover, the bottom horizontal morphism corresponds to an element $z' \in U(\mathcal{C}_{\{0<n\}}(a,b))$ and the commutativity of the diagram comes down to the condition that $\mathcal{C}(\nu_{k,n-k})(z') = \tilde{m}_{\{0<k\},\{0<n-k\}}(p_k \otimes p_{n-k})(x_k)$ and $\mathcal{C}(\delta_i)(z') = p_{n-1}(y_i)$ for all $0 < k, i < n$.

Then by Construction 3.2.8, there exists a unique element $z \in U(\mathcal{C}^{temp}_n(a,b))$ such that $\mu_{k,n-k}(z) = x_k$ for all $0 < k < n$, and $p_n(z) = z'$. Moreover, we have that $d_i(z) = y_i$ for all $0 < i < n$. Indeed, again by Construction 3.2.8, it suffices to note that for all $0 < k, i < n$:

$$\begin{aligned} \mu_{k,n-1-k}(d_i(z)) &= \begin{cases} (d_i \otimes \mathrm{id}_{\mathcal{C}^{temp}_{n-k-1}})(\mu_{k+1,n-k}(z)) & \text{if } i \leq k \\ (\mathrm{id}_{\mathcal{C}^{temp}_k} \otimes d_{i-k})(\mu_{k,n-k}(z)) & \text{if } i > k \end{cases} = \mu_{k,n-1-k}(y_i) \\ p_{n-1}(d_i(z)) &= \mathcal{C}(\delta_i)p_n(z) = \mathcal{C}(\delta_i)(z') = p_{n-1}(y_i) \end{aligned}$$

Hence, the element $z$ determines a morphism $\tilde{F}(\Delta^n)_\bullet(0,n) \to \mathcal{C}^{temp}_\bullet(a,b)$ which is a lift of the above diagram. □

**Corollary 3.2.22.** *Let $\mathcal{C}$ be a necklace category with object set $S$. Suppose that for all $a, b \in S$, $\mathcal{C}_\bullet(a,b)$ lifts inner horns in $\mathcal{V}^{\mathcal{N}ec^{op}}$. Then $(\mathcal{C}^{temp}, S)$ is a quasi-category in $\mathcal{V}$.*

*Proof.* Let $a, b \in S$. By Proposition 3.2.21 and Lemma 2.2.25, the canonical morphism $(\mathcal{C}^{temp})_\bullet(a,b) \to \mathcal{C}_\bullet(a,b)$ has the right lifting property with respect to the inner horn inclusion $\tilde{F}(\Lambda^n_j)_\bullet(a,b) \to \tilde{F}(\Delta^n)_\bullet(a,b)$ for all $0 < j < n$. Thus as $\mathcal{C}_\bullet(a,b)$ lifts inner horns, so does $(\mathcal{C}^{temp})_\bullet(a,b)$. Hence, $(\mathcal{C}^{temp}, S)$ is a quasi-category in $\mathcal{V}$. □

### 3.2.4 Local presentability

Fix a regular cardinal $\lambda$. We show that for $\lambda > \aleph_0$ the category $S_\otimes\mathcal{V}$ of templicial objects is locally $\lambda$-presentable if $\mathcal{V}$ is. This will go via the embedding $(-)^{nec} : S_\otimes\mathcal{V} \hookrightarrow \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ (see Theorem 3.2.29). For background on locally presentable categories, we refer to [AR94]. Further, we will make use of the following results from the literature.

**Proposition 3.2.23** ([KL01], Corollary 3.4). *Let $(\mathcal{W}, \otimes, I)$ be a cocomplete monoidal closed category. Then the forgetful functor*

$$\mathcal{W}\,\mathrm{Cat} \to \mathcal{W}\,\mathrm{Quiv}$$

*preserves filtered colimits.*

**Proposition 3.2.24** ([KL01], Proposition 4.4 and Theorem 4.5). *Let $(\mathcal{W}, \otimes, I)$ a monoidal closed category such that $\mathcal{W}$ is locally $\lambda$-presentable. Then $\mathcal{W}\,\mathrm{Quiv}$ and $\mathcal{W}\,\mathrm{Cat}$ are locally $\lambda$-presentable as well.*

**Theorem 3.2.25** ([Hen20], Theorem A.2). *Assume that $\lambda > \aleph_0$ and let $\mathcal{C}$ be a locally $\lambda$-presentable category. Then for every comonad $T$ on $\mathcal{C}$ which preserves $\lambda$-filtered colimits, the category of coalgebras over $T$ is locally $\lambda$-presentable as well.*

**Corollary 3.2.26.** *Assume that $\mathcal{V}$ is locally $\lambda$-presentable. Then the category of necklace categories $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ is locally $\lambda$-presentable as well.*

*Proof.* By the standing hypotheses, the monoidal product $-\otimes -$ in $\mathcal{V}$ preserves colimits in each variable. So as $\mathcal{V}$ is assumed to be locally presentable, each of the functors $A \otimes -$ and $- \otimes A$ has a right-adjoint for all $A \in \mathcal{V}$, and thus $\mathcal{V}$ is monoidal closed. Then by [Day70], the monoidal category $(\mathcal{V}^{\mathcal{N}ec^{op}}, \otimes_{Day}, \underline{I})$ is also closed. Moreover, as $\mathcal{V}$ is locally $\lambda$-presentable, so is $\mathcal{V}^{\mathcal{N}ec^{op}}$ (see for example [AR94, Corollary 1.54]). So it follows by Proposition 3.2.24 that $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ is locally $\lambda$-presentable as well. □

Recall the description of colimits in $\mathcal{V}\,\mathrm{Quiv}$ given in Remark 1.1.17.

**Lemma 3.2.27.** *Let $D_1, D_2 : \mathcal{J} \to \mathcal{V}\,\mathrm{Quiv}$ be filtered diagrams which coincide after composition with $\mathcal{V}\,\mathrm{Quiv} \to \mathrm{Set}$. Let $S^j$ denote the set of objects of $D_1(j)$ and $D_2(j)$ for each $j \in \mathcal{J}$ and set $S = \mathrm{colim}_{j\in\mathcal{J}} S^j$. Then the canonical morphism in $\mathcal{V}\,\mathrm{Quiv}$:*

$$\underset{j\in\mathcal{J}}{\mathrm{colim}}\,(D_1(j)\otimes_{S^j} D_2(j)) \to \left(\underset{j\in\mathcal{J}}{\mathrm{colim}}\, D_1(j)\right)\otimes_S \left(\underset{j\in\mathcal{J}}{\mathrm{colim}}\, D_2(j)\right)$$

*is an isomorphism.*

*Proof.* For each $j \in \mathcal{J}$ and every $t : i \to j$ in $\mathcal{J}$, let $\iota_j : S^j \to S$ and $\iota_t : S^i \to S^j$ denote the canonical maps. Take $x, y \in S$. Evaluating the above morphism in $(x, y)$, we find the following morphism in $\mathcal{V}$:

$$\underset{j\in\mathcal{J}}{\mathrm{colim}} \coprod_{\substack{a,b,c\in S^j\\ \iota_j(a)=x\\ \iota_j(c)=y}} (D_1(j)(a,b)\otimes D_2(j)(b,c)) \xrightarrow{\varphi} \underset{i,j\in\mathcal{J}}{\mathrm{colim}} \coprod_{\substack{a,b\in S^i, b',c\in S^j\\ \iota_i(a)=x\\ \iota_j(c)=y\\ \iota_i(b)=\iota_j(b')}} (D_1(i)(a,b)\otimes D_2(j)(b',c))$$

Now define a morphism $\psi$ from right to left as follows. Take $i, j \in \mathcal{J}$ and $a, b \in S^i$, $b', c \in S^j$ such that $\iota_i(a) = x$, $\iota_j(c) = y$ and $\iota_i(b) = \iota_j(b')$. As $\mathcal{J}$ is filtered, we can choose a $k \in \mathcal{J}$ with morphisms $s : i \to k$ and $t : j \to k$ such that $\iota_s(b) = \iota_t(b')$. Then set $\overline{a} = \iota_s(a)$, $\overline{c} = \iota_t(c)$ and $\overline{b} = \iota_s(b) = \iota_t(b')$ in $S^k$. Then consider the composite

$$D_1(i)(a,b)\otimes D_2(j)(b',c) \to D_1(k)(\overline{a},\overline{b})\otimes D_2(k)(\overline{b},\overline{c}) \to \underset{j\in\mathcal{J}}{\mathrm{colim}}\,(D_1(j)\otimes_{S^j} D_2(j))\,(x,y)$$

of $D_1(s)_{a,b}\otimes D_2(t)_{b',c}$ with the canonical morphism. It is easy to see that these morphisms define a morphism $\psi$ which is inverse to $\varphi$. □

**Proposition 3.2.28.** *The functor $(-)^{temp} : \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec} \to S_{\otimes}\mathcal{V}$ preserves filtered colimits.*

*Proof.* Let $D : \mathcal{J} \to \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ be a filtered diagram. Let $\mathcal{C}$ denote its colimit and set $S = \mathrm{Ob}(\mathcal{C})$. Then $S \simeq \mathrm{colim}_j\,\mathrm{Ob}(D(j))$. Let $\iota_j$ denote the canonical map $\mathrm{Ob}(D(j)) \to S$ for all $j \in \mathcal{J}$. Then we have for every $T \in \mathcal{N}ec$ that $\mathcal{C}_T \simeq \mathrm{colim}_j\, D(j)_T$ in $\mathcal{V}\,\mathrm{Quiv}$ by Proposition 3.2.23. We will show by induction on $n \geq 0$ that the canonical quiver morphism

$$\underset{j\in\mathcal{J}}{\mathrm{colim}}\, D(j)_n^{temp} \to \mathcal{C}_n^{temp}$$

is an isomorphism in $\mathcal{V}\,\mathrm{Quiv}$.

For $n = 0$, this follows from the isomorphism $S \simeq \operatorname{colim}_j \operatorname{Ob}(D(j))$. Assume futher that $n > 0$. By Construction 3.2.8, $\mathcal{C}_n^{temp}$ is the limit of the following diagram of solid arrows:

$$\begin{tikzcd}
\mathcal{C}_n^{temp} \arrow[r, dashed, "(\mu_{k,l})_{k,l}"] \arrow[d, dashed, "p_n"'] & \prod\limits_{\substack{k,l>0\\k+l=n}} \mathcal{C}_k^{temp} \otimes_S \mathcal{C}_l^{temp} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "\prod_{k,l} p_k \otimes p_l"] & \prod\limits_{\substack{r,s,t>0\\r+s+t=n}} \mathcal{C}_r^{temp} \otimes_S \mathcal{C}_s^{temp} \otimes_S \mathcal{C}_t^{temp} \\
\mathcal{C}_{\{0<n\}} \arrow[dr, "(\mathcal{C}(\nu_{k,l}))_{k,l}"'] & \prod\limits_{\substack{k,l>0\\k+l=n}} \mathcal{C}_{\{0<k\}} \otimes_S \mathcal{C}_{\{0<l\}} \arrow[d, "\prod_{k,l} \tilde{m}_{\{0<k\},\{0<l\}}"] & \\
& \prod\limits_{\substack{k,l>0\\k+l=n}} \mathcal{C}_{\{0<k<k+l\}} &
\end{tikzcd}$$

where $\tilde{m}$ is the reverse composition of $\mathcal{C}$ (see Remark 1.1.22). By the induction hypothesis, $\mathcal{C}_k^{temp} \simeq \operatorname{colim}_j D(j)_k^{temp}$ for all $0 \leq k < n$. By Proposition 3.2.24, finite limits commute with filtered colimits in $\mathcal{V}\,\mathrm{Quiv}$ (see for example [AR94], Proposition 1.59). Thus it follows by Lemma 3.2.27 that also $\mathcal{C}_n^{temp} \simeq \operatorname{colim}_j D(j)_n^{temp}$. $\square$

**Theorem 3.2.29.** *Assume that $\lambda > \aleph_0$ and $\mathcal{V}$ is locally $\lambda$-presentable. Then the category of templicial objects $S_\otimes\mathcal{V}$ is locally $\lambda$-presentable.*

*Proof.* Consider the idempotent comonad $T = (-)^{nec} \circ (-)^{temp}$ on $\mathcal{C}$. Then $S_\otimes\mathcal{V}$ is equivalent to the category of coalgebras on $T$ (see the dual of [Bor94b, Corollary 4.2.4] for example). By Proposition 3.2.28, $T$ preserves $\lambda$-filtered colimits. Thus the result follows from Theorem 3.2.25 and Corollary 3.2.26. $\square$

**Proposition 3.2.30.** *Let $\mathcal{C}$ and $\mathcal{D}$ be cocomplete monoidal categories categories and $H : \mathcal{C} \to \mathcal{D}$ a lax monoidal functor which preserves $\lambda$-filtered colimits. Then the induced functor*

$$\mathcal{H} : \mathcal{C}\,\mathrm{Cat} \to \mathcal{D}\,\mathrm{Cat}$$

*preserves $\lambda$-filtered colimits as well.*

*Proof.* We first show that the induced functor $\mathcal{H} : \mathcal{C}\,\mathrm{Quiv} \to \mathcal{D}\,\mathrm{Quiv}$ preserves $\lambda$-filtered colimits. Let $D : \mathcal{J} \to \mathcal{C}\,\mathrm{Quiv} : j \mapsto (Q^j, S^j)$ be a $\lambda$-filtered diagram. Set $S = \operatorname{colim}_{j\in\mathcal{J}} S^j$. Consider the canonical quiver morphism

$$\varphi : \operatorname*{colim}_{j\in\mathcal{J}} \mathcal{H}(Q^j) \to \mathcal{H}\left(\operatorname*{colim}_{j\in\mathcal{J}} Q^j\right)$$

Then $\varphi$ is given on vertices by the identity on $S$. Take $x, y \in S$, we wish to show that the induced morphism

$$\varphi_{x,y} : \operatorname*{colim}_{j\in\mathcal{J}} \coprod_{\substack{a,b\in S^j\\ \iota_j(a)=x\\ \iota_j(b)=y}} H(Q^j(a,b)) \to H\Big(\operatorname*{colim}_{j\in\mathcal{J}} \coprod_{\substack{a,b\in S^j\\ \iota_j(a)=x\\ \iota_j(b)=y}} Q^j(a,b)\Big)$$

is an isomorphism in $\mathcal{V}$. Define a category $\overline{\mathcal{J}}_{x,y}$ with objects all triples $(j, a, b)$ with $j \in \mathcal{J}$ and $a, b \in S^j$ such that $\iota_j(a) = x$ and $\iota_j(b) = y$. A morphism $(i, a, b) \to (j, c, d)$ in $\overline{\mathcal{J}}_{x,y}$ is given by a morphism $t : i \to j$ in $\mathcal{J}$ such that $D(t)(a) = c$ and $D(t)(b) = d$. Then it is easy to see that $\overline{\mathcal{J}}_{x,y}$ is a $\lambda$-filtered category and $\varphi_{x,y}$ is isomorphic to the canonical morphism

$$\underset{(j,a,b)\in\overline{\mathcal{J}}_{x,y}}{\operatorname{colim}} H(Q^j(a,b)) \to H\Big(\underset{(j,a,b)\in\overline{\mathcal{J}}_{x,y}}{\operatorname{colim}} Q^j(a,b)\Big)$$

Hence, $\varphi_{x,y}$ is an isomorphism by the assumption that $H$ preserves $\lambda$-filtered colimits.

Now consider the following commutative diagram of functors

$$\begin{tikzcd}
\mathcal{C}\,\mathrm{Cat} \arrow[r, "\mathcal{H}"] \arrow[d] & \mathcal{D}\,\mathrm{Cat} \arrow[d] \\
\mathcal{C}\,\mathrm{Quiv} \arrow[r, "\mathcal{H}"'] & \mathcal{D}\,\mathrm{Quiv}
\end{tikzcd}$$

The vertical functors preserve filtered colimits by Proposition 3.2.23 and they are clearly conservative. Thus it follows that the top horizontal functor also preserves $\lambda$-filtered colimits. $\square$

**Proposition 3.2.31.** *Assume that $I$ is $\lambda$-presentable in $\mathcal{V}$. Then the functor $\tilde{U} : S_{\otimes}\mathcal{V} \to \mathrm{SSet}$ preserves $\lambda$-filtered colimits.*

*Proof.* In view of Propositions 3.2.11 and 3.2.28, it suffices to show that the forgetful functor $\mathcal{U} : \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec} \to \mathrm{Cat}_{\mathcal{N}ec}$ preserves $\lambda$-filtered colimits. As $I$ is $\lambda$-presentable in $\mathcal{V}$, the forgetful functor $U = \mathcal{V}(I, -) : \mathcal{V} \to \mathrm{Set}$ preserves $\lambda$-filtered colimits and thus so does the induced functor $\mathcal{V}^{\mathcal{N}ec^{op}} \to \mathrm{Set}^{\mathcal{N}ec^{op}}$. The result now follows from Proposition 3.2.30. $\square$

**Proposition 3.2.32.** *The templicial nerve $N_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat} \to S_{\otimes}\mathcal{V}$ preserves filtered colimits.*

*Proof.* In view of Propositions 3.2.14 and 3.2.28, it suffices to show that the functor $\underline{(-)}$ : $\mathcal{V}\,\mathrm{Cat} \to \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ preserves filtered colimits. As the diagonal functor $\underline{(-)} : \mathcal{V} \to \mathcal{V}^{\mathcal{N}ec^{op}}$ preserves all colimits, this follows from Proposition: 3.2.30. $\square$

Under the conditions of Theorem 3.2.29, the category of templicial objects $S_{\otimes}\mathcal{V}$ is locally presentable and thus complete. This can also be seen more directly using the embedding $(-)^{nec} : S_{\otimes}\mathcal{V} \hookrightarrow \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$.

**Proposition 3.2.33.** *Assume that $\mathcal{V}$ is complete. Then the category of templicial objects $S_{\otimes}\mathcal{V}$ is complete.*

*Proof.* As $\mathcal{V}$ is complete, so is $\mathcal{V}^{\mathcal{N}ec^{op}}$ and thus so is $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$. As a coreflective subcategory of $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$, $S_{\otimes}\mathcal{V}$ is thus also complete and the limits are inherited from $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ by applying $(-)^{temp}$ (by the dual of [Bor94a, Prop 3.5.4] for example). $\square$

We end this section by discussing the limits in $S_{\otimes}\mathcal{V}$ in a little more detail.

*Remark* 3.2.34. Fix a small category $\mathcal{J}$ and a diagram $\mathcal{J} \to S_{\otimes}\mathcal{V} : j \mapsto (X^j, S^j)$. Since $S_{\otimes}\mathcal{V}$ is a coreflective subcategory of $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$, the limit of $X$ can be calculated as

$$L = \lim_{j\in\mathcal{J}}(X^j, S^j) = \left(\lim_{j\in J}(X^j, S^j)^{nec}\right)^{temp} \in \mathcal{V}^{\mathcal{N}ec^{op}}$$

As in any category of enriched categories, the limits in $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ are given pointwise. Concretely, this means that the object set of $\lim_{j\in J}(X^j, S^j)^{nec}$ is the limit $S = \lim_{j\in\mathcal{V}} S^j$ in $\mathrm{Set}$, and for all $T \in \mathcal{N}ec$ we have

$$\left(\lim_{j\in\mathcal{V}}(X^j, S^j)^{nec}\right)_T \simeq \lim_{j\in J} \pi_j^*(X_T^j) \in \mathcal{V}\,\mathrm{Quiv}_S$$

where $\pi_j$ denotes the canonical map $S \to S^j$ for all $j \in \mathcal{J}$.

Further, the composition law of $\lim_{j\in J}(X^j, S^j)$ is determined by the canonical quiver morphisms

$$\tilde{m}_{T,U} : \lim_{j\in J} \pi_j^*(X_T^j) \otimes_S \lim_{j\in J} \pi_j^*(X_U^j) \to \lim_{j\in J}\left(\pi_j^*(X_T^j) \otimes_S \pi_j^*(X_U^j)\right) \to \lim_{j\in J} \pi_j^*(X_{T\vee U}^j)$$

where we used the lax structure of $\pi_j^*$ (Lemma 1.1.18).

Hence, it follows from Construction 3.2.8 that $L_0 = I_S$, and for all $n > 0$, $L_n$ is the limit of the following diagram of solid arrows

$$\begin{tikzcd}
L_n \arrow[r, dashed, "(\mu^L_{k,l})_{k,l}"] \arrow[d, dashed, "p_n"'] & \prod\limits_{\substack{k,l>0\\k+l=n}} L_k \otimes_S L_l \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "\prod_{k,l} p_k\otimes p_l"] & \prod\limits_{\substack{r,s,t>0\\r+s+t=n}} L_r \otimes_S L_s \otimes_S L_t \\
\lim_{j\in\mathcal{J}} \pi_j^* X_n^j \arrow[dr, "(\lim_j \pi_j^*(\mu^j_{k,l}))_{k,l}"'] & \prod\limits_{\substack{k,l>0\\k+l=n}} \lim_{j\in\mathcal{J}} \pi_j^* X_k^j \otimes_S \lim_{j\in\mathcal{J}} \pi_j^* X_l^j \arrow[d] & \\
& \prod\limits_{\substack{k,l>0\\k+l=n}} \lim_{j\in\mathcal{J}} \pi_j^*(X_k^j \otimes_{S^j} X_l^j) &
\end{tikzcd}$$

in $\mathcal{V}\,\mathrm{Quiv}_S$.

So in low dimensions, we have for all $a, b \in S$:

$$L_1(a,b) = \lim_{j\in\mathcal{J}} X_1^j(\pi_j(a), \pi_j(b))$$

while $L_2(a,b)$ is given by the pullback:

$$\begin{tikzcd}
L_2(a,b) \arrow[r] \arrow[d] & \left(\lim_{j\in\mathcal{J}} \pi_j^* X_1^j \otimes_S \lim_{j\in\mathcal{J}} \pi_j^* X_1^j\right)(a,b) \arrow[d] \\
\lim_{j\in\mathcal{J}} X_2^j(\pi_j(a), \pi_j(b)) \arrow[r] & \lim_{j\in\mathcal{J}}\left((X_1^j \otimes_{S^j} X_1^j)(\pi_j(a), \pi_j(b))\right)
\end{tikzcd}$$

where the bottom horizontal morphism is induced by the comultiplications $\mu_{1,1}^j$ of each templicial object $X^j$.

**Example 3.2.35.** Applying Remark 3.2.34 to the empty diagram $\mathcal{J} = \emptyset$, we find that the terminal object $1$ of $S_{\otimes}\mathcal{V}$ is given as follows. Its set of vertices is a singleton $\{*\}$ and for all $n \geq 0$,

$$1_n = \underbrace{1 \otimes ... \otimes 1}_{n \text{ times}}$$

where $1$ represents the terminal object of $\mathcal{V}$. Note that this is isomorphic to the templicial nerve of the terminal object in $\mathcal{V}$-$\mathrm{Cat}$. This isn't surprising as $N_{\mathcal{V}}$ is a right-adjoint by Proposition 2.3.14. Let us consider some special cases:

- In case $\mathcal{V}$ is (semi-)cartesian, the terminal object $1$ of $S_{\otimes}\mathcal{V}$ reduces to the constant functor on the terminal object $1$ of $\mathcal{V}$.
- In case $\mathcal{V} = \mathrm{Mod}(k)$ with $k$ a unital commutative ring, then $1_n = 0$ for all $n > 0$ and $1_0 = k$. Note that this is not isomorphic to the initial object of $S_{\otimes}\mathrm{Mod}(k)$ since the latter has an empty vertex set. So unlike $\mathrm{Mod}(k)$, the category of templicial $k$-modules does not have a zero object.

# Chapter 4

# Examples of quasi-categories in a monoidal category

*"Kate Walker! I see you managed to produce two XZ2005_B models!"*

- Oscar (Syberia)

There are a couple of classical constructions which provide examples of quasi-categories as simplicial sets. In Chapter 1, we already discussed the nerve functor $N : \mathrm{Cat} \to \mathrm{SSet}$ and its templicial analogue $N_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat} \to S_{\otimes}\mathcal{V}$.

Another classical construction is the *homotopy coherent nerve* $N^{hc} : \mathrm{Cat}_{\Delta} \to \mathrm{SSet}$ from simplicial categories (that is, categories enriched in simplicial sets) to simplicial sets. It was introduced by Cordier in [Cor82]. Later, Cordier and Porter showed in [CP86, Theorem 2.1] that the homotopy coherent nerve $N^{hc}(\mathcal{C})$ of a simplicial category $\mathcal{C}$ is a quasi-category if every hom-object $\mathcal{C}(A, B)$ is a Kan complex. They also constructed the left-adjoint $\mathfrak{C} : \mathrm{SSet} \to \mathrm{Cat}_{\Delta}$ to $N^{hc}$. This *categorification functor* was later described in a very elegant way by Dugger and Spivak in [DS11b].

Fixing a unital commutative ring $k$, there is also the *differential graded (dg) nerve* $N^{dg} : k\,\mathrm{Cat}_{dg} \to \mathrm{SSet}$ from differential graded categories over $k$ to simplicial sets, see [Lur16]. Lurie showed that the dg-nerve $N^{dg}(\mathcal{C})$ of any dg-category $\mathcal{C}_{\bullet}$ is always a quasi-category.

The two sections of this chapter are devoted to constructing templicial analogues of the homotopy coherent nerve and the dg-nerve respectively. For the former, the two occurences of simplicial sets are generalized differently. The category $\mathrm{SSet}$ on its own is replaced by $S_{\otimes}\mathcal{V}$. But $\mathrm{SSet}$ as enriching category for $\mathrm{Cat}_{\Delta}$ is replaced by the category $S\mathcal{V}$ of simplicial objects in $\mathcal{V}$. This yields the category $\mathcal{V}\,\mathrm{Cat}_{\Delta}$ of small $S\mathcal{V}$-enriched categories. Inspired by Dugger and Spivak's description of the categorification functor $\mathfrak{C}$, we then construct an adjunction $\mathfrak{C}_{\mathcal{V}} : S_{\otimes}\mathcal{V} \leftrightarrows \mathcal{V}\,\mathrm{Cat}_{\Delta} : N^{hc}_{\mathcal{V}}$ (§4.1.2) which recovers the classical adjunction when $\mathcal{V} = \mathrm{Set}$. To do this, we'll use the category $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ of necklace categories (Definition 3.2.3) as an intermediate step. Moreover, if an $S\mathcal{V}$-enriched category $\mathcal{C}$ is locally Kan, then $N^{hc}_{\mathcal{V}}$ will be a quasi-category in $\mathcal{V}$ (Proposition 4.1.20).

The bulk of this chapter is contained in Section 4.2, where we lift the classical dg-nerve $N^{dg}$ along $\tilde{U}$ to obtain the *linear dg-nerve* $N^{dg}_{k} : k\,\mathrm{Cat}_{dg} \to S_{\otimes}\,\mathrm{Mod}(k)$. This goes in two major steps. The first equates dg-categories with categories enriched in augmented simplicial modules $S^+ \operatorname{Mod}(k)$, through an augmented version of the Dold-Kan correspondence (Proposition 4.2.10). The second is an equivalence between these $S^+ \operatorname{Mod}(k)$-*categories* and templicial $k$-modules equipped with a Frobenius structure (Theorem 4.2.17). Moreover, we will show that $N_k^{dg}(\mathcal{C})$ is a quasi-category in $\operatorname{Mod}(k)$ for any dg-category $\mathcal{C}_\bullet$ (Corollary 4.2.65). Finally, we'll compare the linear dg-nerve to the other nerves defined earlier.

## 4.1 Categories enriched in simplicial objects

In this section we generalize the adjunction between the categorification functor $\mathfrak{C} : \operatorname{SSet} \to \operatorname{Cat}_\Delta$ and the homotopy coherent nerve $N^{hc} : \operatorname{Cat}_\Delta \to \operatorname{SSet}$ to the templicial level, yielding an adjunction $\mathfrak{C}_\mathcal{V} \dashv N^{hc}_\mathcal{V}$ which depends on $\mathcal{V}$. One can quickly see that $\mathfrak{C}$ actually factors through the category $\operatorname{Cat}_{\mathcal{N}ec}$ of Definition 3.2.3. Moreover, the functor $\operatorname{Cat}_{\mathcal{N}ec} \to \operatorname{Cat}_\Delta$ is determined on hom-objects by a functor $\mathfrak{n} : \operatorname{Set}^{\mathcal{N}ec^{op}} \to \operatorname{SSet}$. It is now straightforward to generalize $\mathfrak{n}$ to a functor $\mathcal{V}^{\mathcal{N}ec^{op}} \to S\mathcal{V}$ (Construction 4.1.11), which in turn induces a functor $\mathcal{V}\operatorname{Cat}_{\mathcal{N}ec} \to \mathcal{V}\operatorname{Cat}_\Delta$. Here, $\mathcal{V}\operatorname{Cat}_\Delta$ denotes the category of all small $S\mathcal{V}$-enriched categories. Composing with the embedding $(-)^{nec} : S_\otimes\mathcal{V} \hookrightarrow \mathcal{V}\operatorname{Cat}_{\mathcal{N}ec}$ then yields our templicial categorification $\mathfrak{C}_\mathcal{V} : S_\otimes\mathcal{V} \to \mathcal{V}\operatorname{Cat}_\Delta$. Finally, the templicial homotopy coherent nerve is obtained as the right-adjoint to $\mathfrak{C}_\mathcal{V}$.

We open this section by recalling the classical adjunction $\mathfrak{C} \dashv N^{hc}$. Then we generalize it to $\mathfrak{C}_\mathcal{V} \dashv N^{hc}_\mathcal{V}$, using the approach outlined above. Following Dugger and Spivak's description of $\mathfrak{C}$, also $\mathfrak{C}_\mathcal{V}[X]$ can be reformulated in a similar way, if we assume that the templicial object $X$ is free (Proposition 4.1.28). We conclude the section by comparing the templicial homotopy coherent nerve $N^{hc}_\mathcal{V}$ to the templicial nerve $N_\mathcal{V}$ defined in §2.3.1 (Proposition 4.1.33).

For this section, we impose the additional standing hypotheses that $(\mathcal{V}, \otimes, I)$ is complete and symmetric monoidal closed. In other words, $\mathcal{V}$ is a Bénabou cosmos.

### 4.1.1 The classical homotopy coherent nerve

We recall the definitions of the *categorification functor* $\mathfrak{C} : \operatorname{SSet} \to \operatorname{Cat}_\Delta$ and its right-adjoint, the *homotopy coherent nerve* $N^{hc} : \operatorname{Cat}_\Delta \to \operatorname{SSet}$. This is taken from [Lur09a, §1.1.5] and [DS11b].

**Definition 4.1.1.** Consider the category of simplicial sets $\operatorname{SSet}$ as a monoidal category with the cartesian product: $(\operatorname{SSet}, \times, \Delta^0)$. A *simplicial category* is a category enriched in $\operatorname{SSet}$. A *simplicial functor* is a $\operatorname{SSet}$-enriched functor between simplicial categories. We denote

$$\operatorname{Cat}_\Delta$$

for the category of small simplicial categories and simplicial functors between them.

**Construction 4.1.2.** Let $n \geq 0$, we construct a simplicial category $\mathfrak{C}[\Delta^n]$ as follows. Its objects are given by the set $[n]$ and for all $i, j \in [n]$, we set

$$\mathfrak{C}[\Delta^n] = \begin{cases} N(\mathcal{P}_{j-i}) & \text{if } i \leq j \\ \emptyset & i > j \end{cases}$$

where $\mathcal{P}_{j-i}$ denotes the poset of Notation 2.2.9 and $N : \mathrm{Cat} \to \mathrm{SSet}$ the classical nerve functor (Definition 1.3.22). Note that $N(\mathcal{P}_{j-i}) \simeq (\Delta^1)^{\times j-i-1}$ if $i < j$ and $N(\mathcal{P}_{j-i}) \simeq \Delta^0$ if $i = j$. Further, given $i \leq j \leq k$ in $[n]$, the reverse composition (see Remark 1.1.22)

$$\tilde{m}_{i,j,k} : \mathfrak{C}[\Delta^n](i,j) \times \mathfrak{C}[\Delta^n](j,k) \to \mathfrak{C}[\Delta^n](i,k)$$

is given by applying $N$ to the order morphism

$$\mathcal{P}_{j-i} \times \mathcal{P}_{k-j} \to \mathcal{P}_{k-i} : (T, U) \mapsto T \vee U$$

of Remark 2.2.10. Finally, the identities are given by the unique vertex of $\mathfrak{C}[\Delta^n](i,i) \simeq \Delta^0$ for all $i \in [n]$.

Further, given $f : [m] \to [n]$ in $\mathbf{\Delta}$, we construct a simplicial functor $\mathfrak{C}[\Delta^m] \to \mathfrak{C}[\Delta^n]$ as follows. On objects, it is given by the map $f : [m] \to [n]$ itself. For all $i \leq j$ in $[m]$, $f$ induces a morphism in $\mathbf{\Delta}_f$:

$$f_{i,j} : [j-i] \to [f(j) - f(i)] : k \mapsto f(k+i) - f(i)$$

Then the map

$$\mathfrak{C}[\Delta^m](i,j) \to \mathfrak{C}[\Delta^n](f(i), f(j))$$

is given by applying $N$ to the order morphism of Remark 2.2.10 induced by $f_{i,j}$:

$$\mathcal{P}(f_{i,j}) : \mathcal{P}_{j-i} \to \mathcal{P}_{f(j)-f(i)} : T \mapsto f_{i,j}(T)$$

It is now easy to see that the above constructions define a functor

$$\mathfrak{C}[\Delta^{(-)}] : \mathbf{\Delta} \to \mathrm{Cat}_\Delta$$

**Definition 4.1.3.** Consider the cosimplicial object $\mathfrak{C}[\Delta^{(-)}]$ of Construction 4.1.2. Then Proposition 1.3.11 provides an adjunction

$$\begin{tikzcd} \mathrm{SSet} \arrow[r, shift left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_\Delta \arrow[l, shift left, "N^{hc}"] \end{tikzcd}$$

The left-adjoint $\mathfrak{C}$ is called the *categorification functor* and is given by left Kan extension of $\mathfrak{C}[\Delta^{(-)}]$ along the Yoneda embedding $\mathbf{\Delta} \hookrightarrow \mathrm{SSet}$. The right-adjoint $N^{hc}$ is called the *homotopy coherent nerve functor*. We have for all small simplicial categories $\mathcal{C}$ and $n \geq 0$ that

$$N^{hc}(\mathcal{C})_n \simeq \mathrm{Cat}_\Delta(\mathfrak{C}[\Delta^n], \mathcal{C})$$

**Example 4.1.4.** Given a small simplicial category $\mathcal{C}$, let us describe its homotopy coherent nerve in low dimensions.

- The vertices of $N^{hc}(\mathcal{C})$ are given by the set of objects $\mathrm{Ob}(\mathcal{C})$.

- The edges of $N^{hc}(\mathcal{C})$ are given by the morphisms of $\mathcal{C}$ (that is vertices $f \in \mathcal{C}_0(A, B)$ for some $A, B \in \mathrm{Ob}(\mathcal{C})$).
- A 2-simplex of $N^{hc}(\mathcal{C})$ is given by a (not necessarily commutative) diagram of morphisms in $\mathcal{C}$:

$$\begin{tikzcd} & C \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow\sigma"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & B \end{tikzcd}$$

  along with an edge $\sigma$ of $\mathcal{C}(A, B)$ from $h$ to the composition $g \circ f$.

The first steps of Dugger and Spivak's streamlining of the description of $\mathfrak{C}$ are outlined in the following two propositions. For now this will suffice. We will return to their reduction in §4.1.3.

**Proposition 4.1.5** ([DS11b], Proposition 3.7)**.** *There is an isomorphism of simplicial sets*

$$\mathfrak{C}[T](0, p) \simeq N(\mathcal{P}_T)$$

*that is natural in all necklaces $(T, p) \in \mathcal{N}ec$.*

**Proposition 4.1.6** ([DS11b], Proposition 4.3)**.** *For every simplicial set $K$ with vertices $a$ and $b$, there is an isomorphism of simplicial sets*

$$\mathfrak{C}[K](a, b) \simeq \operatorname*{colim}_{\substack{T \to K_{a,b} \text{ in } \mathrm{SSet}_{*,*} \\ (T,p) \in \mathcal{N}ec}} \mathfrak{C}[T](0, p)$$

We now introduce another way of describing the categorification, this time by means of a weighted colimit. This will make it easier to generalize to the context of templicial objects. For background on weighted (co)limits, we refer to the relevant literature (for example, see [Rie14]). Recall by Proposition 2.1.15 that we may view a simplicial set $K$ as a templicial set and thus we can apply Construction 2.2.16 to obtain a functor $K_\bullet(a, b) : \mathcal{N}ec^{op} \to \mathrm{Set} : T \mapsto K_T(a, b)$ for any two vertices $a$ and $b$ of $K$.

**Proposition 4.1.7.** *For any simplicial set $K$ with vertices $a$ and $b$, $\mathfrak{C}[K](a, b)$ is isomorphic to the weighted colimit in* SSet*:*

$$\mathrm{colim}^{K_\bullet(a,b)} N\mathcal{P}_{(-)}$$

*of $N\mathcal{P}_{(-)} : \mathcal{N}ec \to \mathrm{SSet}$ with weight $K_\bullet(a, b) : \mathcal{N}ec^{op} \to \mathrm{Set}$.*

*Proof.* From Propositions 4.1.5 and 4.1.6, it is clear that $\mathfrak{C}[K](a, b)$ is isomorphic to the following coequalizer in SSet:

$$\coprod_{\substack{T \to U \to K_{a,b} \\ T, U \in \mathcal{N}ec}} N(\mathcal{P}_T) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{\substack{T \to K_{a,b} \\ T \in \mathcal{N}ec}} N(\mathcal{P}_T) \twoheadrightarrow \mathfrak{C}[K](a, b)$$

where $\alpha$ and $\beta$ are given by respectively projecting onto $T \to K_{a,b}$ and applying $N\mathcal{P}_{(-)}$ to $T \to U$ for any $T \to U \to K_{a,b}$ in $\mathrm{SSet}_{*,*}$.

Now note that a morphism $T \to K_{a,b}$ in $\mathrm{SSet}_{*,*}$ with $T$ a necklace, is equivalent to an element of the set $K_T(a,b)$ (which we consider as a constant simplicial set). Hence, we obtain a coequalizer diagram

$$\coprod_{\substack{T \to U \\ \text{in } \mathcal{N}ec}} K_U(a,b) \times N(\mathcal{P}_T) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{T \in \mathcal{N}ec} K_T(a,b) \times N(\mathcal{P}_T) \longrightarrow\!\!\!\!\!\rightarrow \mathfrak{C}[K](a,b)$$

where $\alpha$ and $\beta$ are given by respectively applying $K_\bullet(a,b)$ and $N\mathcal{P}_{(-)}$ to $T \to U$ in $\mathcal{N}ec$. But this coequalizer is precisely the weighted colimit described in the statement. $\square$

### 4.1.2 The templicial homotopy coherent nerve

We quickly recall the pointwise monoidal structure on the category of simplicial objects in $\mathcal{V}$, so that we can define categories enriched over $S\mathcal{V}$.

**Construction 4.1.8.** It is easily seen that the monoidal structure on $\mathcal{V}$ induces a symmetric monoidal structure on $S\mathcal{V}$. Given simplicial objects $X$ and $Y$, their monoidal product $X \otimes Y$ is given by, for all $n \geq 0$:

$$(X \otimes Y)_n = X_n \otimes Y_n$$

The monoidal unit in $S\mathcal{V}$ is then given by the constant functor on the monoidal unit $I$ of $\mathcal{V}$, which is isomorphic to $F\Delta^0$. Note that in particular for $\mathcal{V} = \mathrm{Set}$, we recover the cartesian product on simplicial sets $(\mathrm{SSet}, \times, \Delta^0)$.

Moreover, the category $S\mathcal{V}$ is enriched and tensored over $\mathcal{V}$. Given two simplicial objects $X$ and $Y$, we denote their hom-object by $[X,Y] \in \mathcal{V}$. It is the object of natural tranformations from $X$ to $Y$ and can be realized as the following equalizer in $\mathcal{V}$:

$$[X,Y] \hookrightarrow \prod_{n \geq 0} \underline{\mathcal{V}}(X_n, Y_n) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{\substack{f:[m] \to [n] \\ \text{in } \mathbf{\Delta}}} \underline{\mathcal{V}}(X_n, Y_m)$$

where $\underline{\mathcal{V}}(-,-)$ denotes the internal hom of $\mathcal{V}$, which exists by the standing hypotheses. Further, $\alpha$ and $\beta$ are given by respectively post-composing with $Y(f)$ and pre-composing with $X(f)$ for a morphism $f : [m] \to [n]$ in $\mathbf{\Delta}$.

The tensoring $V \cdot X$ of an object $V \in \mathcal{V}$ with a simplicial object $X$ is given simply by the monoidal product $\underline{V} \otimes X$ where $\underline{V}$ is the constant simplicial object on $V$.

**Definition 4.1.9.** An *$S\mathcal{V}$-category* is a category enriched in the symmetric monoidal category $(S\mathcal{V}, \otimes, F\Delta^0)$ of Construction 4.1.8. An *$S\mathcal{V}$-functor* is an $S\mathcal{V}$-enriched functor. We denote the category of small $S\mathcal{V}$-categories and $S\mathcal{V}$-functors by

$$\mathcal{V}\,\mathrm{Cat}_\Delta = S\mathcal{V}\,\mathrm{Cat}$$

Note that in particular if $\mathcal{V} = \mathrm{Set}$, we recover the category of small simplicial categories $\mathrm{Cat}_\Delta$.

**Notation 4.1.10.** Consider the monoidal adjunction $F : \mathrm{Set} \leftrightarrows \mathcal{V} : U$. It induces a monoidal adjunction $F : \mathrm{SSet} \leftrightarrows S\mathcal{V} : U$ by post-composition. Hence, by Proposition

1.1.23 we have an induced adjunction between simplicial categories and $S\mathcal{V}$-categories which we denote by

$$\mathrm{Cat}_\Delta \underset{\mathcal{U}}{\overset{\mathcal{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{V}\,\mathrm{Cat}_\Delta$$

We are now ready to construct the templicial analogue $\mathfrak{C}_{\mathcal{V}} : S_\otimes\mathcal{V} \to \mathcal{V}\,\mathrm{Cat}_\Delta$ of the categorification functor $\mathfrak{C}$. The templicial homotopy coherent nerve $N^{hc}_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat}_\Delta \to S_\otimes\mathcal{V}$ will then be defined as the right-adjoint to $\mathfrak{C}_{\mathcal{V}}$ (see Definition 4.1.13). Note that by Proposition 4.1.7 it is clear that $\mathfrak{C}$ factors through the functor $(-)^{nec} : \mathrm{SSet} \to \mathrm{Cat}_{\mathcal{N}ec}$ of Construction 3.2.5. Therefore, in order to construct $\mathfrak{C}_{\mathcal{V}}$, we will first build an adjunction between $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ and $\mathcal{V}\,\mathrm{Cat}_\Delta$.

**Construction 4.1.11.** We construct an adjunction

$$\mathcal{V}^{\mathcal{N}ec^{op}} \underset{\mathfrak{n}}{\overset{\mathfrak{s}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} S\mathcal{V}$$

as follows. Given a functor $X : \mathcal{N}ec^{op} \to \mathcal{V}$, consider the weighted colimit in $S\mathcal{V}$:

$$\mathfrak{s}(X) = \operatorname*{colim}_{T\in\mathcal{N}ec}{}^{X_T} FN\mathcal{P}_T$$

of the composite $\mathcal{N}ec \xrightarrow{\mathcal{P}_{(-)}} \mathrm{Cat} \xrightarrow{N} \mathrm{SSet} \xrightarrow{F} S\mathcal{V}$ with weight $X$. Explicitly, $\mathfrak{s}(X)$ may be realized as the following reflexive coequalizer in $S\mathcal{V}$:

$$\coprod_{\substack{f:T\to U\\ \text{in } \mathcal{N}ec}} X_U \otimes FN(\mathcal{P}_T) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{T\in\mathcal{N}ec} X_T \otimes FN(\mathcal{P}_T) \twoheadrightarrow \mathfrak{s}(X) \tag{4.1}$$

where $\alpha$ and $\beta$ are given by respectively applying $X$ and $FN\mathcal{P}_{(-)}$ to a necklace morphism $f : T \to U$, and $\gamma$ is given by selecting the identity $\mathrm{id}_T$ for any necklace $T$.

As a weighted colimit, $\mathfrak{s}(X)$ fits into a canonical bijection of sets

$$S\mathcal{V}(\mathfrak{s}(X), Y) \simeq \mathcal{V}^{\mathcal{N}ec^{op}}(X, [FN\mathcal{P}_{(-)}, Y])$$

which is natural in $Y \in S\mathcal{V}$ (see [Rie14, Definition 7.4.1] for example). Hence, the assignment $X \mapsto \mathfrak{s}(X)$ extends to a functor $\mathfrak{s} : \mathcal{V}^{\mathcal{N}ec^{op}} \to S\mathcal{V}$ which is left-adjoint to the functor

$$\mathfrak{n} : S\mathcal{V} \to \mathcal{V}^{\mathcal{N}ec^{op}} : Y \mapsto [FN\mathcal{P}_{(-)}, Y]$$

**Proposition 4.1.12.** *The adjunction $\mathfrak{s} \dashv \mathfrak{n}$ of Construction 4.1.11 is monoidal in the sense of Definition 1.1.5.*

*Proof.* Consider the monoidal unit $\underline{I}$ of $\mathcal{V}^{\mathcal{N}ec^{op}}$, that is the constant functor on $I$. Let $* : \mathcal{N}ec^{op} \to \mathrm{Set}$ denote the constant functor on a singleton so that $F(*) \simeq \underline{I}$. Note that $\mathcal{P}_{\{0\}}$ contains a single element and thus $N\mathcal{P}_{\{0\}} \simeq \Delta^0$. Then since $F : \mathrm{SSet} \to S\mathcal{V}$ preserves colimits and $\{0\}$ is the terminal object of $\mathcal{N}ec$, we have

$$\mathfrak{s}(\underline{I}) \simeq F\left(\operatorname*{colim}_{T\in\mathcal{N}ec}{}^{*} N\mathcal{P}_T\right) \simeq F\left(\operatorname*{colim}_{T\in\mathcal{N}ec} N\mathcal{P}_T\right) \simeq FN\mathcal{P}_{\{0\}} \simeq F\Delta^0$$

Now let $X$ and $Y$ be functors $\mathcal{N}ec^{op} \to \mathcal{V}$ and consider their Day convolution $X \otimes_{Day} Y$ (Construction 3.2.1). Note that by Remark 2.2.10, we have $FN\mathcal{P}_U \otimes FN\mathcal{P}_V \simeq FN\mathcal{P}_{U\vee V}$ for all necklaces $U$ and $V$. We construct a commutative diagram in $S\mathcal{V}$ as follows:

$$
\begin{tikzcd}
\coprod\limits_{\substack{U\to U'\\ N\to N'\\ V\to V'\\ \text{in } \mathcal{N}ec}} (X_{U'}\otimes Y_{V'})\otimes FN\mathcal{P}_{U\vee N\vee V} \arrow[r, shift left=2, "\alpha"] \arrow[r, shift right=2, "\beta"'] \arrow[d, shift left=2] \arrow[d, shift right=2] & \coprod\limits_{U,N,V\in\mathcal{N}ec} (X_U\otimes Y_V)\otimes FN\mathcal{P}_{U\vee N\vee V} \arrow[l, "\gamma"'] \arrow[r, two heads] \arrow[d, shift left=2, "\beta'"] \arrow[d, shift right=2, "\alpha'"'] & C \arrow[d, shift left=2] \arrow[d, shift right=2] \\
\coprod\limits_{\substack{U\to U'\\ V\to V'\\ \text{in } \mathcal{N}ec}} (X_{U'}\otimes Y_{V'})\otimes FN\mathcal{P}_{U\vee V} \arrow[u] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, two heads] & \coprod\limits_{U,V\in\mathcal{N}ec} (X_U\otimes Y_V)\otimes FN\mathcal{P}_{U\vee V} \arrow[u] \arrow[l] \arrow[r, two heads, "q"'] \arrow[d, two heads] & \mathfrak{s}(X)\otimes\mathfrak{s}(Y) \arrow[u] \arrow[d, two heads] \\
\coprod\limits_{\substack{T\to T'\\ \text{in } \mathcal{N}ec}} (X\otimes_{Day} Y)_{T'}\otimes FN\mathcal{P}_{T} \arrow[r, shift left=2] \arrow[r, shift right=2] & \coprod\limits_{T\in\mathcal{N}ec} (X\otimes_{Day} Y)_{T}\otimes FN\mathcal{P}_{T} \arrow[l] \arrow[r] & \mathfrak{s}(X\otimes_{Day} Y)
\end{tikzcd}
$$

The middle and bottom rows are reflexive coequalizers induced by (4.1). In the top row, $\alpha$ is given by applying $X$ and $Y$ to the morphisms $U \to U'$ and $V \to V'$, and $\beta$ is given by applying $FN\mathcal{P}_{(-)}$ to the morphism $U \vee N \vee V \to U' \vee N' \vee V'$. Also, $\gamma$ selects the identities $\mathrm{id}_U$, $\mathrm{id}_N$ and $\mathrm{id}_V$ for any $U, N, V \in \mathcal{N}ec$. Then clearly $\alpha\gamma = \mathrm{id} = \beta\gamma$ and we define $C$ as the coequalizer of $\alpha$ and $\beta$. The middle column is a reflexive coequalizer provided by Proposition 3.2.2. The left hand column is also a reflexive coequalizer and can be constructed similarly to the proof of Proposition 3.2.2. Consequently, we have an induced reflexive coequalizer in the right hand column (apply the the dual of [Agu97, Lemma 1.1.2] for example).

We claim that the epimorphism $\mathfrak{s}(X)\otimes\mathfrak{s}(Y) \twoheadrightarrow \mathfrak{s}(X\otimes_{Day} Y)$ is an isomorphism. For this, it would suffice that the two morphisms $C \to \mathfrak{s}(X)\otimes\mathfrak{s}(Y)$ coincide, which will follow if we can show that $q\alpha' = q\beta'$. Take $U, N, V \in \mathcal{N}ec$ and set $T = U \vee N \vee V$. Denote by $\sigma$ the unique necklace map $N \to \{0\}$. Then we have

$$
\begin{aligned}
q\alpha'\iota_{U,N,V} &= q\iota_{U\vee N,V}((X(\mathrm{id}_U \vee\sigma)\otimes \mathrm{id}_{Y_V})\otimes \mathrm{id}_{FN\mathcal{P}_T} \\
&= q\iota_{U,V}(\mathrm{id}_{X_U\otimes Y_V} \otimes FN\mathcal{P}(\mathrm{id}_U \vee\sigma\vee \mathrm{id}_V)) \\
&= q\iota_{U,N\vee V}((\mathrm{id}_{X_U} \otimes Y(\sigma\vee \mathrm{id}_V))\otimes \mathrm{id}_{FN\mathcal{P}_T} = q\beta'\iota_{U,N,V}
\end{aligned}
$$

by virtue of the coequalizer in the middle row of the above diagram.

Finally, it is readily verified that the isomorphism $\mathfrak{s}(X)\otimes\mathfrak{s}(Y) \simeq \mathfrak{s}(X\otimes_{Day} Y)$ is natural in $X$ and $Y$, and defines a strong monoidal structure on the functor $\mathfrak{s}$. □

**Definition 4.1.13.** By virtue of Propositions 4.1.12 and 1.1.23, the adjunction $\mathfrak{s} \dashv \mathfrak{n}$ between $\mathcal{V}^{\mathcal{N}ec^{op}}$ and $S\mathcal{V}$ induces an adjunction

$$
\begin{tikzcd}
\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec} \arrow[r, shift left=2, "\mathfrak{s}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\,\mathrm{Cat}_{\Delta} \arrow[l, shift left=2, "\mathfrak{n}"]
\end{tikzcd}
$$

We call the composite

$$\mathfrak{C}_{\mathcal{V}} : S_{\otimes}\mathcal{V} \xrightarrow{(-)^{nec}} \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec} \xrightarrow{\mathfrak{s}} \mathcal{V}\,\mathrm{Cat}_{\Delta}$$

the *categorification functor*. It is left-adjoint to the composite

$$N^{hc}_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat}_{\Delta} \xrightarrow{\mathfrak{n}} \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec} \xrightarrow{(-)^{temp}} S_{\otimes}\mathcal{V}$$

which we call the *templicial homotopy coherent nerve*.

**Example 4.1.14.** Suppose $\mathcal{V} = \mathrm{Set}$. Then the adjunction $\mathfrak{C}_{\mathcal{V}} \dashv N^{hc}_{\mathcal{V}}$ reduces to the classical adjunction $\mathfrak{C} \dashv N^{hc}$. Indeed, it suffices to note that $\mathfrak{C}_{\mathcal{V}}$ reduces to $\mathfrak{C}$, which follows from Proposition 4.1.7 and Construction 4.1.11.

*Remark* 4.1.15. Given a templicial object $(X, S)$, let us make the structure of $\mathfrak{C}_{\mathcal{V}}[X]$ as a $S\mathcal{V}$-enriched category a little more explicit.

The object set of $\mathfrak{C}_{\mathcal{V}}[X]$ is just $S$. By Construction 4.1.11 we have a reflexive coequalizer diagram for all $a, b \in S$:

$$\coprod_{\substack{f:T\to U\\ \text{in } \mathcal{N}ec}} X_U(a,b) \otimes FN(\mathcal{P}_T) \underset{\beta}{\overset{\alpha}{\underset{\leftarrow\gamma-}{\rightrightarrows}}} \coprod_{T\in\mathcal{N}ec} X_T(a,b)\otimes FN(\mathcal{P}_T) \longrightarrow\!\!\!\!\!\rightarrow \mathfrak{C}_{\mathcal{V}}[X](a,b) \tag{4.2}$$

where $\alpha$ and $\beta$ are given by respectively applying $X_{\bullet}(a,b)$ and $FN\mathcal{P}_{(-)}$ to a necklace map $f : T \to U$, and $\gamma$ is given by selecting the identity $\mathrm{id}_T$ for any necklace $T$.

Take $a, b, c \in S$. Then the reverse composition law

$$\tilde{m}_{a,b,c} : \mathfrak{C}_{\mathcal{V}}[X](a,b) \otimes \mathfrak{C}_{\mathcal{V}}[X](b,c) \to \mathfrak{C}_{\mathcal{V}}[X](a,c)$$

of $\mathfrak{C}_{\mathcal{V}}[X]$ is induced by the morphisms $X_T(a,b)\otimes X_U(b,c) \to X_{T\vee U}(a,c)$ and the isomorphisms $\mathcal{P}_T \times \mathcal{P}_U \simeq \mathcal{P}_{T\vee U}$ (see Remark 2.2.10) which are natural in $T, U \in \mathcal{N}ec$.

Take $a \in S$. Then the identity on $a$ in $\mathfrak{C}_{\mathcal{V}}[X]$ is given by

$$u_a : F\Delta^0 \simeq X_0(a,a) \otimes FN(\mathcal{P}_0) \to \mathfrak{C}_{\mathcal{V}}[X](a,a)$$

where the isomorphism is induced by $N(\mathcal{P}_{\{0\}}) \simeq \Delta^0$ and $X_0(a,a) \simeq I$.

**Example 4.1.16.** Let $\mathcal{C}$ be a small $S\mathcal{V}$-category. We describe the templicial object $N^{hc}_{\mathcal{V}}(\mathcal{C})$ in low dimenions. Note the analogy with Example 4.1.4.

- The vertex set of $N^{hc}_{\mathcal{V}}(\mathcal{C})$ is simply $\mathrm{Ob}(\mathcal{C})$.
- Further for any $A, B \in \mathrm{Ob}(\mathcal{C})$, it follows from $N(\mathcal{P}_{\{0<1\}}) \simeq \Delta^0$ that

$$N^{hc}_{\mathcal{V}}(\mathcal{C})_1(A,B) = \mathfrak{n}(\mathcal{C})_{\{0<1\}}(A,B) = [FN(\mathcal{P}_{\{0<1\}}), \mathcal{C}(A,B)] \simeq \mathcal{C}_0(A,B)$$

- In dimension 2, it follows from $N(\mathcal{P}_{\{0<2\}}) \simeq \Delta^1$ and $N(\mathcal{P}_{\{0<1<2\}}) \simeq \Delta^0$ that

$$\mathfrak{n}(\mathcal{C})_{\{0<2\}}(A,B) = [FN(\mathcal{P}_{\{0<2\}}), \mathcal{C}(A,B)] \simeq \mathcal{C}_1(A,B)$$
$$\mathfrak{n}(\mathcal{C})_{\{0<1<2\}}(A,B) = [FN(\mathcal{P}_{\{0<1<2\}}), \mathcal{C}(A,B)] \simeq \mathcal{C}_0(A,B)$$

  The morphism $\mathfrak{n}(\mathcal{C})_{\{0<2\}}(A,B) \to \mathfrak{n}(\mathcal{C})_{\{0<1<2\}}(A,B)$ is induced by inert necklace map $\nu_{1,1} : \{0<1<2\} \hookrightarrow \{0<2\}$ and thus corresponds to the face map

$d_0 : \mathcal{C}_1(A, B) \to \mathcal{C}_0(A, B)$. It follows from (3.3) that we have a pullback diagram:

$$\begin{tikzcd} N^{hc}_{\mathcal{V}}(\mathcal{C})_2(A,B) \arrow[r] \arrow[d] & \coprod_{C \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}_0(A,C) \otimes \mathcal{C}_0(C,B) \arrow[d, "\tilde{m}_{0,0}"] \\ \mathcal{C}_1(A,B) \arrow[r, "d_0"'] & \mathcal{C}_0(A,B) \end{tikzcd}$$

So applying $U$, we see that the underlying set of the object $N^{hc}_{\mathcal{V}}(\mathcal{C})_2(A, B)$ consists of pairs $(\sigma, \alpha)$ with $\alpha \in U(\coprod_{C \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}_0(A, C) \otimes \mathcal{C}_0(C, B))$ and $\sigma$ an edge in $\mathcal{C}(A, B)$ from $h = d_1(\sigma)$ to $\tilde{m}(\alpha)$.

**Proposition 4.1.17.** *There are canonical natural isomorphisms*

$$\mathfrak{C}_{\mathcal{V}} \circ \tilde{F} \simeq \mathcal{F} \circ \mathfrak{C} \quad \textit{and} \quad \tilde{U} \circ N^{hc}_{\mathcal{V}} \simeq N^{hc} \circ \mathcal{U}$$

*with $\mathcal{F} \dashv \mathcal{U}$ the adjunction from Notation 4.1.10.*

*Proof.* As $\mathfrak{C}_{\mathcal{V}} \dashv N^{hc}_{\mathcal{V}}$, $\mathfrak{C} \dashv N^{hc}$, $\tilde{F} \dashv \tilde{U}$ and $\mathcal{F} \dashv \mathcal{U}$, it suffices to only show the first natural isomorphism. Since $F : \mathrm{SSet} \to S\mathcal{V}$ preserves colimits and is strong monoidal, it is clear that

$$\operatorname*{colim}_{T \in \mathcal{N}ec}{}^{FX_T} FN\mathcal{P}_T \simeq F\left(\operatorname*{colim}_{T \in \mathcal{N}ec}{}^{X_T} N\mathcal{P}_T\right)$$

for any functor $X : \mathcal{N}ec^{op} \to \mathrm{Set}$. It follows that we have a natural isomorphism $F \circ \mathfrak{s} \simeq \mathfrak{s} \circ F$ of functors $\mathrm{Set}^{\mathcal{N}ec^{op}} \to S\mathcal{V}$, and thus also $\mathcal{F} \circ \mathfrak{s} \simeq \mathfrak{s} \circ \mathcal{F}$ of functors $\mathrm{Cat}_{\mathcal{N}ec} \to \mathcal{V}\,\mathrm{Cat}_{\Delta}$. Thus by Proposition 3.2.11 we have

$$\mathcal{F} \circ \mathfrak{C} \simeq \mathcal{F} \circ \mathfrak{s} \circ (-)^{nec} \simeq \mathfrak{s} \circ \mathcal{F} \circ (-)^{nec} \simeq \mathfrak{s} \circ (-)^{nec} \circ \tilde{F} \simeq \mathfrak{C}_{\mathcal{V}} \circ \tilde{F}$$

□

*Remark* 4.1.18. The categorification functor $\mathfrak{C}_{\mathcal{V}}$ does not commute with the forgetful functors in the sense that $\mathcal{U} \circ \mathfrak{C}_{\mathcal{V}} \simeq \mathfrak{C} \circ \tilde{U}$. In fact, the canonical simplicial functor $\mathfrak{C}[\tilde{U}(X)] \to \mathcal{U}(\mathfrak{C}_{\mathcal{V}}[X])$ even fails to be a Dwyer-Kan equivalence in general. This is a direct consequence of Example 2.3.18 and will be discussed further in Example 4.1.34.

**Proposition 4.1.19.** *Let $\mathcal{C}$ be a small $S\mathcal{V}$-category. Then the templicial object $N^{hc}_{\mathcal{V}}(\mathcal{C})$ has a Frobenius structure.*

*Proof.* Note that we can extend the strong monoidal functor $\mathcal{P}_{(-)} : \mathcal{N}ec \to \mathrm{Cat}$ to a functor $\overline{\mathcal{P}}_{(-)} : \overline{\mathcal{N}ec} \to \mathrm{Cat}$ by setting

$$\overline{\mathcal{P}}_{(f,U')} : \mathcal{P}_T \to \mathcal{P}_U : T' \mapsto f(T') \cup U'$$

for every morphism $(f, U') : T \to U$ in $\overline{\mathcal{N}ec}$. Then the functor $\mathfrak{n} : S\mathcal{V} \to \mathcal{V}^{\mathcal{N}ec^{op}}$ of Construction 4.1.11 clearly factors as

$$S\mathcal{V} \xrightarrow{\overline{\mathfrak{n}}} \mathcal{V}^{\overline{\mathcal{N}ec}^{op}} \to \mathcal{V}^{\mathcal{N}ec^{op}}$$

where $\overline{\mathfrak{n}}$ sends a simplicial object $Y$ to the functor $[FN\overline{\mathcal{P}}_{(-)}, Y] : \overline{\mathcal{N}ec}^{op} \to \mathcal{V}$. It follows that $\mathfrak{n} : \mathcal{V}\,\mathrm{Cat}_{\Delta} \to \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$ factors through the forgetful functor $\mathcal{V}\,\mathrm{Cat}_{\overline{\mathcal{N}ec}} \to \mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$. Thus the result follows from Proposition 3.2.20. □

**Proposition 4.1.20.** *Let $\mathcal{C}$ be a small $S\mathcal{V}$-category such that for all $A, B \in \mathrm{Ob}(\mathcal{C})$, the simplicial set $U(\mathcal{C}(A,B))$ is a Kan complex. Then the templicial object $N^{hc}_{\mathcal{V}}(\mathcal{C})$ is a quasi-category in $\mathcal{V}$.*

*Proof.* By Corollary 3.2.22, it suffices to check that for all $A, B \in \mathrm{Ob}(\mathcal{C})$, the functor

$$\mathfrak{n}(\mathcal{C}(A,B))_\bullet = [FN\mathcal{P}_{(-)}, \mathcal{C}(A,B)] : \mathcal{N}ec^{op} \to \mathcal{V}$$

lifts inner horns in $\mathcal{V}^{\mathcal{N}ec^{op}}$. By the adjunction $\mathfrak{s} \dashv \mathfrak{n}$, this is equivalent to showing that for all $0 < j < n$, every diagram of solid arrows

$$\begin{tikzcd} \mathfrak{s}\left(\tilde{F}(\Lambda^n_j)_\bullet(0,n)\right) \arrow[r] \arrow[d] & \mathcal{C}(A,B) \\ \mathfrak{s}\left(\tilde{F}(\Delta^n)_\bullet(0,n)\right) \arrow[ur, dashed] & \end{tikzcd}$$

in $S\mathcal{V}$ has a lift given by the dotted arrow. Now by Proposition 4.1.17,

$$\mathfrak{s}\left(\tilde{F}(\Lambda^n_j)_\bullet(0,n)\right) = \mathfrak{C}_{\mathcal{V}}[\tilde{F}(\Lambda^n_j)](0,n) \simeq F\left(\mathfrak{C}[\Lambda^n_j](0,n)\right)$$
$$\mathfrak{s}\left(\tilde{F}(\Delta^n)_\bullet(0,n)\right) = \mathfrak{C}_{\mathcal{V}}[\tilde{F}(\Delta^n)](0,n) \simeq F\left(\mathfrak{C}[\Delta^n](0,n)\right)$$

So by the adjunction $F \dashv U$, the above lifting problem is further equivalent to

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_j](0,n) \arrow[r] \arrow[d] & U(\mathcal{C}(A,B)) \\ \mathfrak{C}[\Delta^n](0,n) \arrow[ur, dashed] & \end{tikzcd}$$

in $\mathrm{SSet}$. Thus as $U(\mathcal{C}(A,B))$ is a Kan complex, it suffices to prove that the left vertical map is anodyne (Definition 1.3.16), which was done in [Lur09a, Proposition 1.1.5.10] and is given in more detail in [Lur18, Tag 00LH] (beware that in the latter, the notation $\mathrm{Path}$ is used instead of $\mathfrak{C}$). $\square$

### 4.1.3 Categorification in terms of flanked flags

In this subsection we continue adapting Dugger and Spivak's simplification of $\mathfrak{C}$ to the templicial setting. Given a simplicial set $K$ with vertices $a$ and $b$, and a fixed $n \geq 0$, they describe the set of $n$-simplices of $\mathfrak{C}[K](a,b)$ much more simply by means of so-called *flags* of a necklace $T$ and *totally non-degenerate* maps $T \to K_{a,b}$. Let us first recall these definitions.

**Definition 4.1.21.** Let $(T, p)$ be a necklace and $n \geq 0$. A *flag of length $n$* on $T$ is defined as an $n$-simplex of the nerve $N(\mathcal{P}_T)$. Explicitly, a flag of length $n$ on $T$ is a sequence of inclusions

$$\vec{T} = (T_0 \subseteq \ldots \subseteq T_n)$$

such that $T \subseteq T_0$ and $T_n \subseteq [p]$. We call a flag $\vec{T}$ on a necklace $T$ *flanked* if $T = T_0$ and $T_n = [p]$.

**Definition 4.1.22.** Let $K$ be a simplicial set with vertices $a$ and $b$, and $T = \Delta^{n_1} \vee ... \vee \Delta^{n_k}$ a necklace. A map $T \to K_{a,b}$ in $\mathrm{SSet}_{*,*}$ is *totally non-degenerate* if for every $i \in \{1, ..., k\}$, the composite map in SSet

$$\Delta^{n_i} \hookrightarrow T \to K_{a,b}$$

represents a non-degenerate $n_i$-simplex of $K$.

As an immediate consequence of Proposition 4.1.6 (see [DS11b, Corollary 4.4]), we see that an $n$-simplex of $\mathfrak{C}[K](a,b)$ consists of an equivalence class

$$[T, T \to K_{a,b}, \vec{T}] \tag{4.3}$$

of triples $(T, T \to K_{a,b}, \vec{T})$ where

- $T$ is a necklace,
- $T \to K_{a,b}$ is a map in $\mathrm{SSet}_{*,*}$ (equivalently, an element of $K_T(a,b)$),
- $\vec{T}$ is a flag of length $n$ on $T$.

The equivalence relation is generated by considering two triples $(T, T \to K_{a,b}, \vec{T})$ and $(U, U \to K_{a,b}, \vec{U})$ to be equivalent if there exists a map of necklaces $f : T \to U$ making the obvious diagram commute, such that $f(T_i) = U_i$ for all $0 \le i \le n$.

It is then shown in [DS11b, Lemma 4.5 and Corollary 4.8] that we can make the following reductions:

1. In every equivalence class (4.3) there exists a triple $(T, T \to K_{a,b}, \vec{T})$ such that $\vec{T}$ is flanked. Moreover, two such triples are equivalent if and only if they can be connected by a zig-zag of morphisms of flagged necklaces in which every triple has a flag that is flanked.
2. In every equivalence class (4.3) there exists a *unique* triple $(T, T \to K_{a,b}, \vec{T})$ such that $\vec{T}$ is flanked and $T \to K_{a,b}$ is totally non-degenerate. In other words, there is a bijection
$$\mathfrak{C}[K]_n(a,b) \simeq \coprod_{\substack{T \in \mathcal{N}ec \\ \vec{T} \text{ flag of length } n}} K_T^{nd}(a,b)$$
where $K_T^{nd}(a,b) \subseteq K_T(a,b)$ is the subset of totally non-degenerate maps $T \to K_{a,b}$.

Generalizing the first of the above reductions to templicial objects is fairly straightforward. This is done in Proposition 4.1.25 and the proof is essentially that of [DS11b]. Because the second reduction involves non-degenerate simplices, we will have to restrict to free templicial objects (see §3.1.3). This is the content of Proposition 4.1.28.

**Notation 4.1.23.** We denote by

$$\mathcal{N}ec^{\urcorner}[n]$$

the category of pairs $(T, \vec{T})$ where $T$ is a necklace and $\vec{T} = (T_0, ..., T_n)$ is a flag of length $n$ on $T$. A morphism $(T, \vec{T}) \to (U, \vec{U})$ in $\mathcal{N}ec^{\urcorner}[n]$ is a necklace map $f : T \to U$ such that $f(T_i) = U_i$ for all $i \in [n]$. Further, we let

$$\mathcal{N}ec^{\urcorner}_f[n]$$

denote the full subcategory of $\mathcal{N}ec^{\urcorner}[n]$ spanned by flagged necklaces whose flags are flanked. Note that a morphism in $\mathcal{N}ec^{\urcorner}_f[n]$ is necessarily active and surjective on vertices.

**Lemma 4.1.24.** *Let $n \geq 0$. The subcategory $\iota : \mathcal{N}ec^{\urcorner}_f[n] \hookrightarrow \mathcal{N}ec^{\urcorner}[n]$ is coreflective. We call the right adjoint to $\iota$ the* flankification functor*.*

*Proof.* We construct the flankification functor $\gamma : \mathcal{N}ec^{\urcorner}[n] \to \mathcal{N}ec^{\urcorner}_f[n]$. For $(T, p) \in \mathcal{N}ec$ and $\vec{T}$ a flag of length $n$ on $T$, there is a unique isomorphism of posets $T_n \simeq [k]$ where $k = \ell(T)$. For all $i \in [n]$, write $T'_i$ for the image of $T_i$ under this isomorphism so that $T'_0 \subseteq ... \subseteq T'_n = [k]$. Further set $T' = T'_0$ so that the flag $\vec{T}'$ is flanked on $T'$. Then define $\gamma(T, \vec{T}) = (T', \vec{T}')$. We moreover obtain a morphism of flagged necklaces $\epsilon : \iota\gamma(T, \vec{T}) \to (T, \vec{T})$ with underlying morphism $[k] \simeq T_n \hookrightarrow [p]$ in $\mathbf{\Delta}_f$.

Given $(U, \vec{U}) \in \mathcal{N}ec^{\urcorner}_f[n]$ with $(U, q)$ a necklace, and a morphism $f : \iota(U, \vec{U}) \to (T, \vec{T})$ in $\mathcal{N}ec^{\urcorner}[n]$, we have in particular that $T_n = f(U_n) = f([q])$. So the morphism $f : [q] \to [p]$ in $\mathbf{\Delta}_f$ factors uniquely through $[k] \hookrightarrow [p]$ as $g : [q] \to [k]$. Moreover, $g$ defines a morphism $(U, \vec{U}) \to \gamma(T, \vec{T})$.

We conclude that the functor $\iota : \mathcal{N}ec^{\urcorner}_f[n] \to \mathcal{N}ec^{\urcorner}[n]$ has a right adjoint which is given on objects by $(T, \vec{T}) \mapsto \gamma(T, \vec{T})$. ☐

**Proposition 4.1.25.** *Let $(X, S)$ be a templicial object and $a, b \in S$. Then for every $n \geq 0$, we have a canonical isomorphism*

$$\mathfrak{C}_{\mathcal{V}}[X]_n(a, b) \simeq \operatorname*{colim}_{(T, \vec{T}) \in \mathcal{N}ec^{\urcorner}_f[n]} X_T(a, b)$$

*Proof.* We can rewrite the coequalizer (4.2) in dimension $n$ as

$$\coprod_{\substack{f : T \to U \\ \vec{T} \text{ flag on } T \\ \text{of length } n}} X_U(a, b) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{\substack{T \in \mathcal{N}ec \\ \vec{T} \text{ flag on } T \\ \text{of length } n}} X_T(a, b) \twoheadrightarrow \mathfrak{C}_{\mathcal{V}}[X]_n(a, b)$$

where $\alpha$ is given by $X(f)$ and $\beta$ is given by applying $f$ to $\vec{T}$, for a necklace morphism $f : T \to U$. We thus have a canonical isomorphism

$$\mathfrak{C}_{\mathcal{V}}[X]_n(a, b) \simeq \operatorname*{colim}_{(T, \vec{T}) \in \mathcal{N}ec^{\urcorner}[n]} X_T(a, b)$$

Now as the inclusion $\mathcal{N}ec^{\urcorner}_f[n] \hookrightarrow \mathcal{N}ec^{\urcorner}[n]$ is a left adjoint by Lemma 4.1.24, the corresponding functor between opposite categories is a right adjoint and thus a final functor. Hence, the result follows. ☐

*Remark* 4.1.26. The simplicial structure of $\mathfrak{C}_{\mathcal{V}}[X](a,b) = \operatorname{colim}^{X_\bullet(a,b)} N\mathcal{P}_{(-)}$ is given by that of $N\mathcal{P}_T$, i.e. by deleting and copying terms in a flag, but the simplicial structure on $\operatorname{colim}_{(T,\vec{T})\in\mathcal{N}ec_f^{\urcorner}} X_T(a,b)$ is slightly more difficult. The degeneracy maps and inner face maps are still given by respectively copying and deleting terms in the flags. The outer face maps however are given by first deleting the term $T_0$ or $T_n$ from a flag $(T_0,...,T_n)$ and then applying the flankification functor.

**Notation 4.1.27.** Let $(X,S)$ be a free templicial object and $T$ a necklace which we write as $\{0 = t_0 < t_1 < t_2 < ... < t_k = p\}$. Then we denote

$$X_T^{nd} = X_{t_1}^{nd} \otimes_S X_{t_2-t_1}^{nd} \otimes_S ... \otimes_S X_{p-t_{k-1}}^{nd} \in \mathcal{V}\operatorname{Quiv}_S$$

where $X_n^{nd}$ denotes the quiver of non-degenerate simplices of Definition 3.1.38.

**Proposition 4.1.28.** *Let $(X,S)$ be a free templicial object. For all $n \geq 0$ and $a,b \in S$, we have an isomorphism in $\mathcal{V}$:*

$$\mathfrak{C}_{\mathcal{V}}[X]_n(a,b) \simeq \coprod_{(T,\vec{T})\in\mathcal{N}ec_f^{\urcorner}[n]} X_T^{nd}(a,b)$$

*Proof.* By Proposition 4.1.25 and Lemma 3.1.39, we have an isomorphism

$$\mathfrak{C}_{\mathcal{V}}[X]_n(a,b) \simeq \operatorname*{colim}_{(T,\vec{T})\in\mathcal{N}ec_f^{\urcorner}[n]} \coprod_{\substack{f_i:[t_i-t_{i-1}]\twoheadrightarrow[n_i]\\ i\in\{1,...,k\}}} \left(X_{n_1}^{nd}\otimes_S ... \otimes_S X_{n_k}^{nd}\right)(a,b)$$

where we've written $T = \{0 = t_0 < t_1 < ... < t_k = p\}$ for any $(T,\vec{T}) \in \mathcal{N}ec_f^{\urcorner}[n]$. Now let $f : (T,p) \to (U,q)$ be an active necklace map such that its underlying morphism $f : [p] \to [q]$ in $\mathbf{\Delta}_f$ is surjective. By Remark 2.1.4 we can uniquely decompose $f = f_1 + ... + f_k$ with $f_i : [t_i - t_{i-1}] \twoheadrightarrow [n_i]$ in $\mathbf{\Delta}_{surj}$ for all $i \in \{1,...,n\}$. Moreover, given a flag $\vec{T}$ of length $n$ on $T$, there is a unique flanked flag $\vec{U} = (U_0,...,U_n)$ on $U$ such that $f : T \to U$ lifts to a morphism $f : (T,\vec{T}) \to (U,\vec{U})$ in $\mathcal{N}ec_f^{\urcorner}[n]$ (simply set $U_i = f(T_i)$). It follows that

$$\begin{aligned}\mathfrak{C}_{\mathcal{V}}[X]_n(a,b) &\simeq \operatorname*{colim}_{(T,\vec{T})\in\mathcal{N}ec_f^{\urcorner}[n]} \coprod_{\substack{(T,\vec{T})\to(U,\vec{U})\\ \text{in } \mathcal{N}ec_f^{\urcorner}[n]}} X_U^{nd}(a,b)\\ &\simeq \coprod_{(U,\vec{U})\in\mathcal{N}ec_f^{\urcorner}[n]} \operatorname*{colim}_{\substack{(T,\vec{T})\to(U,\vec{U})\\ \text{in } \mathcal{N}ec_f^{\urcorner}[n]}} X_U^{nd}(a,b) \simeq \coprod_{(U,\vec{U})\in\mathcal{N}ec_f^{\urcorner}[n]} X_U^{nd}(a,b)\end{aligned}$$

The last isomorphism is obtained by noting that the colimit on the left hand side is indexed over the category $\left((\mathcal{N}ec_f^{\urcorner}[n])_{/(U,\vec{U})}\right)^{op}$, which is connected, and the functor involved is constant on $X_U^{nd}(a,b)$. $\square$

### 4.1.4 Comparison with the templicial nerve

Analogous to the classical homotopy coherent nerve, we show that the templicial homotopy coherent nerve $N_{\mathcal{V}}^{hc}$ restricts to the templicial nerve $N_{\mathcal{V}}$ (Construction 2.3.4) when applied to ordinary $\mathcal{V}$-enriched categories. This is the content of Proposition 4.1.33.

**Definition 4.1.29.** Let $Y$ be a simplicial object. The *object of connected components* of $Y$ is defined to be the colimit $\pi_0(Y) \in \mathcal{V}$ of $Y$ as a functor $\mathbf{\Delta}^{op} \to \mathcal{V}$. Equivalently, it is given by the reflexive coequalizer:

$$\begin{tikzcd} Y_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & Y_0 \arrow[l, "s_0" description] \arrow[r, two heads] & \pi_0(Y) \end{tikzcd} \tag{4.4}$$

The assignment $Y \mapsto \pi_0(Y)$ clearly extends to a functor

$$\pi_0 : S\mathcal{V} \to \mathcal{V}$$

**Proposition 4.1.30.** *We have a canonical natural isomorphism*

$$\pi_0 \circ F \simeq F \circ \pi_0$$

*where $\pi_0$ on the right hand side denotes the connected component functor of Definition 1.3.19.*

*Moreover, if $U$ preserves reflexive coequalizers, then we also have*

$$\pi_0 \circ U \simeq U \circ \pi_0$$

*Proof.* The first claim follows from the fact that $F$ preserves colimits. The second claim is trivial. $\square$

**Proposition 4.1.31.** *The functor $\pi_0 : S\mathcal{V} \to \mathcal{V}$ is strong monoidal and left adjoint to the constant simplicial object functor:*

$$\underline{(-)} : \mathcal{V} \to S\mathcal{V}$$

*Proof.* Consider the monoidal unit $F(\Delta^0)$, which is the constant functor on $I \in \mathcal{V}$. Then clearly, $\pi_0(F(\Delta^0)) \simeq I$. Further, as the coequalizer (4.4) is reflexive, we have a canonical isomorphism

$$\pi_0(X \otimes Y) \xrightarrow{\sim} \pi_0(X) \otimes \pi_0(Y)$$

which is natural in $X, Y \in S\mathcal{V}$. It follows that these isomorphisms provide $\pi_0$ with the structure of a strong monoidal functor.

In general, the functor $\mathcal{V}^{\mathcal{J}} \to \mathcal{V}$ taking the colimit of a diagram $\mathcal{J} \to \mathcal{V}$ is left adjoint to the functor sending an object to the constant diagram $\mathcal{J} \to \mathcal{V}$. $\square$

*Remark* 4.1.32. By Propositions 4.1.31 en 1.1.23, there is an induced adjunction

$$\begin{tikzcd} \mathcal{V}\,\mathrm{Cat}_\Delta \arrow[r, shift left=2, "\pi_0"] \arrow[r, phantom, "\perp"] & \mathcal{V}\,\mathrm{Cat} \arrow[l, shift left=2, "\underline{(-)}"] \end{tikzcd}$$

which we will denote by the same symbols.

Recall the homotopy functor $h_{\mathcal{V}} : S_{\otimes}\mathcal{V} \to \mathcal{V}\,\mathrm{Cat}$ of subsection §2.3.2.

**Proposition 4.1.33.** *We have a canonical natural isomorphisms*

$$N_{\mathcal{V}}^{hc} \circ \underline{(-)} \simeq N_{\mathcal{V}} \quad \textit{and} \quad \pi_0 \circ \mathfrak{C}_{\mathcal{V}} \simeq h_{\mathcal{V}}$$

*Proof.* By comparing left-adjoints, we are reduced to showing the first isomorphism. By definition, $N^{hc}_{\mathcal{V}} = (-)^{temp} \circ \mathfrak{n}$, so by Proposition 3.2.14, it suffices to show that we have a natural isomorphism

$$\mathfrak{n} \circ \underline{(-)} \simeq \underline{(-)}$$

where $\underline{(-)}$ on the right hand side denotes the diagonal functor $\mathcal{V} \to \mathcal{V}^{\mathcal{N}ec^{op}}$. Take an object $A \in \mathcal{V}$. Since the simplicial set $N(\mathcal{P}_T)$ is clearly only has one connected component, it follows from Proposition 4.1.30 that

$$[FN\mathcal{P}_T, \underline{A}] \simeq \underline{\mathcal{V}}(\pi_0 FN\mathcal{P}_T, A) \simeq \underline{\mathcal{V}}(F(\pi_0 N\mathcal{P}_T), A) \simeq \underline{\mathcal{V}}(F(\{*\}), A) \simeq A$$

for all necklaces $T$. It follows that $\mathfrak{n}(\underline{A})$ is isomorphic to the constant functor on $A$, that is $\underline{A} : \mathcal{N}ec^{op} \to \mathcal{V}$. Clearly, this isomorphism is natural in $A$ as desired. $\square$

**Example 4.1.34.** For a general templicial object $X$, consider the canonical map of simplicial sets

$$\tilde{U}(X) \to \tilde{U}(N^{hc}_{\mathcal{V}}\mathfrak{C}_{\mathcal{V}}[X])$$

By Proposition 4.1.17, $\tilde{U}(N^{hc}_{\mathcal{V}}\mathfrak{C}_{\mathcal{V}}[X]) \simeq N^{hc}\mathcal{U}(\mathfrak{C}_{\mathcal{V}}[X])$ and thus by the adjunction $\mathfrak{C} \dashv N^{hc}$, we have a canonical $\mathrm{SSet}$-enriched functor

$$\mathfrak{C}[\tilde{U}(X)] \to \mathcal{U}(\mathfrak{C}_{\mathcal{V}}[X])$$

By construction, this functor is bijective on objects, but it is not a Dwyer-Kan equivalence in the sense of [Ber07], i.e. it does not induce weak homotopy equivalences (Definition 1.3.21) on hom-objects. More explicitly, given vertices $a$ and $b$ of $X$, the induced map of simplicial sets

$$\mathfrak{C}[\tilde{U}(X)](a, b) \to U\left(\mathfrak{C}_{\mathcal{V}}[X](a, b)\right)$$

is not a weak homotopy equivalence in general. Indeed, this already fails on the level of connected components. Consider the induced functor

$$\pi_0\mathfrak{C}[\tilde{U}(X)] \to \pi_0\mathcal{U}\mathfrak{C}_{\mathcal{V}}[X]$$

If $\mathcal{V} = \mathrm{Mod}(k)$, then $U$ preserves reflexive coequalizers and thus by Proposition 4.1.30 and Proposition 4.1.33, we would have an equivalence of categories

$$h\tilde{U}(X) \to \mathcal{U}(h_{\mathcal{V}}(X))$$

which fails to be the case in general by Example 2.3.18.

## 4.2 Differential graded categories

We fix a unital commutative ring $k$ for the entirety of this section. Consider the monoidal category $\mathrm{Mod}(k)$ with monoidal product given by the tensor product $\otimes_k$ over $k$ and with monoidal unit given by $k$. Note that $(\mathrm{Mod}(k), \otimes_k, k)$ satisfies all the hypotheses imposed on $\mathcal{V}$ so far. That is, $\mathrm{Mod}(k)$ is a Bénabou cosmos and $k$ is projective and finitely presentable (and thus small in the sense of Definition 1.2.9). Moreover the forgetful functor $U : \mathrm{Mod}(k) \to \mathrm{Set}$ is faithful and conservative and $U$ preserves and reflects reflexive coequalizers (thus also regular epimorphisms). We may sometimes simplify notation by replacing $\mathrm{Mod}(k)$ by $k$ in certain expressions. For example, we will write $k\,\mathrm{Quiv}$ for $\mathrm{Mod}(k)\,\mathrm{Quiv}$ and $N_k^{hc}$ for $N_{\mathrm{Mod}(k)}^{hc}$.

The main goal of this section is to introduce and study a $k$-linear version of the differential graded nerve $N^{dg} : k\,\mathrm{Cat}_{dg} \to \mathrm{SSet}$, which we aptly call the *linear dg-nerve* $N_k^{dg}$. Note that for a given dg-category $\mathcal{C}_\bullet$, its dg-nerve $N^{dg}(\mathcal{C})$ is just a simplicial set and thus loses all reference to the ring $k$. Intuitively one might consider the linear dg-nerve $N_k^{dg}(\mathcal{C})$ as a way to retain the $k$-linear structure of $\mathcal{C}_\bullet$ while still resembling the original dg-nerve.

To construct the linear dg-nerve, we first show a two-step equivalence of categories between small, non-negatively graded dg-categories over $k$ and templicial $k$-modules with a Frobenius structure:

$$k\,\mathrm{Cat}_{dg,\geq 0} \simeq k\,\mathrm{Cat}_{\Delta_+} \simeq S_\otimes^{Frob}\,\mathrm{Mod}(k)$$

Here, the middle category has as objects all small categories enriched in augmented simplicial $k$-modules $S^+\,\mathrm{Mod}(k)$ with the join operation. Each equivalence is dealt with in its own subsection (see §4.2.1 and §4.2.2). The first of these is a consequence of a monoidal equivalence on the level of hom-objects, which we call the *augmented Dold-Kan correspondence*. The second is inspired by the tensor-algebra of a graded $k$-module, which as it turns out can always be viewed as a Frobenius monoidal functor. Finally, the linear dg-nerve is obtained by composing these equivalences with the obvious forgetful functor $S_\otimes^{Frob}\,\mathrm{Mod}(k) \to S_\otimes\,\mathrm{Mod}(k)$ (Definition 4.2.46).

The remaining subsections are devoted to showing three important results concerning $N_k^{dg}$. In §4.2.4, we show that we can recover the classical dg-nerve $N^{dg}$ from $N_k^{dg}$ by composing with the functor $\tilde{U} : S_\otimes\,\mathrm{Mod}(k) \to \mathrm{SSet}$ of Proposition 2.1.25. Then in §4.2.3 we show that $N_k^{dg}(\mathcal{C})$ is a quasi-category in $\mathrm{Mod}(k)$ for any dg-category $\mathcal{C}_\bullet$, as is the case for the classical situation. In fact, we'll show that any templicial $k$-module with a (non-associative) Frobenius structure is already a quasi-category in $\mathrm{Mod}(k)$. Finally, in §4.2.5 we show that a classical map comparing the homotopy coherent and dg-nerves can be lifted to a comparison map between the templicial homotopy coherent and linear dg-nerves.

### 4.2.1 The augmented Dold-Kan correspondence

The classical Dold-Kan correspondence (which we will recall here shortly, see Proposition 4.2.4) provides an equivalence between the categories of simplicial $k$-modules $S\,\mathrm{Mod}(k)$

and non-negatively graded chain complexes $\mathrm{Ch}_{\geq 0}(k)$ through the normalized chain functor $N_\bullet : S\,\mathrm{Mod}(k) \to \mathrm{Ch}_{\geq 0}(k)$. Famously, it is not an equivalence of monoidal categories. The normalized chain functor does carry both a lax and colax monoidal structure however (usually called the *Eilenberg-Zilber* and *Alexander-Whitney* maps respectively, see [May67, §29] for details). These promote the Dold-Kan equivalence to a weak Quillen monoidal equivalence between monoidal model categories in the sense of [SS03].

Alternatively, we can cheat our way out of the non-monoidality of the equivalence $\mathrm{Ch}_{\geq 0}(k) \simeq S\,\mathrm{Mod}(k)$ by replacing $S\,\mathrm{Mod}(k)$ with the category of augmented simplicial $k$-modules $S^+\,\mathrm{Mod}(k)$. Then we still have an equivalence of categories $\mathrm{Ch}_{\geq 0}(k) \simeq S^+\,\mathrm{Mod}(k)$ (Proposition 4.2.10) which we call the *augmented Dold-Kan correspondence*. Now, equipping $S^+\,\mathrm{Mod}(k)$ with the monoidal product of the join (Construction 4.2.12), this equivalence does become monoidal "on the nose" (Theorem 4.2.14). It is important to note that the join operation is very different from the usual monoidal product on simplicial objects, which is pointwise (see Construction 4.1.8). It was chosen specifically so that it would make the augmented Dold-Kan correspondence monoidal.

**Definition 4.2.1.** A *chain complex* $C_\bullet$ over $k$ is a diagram of $k$-modules

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \xrightarrow{\partial_{-2}} \cdots$$

such that for all $n \in \mathbb{Z}$, we have $\partial_{n-1} \circ \partial_n = 0$. The maps $\partial = (\partial_n)_{n\in\mathbb{Z}}$ are called the *differential* of $C_\bullet$. A *chain map* $f : C_\bullet \to D_\bullet$ between chain complexes is a collection of $k$-linear maps $(f_n : C_n \to D_n)_{n\geq 0}$ such that the following square commutes for all $n \in \mathbb{Z}$:

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "\partial_{n-1}"'] & D_{n-1} \end{tikzcd}$$

We denote the category of all chain complexes and chain maps by

$$\mathrm{Ch}(k)$$

Note that $\mathrm{Ch}(k)$ is enriched over $\mathrm{Mod}(k)$ in an obvious way.

We call a chain complex $C_\bullet$ *non-negatively graded* if $C_n = 0$ for all $n < 0$. We denote by

$$\mathrm{Ch}_{\geq 0}(k)$$

the subcategory of $\mathrm{Ch}(k)$ spanned by all non-negatively graded chain complexes.

For more details on chain complexes, we refer to the literature (see [Wei94] for example).

**Construction 4.2.2.** Let $A$ be a simplicial $k$-module. We construct a chain complex $N_\bullet(A)$ as follows. For all $n \geq 0$, define

$$N_n(A) = \frac{A_n}{\sum_{i=0}^{n-1} s_i(A_{n-1})}$$

It follows from the simplicial identities (1.3) that the $i$th face map $d_i : A_n \to A_{n-1}$ induces a map $\overline{d}_i : N_n(A) \to N_{n-1}(A)$. Then for all $n \geq 1$, set

$$\partial_n = \sum_{i=0}^{n} (-1)^i \overline{d}_i : N_n(A) \to N_{n-1}(A)$$

Again by the simplicial identities it follows that $\partial$ squares to zero.

Given a morphism $f : A \to B$ of simplicial $k$-modules, it follows from the naturality of $f$, that $f_n : A_n \to B_n$ induces a map $N_n(f) = \overline{f} : N_n(A) \to N_n(B)$. This defines a chain map

$$N_\bullet(f) : N_\bullet(A) \to N_\bullet(B)$$

We thus obtain a functor

$$N_\bullet : S\,\mathrm{Mod}(k) \to \mathrm{Ch}_{\geq 0}(k)$$

Given a simplicial set $K$, we will also write $N_\bullet(K;k) = N_\bullet(FK)$ with $F : \mathrm{Set} \to \mathrm{Mod}(k) : S \mapsto \oplus_{a\in S} k$ the free module functor.

**Definition 4.2.3.** We call the functor $N_\bullet : S\,\mathrm{Mod}(k) \to \mathrm{Ch}_{\geq 0}(k)$ the *normalized chain functor.*

**Proposition 4.2.4** ([Dol58], [Kan58]). *The normalized chain functor $N_\bullet$ is an equivalence of categories.*

We will now discuss the analogue of the Dold-Kan correspondence for augmented simplicial $k$-modules. Recall the augmented simplex category $\mathbf{\Delta}_+$ from Definition 2.1.1.

**Definition 4.2.5.** We denote $S^+\,\mathrm{Mod}(k) = \mathrm{Fun}(\mathbf{\Delta}_+^{op}, \mathrm{Mod}(k))$ for the category of *augmented simplicial (k-)modules*, i.e. functors $\mathbf{\Delta}_+^{op} \to \mathrm{Mod}(k)$, and *augmented simplicial maps*, i.e. natural transformations, between them.

**Notation 4.2.6.** For any $n \geq -1$, we denote

$$\Delta^n = \mathbf{\Delta}_+(-,[n]) : \mathbf{\Delta}_+^{op} \to \mathrm{Set}$$

Note that for $n \geq 0$, the restriction of $\Delta^n$ to $\mathbf{\Delta}^{op}$ is precisely the standard $n$-simplex in $\mathrm{SSet}$, and that $\Delta^n$ has a single $(-1)$-simplex.

Further, $(\Delta^{-1})_n = \emptyset$ for all $n \geq 0$ and $(\Delta^{-1})_{-1} = \{*\}$.

**Construction 4.2.7.** Given an augmented simplicial $k$-module $A$, we construct a non-negatively graded chain complex $N^+_\bullet(A)$ as follows. For all $n \geq 0$, set

$$N^+_n(A) = \frac{A_{n-1}}{\sum_{i=0}^{n-2} s_i(A_{n-2})}$$

for all $n \geq 0$. So in low degrees: $N^+_0(A) = A_{-1}$, $N^+_1(A) = A_0$ and $N^+_2(A) = A_1/s_0(A_0)$. The differential is given by, for all $n \geq 0$:

$$\partial_{n+1} = \sum_{i=0}^{n} (-1)^i \overline{d}_i : N^+_{n+1}(A) \to N^+_n(A)$$

where $\overline{d}_i$ is induced by the $i$th face map $d_i : A_n \to A_{n-1}$ of $A$. It follows from the simplicial identities (1.3) that this differential is well-defined and squares to zero.

Given an augmented simplicial map $f : A \to B$, set $N^+_n(f) = \overline{f}_{n-1}$ to be the map $N^+_n(A) \to N^+_n(B)$ induced by $f_{n-1} : A_{n-1} \to B_{n-1}$. This defines a chain map

$$N^+_\bullet(f) : N^+_\bullet(A) \to N^+_\bullet(B)$$

by the naturality of $f$. It is clear that we get a functor

$$N_\bullet^+ : S^+ \operatorname{Mod}(k) \to \operatorname{Ch}_{\geq 0}(k)$$

Given an augmented simplicial set $K$, i.e. a functor $K : \mathbf{\Delta}_+^{op} \to \operatorname{Set}$, we will also write $N_\bullet^+(K;k) = N_\bullet^+(FX)$, analogously to the classical normalized chain complex.

**Definition 4.2.8.** We call the functor $N_\bullet^+ : S^+ \operatorname{Mod}(k) \to \operatorname{Ch}_{\geq 0}(k)$ of Construction 4.2.7 the *augmented normalized chain functor*.

*Remark* 4.2.9. Pre-composition with the inclusion $\mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_+$ induces a functor

$$(-)_{\geq 0} : S^+ \operatorname{Mod}(k) \to S \operatorname{Mod}(k)$$

which forgets the module $A_{-1}$ and the face map $d_0 : A_0 \to A_{-1}$ of a given augmented simplicial module $A$.

Further, consider the following isomorphism of categories

$$s : \operatorname{Ch}_{\geq 0}(k) \xrightarrow{\sim} \operatorname{Ch}_{>0}(k)$$

with $sC_n = C_{n-1}$ and $\partial_n^{sC_\bullet} = \partial_{n-1}^{C_\bullet}$ for all $n > 0$ and any non-negatively graded chain complex $C_\bullet$.

Then we have an isomorphism of non-negatively graded chain complexes

$$N_\bullet^+(A) \simeq \left( sN_\bullet(A_{\geq 0}) \xrightarrow{d_0} A_{-1} \right)$$

which is natural in all augmented simplicial objects $A$.

**Proposition 4.2.10.** *The augmented normalized chain functor $N^+ : S^+ \operatorname{Mod}(k) \to \operatorname{Ch}(k)$ has a right-adjoint $\Gamma^+ : \operatorname{Ch}(k) \to S^+ \operatorname{Mod}(k)$ which is given by, for all chain complexes $C_\bullet$:*

$$\Gamma^+(C_\bullet) = \operatorname{Ch}(k) \left( N_\bullet^+(\Delta^{(-)}; k), C_\bullet \right) : \mathbf{\Delta}_+^{op} \to \operatorname{Mod}(k)$$

*Moreover, the restriction*

$$\begin{tikzcd} S^+ \operatorname{Mod}(k) \arrow[r, shift left, "N_\bullet^+"] \arrow[r, phantom, "\sim"] & \operatorname{Ch}_{\geq 0}(k) \arrow[l, shift left, "\Gamma^+"] \end{tikzcd}$$

*is an adjoint equivalence of categories.*

*Proof.* Because $s$, $(-)_{\geq 0}$ and $N$ all clearly preserve colimits, it follows from Remark 4.2.9 that $N^+$ preserves colimits as well. Thus the first statement follows from a general nerve construction applied to the the functor $N_\bullet^+(\Delta^{(-)}; k) : \mathbf{\Delta}_+ \to \operatorname{Ch}(k)$.

It remains to show that $N_\bullet^+ : S^+ \operatorname{Mod}(k) \to \operatorname{Ch}_{\geq 0}(k)$ is an equivalence, which will follow from Proposition 4.2.4. Given augmented simplicial modules $A$ and $B$, it follows from Remark 4.2.9 that

$$\begin{aligned} &\operatorname{Ch}(k)(N_\bullet^+(A), N_\bullet^+(B)) \\ &\simeq \operatorname{Ch}(k)(N_\bullet(A_{\geq 0}), N_\bullet(B_{\geq 0})) \times_{\operatorname{Mod}(k)(A_0, B_{-1})} \operatorname{Mod}(k)(A_{-1}, B_{-1}) \\ &\simeq S \operatorname{Mod}(k)(A_{\geq 0}, B_{\geq 0}) \times_{\operatorname{Mod}(k)(A_0, B_{-1})} \operatorname{Mod}(k)(A_{-1}, B_{-1}) \\ &\simeq S^+ \operatorname{Mod}(k)(A, B) \end{aligned}$$

This proves that $N_\bullet^+ : S_+\,\mathrm{Mod}(k) \to \mathrm{Ch}(k)$ is fully faithful. Further, let $C_\bullet$ be a non-negatively graded chain complex and let $C_{>0} \in \mathrm{Ch}_{>0}(k)$ be obtained by forgetting $C_0$ and $\partial_1 : C_1 \to C_0$. Then choose a simplicial module $A$ so that $N_\bullet(A) \simeq s^{-1}C_{>0}$. Note that $A_0 \simeq C_1$, so we can promote $A$ to an augmented simplicial module $A^+$ by setting $A_{-1} = C_0$ and $d_0 = \partial_1 : A_0 \to A_{-1}$. It follows that $N_\bullet^+(A) \simeq C_\bullet$. Thus $N_\bullet^+$ is essentially surjective as well. □

*Remark* 4.2.11. Let us make the functor $\Gamma^+$ a little more explicit. Let $C_\bullet$ be a chain complex. For all $n \geq -1$, the $k$-module $\Gamma^+(C_\bullet)_n$ consists of all collections

$$(a_I)_{I\subseteq[n]} \in \bigoplus_{I\subseteq[n]} C_{|I|}$$

that satisfy, for all $I = \{i_1 < \ldots < i_m\} \subseteq [n]$:

$$\partial(a_I) = \sum_{j=1}^{m} (-1)^{j-1} a_{I\setminus\{i_j\}}$$

For $h : [m] \to [n]$ in $\mathbf{\Delta}_+$, the map

$$\Gamma^+(C_\bullet)(h) : \Gamma^+(C_\bullet)_n \to \Gamma^+(C_\bullet)_m : (a_I)_{I\subseteq[n]} \mapsto (b_J)_{J\subseteq[m]}$$

is given by $b_J = a_{h(J)}$ if $h|_J$ is injective and $b_J = 0$ otherwise.

Finally, if $f : C_\bullet \to D_\bullet$ is a chain map, then

$$\Gamma^+(f)_n : \Gamma^+(C_\bullet)_n \to \Gamma^+(D_\bullet)_n : (a_I)_{I\subseteq[n]} \mapsto (f(a_I))_{I\subseteq[n]}$$

for all $n \geq -1$.

**Construction 4.2.12.** As both $\mathrm{Mod}(k)$ and $\mathbf{\Delta}_+$ are monoidal categories (see 2.1.3), we can endow $S^+\,\mathrm{Mod}(k)$ with the monoidal structure given by Day convolution (see [Day70]). This is also known as the *join* of augmented simplicial objects. We denote the resulting monoidal closed category by $(S^+\,\mathrm{Mod}(k), \star, F(\Delta^{-1}))$.

Explicitely, the join of two augmented simplicial modules $A$ and $B$ is given by

$$(A \star B)_n = \bigoplus_{\substack{k,l\geq -1\\ k+l+1=n}} A_k \otimes B_l$$

for all $n \geq -1$. Given $f : [m] \to [n]$, and $k, l \geq -1$ such that $k + l + 1 = n$, there exist unique $f_1^k : [p] \to [k]$ and $f_2^k : [q] \to [l]$ with $p + q + 1 = m$ and $f_1^k \star f_2^k = f$. With these notations, we have for all $k, l \geq -1$ with $k + l + 1 = n$:

$$(A \star B)(f) \circ \iota_{k,l} = \iota_{p,q} \circ \left(A(f_1^k) \otimes B(f_2^k)\right)$$

The monoidal unit is given by $F(\Delta^{-1})$. Thus $F(\Delta^{-1})_{-1} = k$ and $F(\Delta^{-1})_n = 0$ for all $n \geq 0$.

**Construction 4.2.13.** Let us recall the usual symmetric monoidal closed structure $(\otimes, k[0])$ on $\mathrm{Ch}(k)$. It is defined as follows.

Given two chain complexes $C_\bullet$ and $D_\bullet$, their tensor product $C_\bullet \otimes D_\bullet$ is given by

$$(C \otimes D)_n = \bigoplus_{\substack{p,q \in \mathbb{Z} \\ p+q=n}} C_p \otimes D_q$$

with differential $\partial^{C\otimes D}$ determined by

$$\partial_n^{C\otimes D}(x \otimes y) = \partial_p(x) \otimes y + (-1)^p x \otimes \partial_q(y)$$

for all $p, q \geq 0$ with $n = p + q$ and $x \in C_p$, $y \in D_q$.

The monoidal unit $k[0]$ is the chain complex with

$$k[0]_n = \begin{cases} k & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

The symmetry in $\mathrm{Ch}(k)$ is defined as follows. For chain complexes $C_\bullet$ and $D_\bullet$, consider the isomorphism

$$\sigma_{C,D} : C_\bullet \otimes D_\bullet \xrightarrow{\sim} D_\bullet \otimes C_\bullet$$

determined by

$$(\sigma_{C,D})_n(x \otimes y) = (-1)^{p\cdot q} y \otimes x$$

for all $p, q \geq 0$ with $n = p + q$ and $x \in C_p$, $y \in D_q$.

Finally, the subcategory $\mathrm{Ch}_{\geq 0}(k)$ inherits a symmetric monoidal structure from $\mathrm{Ch}(k)$.

**Theorem 4.2.14.** *The adjunction $N_\bullet^+ : S^+\,\mathrm{Mod}(k) \leftrightarrows \mathrm{Ch}(k) : \Gamma^+$ is monoidal.*

*Moreover, the restriction $N_\bullet^+ : S^+\,\mathrm{Mod}(k) \leftrightarrows \mathrm{Ch}_{\geq 0}(k) : \Gamma^+$ is a monoidal equivalence.*

*Proof.* For both statements it suffices to show that $N_\bullet^+$ has the structure of a strong monoidal functor. Let $A$ and $B$ be augmented simplicial modules. For all $n > 0$ and $i \in [n-2]$, the degeneracy map $s_i : (A \star B)_{n-2} \to (A \star B)_{n-1}$ is given by

$$s_i|_{A_k \otimes B_{n-k-3}} = \begin{cases} s_i^A \otimes \mathrm{id}_{B_{n-k-3}} & \text{if } i \leq k \\ \mathrm{id}_{A_k} \otimes s_{i-k-1}^B & \text{if } i > k \end{cases}$$

It follows that the submodule $\sum_{i=0}^{n-2} s_i((A \star B)_{n-2})$ of $(A \otimes B)_{n-1}$ is equal to

$$\bigoplus_{\substack{p,q \geq 0 \\ p+q=n}} \left( \sum_{i=0}^{p-2} (s_i^A \otimes \mathrm{id}_{B_{q-1}})(A_{p-2} \otimes B_{q-1}) + \sum_{i=0}^{q-2} (\mathrm{id}_{A_{p-1}} \otimes s_i^B)(A_{p-1} \otimes B_{q-2}) \right)$$

Consequently, we have an isomorphism

$$N_n^+(A \star B) \simeq \bigoplus_{\substack{p,q \geq 0 \\ p+q=n}} (N_p^+(A) \otimes N_q^+(B)) = (N^+(A) \otimes N^+(B))_n$$

Moreover this isomorphism is a chain map. This follows from the fact that for all $n \geq 0$ and $i \in [n]$, the face map $d_i : (A \star B)_n \to (A \star B)_{n-1}$ is given by,

$$d_i|_{A_k \otimes B_l} = \begin{cases} d_i^A \otimes \mathrm{id}_{B_l} & \text{if } i \leq k \\ \mathrm{id}_{A_k} \otimes d_{i-k-1}^B & \text{if } i > k \end{cases}$$

So, we get an isomorphism

$$\mu_{A,B} : N_\bullet^+(A \star B) \xrightarrow{\sim} N_\bullet^+(A) \otimes N_\bullet^+(B)$$

It is a direct verification that this isomorphism is natural in $A$ and $B$, and coassociative.

We clearly have an isomorphism $\epsilon : N_\bullet^+(\Delta^{-1}; k) \xrightarrow{\sim} k[0]$ and it follows easily that $\mu$ is counital with respect to $\epsilon$. □

### 4.2.2 Frobenius structures and $S^+ \operatorname{Mod}(k)$-categories

This subsection is entirely devoted to showing that there is an equivalence of categories between small categories enriched in augmented simplicial $k$-modules $S^+ \operatorname{Mod}(k)$ and templicial $k$-modules with a Frobenius structure (Theorem 4.2.17). We will achieve this by very explicitly defining the functors in both directions, and showing they are inverse to each other.

**Definition 4.2.15.** An $S^+ \operatorname{Mod}(k)$-*category* is a category enriched in the monoidal category $(S^+ \operatorname{Mod}(k), \star, F(\Delta^{-1}))$ of Construction 4.2.12. An $S^+ \operatorname{Mod}(k)$-*functor* is an $S^+ \operatorname{Mod}(k)$-enriched functor. We denote the category of small $S^+ \operatorname{Mod}(k)$-categories and $S^+ \operatorname{Mod}(k)$-functors by

$$k \operatorname{Cat}_{\Delta_+} = S^+ \operatorname{Mod}(k)\text{-}\operatorname{Cat}$$

**Definition 4.2.16.** A *Frobenius templicial (k-)module* is a pair $(X, Z)$ with $(X, S)$ a templicial $k$-module and $Z$ a Frobenius structure on $X : \mathbf{\Delta}_f^{op} \to k \operatorname{Quiv}_S$. Recall that then $X$ is in particular a lax monoidal functor.

Let $(\alpha, f) : (X, S) \to (Y, T)$ be a templicial map and assume that $X$ and $Y$ have Frobenius structures $Z_X$ and $Z_Y$ respectively. Then $f^* : k \operatorname{Quiv}_T \to k \operatorname{Quiv}_S$ is lax monoidal by Lemma 1.1.18. We call $(\alpha, f)$ a *Frobenius templicial map* if the induced natural transformation $X \to f^*Y$ is monoidal with respect to the lax structures on $X$ and $f^*Y$. This is equivalent to requiring the following diagram to commute for all $k, l \geq 0$:

$$\begin{tikzcd} f_!(X_{k+l}) \arrow[rr, "\alpha_{k+l}"] & & Y_{k+l} & \\ f_!(X_k \otimes_S X_l) \arrow[u, "f_!(Z_X^{k,l})"] \arrow[r] & f_!(X_k) \otimes_T f_!(X_l) \arrow[rr, "\alpha_k \otimes_T \alpha_l"'] & & Y_k \otimes_T Y_l \arrow[ul, "Z_Y^{k,l}"'] \end{tikzcd} \tag{4.5}$$

We denote the category of Frobenius templicial objects and Frobenius templicial maps between them by

$$S_{\otimes}^{Frob} \operatorname{Mod}(k)$$

Note that there is an obvious forgetful functor $S_{\otimes}^{Frob} \operatorname{Mod}(k) \to S_{\otimes} \operatorname{Mod}(k)$.

**Theorem 4.2.17.** *There is an adjoint equivalence of categories*

$$k\,\mathrm{Cat}_{\Delta_+} \underset{\mathcal{K}}{\overset{\mathcal{T}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} S_{\otimes}^{Frob}\,\mathrm{Mod}(k)$$

Let us illucidate these functors a little bit before delving into the details. Consider a $\mathbb{Z}$-graded $k$-module $M_\bullet$ that is concentrated in degree $\geq 1$ (so $M_n = 0$ for all $n \leq 0$). Then consider its tensor algebra

$$TM_\bullet = \bigoplus_{p\geq 0} M_\bullet^{\otimes p}$$

Note that for all $n \geq 0$, we have:

$$TM_n \simeq \bigoplus_{\substack{p\geq 0 \\ n_1+...+n_p=n \\ n_i>0}} M_{n_1} \otimes ... \otimes M_{n_p} \simeq \bigoplus_{T\in\mathcal{P}_n} M_T$$

where $M_T = M_{t_1} \otimes M_{t_2-t_1} \otimes ... \otimes M_{n-t_{p-1}}$ if $T = \{0 < t_1 < t_2 < ... < t_{p-1} < n\}$. Consider the monoid of natural numbers $\mathbb{N}$ (including 0). We can view $\mathbb{N}$ as a discrete monoidal category. Then the non-negatively graded module $TM_\bullet$ can be identified with a functor

$$TM_\bullet : \mathbb{N} \to \mathrm{Mod}(k)$$

Moreover, this functor is Frobenius monoidal in the sense of Definition 2.2.34. The lax structure is given by concatenating tensors as is usually done in the tensor algebra. The colax structure is given by separating tensors.

This construction supplies a functor

$$T : gr_{\geq 1}(\mathrm{Mod}(k)) \to \mathrm{Frob}(\mathbb{N}, \mathrm{Mod}(k))$$

from the category of graded $k$-modules in degree $\geq 1$ to the category of Frobenius monoidal functors $\mathbb{N} \to \mathrm{Mod}(k)$. In fact, by analogous arguments as presented below, one can show that this functor is an equivalence of categories. Its inverse is the functor

$$K : \mathrm{Frob}(\mathbb{N}, \mathrm{Mod}(k)) \to gr_{\geq 1}\,\mathrm{Mod}(k)$$

which sends any Frobenius monoidal functor $X : \mathbb{N} \to \mathrm{Mod}(k)$ to the graded $k$-module $K(X)_\bullet$ given by

$$K(X)_n = \bigcap_{k=1}^{n-1} \ker(\mu_{k,n-k})$$

for all $n \geq 1$, where $\mu_{k,n-k} : X_n \to X_k \otimes X_{n-k}$ denote the comultiplication maps of $X$.

The functors $\mathcal{T}$ and $\mathcal{K}$ in the equivalence of Theorem 4.2.17 are constructed in essentially the same way as the functors $T$ and $K$ above. We can interpret this as an upgrade of the above equivalence, obtained by equipping both sides with a certain simplicial structure and allowing them to vary over different sets of objects.

For what follows, it will be convenient to extend the augmented simplex category $\mathbf{\Delta}_+$ to the equivalent category of finite linearly ordered sets.

**Notation 4.2.18.** We denote by $\mathrm{Lin}$ the category of all finite linearly ordered sets and order morphisms between them.

**Construction 4.2.19.** Given finite linearly ordered sets $I$ and $J$, we denote by $I \sqcup J$ the disjoint union of $I$ and $J$ endowed with the partial order defined as follows. For all $i, j \in I \sqcup J$,

$$i \leq j \Leftrightarrow (i \leq j \text{ in } I) \text{ or } (i \leq j \text{ in } J) \text{ or } (i \in I \text{ and } j \in J)$$

Given morphisms $f : I \to I'$ and $g : J \to J'$ in $\mathrm{Lin}$, we have the following induced morphism in $\mathrm{Lin}$:

$$f \sqcup g : I \sqcup J \to I' \sqcup J' : i \mapsto \begin{cases} f(i) & \text{if } i \in I \\ g(i) & \text{if } i \in J \end{cases}$$

We thus have a functor $- \sqcup - : \mathrm{Lin} \times \mathrm{Lin} \to \mathrm{Lin}$.

It is readily verified that this defines a monoidal structure on $\mathrm{Lin}$ with monoidal unit given by the empty poset $\emptyset$.

If $I_1, I_2 \subseteq J$ are subsets of a finite linearly ordered subset $J$, we'll also write $I_1 \sqcup I_2$ for the union $I_1 \cup I_2$ to indicate that $i < j$ for all $i \in I_1$ and $j \in I_2$. Note that up to isomorphism this coincides with the above definition.

*Remark* 4.2.20. Note that for any finite linearly ordered set $J = \{j_0 < ... < j_k\}$, there is a unique order isomorphism $J \simeq [k]$ so that we have a canonical equivalence of monoidal categories

$$(\mathrm{Lin}, \sqcup, \emptyset) \simeq (\boldsymbol{\Delta}_+, \star, [-1])$$

By pre-composing with this equivalence, we can extend every augmented simplicial object $A \in S^+ \operatorname{Mod}(k)$ to a functor $\mathrm{Lin}^{op} \to \operatorname{Mod}(k)$. Concretely, we set

$$A_J = A_k$$

for all $J \in \mathrm{Lin}$ with $k \geq -1$ as above. Given an order morphism $f : I \to J$ between finite linearly ordered sets, consider the unique isomorphisms $I \simeq [k]$ and $J \simeq [l]$ for some $k, l \geq -1$ and let $g : [k] \to [l]$ be the induced morphism of $\boldsymbol{\Delta}_+$. Then we write

$$A(f) = A(g) : A_I \to A_J$$

Further, the join of two augmented simplicial objects $X$ and $Y$ can be rewritten as follows. Given a finite linearly ordered set $J$, we have

$$(A \star B)_J = \bigoplus_{\substack{I_1, I_2 \subseteq J \\ I_1 \sqcup I_2 = J}} A_{I_1} \otimes B_{I_2}$$

Given an order morphism between finite linearly ordered sets $f : J \to J'$, we have for all $I_1, I_2 \subseteq J'$ with $I_1 \sqcup I_2 = J$ that

$$(A \star B)(f) \circ \iota_{I_1, I_2} = \iota_{f^{-1}(I_1), f^{-1}(I_2)} \circ \big(A(f|_{f^{-1}(I_1)}) \otimes B(f|_{f^{-1}(I_2)})\big)$$

**Notation 4.2.21.** Let $(T, p)$ be a necklace and write $T = \{0 = t_0 < t_1 < ... < t_k = p\}$. For all $i \in \{1, ..., k\}$, we denote

$$T_i^c = \{t_{i-1} + 1, t_{i-1} + 2, ..., t_i - 1\}$$

considered as an object of $\mathrm{Lin}$. Note that

$$T_1^c \sqcup ... \sqcup T_k^c = [p] \setminus T$$

**From $S^+ \operatorname{Mod}(k)$-categories to Frobenius templicial modules**

We start by constructing the functor $\mathcal{T} : k\operatorname{Cat}_{\Delta_+} \to S^{Frob}_{\otimes} \operatorname{Mod}(k)$.

*Remark* 4.2.22. Let $\mathcal{C} \in k\operatorname{Cat}_{\Delta_+}$ and set $S = \operatorname{Ob}(\mathcal{C})$. Its underlying $S^+ \operatorname{Mod}(k)$-enriched quiver can be identified with a functor

$$\mathcal{C} : \mathbf{\Delta}_+^{op} \to k\operatorname{Quiv}_S$$

which we can extend to $\operatorname{Lin}^{op}$ via the equivalence $\operatorname{Lin} \simeq \mathbf{\Delta}_+$ as in Remark 4.2.20. So concretely, we have a quiver

$$\mathcal{C}_J \in k\operatorname{Quiv}_S$$

for every finite linearly ordered set $J$, and a quiver morphism

$$\mathcal{C}(f) : \mathcal{C}_J \to \mathcal{C}_I$$

for every morphism $f : I \to J$ in Lin.

Further, the identities in $\mathcal{C}$ may be identified with a quiver morphism

$$u : k_S \to \mathcal{C}_\emptyset = \mathcal{C}_{-1}$$

where $k_S$ is the monoidal unit of $k\operatorname{Quiv}_S$. The reverse composition law of $\mathcal{C}$ (see Remark 1.1.22) is determined by quiver morphisms

$$\tilde{m}_{I_1,I_2} : \mathcal{C}_{I_1} \otimes_S \mathcal{C}_{I_2} \to \mathcal{C}_J$$

for all finite linearly ordered sets $J$ with $I_1, I_2 \subseteq J$ such that $I_1 \sqcup I_2 = J$.

Because of the associativity of the composition in $\mathcal{C}$, we also have an induced quiver morphism, for all $p \geq 2$:

$$\tilde{m}_{I_1,\dots,I_p} : \mathcal{C}_{I_1} \otimes_S \dots \otimes_S \mathcal{C}_{I_p} \to \mathcal{C}_J$$

for all finite linearly ordered sets $J$ with $I_1, \dots, I_p \subseteq J$ such that $I_1 \sqcup \dots \sqcup I_p = J$. Further, we write $\tilde{m}_{I_1,\dots,I_p} = u$ if $p = 0$ and $\tilde{m}_{I_1,\dots,I_p} = \mathrm{id}_{\mathcal{C}_{I_1}}$ if $p = 1$.

**Construction 4.2.23.** Let $\mathcal{C}$ be a $S^+ \operatorname{Mod}(k)$-category with object set $S = \operatorname{Ob}(\mathcal{C})$.

- Given an $n \geq 0$, consider the quiver

  $$\mathcal{T}(\mathcal{C})_n = \bigoplus_{T \in \mathcal{P}_n} \mathcal{T}(\mathcal{C})_T$$

  where, for every necklace $T = \{0 = t_0 < t_1 < \dots < t_k = n\}$:

  $$\begin{aligned}\mathcal{T}(\mathcal{C})_T &= \mathcal{C}_{T_1^c} \otimes_S \mathcal{C}_{T_2^c} \otimes_S \dots \otimes_S \mathcal{C}_{T_k^c} \\ &= \mathcal{C}_{t_1-2} \otimes_S \mathcal{C}_{t_2-t_1-2} \otimes_S \dots \otimes_S \mathcal{C}_{p-t_{k-1}-2} \quad \in k\operatorname{Quiv}_S\end{aligned}$$

- Given a morphism $f : [m] \to [n]$ in $\mathbf{\Delta}_f$, we define a quiver morphism

  $$\mathcal{T}(\mathcal{C})(f) : \mathcal{T}(\mathcal{C})_n \to \mathcal{T}(\mathcal{C})_m$$

as follows. For $T \in \mathcal{P}_n$, write $U = f^{-1}(T) = \{0 = u_0 < u_1 < ... < u_l = m\}$. Then for every $j \in \{0, ..., l\}$, $f(u_j) = t_{p_j}$ for some $p_j \in \{0, ..., k\}$. Thus we can restrict $f$ to

$$f|_{U_j^c} : U_j^c \to T^c_{p_{j-1}+1} \sqcup ... \sqcup T^c_{p_j}$$

in Lin for all $j \in \{1, ..., l\}$. Now define a quiver morphism $\mathcal{T}(\mathcal{C})_U \to \mathcal{T}(\mathcal{C})_T$ as

$$\mathcal{T}(\mathcal{C})(f)_T = \mathcal{C}(f|_{U_1^c})\tilde{m}_{T_1^c,...,T^c_{p_1}} \otimes_S ... \otimes_S \mathcal{C}(f|_{U_l^c})\tilde{m}_{T^c_{p_{l-1}+1},...,T^c_k}$$

Then $\mathcal{T}(\mathcal{C})(f)$ is defined by setting, for all $T \in \mathcal{P}_n$

$$\mathcal{T}(\mathcal{C})(f) \circ \iota_T = \iota_{f^{-1}(T)} \circ \mathcal{T}(\mathcal{C})(f)_T$$

**Example 4.2.24.** Let $0 < j < n$ and consider the coface map $\delta_j : [n-1] \to [n]$ in $\mathbf{\Delta}_f$. For a necklace $T = \{0 = t_0 < t_1 < ... < t_k = n\}$, we have

$$\mathcal{T}(\mathcal{C})(\delta_j)_T = \begin{cases} \mathrm{id}_{\mathcal{C}_{t_1-2}} \otimes ... \otimes d_{j-t_{p-1}-1} \otimes ... \otimes \mathrm{id}_{\mathcal{C}_{n-t_{k-1}-2}} & \text{if } j \notin T \\ \mathrm{id}_{\mathcal{C}_{t_1-2}} \otimes ... \otimes \tilde{m}_{j-t_{p-1}-2, t_{p+1}-j-2} \otimes ... \otimes \mathrm{id}_{\mathcal{C}_{n-t_{k-1}-2}} & \text{if } j \in T \end{cases}$$

where $p \in \{1, ..., k\}$ is the unique integer such that $t_{p-1} < j \le t_p$.

Similarly, consider the codegeneracy map $\sigma_i : [n+1] \to [n]$ in $\mathbf{\Delta}_f$ with $0 \le i \le n$. For a necklace $T = \{0 = t_0 < t_1 < ... < t_k = n\}$, we have

$$\mathcal{T}(\mathcal{C})(\sigma_i)_T = \begin{cases} \mathrm{id}_{\mathcal{C}_{t_1-2}} \otimes ... \otimes s_{i-t_{p-1}-1} \otimes ... \otimes \mathrm{id}_{\mathcal{C}_{n-i_{k-1}-2}} & \text{if } i \notin T \\ \mathrm{id}_{\mathcal{C}_{t_1-2}} \otimes ... \otimes \mathrm{id}_{\mathcal{C}_{i-t_{p-1}-2}} \otimes u \otimes \mathrm{id}_{\mathcal{C}_{t_{p+1}-i-2}} \otimes ... \otimes \mathrm{id}_{\mathcal{C}_{n-t_{k-1}-2}} & \text{if } i \in T \end{cases}$$

where $p \in \{1, ..., k\}$ is the unique integer such that $t_{p-1} < i \le t_p$.

**Proposition 4.2.25.** *Let $\mathcal{C}$ be a small $S^+ \operatorname{Mod}(k)$-category. Then the assignments $n \mapsto \mathcal{T}(\mathcal{C})_n$ and $f \mapsto \mathcal{T}(\mathcal{C})(f)$ of Construction 4.2.23 define a Frobenius monoidal functor*

$$\mathcal{T}(\mathcal{C}) : \mathbf{\Delta}_f^{op} \to k\,\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$$

*and thus $(\mathcal{T}(\mathcal{C}), \mathrm{Ob}(\mathcal{C}))$ is a Frobenius templicial $k$-module.*

*Proof.* Set $S = \mathrm{Ob}(\mathcal{C})$. We first show that $\mathcal{T}(\mathcal{C})$ is a well-defined functor. Take morphisms $f : [m] \to [n]$ and $g : [n] \to [p]$ in $\mathbf{\Delta}_f$ and $T \in \mathcal{P}_p$. Setting $U = g^{-1}(T)$ and $V = f^{-1}(U)$, we must show that

$$\mathcal{T}(\mathcal{C})(f)_U \circ \mathcal{T}(\mathcal{C})_T(g) = \mathcal{T}(\mathcal{C})(gf)_T$$

By the functoriality of the monoidal product $- \otimes_S -$, we may assume that $\ell(V) = 1$. Now write $U = \{0 = u_0 < u_1 < ... < u_l = n\}$ and let $k = \ell(T)$. It follows from the naturality and the associativity of $\tilde{m}$ that

$$\begin{aligned} &\mathcal{T}(\mathcal{C})(f)_U \circ \mathcal{T}(\mathcal{C})(g)_T \\ &= \mathcal{C}(f|_{V_1^c})\tilde{m}_{U_1^c,...,U_l^c}\left(\mathcal{C}(g|_{U_1^c})\tilde{m}_{T_1^c,...,T^c_{p_1}} \otimes_S ... \otimes_S \mathcal{C}(g|_{U_l^c})\tilde{m}_{T^c_{p_{l-1}+1},...,T^c_k}\right) \\ &= \mathcal{C}((g|_{U_1^c} \sqcup ... \sqcup g|_{U_l^c})f|_{V_1^c})\tilde{m}_{T_1^c,...,T_k^c} \\ &= \mathcal{C}(gf|_{V_1^c})\tilde{m}_{T_1^c,...,T_k^c} = \mathcal{T}(\mathcal{C})(gf)_T \end{aligned}$$

Further, for a necklace $(T, n)$ with $k = \ell(T)$ we have

$$\mathcal{T}(\mathcal{C})(\mathrm{id}_{[n]})_T = \mathcal{C}(\mathrm{id}_{T_1^c})\tilde{m}_{T_1^c} \otimes_S \ldots \otimes_S \mathcal{C}(\mathrm{id}_{T_k^c})\tilde{m}_{T_k^c} = \mathrm{id}_{\mathcal{C}_{T_1^c}} \otimes_S \ldots \otimes_S \mathrm{id}_{\mathcal{C}_{T_k^c}}$$

Next we equip $\mathcal{T}(\mathcal{C})$ with colax and Frobenius structures. First note that by definition, $\mathcal{T}(\mathcal{C})_0 \simeq k_S$ is the monoidal unit of $k\,\mathrm{Quiv}_S$ and $\mathcal{T}(\mathcal{C})_T \otimes_S \mathcal{T}(\mathcal{C})_U \simeq \mathcal{T}(\mathcal{C})_{T\vee U}$ for all necklaces $T$ and $U$. Then take $k, l \geq 0$. Every necklace $T \in \mathcal{P}_{k+l}$ containing $k$ can be uniquely split as $T = T_1 \vee T_2$ with $T_1 \in \mathcal{P}_k$ and $T_2 \in \mathcal{P}_l$ so that we have the following canonical projection and coprojection maps:

$$\mu_{k,l} : \mathcal{T}(\mathcal{C})_{k+l} = \bigoplus_{T\in\mathcal{P}_{k+l}} \mathcal{T}(\mathcal{C})_T \to \bigoplus_{\substack{T\in\mathcal{P}_{k+l}\\ k\in T}} \mathcal{T}(\mathcal{C})_T \simeq \mathcal{T}(\mathcal{C})_k \otimes_S \mathcal{T}(\mathcal{C})_l$$

$$Z^{k,l} : \mathcal{T}(\mathcal{C})_k \otimes_S \mathcal{T}(\mathcal{C})_l \simeq \bigoplus_{\substack{T\in\mathcal{P}_{k+l}\\ k\in T}} \mathcal{T}(\mathcal{C})_T \to \bigoplus_{T\in\mathcal{P}_{k+l}} \mathcal{T}(\mathcal{C})_T = \mathcal{T}(\mathcal{C})_{k+l}$$

Now take $f : [k] \to [p]$ and $g : [l] \to [q]$ in $\mathbf{\Delta}_f$, and $T \in \mathcal{P}_p$, $U \in \mathcal{P}_q$. Then we have $f^{-1}(T) \vee g^{-1}(U) = (f+g)^{-1}(T \vee U)$ and it follows from Construction 4.2.23 that

$$\mathcal{T}(\mathcal{C})(f+g)_{T\vee U} = \mathcal{T}(\mathcal{C})(f)_T \otimes_S \mathcal{T}(\mathcal{C})(g)_U$$

From this it is easy to see that $\mu_{k,l}$ and $Z^{k,l}$ are natural in $k, l \geq 0$.

We complete the proof by showing that the maps $\mu_{k,l}$ and $Z^{k,l}$ satisfy the Frobenius equation (2.9). Take $k, l, p, q \geq 0$ such that $k + l = p + q$ and assume that $k \geq p$. Then for all $T \in \mathcal{P}_{p+q}$ with $p \in T$ we have

$$(Z^{p,k-p} \otimes_S \mathrm{id}_{\mathcal{T}(\mathcal{C})_l})(\mathrm{id}_{\mathcal{T}(\mathcal{C})_p} \otimes_S \mu_{k-p,l})\iota_T = \begin{cases} \iota_T & \text{if } k \in T \\ 0 & \text{if } k \notin T \end{cases} = \mu_{k,l} Z^{p,q} \iota_T$$

A similar proof shows the case $k \leq p$. □

**Construction 4.2.26.** Let $H : \mathcal{C} \to \mathcal{D}$ be an $S^+\,\mathrm{Mod}(k)$-functor between small $S^+\,\mathrm{Mod}(k)$-categories and let $f : S \to T$ denote its object map. For every $n \geq 0$, we construct a quiver map in $k\,\mathrm{Quiv}_T$:

$$\mathcal{T}(H)_n : f_!(\mathcal{T}(\mathcal{C})_n) \to \mathcal{T}(\mathcal{D})_n$$

For every finite linearly ordered set $J$, we have a quiver map $H : \mathcal{C}_J \to f^*(\mathcal{D}_J)$ in $k\,\mathrm{Quiv}_S$. Denote its adjoint in $k\,\mathrm{Quiv}_T$ by

$$H'_J : f_!(\mathcal{C}_J) \to \mathcal{D}_J$$

Then define, for all necklaces $(U, n)$ with $k = \ell(U)$:

$$\mathcal{T}(H)_U : f_!(\mathcal{T}(\mathcal{C})_U) \to f_!(\mathcal{C}_{U_1^c}) \otimes_T \ldots \otimes_T f_!(\mathcal{C}_{U_k^c}) \xrightarrow{H'_{U_1^c} \otimes_T \ldots \otimes_T H'_{U_k^c}} \mathcal{T}(\mathcal{D})_U$$

Finally, for $n \geq 0$, set

$$\mathcal{T}(H)_n : f_!(\mathcal{T}(\mathcal{C})_n) \simeq \bigoplus_{U\in\mathcal{P}_n} f_!(\mathcal{T}(\mathcal{C})_U) \xrightarrow{\bigoplus_U \mathcal{T}(H)_U} \bigoplus_{U\in\mathcal{P}_n} \mathcal{T}(\mathcal{D})_U = \mathcal{T}(\mathcal{D})_n$$

**Lemma 4.2.27.** *Let $H : \mathcal{C} \to \mathcal{D}$ be an $S^+ \operatorname{Mod}(k)$-functor between small $S^+ \operatorname{Mod}(k)$-categories. Then the quiver maps $(\mathcal{T}(H)_n)_{n\geq 0}$ of Construction 4.2.26 define a Frobenius templicial map $\mathcal{T}(\mathcal{C}) \to \mathcal{T}(\mathcal{D})$ between Frobenius templicial $k$-modules.*

*Proof.* We employ the same notations as in Construction 4.2.26. It follows from the enriched functoriality of $H$ that

- $H'_J f_!(\mathcal{C}(h)) = \mathcal{D}(h) H'_I$ for all $h : I \to J$ in Lin, and
- for all $I, J \in \text{Lin}$, the following diagram commutes

$$\begin{tikzcd}
f_!(\mathcal{C}_{I\sqcup J}) \arrow[r, "H'_{I\sqcup J}"] & \mathcal{D}_{I\sqcup J} & \\
f_!(\mathcal{C}_I \otimes_S \mathcal{C}_J) \arrow[u, "\tilde{m}^{\mathcal{C}}_{I,J}"] \arrow[r] & f_!(\mathcal{C}_I) \otimes_T f_!(\mathcal{C}_J) \arrow[r, "H'_I \otimes_T H'_J"'] & \mathcal{D}_I \otimes_T \mathcal{D}_J \arrow[ul, "\tilde{m}^{\mathcal{D}}_{I,J}"']
\end{tikzcd}$$

From this it easily follows that the quiver maps $(\mathcal{T}(H)_n)_{n\geq 0}$ define a natural transformation $\mathcal{T}(H) : f_!\mathcal{T}(C) \to \mathcal{T}(\mathcal{D})$ between functors $\mathbf{\Delta}_f^{op} \to k \operatorname{Quiv}_T$.

Further, it is clear that for all necklaces $U$ and $V$, $\mathcal{T}(H)_{U\vee V}$ is equal to the composite

$$f_!(\mathcal{T}(\mathcal{C})_{U\vee V}) \to f_!(\mathcal{T}(\mathcal{C})_U) \otimes_T f_!(\mathcal{T}(\mathcal{C})_V) \xrightarrow{\mathcal{T}(H)_U \otimes \mathcal{T}(H)_V} \mathcal{T}(\mathcal{D})_U \otimes_T \mathcal{T}(\mathcal{D})_V \simeq \mathcal{T}(\mathcal{D})_{U\vee V}$$

From this it is easy to see that the natural transformation $\mathcal{T}(H)$ is monoidal and satisfies (4.5). Thus $(\mathcal{T}(H), f) : (\mathcal{T}(C), S) \to (\mathcal{T}(\mathcal{D}), T)$ a Frobenius templicial map. □

**Proposition 4.2.28.** *The assignments $\mathcal{C} \mapsto (\mathcal{T}(\mathcal{C}), \operatorname{Ob}(\mathcal{C}))$ of Proposition 4.2.25 and $H \mapsto \mathcal{T}(H)$ of Lemma 4.2.27 define a functor*

$$\mathcal{T} : k \operatorname{Cat}_{\Delta_+} \to S_{\otimes}^{Frob} \operatorname{Mod}(k)$$

*Proof.* This is now immediate from the definitions. □

### From Frobenius templicial modules to $S^+ \operatorname{Mod}(k)$-categories

We will now construct the inverse $\mathcal{K} : S_{\otimes}^{Frob} \operatorname{Mod}(k) \to k \operatorname{Cat}_{\Delta_+}$ to $\mathcal{T}$.

**Lemma 4.2.29.** *Let $(X, S)$ be a templicial $k$-module with comultiplication $\mu$ and $m, n \geq 1$. Let $f : [m] \to [n]$ be an order morphism such that $f^{-1}(\{0\}) = \{0\}$ and $f^{-1}(\{n\}) = m$. Then the quiver map $X(f) : X_n \to X_m$ restricts to*

$$\bigcap_{k=1}^{n-1} \ker(\mu_{k,n-k}) \to \bigcap_{p=1}^{m-1} \ker(\mu_{p,m-p})$$

*Proof.* Take $a, b \in S$ and $x \in X_n(a, b)$ such that $\mu_{k,n-k}(x) = 0$ for all $0 < k < n$. Then for all $0 < p < m$, there exist unique morphisms $f_1 : [p] \to [k]$ and $f_2 : [m-p] \to [n-k]$ in

$\mathbf{\Delta}_f$ such that $f_1 + f_2 = f$ with $k = f(p)$ (Remark 2.1.4). By the hypothesis on $f$, we have that $0 < k < n$ as well. Now

$$\mu_{p,m-p}(X(f)(x)) = (X(f_1) \otimes_S X(f_2))\mu_{k,n-k}(x) = 0$$

and the result follows. □

**Construction 4.2.30.** Let $(X, S)$ be a templicial $k$-module with a Frobenius structure $Z$. We construct an $S^+ \operatorname{Mod}(k)$-enriched quiver $\mathcal{K}(X)$ as follows.

Set $\operatorname{Ob}(\mathcal{K}(X)) = S$. Take $n \geq -1$ and consider the subquiver

$$\mathcal{K}(X)_n = \bigcap_{k=1}^{n+1} \ker(\mu_{k,n+2-k}) \subseteq X_{n+2}$$

So for example $\mathcal{K}(X)_{-1} = X_1$ and $\mathcal{K}(X)_0 = \ker(\mu_{1,1})$. Given $f : [m] \to [n]$ in $\mathbf{\Delta}_+$, the morphism

$$[0] \star f \star [0] : [m+2] \to [n+2]$$

satisfies the hypothesis of Lemma 4.2.29 and thus induces a quiver map

$$\mathcal{K}(X)(f) : \mathcal{K}(X)_n \to \mathcal{K}(X)_m$$

It is clear that this defines a functor $\mathcal{K}(X) : \mathbf{\Delta}_+^{op} \to k\operatorname{Quiv}_S$, or equivalently a quiver

$$\mathcal{K}(X) \in S^+ \operatorname{Mod}(k)\text{-}\operatorname{Quiv}_S$$

**Lemma 4.2.31.** *Let $f : [k] \to [p]$ and $g : [l] \to [q]$ be morphisms in $\mathbf{\Delta}_+$. Then*

$$\delta_{p+2}([0] \star f \star g \star [0]) = ([0] \star f \star [0] + [0] \star g \star [0])\delta_{k+2}$$

*Proof.* Clearly the morphisms on both sides of the equation preserve the endpoints. Evaluating either side in $0 < i < k + l + 3$, we obtain

$$\begin{cases} f(i-1) + 1 & \text{if } i \leq k+1 \\ g(i-k-2) + p + 3 & \text{if } i \geq k+2 \end{cases}$$

□

**Proposition 4.2.32.** *Let $(X, S)$ be a templicial $k$-module with Frobenius structure $Z$. Then the quiver maps*

$$\left(\tilde{m}_{p,q} = d_{p+2} Z^{p+2,q+2}|_{\mathcal{K}(X)_p \otimes_S \mathcal{K}(X)_q} : \mathcal{K}(X)_p \otimes_S \mathcal{K}(X)_q \to \mathcal{K}(X)_{p+q+1}\right)_{p,q \geq -1}$$

*define a reverse composition law on the quiver $\mathcal{K}(X)$ of Construction 4.2.30 with identities determined by the quiver map*

$$u : k_S \simeq X_0 \xrightarrow{s_0} X_1 = \mathcal{K}(X)_{-1}$$

*Consequently, $\mathcal{K}(X)$ has the structure of an $S^+ \operatorname{Mod}(k)$-category.*

*Proof.* Let $p, q \geq -1$ and set $n = p + q + 3$. consider the quiver map

$$d_{p+2}Z^{p+2,q+2} : X_{p+2} \otimes_S X_{q+2} \to X_{p+q+3}$$

For all $0 < k < n$, we have

$$\mu_{k,n-k}d_{p+2}Z^{p+2,q+2} = \begin{cases} (d_{p+2} \otimes_S \mathrm{id}_{X_{n-k}})\mu_{k+1,n-k}Z^{p+2,q+2} & \text{if } p+2 \leq k \\ (\mathrm{id}_{X_k} \otimes_S d_{p+2-k})\mu_{k,n-k+1}Z^{p+2,q+2} & \text{if } p+2 > k \end{cases}$$

$$= \begin{cases} (d_{p+2}Z^{p+2,k-p-1} \otimes_S \mathrm{id}_{X_{n-k}})(\mathrm{id}_{X_{p+2}} \otimes_S \mu_{k-p-1,n-k}) & \text{if } p+2 \leq k \\ ((\mathrm{id}_{X_k} \otimes_S d_{p+2-k}Z^{p+2-k,q+2})(\mu_{k,p+2-k} \otimes_S \mathrm{id}_{X_{q+2}}) & \text{if } p+2 > k \end{cases}$$

which implies that $d_{p+2}Z^{p+2,q+2}$ restricts to a quiver map

$$\tilde{m}_{p,q} : \mathcal{K}(X)_p \otimes_S \mathcal{K}(X)_q \to \mathcal{K}(X)_{p+q+1}$$

Take morphisms $f : [k] \to [p]$ and $g : [l] \to [q]$ in $\mathbf{\Delta}_+$. By Lemma 4.2.31, we have that

$$X([0] \star f \star g \star [0])d_{p+2}Z^{p+2,q+2} = d_{k+2}Z^{k+2,l+2}\,(X([0] \star f \star [0]) \otimes_S X([0] \star g \star [0]))$$

It follows that the quiver maps $(\tilde{m}_{p,q})_{p,q \geq -1}$ define a quiver map in $S^+\,\mathrm{Mod}(k)\text{-}\,\mathrm{Quiv}_S$:

$$\tilde{m} : \mathcal{K}(X) \star_S \mathcal{K}(X) \to \mathcal{K}(X)$$

It remains to show that $\tilde{m}$ is associative and unital with respect to $u$. For this it suffices to note that for all $p, q, r \geq -1$:

$$\begin{aligned} &d_{p+q+3}Z^{p+q+3,r+2}(d_{p+2}Z^{p+2,q+2} \otimes_S \mathrm{id}_{X_{r+2}}) \\ &= d_{p+q+3}d_{p+2}Z^{p+q+4,r+2}(Z^{p+2,q+2} \otimes_S \mathrm{id}_{X_{r+2}}) \\ &= d_{p+2}d_{p+q+4}Z^{p+2,q+r+4}(\mathrm{id}_{X_{p+2}} \otimes_S Z^{q+2,r+2}) \\ &= d_{p+2}Z^{p+2,q+r+3}(\mathrm{id}_{X_{p+2}} \otimes_S d_{q+2}Z^{q+2,r+2}) \end{aligned}$$

$$d_{p+2}Z^{p+2,1}(\mathrm{id}_{X_{p+2}} \otimes_S u) = d_{p+2}s_{p+2}Z^{p+2,0}(\mathrm{id}_{X_{p+2}} \otimes_S \epsilon^{-1}) = \mathrm{id}_{X_{p+2}}$$
$$d_1Z^{1,p+2}(u \otimes_S \mathrm{id}_{X_{p+2}}) = d_1s_0Z^{0,p+2}(\epsilon^{-1} \otimes_S \mathrm{id}_{X_{p+2}}) = \mathrm{id}_{X_{p+2}}$$

where $\epsilon : X_0 \xrightarrow{\sim} k_S$ is the counit of $X$ and $k_S$ is the monoidal unit of $k\,\mathrm{Quiv}_S$. ∎

**Proposition 4.2.33.** *The assignment $(X, Z) \mapsto \mathcal{K}(X)$ of Proposition 4.2.32 extends to a functor*

$$\mathcal{K} : S_\otimes^{Frob}\,\mathrm{Mod}(k) \to k\,\mathrm{Cat}_{\Delta_+}$$

*Proof.* Take a Frobenius templicial map $\alpha : X \to Y$ with vertex map $f : S \to T$. Consider the underlying natural transformation $\alpha : f_!X \to Y$ and its adjoint $\alpha' : X \to f^*Y$ (see Construction 1.1.16). Then $\alpha'$ clearly restricts to a natural transformation

$$\mathcal{K}(\alpha) : \mathcal{K}(X) \to f^*\mathcal{K}(Y)$$

between functors $\mathbf{\Delta}_+^{op} \to k\,\mathrm{Quiv}_S$, which can equivalently be considered as a morphism in $S^+\,\mathrm{Mod}(k)\text{-}\,\mathrm{Quiv}_S$. It then follows from the compatibilty of $\alpha$ with the Frobenius structures that $\mathcal{K}(\alpha)$ is an $S^+\,\mathrm{Mod}(k)$-functor.

It immediately follows from the definitions that this defines a functor

$$\mathcal{K} : S_\otimes^{Frob}\,\mathrm{Mod}(k) \to k\,\mathrm{Cat}_{\Delta_+}$$

∎

**Proving the equivalence**

We finish this subsection by showing that the functors $\mathcal{T}$ and $\mathcal{K}$ are each other's inverses.

**Construction 4.2.34.** Let $(X, S)$ be a templicial $k$-module with Frobenius structure $Z$. Then from the inclusions $(\mathcal{K}(X)_p \hookrightarrow X_{p+2})_{p\geq -1}$, we obtain a quiver map

$$\mathcal{T}\mathcal{K}(X)_T = \mathcal{K}(X)_{t_1-2} \otimes_S \dots \otimes_S \mathcal{K}(X)_{n-t_{k-1}-2} \to X_{t_1} \otimes_S \dots \otimes_S X_{n-t_{k-1}} = X_T$$

for any necklace $T = \{0 = t_0 < t_1 < \dots < t_k = n\}$. Thus for any $n$, we can consider the quiver map

$$\epsilon_{X_n} : \mathcal{T}\mathcal{K}(X)_n = \bigoplus_{T\in\mathcal{P}_n} \mathcal{T}\mathcal{K}(X)_T \to \bigoplus_{T\in\mathcal{P}_n} X_T \xrightarrow{(Z^T)_T} X_n$$

**Proposition 4.2.35.** *Let $X$ be a Frobenius templicial $k$-module. The quiver morphisms $(\epsilon_{X_n})_{n\geq 0}$ of Construction 4.2.34 define a Frobenius templicial map*

$$\epsilon_X : \mathcal{T}\mathcal{K}(X) \to X$$

*which is natural in $X$.*

*Proof.* Let $T = \{0 = t_0 < t_1 < t_2 < \dots < t_k = n\}$ be a necklace. Writing $\tilde{m}$ for the reverse composition law of $\mathcal{K}(X)$ (see Proposition 4.2.32), note that

$$\tilde{m}_{T_1^c,\dots,T_k^c} = X(\delta_T)Z^T|_{\mathcal{T}\mathcal{K}(X)_T} : \mathcal{K}(X)_{T_1^c} \otimes_S \dots \otimes_S \mathcal{K}(X)_{T_k^c} \to \mathcal{K}(X)_{[n]\setminus T}$$

where we denoted $\delta_T = \delta_{t_{k-1}} \cdots \delta_{t_2}\delta_{t_1}$. Now take a morphism $f : [m] \to [n]$ in $\mathbf{\Delta}_f$ with a necklace $T \in \mathcal{P}_n$ and set $U = f^{-1}(T)$. Write $U = \{0 = u_0 < u_1 < \dots < u_l = m\}$ and let $(T_1, \dots, T_l)$ be the splitting of $T$ over $f(U)$. Further, $f = f_1 + \dots + f_l$ for some unique $f_i : [u_i - u_{i-1}] \to [f(u_i) - f(u_{i-1})]$ in $\mathbf{\Delta}_f$. In fact, we have the following equality of morphisms in $\mathbf{\Delta}_+$:

$$f_i = \delta_{T_i}([0] \star f|_{U_i^c} \star [0])$$

where we identified $U_i^c$ with $[u_i - u_{i-1} - 2]$. Hence, we find that

$$\begin{aligned}
&Z^U|_{\mathcal{T}\mathcal{K}(X)_U} \circ \mathcal{T}\mathcal{K}(X)(f)_T \\
&\quad = Z^U\left(\mathcal{K}(X)(f|_{U_1^c})X(\delta_{T_1})Z^{T_1} \otimes_S \dots \otimes_S \mathcal{K}(X)(f|_{U_l^c})X(\delta_{T_l})Z^{T_l}\right)|_{\mathcal{T}\mathcal{K}(X)_T} \\
&\quad = Z^U(X(f_1)Z^{T_1} \otimes_S \dots \otimes_S X(f_l)Z^{T_l})|_{\mathcal{T}\mathcal{K}(X)_T} \\
&\quad = X(f)Z^{f(U)}(Z^{T_1} \otimes_S \dots \otimes_S Z^{T_l})|_{\mathcal{T}\mathcal{K}(X)_T} = X(f) \circ Z^T|_{\mathcal{T}\mathcal{K}(X)_T}
\end{aligned}$$

where we used the associativity of $Z$ in the last equality. We have thus shown that $\epsilon_X : \mathcal{T}\mathcal{K}(X) \to X$ is a natural transformation between functors $\mathbf{\Delta}_f^{op} \to k\,\mathrm{Quiv}_S$.

Next, it follows from Proposition 2.2.40 that for all necklaces $(T, n)$ and $0 < k < n$:

$$\mu_{k,n-k}Z^T|_{\mathcal{T}\mathcal{K}(X)_T} = \begin{cases} Z^{T_1}|_{\mathcal{T}\mathcal{K}(X)_{T_1}} \otimes_S Z^{T_2}|_{\mathcal{T}\mathcal{K}(X)_{T_2}} & \text{if } k \in T \\ 0 & \text{if } k \notin T \end{cases}$$

where $(T_1, T_2)$ is the splitting of $T$ over $\{0 < k < n\}$. From this is it easy to see that $\epsilon_X$ is respects the comultiplications of $\mathcal{T}\mathcal{K}(X)$ and $X$ as well. Then it is clear from the definitions that $\epsilon_X$ is in fact a Frobenius templicial map.

Finally, the naturality of $\epsilon_X$ in $X$ quickly follows from the definitions of $\mathcal{T}$ and $\mathcal{K}$, and the diagram (4.5). ☐

**Lemma 4.2.36.** *Let $X$ be a Frobenius templicial object and $n \geq 1$. The inclusion of quivers $\mathcal{K}(X)_{n-2} \hookrightarrow X_n$ has a retraction*

$$\xi_n = \sum_{T \in \mathcal{P}_n} (-1)^{\ell(T)+1} Z^T \mu_T$$

*Proof.* Take $0 < k < n$, then by Corollary 2.2.41.3, we have

$$\begin{aligned} \mu_{k,n-k}\left(\sum_{T \in \mathcal{P}_n} (-1)^{\ell(T)} Z^T \mu_T\right) &= \sum_{U \in \mathcal{P}_n} \sum_{\substack{T \in \mathcal{P}_n \\ T \cup \{k\} = U}} (-1)^{\ell(T)} \mu_{k,n-k} Z^U \mu_U \\ &= \sum_{\substack{U \in \mathcal{P}_n \\ k \in U}} \left((-1)^{\ell(U \setminus \{k\})} + (-1)^{\ell(U)}\right) \mu_{k,n-k} Z^U \mu_U = 0 \end{aligned}$$

This shows that $\xi_n : X_n \to \mathcal{K}(X)_{n-2}$ is well-defined.

Further note that for $T \in \mathcal{P}_n$, we have $\mu_T|_{\mathcal{K}(X)_{n-2}} = 0$ unless $T = \{0, n\}$, by definition of $\mathcal{K}(X)_{n-2}$. It follows that $\xi|_{\mathcal{K}(X)_{n-2}} = \mathrm{id}_{\mathcal{K}(X)_{n-2}}$ as desired. $\square$

**Lemma 4.2.37.** *Let $n \geq 0$ and $V \subsetneq U$ necklaces in $\mathcal{P}_n$. Then*

$$\sum_{\substack{T \in \mathcal{P}_n \\ V \subseteq T \subseteq U}} (-1)^{\ell(T)} = 0$$

*Proof.* Choose $k \in U \setminus V$, then $T \mapsto T \setminus \{k\}$ defines a bijection

$$\{T \in \mathcal{P}_n \mid V \subseteq T \subseteq U, k \in T\} \xrightarrow{\sim} \{T \in \mathcal{P}_n \mid V \subseteq T \subseteq U, k \notin T\}$$

Moreover, if $k \in T$, then $\ell(T \setminus \{k\}) = \ell(T) - 1$. The result follows. $\square$

**Proposition 4.2.38.** *The natural transformation of Proposition 4.2.35*

$$\epsilon : \mathcal{T} \circ \mathcal{K} \to \mathrm{id}_{S_\otimes^{Frob} \mathrm{Mod}(k)}$$

*is an isomorphism.*

*Proof.* Fix $n \geq 0$. Given a necklace $T = \{0 = t_0 < t_1 < ... < t_k = p\}$ we set

$$\xi_T = \xi_{t_1} \otimes_S ... \otimes_S \xi_{n-t_{k-1}} : X_T \to \mathcal{T}\mathcal{K}(X)_T$$

where for each $p \geq 1$, $\xi_p$ denotes the retraction from Lemma 4.2.36. We claim that the quiver morphism

$$(\xi_T \mu_T)_T : X_n \to \bigoplus_{T \in \mathcal{P}_n} \mathcal{T}\mathcal{K}(X)_T = \mathcal{T}\mathcal{K}(X)_n$$

is inverse to $\epsilon_{X_n}$.

It follows from Remark 2.2.4 and the coassociativity of $\mu$ that

$$\xi_T \mu_T = \sum_{\substack{U \in \mathcal{P}_n \\ T \subseteq U}} (-1)^{\ell(U)+k} (Z^{U_1} \otimes_S ... \otimes_S Z^{U_k}) \mu_U : X_n \to \mathcal{T}\mathcal{K}(X)_T$$

where $(U_1, ..., U_k)$ denotes the splitting of $U$ over $T$. It follows from Lemma 4.2.37 (with $V = \{0 < n\}$) and the associativity of $Z$ that

$$\sum_{T\in\mathcal{P}_n} Z^T|_{\mathcal{T}\mathcal{K}(X)_T}\xi_T\mu_T = \sum_{\substack{U\in\mathcal{P}_n\\ T\subseteq U}} (-1)^{\ell(U)+\ell(T)} Z^U \mu_U$$
$$= \sum_{U\in\mathcal{P}_n} (-1)^{\ell(U)} \left( \sum_{\substack{T\in\mathcal{P}_n\\ T\subseteq U}} (-1)^{\ell(T)} \right) Z^U \mu_U = \mathrm{id}_{X_n}$$

Hence, $(\xi_T\mu_T)_T$ is a section of $\epsilon_{X_n}$.

Conversely, take necklaces $T, U \in \mathcal{P}_n$. Then by Proposition 2.2.40:

$$\xi_U\mu_U Z^T|_{\mathcal{K}(X)_T} = (\xi_{u_1} Z^{T_1} \otimes_S ... \otimes_S \xi_{n-u_{l-1}} Z^{T_l})(\mu_{U_1} \otimes_S ... \otimes_S \mu_{U_k})|_{\mathcal{T}\mathcal{K}(X)_T}$$

where $U = \{0 = u_0 < u_1 < ... < u_l = n\}$ and $(T_1, ..., T_l)$ and $(U_1, ..., U_k)$ are the splittings of $T$ over $U$ and $U$ over $T$ respectively. Now since $Z$ is associative, it follows completely dually to the proof of Lemma 4.2.36 that for all $j \in \{1, ..., l\}$,

$$\xi_{u_j-u_{j-1}} Z^{T_j} = 0$$

whenever $\ell(T_j) > 1$. Hence the right hand side of the above equation vanishes unless the length of every $U_i$ and $T_j$ is 1. In the latter case, we have $T = U$ and thus

$$\xi_U\mu_U Z^T|_{\mathcal{T}\mathcal{K}(X)_T} = \xi_T|_{\mathcal{T}\mathcal{K}(X)_T} = \mathrm{id}_{\mathcal{T}\mathcal{K}(X)_T}$$

Thus $(\xi_T\mu_T)_T$ is also a post-inverse of $\epsilon_{X_n}$. □

**Construction 4.2.39.** Let $\mathcal{C}$ be an $S^+\operatorname{Mod}(k)$-category with object set $S = \operatorname{Ob}(\mathcal{C})$. Note that for all $n \geq -1$, we can consider the following coprojection in $k\operatorname{Quiv}_S$:

$$\mathcal{C}_n = \mathcal{C}_{\{0<n+2\}^c} \hookrightarrow \bigoplus_{T\in\mathcal{P}_{n+2}} \mathcal{C}_{T_1^c} \otimes_{\mathcal{S}} ... \otimes_S \mathcal{C}_{T_k^c} = \mathcal{T}(\mathcal{C})_{n+2}$$

It immediately follows from the definition of the comultiplication of $\mathcal{T}(\mathcal{C})$ that this quiver map factors as

$$\mathcal{C}_n \xrightarrow{\eta_{\mathcal{C}_n}} \mathcal{K}\mathcal{T}(\mathcal{C})_n \hookrightarrow \mathcal{T}(\mathcal{C})_n$$

where $\eta_{\mathcal{C}_n}$ is an isomorphism and the second quiver map is the canonical inclusion.

**Proposition 4.2.40.** *Let $\mathcal{C}$ be a small $S^+\operatorname{Mod}(k)$-category. Then the quiver morphisms $(\eta_{\mathcal{C}_n})_{n\geq-1}$ of Construction 4.2.39 define an isomorphism in $k\operatorname{Cat}_{\Delta_+}$:*

$$\eta : \mathcal{C} \to \mathcal{K}\mathcal{T}(\mathcal{C})$$

*that is natural in $\mathcal{C}$.*

*Proof.* If $f : [m] \to [n]$ is a morphism in $\mathbf{\Delta}_+$, then $g = [0]\star f\star[0] : [m+2] \to [n+2]$ belongs to $\mathbf{\Delta}_f$ and $g^{-1}(\{0 < n+2\}) = \{0 < m+2\}$. Thus $\mathcal{T}(\mathcal{C})(f)_{\{0<n+2\}} = \mathcal{C}(f)$. It follows that the quiver maps $(\eta_{\mathcal{C}_n})_{n\geq-1}$ define a map $\eta_{\mathcal{C}} : \mathcal{C} \to \mathcal{K}\mathcal{T}(\mathcal{C})$ in $S^+\operatorname{Mod}(k)\text{-}\operatorname{Quiv}_S$ where $S = \operatorname{Ob}(\mathcal{C})$.

To show that $\eta_{\mathcal{C}}$ is also an $S^+\operatorname{Mod}(k)$-functor, let $\tilde{m}$ denote the reverse composition law of $\mathcal{C}$. Also, let the quiver map $u : k_S \to \mathcal{C}_{-1}$ represent the identities of $\mathcal{C}$ with $k_S$ the monoidal unit of $k\operatorname{Quiv}_S$. For $p, q \geq -1$, consider the coface map $\delta_{p+2} : [p+q+3] \to [p+q+4]$. Then it suffices to note that $\tilde{m}_{p,q}$ is precisely

$$\mathcal{T}(\mathcal{C})(\delta_{p+2})_{\{0<p+2<p+q+4\}} : \mathcal{C}_p \otimes_S \mathcal{C}_q \to \mathcal{C}_{p+q+1}$$

and that the degeneracy map $s_0 : \mathcal{T}(\mathcal{C})_0 \to \mathcal{T}(\mathcal{C})_1$ coincides with $u$.

Finally, the naturality of $\eta_{\mathcal{C}}$ follows immediately from the definitions. □

*Proof of Theorem 4.2.17.* In view of Propositions 4.2.38 and 4.2.40, it remains to verify the triangle identities for the unit $\eta$ and the counit $\epsilon$.

Let $(X, Z)$ be a Frobenius templicial module. Then $\mathcal{K}(\epsilon_X) \circ \eta_{\mathcal{K}(X)} = \mathrm{id}_{\mathcal{K}(X)_n}$ follows from the fact that for all $n \geq -1$, $Z^{\{0<n+2\}}|_{\mathcal{K}(X)_n}$ is the identity on $\mathcal{K}(X)_n$.

Let $\mathcal{C}$ be an $S^+\operatorname{Mod}(k)$-category with object set $S = \operatorname{Ob}(\mathcal{C})$. Then to prove $\epsilon_{\mathcal{T}(\mathcal{C})} \circ \mathcal{T}(\eta_{\mathcal{C}}) = \mathrm{id}_{\mathcal{T}(\mathcal{C})}$, it suffices to note that the composite

$$\mathcal{T}(\mathcal{C})_n = \bigoplus_{T \in \mathcal{P}_n} \mathcal{C}_{T_1^c} \otimes_S \dots \otimes_S \mathcal{C}_{T_k^c} \hookrightarrow \bigoplus_{T \in \mathcal{P}_n} \mathcal{T}(\mathcal{C})_{t_1} \otimes_S \dots \otimes_S \mathcal{T}(\mathcal{C})_{n-t_{k-1}} \xrightarrow{(Z^T)_T} \mathcal{T}(\mathcal{C})_n$$

is the identity for all $n \geq 0$, where $Z$ is the Frobenius structure of $\mathcal{T}(\mathcal{C})$. The latter follows quickly from the definition of $Z$. □

### 4.2.3 The linear differential graded nerve

We are now ready to define the linear dg-nerve $N_k^{dg} : k\operatorname{Cat}_{dg} \to S_{\otimes}\operatorname{Mod}(k)$. It is constructed using the two equivalences from the previous subsections (Definition 4.2.46). The remainder of this subsection is devoted to showing that the linear dg-nerve $N_k^{dg}$ lifts the classical dg-nerve $N^{dg}$ along $\tilde{U} : S_{\otimes}\operatorname{Mod}(k) \to \operatorname{SSet}$ (Corollary 4.2.54). We will achieve this by proving the more general Proposition 4.2.53 which characterizes templicial maps into the linear dg-nerve.

**Definition 4.2.41.** A *differential graded category* or *dg-category* over $k$ is a category enriched in the monoidal category $(\operatorname{Ch}(k), \otimes, k[0])$ of Construction 4.2.13. A *dg-functor* is a $\operatorname{Ch}(k)$-enriched functor. We denote the category of small dg-categories over $k$ and dg-functors between them by

$$k\operatorname{Cat}_{dg} = \operatorname{Ch}(k)\text{-}\operatorname{Cat}$$

We call a dg-category $\mathcal{C}$ *non-negatively graded* if for all $A, B \in \operatorname{Ob}(\mathcal{C})$, the chain complex $\mathcal{C}_\bullet(A, B)$ is non-negatively graded. We denote

$$k\operatorname{Cat}_{dg,\geq 0}$$

for the full subcategory of $k\operatorname{Cat}_{dg}$ spanned by all non-negatively graded dg-categories. Equivalently, we can define $k\operatorname{Cat}_{dg,\geq 0}$ as the category $\operatorname{Ch}_{\geq 0}(k)\text{-}\operatorname{Cat}$ of small categories enriched in $(\operatorname{Ch}_{\geq 0}(k), \otimes, k[0])$.

For more details on dg-categories, we refer to the literature. See [Kel06] and [Toë11] for example.

*Remark* 4.2.42. Let $\mathcal{C}_\bullet$ be a dg-category. As noted in Remark 1.1.22, $\mathcal{C}_\bullet$ comes equipped with a chain map

$$\tilde{m} : \mathcal{C}_\bullet(A, B) \otimes \mathcal{C}_\bullet(B, C) \to \mathcal{C}_\bullet(A, C)$$

for all $A, B, C \in \mathrm{Ob}(\mathcal{C})$, while the composition law of a dg-category is conventionally given by chain maps

$$m : \mathcal{C}_\bullet(B, C) \otimes \mathcal{C}_\bullet(A, B) \to \mathcal{C}_\bullet(A, C)$$

We can of course easily pass from one to the other by composing with the symmetry in $\mathrm{Ch}(k)$: $\sigma : \mathcal{C}_\bullet(A, B) \otimes \mathcal{C}_\bullet(B, C) \xrightarrow{\sim} \mathcal{C}_\bullet(B, C) \otimes \mathcal{C}_\bullet(A, B)$. But beware that this introduces a sign; for all $f \in \mathcal{C}_p(A, B)$ and $g \in \mathcal{C}_q(B, C)$ we have

$$\sigma(f \otimes g) = (-1)^{pq} g \otimes f$$

Further, we reserve the notation

$$g \circ f = m(g \otimes f)$$

for the conventional composition. Thus we have

$$\tilde{m}(f \otimes g) = (-1)^{pq} g \circ f$$

**The classical dg-nerve**

Let us first recall the classical dg-nerve functor

$$N^{dg} : k\,\mathrm{Cat}_{dg} \to \mathrm{SSet}$$

which implicitly goes back to Block and Smith [BS14], but was formally constructed and named by Lurie [Lur16, Construction 1.3.1.6]. A few different versions of $N^{dg}$ exist in the literature, with varying sign conventions. Most notably there is Faonte's "small dg-nerve" [Fao17, Definition 2.2.8] and a second version by Lurie [Lur18, Tag 00PL]. They are however all isomorphic to each other. For our purposes, Faonte's version is the most convenient, which is why we will use it here. It is defined as follows.

Given a small dg-category $\mathcal{C}_\bullet$, the *differential graded (dg) nerve* $N^{dg}(\mathcal{C})$ is the simplicial set where for every $n \geq 0$, an $n$-simplex is a pair

$$\left( (A_i)_{i=0}^n, (f_I)_{\substack{I \subseteq [n] \\ |I| \geq 2}} \right)$$

where $A_0, ..., A_n \in \mathrm{Ob}(\mathcal{C})$ and for each subset $I = \{i_0 < ... < i_m\} \subseteq [n]$ with $m \geq 1$, $f_I \in \mathcal{C}_{m-1}(A_{i_0}, A_{i_m})$ such that

$$\partial(f_I) = \sum_{j=1}^{m-1} \left( (-1)^{j-1} f_{I \setminus \{i_j\}} + (-1)^{m(j-1)+1} f_{\{i_j < ... < i_m\}} \circ f_{\{i_0 < ... < i_j\}} \right)$$

or, when employing the reverse composition law of $\mathcal{C}_\bullet$ (see Remark 4.2.42):

$$\partial(f_I) = \sum_{j=1}^{m-1} (-1)^{j-1} \left(f_{I\setminus\{i_j\}} - \tilde{m}(f_{\{i_0<\ldots<i_j\}} \otimes f_{\{i_j<\ldots<i_m\}})\right)$$

For any $h : [m] \to [n]$ in $\mathbf{\Delta}_f$, the map $N^{dg}(\mathcal{C})_n \to N^{dg}(\mathcal{C})_m$ is given by

$$\left((A_i)_{i=0}^n, (f_I)_{\substack{I\subseteq[n]\\|I|\geq 2}}\right) \mapsto \left((A_{h(i)})_{i=0}^m, (h^* f_J)_{\substack{J\subseteq[m]\\|J|\geq 2}}\right)$$

where

$$h^* f_J = \begin{cases} f_{h(J)} & \text{if } h \text{ is injective on } J \\ \mathrm{id}_{A_i} & \text{if } J = \{j_0 < j_1\} \text{ with } h(j_0) = i = h(j_1) \\ 0 & \text{otherwise} \end{cases}$$

**Example 4.2.43.** Given a small dg-category $\mathcal{C}_\bullet$, let us decribe the dg-nerve $N^{dg}(\mathcal{C})$ in low dimensions.

- The vertices of $N^{dg}(\mathcal{C})$ are given by the object set $\mathrm{Ob}(\mathcal{C})$.
- The edges of $N^{dg}(\mathcal{C})$ are given by the 0-cycles of the chain complex $\mathcal{C}_\bullet(A_0, A_1)$ for some $A_0, A_1 \in \mathrm{Ob}(\mathcal{C})$, i.e. $f_{01} \in \mathcal{C}_0(A_0, A_1)$ such that $\partial(f) = 0$.
- A 2-simplex of $N^{dg}(\mathcal{C})$ is given by a (not necessarily commutative) diagram of 0-cycles:

$$\begin{tikzcd} & A_1 \arrow[dr, "f_{12}"] \arrow[d, phantom, "\Uparrow f_{012}"] & \\ A_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & {} & A_2 \end{tikzcd}$$

  with $f_{012} \in \mathcal{C}_1(A_0, A_2)$ such that $\partial(f_{012}) = f_{02} - f_{12} \circ f_{01}$. So $f_{012}$ is a homotopy in $\mathcal{C}_\bullet(A_0, A_2)$ from $f_{02}$ to $f_{12} \circ f_{01}$.

### The linear dg-nerve

**Proposition 4.2.44.** *The adjunction $N_\bullet^+ : S^+\,\mathrm{Mod}(k) \leftrightarrows \mathrm{Ch}(k) : \Gamma^+$ of Proposition 4.2.10 induces an adjunction*

$$\begin{tikzcd} k\,\mathrm{Cat}_{\Delta_+} \arrow[r, shift left, "N_\bullet^+"] \arrow[r, phantom, "\perp"] & k\,\mathrm{Cat}_{dg} \arrow[l, shift left, "\Gamma^+"] \end{tikzcd}$$

*Moreover, the restriction*

$$\begin{tikzcd} k\,\mathrm{Cat}_{\Delta_+} \arrow[r, shift left, "N_\bullet^+"] \arrow[r, phantom, "\sim"] & k\,\mathrm{Cat}_{dg,\geq 0} \arrow[l, shift left, "\Gamma^+"] \end{tikzcd}$$

*is an equivalence of categories.*

*Proof.* This immediately follows from Theorem 4.2.14. □

**Corollary 4.2.45.** *There is an equivalence of categories*

$$S_{\otimes}^{Frob} \operatorname{Mod}(k) \simeq k \operatorname{Cat}_{dg,\geq 0}$$

*Proof.* Combine Proposition 4.2.44 and Theorem 4.2.17. □

**Definition 4.2.46.** We define the *(k-)linear differential graded (dg) nerve* as the composite

$$N_k^{dg} : k \operatorname{Cat}_{dg} \xrightarrow{\Gamma^+} k \operatorname{Cat}_{\Delta_+} \xrightarrow{\mathcal{T}} S_{\otimes}^{Frob} \operatorname{Mod}(k) \to S_{\otimes} \operatorname{Mod}(k)$$

where $\Gamma^+$ is the right-adjoint from Proposition 4.2.44, $\mathcal{T}$ is the equivalence from Proposition 4.2.28 and the third arrow represents the forgetful functor.

**Notation 4.2.47.** Given $n > 0$, a subset $I \subseteq \{0 < n\}^c = \{1, ..., n-1\}$ and $k \in \{1, ..., n-1\} \setminus I$, we write

$$I_{<k} = \{i \mid i < k\} \quad \text{and} \quad I_{>k} = \{i - k \mid i \in I, i > k\}$$

and consider $I_{<k}$ and $I_{>k}$ as subsets of $\{0 < k\}^c = \{1, ..., k-1\}$ and $\{0 < n-k\}^c = \{1, ..., n-k-1\}$ respectively. Note that $I_{<k} \sqcup I_{>k} \simeq I$ as linearly ordered sets (see Construction 4.2.19).

*Remark* 4.2.48. Given a small dg-category $\mathcal{C}_\bullet$, let us make the templicial object $N_k^{dg}(\mathcal{C})$ a little more explicit. The vertex set of $N_\bullet^{dg}(\mathcal{C})$ is simply $S = \operatorname{Ob}(\mathcal{C})$.

Take $n \geq 0$. From Construction 4.2.23 we have (also see Notation 4.2.21):

$$N_k^{dg}(\mathcal{C})_n = \bigoplus_{T \in \mathcal{P}_n} \mathcal{T}(\Gamma^+(\mathcal{C}))_T = \bigoplus_{T \in \mathcal{P}_n} \Gamma^+(\mathcal{C})_{T_1^c} \otimes_S ... \otimes_S \Gamma^+(\mathcal{C})_{T_k^c} \in k \operatorname{Quiv}_S$$

For all $A, B \in \operatorname{Ob}(\mathcal{C})$, $\Gamma^+(\mathcal{C})_{T_i^c}(A, B) = \Gamma^+(\mathcal{C}_\bullet(A, B))_{T_i^c}$ is the $k$-module

$$\left\{ (a_I)_I \in \bigoplus_{I \subseteq T_i^c} \mathcal{C}_{|I|}(A, B) \;\middle|\; \partial(a_I) = \sum_{j=1}^{p} (-1)^{j-1} a_{I \setminus \{i_j\}} \right\}$$

where we've written $I = \{i_1 < ... < i_p\} \subseteq T_i^c$.

In view of Proposition 4.2.25, the counit of $N_k^{dg}(\mathcal{C})$ is just the identity $N_k^{dg}(\mathcal{C})_0 = k_S$, the monoidal unit of $k \operatorname{Quiv}_S$. The comultiplication maps $\mu_{p,q}$ and Frobenius structure maps $Z^{p,q}$ are defined by the canonical projections and coprojections respectively:

$$\mu_{p,q} : N_k^{dg}(\mathcal{C})_{p+q} = \bigoplus_{T \in \mathcal{P}_{p+q}} \mathcal{T}(\Gamma^+(\mathcal{C}))_T \twoheadrightarrow \bigoplus_{\substack{T \in \mathcal{P}_{p+q} \\ p \in T}} \mathcal{T}(\Gamma^+(\mathcal{C}))_T \simeq N_k^{dg}(\mathcal{C})_p \otimes_S N_k^{dg}(\mathcal{C})_q$$

$$Z^{p,q} : N_k^{dg}(\mathcal{C})_p \otimes_S N_k^{dg}(\mathcal{C})_q \simeq \bigoplus_{\substack{T \in \mathcal{P}_{p+q} \\ p \in T}} \mathcal{T}(\Gamma^+(\mathcal{C}))_T \hookrightarrow \bigoplus_{T \in \mathcal{P}_{p+q}} \mathcal{T}(\Gamma^+(\mathcal{C}))_T = N_k^{dg}(\mathcal{C})_{p+q}$$

Finally, the inner face maps and degeneracy maps of $N_k^{dg}(\mathcal{C})$ are completely determined by projection onto the component $\Gamma^+(\mathcal{C})_{\{0<p\}^c}$ of $N_k^{dg}(\mathcal{C})_p$ corresponding to the necklace $T = \{0 < p\} \in \mathcal{P}_p$. More precisely:

- For all $0 < j < n$, the composite of $d_j : N_k^{dg}(\mathcal{C})_n \to N_k^{dg}(\mathcal{C})_{n-1}$ with the canonical projection $N_k^{dg}(\mathcal{C})_{n-1} \twoheadrightarrow \Gamma^+(\mathcal{C})_{\{0<n-1\}^c}$ is equal to the composite

$$N_k^{dg}(\mathcal{C})_n \twoheadrightarrow \Gamma^+(\mathcal{C})_{\{0<n\}^c} \oplus (\Gamma^+(\mathcal{C})_{\{0<j\}^c} \otimes_S \Gamma^+(\mathcal{C})_{\{j<n\}^c}) \xrightarrow{(d_j^+, m^+)} \Gamma^+(\mathcal{C})_{\{0<n-1\}^c}$$

If $\tilde{m}$ is the reverse composition law of $\mathcal{C}_\bullet$, then $d_j^+$ and $m^+$ are defined by:

$$d_j^+ \left((a_I)_{I \subseteq \{0<n\}^c}\right) = (a_{\delta_j(J)})_{J \subseteq \{0<n-1\}^c}$$
$$m^+ \left(((a_I)_{I \subseteq \{0<j\}^c} \otimes (b_J)_{J \subseteq \{0<n-j\}^c}\right) = \left(\tilde{m}(a_{\delta_j(K)_{<j}} \otimes b_{\delta_j(K)_{>j}})\right)_{K \subseteq \{0<n-1\}^c}$$

where we used Notation 4.2.47.

- For all $0 \leq i \leq n$, the composite of $s_i : N_k^{dg}(\mathcal{C})_n \to N_k^{dg}(\mathcal{C})_{n+1}$ with the canonical projection $N_k^{dg}(\mathcal{C})_{n+1} \twoheadrightarrow \Gamma^+(\mathcal{C})_{\{0<n+1\}^c}$ is equal to

$$\begin{cases} N_k^{dg}(\mathcal{C})_n \twoheadrightarrow \Gamma^+(\mathcal{C})_{\{0<n\}^c} \xrightarrow{s_i^+} \Gamma^+(\mathcal{C})_{\{0<n+1\}^c} & \text{if } 0 < i < n \\ N_k^{dg}(\mathcal{C})_0 = I_{\mathrm{Ob}(\mathcal{C})} \xrightarrow{u} Z_0(\mathcal{C}) = \Gamma^+(\mathcal{C})_\emptyset & \text{if } n = i = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $u : k_S \to \mathcal{C}_0$ represents the identities in $\mathcal{C}_\bullet$, $Z_0$ denotes the functor taking 0-cycles and

$$s_i^+ \left((a_I)_{I \subseteq \{0<n\}^c}\right) = (b_J)_{J \subseteq \{0<n+1\}^c}$$

with $b_J = a_{\sigma_i(J)}$ if $\{i, i+1\} \nsubseteq J$ and $b_J = 0$ if $\{i, i+1\} \subseteq J$.

**Example 4.2.49.** Given small a dg-category $\mathcal{C}_\bullet$, let us describe the templicial object $N_k^{dg}(\mathcal{C})$ in low dimensions. Note the analogy with Example 4.2.43.

- The vertices of $N_k^{dg}(\mathcal{C})$ are given by the object set $\mathrm{Ob}(\mathcal{C})$.
- Take objects $A, B \in \mathrm{Ob}(\mathcal{C})$. Then

$$N_k^{hc}(\mathcal{C})_1(A, B) = \Gamma^+(\mathcal{C}_\bullet(A, B))_{-1} = Z_0(\mathcal{C}_\bullet(A, B))$$

is the submodule of $\mathcal{C}_0(A, B)$ of 0-cycles.

- In two dimensions, we have

$$\begin{aligned} N_k^{dg}(\mathcal{C})_2(A, B) &= \Gamma^+(\mathcal{C}_\bullet(A, B))_0 \oplus \bigoplus_{C \in \mathrm{Ob}(\mathcal{C})} \Gamma^+(\mathcal{C}_\bullet(A, C))_{-1} \otimes \Gamma^+(\mathcal{C}_\bullet(C, B))_{-1} \\ &\simeq \mathcal{C}_1(A, B) \oplus \bigoplus_{C \in \mathrm{Ob}(\mathcal{C})} Z_0(\mathcal{C}_\bullet(A, C)) \otimes Z_0(\mathcal{C}_\bullet(C, B)) \end{aligned}$$

The comultiplication map $\mu_{1,1} : N_k^{dg}(\mathcal{C})_2 \to N_k^{dg}(\mathcal{C})_1 \otimes N_k^{dg}(\mathcal{C})_1$ is given by projection onto the second term in the expression above. On the other hand, the face map $d_1 : N_k^{dg}(\mathcal{C})_2 \to N_k^{dg}(\mathcal{C})_1$ is defined as follows. Given a pair $(h, \alpha)$ with $h \in \mathcal{C}_1(A, B)$ and $\alpha$ a tensor belonging to the second term in the expression above, we have

$$d_1(h, \alpha) = \partial(h) + \tilde{m}(\alpha)$$

where $\partial : \mathcal{C}_1(A, B) \to Z_0(\mathcal{C}_\bullet(A, B))$ is the differential. Setting $f = d_1(h, \alpha)$, we thus find that $h$ describes a homotopy in $\mathcal{C}_\bullet(A, B)$ between $f$ and $\tilde{m}(\alpha)$.

**Templicial maps into the linear dg-nerve**

The description of the simplices of the dg-nerve in Example 4.2.43 can be generalized to the following remark from [Lur18].

*Remark* 4.2.50 ([Lur18], Tag 00PV). Let $\mathcal{C}_\bullet$ be a small dg-category over $k$ and let $K$ be a simplicial set. A map of simplicial sets $f : K \to N^{dg}(\mathcal{C})$ is equivalent to the following data:

- A map of sets $f_0 : S \to \mathrm{Ob}(\mathcal{C})$.
- For all $a, b \in K_0$ and $n > 0$, a map

$$f_n : K_n(a,b) \to U\left(\mathcal{C}_{n-1}(f_0(a), f_0(b))\right)$$

Moreover, this data must satisfy the following conditions:

(a) For all $a, b \in K_0$, $0 \leq i \leq n$ and $\sigma \in K_n(a,b)$,

$$f_{n+1}(s_i^K(\sigma)) = \begin{cases} \mathrm{id}_{f(a)} & \text{if } n = 0, a = b \\ 0 & \text{otherwise} \end{cases}$$

(b) For all $a, b \in K_0$, $n > 0$ and $\sigma \in K_n(a,b)$,

$$\partial(f_n(\sigma)) = \sum_{j=1}^{n-1} (-1)^{j-1} \left( f_{n-1}(d_j^K(\sigma)) - \tilde{m}(f_j(d_{j+1}^K ... d_n^K(\sigma)) \otimes f_{n-j}(d_0^K ... d_0^K(\sigma))) \right)$$

We will now show the templicial analogue of Remark 4.2.50 (see Proposition 4.2.53). In the following lemma we again make use of Notation 4.2.47.

**Lemma 4.2.51.** *Let $\mathcal{C}_\bullet$ be a dg-category over $k$ with object set $S$ and denote its reverse composition law and identities by $\tilde{m}$ and $u$ respectively. Let $(X, S)$ be a templicial $k$-module with comultiplication $\mu$. Define*

- *the set $\mathcal{S}_1$ of all collections of morphisms in $k\,\mathrm{Quiv}_S$:*

$$(\beta_n : X_n \to \mathcal{C}_{n-1})_{n>0}$$

- *the set $\mathcal{S}_2$ of all collections of morphisms in $k\,\mathrm{Quiv}_S$:*

$$\left(\alpha_I : X_n \to \mathcal{C}_{|I|}\right)_{I \subseteq \{1,...,n-1\}, n>0}$$

  *such that for all $n > 0$, $I \subseteq \{1, ..., n-1\}$ and $j \in \{1, ..., n-1\} \setminus I$, we have:*

$$\alpha_I = \alpha_{\delta_j^{-1}(I)} d_j^X - \tilde{m}(\alpha_{I_{<j}} \otimes_S \alpha_{I_{>j}}) \mu_{j,n-j}^X \tag{4.6}$$

  *where $\delta_j$ is the coface map $[n-1] \to [n]$.*

*Then the map*

$$\mathcal{S}_2 \to \mathcal{S}_1 : (\alpha_I)_{I \subseteq \{1,...,n-1\}, n>0} \mapsto (\alpha_{\{1,...,n-1\}})_{n>0}$$

*is a bijection.*

*Proof.* It follows from equation (4.6) that a collection $(\alpha_I)_I$ in $\mathcal{S}_2$ is completely determined by the morphisms $\alpha_{\{1,...,n-1\}}$ for $n > 0$. Thus the above map is injective. We now show the surjectivity.

Let $(\beta_n : X_n \to \mathcal{C}_{n-1})_{n>0}$ be a collection of quiver morphisms. For every $n > 0$, set $\alpha_{\{1,...,n-1\}} = \beta_n$. Given $I \subsetneq \{1, ..., n-1\}$, choose some $j \in \{1, ..., n-1\} \setminus I$ and define $\alpha_I : X_n \to \mathcal{C}_{\ell(I)-1}$ by equation (4.6), inductively on $n$. Note that this doesn't depend on the choice of $j$. Indeed, if $j < k$ in $\{1, ..., n-1\} \setminus I$, then we have by induction that:

$$\begin{aligned}
&\alpha_{\delta_j^{-1}(I)} d_j^X - \tilde{m}(\alpha_{I_{<j}} \otimes_S \alpha_{I_{>j}})\mu^X_{j,n-j} \\
&= \alpha_{\delta_{k-1}^{-1}\delta_j^{-1}(I)} d^X_{k-1} d^X_j - \tilde{m}(\alpha_{\delta_j^{-1}(I)_{<k-1}} \otimes_S \alpha_{\delta_j^{-1}(I)_{>k-1}})\mu^X_{k-1,n-k} d^X_j \\
&- \tilde{m}\left(\alpha_{I_{<j}} \otimes_S (\alpha_{\delta_{k-j}^{-1}(I_{>j})} d^X_{k-j} - \tilde{m}\left(\alpha_{(I_{>j})_{<k-j}} \otimes_S \alpha_{(I_{>j})_{>k-j}}\right)\mu^X_{k-j,n-k})\right)\mu^X_{j,n-j} \\
&= \alpha_{(\delta_k\delta_j)^{-1}(I)} d^X_j d^X_k - \tilde{m}(\alpha_{\delta_j^{-1}(I_{<k})} \otimes_S \alpha_{I_{>k}})\mu^X_{k-1,n-k} d^X_j \\
&- \tilde{m}\left(\alpha_{I_{<j}} \otimes_S \alpha_{\delta_{k-j}^{-1}(I_{>j})}\right)\mu^X_{j,n-j-1} d^X_k + \tilde{m}^{(3)}\left(\alpha_{I_{<j}} \otimes_S \alpha_{(I_{<k})_{>j}} \otimes_S \alpha_{I_{>k}}\right)\mu^X_{j,k-j,n-k}
\end{aligned}$$

which is can be seen to equal $\alpha_{\delta_k^{-1}(I)} d^X_k - \tilde{m}(\alpha_{I_{<k}} \otimes_S \alpha_{I_{>k}})\mu^X_{k,n-k}$ by a similar calculation. Hence, $(\alpha_I)_I$ belongs to the set $\mathcal{S}_2$. $\square$

**Lemma 4.2.52.** *Let $\mathcal{C}_\bullet$ be a dg-category over $k$ with object set $S$ and denote its reverse composition law and identities by $\tilde{m}$ and $u$ respectively. Let $(X, S)$ be a templicial $k$-module with comultiplication $\mu$. Consider the bijection $\mathcal{S}_2 \xrightarrow{\sim} \mathcal{S}_1$ of Lemma 4.2.51. Then for all $(\alpha_I)_I \in \mathcal{S}_2$ with $(\beta_n)_{n>0} = (\alpha_{\{1,...,n-1\}})_{n>0}$:*

1. *The following statements are equivalent:*

   *(i) for all $n > 0$ and $I = \{i_1 < ... < i_m\} \subseteq \{1, ..., n-1\}$,*

$$\partial\alpha_I = \sum_{j=1}^{m} (-1)^{j-1} \alpha_{I\setminus\{i_j\}}$$

   *(ii) for all $n > 0$,*

$$\partial\beta_n = \sum_{j=1}^{n-1} (-1)^{j-1} \left(\beta_{n-1} d^X_j - \tilde{m}(\beta_j \otimes_S \beta_{n-j})\mu^X_{j,n-j}\right)$$

2. *The following statements are equivalent:*

   *(i) for all $0 \leq i \leq n$ and $I \subseteq \{1, ..., n\}$,*

$$\alpha_I s^X_i = \begin{cases} u & \text{if } n = 0 \\ \alpha_{\sigma_i(I)} & \text{if } 0 < i < n, \{i, i+1\} \not\subseteq I \\ 0 & \text{otherwise} \end{cases}$$

*(ii) for all* $0 \leq i \leq n$,

$$\beta_{n+1} s_i^X = \begin{cases} u & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

*Proof.* Fix an element $(\alpha_I)_I$ of $\mathcal{S}_2$ and let $(\beta_n)_{n>0} = (\alpha_{\{1,...,n-1\}})_{n>0}$. Let us first prove 1. It is immediate from the definitions that $(i)$ implies $(ii)$. Conversely, assume that $(ii)$ holds. Let $I = \{i_1 < ... < i_m\} \subseteq \{1, ..., n-1\}$ with $n > 0$. We show by induction on $n$ that $\partial\alpha_I = \sum_{j=1}^{m} (-1)^{j-1} \alpha_{I \setminus \{i_j\}}$. If $I = \{1, ..., n-1\}$, this follows directly from $(ii)$. Note that this also covers the case $n = 1$. Otherwise, choose $k \in \{1, ..., n-1\} \setminus I$ and let $p = |I_{<k}|$. Then:

$$\begin{aligned}
\partial\alpha_I &= \partial\alpha_{\delta_k^{-1}(I)} d_k^X - \partial\tilde{m}(\alpha_{I_{<k}} \otimes_S \alpha_{I_{>k}}) \mu_{k,n-k}^X \\
&= \sum_{j=1}^{p} (-1)^{j-1} \alpha_{\delta_k^{-1}(I)\setminus\{i_j\}} d_k^X + \sum_{j=p+1}^{m} (-1)^{j-1} \alpha_{\delta_k^{-1}(I)\setminus\{i_j - 1\}} d_k^X \\
&\quad - \tilde{m}(\partial\alpha_{I_{<k}} \otimes_S \alpha_{I_{>k}}) \mu_{k,n-k}^X - (-1)^p \tilde{m}(\alpha_{I_{<k}} \otimes_S \partial\alpha_{I_{>k}}) \mu_{k,n-k}^X \\
&= \sum_{j=1}^{p} (-1)^{j-1} \left( \alpha_{\delta_k^{-1}(I\setminus\{i_j\})} d_j^X - \tilde{m}(\alpha_{I_{<k}\setminus\{i_j\}} \otimes_S \alpha_{I_{>k}}) \mu_{k,n-k}^X \right) \\
&\quad + \sum_{j=p+1}^{m} (-1)^{j-1} \left( \alpha_{\delta_k^{-1}(I\setminus\{i_j\})} d_j^X - \tilde{m}(\alpha_{I_{<k}} \otimes_S \alpha_{I_{>k}\setminus\{i_j - k\}}) \mu_{k,n-k}^X \right) \\
&= \sum_{j=1}^{m} (-1)^{j-1} \alpha_{I\setminus\{i_j\}}
\end{aligned}$$

It remains to show 2. As before, $(i)$ trivially implies $(ii)$. Assume now that $(ii)$ holds and take $0 \leq i \leq n$ and $I \subseteq \{1, ..., n\}$. When $I = \{1, ..., n\}$, $(i)$ follows directly from $(ii)$. In particular, this covers the case $n = 0$. Otherwise, choose $j \in \{1, ..., n\} \setminus I$. Without loss of generality, we may assume that $j \leq i$. We proceed by induction on $n > 0$ using equation (4.6). Consider the following four cases:

- If $j < i < n$, then the statements $\{i, i+1\} \subseteq I$, $\{i-1, i\} \subseteq \delta_j^{-1}(I)$ and $\{i, i+1\} \subseteq I_{>j}$ are all equivalent and thus

$$\begin{aligned}
\alpha_I s_i^X &= \alpha_{\delta_j^{-1}(I)} s_{i-1}^X d_j^X - \tilde{m}(\alpha_{I_{<j}} \otimes_S \alpha_{I_{>j}} s_{i-j}^X) \mu_{j,n-j}^X \\
&= \begin{cases} \alpha_{\sigma_{i-1}(\delta_j^{-1}(I))} d_j^X - \tilde{m}(\alpha_{I_{<j}} \otimes_S \alpha_{\sigma_{i-j}(I_{>j})}) \mu_{j,n-j}^X = \alpha_{\sigma_i(I)} & \text{if } \{i, i+1\} \not\subseteq I \\ 0 & \text{if } \{i, i+1\} \subseteq I \end{cases}
\end{aligned}$$

- If $j = i < n$, then $\{i, i+1\} \not\subseteq I$ and thus

$$\alpha_I s_i^X = \alpha_{\delta_i^{-1}(I)} - \tilde{m}(\alpha_{I_{<i}} \otimes_S \alpha_{I_{>i}} s_0^X) \mu_{i,n-i}^X = \alpha_{\delta_i^{-1}(I)} = \alpha_{\sigma_i(I)}$$

- If $j < i = n$,

$$\alpha_I s_n^X = \alpha_{\delta_j^{-1}(I)} s_{n-1}^X d_j^X - \tilde{m}(\alpha_{I_{<j}} \otimes_S \alpha_{I_{>j}} s_{n-j}^X) \mu_{j,n-j}^X = 0$$

- If $j = i = n$,
$$\alpha_I s_n^X = \alpha_{\delta_n^{-1}(I)} - \tilde{m}(\alpha_{I_{<n}} \otimes_S \alpha_\emptyset s_0^X)\mu_{n,0}^X = \alpha_{\delta_n^{-1}(I)} - \alpha_{I_{<n}} = 0$$

Hence we have shown $(i)$. □

**Proposition 4.2.53.** *Let $\mathcal{C}_\bullet$ be a small dg-category over $k$ and $(X, S)$ a templicial $k$-module. A templicial map $(\alpha, f) : X \to N_k^{dg}(\mathcal{C})$ is equivalent to the following data:*

- *A map of sets $f : S \to \mathrm{Ob}(\mathcal{C})$.*
- *For all $n > 0$, a quiver map*
$$\beta_n : X_n \to f^*\mathcal{C}_{n-1}$$

*satisfying the following properties:*

*(a) For all $0 \leq i \leq n$,*
$$\beta_{n+1} s_i^X = \begin{cases} u & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases} \tag{4.7}$$
*where $u$ denotes the identities of the dg-category $f^*\mathcal{C}_\bullet$.*

*(b) For all $n > 0$,*
$$\partial\beta_n = \sum_{j=1}^{n-1} (-1)^{j-1} \left(\beta_{n-1} d_j^X - \tilde{m}(\beta_j \otimes_S \beta_{n-j})\mu_{j,n-j}^X\right) \tag{4.8}$$
*where $\tilde{m}$ and $\partial$ are respectively the reverse composition law and the differential of the dg-category $f^*\mathcal{C}_\bullet$ (induced by the lax structure of $f^*$, see Lemma 1.1.18).*

*Moreover, for all $n > 0$, $\beta_n$ is adjoint to the composite*
$$f_! X_n \xrightarrow{\alpha_n} N_k^{dg}(\mathcal{C})_n \to \Gamma^+(\mathcal{C})_{\{0<n\}^c} \xrightarrow{\pi_{\{0<n\}^c}} \mathcal{C}_{n-1}$$

*Proof.* Note that $f^*\mathcal{C}_\bullet$ is a dg-category with set of objects $S$, we may replace $\mathcal{C}_\bullet$ by $f^*\mathcal{C}_\bullet$ and assume that $f$ is the identity on $S$.

We can translate the data of a templicial map $(\alpha, \mathrm{id}_S) : X \to N_k^{dg}(\mathcal{C})$ as follows. Let $(\alpha_n : X_n \to N_k^{dg}(\mathcal{C})_n)_{n>0}$ be a collection of morphisms in $k\,\mathrm{Quiv}_S$. For any $n > 0$ and $I \subseteq \{0 < n\}^c$, consider the following composite:
$$\alpha_I : X_n \xrightarrow{\alpha_n} N_k^{dg}(\mathcal{C})_n \to \Gamma^+(\mathcal{C})_{\{0<n\}^c} \xrightarrow{\pi_I} \mathcal{C}_{|I|}$$
It follows from the templicial structure of the linear dg-nerve (Remark 4.2.48) that the assignment $(\alpha_n)_{n>0} \mapsto (\alpha_I)_{I\subseteq\{0<n\}^c, n>0}$ induces a bijection:
$$\left\{(\alpha_n : X_n \to N_k^{dg}(\mathcal{C})_n)_{n>0} \,\middle|\, \forall 0 < j < n : \mu_{j,n-j}^{N_k^{dg}(\mathcal{C})}\alpha_n = (\alpha_j \otimes_S \alpha_{n-j})\mu_{j,n-j}^X\right\}$$
$$\simeq$$
$$\left\{(\alpha_I : X_n \to \mathcal{C}_{|I|})_{\substack{n>0 \\ I\subseteq\{0<n\}^c}} \,\middle|\, \forall n, \forall I : \partial\alpha_I = \sum_{j=1}^{m} (-1)^{j-1}\alpha_{I\setminus\{i_j\}}\right\}$$

where we have denoted $I = \{i_1 < ... < i_m\} \subseteq \{1, ..., n-1\}$.

Further, it follows that the morphisms $(\alpha_n)_{n>0}$ are compatible with the degeneracy maps if and only if for all $0 \leq i \leq n$ and $I \subseteq \{1, ..., n\}$,

$$\alpha_I s_i^X = \begin{cases} u & \text{if } n = 0 \\ \alpha_{\sigma_i(I)} & \text{if } 0 < i < n, \{i, i+1\} \not\subseteq I \\ 0 & \text{otherwise} \end{cases}$$

and $(\alpha_n)_{n>0}$ are compatible with the face maps if and only if for all $0 < j < n$ and $I \subseteq \{1, ..., n-2\}$:

$$\alpha_I d_j^X = \alpha_{\delta_j(I)} + \tilde{m}(\alpha_{\delta_j(I)_{<j}} \otimes_S \alpha_{\delta_j(I)_{>j-1}}) \mu_{j,n-j}^X$$

Hence the result follows from Lemma 4.2.52. □

**Corollary 4.2.54.** *There is a natural isomorphism $\tilde{U} \circ N_k^{dg} \simeq N^{dg}$.*

*Proof.* Let $\mathcal{C}_\bullet$ be a small dg-category over $k$ and $n \geq 0$. By Proposition 4.2.53, a templicial map $\tilde{F}(\Delta^n) \to N_k^{dg}(\mathcal{C})$ is equivalent to a map of sets $f : [n] \to \mathrm{Ob}(\mathcal{C})$ with a collection of quiver morphisms $\beta_m : \tilde{F}(\Delta^n)_m \to f^*\mathcal{C}_{m-1}$ for $m > 0$ satisfying properties (4.7) and (4.8). The map $f$ is equivalent to a choice of objects $A_0, ..., A_n \in \mathrm{Ob}(\mathcal{C})$. Further, for $i, j \in [n]$ we have

$$\tilde{F}(\Delta^n)_m(i,j) = F(\{h \in \mathbf{\Delta}([m],[n]) \mid h(0) = i, h(m) = j\})$$

and thus we may represent $\beta_m$ by a collection of elements $\beta_{i_0,...,i_m} \in \mathcal{C}_{m-1}(A_{i_0}, A_{i_m})$ for $0 \leq i_0 \leq ... \leq i_m \leq n$. Then by property (4.7), $\beta_{i,i} = \mathrm{id}_i$ and $\beta_{i_0,...,i_m} = 0$ whenever $m \geq 2$ and $i_p = i_{p+1}$ for some $p \in [m-1]$. Hence, $\beta_m$ is completely determined by the elements $\beta_{\{i_0<...<i_m\}}$ with $i_0 < ... < i_m$. Moreover, property (4.8) translates to

$$\partial(\beta_{\{i_0<...<i_m\}}) = \sum_{j=1}^{m-1} (-1)^{j-1} \left( \beta_{I \setminus \{i_j\}} - \tilde{m}(\beta_{\{i_0<...<i_j\}} \otimes \beta_{\{i_j<...<i_m\}}) \right)$$

Hence, the pair $((a_i)_i, (\beta_I)_I)$ is precisely an $n$-simplex of $N^{dg}(\mathcal{C})_n$. We have thus obtained a bijection between $\tilde{U}(N_k^{dg}(\mathcal{C}))_n$ and $N^{dg}(\mathcal{C})_n$.

It now follows easily from the definitions that this bijection is natural in $n$ and $\mathcal{C}_\bullet$. □

### 4.2.4 Quasi-categories and Frobenius structures

Our main result in this subsection is Corollary 4.2.65 which states that the linear dg-nerve of any dg-category is a quasi-category in $\mathrm{Mod}(k)$. This will be a consequence of the more general Theorem 4.2.62 that every templicial $k$-module with a naF-structure is already a quasi-category in $\mathrm{Mod}(k)$. As a byproduct, we find that this applies to $\tilde{F}(\mathcal{C})$ for any ordinary quasi-category $\mathcal{C}$ as well (Corollary 4.2.63).

We start by introducing the *wings* $W^n$ of a simplex $\Delta^n$ for $n \geq 2$, which are defined as the union of its two outer faces. Given a necklace $(T, n)$ and $0 < j < n$, the unique inert necklace map $T \hookrightarrow \{0 < n\}$ can thus be identified with a composite of inclusions of bipointed simplicial sets:

$$T \subseteq W^n \subseteq \Lambda^n_j \subseteq \Delta^n$$

By design, a naF-structure on a templicial object $X$ allows to fill up any necklace $T$ in $X$ to a simplex via the morphism $Z^T : X_T \to X_n$. For general monoidal categories $\mathcal{V}$, this is all we can say. For $\mathcal{V} = \mathrm{Mod}(k)$ however, we can use an alternating sum of the maps $Z^T$ to show that also all wings $W^n$ in $X$ can be filled to a simplex (see Proposition 4.2.59). Finally, from this also all inner horns in $X$ can be filled by appropriately adding and subtracting degenerate simplices (Proposition 4.2.60). This last argument employs the same technique as the one used to show that every simplicial group is a Kan complex (see for instance [Moo58]).

**Definition 4.2.55.** Let $n \geq 2$. We write $W^n$ for the simplicial subset of $\Delta^n$ defined by

$$W^n([m]) = \{f : [m] \to [n] \mid f(m) \leq n-1 \text{ or } f(0) \geq 1\}$$

for all $m \geq 0$. We call $W^n$ the *wings* of $\Delta^n$. It consists of the 0th and $n$th face of $\Delta^n$.

We say a functor $Y_\bullet : \mathcal{N}ec^{op} \to \mathcal{V}$ *lifts wings* if for all $n \geq 2$, any lifting problem in $\mathcal{V}^{\mathcal{N}ec^{op}}$:

$$\begin{tikzcd} \tilde{F}(W^n)_\bullet(0,n) \arrow[r] \arrow[d] & Y_\bullet \\ \tilde{F}(\Delta^n)_\bullet(0,n) \arrow[ur, dashed] & \end{tikzcd}$$

where the vertical morphism is induced by the inclusion $W^n \subseteq \Delta^n$, has a solution.

**Proposition 4.2.56.** *For all $n \geq 2$, we have*

$$W^n_\bullet(0,n) = \bigcup_{k=1}^{n-1} (\Delta^k \vee \Delta^{n-k})_\bullet(0,n)$$

*as a subfunctor of $\Delta^n_\bullet(0,n)$. In particular, we have for all necklaces $(T,p)$:*

$$W^n_T(0,n) = \{f : T \to \Delta^n \text{ in } \mathcal{N}ec \mid \{0 < n\} \subsetneq f(T)\}$$

*Proof.* This is shown similarly to Proposition 2.2.18 and Corollary 2.2.20. ☐

**Lemma 4.2.57.** *For all $n \geq 2$, the inclusion $W^n_\bullet(0,n) \hookrightarrow \Delta^n_\bullet(0,n)$ belongs to $\overline{\mathrm{Horn}}$.*

*Proof.* Given $0 < k < n$, let us denote by $A^n_k$ the simplicial subset of $\Delta^n$ given by the union of the faces $0, ..., k-1$ and $n$. Then $A^n_k$ contains all vertices of $\Delta^n$ and

$$(A^n_k)_\bullet(0,n) = W^n_\bullet(0,n) \cup \bigcup_{j=1}^{k-1} \delta_j(\Delta^{n-1})_\bullet(0,n)$$

by Proposition 4.2.56. We will show by double induction on $n \geq 2$ and $0 < k < n$ that the inclusion

$$(A^n_k)_\bullet(0,n) \hookrightarrow \Delta^n_\bullet(0,n) \tag{4.9}$$

belongs to $\overline{\text{Horn}}$. The result then follows by choosing $k = 1$.

If $k = n - 1$, then (4.9) coincides with the horn inclusion $(\Lambda^n_{n-1})_\bullet(0, n) \hookrightarrow \Delta^n_\bullet(0, n)$ by Proposition 2.2.18. Note that this covers the entire case $n = 2$.

Assume further that $k < n-1$ and let $(T, p)$ be a necklace. Recall that for a map $h : T \to \Delta^n$ in $\mathrm{SSet}_{*,*}$ and $0 < i < n$, $h$ factors through $(\Delta^i \vee \Delta^{n-i})_\bullet(0, n)$ if and only if $i \in h(T)$, and $h$ factors through $\delta_i(\Delta^{n-1})$ if and only if $h([p]) \subseteq [n] \setminus \{i\}$. Now take a map $g : T \to \Delta^{n-1}$ in $\mathrm{SSet}_{*,*}$. It follows that $\delta_k g : T \to \Delta^n$ factors through $(A^n_k)_{0,n} \cap \delta_k(\Delta^{n-1})_{0,n}$ if and only if $g$ factors through $(A^{n-1}_k)_{0,n-1}$. Hence, we obtain a pushout diagram in $\mathrm{Set}^{\mathcal{N}ec^{op}}$:

$$\begin{tikzcd}
(A^{n-1}_k)_\bullet(0, n-1) \arrow[r, "\delta_k"] \arrow[d, hook] & (A^n_k)_\bullet(0, n) \arrow[d, hook] \\
\Delta^{n-1}_\bullet(0, n-1) \arrow[r, "\delta_k"'] & (A^n_{k+1})_\bullet(0, n)
\end{tikzcd}$$

By the induction hypothesis, the left vertical map belongs to $\overline{\text{Horn}}$ and thus so is the right vertical map. By the induction hypothesis, the inclusion $(A^n_{k+1})_\bullet(0, n) \hookrightarrow \Delta^n_\bullet(0, n)$ also belongs to $\overline{\text{Horn}}$. This completes the proof. □

**Proposition 4.2.58.** *Let $(X, S)$ be a quasi-category in $\mathcal{V}$. Then for every $a, b \in S$, the functor $X_\bullet(a, b) : \mathcal{N}ec^{op} \to \mathcal{V}$ lifts wings.*

*Proof.* This is an immediate consequence of Lemma 4.2.57. □

**Proposition 4.2.59.** *Let $(X, S)$ be a templicial $k$-module with a naF-structure. Then for every $a, b \in S$, the functor $X_\bullet(a, b) : \mathcal{N}ec^{op} \to \mathrm{Mod}(k)$ lifts wings.*

*Proof.* Let $Z$ denote the naF-structure of $X$. Take $n \geq 2$ and $a, b \in S$. By Proposition 4.2.56, a morphism $\tilde{F}(W^n)_\bullet(0, n) \to X_\bullet(a, b)$ in $\mathrm{Mod}(k)^{\mathcal{N}ec^{op}}$ corresponds to a collection $(x_k)_{k=1}^{n-1}$ of elements with $x_k \in U((X_k \otimes_S X_{n-k})(a, b))$ for all $0 < k < n$ such that for all $0 < k < l < n$ we have

$$(\mathrm{id}_{X_k} \otimes \mu_{l-k,n-l})(x_k) = (\mu_{k,l-k} \otimes \mathrm{id}_{X_{n-l}})(x_l) \tag{4.10}$$

To extend the above morphism to $\tilde{F}(\Delta^n)_\bullet(0, n)$, we must find an element $z \in X_n(a, b)$ such that $\mu_{k,n-k}(z) = x_k$ for all $0 < k < n$.

Given $T \in \mathcal{P}_n$ with $\ell(T) \geq 2$, we can choose $k \in T \setminus \{0, n\}$. Consider the splitting $(T_1, T_2)$ of $T$ over $\{0 < k < n\}$. Then set

$$x_T = (\mu_{T_1} \otimes \mu_{T_2})(x_k) \in U(X_T(a, b))$$

Note that by (4.10), this expression does not depend on the choice of $k$. Then it follows by Proposition 2.2.40 and Corollary 2.2.41.1 that

$$\begin{aligned}
\mu_{k,n-k} Z^T(x_T) &= (Z^{T_1} \otimes Z^{T_2})(\mathrm{id} \otimes \mu_{T'} \otimes \mathrm{id})(x_T) \\
&= (Z^{T_1} \otimes Z^{T_2})(x_{T \cup \{k\}}) = \mu_{k,n-k} Z^{T \cup \{k\}}(x_{T \cup \{k\}})
\end{aligned}$$

where $T'$ is some necklace with $\ell(T') = 2$ and the second equality follows from (4.10). Now consider

$$z = \sum_{\substack{T\in\mathcal{P}_n \\ \ell(T)\geq 2}} (-1)^{\ell(T)} Z^T(x_T)$$

Then we have for all $0 < k < n$ that $\mu_{k,n-k}(z)$ is equal to

$$\begin{aligned}
&\sum_{\substack{T\in\mathcal{P}_n \\ \ell(T)\geq 2 \\ k\in T}} (-1)^{\ell(T)} \mu_{k,n-k} Z^T(x_T) + \sum_{\substack{T\in\mathcal{P}_n \\ \ell(T)\geq 2 \\ k\notin T}} (-1)^{\ell(T)} \mu_{k,n-k} Z^{T\cup\{k\}}(x_{T\cup\{k\}}) \\
&= \sum_{\substack{T\in\mathcal{P}_n \\ \ell(T)\geq 2 \\ k\in T}} (-1)^{\ell(T)} \mu_{k,n-k} Z^T(x_T) + \sum_{\substack{U\in\mathcal{P}_n \\ \ell(U)\geq 3 \\ k\in U}} (-1)^{\ell(U)-1} \mu_{k,n-k} Z^U(x_U) \\
&= \mu_{k,n-k} Z^{k,n-k}(x_{\{0<k<n\}}) = x_k
\end{aligned}$$

□

**Proposition 4.2.60.** *Let $(X, S)$ be a templicial $k$-module. Then the following statements are equivalent.*

*(1) $X$ is a quasi-category in* $\mathrm{Mod}(k)$.

*(2) For all $a, b \in S$, the functor $X_\bullet(a, b) : \mathcal{N}ec^{op} \to \mathrm{Mod}(k)$ lifts wings.*

*Proof.* If $X$ is a quasi-category in $\mathrm{Mod}(k)$, then (2) holds by Proposition 4.2.58. Conversely, take $0 < j < n$, $a, b \in S$ and let $(x_k)_{k=1}^{n-1}$ and $(y_i)_{i=1,i\neq j}^{n-1}$ be collections of elements satisfying the conditions of Corollary 2.2.22.3. Consider the following condition on elements $z \in U(X_n(a, b))$:

$$\mu_{k,l}(z) = x_k \quad (\text{for all } 0 < k < n) \tag{4.11}$$

Let us start by noting that if $z \in X_n$ satisfies (4.11), then we have for all $0 < k < n$ that

$$\begin{aligned}
\mu_{k,n-k}(s_i(y_i - d_i(z))) &= 0 &&(\text{for all } 0 < i < j) \\
\mu_{k,n-k}(s_{i-1}(y_i - d_i(z))) &= 0 &&(\text{for all } j < i < n)
\end{aligned}$$

Indeed, for the first equation, there are three cases:

$$\begin{aligned}
&\mu_{k,n-k}(s_i(y_i - d_i(z))) \\
&= \begin{cases} (s_i \otimes \mathrm{id}_{X_{n-k}})(\mu_{k-1,n-k}(y_i) - (d_i \otimes \mathrm{id}_{X_{n-k}})(x_k)) & \text{if } i < k \\ (\mathrm{id}_{X_k} \otimes s_0)(\mu_{k,n-k-1}(y_k) - (d_k \otimes \mathrm{id}_{X_{n-k-1}})(x_{k+1})) & \text{if } i = k \\ (\mathrm{id}_{X_k} \otimes s_{i-k})(\mu_{k,n-k-1}(y_i) - (\mathrm{id}_{X_k} \otimes d_{i-k})(x_k)) & \text{if } i > k \end{cases} \\
&= 0
\end{aligned}$$

The second equation follows similarly.

Now assuming (2), there exists an element $z^0 \in U(X_n(a, b))$ satisfying condition (4.11). Then define, inductively on $l \in \{1, ..., j-1\}$:

$$z^l = z^{l-1} + s_l(y_l - d_l(z^{l-1})) \in U(X_n(a, b))$$

By the previous remarks, each $z^l$ satisfies (4.11). We then prove by induction on $l$ that for all $0 < i \leq l$:

$$d_i(z^l) = y_i$$

Indeed, for $l = 0$ this is trivial and if $l > 0$ we have:

$$d_i(z^l) = \begin{cases} y_i - s_{l-1}(d_i(y_l) - d_{l-1}(y_i)) & \text{if } i < l \\ d_l(z^{l-1}) + y_l - d_l(z^{l-1}) & \text{if } i = l \end{cases} = y_i$$

Finally, set $z^n = z^{j-1}$ and define inductively on $l \in \{j+1, ..., n-1\}$:

$$z^l = z^{l+1} + s_{l-1}(y_l - d_l(z^{l+1}))$$

Then again $z^l$ satisfies (4.11) for all $j < l \leq n$. We prove by induction on $l$ that for all $i \in \{1, ..., j-1\} \cup \{l, ..., n-1\}$:

$$d_i(z^l) = y_i$$

Again this is trivial for $l = n$ and if $l < n$ we have

$$d_i(z^l) = \begin{cases} y_i - s_{l-2}(d_i(y_l) - d_{l-1}(y_i)) & \text{if } i < l-1 \\ d_l(z^{l+1}) + y_l - d_l(z^{l+1}) & \text{if } i = l \\ y_i - s_{l-1}(d_{i-1}(y_l) - d_l(y_i)) & \text{if } i > l \end{cases} = y_i$$

Note that the case $i = l - 1$ does not occur. Following Proposition 2.2.28, it thus suffices to set $z = z^{j+1}$. □

The previous proposition does not hold for ordinary simplicial sets, as the following example shows.

**Example 4.2.61.** Consider the simplicial set $X = \Delta^3 \amalg_{\partial\Delta^2} \Delta^2$, gluing an extra 2nd face to the standard 3-simplex. Formally, it is the pushout of the inclusion $\partial\Delta^2 \subseteq \Delta^2$ along the map $\partial\Delta^2 \to \Delta^3$ sending vertices $0 \mapsto 0$, $1 \mapsto 1$ and $2 \mapsto 3$. Denote the simplices of $\Delta^3$ by ordered sequences $[i_0, ..., i_m]$ and denote the extra face by $x \in X_2$. We then have $d_0(x) = [1, 3]$, $d_1(x) = [0, 3]$ and $d_2(x) = [0, 1]$, but $x \neq [0, 1, 3]$.

Then $X$ is certainly not a quasi-category as there exists no 3-simplex $z$ with $d_0(z) = [1, 2, 3]$, $d_2(z) = x$ and $d_3(z) = [0, 1, 2]$.

However, all wings in $X$ can be filled. Indeed, a map $\alpha : W^n \to X$ is uniquely determined by simplices $y, z \in X_{n-1}$ such that $d_0(y) = d_{n-1}(z)$. If either $y$ or $z$ is degenerate, $\alpha$ extends trivially to $\Delta^n$. Assuming they are both non-degenerate, we have either $n = 2$ or $n = 3$. As $W^2 = \Lambda^2_1$ and the quasi-category $\Delta^3$ contains all edges of $X$, the case $n = 2$ is covered. If $n = 3$, we must have $y = [0, 1, 2]$ and $z = [1, 2, 3]$, which can be filled by $[0, 1, 2, 3]$ itself.

**Theorem 4.2.62.** *Let $X$ be a templicial $k$-module with a naF-structure. Then $X$ is a quasi-category in* $\mathrm{Mod}(k)$.

*Proof.* Combine Propositions 4.2.59 and 4.2.60. □

**Corollary 4.2.63.** *Let $\mathcal{C}$ be an ordinary quasi-category. Then $\tilde{F}(\mathcal{C})$ is a quasi-category in* $\mathrm{Mod}(k)$.

*Proof.* This follows from Proposition 2.2.37, Example 3.1.33 and Theorem 4.2.62. □

**Corollary 4.2.64.** *Let $\mathcal{C}$ be a small $S^+\operatorname{Mod}(k)$-category. Then the underlying templicial $k$-module of $\mathcal{T}(\mathcal{C})$ is a quasi-category in $\operatorname{Mod}(k)$.*

*Proof.* This immediately follows from Theorem 4.2.62. □

**Corollary 4.2.65.** *Let $\mathcal{C}_\bullet$ be a small dg-category over $k$. Then its linear dg-nerve $N_k^{dg}(\mathcal{C})$ is a quasi-category in $\operatorname{Mod}(k)$.*

*Proof.* Apply Corollary 4.2.64 to the $S^+\operatorname{Mod}(k)$-category $\Gamma^+(\mathcal{C}_\bullet)$. □

### 4.2.5 Comparison with other nerves

In this final subsection, we compare the linear dg-nerve $N_k^{dg}$ to the two other nerves we defined so far, namely the templicial nerve functor $N_k$ (Construction 2.3.4) and templicial homotopy coherent nerve functor $N_k^{hc}$ (Definition 4.1.13).

**Notation 4.2.66.** Any small $k$-linear category can be considered as a small dg-category concentrated in degree 0. We denote this embedding by $\iota : k\operatorname{Cat} \to k\operatorname{Cat}_{dg}$. Conversely, we can apply the 0th homology functor to all hom-complexes of a small dg-category to get a functor $H_0 : k\operatorname{Cat}_{dg} \to k\operatorname{Cat}$.

**Proposition 4.2.67.** *We have natural isomorphisms*

$$N_k^{dg} \circ \iota \simeq N_k \quad \text{and} \quad h_k \circ N_k^{dg} \simeq H_0$$

*Proof.* Let us denote the functor from left to right in the equivalence of Corollary 4.2.45 by $DG = N_\bullet^+\mathcal{K} : S_\otimes^{Frob}\operatorname{Mod}(k) \to k\operatorname{Cat}_{dg,\geq 0}$. Clearly, $\iota$ factors through $k\operatorname{Cat}_{dg,\geq 0}$ and the templicial nerve functor $N_k$ factors through $S_\otimes^{Frob}\operatorname{Mod}(k)$ by Corollary 2.3.9. Therefore, it suffices to show that we have natural isomorphisms

$$\iota \simeq DG \circ N_k \quad \text{and} \quad h_k \simeq H_0 \circ DG$$

Let $\mathcal{C}$ be a small $k$-linear category. Since the comultiplication maps of $N_k(\mathcal{C})$ are invertible, we have that the $S^+\operatorname{Mod}(k)$-category $\mathcal{K}(N_k(\mathcal{C}))$ is concentrated in degree $-1$ and thus $DG_\bullet(N_k(\mathcal{C}))$ is concentrated in degree 0. It follows that $DG \circ N_k$ is naturally isomorphic to $\iota$.

Let $(X, S)$ be a Frobenius templicial $k$-module. Boiling down the definitions, we see that the set of objects of $DG_\bullet(X)$ is $S$ as well and that for every $a \in S$, the degenerate 1-simplex $s_0(a)$ represents the identity in both $h_kX$ and $H_0(DG_\bullet(X))$. Take $a, b, c \in S$. Then the differential $\partial : DG_1(X)(a,c) \to DG_0(X)(a,c)$ is just the restriction $d_1|_{\ker(\mu_{1,1})} : \ker(\mu_{1,1})(a,c) \to X_1(a,c)$. Hence, for any three $f \in X_1(a,b)$, $g \in X_1(b,c)$ and $h \in X_1(a,c)$, the composition $gf$ is homologous to $h$ in $DG_\bullet(X)$ if and only if there exists a $w \in \ker(\mu_{1,1})(x,z)$ such that $d_1(w) = h - gf$. This is equivalent to the existence of a templicial map $\alpha : \tilde{F}(\Delta^2) \to X$ with $\alpha_{0,1} = 0$, $\alpha_{1,2} = s_0(x)$ and $\alpha_{0,2} = h - gf$ (using the notation of Corollary 2.1.27). In other words, $[g] \circ [f] = [h]$ in $h_kX$. Specializing to the case $f = s_0(x)$, we find that $[g] = [h]$ in $H_0(DG_\bullet(X))$ if and only if $[g] = [f]$ in $h_k X$. This shows that $[f] \mapsto [f]$ defines an isomorphism of $k$-linear categories

$$h_k X \simeq H_0(DG_\bullet(X))$$

It follows easily that this isomorphism is natural in $X$. $\square$

Given a dg-category, there is a classical comparison map between its dg-nerve and the homotopy coherent nerve of its associated simplicial category. We will lift this map to a templicial analogue in Corollary 4.2.70. To achieve this, we'll first prove a general lifting result for Frobenius templicial $k$-modules (Proposition 4.2.69).

**Lemma 4.2.68.** *Let $(X, S)$ be a Frobenius templicial $k$-module and let $\epsilon : \tilde{F}\tilde{U}(X) \to X$ be the canonical templicial map. For all $n \geq 0$ and $a, b \in S$, the induced $k$-linear map*

$$\epsilon_n : F(\tilde{U}(X)_n(a, b)) \to X_n(a, b)$$

*is surjective.*

*Proof.* Fix some $a, b \in S$ and $n \geq 0$. As $\epsilon_0$ clearly is an isomorphism, we may assume $n > 0$. Let us call an element $x \in X_n(a, b)$ *pure* if there is an $n$-simplex $\alpha \in \tilde{U}(X)_n(a, b)$ such that $\alpha_{0,n} = x$ (using the notation of Corollary 2.1.27). We wish to show that the $k$-module $X_n(a, b)$ is generated by pure elements.

Denote the Frobenius structure of $X$ by $Z$. We first make some observations:

1. If $x \in X_n(a, b)$ satisfies $\mu_{k,l}(x) = 0$ for all $k, l > 0$ with $k + l = n$, then $x$ is pure.

   Indeed, we can simply define $\alpha \in \tilde{U}(X)_n(a, b)$ as follows:

   $$\alpha_{i,j} = \begin{cases} x & \text{if } i = 0, j = n \\ 0 & \text{otherwise} \end{cases}$$

   for all $0 \leq i < j \leq n$.

2. Let $T = \{0 = t_0 < t_1 < ... < t_p = n\}$ be a necklace and $a_{t_1}, ..., a_{t_{p-1}} \in S$, and set $a_0 = a, a_n = b$. If we have pure elements $x_i \in X_{t_i - t_{i-1}}(a_{t_{i-1}}, a_{t_i})$ for all $i \in \{1, ..., p\}$, then

   $$Z^T(x_1 \otimes ... \otimes x_p) \in X_n(a, b)$$

   is pure as well.

   Indeed, take an $r \in \{1, ..., p\}$. Then we can choose elements

   $$a_{t_{r-1}+1}, ..., a_{t_r - 1} \in S \quad \text{and} \quad \alpha_{i,j} \in X_{j-i}(a_i, a_j) \text{ for all } t_{r-1} \leq i < j \leq t_r$$

   such that $\alpha_{t_{r-1},t_r} = x_r$ and $\mu_{k-i,j-k}(\alpha_{i,j}) = \alpha_{i,k} \otimes \alpha_{k,j}$ for all $i < k < j$. Now given any $0 \leq i < j \leq n$, let $r, s \in \{1, ..., p\}$ be such that $t_{r-1} \leq i < t_r$ and $t_{s-1} < j \leq t_s$ and define

   $$\alpha_{i,j} = Z^{T'}(\alpha_{i,t_r} \otimes x_{r+1} \otimes ... \otimes x_{s-1} \otimes \alpha_{t_{s-1},j})$$

   where $(T_1, T', T_2)$ is the splitting of $T$ over $\{0 < i < j < n\}$. Note that whenever $t_{r-1} \leq i < j \leq t_r$ for some $r \in \{1, ..., p\}$, this definition coincides with the $\alpha_{i,j}$

already defined. Further, when $(i,j)=(0,n)$, we get $\alpha_{0,n}=Z^T(x_1\otimes\ldots\otimes x_p)$. Now take any $i<k<j$. Then by Proposition 2.2.40,

$$\mu_{k-i,j-k}(\alpha_{i,j})=\alpha_{i,k}\otimes\alpha_{k,j}$$

showing that $\alpha=((a_i)_{i=0}^n,(\alpha_{i,j})_{0\le i<j\le n})$ is an $n$-simplex of $\tilde{U}(X)$.

We proceed by induction on $n>0$ to show that $X_n(a,b)$ is generated by pure elements. If $n=1$, then every element of $X_n(a,b)$ is pure. Now let $n\ge 2$ and $x\in X_n(a,b)$. Given a necklace $T=\{0=t_0<t_1<\ldots<t_p=n\}$ with $p\ge 2$, we have

$$\mu_T(x)=\sum_{i=1}^N x_1^i\otimes\ldots\otimes x_p^i$$

for some $N\in\mathbb{N}$, $a=a_0^i,a_1^i,\ldots,a_{p-1}^i,a_p^i=b\in S$ and $x_j^i\in X_{t_j-t_{j-1}}(a_{j-1}^i,a_j^i)$. By the induction hypothesis, we may assume that all $x_j^i$ are pure. Set

$$y=\sum_{\substack{T\in\mathcal{P}_n\\ \ell(T)\ge 2}}(-1)^{\ell(T)+1}Z^T\mu_T(x)$$

Then by observation 2, $y$ is a linear combination of pure elements. By the same argument as in the proof of Lemma 4.2.36, we find that

$$\mu_{k,l}(x+y)=\mu_{k,l}\left(\sum_{T\in\mathcal{P}_n}(-1)^{\ell(T)+1}Z^T\mu_T(x)\right)=0$$

for all $k,l>0$ with $k+l=n$. Hence, by observation 1, $x+y$ is pure as well and thus $x=(x+y)-y$ is a linear combination of pure elements. □

**Proposition 4.2.69.** *Let $(X,S)$ be a Frobenius templicial $k$-module. Let $\mathcal{C}_\bullet$ be a small dg-category over $k$. Suppose $f:\tilde{U}(X)\to N^{dg}(\mathcal{C})$ is a simplicial map that (as in Remark 4.2.50) corresponds to a map $f_0:S\to\mathrm{Ob}(\mathcal{C})$ along with the following composites for all $a,b\in S$ and $n>0$:*

$$f_n:\tilde{U}(X)_n(a,b)\to U(X_n(a,b))\xrightarrow{U(\beta_n)}U\left(\mathcal{C}_{n-1}(f_0(a),f_0(b))\right)$$

*with $(\beta_n:X_n\to f_0^*\mathcal{C}_{n-1})_{n>0}$ some collection of morphisms in $k\,\mathrm{Quiv}_S$.*

*Then there is a unique templicial map $(\alpha,f_0):X\to N_k^{dg}(\mathcal{C})$ such that $f$ coincides with*

$$\tilde{U}(X)\xrightarrow{\tilde{U}(\alpha)}\tilde{U}(N_k^{dg}(\mathcal{C}))\simeq N^{dg}(\mathcal{C})$$

*where the isomorphism is provided by Corollary 4.2.54.*

*Proof.* In order to construct $\alpha$, we want to show that the morphisms $(\beta_n)_{n>0}$ satisfy the properties (4.7) and (4.8) of Proposition 4.2.53. Consider the counit $\epsilon:\tilde{F}\tilde{U}(X)\to X$ of the adjunction $\tilde{F}\dashv\tilde{U}$. The hypothesis on $f$ guarantees that for any $n$-simplex $\sigma$ of $\tilde{U}(X)$, considered as a simplex of $\tilde{F}\tilde{U}(X)$, we have

$$U(\beta_n\epsilon_n)(\sigma)=f_n(\sigma)$$

As $\epsilon_n$ is surjective (Lemma 4.2.68) and $U$ is faithful, $\beta_n$ is uniquely determined by $f_n$. Further, to verify properties (4.7) and (4.8), it suffices to check that they hold after evaluating in $\epsilon_n(\sigma)$ for arbitrary $\sigma \in \tilde{U}(X)_n$ and $n > 0$. This now follows from Remark 4.2.50.

Moreover, it follows from Remark 4.2.50 and Corollary 4.2.54 that $(\alpha, f_0)$ is the unique templicial map such that $\tilde{U}(\alpha) : \tilde{U}(X) \to \tilde{U}(N_k^{dg}(\mathcal{C})) \simeq N^{dg}(\mathcal{C})$ is precisely $f$. $\square$

The classical normalized chain functor $N_\bullet : S\,\mathrm{Mod}(k) \to \mathrm{Ch}(k)$ has a colax monoidal structure given by the Alexander-Whitney homomorphism (see [May67, §29] for example). Thus the right-adjoint to $N_\bullet$:

$$\Gamma : \mathrm{Ch}(k) \to S\,\mathrm{Mod}(k)$$

has an induced lax monoidal structure by Lemma 1.1.4. Given a small dg-category $\mathcal{C}_\bullet$, we can consequently apply $\Gamma$ to the hom-complexes of $\mathcal{C}_\bullet$ to obtain an $S\,\mathrm{Mod}(k)$-category $\mathcal{C}^\triangle$ with the same objects as $\mathcal{C}_\bullet$. In [Lur16, Proposition 1.3.1.17], Lurie shows that there is an equivalence of quasi-categories (which was later shown to even be a trivial fibration [Lur18, Tag 00SV]):

$$\mathfrak{Z} : N^{hc}(\mathcal{U}(\mathcal{C}^\triangle)) \to N^{dg}(\mathcal{C})$$

which satisfies the following two properties:

- The map $\mathfrak{Z}$ is given by the identity on vertices.
- For any $A, B \in \mathrm{Ob}(\mathcal{C})$, $n > 0$ and $\sigma \in N^{hc}(\mathcal{U}(\mathcal{C}^\triangle))_n(A, B)$, the $(n-1)$-chain $\mathfrak{Z}(\sigma)_I \in \mathcal{C}_{n-1}(A, B)$ for $I = [n]$ is given as follows.

  The $n$-simplex $\sigma$ can be identified with a simplicial functor $\mathfrak{C}[\Delta^n] \to \mathcal{U}(\mathcal{C}^\triangle)$ (see §4.1.2), which induces a simplicial map

  $$f : \square^{n-1} \doteqdot N(\mathcal{P}_n) \simeq \mathfrak{C}[\Delta^n](0, n) \to U(\Gamma(\mathcal{C}_\bullet(A, B)))$$

  As $\Gamma$ is right-adjoint to the normalized chain complex $N_\bullet : S\,\mathrm{Mod}(k) \to \mathrm{Ch}(k)$, this map determines a chain map

  $$f' : N_\bullet(\square^{n-1}; k) \to \mathcal{C}_\bullet(A, B)$$

  Then $\mathfrak{Z}(\sigma)_I$ is defined as the image under $f'$ of the following $(n-1)$-chain:

  $$[\square^{n-1}] = \sum_{\tau \in \Sigma_{n-1}} \mathrm{sgn}(\tau)\hat{\tau} \in N_{n-1}(\square^{n-1}; k)$$

  where $\mathrm{sgn}(\tau) \in \{-1, 1\}$ denotes the sign of a permutation $\tau \in \Sigma_{n-1}$ and

  $$\hat{\tau} = (\{0, n\} \subseteq \{0, \tau(1), n\} \subseteq \{0, \tau(1), \tau(2), n\} \subseteq ... \subseteq \{0, ..., n\})$$

  is a non-degenerate $(n-1)$-simplex of $\square^{n-1} = N(\mathcal{P}_n)$.

In fact, the map $\mathfrak{Z}$ is unique with these two properties as is shown in [Lur18, Tag 00SN], where $[\square^{n-1}]$ is called the *fundamental chain* of $\square^{n-1}$. For more details on the map $\mathfrak{Z}$, see [Fao17] or [Lur18]. Note however that they use slightly different versions of the homotopy coherent nerve.

**Corollary 4.2.70.** *Let $\mathcal{C}_\bullet$ be a small dg-category over $k$. Then there is a unique templicial map $N_k^{hc}(\mathcal{C}^\triangle) \to N_k^{dg}(\mathcal{C})$ such that $\mathfrak{Z}$ is equal to the composite*

$$N^{hc}(\mathcal{U}(\mathcal{C}^\triangle)) \simeq \tilde{U}N_k^{hc}(\mathcal{C}^\triangle) \to \tilde{U}N_k^{dg}(\mathcal{C}) \simeq N^{dg}(\mathcal{C})$$

*where the isomorphisms are provided by Proposition 4.1.17 and Corollary 4.2.54.*

*Proof.* By Proposition 4.1.17, we have an isomorphism of simplicial sets $N^{hc}(\mathcal{U}(\mathcal{C}^\triangle)) \simeq \tilde{U}N_k^{hc}(\mathcal{C})$. It follows from the properties of $\mathfrak{Z}$ that for all $A, B \in \mathrm{Ob}(\mathcal{C})$ and $n > 0$, we have a commutative diagram:

$$\begin{array}{ccc}
\tilde{U}N_k^{hc}(\mathcal{C}^\triangle)_n(A,B) & \xrightarrow{\mathfrak{Z}_n} & N^{dg}(\mathcal{C})_n(A,B) \\
\downarrow & & \downarrow \\
U(N_k^{hc}(\mathcal{C}^\triangle)_n(A,B)) & \xrightarrow[U(\beta_n)]{} & U(\mathcal{C}_{n-1}(A,B))
\end{array}$$

where $\beta_n$ is the $k$-linear map given by the composite of

$$N_k^{hc}(\mathcal{C}^\triangle)_n(A,B) \xrightarrow{p_n} [FN(\mathcal{P}_n), \Gamma(\mathcal{C}_\bullet(A,B))] \simeq [N_\bullet(\square^{n-1}; k), \mathcal{C}_\bullet(A,B)]$$

with the evaluation map at $[\square^{n-1}]$:

$$ev_{[\square^{n-1}]} : [N_\bullet(\square^{n-1}; k), \mathcal{C}_\bullet(A,B)] \to \mathcal{C}_{n-1}(A,B)$$

Hence, the result follows from Propositions 4.1.19 and 4.2.69. $\square$

# Chapter 5

# Future research

*"And so I close, realizing that perhaps the ending has not yet been written."*

— Atrus (Myst)

In this final chapter we pose some open questions and discuss possible avenues for answering them.

## Model structure

The most glaring open problem regarding templicial objects is the lack of any homotopy theory. Specifically, it is desirable to have a model structure on the category $S_{\otimes}\mathcal{V}$. For details on model categories we refer to the relevant literature (see [Hov99] or [Hir03] for example).

The *Joyal model structure* (originally constructed in [Joy08, Chapter 6], a modern account is given in [Cis19, §3.3]) is the unique model structure on $\mathrm{SSet}$ whose cofibrations are the monomorphisms and whose fibrant objects are the quasi-categories. Let us denote it by $\mathrm{SSet}_J$. By analogy, we can ask the following question:

**Question 1.** *Given a suitable monoidal category $\mathcal{V}$, does there exist a model structure on $S_{\otimes}\mathcal{V}$ whose fibrant objects are the quasi-categories in $\mathcal{V}$ (Definition 2.2.26) and whose cofibrations are the projective templicial morphisms (Definition 3.1.24)?*

By a result of Joyal [Joy04, p. 50.10], this would fully determine the model category structure. Of course the issue is showing existence.

One major source of difficulty is the fact that we do not have a set which generates the class of projective templicial morphisms as a weakly saturated class. Unlike in $\mathrm{SSet}$, where the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 0$ generate the monomorphisms, the induced morphisms $\tilde{F}(\partial\Delta^n) \to \tilde{F}(\Delta^n)$ in $S_{\otimes}\mathcal{V}$ do not generate the projective templicial morphisms. Thus we currently do not have access to recognition theorems for cofibrantly generated model categories like [Bek00, Theorem 1.7] or [Hir03, Theorem 11.3.1].

Let us first consider what doesn't work. In [Qui67, II.3, II.4], Quillen constructed a model structure on $\mathrm{SSet}$ with fibrant objects the Kan complexes, and showed that it can be right transferred along the forgetful functor $U : S\mathcal{V} \to \mathrm{SSet}$ (for suitable $\mathcal{V}$). Similarly, one could hope to construct a model structure on $S_\otimes\mathcal{V}$ via right transfer along $\tilde{U} : S_\otimes\mathcal{V} \to \mathrm{SSet}_J$. However, it is possible to adapt Example 2.3.18 to show that this model structure does not exist.

An alternative approach is the following. In [Ber07], Bergner constructed a model structure on the category $\mathrm{Cat}_\Delta$ of simplicial categories. It was shown by Lurie in [Lur09a] that the categorification functor $\mathfrak{C} : \mathrm{SSet} \to \mathrm{Cat}_\Delta$ is the left-adjoint in a Quillen equivalence between the model categories $\mathrm{Cat}_\Delta$ and $\mathrm{SSet}_J$. Moreover, $\mathfrak{C}$ preserves and reflects weak equivalences (see [DS11a, Proposition 8.1] for example). It is further not difficult to see that $\mathfrak{C}$ also preserves and reflects cofibrations. So in fact, the Joyal model structure $\mathrm{SSet}_J$ is given via left transfer along $\mathfrak{C} : \mathrm{SSet} \to \mathrm{Cat}_\Delta$. From Quillen's model category $S\mathcal{V}$, several results [BM13][Sta14][Mur15] produce an induced model structure on $\mathcal{V}\,\mathrm{Cat}_\Delta$, generalizing the one of Bergner. So one could similarly try to equip $S_\otimes\mathcal{V}$ with a model structure by left transfer along $\mathfrak{C}_\mathcal{V} : S_\otimes\mathcal{V} \to \mathcal{V}\,\mathrm{Cat}_\Delta$ (Definition 4.1.13). Recall that $S_\otimes\mathcal{V}$ is locally presentable if $\mathcal{V}$ is, by Theorem 3.2.29. So in this case, the results of [HKRS17] would become available. At present it is still unclear whether such a model structure exists, or whether it would have the same cofibrations and fibrant objects as posited in Question 1.

A third, less straightforward approach is through the use of necklace categories $\mathcal{V}\,\mathrm{Cat}_{\mathcal{N}ec}$. The author believes it should be possible to construct a model structure on $\mathcal{V}^{\mathcal{N}ec^{op}}$ whose cofibrations and trivial cofibrations are generated by $\mathrm{Cell}$ and $\mathrm{Horn}$ respectively (see Notation 2.2.23 and Remark 2.2.24). In that case, a templicial object $(X, S)$ would be a quasi-category in $\mathcal{V}$ if and only if the $X_\bullet(a,b)$ is fibrant in $\mathcal{V}^{\mathcal{N}ec^{op}}$ for all $a, b \in S$. In any case, a more thorough investigation of the category $\mathcal{V}^{\mathcal{N}ec^{op}}$ is necessary.

## Morphism spaces

Consider a simplicial set $K$ with vertices $a, b \in K_0$. In [Lur09a, §1.2.2], Lurie constructed simplicial sets $\mathrm{Hom}^L_K(a,b)$, $\mathrm{Hom}^{cyl}_K(a,b)$ and $\mathrm{Hom}^R_K(a,b)$, called the *left-pinched morphism space*, *morphism space* and *right-pinched morphism space* respectively. If $K$ is a quasi-category, then these are all homotopy equivalent Kan complexes. As noted by Dugger and Spivak in [DS11a], these morphism spaces can be described by certain cosimplicial objects $C_\pitchfork : \mathbf{\Delta} \to \mathrm{SSet}_{*,*}$ where $\pitchfork \in \{L, cyl, R\}$. Indeed:

$$\mathrm{Hom}^\pitchfork_K(a,b) \simeq \mathrm{SSet}_{*,*}(C^{(-)}_\pitchfork, K_{a,b})$$

Moreover, they constructed a zig-zag of weak homotopy equivalences in $\mathrm{SSet}$:

$$\mathfrak{C}[K](a,b) \overset{\sim}{\longleftrightarrow} \mathrm{Hom}^\pitchfork_K(a,b) \tag{5.1}$$

assuming $K$ is a quasi-category.

In fact, for every $n \geq 0$, the simplicial set $C^n_\pitchfork$ has exactly two vertices $0$ and $1$. It is not difficult to see that also

$$\mathrm{Hom}^\pitchfork_K(a,b) \simeq \mathrm{Set}^{\mathcal{N}ec^{op}}((C^{(-)}_\pitchfork)_\bullet(0,1), K_\bullet(a,b))$$

Given a templicial object $(X, S)$ with $a, b \in S$, we can now define

$$\mathrm{Hom}_X^{\pitchfork}(a,b) = [\tilde{F}(C_{\pitchfork}^{(-)})_\bullet(0,1), X_\bullet(a,b)] \in S\mathcal{V}$$

where $[-,-]$ denotes the canonical enrichment of $\mathcal{V}^{\mathcal{N}ec^{op}}$ over $\mathcal{V}$. It follows quickly that there is an isomorphism of simplicial sets

$$U(\mathrm{Hom}_X^{\pitchfork}(a,b)) \simeq \mathrm{Hom}_{\tilde{U}(X)}^{\pitchfork}(a,b) \tag{5.2}$$

Thus if $X$ is a quasi-category in $\mathcal{V}$, then $\tilde{U}(X)$ is a quasi-category by Proposition 2.2.31 and thus $\mathrm{Hom}_X^L(a,b)$, $\mathrm{Hom}_X^{cyl}(a,b)$ and $\mathrm{Hom}_X^R(a,b)$ are weakly equivalent fibrant objects in Quillen's model structure on $S\mathcal{V}$ mentioned above. The following question now presents itself.

**Question 2.** *Given a quasi-category $X$ in $\mathcal{V}$ with vertices $a$ and $b$, does there exist a zig-zag of weak equivalences in $S\mathcal{V}$:*

$$\mathfrak{C}_{\mathcal{V}}[X](a,b) \xleftrightarrow{\sim} \mathrm{Hom}_X^{\pitchfork}(a,b)$$

*which specializes to* (5.1) *when* $\mathcal{V} = \mathrm{Set}$?

If this can be shown, then it would follow from (5.2) that the canonical map

$$\mathfrak{C}[\tilde{U}(X)](a,b) \to U(\mathfrak{C}_{\mathcal{V}}[X](a,b))$$

is a weak homotopy equivalence for all quasi-categories $X$ in $\mathcal{V}$. We have already shown that this holds on the level of connected components (Corollary 2.3.26), and that it fails if $X$ is not assumed to be a quasi-category in $\mathcal{V}$ (Example 4.1.34).

# Comparisons with other models

For the following comparisons to make sense, let us assume that the model structure of Question 1 exists.

## Categories weakly enriched in simplicial objects

As mentioned above, the categorification functor $\mathfrak{C} : \mathrm{SSet}_J \to \mathrm{Cat}_\Delta$ is the left-adjoint in a Quillen equivalence. The following question is a natural one.

**Question 3.** *Is the adjunction $\mathfrak{C}_{\mathcal{V}} : S_\otimes\mathcal{V} \leftrightarrows \mathcal{V}\,\mathrm{Cat}_\Delta : N_{\mathcal{V}}^{hc}$ a Quillen equivalence?*

In [GH15] Gepner and Haugseng developed a very extensive theory of $\infty$-categories enriched in a monoidal $\infty$-category $\mathcal{W}$ which can be organized as the objects of an $\infty$-category $\mathrm{Cat}_\infty^{\mathcal{W}}$. In the process, they formalized the idea of categories "weakly or homotopy-coherently enriched in $\mathcal{W}$". For example, when $\mathcal{W}$ is the monoidal $\infty$-category $\mathcal{S}$ of spaces, then $\mathrm{Cat}_\infty^{\mathcal{S}}$ recovers the $\infty$-category of $\infty$-categories $\mathrm{Cat}_\infty$ introduced by Lurie in [Lur09a]. Thus quasi-categories are categories weakly enriched in spaces. Given a monoidal model category $\mathcal{M}$ with class of weak equivalences $W$, then there is an associated monoidal $\infty$-category $\mathcal{M}[W^{-1}]$ and thus one can consider the $\infty$-category $\mathrm{Cat}_{\infty}^{\mathcal{M}[W^{-1}]}$. Haugseng then proved in [Hau15] that $\mathrm{Cat}_{\infty}^{\mathcal{M}[W^{-1}]}$ is equivalent to the $\infty$-category associated to the model category $\mathcal{M}\,\mathrm{Cat}$. Hence, if the answer to the above question is affirmative, we may view quasi-categories in $\mathcal{V}$ as categories weakly enriched in simplicial objects in $\mathcal{V}$.

Let us specialize to the case $\mathcal{V} = \mathrm{Mod}(k)$ for a moment. Assuming the question above has a positive answer, we would thus have a Quillen equivalence $S_{\otimes}\,\mathrm{Mod}(k) \leftrightarrows k\,\mathrm{Cat}_{\Delta}$. In [Tab05b][Tab10], Tabuada constructed a model structure on the category $k\,\mathrm{Cat}_{dg}$ of small dg-categories over $k$, and showed that the induced model category $k\,\mathrm{Cat}_{dg,\geq 0}$ is Quillen equivalent to the model category $k\,\mathrm{Cat}_{\Delta}$. Thus we would get a zig-zag of Quillen equivalences between $S_{\otimes}\,\mathrm{Mod}(k)$ and $k\,\mathrm{Cat}_{dg}$ as well. How does this zig-zag relate to the linear dg-nerve $N_k^{dg} : k\,\mathrm{Cat}_{dg,\geq 0} \to S_{\otimes}\,\mathrm{Mod}(k)$ (Definition 4.2.46)? By analogy with the classical situation, we arrive at the following question.

**Question 4.** *Let $\mathcal{C}_{\bullet}$ be a dg-category. Is the templicial morphism $N_k^{hc}(\mathcal{C}^{\Delta}) \to N_k^{dg}(\mathcal{C})$ of Corollary 4.2.70 contractible in $S_{\otimes}\,\mathrm{Mod}(k)$?*

## Segal enriched categories

Recall that a *Segal precategory* is a bisimplicial set $X : \mathbf{\Delta}^{op} \to \mathrm{SSet}$ such that the simplicial set $X_0$ is discrete. A Segal precategory $X$ is called a *Segal category* if it satisfies the *Segal condition*, that is the canonical map

$$X_n \to X_1 \times_{X_0} \dots \times_{X_0} X_1$$

is a weak homotopy equivalence of simplicial sets for all $n \geq 0$.

Segal categories were originally introduced by Dwyer, Kan and Smith in [DKS89] (under a different name). They were extensively studied by Hirschowitz and Simpson in [HS01] who also put a model structure on the category of Segal precategories $\mathrm{SePC}$ with as fibrant objects the Reedy fibrant Segal categories. In [JT07], Joyal and Tierney constructed two Quillen equivalences between $\mathrm{SSet}_J$ and $\mathrm{SePC}$.

Later, Bacard defined Segal categories enriched in a non-cartesian monoidal model category $\mathcal{M}$. These are many-object versions of the *homotopy monoids* appearing in [Lei00]. In fact, Leinster's observation that we used to define templicial objects (Proposition 2.1.6) originally appeared in this context as well. Following the same philosophy, Bacard replaced the bisimplicial object $X$ by a colax monoidal functor

$$X : \mathcal{P}_{\overline{S}} \to \mathcal{M}$$

where $S$ is a set and $\mathcal{P}_{\overline{S}}$ is a labelled version of $\mathbf{\Delta}_{+} \simeq \mathbf{\Delta}_f^{op}$ to allow for a discrete set of vertices. Let us call such a functor $X$ a *Segal $\mathcal{M}$-precategory*. Then $X$ is called a *Segal $\mathcal{M}$-category* if it satisfies the *Segal condition* which imposes that the comultiplication and counit morphisms

$$X_{k+l}(a,b,c) \to X_k(a,b) \otimes X_l(b,c) \quad \text{and} \quad X_0(a) \to I \tag{5.3}$$

are weak equivalences in $\mathcal{M}$ for all $a, b, c \in S$ and $k, l \geq 0$. If $\mathcal{M} = \mathrm{SSet}$, this recovers the classical Segal categories. If the morphisms (5.3) are all isomorphisms, this recovers $\mathcal{M}$-enriched categories. In this sense, Segal enriched categories can also be seen as categories "weakly enriched in $\mathcal{M}$".

In view of the discussion above on Gepner and Haugseng's enriched $\infty$-categories, the author expects quasi-categories in $\mathcal{V}$ to relate to Segal $S\mathcal{V}$-categories in the same way that ordinary quasi-categories relate to ordinary Segal categories. At the time of writing, the author is unaware of the existence of any model structure for Segal $\mathcal{M}$-categories for general (non-cartesian) $\mathcal{M}$. But we can still ask the following.

**Question 5.** *Can the Quillen equivalences of [JT07] be generalized to adjunctions between the categories of templicial objects in $\mathcal{V}$ and Segal $S\mathcal{V}$-precategories?*

## Enhancements of triangulated categories

*Stable $\infty$-categories* were introduced by Lurie in [Lur09b] as quasi-categories $\mathcal{C}$ with a zero object and a good notion of loop and suspension functors $\Omega, \Sigma : \mathcal{C} \to \mathcal{C}$. The homotopy category $h\mathcal{C}$ of a stable $\infty$-category $\mathcal{C}$ always comes equipped with the structure of a triangulated category in the sense of [Ver96]. As such, stable $\infty$-categories are often called *enhancements* of triangulated categories.

A different enhancement of triangulated categories are pretriangulated dg-categories in the sense of [BK90]. In [Coh16], Cohn showed that these two types of enhancements are equivalent. More precisely, Tabuada [Tab05a] constructed a model structure on $k\,\mathrm{Cat}_{dg}$ whose weak equivalences are given by the Morita equivalences and the fibrant objects are in particular pretriangulated dg-categories. Cohn proved that the $\infty$-category associated to $k\,\mathrm{Cat}_{dg}$ is equivalent to the $\infty$-category of idempotent-complete $k$-linear stable $\infty$-categories.

We have already related dg-categories over $k$ with quasi-categories in $\mathrm{Mod}(k)$ via the linear dg-nerve $N_k^{dg} : k\,\mathrm{Cat}_{dg} \to S_{\otimes}\,\mathrm{Mod}(k)$. It would be interesting to see which quasi-categories $X$ in $\mathrm{Mod}(k)$ correspond to pretriangulated dg-categories and what conditions on $X$ induce a stable $\infty$-category $\tilde{U}(X)$.

**Question 6.** *What is the relation between quasi-categories in $\mathrm{Mod}(k)$ and $k$-linear stable $\infty$-categories or pretriangulated dg-categories over $k$? Moreover, what is the relation between the linear dg-nerve $N_k^{dg}$ and Cohn's result [Coh16]?*

# Some smaller questions

## General nerve constructions

Recall from Proposition 1.3.11 that any cosimplicial object $C : \mathbf{\Delta} \to \mathcal{D}$ in a cocomplete category $\mathcal{D}$ gives rise to an adjunction $\overline{C} : \mathrm{SSet} \leftrightarrows \mathcal{D} : N^C$. Many examples of simplicial sets, like the nerve of a category, the homotopy-coherent nerve of a simplicial category and the singular set of a topological space arise in this way. Even if $\mathcal{D}$ is not cocomplete, the formula (1.4) still makes sense. Similarly, we can ask:

**Question 7.** *What structure on a category $\mathcal{D}$ and what small amount of data in $\mathcal{D}$ determines an adjunction $S_\otimes\mathcal{V} \leftrightarrows \mathcal{D}$, or simply a functor $\mathcal{D} \to S_\otimes\mathcal{V}$?*

## Monoidal closure

Because the category of simplicial sets is just a category of presheaves, it is cartesian closed and its internal hom-objects are very easy to describe. Assume $\mathcal{V}$ is a Bénabou cosmos as in Section 4.1. Given templicial objects $(X, S)$ and $(Y, T)$, we can construct their pointwise monoidal product $(X, S) \boxtimes (Y, T)$ as follows. It has vertex set $S \times T$ and for all $(a, b), (c, d) \in S \times T$ and $n \geq 0$, we set

$$(X \boxtimes Y)_n((a, b), (c, d)) = X_n(a, c) \otimes Y_n(b, d)$$

It is relatively painless to see that this defines a monoidal structure $(\boxtimes, \tilde{F}(\Delta^0))$ on $S_\otimes\mathcal{V}$. Moreover, it immediately follows that $- \boxtimes -$ preserves colimits in each variable.

Assuming $\mathcal{V}$ is locally presentable, then so is $S_\otimes\mathcal{V}$ by Theorem 3.2.29. Hence, $S_\otimes\mathcal{V}$ is monoidal closed. Unfortunately, this does not give us an explicit description of the internal hom-objects.

**Question 8.** *Can the internal hom-objects of $S_\otimes\mathcal{V}$ be described more explicitly?*

## Linear $A_\infty$-nerve

In [Fao17], Faonte extended the dg-nerve $N^{dg}$ to an $A_\infty$-*nerve* $N^{A_\infty} : k\,\mathrm{Cat}_{A_\infty} \to \mathrm{SSet}$ from the category of small $A_\infty$-categories over $k$ to simplicial sets. For background on $A_\infty$-categories, we refer to the literature (see [Kel01] for example).

**Question 9.** *Does the linear dg-nerve $N_k^{dg} : k\,\mathrm{Cat}_{dg} \to S_\otimes \mathrm{Mod}(k)$ (see §4.2.3) extend to a functor $N_k^{A_\infty} : k\,\mathrm{Cat}_{A_\infty} \to S_\otimes \mathrm{Mod}(k)$ so that there is an isomorphism $\tilde{U} \circ N_k^{A_\infty} \simeq N^{A_\infty}$?*

Appendix A

# Alternative definition of templicial objects

We discuss an alternative definition of templicial objects, which we'll also call *based colax monoidal functors* (Definition A.1.2). They are conceptually simpler than Definition 2.1.9 and don't rely on quivers, but for our purposes they turned out to be less practical. As in Chapter 2, we fix a monoidal category $\mathcal{V}$ which is bicomplete such that the monoidal product $- \otimes -$ preserves colimits in each variable. We will identify some conditions on $\mathcal{V}$ for which based colax monoidal functors coincide with templicial objects (see Definition A.2.8 and Theorem A.2.10).

## A.1 Based colax monoidal functors

*Remark* A.1.1. Let $S$ be a set. Note that $S$ has a unique comonoid structure in $(\mathrm{Set}, \times, \{*\})$ with the diagonal $\Delta : S \to S \times S$ as comultiplication and the terminal map $t : S \to \{*\}$ as counit. This extends to an equivalence of categories:

$$\mathrm{Set} \simeq \mathrm{Comon}(\mathrm{Set})$$

As the free functor $F : \mathrm{Set} \to \mathcal{V} : S \mapsto \coprod_{a \in S} I$ is strong monoidal, we have an induced functor

$$F : \mathrm{Set} \simeq \mathrm{Comon}(\mathrm{Set}) \to \mathrm{Comon}(\mathcal{V})$$

**Definition A.1.2.** Let $(X, \mu, \epsilon) : \mathbf{\Delta}_f^{op} \to \mathcal{V}$ be a colax monoidal functor. Then $X_0$ has the structure of a comonoid with comultiplication given by $\mu_{0,0} : X_0 \to X_0 \otimes X_0$ and counit given by $\epsilon : X_0 \to I$. We call a set $S$ a *base* of $X$ if it comes equipped with an isomorphism of comonoids $\varphi : X_0 \xrightarrow{\sim} F(S)$. We call the triple $(X, S, \varphi)$ a *based colax monoidal functor*.

Consider the functor

$$(-)_0 : \mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathcal{V}) \to \mathrm{Comon}(\mathcal{V}) : (X, \mu, \epsilon) \mapsto (X_0, \mu_{0,0}, \epsilon)$$

We define the category $\mathrm{Colax}_b(\mathbf{\Delta}_f^{op}, \mathcal{V})$ by the 2-pullback

$$\begin{tikzcd}
\mathrm{Colax}_b(\mathbf{\Delta}_f^{op}, \mathcal{V}) \arrow[r] \arrow[d] & \mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathcal{V}) \arrow[d, "(-)_0"] \\
\mathrm{Set} \arrow[r, "F"'] & \mathrm{Comon}(\mathcal{V})
\end{tikzcd}$$

Note that its objects are precisely the based colax monoidal functors.

A morphism $X \to Y$ in $\mathrm{Colax}_b(\mathbf{\Delta}_f^{op}, \mathcal{V})$ with respective bases $S$ and $T$ is a monoidal natural transformation $\alpha$ such that through the isomorphisms $X_0 \simeq F(S)$ and $Y_0 \simeq F(T)$, $\alpha_0$ is induced by some map of sets $f : S \to T$.

We now describe a comparison functor from templicial objects to based colax monoidal functors. In the next subsection, we will give sufficient conditions on $\mathcal{V}$ for this functor to be an equivalence.

**Construction A.1.3.** Consider the natural transformation $t : \mathrm{id}_{\mathrm{Set}} \to *$ given by the terminal map $t_S : S \to \{*\}$ for every set $S$. This induces a pseudonatural transformation

$$\Phi_{\mathcal{V}} t : \Phi_{\mathcal{V}} \to \Phi_{\mathcal{V}} \circ *$$

between pseudofunctors $\mathrm{Set} \to \mathrm{Cat}$, where $\Phi_{\mathcal{V}} = \mathrm{Colax}(\mathbf{\Delta}_f^{op}, (-)_!)$ is as in Proposition 2.1.18. Through the Grothendieck construction, we obtain a functor

$$\mathfrak{c} : \int \Phi_{\mathcal{V}} \to \int \Phi_{\mathcal{V}} \circ * \simeq \mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathcal{V}) \times \mathrm{Set}$$

Explicitly, this functor sends a pair $(X, S)$ with $S$ a set and $X : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$ colax monoidal to the pair $(\mathfrak{c}X, S)$, where

$$\mathfrak{c}X_n = (t_S)_!(X_n) = \coprod_{a,b \in S} X_n(a, b)$$

for all $n \geq 0$. The comultiplication and counit are induced by those of $X$. Moreover, a templicial morphism $(\alpha, f) : (X, S) \to (Y, T)$ is sent to the pair $(\mathfrak{c}\alpha, f)$, where for every $n \geq 0$,

$$\mathfrak{c}\alpha_n : \coprod_{a,b \in S} X_n(a, b) \to \coprod_{x,y \in T} Y_n(x, y)$$

factors through $(\alpha_n)_{a,b} : X_n(a, b) \to Y_n(f(a), f(b))$ for all $a, b \in S$.

Note that, up to equivalence, we may consider $\mathrm{Colax}_b(\mathbf{\Delta}_f^{op}, \mathcal{V})$ as a subcategory of $\mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathcal{V}) \times \mathrm{Set}$.

**Proposition A.1.4.** *The functor* $\mathfrak{c} : \int \Phi_{\mathcal{V}} \to \mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathcal{V}) \times \mathrm{Set}$ *of Construction A.1.3 restricts to a functor*

$$\mathfrak{c} : S_{\otimes}\mathcal{V} \to \mathrm{Colax}_b(\mathbf{\Delta}_f^{op}, \mathcal{V})$$

*Proof.* Note that for any set $S$, $(t_S)_!(I_S) \simeq \coprod_{x\in S} I = F(S)$. Take an object $(X,S)$ of $\int \Phi_{\mathcal{V}}$, then the counit $\epsilon : X_0 \to I_S$ induces a morphism

$$\varphi_{(X,S)} : \mathfrak{c}X_0 = (t_S)_!(X_0) \to F(S)$$

in $\mathcal{V}$. It easily follows that $\varphi_{(X,S)}$ is a comonoid morphism which is natural in $(X,S)$. Moreover, if $(X,S)$ is a templicial object, then $\epsilon$ and thus $\varphi_{(X,S)}$ is an isomorphism. $\square$

## A.2 Decomposing monoidal categories

We now describe how to invert the comparison functor $\mathfrak{c} : S_{\otimes}\mathcal{V} \to \mathrm{Colax}_b(\mathbf{\Delta}_f^{op}, \mathcal{V})$. For this we need to "pull apart" the objects $X_n \in \mathcal{V}$ of a based colax monoidal functor to form a quiver. This goes as follows.

**Construction A.2.1.** Let $X : \mathbf{\Delta}_f^{op} \to \mathcal{V}$ be a based colax monoidal functor with comultiplication $\mu$ and base $S$. Via the isomorphism $X_0 \simeq F(S) \simeq \coprod_{a\in S} I$, we have for every $n \geq 0$, a morphism

$$\mu_{0,n,0} : X_n \to X_0 \otimes X_n \otimes X_0 \simeq \coprod_{a,b\in S} X_n$$

which assemble into a natural transformation $\mu_{0,-,0} : X \to \coprod_{a,b\in S} X$.

Then define $X(a,b)$ as the equalizer

$$X(a,b) \xrightarrow{e_{a,b}} X \underset{c_{a,b}}{\overset{\mu_{0,-,0}}{\rightrightarrows}} \coprod_{a,b\in S} X$$

in $\mathrm{Fun}(\mathbf{\Delta}_f^{op}, \mathcal{V})$, where $c_{a,b}$ is the $(a,b)$th coprojection.

**Lemma A.2.2.** *Let $X$ be a based colax monoidal functor with comultiplication $\mu$ and base $S$. Let $\nabla : \coprod_{a,b\in S} X \to X$ denote the codiagonal. Then*

$$\left(\coprod_{a,b\in S} \mu_{0,-,0}\right)\mu_{0,-,0} = \left(\coprod_{a,b\in S} c_{a,b}\right)\mu_{0,-,0} \quad \textit{and} \quad \nabla\mu_{0,-,0} = \mathrm{id}_X$$

*Proof.* Note that through the isomorphism $X_0 \simeq \coprod_{a\in S} I$, the counit $\epsilon : X_0 \to I$ becomes the codiagonal. Moreover, for all $n \geq 0$, the morphisms $\mathrm{id}_{X_0} \otimes \mu_{0,n,0} \otimes \mathrm{id}_{X_0}$ and $\mu_{0,0} \otimes \mathrm{id}_{X_n} \otimes \mu_{0,0}$ become $\coprod_{a,b} \mu_{0,n,0}$ and $\coprod_{a,b} c_{a,b}$ respectively. Thus the result follows from the coassociativity of $\mu$ and its counitality with $\epsilon$. $\square$

The previous lemma leads us to define the following.

**Definition A.2.3.** Let $\mathcal{C}$ be a category with coproducts. Let $S$ be a set and $A \in \mathcal{C}$. We denote $\iota_j : A \to \coprod_{j\in S} A$ for the $j$th coprojection and $\nabla : \coprod_{i\in S} A \to A$ for the codiagonal. A morphism $f : A \to \coprod_{i\in S} A$ is called *decomposing* if

$$\left(\coprod_{i\in S} f\right) f = \left(\coprod_{i\in S} \iota_i\right) f \quad \text{and} \quad \nabla f = \mathrm{id}_A$$

A *decomposing equalizer* is the equalizer of a decomposing morphism with a coprojection $\iota_j$ for some $j \in S$.

**Examples A.2.4.** 1. Any coprojection $\iota_j : A \to \coprod_i A$ is itself decomposing.

2. By Lemma A.2.2, the natural transformation $\mu_{0,-,0}$ is a decomposing morphism in $\mathrm{Fun}(\mathbf{\Delta}_f^{op}, \mathcal{V})$ and the equalizer of Construction A.2.1 is a decomposing equalizer.

*Remark* A.2.5. Note that because of the condition $\nabla f = \mathrm{id}_A$, a decomposing equalizer is always coreflexive.

**Lemma A.2.6.** *Let $\mathcal{C}$ be a category with coproducts and consider a decomposing morphism $f : A \to \coprod_{i\in S} A$. Then*

$$A \xrightarrow{f} \coprod_{i\in S} A \underset{\coprod_i \iota_i}{\overset{\coprod_i f}{\rightrightarrows}} \coprod_{i\in S}\coprod_{j\in S} A$$

*is a split equalizer.*

*Proof.* Let $\overline{\nabla} : \coprod_i \coprod_j A \to \coprod_j A$ denote the codiagonal which collapses the outer coproduct. Then it immediately follows that $\overline{\nabla} \coprod_i f = f\nabla$ and $\overline{\nabla} \coprod_i \iota_i = \mathrm{id}$. By hypothesis, we also have $\nabla f = \mathrm{id}_A$. $\square$

**Proposition A.2.7.** *Suppose that coproducts commute with decomposing equalizers in $\mathcal{V}$. Let $X$ be a based colax monoidal functor with base $S$, comultiplication $\mu$ and counit $\epsilon$. Then:*

1. *The canonical natural transformation*

$$(e_{a,b})_{a,b} : \coprod_{a,b\in S} X(a,b) \to X$$

   *is an isomorphism.*

2. *If coproducts are disjoint in $\mathcal{V}$, then for all $a, b \in S$, the composition*

$$X_0(a,a) \xrightarrow{e_{a,a}} X_0 \xrightarrow{\epsilon} I$$

   *is an isomorphism, and $X_0(a,b) \simeq 0$ if $a \neq b$.*

3. *If the monoidal product $- \otimes -$ of $\mathcal{V}$ preserves decomposing equalizers in each variable, then for all $k, l \geq 0$ and $a, b \in S$, the composite $\mu_{k,l}e_{a,b}$ factorizes uniquely as*

$$X_{k+l}(a,b) \xrightarrow{\mu_{k,l}^{a,b}} \coprod_{c\in S} X_k(a,c) \otimes X_l(c,b) \xrightarrow{(e_{a,c}\otimes e_{c,b})_c} X_k \otimes X_l$$

*Proof.* 1. By Example A.2.4.2, $\mu_{0,-,0}$ is decomposing and thus $\coprod_{a,b\in S} X(a,b)$ is the equalizer of $\coprod_{a,b} \mu_{0,-,0}$ and $\coprod_{a,b} c_{a,b}$. Hence by Lemma A.2.6, it is isomorphic to $X$. More precisely, for the isomorphism $\varphi : \coprod_{a,b} X(a,b) \xrightarrow{\sim} X$ we have $\coprod_{a,b} e_{a,b} = \mu_{0,-,0}\varphi$ and thus as $\epsilon$ coincides with the codiagonal $\nabla : \coprod_a I \to I$, we get $\varphi = (e_{a,b})_{a,b}$.

2. As coproducts are disjoint we have an equalizer diagram

$$I_{a,x,b} \xrightarrow{f} I \underset{\iota_{a,b}}{\overset{\iota_{x,x}}{\rightrightarrows}} \coprod_{y,z\in S} I$$

where $I_{a,x,b} = I$ if $a = b = x$ and $I_{a,x,b} = 0$ otherwise. Taking the coproduct of this diagram over all $x \in S$, we find an equalizer

$$I_{a,b} \xrightarrow{f} \coprod_{x\in S} I \underset{\coprod_x \iota_{a,b}}{\overset{\coprod_x \iota_{x,x}}{\rightrightarrows}} \coprod_{y,x,z\in S} I$$

where $I_{a,b} = I$ if $a = b$ and $I_{a,b} = 0$ if $a \neq b$. Now via the isomorphism $X_0 \simeq \coprod_x I$, $\mu_{0,0,0}$ becomes $\coprod_x \iota_{x,x}$ and thus we have an isomorphism $\varphi : X_0(a,b) \to I_{a,b}$ such that $\iota_{a,b}\varphi = e_{a,b}$. As $\epsilon$ coincides with the codiagonal $\nabla$, we find that $\varphi = \epsilon e_{a,b}$.

3. Note that since decomposing equalizers are coreflexive, and they are preserved by $- \otimes -$ in each variable, they are also preserved in both variables simultaneously. It then follows from Example A.2.4.2 that the morphism

$$\coprod_{c\in S} X_k(a,c) \otimes X_l(c,b) \xrightarrow{\coprod_c e_{a,c}\otimes e_{c,b}} \coprod_{c\in S} X_k \otimes X_l$$

is the equalizer of $\coprod_c \mu_{0,k,0} \otimes \mu_{0,l,0}$ and $\coprod_c c_{a,c} \otimes c_{c,b}$. Using the isomorphism $X_0 \simeq \coprod_c I$, we see that this is equivalently the equalizer of $\mu_{0,k,0} \otimes \mathrm{id}_{X_0} \otimes \mu_{0,l,0}$ and $c_{a,*} \otimes \mu_{0,0,0} \otimes c_{*,b}$, where

$$c_{a,*} : X_k \simeq I \otimes X_k \xrightarrow{\iota_a \otimes \mathrm{id}_{X_k}} \coprod_{a\in S} I \otimes X_k \simeq X_0 \otimes X_k$$

and similarly for $c_{*,b}$.

Now note that for the morphisms $c_{a,b} : X_{k+l} \to X_0 \otimes X_{k+l} \otimes X_0$ and $e_{a,b} : X_{k+l}(a,b) \to X_{k+l}$, we have

$$\begin{aligned}(\mu_{0,k,0} \otimes \mathrm{id}_{X_0} \otimes \mu_{0,l,0})\mu_{k,0,l}e_{a,b} &= (\mathrm{id}_{X_0} \otimes \mu_{k,0,0,0,l} \otimes \mathrm{id}_{X_0})\mu_{0,k+l,0}e_{a,b} \\ &= (\mathrm{id}_{X_0} \otimes \mu_{k,0,0,0,l} \otimes \mathrm{id}_{X_0})c_{a,b}e_{a,b} = (c_{a,*} \otimes \mu_{0,0,0} \otimes c_{*,b})\mu_{k,0,l}e_{a,b}\end{aligned}$$

Thus there is a unique $\mu_{k,l}^{a,b} : X_{k+l}(a,b) \to \coprod_{c\in S} X_k(a,c) \otimes X_l(c,b)$ such that $(\coprod_c e_{a,c} \otimes e_{c,b})\mu_{k,l}^{a,b} = \mu_{k,0,l}e_{a,b}$. Composing this equality with the codiagonal $\coprod_c X_k \otimes X_l \to X_k \otimes X_l$, the result follows.

□

**Definition A.2.8.** We call $\mathcal{V}$ *decomposing* if it satisfies the hypotheses of Proposition A.2.7, that is:

(a) coproducts commute with decomposing equalizers in $\mathcal{V}$,

(b) coproducts are disjoint in $\mathcal{V}$,

(c) the monoidal product $- \otimes -$ of $\mathcal{V}$ preserves decomposing equalizers in each variable.

**Construction A.2.9.** Let $\mathcal{V}$ be decomposing. We construct a functor

$$\mathfrak{d} : \mathrm{Colax}_b(\mathbf{\Delta}_f^{op}, \mathcal{V}) \to S_\otimes \mathcal{V}$$

Take a based colax monoidal functor $X$ of $\mathcal{V}$ with base $S$, comultiplication $\mu$ and counit $\epsilon$. From Construction A.2.1, we have a collection of functors $(X(a,b) : \mathbf{\Delta}_f^{op} \to \mathcal{V})_{a,b\in S}$, which we can regard as a functor

$$\tilde{X} : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_S$$

By Proposition A.2.7.2, we have a quiver isomorphism $\tilde{\epsilon} : \tilde{X}_0 \xrightarrow{\sim} I_S$, and the morphisms $\mu_{k,l}^{a,b}$ of Proposition A.2.7.3 combine to give a quiver morphism

$$\tilde{\mu}_{k,l} : \tilde{X}_{k+l} \to \tilde{X}_k \otimes_S \tilde{X}_l$$

It follows from the coassociativity and counitality of $\mu$ and $\epsilon$ that $\tilde{\mu}$ and $\tilde{\epsilon}$ define a strongly unital, colax monoidal structure on $\tilde{X}$ and thus $(\tilde{X}, S)$ is a templicial object in $\mathcal{V}$.

Next, let $X$ and $Y$ be based colax monoidal functors of $\mathcal{V}$ with respective bases $S$ and $T$. Let $\alpha : X \to Y$ be a morphism of based colax monoidal functors. As $\alpha$ is a monoidal natural transformation, there exist unique morphisms $\alpha^{a,b} : X(a,b) \to Y(f(a), f(b))$ such that $e_{f(a),f(b)}\alpha^{a,b} = \alpha e_{a,b}$, for all $a, b \in S$. This defines a natural transformation $\tilde{X} \to f^*\tilde{Y}$. It further follows from the monoidality of $\alpha$ that the corresponding natural transformation $\tilde{\alpha} : f_!\tilde{X} \to \tilde{Y}$ is monoidal. Hence, $(\tilde{\alpha}, f)$ is a morphism of templicial objects $\tilde{X} \to \tilde{Y}$.

If further $\beta : Y \to Z$ is a morphism of based colax monoidal functors, then by uniqueness, $(\beta \circ \alpha)^{a,b} = \beta^{f(a),f(b)} \circ \alpha^{a,b}$ for all $a, b \in S$. It follows that the assignments $X \mapsto (\tilde{X}, S)$ and $\alpha \mapsto (\tilde{\alpha}, f)$ define a functor.

**Theorem A.2.10.** *Suppose $\mathcal{V}$ is decomposing. Then we have an adjoint equivalence of categories*

$$S_\otimes \mathcal{V} \underset{\mathfrak{d}}{\overset{\mathfrak{c}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathrm{Colax}_b(\mathbf{\Delta}_f^{op}, \mathcal{V})$$

*Proof.* The isomorphism of Proposition A.2.7.1 is monoidal by 2. and 3. of the same Proposition. Moreover, it is directly seen to be natural in $X$. Thus $\mathfrak{c} \circ \mathfrak{d} \simeq \mathrm{id}$.

Let $(X, S)$ be a templicial object of $\mathcal{V}$. We have a functor $X(a,b) : \mathbf{\Delta}_f^{op} \to \mathcal{V}$ for every $a, b \in S$. As coproducts are disjoint in $\mathcal{V}$, the equalizer of $\iota_{a,b}, \iota_{c,d} : X(c,d) \to \coprod_{x,y\in S} X(c,d)$ in $\mathrm{Fun}(\mathbf{\Delta}_f^{op}, \mathcal{V})$ is $X(a,b)$ if $(c,d) = (a,b)$ and 0 otherwise. Because coproducts commute with decomposing equalizers, we get an equalizer diagram

$$X(a,b) \xrightarrow{\iota_{a,b}} \coprod_{c,d\in S} X(c,d) \underset{\coprod_{c,d}\iota_{a,b}}{\overset{\coprod_{c,d}\iota_{c,d}}{\rightrightarrows}} \coprod_{c,d\in S} \coprod_{x,y\in S} X(c,d)$$

Now $\coprod_{c,d} X(c,d)$ is the functor underlying $\mathfrak{c}(X,S)$ and the morphisms $\coprod_{c,d}\iota_{c,d}$ and $\coprod_{c,d}\iota_{a,b}$ correspond to the induced morphisms $\mu_{0,-,0}$ and $c_{a,b}$ on $\mathfrak{c}(X,S)$ respectively. Consequently, we have an isomorphism between the underlying functors of $(X,S)$ and

$\mathfrak{dc}(X, S)$. It follows from the definitions that this isomorphism is monoidal and that it is natural in $(X, S)$. Therefore $\mathfrak{d} \circ \mathfrak{c} \simeq \mathrm{id}$.

Finally, the triangle identities are easily verified. $\square$

We finish this section by giving some examples of monoidal categories that are decomposing, and thus for which Theorem A.2.10 is applicable.

**Example A.2.11.** In a cartesian category $\mathcal{V}$, the product $- \times -$ commutes with all equalizers. So if we assume that coproducts are disjoint and commute with equalizers, then $\mathcal{V}$ is decomposing.

This is the case for the categories $\mathrm{Set}$ of sets, $\mathrm{Top}$ of topological spaces, $\mathrm{Cat}$ of small categories and $\mathrm{Poset}$ of posets for example.

**Lemma A.2.12.** *Let $\mathcal{C}$ be a category enriched over abelian groups. Then any decomposing equalizer in $\mathcal{C}$ is split.*

*Proof.* Let $f : A \to \bigoplus_{i \in S} A$ be a decomposing morphism in $\mathcal{C}$ and fix $j \in S$. Consider the equalizer $e : E \to A$ of $f$ and $\iota_j$. Then for the $j$th projection $p : \bigoplus_{i \in S} A \to A$ we have $p\iota_j = \mathrm{id}_A$ and

$$fpf = p' \left( \bigoplus_{i \in S} f \right) f = p' \left( \bigoplus_{i \in S} \iota_i \right) f = \iota_j p f$$

where $p' : \bigoplus_{i,k} A \to \bigoplus_k A$ is the projection onto the component $i = j$. So there exists a unique $s : A \to E$ such that $es = pf$. Then, $ese = pfe = p\iota_j e = e$ and thus $se = \mathrm{id}_E$ because $e$ is a monomorphism. $\square$

**Proposition A.2.13.** *If $\mathcal{V}$ is enriched over abelian groups, then $\mathcal{V}$ is decomposing.*

*Proof.* By Lemma A.2.12, decomposing equalizers in $\mathcal{V}$ are split equalizers and are thus preserved by all functors. In particular, both the coproduct functor $\mathcal{V}^S \to \mathcal{V}$ and the monoidal product $- \otimes -$ preserve decomposing equalizers. Further, in an $\mathrm{Ab}$-enriched category, coproducts are always disjoint. $\square$

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