Weak units, higher operads, and deformations of monoidal categories

Zwakke eenheden, hogere operads, en deformaties van monoïdale categorieën

Piergiorgio Panero

Supervisors **Prof. dr. Boris Shoikhet — Prof. dr. Wendy Lowen**

Thesis submitted in fulfilment of the requirements for the degree of Doctor of Science: Mathematics
Faculty of Science — Antwerpen, 2022

University of Antwerp

University of Antwerp

Faculty of Science
Doctor of Science: Mathematics

Weak units, higher operads, and deformations of monoidal categories

Zwakke eenheden, hogere operads, en deformaties van monoïdale categorieën

Thesis submitted in fulfilment of the requirements for the degree of
Doctor of Science: Mathematics
at University of Antwerp

Piergiorgio Panero

Supervisors
Prof. dr. Boris Shoikhet
Prof. dr. Wendy Lowen

Antwerpen, 2022

Jury

Chairman
Prof. dr. Sonja Hohloch, University of Antwerp, Belgium

Supervisors
Prof. dr. Boris Shoikhet, Euler International Mathematical Institute, Russia
Prof. dr. Wendy Lowen, University of Antwerp, Belgium

Members
Prof. dr. Michail Batanin, Mathematical Institute of Charles University, Czech Republic
Prof. dr. Karim Johannes Becher, University of Antwerp, Belgium
Prof. dr. Bernhard Keller, Université Paris Cité, France
Prof. dr. Dmitry Tamarkin, Northwestern University, USA

Contact
Piergiorgio Panero
University of Antwerp
Faculty of Sciences
Middelheimlaan 1, 2020 Antwerpen, België

Contents

Acknowledgements

I cannot express how grateful I am to you, Boris. Thank you for all the time you dedicated to me, teaching me how to do research, and thank you for all our discussions, both mathematical and not. I owe you a lot, professionally and personally.

Thank you, Wendy, for all the support you gave me in the last year and for always sharing your profound mathematical knowledge.
I would like to thank all the members of the jury for their careful reading and their relevant remarks and suggestions: with your help, this work has improved considerably.

I would also like to thank Tania Beyers and Robby Standaert, without whom I would never have overcome all the bureaucratic problems.

Thanks to all my colleagues at the Math Department: the Category people, in particular Arne, Julia, Lander and Matt, for having such a good time during seminars and conferences (too few, alas!) and for organizing the Kerodon for so long; the Algebra people, especially Marco and Nicolas, for sharing endless discussions on a huge variety of topics; the Differential people, Tom and Sándor, for the indubitable help and support during these years.

Eterna gratitudine al gruppo ANFFAS di Torino: Francesco e Martino, grazie di cuore per essere sempre al mio fianco e per la vostra sincera amicizia: avete reso più lieti questi quattro anni.
Un grazie ai compagni della VªC, Carlo, Davide, Federico, Mattia e Stefano, per accogliermi ogni volta e farmi sentire a casa.
Un caloroso grazie alla famiglia dell'A4 Verzuolo e soprattutto a Franco e Giovanni, compagni infaticabili su per i monti piemontesi (e non!).

Un enorme ringraziamento va a tutta la mia famiglia, per il supporto costante e l'amore incondizionato.
Grazie Mamma, per avermi fatto scoprire tu per prima l'amore per la matematica: non sarei nulla senza te.
Grazie Zia, per tutto il sostegno e tutte le chiamate in questi anni lontano da casa: mi hai fatto sentire come se non me ne fossi mai andato.
Grazie Mauro, per la genuina bontà d'animo e l'enorme generosità di cuore.

Et enfin, merci à toi, Noémie, pour toutes ces années passées ensemble, pour tout ton soutien, pour tout ton amour et pour avoir toujours cru en moi, même dans les moments les plus dures : cette thèse t'est dédiée.

Introduction

Gli uomini universalmente, volendo vivere, conviene credano la vita bella e pregevole; e tale la credono; e si adirano contro chi pensa altrimenti. Perché in sostanza il genere umano crede sempre, non il vero, ma quello che è, o pare che sia, più a proposito suo. Il genere umano, che ha creduto e crederà tante scempiataggini, non crederà mai né di non saper nulla, né di non esser nulla, né di non aver nulla a sperare.

G. Leopardi,
Operette morali

Differential graded categories (or dg-categories for short) are fundamental objects in algebraic geometry and higher category theory. After their introduction by Kelly [Ke] in the context of homological algebra, Bondal-Kapranov [BK] used dg-categories in order to “enhance” triangulated categories and since then the study of dg-categories and their category $\mathbb{C}_{dg}(k)$ has increased intensively, see e.g. [Dr], [Kel2], [Tab], [To2].

Dg-categories

Let us recall the basic data of a dg-category $\mathcal{C}$ over a commutative ring k: a set of objects $Ob(\mathcal{C})$ and, for any $X, Y \in Ob(\mathcal{C})$, a cochain complex of k-modules $\mathcal{C}(X, Y)$ of morphisms from X to Y. These data alone form a dg-quiver, and to turn this into a dg-category one needs more structure: for any triple of objects X, Y, Z a strictly associative composition operation

$$\circ_{X,Y,Z} \colon \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

and for any object $X \in Ob(\mathcal{C})$ a degree 0 morphism $\mathrm{id}_X \in \mathcal{C}(X, X)$, which is the strict identity for the composition:

$$f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$$

for any $f \in \mathcal{C}(X, Y)$. By the Leibniz rule, it follows that $d(\mathrm{id}_X) = 0$, i.e. the identity morphisms are closed.
But why should one care about these gadgets?

One motivation comes from algebraic geometry: given a scheme X, one can recover its geometrical and higher cohomological informations from the derived category $\mathcal{D}(QC(X))$

of quasi-coherent sheaves on X. This latter is obtained from the category $\mathrm{K}(QC(X))$ of complexes of quasi-coherent sheaves, by localizing it over the quasi-isomorphisms.
Moreover, derived categories are example of triangulated category, whose theory was introduced and studied by Verdier. However, many geometrical constructions behave badly in this setting: functor categories and tensor products of triangulated categories are not triangulated anymore, the cone is not functorial, et cetera.

One way to overcome these problems is to consider a "dg-enhancement" $\mathcal{C}$ of a triangulated category $\mathcal{T}$, i.e. a (pretriangulated) dg-category $\mathcal{C}$ such that $H^0(\mathcal{C}) \to \mathcal{T}$ is an equivalence of triangulated categories. Such a dg-enhancement has been shown to exist and to be unique in many geometrical examples. In addition to that, such a dg-category allows to compute many invariants (K-theory, Hochschild co/homology), that the derived category alone cannot.

Now let us turn back to the definition of the derived category $\mathcal{D}(QC(X))$: it is obtained from the category of complexes $\mathrm{K}(QC(X))$ by formally inverting the quasi-isomorphisms of complexes. This procedure is known as the Gabriel-Zisman localization and it is relevant in homotopy theory and homological algebra. Whenever one has a category with weak equivalences $(\mathcal{C}, \mathcal{W})$, this construction gives, if it exists, the localized category $\mathcal{C}[\mathcal{W}^{-1}]$. Thus $\mathcal{D}(QC(X)) = \mathrm{K}(QC(X))[\mathcal{W}^{-1}]$, where $\mathcal{W}$ is the class of quasi-isomorphisms.

However, the Gabriel-Zisman localization is known to be problematic, as the resulting category may not be locally small. One method to avoid this issue is to construct a model structure on the category $\mathcal{C}$ of interest, specifying two other classes of morphisms (cofibrations and fibrations) in addition to the weak equivalences $\mathcal{W}$, and showing that these three classes satisfy certain properties and the small co/completeness of $\mathcal{C}$. Quillen proved in [Q] that to any model category $\mathcal{C}$, one can assign an homotopy category $\mathrm{Ho}(\mathcal{C})$, and this turns out to be equivalent to the localized category $\mathcal{C}[\mathcal{W}^{-1}]$ obtained via the Gabriel-Zisman construction.

Tabuada in [Tab] showed that $\mathbb{C}_{dg}(k)$ admits a model structure where the weak equivalences are the quasi-equivalences of dg-categories, paving the way for Toën to develop derived Morita theory.

Weakly unital dg-categories

From now on let us assume $\Bbbk$ to be a field of characteristic 0. Along with $\mathbb{C}_{dg}(\Bbbk)$, people introduced a "relaxed" version of dg-categories, known as A_∞-categories: if a dg-category is a dg-quiver with an associative and unital composition operation, a (strictly unital) A_∞-category is a graded-quiver with (unital) composition which is associative only up to homotopy. This amounts to ask for n-ary operations m_n, $n \geq 1$, satisfying some constraints which in low dimension read as:

$$\begin{aligned} m_1 \circ m_1 = 0, &\quad \text{i.e. } m_1 \text{ is a differential,} \\ m_2(\mathrm{id} \otimes m_1) + m_2(m_1 \otimes \mathrm{id}) = m_1(m_2) &\quad \text{i.e. } m_1 \text{ is a derivation w.r.t the composition } m_2, \\ m_2(\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) = [m_1, m_3] &\quad \text{i.e. } m_2 \text{ is associative up to homotopy,} \end{aligned}$$

where $[-,-]$ is the commutator.

In particular, any dg-category can be seen as an A_∞-category by setting m_1 to be the differential, m_2 the composition, and all the higher m_n to be 0, and so there is a canonical embedding: $i\colon \mathbb{C}_{dg}(\Bbbk) \to \mathbb{C}_{A_\infty}(\Bbbk)$. The drawback is that the category of (unital) A_∞-categories $\mathbb{C}_{A_\infty}(\Bbbk)$ does *not* admit a model category structure, since it does not have all coequalizers.

By setting the unitality condition to hold up to homotopy as well, Fukaya introduced weakly unital A_∞-categories in the context of Homological mirror symmetry [Fu]. Since then, weakly unital dg- and A_∞-categories have been studied by many authors, e.g. [LyMa], [Ly1], [LH], [KS2], [COS] among others.
But what does it mean in the dg-setting? Any object $X \in Ob(\mathcal{C})$ has its own closed morphism id_X of degree 0, but

$$\mathrm{id}_Y \circ f \neq f \neq f \circ \mathrm{id}_X\,.$$

These equalities hold only up to homotopy, i.e. there exist morphisms $p(f,1)$, $p(1,f) \in \mathcal{C}(X,Y)$ such that

$$d(p(f,1)) = f - f \circ \mathrm{id}_X \quad d(p(1,f)) = f - \mathrm{id}_Y \circ f$$

There are many ways to phrase this homotopy unitality out and up to now there are three different definitions of a weakly unital A_∞- (or dg-) category, which are due to Fukaya, to Lyubashenko, and to Kontsevich-Soibelman. It was proven in [LyMa] that the three definitions are equivalent, which means that if a given A_∞-category is weakly unital in one sense, it is also weakly unital in the other ones. Nevertheless, the three categories of weakly unital A_∞-categories are not equivalent. However, their homotopy categories are expected to be equivalent, and equivalent to the homotopy category of strictly unital dg-categories (and in fact Theorem 2.2 of [COS] confirms this claim for the category of Lyubashenko weakly unital dg-categories, even though it seemingly does not admit a model structure, and the proof in loc.cit. is direct).

Despite the seeming oddity of weakly unital dg-categories, many algebraic constructions give rise to weakly ones. The simplest example is the Cobar-Bar resolution $R(A) = \mathrm{Cobar}(\mathrm{Bar}(A))$ of a dg-algebra A (or of a dg-category $\mathcal{A}$): given a unital dg-algebra A, $R(A)$ is a Kontsevich-Soibelman weakly unital dg-algebra, see Example 2.1.6. Furthermore, the Cobar-Bar construction is a very natural resolution and one would like to consider it as a cofibrant replacement of A, when computing Hom-sets in the homotopy category. However, $\mathbb{C}_{dg}(\Bbbk)$ does not give room to such a construction!

Certainly one could consider $\mathrm{Hom}(\mathrm{Cobar}(\mathrm{Bar}(A)), B)$ in the non-unital setting, and then he would get the set of *all* A_∞-maps from A to B (or A_∞-functors, for the case of dg-categories), where an A_∞-map F between dg-algebras A and B is a sequence of homogeneous maps $F_n\colon A^{\otimes n} \to B$, $n \geq 1$, of degree $1-n$, such that for all $n \geq 1$ we have:

$$[d, F_n] = \sum_{i+j=n} \pm m_B(F_i, F_j) + \sum_{\kappa+\ell=n-2} \pm F_{n-1}(\mathrm{id}^{\otimes\kappa} \otimes m_A \otimes \mathrm{id}^{\otimes\ell}).$$

Looking at these equations for $n = 1, 2$, we see that F_1 induces a morphism of complexes from A to B, compatible with the algebra structure up to an homotopy given by F_2.

However, it is well-known [LH] that the correct Hom-set in the homotopy category is defined via the *unital* A_∞-maps (respectively, unital A_∞-functors), i.e. those A_∞-maps

$F: A \to B$ such that $F_1(1_A) = 1_B$ and $F_k(\dots, 1_A, \dots) = 0$ for $k \geq 2$. The reason is that one has to take $\mathrm{Hom}(\mathrm{Cobar}(\mathrm{Bar}(A)), B)$ in the category of (Kontsevich-Soibelman) *weakly unital* dg-categories, see Definition 2.1.1, that gives rise exactly to the unital A_∞-functors $A \to B$, see Example 2.1.7.

Another motivating example is a further generalisation of the twisted tensor product of small dg-categories [Sh2], [Sh3], which is supposed to have better homotopical and monoidal properties, and which exists only in the weakly unital context. Therefore, it would be beneficial to have a model category structure on the category $\mathbb{C}_{dgwu}(\Bbbk)$ of small weakly unital dg-categories, and to show that this is Quillen equivalent to the model category of $\mathbb{C}_{dg}(\Bbbk)$ small dg-categories. In the first part of this work, we completely solve this problem: we endow $\mathbb{C}_{dgwu}(\Bbbk)$ with a model structure where the weak equivalences are the quasi-equivalences (as in the Tabuada model structure on $\mathbb{C}_{dg}(\Bbbk)$), and we show that the two model categories $\mathbb{C}_{dgwu}(\Bbbk)$ and $\mathbb{C}_{dg}(\Bbbk)$ are Quillen equivalent.

Note that, among the three definitions of a weakly unital dg-category recalled above, only the one given by Kontsevich and Soibelman [[KS2], Sect. 4.2] seems to admit a model structure.

Monoidal $\Bbbk$-linear categories

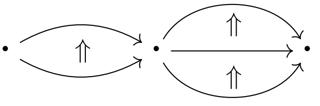
In the second half of this thesis, we address a second problem. Given a monoidal $\Bbbk$-linear category $\mathcal{C}$, one has two deformation complexes attached to it: the first is the well-known Hochschild cohomological complex $\mathrm{CH}^*(\mathcal{C}, \mathcal{C})$ (associated to any $\Bbbk$-linear or dg-category) whose cohomology governs the deformations of the $\Bbbk$-linear structure (i.e. the composition); the second is the Davydov-Yetter complex $\mathrm{C_{DY}}^*(\mathrm{Id}_\mathcal{C})$, introduced independently by Davydov [Da], Crane and Yetter [CY], [Ye1], [Ye2] at the end of the nineties. The cohomology of this complex describes the infinitesimal deformations of the monoidal structure of a $\Bbbk$-linear monoidal functor $F: \mathcal{C} \to \mathcal{D}$ or the tensor structure of a $\Bbbk$-linear monoidal category (when $F = \mathrm{Id}_\mathcal{C}$). Both these complexes can be obtained as the totalization of functors from Δ to $\mathrm{Ch}(\Bbbk)$, where Δ is the subcategory of Cat whose objects are finite ordinals $[n]$ (remember that any partial order set defines a category in a canonical way), and functors between $[n]$ and $[m]$ are order-preserving maps.

Thus, in order to control the complete deformation theory of $\mathcal{C}$, one should "pack" together these two complexes. In this work we do so by constructing a functor (See Definition 3.2.3)

$$A(F, F): \Theta_2 \to \mathrm{Vect}(\Bbbk)$$

for each $\Bbbk$-linear monoidal functor $F: \mathcal{C} \to \mathcal{D}$. Θ_2 should be considered as the analogue of Δ for 2-categories (for a precise definition of what Θ_n are, for each $n \geq 1$, see Section 1.4). The objects of Θ_2 are tuples $([n]; [\ell_1], \dots, [\ell_n])$, with $[n], [\ell_i] \in Ob(\Delta)$ and morphisms between two such tuples $([n]; [\ell_1], \dots, [\ell_n])$ and $([m]; [\kappa_1], \dots, [\kappa_m])$ are again tuples $(\phi; \phi^i_j)$, where $\phi: [n] \to [m]$ and $\phi^i_j: [\ell_i] \to [\kappa_j]$ are morphisms in Δ.

As an example of an object of Θ_2, $S = ([2]; [1], [2])$ represents the free 2-category over the following diagram:



Given an object $T = ([k]; [n_1], \dots, [n_k]) \in Ob(\Theta_2)$, the value of $A(F,F)_T$ is defined as a ***subcomplex*** of:

$$\hat{A}(F,F)_T = \prod_{\substack{X_i \in \mathcal{C} \\ Y_i \in \mathcal{C}}} \underline{\mathrm{Hom}}_{\Bbbk}\Big(\mathrm{Mor}_{n_0}(X_0,Y_0) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \mathrm{Mor}_{n_k}(X_k,Y_k),\ \mathcal{D}(FX_0 \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} FX_k, FY_0 \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} FY_k)\Big) \tag{0.0.1}$$

where $\mathrm{Mor}_n(X_0, X_n)$ is the $\Bbbk$-vector space defined as:

$$\mathrm{Mor}_n(X_0, X_n) = \bigoplus_{X_1,\dots,X_{n-1} \in Ob(\mathcal{C})} \mathcal{C}(X_{n-1}, X_n) \otimes \mathcal{C}(X_{n-2}, X_{n-1}) \otimes \cdots \otimes \mathcal{C}(X_0, X_1).$$

This functor is such that its 2-cocellular totalization $\mathrm{Tot}_{\Theta_2}(A(F,F))$ is the desired deformation complex.

Example 0.0.1. Let $S = ([2]; [1], [2])$ as above, then a S-diagram in our monoidal $\Bbbk$-linear category $\mathcal{C}$ is of the following form:

$$\begin{array}{cc} & X_{2,2} \\ X_{1,1} & \uparrow f_{2,2} \\ f_{1,1}\uparrow & X_{2,1} \\ X_{1,0} & \uparrow f_{2,1} \\ & X_{2,0} \end{array}$$

where we interpret the "1-morphisms" of Θ_2 as objects $X_{i,j}$ of our monoidal dg-category $\mathcal{C}$ and the "2-morphisms" of Θ_2 as 1-morphisms $f_{i,j}$ of $\mathcal{C}$. A cochain $\Xi \in A(F,F)_S$ produces a morphism in $\mathcal{D}$ out of any such S-diagram in $\mathcal{C}$:

$$\begin{array}{cc} & X_{2,2} \\ X_{1,1} & \uparrow f_{2,2} \\ f_{1,1}\uparrow & X_{2,1} \\ X_{1,0} & \uparrow f_{2,1} \\ & X_{2,0} \end{array} \quad \xmapsto{\Xi} \quad \begin{array}{c} FX_{1,1} \otimes FX_{2,2} \\ \uparrow \\ FX_{1,0} \otimes FX_{2,0} \end{array}$$

The idea is to pack the Hochschild cochain complex "vertically", i.e. considering the restriction along the vertical embedding of Δ into Θ_2

$$v \colon \Delta \to \Theta_2 \colon [n] \longmapsto ([1]; [n]),$$

we get:

$$(\hat{A}(F,F) \circ v)([n]) = \prod_{X_i \in Ob(\mathcal{C})} \underline{\mathrm{Hom}}_{\Bbbk}\left(\mathrm{Mor}_n(X_0, X_n), \mathcal{D}(FX_0, FX_n)\right)$$

which is nothing but $\mathrm{CH}^n(\mathcal{C},\mathcal{C})$ when $F=\mathrm{Id}_{\mathcal{C}}$.

Similarly we pack the Davydov-Yetter complex "horizontally" i.e. considering the restriction along the horizontal embedding of Δ into Θ_2

$$h\colon \Delta \to \Theta_2\colon [n] \longmapsto ([n];[0],\dots,[0]),$$

we get:

$$(\hat{A}(F,F)\circ h)([n]) = \prod_{X_i\in Ob(\mathcal{C})} \mathcal{D}(FX_1\otimes_{\mathcal{D}}\cdots\otimes_{\mathcal{D}} FX_n, FX_1\otimes_{\mathcal{D}}\cdots\otimes_{\mathcal{D}} FX_n),$$

which is the "unnatural"[1] version of

$$\mathrm{C_{DY}}^n(F) \coloneqq End(F^{\otimes n}),$$

the endomorphism algebra of natural transformations from $F^{\otimes n}$ to itself.

As we are interested in the deformation theory of $\Bbbk$-linear monoidal categories, the general "deformation problem mantra" tells us that the cohomology of our complex $\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id}_{\mathcal{C}},\mathrm{Id}_{\mathcal{C}}))$ (appropriately shifted) should be a dg-Lie algebra (or its homotopy analogue L_∞-algebra). This structure could be inherited by an homotopy n-algebra structure on $\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id}_{\mathcal{C}},\mathrm{Id}_{\mathcal{C}}))$ (i.e. an algebra over the $\mathrm{Ch}_\bullet(\mathrm{E}_n,\Bbbk)$, the chain operad of n-disks).
It was shown by Batanin and Davydov in [BD] that the Davydov-Yetter complex $\mathrm{C_{DY}}^*(\mathcal{C})$ is an homotopy 3-algebra and therefore the infinitesimal deformations of the tensor structure are controlled by $H^3_{DY}(\mathcal{C})$. The Hochschild cochain complex $\mathrm{CH}^*(\mathcal{C},\mathcal{C})$ is known to be an homotopy 2-algebra by the Deligne conjecture (now a theorem), and so the infinitesimal deformations of the $\Bbbk$-linear structure are controlled by $\mathrm{HH}^2(\mathcal{C},\mathcal{C})$.
This asymmetry is a feature rather than a bug in our complex, by the following argument: as the dimension of a Θ_2 object $T=([k];[n_1],\dots,[n_k])$ is defined as

$$dim(T) \coloneqq k+\sum_{i=1}^{k} n_i,$$

it follows that both Davydov-Yetter $\mathrm{C_{DY}}^3(\mathcal{C})$ and Hochschild cochains $\mathrm{CH}^3(\mathcal{C},\mathcal{C})$ sit in $\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id}_{\mathcal{C}},\mathrm{Id}_{\mathcal{C}}))^3$ (since the horizontal embedding h maps [3] to ([3]; [0], [0], [0]) and the vertical embedding v maps [2] to ([1]; [2])). As a consequence to this fact, we expect $\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id}_{\mathcal{C}},\mathrm{Id}_{\mathcal{C}}))$ (respectively, $\mathrm{Tot}_{\Theta_2}(A(F,F))$) to be an homotopy 3-algebra (respectively, an homotopy 2-algebra).

As we said above, Batanin and Davydov proved that $\mathrm{C_{DY}}^*(\mathrm{Id}_{\mathcal{C}})$ (respectively, $\mathrm{C_{DY}}^*(F)$) is an homotopy 3-algebra (respectively, an homotopy 2-algebra), by showing that the corresponding cosimplicial monoids are 2-commutative (respectively, 1-commutative) (see Subsection 1.3.3). Taking inspiration from their work, we consider the Δ-totalization $\mathrm{Tot}_\Delta(A(F,F))$ of $A(F,F)$, which is a cosimplicial monoid, and show that it is 1-commutative. This implies that $\mathrm{Tot}_{\Theta_2}(A(F,F))$ is an homotopy 2-algebra. Unfortunately, in the case $F=\mathrm{Id}_{\mathcal{C}}$ the cosimplicial monoid $\mathrm{Tot}_\Delta(A(\mathrm{Id}_{\mathcal{C}},\mathrm{Id}_{\mathcal{C}}))$ fails to be 2-commutative.

[1] This is a delicate point of the story: if we did not consider the sub-complex $A(F,F)$ as we do in Definition 3.2.3, we would fail to have a 2-cocellular object or, alternatively, $\hat{A}(F,F)$ does not define a functor $\Theta_2\to\mathrm{Vect}(\Bbbk)$. For a more detailed explanation, see Example 3.2.4 and Remark 3.2.5.

The reason for this failure is due to the nature of our cochains: in the Davydov-Yetter complex case the cochains are natural transformations, and the naturality is a crucial condition for the 2-commutativity to be satisfied. As our cochains lack full naturality, our cosimplicial monoid $\mathrm{Tot}_\Delta(A(\mathrm{Id}_\mathcal{C}, \mathrm{Id}_\mathcal{C}))$ is not 2-commutative, but it satisfies an "homotopy" 2-commutativity. However, this concept is hard to phrase out precisely and will be object of further study.

Overview

Chapter 1 provides a basic and essential summary of the theory and properties of the mathematical tools that we will employ in this work: none of the results present there is new. In Section 1.1 we give the basic definitions of dg- and $\Bbbk$-linear category, dg- and $\Bbbk$-linear functors and describe the family of dg-functors considered by Tabuada in [Tab] to construct the closed model structure of $\mathbb{C}_{dg}(\Bbbk)$. In Section 1.2 we revise the basic notions of model categories, together with some examples. In Section 1.3 we give a brief introduction to the world of operads and higher operads, together with their algebras and some examples, notably the lattice paths operad $\mathcal{L}$ and the paths operad $\mathcal{M}$. In Section 1.4 we describe the three equivalent ways of defining the categories Θ_n. In Section 1.5 we describe the basics of monad theory. In Section 1.6 we recall the basics of monoidal categories and fix the notation.

Chapter 2 is based on the two articles [PS1] and [PS2] by the author and the supervisor. In Section 2.1 we define Kontsevich-Soibelman weakly unital dg-categories and their category $\mathbb{C}_{dgwu}(\Bbbk)$. In Section 2.2 we prove the small co/completeness of $\mathbb{C}_{dgwu}(\Bbbk)$ (Proposition 2.2.9), together with the monadicity Theorem 2.2.11 for the monad T induced by the dg-operad $\mathcal{O}'$ (telling us that $\mathbb{C}_{dgwu}(\Bbbk)$ is equivalent to the category of T-algebras in $\mathbb{G}_{dgu}(\Bbbk)$). In Section 2.3 we construct a technical and essential tool in order to prove the closed model structure on $\mathbb{C}_{dgwu}(\Bbbk)$, i.e. the pretriangulated hull of a weakly unital dg-category (Definition 2.3.4). In Section 2.4 we prove the first main theorem of the chapter:

Theorem 0.0.2 (proven in Theorem 2.4.14)**.** *For a field $\Bbbk$, there is a cofibrantly generated Quillen model structure on $\mathbb{C}_{dgwu}(\Bbbk)$.*

We do so by introducing a weakly unital replacement of the Kontsevich dg-category $\mathcal{K}$ (Subsection 2.4.1). In Section 2.5 we prove the second main theorem of the chapter:

Theorem 0.0.3 (proven in Prop. 2.5.2 and Theorem 2.5.3)**.** *There is a Quillen equivalence*

$$L\colon \mathbb{C}_{dgwu}(\Bbbk) \rightleftarrows \mathbb{C}_{dg}(\Bbbk)\colon R$$

where $\mathbb{C}_{dg}(\Bbbk)$ is endowed with the Tabuada closed model structure [Tab].

In Section 2.6 we provide an explicit and canonical cofibrant resolution of a unital dg-algebra in Proposition 2.6.2. Sections 2.7 and 2.8 serve as an appendix to the chapter: in the former we give a proof of a technical Proposition and in the latter we compute the cohomology of the dg-operad $\mathcal{O}'$, proving Theorem 2.2.3. This is a fundamental result in the proof of the Quillen equivalence stated above: if there were not a quasi-isomorphism of operads $\mathcal{O}' \to Assoc_+$, $\mathbb{C}_{dgwu}(\Bbbk)$ would not be Quillen equivalent to $\mathbb{C}_{dg}(\Bbbk)$.

Chapter 3 is based on the preprint [PS3] by the author and the supervisor. In Section 3.1 we recall some basics of deformation theory and define the complexes of our interest, namely the Hochschild and the Davydov-Yetter complexes, describing the deformations of an associative algebra (or dg-category) and the deformations of the tensor structure of a monoidal $\mathbb{k}$-linear category, respectively. In Section 3.2 we define our main object of interest: a 2-cocellular vector space, named $A(F, F)$, associated to a monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$, for $\mathcal{C}, \mathcal{D}$ $\mathbb{k}$-linear monoidal categories. We also give an explicit list of degeneracy and face maps in Θ_2 in Subsection 3.2.1, together with their action on $A(F, F)$ in Subsection 3.2.6. Moreover we describe explicitly the totalization of a 2-cocellular chain complex $X_\bullet$ in Subsection 3.2.2. In Section 3.3 we introduce an abelian category 2-Bimod($\mathcal{C}$) of 2-bimodules over any $\mathbb{k}$-linear bicategory $\mathcal{C}$. Moreover we give an intrinsic homological algebra interpretation of our complex $A(\mathcal{C}, \mathcal{D})(F, G)(\eta, \theta)$ in Subsection 3.3.3. In Section 3.4 we explicitly compute a relative Δ-totalization of $A(F, F)$ and we prove the main theorem of the chapter:

Theorem 0.0.4 (proven in Theorem 3.4.9)**.** *Let $\mathcal{C}, \mathcal{D}$ be $\mathbb{k}$-linear monoidal categories, $F\colon \mathcal{C} \to \mathcal{D}$ a $\mathbb{k}$-linear monoidal functor. Then the 2-cocellular totalization* $\mathrm{Tot}_{\Theta_2}(A(F, F))$ *has a structure of an algebra over an operad homotopically equivalent to* $\mathrm{Ch}_\bullet(\mathrm{E}_2; \mathbb{k})$.

In Section 3.5 we describe the totalizations of $A(F, F)$ and $A(\mathrm{Id}_\mathcal{C}, \mathrm{Id}_\mathcal{C})$ as deformation complexes of monoidal functors and monoidal $\mathbb{k}$-linear categories, giving an example when these deformations are relevant. In addition, we prove the following theorems:

Theorem 0.0.5 (proven in Theorem 3.5.3)**.** *Let $\mathcal{C}$ be a $\mathbb{k}$-linear (or a dg- over $\mathbb{k}$) monoidal category. The third cohomology* $H^3(\mathrm{Tot}_{\Theta_2} A(\mathrm{Id}_\mathcal{C}, \mathrm{Id}_\mathcal{C}))$ *is isomorphic to the equivalence classes of infinitesimal deformations of the monoidal $\mathbb{k}$-linear (or dg-) category $\mathcal{C}$.*

and

Theorem 0.0.6 (proven in Theorem 3.5.5)**.** *Let $\mathcal{C}, \mathcal{D}$ be $\mathbb{k}$-linear (or dg- over $\mathbb{k}$) monoidal categories, $F\colon \mathcal{C} \to \mathcal{D}$ a monoidal dg-functor. The second cohomology* $H^2(\mathrm{Tot}_{\Theta_2} A(F, F))$ *is isomorphic to the equivalence classes of infinitesimal deformations of the functor F.*

Sections 3.6 and 3.7 serve as an appendix to the chapter: in the former we explicitly give a list of relations for the degeneracy and face maps in Θ_2, while in the latter we give a proof of a technical Proposition.

Chapter 4 discusses a possible way to show that $\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id}_\mathcal{C}, \mathrm{Id}_\mathcal{C}))$ is a homotopy 3-algebra. We do so by introducing a 2-dimensional generalization of the lattice path operad.

Notations and Assumptions

(i) We denote by calligraphic letters $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{D}$ generic (enriched) categories and by capital letters $X, Y, Z, T, \ldots$ the objects of a category. We will denote by $\mathcal{C}(X, Y)$ the hom-object of an enriched category $\mathcal{C}$ between two objects X, Y and by capital letters F, G generic (enriched) functors from $\mathcal{C}$ to $\mathcal{D}$.

(ii) We denote by $\mathcal{E}$ a symmetric monoidal (closed) category.

(iii) We denote by $\mathcal{O}$ a generic (non-)symmetric operad.

We assume that $\Bbbk$ is a field of characteristic 0.

List of Symbols

$\mathrm{Ch}(\Bbbk)$ - the category of cochain complexes of vector spaces over $\Bbbk$

$\mathrm{Vect}(\Bbbk)$ - the category of vector spaces over $\Bbbk$

$\mathbb{C}_{dg}(\Bbbk)$ - the category of *small* dg-categories over $\Bbbk$

$\mathbb{C}_{dgwu}(\Bbbk)$ - the category of *small* weakly unital dg-categories over $\Bbbk$

$\mathbb{C}(\Bbbk)$ - the category of *small* $\Bbbk$-linear categories

$\mathbb{C}_{A_\infty}(\Bbbk)$ - the category of *small* A_∞-categories over $\Bbbk$

$\mathbb{G}_{dg}(\Bbbk)$ - the category of dg-quivers over $\Bbbk$

$\mathbb{G}_{dgu}(\Bbbk)$ - the category of unital dg-quivers over $\Bbbk$

$\mathcal{K}$ - the Kontsevich strictly unital dg-category

$\mathcal{K}'$ - the Kontsevich weakly unital dg-category

Δ - the category of finite ordinals $[n]$ and order-preserving morphisms

$\mathcal{I}$ - the category of finite intervals $\langle n\rangle$ and morphisms preserving both the order and the endpoints

Θ_2 - the category of Θ_2-objects and Θ_2-morphisms

Ord_n - the category of n-ordinals

Ω_n - the category of n-stage trees

$\Omega_n^{(p)}$ - the category of pruned n-stage trees

Cat - the category of *small* strict categories

$\mathrm{Cat}_{*,*}$ - the category of bipointed categories

Cat_n - the category of *small* strict n-categories

Disk_n - the category of n-disks

$\mathcal{L}$ - the lattice path operad

$\mathcal{M}$ - the commutative path operad

E_n - the topological operad of little n-disks

$\mathrm{Ch}_\bullet(\mathrm{E}_n, \Bbbk)$ - the dg-operad of little n-disks

e_n - the homology dg-operad of $\mathrm{Ch}_\bullet(\mathrm{E}_n, \Bbbk)$

Pois_n - the dg-operad of Poisson n-algebras

$\mathrm{CH}^*(\mathcal{A},\mathcal{A})$ - the Hochschild cochain complex of a $\Bbbk$-linear category $\mathcal{A}$

$\mathrm{HH}^*(\mathcal{A},\mathcal{A})$ - the Hochschild cohomology of a $\Bbbk$-linear category $\mathcal{A}$

${\mathrm{C}_{\mathrm{DY}}}^*(F)$ - the Davydov-Yetter cochain complex of a monoidal functor F

${\mathrm{H}_{\mathrm{DY}}}^*(F)$ - the Davydov-Yetter cohomology of a monoidal functor F

$\mathrm{Tot}(X_\bullet)$ - the totalization of a cosimplicial object $X_\bullet$

$\mathrm{Tot}_{\Theta_2}(Y_\bullet)$ - the totalization of a 2-cocellular object $Y_\bullet$

Nederlandse Samenvatting

Differentiaal gegradeerde categorieën (afgekort: dg-categorieën) zijn fundamentele objecten in de algebraïsche meetkunde en hogere categorietheorie. Na hun introductie door Kelly in de homologe algebra, werden dg-categorieën gebruikt door Bondal-Kapranov om getrianguleerde categorieën te veredelen. Sindsdien is de studie van dg-categorieën en hun categorie $\mathbb{C}_{dg}(\mathbb{k})$ intensief toegenomen.

Samen met $\mathbb{C}_{dg}(\mathbb{k})$ werden er "gerelaxeerde" versies van dg-categorieën ingevoerd, onder de naam A_∞-categorieën. Waar een dg-categorie een dg-graf is met een associatieve en unitale samenstellingsoperatie, is een A_∞-categorie een dg-graf met een unitale samenstelling die enkel associatief is op homotopie na. Hierop voortbouwend introduceerde Fukaya, in zijn werk over Homologe Spiegelsymmetrie, zwak unitale A_∞-categorieën door te vereisen dat ook de unitaliteit enkel geldt op homotopie na. Sindsdien werden zwak unitale dg- en A_∞-categorieën bestudeerd door vele auteurs.

Het zou voordelig zijn om een meer gerelaxeerde model categorie te hebben (dan de model categorie van kleine dg-categorieën), die bestaat uit de kleine, zwak unitale dg-categorieën, en die Quillen equivalent is met de modelcategorie van kleine dg-categorieën. Dit probleem lossen we volledig op in het eerste deel van dit werk.

In de tweede helft van deze thesis behandelen we een tweede probleem. Gegeven een monoïdale $\mathbb{k}$-lineaire categorie $\mathcal{C}$, dan zijn er twee deformatiecomplexen die ermee geassocieerd zijn: de eerste is het bekende cohomologe Hochschild complex $\mathrm{CH}^\bullet(\mathcal{C},\mathcal{C})$ (geassocieerd aan elke $\mathbb{k}$-lineaire of dg-categorie) waarvan de cohomologie de deformaties van de $\mathbb{k}$-lineaire structuur (i.e. de samenstelling) controleert; de tweede is het Davydov-Yetter complex. De cohomologie van dit complex beschrijft de infinitesimale deformaties van de monoïdale structuur van een $\mathbb{k}$-lineaire monoïdale functor $F\colon \mathcal{C}\to\mathcal{D}$ of de tensorstructuur van een $\mathbb{k}$-lineaire monoïdale categorie (wanneer $F=\mathrm{Id}_{\mathcal{C}}$).

Dus om de volledige deformatietheorie van $\mathcal{C}$ the controleren, moet men deze twee complexen "samenvoegen". We bereiken dit in dit werk door een functor $A(F,F)\colon \Theta_2\to \mathrm{Vect}(\mathbb{k})$ te construeren voor een $\mathbb{k}$-lineaire monoïdale functor $F\colon\mathcal{C}\to\mathcal{D}$ zodanig dat de 2-cocellulaire totalisatie $\mathrm{Tot}_{\Theta_2}(A(F,F))$ het gewenste deformatiecomplex is.

De thesis is als volgt gestructureerd:

Hoofdstuk 1 geeft een elementaire en essentiële samenvatting van de theorie en eigenschappen van het wiskundige gereedschap dat we zullen gebruiken in dit werk. Geen enkel van de resultaten daar zijn nieuw.

Hoofdstuk 2 is gebaseerd op twee artikels [PS1] en [PS2] van de auteur en de promotor. In Sectie 2.1 definiëren we Kontsevich-Soibelman zwak unitale dg-categorieën en hun categorie $\mathbb{C}_{dgwu}(\mathbb{k})$, en we bewijzen de kleine (co)compleetheid van $\mathbb{C}_{dgwu}(\mathbb{k})$ en een monadiciteitsstelling. In Sectie 2.2 construeren we een technisch en essentieel hulpmiddel om de modelstructuur op $\mathbb{C}_{dgwu}(\Bbbk)$ te bewijzen, namelijk het gepretrianguleerd omhulsel van een zwak unitale dg-categorie. In Sectie 2.3 bewijzen we één van de hoofdstellingen: for een lichaam $\Bbbk$ bestaat er een cofibrant voortgebrachte Quillen-modelstructuur op $\mathbb{C}_{dgwu}(\Bbbk)$. We doen dit een zwak unitale vervanging voor de Kontsevich dg-categorie $\mathcal{K}'$ te introduceren. In Sectie 2.4 bewijzen we de tweede hoofdstelling van dit hoofdstuk: er is een Quillen-equivalentie tussen $\mathbb{C}_{dgwu}(\Bbbk)$ en $\mathbb{C}_{dg}(\Bbbk)$, uitgerust met de Tabuada-modelstructuur. In Sectie 2.5 geven we een expliciete en canonieke cofibrante resolutie van een unitale dg-algebra. Secties 2.6 en 2.7 zijn technischer: in de eerstgenoemde geven we een bewijs van één propositie en in de laatstgenoemde berekenen we de cohomologie van de dg-operad $\mathcal{O}'$.

Hoofdstuk 3 is gebaseerd op de preprint [PS3] van de auteur en de promotor. In Sectie 3.1 brengen we de basis van deformatietheorie in herinnering en beschrijven we de Hochschild en Davydov-Yetter complexen. In Sectie 3.2 definiëren we het belangrijkste object waarin we geïnteresseerd zijn: een 2-cocellulaire vectorruimte genaamd $A(F,F)$, geassocieerd aan een monoïdale functor $F\colon \mathcal{C} \to \mathcal{D}$ tussen $\Bbbk$-lineaire monoïdale categorieën $\mathcal{C}$ en $\mathcal{D}$. In Sectie 3.3 berekenen we expliciet de Δ-totalisatie van $A(F,F)$ en we bewijzen de hoofdstelling van dit hoofdstuk: voor $\mathcal{C}$ en $\mathcal{D}$ $\Bbbk$-lineaire monoïdale categorieën en $F\colon \mathcal{C} \to \mathcal{D}$ een $\Bbbk$-lineaire monoïdale functor, heeft de 2-cocellulaire totalisatie $\mathrm{Tot}_{\Theta_2}(A(F,F))$ de structuur van een homotopie-2-algebra. In Sectie 3.4 beschrijven we de totalisaties van $A(F,F)$ en $A(\mathrm{Id},\mathrm{Id})$ als deformatiecomplexen van monoïdale functoren en $\Bbbk$-lineaire monoïdale categorieën, en geven we een voorbeeld waar deze deformaties relevant zijn. Secties 3.5 en 3.6 zijn technischer: in de eerstgenoemde geven we een expliciete lijst van relaties voor de ontaardings- en zijvlakafbeeldingen in Θ_2 en in de laatstgenoemde geven we een bewijs van één propositie.

Hoofdstuk 4 bespreekt een mogelijke manier om te tonen dat $\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id}_C,\mathrm{Id}_C))$ een homotopie-3-algebra is. We doen dit door het introduceren van een 2-dimensionale veralgemening van de roosterpadoperad.

Chapter 1

Preliminary results

Eine Seele ohne Körper ist so unmenschlich und entsetzlich wie ein Körper ohne Seele, und übrigens ist das erstere die seltene Ausnahme und das zweite die Regel.

T. Mann,
Den Zauberberg

This chapter serves as a brief (and far from being exhaustive) overview of the theory needed in the two main chapters. We begin by introducing the theory of dg-categories:

1.1 Dg-categories

The reader is referred to the survey [Kel2] and the lecture notes [To2].

Definition 1.1.1. A $\Bbbk$-linear (respectively, dg-) category $\mathcal{C}$ consists of the following data:

- a set of objects ($Ob(\mathcal{C})$)
- for any pair of objects X, Y, a $\Bbbk$-vector space (respectively, a dg-vector space) $\mathcal{C}(X, Y) \in Ob(\mathrm{Vect}(\Bbbk))$ (respectively, $\in Ob(\mathrm{Ch}(\Bbbk))$),
- for any triple of objects X, Y, Z a composition morphism $\circ_{X,Y,Z} \colon \mathcal{C}(X, Y) \otimes \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$, which is a $\Bbbk$-linear (respectively, chain) map,
- for any object $X \in Ob(\mathcal{C})$, a unit morphism $e \colon \Bbbk \to C(X, X)$. (In the dg-setting, $\Bbbk$ denotes the dg-vector space concentrated in degree 0).

These data satisfy the usual associativity and unit conditions.

Definition 1.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be $\Bbbk$-linear (respectively, dg-) categories, a $\Bbbk$-linear (respectively, dg-) functor $F \colon \mathcal{C} \to \mathcal{D}$ consists of the following data:

- a map of sets $F_0 \colon Ob(\mathcal{C}) \to Ob(\mathcal{D})$

- for any pair of objects $X, Y \in Ob(\mathcal{C})$, a $\mathbb{k}$-linear map (respectively, a cochain complex map) $F_{X,Y} : \mathcal{C}(X, Y) \to \mathcal{D}(F_0(X), F_0(Y))$

These data satisfy the usual associativity and unit conditions.

The collection of *small* dg-categories together with dg-functors form a category $\mathbb{C}_{dg}(\mathbb{k})$, and analogously the collection of *small* $\mathbb{k}$-linear categories together with $\mathbb{k}$-linear functors form a category $\mathbb{C}(\mathbb{k})$. Taking the 0th-cohomology of complexes defines a functor:

$$H^0(-) : \mathbb{C}_{dg}(\mathbb{k}) \to \mathbb{C}(\mathbb{k})$$

which is the identity on the objects' level, and $H^0(\mathcal{C})(X, Y) := H^0(\mathcal{C}(X, Y))$ for each $X, Y \in \mathcal{C}$.

We can endow the category $\mathbb{C}_{dg}(\mathbb{k})$ with a tensor product:

$$\otimes : \mathbb{C}_{dg}(\mathbb{k}) \times \mathbb{C}_{dg}(\mathbb{k}) \to \mathbb{C}_{dg}(\mathbb{k}),$$

where

$$Ob(\mathcal{C} \otimes \mathcal{D}) := Ob(\mathcal{C}) \times Ob(\mathcal{D})$$

and

$$\mathcal{C} \otimes \mathcal{D}((X, Y), (X', Y')) := \mathcal{C}(X, X') \otimes_{\mathbb{k}} \mathcal{D}(Y, Y')$$

and it turns out that $(\mathbb{C}_{dg}(\mathbb{k}), \otimes, I, \mathcal{H}om)$ is a closed symmetric monoidal category, where the unit is the dg-category I with only one object $*$ and as hom-complex $I(*, *) := \mathbb{k}$, where $\mathbb{k}$ is concentrated in degree 0.

Definition 1.1.3. A **quasi-equivalence** of dg-categories is a dg-functor $F : \mathcal{C} \to \mathcal{D}$, such that:

- for any two objects $X, Y \in \mathcal{C}$, the map of complexes $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is a quasi-isomorphism of complexes,
- the functor $H^0(F) : H^0(\mathcal{C}) \to H^0(\mathcal{D})$ is an equivalence of $\mathbb{k}$-linear categories.

The collection of quasi-equivalences in $\mathbb{C}_{dg}(\mathbb{k})$ includes all the isomorphisms and is closed under the 2-of-3 property.

Let us introduce also another class of morphisms in $\mathbb{C}_{dg}(\mathbb{k})$, whose role will become clear later:

Definition 1.1.4. An **isofibration** of dg-categories is a dg-functor $F : \mathcal{C} \to \mathcal{D}$, such that:

- for any two objects $X, Y \in Ob(\mathcal{C})$, the map of complexes $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is a surjection of complexes,
- for any $X \in Ob(\mathcal{C})$ and for any isomorphism $g : FX \to Z$ in $H^0(\mathcal{D})$, there exists an isomorphism $f : X \to Y$ in $H^0(\mathcal{C})$ such that $F(f) = g$ and $FY = Z$.

It was shown by Tabuada in [Tab] that the category $\mathbb{C}_{dg}(\Bbbk)$ is endowed with a model category structure (and we will describe in the next Section 1.2), thus giving a rigorous and clean tool to define the homotopy category $\mathrm{Ho}(\mathbb{C}_{dg}(\Bbbk))$. However, $\mathbb{C}_{dg}(\Bbbk)$ fails to be a monoidal model category, since the tensor product of cofibrant objects is not necessarily cofibrant. Thus the internal hom $\mathcal{H}om$ of $\mathbb{C}_{dg}(\Bbbk)$ cannot induce an internal hom in $\mathrm{Ho}(\mathbb{C}_{dg}(\Bbbk))$. Even so, Bertrand Toën proved in [To1] that the homotopy category $\mathrm{Ho}(\mathbb{C}_{dg}(\Bbbk))$ is a closed symmetric monoidal category.

For later use we introduce the A_∞-functors between dg-categories and we need to specify the signs in the A_∞-identity. As we adapt here to the right to left formalism, our signs agree with the ones in [Ly1].

Definition 1.1.5. Given $\mathcal{C}, \mathcal{D}$ (non-unital) dg-categories over $\Bbbk$, an A_∞-functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of the following data:

- a map of sets $F_0\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$,
- for any $n \geq 1$ and a sequence of objects $X_0, \dots, X_n \in \mathrm{Ob}(\mathcal{C})$ of degree $1-n$

  $$F_n\colon \mathcal{C}(X_{n-1}, X_n) \otimes \cdots \otimes \mathcal{C}(X_1, X_2) \otimes \mathcal{C}(X_0, X_1) \to \mathcal{D}(F_0(X_0), F_0(X_n))[1-n]$$

  such that one has:

  $$\begin{aligned} & d(F_n(f_n \otimes \cdots \otimes f_1))+ \\ & \sum_{a+b=n} (-1)^{b-1+(a-1)(|f_1|+\cdots+|f_b|)} F_a(f_n \otimes \cdots \otimes f_{b+1}) \cdot F_b(f_b \otimes \cdots \otimes f_1) = \\ & \sum_{k=0}^{n-1} (-1)^{n-1+|f_1|+\cdots+|f_k|} F_n(f_n \otimes \cdots \otimes f_{k+2} \otimes d(f_{k+1}) \otimes f_k \otimes \cdots \otimes f_1)+ \\ & \sum_{k=0}^{n-2} (-1)^k F_{n-1}(f_n \otimes \cdots \otimes f_{k+3} \otimes (f_{k+2} \circ f_{k+1}) \otimes f_k \otimes \cdots \otimes f_1) \end{aligned} \tag{1.1.1}$$

Definition 1.1.6. Let $\mathcal{C}$, $\mathcal{D}$ be (unital) dg-categories over $\Bbbk$, $F\colon \mathcal{C} \to \mathcal{D}$ an A_∞-functor, $\{F_i\}_{i\geq 1}$ its Taylor components.

(1) F is called *strongly unital* if $F_1(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F_n(\dots, \mathrm{id}_X, \dots) = 0$ for any object X and $n \geq 2$,

(2) F is called *weakly unital* if $F_1(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any object $X \in \mathcal{C}$ (and the second condition is dropped).

1.1.1 Hochschild cohomology of $\Bbbk$-linear categories

Let $\mathcal{A}$ be a $\Bbbk$-linear category, the cohomological Hochschild complex $\mathrm{CH}^*(\mathcal{A}, \mathcal{A})$ of $\mathcal{A}$ is defined as:

$$\begin{aligned} \mathrm{CH}^0(\mathcal{A}, \mathcal{A}) &:= \prod_{X \in Ob(\mathcal{A})} \mathcal{A}(X, X), \\ \mathrm{CH}^n(\mathcal{A}, \mathcal{A}) &:= \prod_{X_i \in Ob(\mathcal{A})} \mathrm{Hom}_\Bbbk\left(\mathcal{A}(X_{n-1}, X_n) \otimes \cdots \otimes \mathcal{A}(X_0, X_1), \mathcal{A}(X_0, X_n)\right) \text{ if } n \geq 1, \end{aligned} \tag{1.1.2}$$

and the differential is the Hochschild differential, $d_n\colon \mathrm{CH}^n(\mathcal{A},\mathcal{A}) \to \mathrm{CH}^{n+1}(\mathcal{A},\mathcal{A})$:

$$\begin{aligned} d_n(\Psi)(f_{n+1},\dots,f_1) \coloneqq\ & f_{n+1}\circ\Psi(f_n,\dots,f_1)+ \\ & \sum_{i=1}^{n}(-1)^i\Psi(f_{n+1},\dots,f_{i+2},f_{i+1}\circ f_i,f_{i-1},\dots,f_1) \\ & -\Psi(f_{n+1},\dots,f_2)\circ f_1 \end{aligned} \tag{1.1.3}$$

It is standard that cohomological Hochschild complex $\mathrm{CH}^*(\mathcal{A},\mathcal{A})$ can be interpreted as

$$\mathrm{RHom}^\bullet_{\mathcal{A}\text{-Bimod}}(\mathcal{A},\mathcal{A}),$$

where $\mathcal{A}$ is the tautological $\mathcal{A}$-bimodule.
Thus the Hochschild Cohomology $\mathrm{HH}^*(\mathcal{A},\mathcal{A})$ of $\mathcal{A}$ can be interpreted as:

$$\mathrm{Ext}^\bullet_{\mathcal{A}\text{-Bimod}}(\mathcal{A},\mathcal{A})$$

The Hochschild cohomology has a rich algebraic structure which we will investigate in Subsection 3.1.1.

1.2 Model categories

Model categories were introduced by Quillen (under the name “closed model categories” in [Q]) in order to tackle the localization problem: i.e. given a category $\mathcal{A}$ with a class of weak equivalences $\mathcal{W} \subset \mathrm{Mor}(\mathcal{A})$, how do we construct its localization $\mathcal{A}[\mathcal{W}^{-1}]$ along $\mathcal{W}$ in a way that it is locally small? All the results of this Section can be found in [DS], [GS], [Hi] and [Ho].

Definition 1.2.1. A **model structure** on a category $\mathcal{C}$ is a choice of three distinguished classes of morphisms: cofibrations $\mathcal{C}of \subset \mathrm{Mor}(\mathcal{C})$, fibrations $\mathcal{F}ib \subset \mathrm{Mor}(\mathcal{C})$ and weak equivalences $\mathcal{W} \subset \mathrm{Mor}(\mathcal{C})$ satisfying the following conditions:

- $\mathcal{W}$ makes $\mathcal{C}$ into a category with weak equivalences, i.e. $\mathcal{W}$ contains all isomorphisms and is closed under 2-of-3: given a composable pair of morphisms f, g, if two out of the three morphisms $(f, g, g\circ f)$ are in $\mathcal{W}$, so is the third;
- $(\mathcal{C}of, \mathcal{F}ib\cap\mathcal{W})$ and $(\mathcal{C}of\cap\mathcal{W}, \mathcal{F}ib)$ are two weak factorization systems on $\mathcal{C}$, i.e. see the definition below.

Definition 1.2.2. A **weak factorization system** on a category $\mathcal{C}$ is a pair $(\mathcal{L},\mathcal{R})$ of classes of morphisms of $\mathcal{C}$ such that

- Every morphism $f\colon X \to Y$ of $\mathcal{C}$ may be factored as the composition of a morphism in $\mathcal{L}$ followed by one in $\mathcal{R}$:
$$f\colon X \xrightarrow{\in\mathcal{L}} Z \xrightarrow{\in\mathcal{R}} Y$$
- The classes are closed under having the lifting property against each other:
  - $\mathcal{L}$ is precisely the class of morphisms having the left lifting property against every morphism in $\mathcal{R}$;

– $\mathcal{R}$ is precisely the class of morphisms having the right lifting property against every morphism in $\mathcal{L}$.

Definition 1.2.3. A **model category** is a small complete and small cocomplete category $\mathcal{C}$ equipped with a model structure.

Remark 1.2.4. It is a well-known fact that $\mathcal{C}of$ (respectively, $\mathcal{F}ib$) and $\mathcal{W}$ determine completely $\mathcal{F}ib$ (respectively, $\mathcal{C}of$).

Given a model category $\mathcal{C}$, we can choose two functors Q and R, called cofibrant and fibrant replacement respectively. Q assigns to any object X a cofibrant object QX, such that there is a natural morphism $QX \to X \in \mathcal{F}ib \cap \mathcal{W}$, i.e. an acyclic fibration. Analogously, R assigns to any object X a fibrant object RX, such that there is a natural morphism $X \to RX \in \mathcal{C}of \cap \mathcal{W}$, i.e. an acyclic cofibration.

Quillen showed in [Q] that the category $\mathrm{Ho}(\mathcal{C})_{cf}$ whose objects are objects of $\mathcal{C}$ both fibrant and cofibrant and whose morphisms are homotopy classes of morphisms in $\mathcal{C}$, is equivalent to the Gabriel-Zisman localization $\mathcal{C}[\mathcal{W}^{-1}]$ of $\mathcal{C}$ along the weak equivalences.

Given two model categories $\mathcal{A}$ and $\mathcal{C}$, there is a convenient notion of morphisms, i.e. Quillen adjunction/pair:

Definition 1.2.5. For $\mathcal{A}$ and $\mathcal{C}$ two model categories, a **Quillen pair** (L, R):

$$L\colon \mathcal{A} \leftrightarrows \mathcal{C}\colon R$$

is an adjoint pair of functors (L, R) such that L preserves cofibrations and acyclic cofibrations, or equivalently, R preserves fibrations and acyclic fibrations.

Definition 1.2.6. A Quillen pair $L\colon \mathcal{A} \leftrightarrows \mathcal{C}\colon R$ is a **Quillen equivalence** if for any cofibrant object $X \in \mathcal{A}$ and any fibrant object $Y \in \mathcal{C}$, a morphism $f\colon LX \to Y$ is a weak equivalence in $\mathcal{C}$ if and only if its adjoint morphism $X \to RY$ is a weak equivalence in $\mathcal{A}$.

Example 1.2.7. Here are some examples of model categories:

1. the category of topological spaces $\mathscr{T}op$ is a model category, where weak equivalences $\mathcal{W}$ are weak homotopy equivalences, fibrations $\mathcal{F}ib$ are Serre fibration;
2. the category of simplicial sets $\mathscr{S}\mathscr{S}ets$ is a model category, where where weak equivalences $\mathcal{W}$ are simplicial weak equivalences and fibrations $\mathcal{F}ib$ are Kan fibrations;
3. the category Cat of small categories is a model category, where weak equivalences $\mathcal{W}$ are equivalences of categories, fibrations $\mathcal{F}ib$ are isofibrations.

Remark 1.2.8. The singular simplicial complex/geometric realization $(S_\bullet(-), |-|)$ adjunction constitutes a Quillen equivalence between $\mathscr{T}op$ and $\mathscr{S}\mathscr{S}ets$, and it was shown by Quillen in [Q].

In [Tab], Gonçalo Tabuada constructed a model structure for $\mathbb{C}_{dg}(\Bbbk)$:

Theorem 1.2.9. *The category $\mathbb{C}_{dg}(\Bbbk)$ of small dg-categories admits a model category structure, where weak equivalences $\mathcal{W}$ are quasi-equivalences, fibrations $\mathcal{F}ib$ are isofibrations of dg-categories.*

By Remark 1.2.4, these data determine a model structure on $\mathbb{C}_{dg}(\mathbb{k})$.

The model categories $\mathscr{T}op$, $\mathscr{S}\mathscr{S}ets$, $\mathbb{C}_{dg}(\mathbb{k})$ are all cofibrantly generated model categories, i.e. such that there is a set of cofibrations and one of trivial cofibrations, such that all other (trivial) cofibrations are generated from these.

Definition 1.2.10. Let $\mathcal{C}$ be a category with all colimits and let $S \subset \mathrm{Mor}(\mathcal{C})$ a class of morphisms. We write

- rlp(S) for the collection of morphisms with the right lifting property with respect to S,
- llp(S) for the collection of morphisms with the left lifting property with respect to S

Moreover, we also write, now for $I \subset \mathrm{Mor}(\mathcal{C})$:

- *I-cell* for the relative cell complexes, the class of morphisms obtained by transfinite composition of pushouts of coproducts of elements in I;
- *I-cof* for the class of retracts (in the arrow category $Arr(\mathcal{C})$ of elements in $cell(I)$
- $I\text{-}inj := rlp(I)$ for the class of morphisms with the right lifting property with respect to I, the I-injective morphisms.

Definition 1.2.11. A model category $\mathcal{C}$ is cofibrantly generated if there are small sets of morphisms $I, J \subset \mathrm{Mor}(\mathcal{C})$ such that:

- *I-cof* is precisely the collection of cofibrations of $\mathcal{C}$;
- *j-cof* is precisely the collection of acyclic cofibrations in $\mathcal{C}$;
- I and J permit the small object argument.

In many situations, it is easier to show that a category admits a cofibrantly generated model structure by using a well-known theorem, [[Ho], Theorem 2.1.19]:

Theorem 1.2.12. *Let $\mathcal{C}$ be a small complete and cocomplete category. Suppose that W is a subcategory of $\mathcal{C}$, and I and J are sets of maps. Assume that the following conditions hold:*

1. *the subcategory W has 2-of-3 property and is closed under retracts,*
2. *the domains of I are small relative to I-cell,*
3. *the domains of J are small relative to J-cell,*
4. *$J\text{-}cell \subset W \cap I\text{-}cof$,*
5. *$I\text{-}inj = W \cap J\text{-}inj$.*

Then there is a cofibrantly generated closed model structure on $\mathcal{C}$, for which the morphisms $\mathcal{W}$ of W are weak equivalences, I are generating cofibrations, J are generating acyclic cofibrations. Its fibrations are defined as J-inj.

In [Tab], the author describes explicitly the sets I and J and shows that I and J satisfy the conditions of the theorem above. Eventually he describes the class of isofibrations (the fibrations in $\mathbb{C}_{dg}(\mathbb{k})$) as J-*inj*.

1.3 Operads and higher operads

Operads are collections of abstract operations, together with the compositions of these, and are particularly important and useful in all the categories with a good notion of "homotopy".

All the results of this Section can be found in [Ba2], [Ba3], [Ba4], [BB], [BD], [GK] and [Yau].

1.3.1 Definitions and basic examples

Let $(\mathcal{E}, \otimes, e, \underline{\mathrm{Hom}}_{\mathcal{E}})$ be a symmetric monoidal closed category.

Definition 1.3.1. A **symmetric operad** $\mathcal{O}$ in $\mathcal{E}$ consists of objects $\mathcal{O}(n) \in Ob(\mathcal{E}), n \in \mathbb{N}, n \geq 0$, together with the following structure:

- composition operations:

$$\circ\colon \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k)$$

- a unit operation $1\colon e \to \mathcal{O}(1)$
- for each $n \in \mathbb{N}$, right actions of the symmetric group: $\rho_n\colon \Sigma_n \to \mathrm{Hom}_{\mathcal{E}}(\mathcal{O}(n), \mathcal{O}(n))$

satisfying conditions of associativity and unitality of composition and the compatibility with the symmetric group actions.

Example 1.3.2. Here are some examples of symmetric operads:

(i) every monoid N of $\mathcal{E}$ defines an operad $\mathcal{N}$, with $\mathcal{N}(1) = N$ and $\mathcal{N}(i) = \emptyset$ otherwise, where the only non trivial operadic composition $\circ\colon \mathcal{N}(1) \otimes \mathcal{N}(1) \to \mathcal{N}(1)$ amounts to the multiplication of the monoid N;

(ii) If $\mathcal{E} = Set$, we can define the commutative operad Com with $Com(n) = \{*\}$ for each $n \geq 0$.

(iii) If $\mathcal{E} = Set$, we can define the associative operad $Assoc$ with $Assoc(n) = \Sigma_n$, the n-th symmetric group.

(iv) Let $\mathcal{E} = \mathrm{Ch}(\Bbbk)$, Pois_n is the operad generated by two binary operations $- \cdot -$ (of degree 0) and $[-,-]$ (of degree $1-n$). The $- \cdot -$ satisfies associativity and graded commutativity relations:

$$(- \cdot (- \cdot -)) = ((- \cdot -) \cdot -), \quad (- \cdot -) = (-1)^{(|-|+1-n)(|-|+1-n)}(- \cdot -) \circ \tau,$$

where $\tau \in \Sigma_2$ is a 2-cycle. The $[-,-]$ satisfies Jacobi identity, Leibniz rule and alternating relations:

$$[-,[-,-]] + [-,[-,-]] \circ \sigma + (-1)^{(|-|+1-n)(|-|+1-n)}[-,[-,-]] \circ \sigma^2 = 0,$$

$$d[-,-] = [d(-),-] + (-1)^{|-|+1-n}[-,d(-)]$$

$$[-,-] + (-1)^{(|-|+1-n)(|-|+1-n)}[-,-] \circ \tau = 0,$$

where $\sigma \in \Sigma_3$ is a 3-cycle and $\tau \in \Sigma_2$ is a 2-cycle. Moreover these binary operations satisfy the Poisson relation:

$$[-, - \cdot -] = (([-,-]) \cdot -) + (-1)^{(|-|+1-n)|-|}(- \cdot ([-,-]))$$

We call Pois_n the n-Poisson operad.

Definition 1.3.3. For any object X of $\mathcal{E}$ we can define its **endomorphism operad** $End(X)$ as:

$$End(X)(n) := \underline{\mathrm{Hom}}_{\mathcal{E}}(X^{\otimes n}, X)$$

where the composition, unit and symmetric group operations follow from the enriched functoriality of iterated tensor product.

Definition 1.3.4. Given two operad in $\mathcal{E}$ $\mathcal{O}, \mathcal{P}$, a morphism of operads $F\colon \mathcal{O} \to \mathcal{P}$ is a sequence of Σ_n-equivariant maps $F_n\colon \mathcal{O}(n) \to \mathcal{P}(n)$, compatible with the composition and unit operations.

Definition 1.3.5. An **algebra over an operad** $\mathcal{O}$ is an object X of $\mathcal{E}$ endowed with a morphism of operad: $F\colon \mathcal{O} \to End(X)$.

Example 1.3.6. Considering the operads of Example 1.3.2 above:

(i) the algebras over $\mathcal{N}$ are objects X endowed with a monoid action of N;

(ii) the algebras over Com are commutative monoids.

(iii) the algebras over $Assoc$ are associative monoids.

(iv) the algebras over Pois_n are Poisson n-algebras: if $n = 1$ these are simply Poisson algebras; if $n = 2$, Poisson 2-algebras are known as Gerstenhaber algebras.

Example 1.3.7. Let $n \geq 1$ be fixed and let $k\mathscr{T}op$ be the category of compactly generated spaces, which is a nice category of spaces, i.e cartesian closed category of topological spaces. The operad of little n-disks is the collection of topological spaces $\mathrm{E}_n(k)$ of rectilinear embeddings of k little disks in the n-dimensional unit disk, where rectilinear means that the embedding may rescale and translate the little disks, but not rotate or deform them otherwise. The operadic compositions are defined through the grafting operation, i.e. the gluing of configurations of disks, with (properly rescaled) configurations being inserted in place of small disks.

These family of topological operads E_n were introduced by Boardman and Vogt, and later studied by May in the context of n-fold loop spaces.

Given a symmetric operad $\mathcal{O}$ in $\mathcal{E}$ and a lax symmetric monoidal functor $F\colon \mathcal{E} \to \mathcal{V}$ between symmetric monoidal closed categories, we can consider the collection of objects in $\mathcal{V}$:

$$F(\mathcal{O})(n) := F(\mathcal{O}(n))$$

It is straightforward to check that $F(\mathcal{O})$ is a symmetric operad in $\mathcal{V}$, the operadic composition induced by the one for $\mathcal{O}$.

If we apply the singular chain complex functor to the topological operads E_n, we get a family of dg-operads $\mathrm{Ch}_\bullet(\mathrm{E}_n, \mathbb{k})$. Similarly we can apply the homology functor to the dg-operad $\mathrm{Ch}_\bullet(\mathrm{E}_n, \mathbb{k})$: the resulting dg-operad (with trivial differential) is $H_\bullet(\mathrm{E}_n, \mathbb{k})$, commonly denoted by e_n. Both these families of operads $C_\bullet(\mathrm{E}_n, \mathbb{k})$ and e_n have been studied thoroughly in the last decades, and people made a link between these operads and deformation theory.

The first result in this direction was given in 1976 by Cohen [Co]:

Theorem 1.3.8. *Let $\mathbb{k}$ be a field, char$(\mathbb{k}) = 0$, then for all $n \geq 2$ the operads e_n and Pois_n are naturally isomorphic.*

As we will recall in Subsection 3.1.1, Gerstenhaber proved that, for any associative algebra A, the Hochschild cohomology complex $\mathrm{HH}^*(A, A)$ is a Gerstenhaber algebra (hence the name), i.e. it is a Pois_2-algebra in operadic terms, with the cup product as degree 0 operation and the Gerstenhaber bracket as degree -1 operation.

Then Theorem 1.3.8 states that the homology dg-operad e_2 of the little disks dg-operad $\mathrm{Ch}_\bullet(\mathrm{E}_2, \mathbb{k})$ acts on $\mathrm{HH}^*(A, A)$, and this led Deligne to raise the following question: "Does the dg-operad $\mathrm{Ch}_\bullet(\mathrm{E}_2, \mathbb{k})$ act on the Hochschild cochain complex of an associative algebra?"

This question, known as the Deligne conjecture, led to the development of operadic theory and it fundamentally influenced modern deformation theory. This conjecture was eventually proven to be true, now by many authors, see [MS1], [Tam1], [KS1], [Sh3].

For further need, let us introduce another class of operads:

Definition 1.3.9. Given a set C of colours, a **coloured operad** $\mathcal{O}$ in $\mathcal{E}$ consists of:

- an object $\mathcal{O}(k_1, \ldots, k_n; n)$ of $\mathcal{E}$ for each $n \geq 0$ and for each $n+1$-tuple $(k_1, \ldots, k_n; n)$ of colours of C,

together with:

- a unit morphism $1_c\colon e \to \mathcal{O}(c, c)$ for every $c \in C$,
- a composition operation:

  $$\mathcal{O}(c_1, \ldots, c_n; c) \otimes \mathcal{O}(c_1^1, \ldots, c_{k_1}^1; c_1) \otimes \cdots \otimes \mathcal{O}(c_1^n, \ldots, c_{k_n}^n; c_n) \to \mathcal{O}(c_1^1, \ldots, c_{k_n}^n; c)$$

  for every $(n+1)$-tuple $(c_1, \ldots, c_n; c)$ and n other tuples $(c_1^i, \ldots, c_{k_i}^i; c_i)$, $1 \leq i \leq n$,

- a morphism:
$$\sigma_*\colon \mathcal{O}(c_1,\dots,c_n;c) \to \mathcal{O}(c_{\sigma(1)},\dots,c_{\sigma(n)};c)$$
for all n, for all tuples and for all $\sigma \in \Sigma_n$,

satisfying all the evident conditions.

Remark 1.3.10. If $\mathcal{O}$ is a one-coloured operad (i.e. $C = \{*\}$), $\mathcal{O}$ is nothing but a symmetric operad as in Definition 1.3.1.

Remark 1.3.11. One can consider the unary maps of a coloured operad $\mathcal{O}$: $\mathcal{O}_u(c_1;c_2)$ for any couple of colours c_1 and c_2. Then, thanks to the operadic composition and the associativity and unitality of this, we get that $\mathcal{O}_u$ is actually an $\mathcal{E}$-category, with objects the colours of $\mathcal{O}$.

1.3.2 Higher operads

So far we have treated operads acting at most on a coloured collection of objects X_c, $c \in C$, of a monoidal category $\mathcal{E}$.

In [Ba2], [Ba3] and [Ba4], Batanin developed a whole theory of higher operads acting on objects of a monoidal n-globular category, with the purpose to develop the theory of *weak n-categories.*

Let us start by recalling the basic definitions:

Definition 1.3.12. An n**-globular category** $\mathcal{C}_\bullet$ is a sequence of categories $\mathcal{C}_i$, $0 \le i \le n$, together with functors $s_{i,k}, t_{i,k}\colon \mathcal{C}_i \to \mathcal{C}_k$, $k < i \le n$, called source and target functors satisfying the following conditions:

$$s_{k,m} \circ s_{l,k} = s_{l,m}, \quad t_{k,m} \circ t_{l,k} = t_{l,m},$$

$$s_{l,l-1} \circ t_{l,l+1} = s_{l+1,l-1}, \quad t_{l,l-1} \circ s_{l+1,l} = t_{l+1,l-1}.$$

Definition 1.3.13. A **monoidal** n**-globular category** is an n-globular category $\mathcal{C}_\bullet$ together with:

- composition functors
$$\otimes_k\colon \mathcal{C}_i \times_k \mathcal{C}_i \to \mathcal{C}_i, \quad k < i \le n,$$
- cylinder functors
$$Z\colon \mathcal{C}_{i-1} \to \mathcal{C}_i, \quad i \le n,$$
such that
$$s_{i,j}(T \otimes_k S) = s_{i,j}T, \qquad t_{i,j}(T \otimes_k S) = t_{i,j}S, \quad \text{if} \quad j < i \le k,$$
$$s_{i,i-1}Z = t_{i,i-1}Z = \mathrm{Id}_{C_{i-1}}$$
$$s_{j,i}(T \otimes_k S) = s_{j,i}T \otimes_k s_{j,i}S, \qquad t_{j,i}(T \otimes_k S) = t_{j,i}T \otimes_k t_{j,i}S, \quad \text{if} \quad k < i < j,$$
$$Z^kT \otimes_j Z^kS = Z^k(T \otimes_j S);$$

together with associativity isomorphisms, left unity isomorphisms, right unity isomorphisms and interchange isomorphisms, satisfying the usual diagrams (pentagon condition for associativity, triangle condition for associativity and left and right unity isomorphisms, etc..)

Example 1.3.14. There are lots of interesting examples:

(i) Strict n-categories are monoidal n-globular categories where every $\mathcal{C}_i$ is a discrete category.

(ii) Bicategories are monoidal 1-globular categories, with $\mathcal{C}_0$ a discrete category. In particular, every monoidal category $\mathcal{C}$ gives rise to a monoidal 1-globular category, with $\mathcal{C}_0 = \{*\}$ the terminal category.

(iii) If $\mathcal{C}$ is a category with pull-backs, the category $Span_n(\mathcal{C})$ of n-spans is a monoidal n-globular category (see [[Ba2], Sec. 2]).

(iv) For every symmetric monoidal category $\mathcal{E}$, the sequence $\mathcal{N}_\bullet = \Sigma^n\mathcal{E}$, where $\mathcal{N}_i$ is the terminal category for each $0 \leq i \leq n-1$ and $\mathcal{N}_n = \mathcal{E}$ is a monoidal n-globular category.

(v) The category Ω_n of n-stage trees is a monoidal n-globular category (see [[Ba2], Sec. 2] and following Definition).

Let us recall the category Ω_n of n-stage trees:

Definition 1.3.15. A n-**stage tree** (or simply n-tree) is a chain of order preserving maps of ordinals

$$[k_n] \xrightarrow{\rho_{n-1}} [k_{n-1}] \xrightarrow{\rho_{n-2}} \dots \xrightarrow{\rho_1} [k_1] \xrightarrow{\rho_0} [0]$$

If $i \in [k_m]$ and there is no $j \in [k_{m+1}]$ such that $\rho_m(j) = i$ then we call i a **leaf** of T of height i. We will call the leaves of T of height n the **tips** of T. If for an n-tree T all its leaves are tips we call such a tree **pruned**.

Definition 1.3.16. The category Ω_n has as objects the trees of height n. The morphisms of Ω_n are commutative diagrams in *Sets*

$$\begin{array}{ccccccccc} [k_n] & \xrightarrow{\rho_{n-1}} & [k_{n-1}] & \xrightarrow{\rho_{n-2}} & \dots & \xrightarrow{\rho_1} & [k_1] & \xrightarrow{\rho_0} & [0] \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \mathrm{id}} \\ [\ell_n] & \xrightarrow{\rho'_{n-1}} & [\ell_{n-1}] & \xrightarrow{\rho'_{n-2}} & \dots & \xrightarrow{\rho'_1} & [\ell_1] & \xrightarrow{\rho'_0} & [0] \end{array} \tag{1.3.1}$$

and the functions f_ℓ are such that for all i and all $j \in [k_{i-1}]$ the restriction of f_i to $\rho^{-1}(j)$ preserves the natural order on it.

Moreover it is shown that:

Theorem 1.3.17. *The monoidal n-globular category Ω_n is the free n-category generated by the n-globular set U_n with one-element set in every dimension.*

Remark 1.3.18. This result is analogous to the fact that the set of natural numbers can be interpreted as the set of 1-cells in the free category generated by one object and one non-identity endomorphism of this object.

Actually we can associate to any n-tree its maximal pruned n-subtree, and this construction is functorial:

$$(-)^{(p)}\colon \Omega_n \to \Omega_n^{(p)},$$

where $\Omega_n^{(p)}$ denotes the full subcategory of pruned n-trees.

Given a morphism $\sigma\colon T \to S$ of n-trees, for any tip $i \in S$ we can consider its fiber $\sigma^{-1}(i)$ together with its induced n-tree structure, and we will denote it by T_i. In the case when S and T are pruned, we will consider the maximal pruned subtree of T_i, denoted by $T_i^{(p)}$.

Batanin showed the following:

Theorem 1.3.19. *The category* Ord_n *of n-ordinals and their order preserving maps is isomorphic to the category* $\Omega_n^{(p)}$ *of pruned n-trees and their morphisms.*

where:

Definition 1.3.20. An *n-ordinal* S is a sequence of maps in Δ:

$$[k_n] \xrightarrow{\rho_{n-1}} [k_{n-1}] \xrightarrow{\rho_{n-2}} \dots \xrightarrow{\rho_1} [k_1] \xrightarrow{\rho_0} [0]$$

The category Ord_n of n-ordinals has all n-ordinals as its objects, and the morphisms $S \to T$ are commutative diagrams as (1.3.1) above. An object of Ord_n is a non empty n-ordinal.

Example 1.3.21. We denote by U_n the terminal n-ordinal, i.e. the n-ordinal where $[k_i] = [0]$ for each $i \geq 1$.
We denote instead by $z^n U_0$ the initial n-ordinal, i.e. the n-ordinal where $[k_i] = \emptyset$ for each $i \geq 1$.

Similarly to what we did for (non-)symmetric operads, we can use n-ordinals to define higher operads:

Definition 1.3.22. A **pruned** $(n-1$**-terminal)** n**-operad** in $\Sigma^n\mathcal{E}$ is a collection A_T, $T \in \mathrm{Ord}_n$, of objects of $\mathcal{E}$ equipped with the following structure:

- a morphism $I\colon e \to A_{U_n}$ (the unit);
- for every morphism $\sigma\colon T \to S$ in Ord_n, a morphism $m_\sigma\colon A_S \otimes A_{T_1^{(p)}} \otimes \cdots \otimes A_{T_k^{(p)}} \to A_T$ (the multiplication).

satisfying the proper associativity and unitality conditions.

Definition 1.3.23. A pruned $(n-1)$-terminal n-operad A is called **reduced** if

$$A_{z^n U_0} = A_{U_n} = e$$

and its unit is given by the identity. A morphism between two reduced n-operads is an n-operadic morphism which induces identity morphisms in arity $z^n U_0$ and U_n.

Definition 1.3.24. Let V be a cartesian monoidal model category. A pruned n-operad A will be called **contractible** provided the unique map to the terminal n-operad is a weak equivalence i.e. every A_T is a contractible object.
If V is not cartesian monoidal, let A be a pruned n-operad equipped with a map of operads $A \to I$, where I is an n-operad whose components are tensor unit of your symmetric monoidal category. Such an A is **contractible** if this augmentation map is a weak equivalence for every A_T.

In [Ba3], Batanin showed the following result:

Theorem 1.3.25. *Let A be a contractible reduced n-operad in* $\mathrm{Ch}(\Bbbk)$ *such that A_T is a chain complex of projective $\Bbbk$-vector spaces for every T (i.e. A_T are cofibrant objects in* $\mathrm{Ch}(\Bbbk)$*). Let X be an algebra of A. Then X admits an action of a symmetric reduced operad weakly equivalent to the operad of $\Bbbk$-chains of the little n-disk operad.*

In [Tam2], Tamarkin constructed a 2-operad (denoted *seq*) in $\mathrm{Ch}(\Bbbk)$ coloured in $\mathbb{N}$, acting on the 2-globular object $\mathcal{C}$ of small dg-categories $\mathbb{C}_{dg}(\Bbbk)$: $\mathcal{C}_0$ is the set of small dg-categories, $\mathcal{C}_1$ is the set of dg-functors and $\mathcal{C}_2$ is the set of chain complexes $Coh(F,G)$ of *derived dg-transformation*, see [[Tam2] Sec. 2],[[Sh2] Sec. 2.1]. Tamarkin showed that the δ-condensation (see Definition 1.3.35) of *seq* is contractible, so by Theorem 1.3.25 he proved a "global" Deligne conjecture.

In [BM1] Batanin and Markl introduced a theory of centers and homotopy centers of monoids M inside monoidal categories $\mathcal{K}$ enriched in duoidal categories $\mathcal{D}$. An example of such a center is the 2-category of categories, while examples of homotopy centers include the *Gray*-category of 2-categories, 2-functors and pseudonatural transformations and Tamarkin's homotopy 2-category of dg-categories, dg-functors and derived dg-transformations.

In [BM2] the authors proved the corresponding Duoidal Deligne conjecture:

Theorem 1.3.26. *The homotopy center of a monoid M in a multiplicative $\mathcal{D}$-category admits an action of a contractible 2-operad that lifts the duoid structure on the center $Z(M)$.*

generalizing Tamarkin's result to all such homotopy centers.

1.3.3 (Lattice) paths operads

In the attempt of understanding and generalizing Tamarkin's construction, Batanin and Berger introduced the lattice path operad $\mathcal{L}$. The construction involves Joyal duality and the funny tensor product $\Box$ of 1-categories so let us briefly introduce these tools.

$\mathrm{Cat}_{*,*}$ is the category which has :

- as objects: bipointed categories, i.e. categories with two distinguished objects;
- as morphisms: functors preserving the two distinguished objects.

In $\mathrm{Cat}_{*,*}$ we have precisely two closed symmetric monoidal structures: the cartesian product $\times$ and the funny tensor product $\square$. As the fundamental objects of Cat are the ordinals $[n], n \geq 0$, categories freely generated by the linear graphs $\ell_n = (0 \to \cdots \to n)$ of length n, the fundamental objects of $\mathrm{Cat}_{*,*}$ are the intervals $\langle n\rangle, n \geq 1$, bipointed in $(0,n)$. An interval $\langle n\rangle$ has the same set of objects of $[n]$, but we keep track of its minimum and maximal objects $(0,n)$. The collection of intervals $\langle n\rangle$, $n \geq 1$, together with maps preserving the order and the maximum and minimum objects define a category, which we will denote by $\mathcal{I}$ (note that in our setting we have not included the final object $\langle 0\rangle$). In particular, the category $\mathcal{I}$ is a subcategory of $\mathrm{Cat}_{*,*}$.
There is a well known duality among ordinals and intervals, due to Joyal [J]:

Theorem 1.3.27. *There is an equivalence of categories*

$$\Delta \xrightarrow{\sim} (\mathcal{I})^{op}.$$

In particular, for any $n, m \geq 0$:

$$\mathrm{Cat}([n],[m]) \cong \mathrm{Cat}_{*,*}(\langle m+1\rangle, \langle n+1\rangle).$$

Now let us recall the funny tensor product $\square$ on $\mathrm{Cat}_{*,*}$: given two small bipointed categories $(A, a_\perp, a_\top)$, $(B, b_\perp, b_\top)$, then $Ob(A\square B) = Ob(A)\times Ob(B)$, bipointed in $((a_\perp, b_\perp),(a_\top, b_\top))$. Morphisms are generated by the expressions (f, id) and (id, g) where $f\colon a \to a'$ in A and $g\colon b \to b'$ in B. However, instead of factorizing over $(f,id)\circ(id,g) = (id,g)\circ(f,id)$ as for the cartesian product, we factorize by relations

$$(f,id)\circ(id,g)\circ(id,g') = (f,id)\circ(id,g\circ g') \text{ and } (f',id)\circ(f,id)\circ(id,g) = (f'\circ f,id)\circ(id,g)$$

and similarly on the other side. The result is that in $A\square B$ there are two different morphisms $(f,id)\circ(id,g)$ and $(id,g)\circ(f,id)$ from (a,b) to (a',b') unless one of f or g is the identity.

Now we can define:

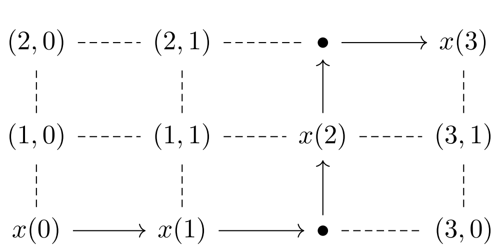
Definition 1.3.28. The **lattice paths operad** is the $\mathbb{N}$-coloured operad in *Sets* defined by:

$$\mathcal{L}(n_1,\dots,n_k;n) \coloneqq \mathrm{Cat}_{*,*}(\langle n+1\rangle, \langle n_1+1\rangle\square\dots\square\langle n_k+1\rangle).$$

The operadic composition maps are induced by tensoring and composing in $\mathrm{Cat}_{*,*}$.

Remark 1.3.29. By Theorem 1.3.27, it follows that the underlying category $\mathcal{L}_u$ of $\mathcal{L}$ is Δ.

Example 1.3.30. In order to help visualization, this is an example of a lattice path $x \in \mathcal{L}(2,1;2)$:



Following ideas from [MS1], they constructed a filtration of operads, based on the complexity of a lattice path: for each $1 \leq i < j \leq k$, there are canonical projection functors

$$p_{ij} \colon \langle n_1+1\rangle\Box\dots\Box\langle n_k+1\rangle \to \langle n_i+1\rangle\Box\langle n_j+1\rangle.$$

These functors, together with the unique functor in $\mathrm{Cat}_{*,*}(< 1 >, < n+1 >)$, induce maps

$$\phi_{ij} \colon \mathcal{L}(n_1,\dots,n_k;n) \to \mathcal{L}(n_i,n_j;0) \quad 1 \leq i < j \leq k$$

Definition 1.3.31. For each $x \in \mathcal{L}(n_1,\dots,n_k;n)$ and each $1 \leq i < j \leq k$, let $c_{ij}(x)$ be the number of changes of directions (i.e. the corners of the corresponding lattice path, as in Example 1.3.30) in the lattice path $\phi_{ij}(x)$. The **complexity index** $c(x)$ of $x \in \mathcal{L}(n_1,\dots,n_k;n)$ is defined by

$$c(x) = \max_{1\leq i<j\leq k} c_{ij}(x)$$

The m-th filtration stage $\mathcal{L}^{(m)}$ of the lattice paths operad $\mathcal{L}$ is defined by

$$\mathcal{L}^{(m)}(n_1,\dots,n_k;n) = \{x \in \mathcal{L}(n_1,\dots,n_k;n) | c(x) \leq m\}$$

It is easy to check that, for all $m \geq 0$, $\mathcal{L}^{(m)}$ is a suboperad of $\mathcal{L}$, and moreover these suboperads make up an exhaustive filtration of $\mathcal{L}$:

$$\Delta = \mathcal{L}^{(0)} \subset \mathcal{L}^{(1)} \subset \mathcal{L}^{(2)} \subset \dots \subset \mathcal{L}.$$

Definition 1.3.32. A **functor-operad** ξ on an $\mathcal{E}$-category $\mathcal{C}$ consists of a sequence of twisted-symmetric $\mathcal{E}$-functors $\xi_k \colon \mathcal{C}^{\otimes k} \to \mathcal{C}$, $k \geq 0$, together with $\mathcal{E}$-natural transformations

$$\mu_{i_1,\dots,i_k} \colon \xi_k \circ (\xi_{i_1} \otimes \dots \otimes \xi_{i_k}) \to \xi_{i_1+\dots+i_k}, \quad i_1,\dots,i_k \geq 0,$$

satisfying some constraints similar to the operadic ones.
A ξ-algebra is an object X of $\mathcal{C}$ equipped with a sequence of morphisms

$$\alpha_k : \xi_k(X,\dots,X) \to X, \quad k \geq 0,$$

satisfying identities similar to those of an algebra over an operad.

Proposition 1.3.33. *Let X, Y be objects of an $\mathcal{E}$-category $\mathcal{C}$ with functor-operad ξ, and let Y be a ξ-algebra. Then $\underline{\mathcal{C}}(X,Y)$ is a $Coend_\xi(X)$-algebra where the coendomorphism operad is given by*

$$Coend_\xi(X)(k) = \underline{\mathcal{C}}(X, \xi_k(X,\dots,X)), \quad k \geq 0.$$

Given a coloured operad $\mathcal{O}$ in $\mathcal{E}$, with set of colours C, we can first consider its underlying category: $\mathcal{O}_u$. As we saw before, $\mathcal{O}_u$ is a $\mathcal{E}$-category. Since the unary operations act contravariantly on the inputs and covariantly on the output, any coloured operad $\mathcal{O}$ in $\mathcal{E}$ can be considered as a sequence of functors

$$\mathcal{O}(-,\dots,-;-) \colon \mathcal{O}_u^{op} \otimes \dots \otimes \mathcal{O}_u^{op} \otimes \mathcal{O}_u \to \mathcal{E}, \quad k \geq 0$$

The category of $\mathcal{E}$-functors $\mathcal{O}_u \to \mathcal{E}$ and $\mathcal{E}$-natural transformations is the underlying category of a $\mathcal{E}$-category which we shall denote by $\mathcal{E}^{\mathcal{O}_u}$. Each coloured operad $\mathcal{O}$ in $\mathcal{E}$ induces a sequence of $\mathcal{E}$-functors

$$\xi(\mathcal{O})_k \colon \mathcal{E}^{\mathcal{O}_u} \otimes \dots \otimes \mathcal{E}^{\mathcal{O}_u} \to \mathcal{E}^{\mathcal{O}_u} \quad k \geq 0,$$

by the familiar coend formulas, given by the Day-Street convolution of the operad:

$$\xi(\mathcal{O})_k(X_1,\dots,X_k)(c) = \mathcal{O}(-,\dots,-;c)\otimes_{\mathcal{O}_u\otimes\cdots\otimes\mathcal{O}_u} X_1(-)\otimes\cdots\otimes X_k(-).$$

Proposition 1.3.34. *The sequence $\xi(\mathcal{O})_k$, $k\geq 0$, extends to a functor-operad on the diagram category $\mathcal{E}^{\mathcal{O}_u}$ in such a way that the categories of $\mathcal{O}$-algebras and of $\xi(\mathcal{O})$-algebras are canonically isomorphic.*

Definition 1.3.35. The composite construction of Propositions 1.3.33 and 1.3.34 above, assigning a single coloured operad to a coloured one

$$\mathcal{O}\longmapsto Coend_{\xi(\mathcal{O})}(\delta)$$

for any choice of V-functor $\delta\colon \mathcal{O}_u\to\mathcal{E}$ is called δ-**condensation**. Moreover, by Proposition 1.3.33 there is a parallel δ-**totalization** functor:

$$\underline{\mathrm{Hom}}_{\mathcal{O}_u}(\delta,-)\colon Alg_{\mathcal{O}}\to Alg_{Coend_{\xi(\mathcal{O})}(\delta)}$$

which sends an $\mathcal{O}$-algebra A to a $Coend_{\xi(\mathcal{O})}(\delta)$-algebra $\underline{\mathrm{Hom}}_{\mathcal{O}_u}(\delta,A)$.

This is the main result of their paper ([BB], Theorem 3.8):

Theorem 1.3.36. *Let δ be a standard system of simplices in a monoidal model category $\mathcal{E}$. If the lattice paths operad $\mathcal{L}$ is strongly δ-reductive, then δ-condensation of the different filtration stages $\mathcal{L}^{(m)}$ of $\mathcal{L}$ yields E_m-operads $Coend_{\mathcal{L}^{(m)}}(\delta)$ in $\mathcal{E}$.*

By this theorem it follows:

Corollary 1.3.37. *If $\mathcal{E}$ is the category of chain complexes over a commutative ring and $\delta(k)$, $k\geq 0$ is the chain complex of simplicial chains of $C_*(\Delta(k))$, the totalization $\mathrm{Tot}_\delta(X)$ of a $\mathcal{L}^{(n)}$-algebra X has a natural E_n-algebra structure.*

Remark 1.3.38. Something more general than the corollary above holds (i.e. it holds for any monoidal model category V and for any standard system of simplices δ such that $\mathcal{L}$ is strongly δ-reductive), but we are adapting the results to our needs.

Similarly to what was done with the $\square$ funny tensor product, Batanin and Davydov in [BD] introduced the following:

Definition 1.3.39. The **paths operad** $\mathcal{M}$ is the $\mathbb{N}$-coloured operad in *Sets* defined by:

$$\mathcal{M}(n_1,\dots,n_k;n) = \mathrm{Cat}_{*,*}(\langle n+1\rangle,\langle n_1+1\rangle\times\cdots\times\langle n_k+1\rangle).$$

The operadic substitution maps are induced by cartesian product and composition in $\mathrm{Cat}_{*,*}$.

Let now $\mathcal{M}^{(0)}=\mathcal{M}\times Assoc$ be the product in the category of symmetric colored operads, where *Assoc* is the one coloured *Set*-operad for associative monoids. By definition

$$\mathcal{M}^{(0)}(n_1,\dots,n_k;n) = \mathrm{Cat}_{*,*}(\langle n+1\rangle,\langle n_1+1\rangle\times\cdots\times\langle n_k+1\rangle)\times\Sigma_k,$$

and the operadic composition is induced by the operadic composition in $\mathcal{M}$ by the first variable and the operadic composition on symmetric groups Σ_k in the second variable.

Theorem 1.3.40. *The category of algebras of* $\mathcal{M}^{(0)}$ *in any cocomplete symmetric monoidal category* V *is isomorphic to the category of cosimpicial monoids in* V. *The natural projection* $\mathcal{M}^{(0)} \to \mathcal{M}$ *is an operadic morphism which induces the forgetful functor from commutative cosimplicial monoids to cosimplicial monoids.*

Definition 1.3.41. Let $\tau\colon [p] \to [m]$ and $\pi\colon [q] \to [m]$ be two maps in Δ. A shuffling of length n of τ, π is a decomposition of the images of τ and π into disjoint union of connected intervals

$$\begin{aligned} Im(\tau) &= A_1 \cup A_2 \cup \dots \cup A_s, \qquad A_1 < A_2 < \dots < A_s \\ Im(\pi) &= B_1 \cup B_2 \cup \dots \cup B_t, \qquad B_1 < B_2 < \dots < B_t \end{aligned} \tag{1.3.2}$$

with $s + t = n + 1$, satisfying either:

$$A_1 \leq B_1 \leq A_2 \leq B_2 \leq \dots \tag{1.3.3}$$

or

$$B_1 \leq A_1 \leq B_2 \leq A_2 \leq \dots \tag{1.3.4}$$

(that is, the rightmost end-point of A_i may coincide with the leftmost end-point of the sequel B). The **linking number** $lk(\tau, \pi)$ is defined as n is the minimal possible shuffling of τ, π has length n.

Definition 1.3.42. Let X be a cosimplicial monoid,$n \geq 0$. X is called n-commutative if for any $\tau\colon [p] \to [m], \pi\colon [q] \to [m]$ in Δ with $lk(\tau, \pi) \leq n$, the diagram below commutes:

$$\begin{array}{ccccc} X(p) \otimes X(q) & & \xrightarrow{X(\tau)\otimes X(\pi)} & & X(m) \otimes X(m) \\ \downarrow & & & & \downarrow \mu \\ X(q) \otimes X(p) & \xrightarrow{X(\pi)\otimes X(\tau)} & X(m) \otimes X(m) & \xrightarrow{\mu} & X(m) \end{array} \tag{1.3.5}$$

We realize $\mathcal{M}^{(n)}$ as a quotient of $\mathcal{M}^{(0)}$. For this we introduce a relation on $\mathcal{M}^{(0)}(p, q; k)$. Let (ϕ, σ) be an element of $\mathcal{M}^{(0)}(p, q; k)$:

$$\phi\colon \langle k+1 \rangle \to \langle p+1 \rangle \times \langle q+1 \rangle \text{ and } \sigma \in \Sigma_2 = \{e, t\}.$$

We say that (ϕ, e) is *n-equivalent* to (ϕ, t) if $lk(\phi) \leq n$. Let $\mathcal{M}^{(n)}$ be the quotient of $\mathcal{M}^{(0)}$ by the equivalence relation generated by n-equivalence relation.

Theorem 1.3.43. *The category of* n*-commutative cosimplicial monoids is equivalent to the category of* $\mathcal{M}^{(n)}$*-algebras.*

Theorem 1.3.44. *There are morphisms of operads* $p^{(n)}\colon \mathcal{L}^{(n)} \to \mathcal{M}^{(n-1)}$ $n \geq 1$ *making the following diagram commutative:*

$$\begin{array}{ccccccccc} \mathcal{L}^{(1)} & \to & \mathcal{L}^{(2)} & \to \dots \to & \mathcal{L}^{(n)} & \to \dots \to & \mathcal{L} \\ \downarrow p^{(1)} & & \downarrow p^{(2)} & & \downarrow p^{(n)} & & \downarrow p^{(\infty)} \\ \mathcal{M}^{(0)} & \to & \mathcal{M}^{(1)} & \to \dots \to & \mathcal{M}^{(n-1)} & \to \dots \to & \mathcal{M} \end{array} \tag{1.3.6}$$

Corollary 1.3.45. *Let* $X_\bullet$ *be an* n*-commutative cosimplicial monoid in* $\mathrm{Ch}(\Bbbk)$. *Then there is an action of an operad homotopy equivalent to* $\mathrm{Ch}_\bullet(E_{n+1}, \Bbbk)$ *on the totalization* $Tot(X_\bullet) \in \mathrm{Ch}(\Bbbk)$.

1.4 The categories Θ_n

The categories Θ_n were introduced by Joyal in the influential preprint [J], in order to develop a theory of infinite dimensional (weak) categories. Here we recall the definition of the categories $\Theta_n, n \geq 1$, and some of its properties, based on the results of Berger [Be1], [Be2] and Joyal [J]. We also recall the realization $|X|$ of a cellular set $X: \Theta^{op} \to Sets$, and discuss the totalization of a cocellular set $Y: \Theta_n \to Sets$. We denote by $[n]$ the ordinal $0 < 1 < \cdots < n$ having $n+1$ elements. Recall that the simplicial category Δ has objects $[n], n \geq 0$, that is, all non-empty finite ordinals. Its morphisms are the monotonous maps $f: [k] \to [l]$, i.e. $f(i) \leq f(j)$ if $i \leq j$. Recall the relations between the standard elementary face operators $\partial^i: [n-1] \to [n]$ and the elementary degeneracy operators $\epsilon^i: [n+1] \to [n]$, $i = 0, \dots, n$, in Δ:

$$\begin{aligned} &\partial^j \partial^i = \partial^i \partial^{j-1} \text{ if } i < j \\ &\epsilon^j \epsilon^i = \epsilon^i \epsilon^{j+1} \text{ if } i \leq j \\ &\epsilon^i \partial^j = \begin{cases} \partial^i \epsilon^{j-1} & \text{if } i < j; \\ \text{id} & \text{if } i = j, j+1; \\ \partial^{i-1} \epsilon^j & \text{if } i > j+1 \end{cases} \end{aligned} \tag{1.4.1}$$

1.4.1 n-levelled trees

Let us first define the notion of n-levelled tree, introduced by Berger in [Be1].

Definition 1.4.1. A n-levelled tree T is a collection of finite sets $\{T(i)\}$, $0 \leq i \leq n$, endowed with a map $i_T: T_{\geq 1} \to T$ which lowers the level by 1, such that $T(0)$ is a 1-element set, and such that the sets $i_T^{-1}(x)$, $x \in T$, are linearly ordered.

Let us introduce some terminology related to levelled trees, which will be used later on. For $x \in T(i)$ we write $ht(x) = i$, for the height of x. By definition, $n = ht(T) = \max_{x \in T} ht(x)$. A vertex x of a levelled tree is called an input, or a leaf, if $i^{-1}(x) = \emptyset$. Note that for an n-levelled tree T, the height of an input may be smaller than n. Moreover, $T(i)$ can be empty for $i > 0$.
An edge is a pair (x, y) with $x = i_T(y)$. The set of edges of T is denoted by $e(T)$. We define the dimension $d(T) = \sharp e(T)$. A levelled tree is called linear if $d(T) = ht(T)$. For each vertex $x \in T$, the ordered set of incoming edges $e_x(T)$ is defined as $i_T^{-1}(x)$.
For a levelled tree T define a levelled tree $\bar{T}$ as follows: for each $x \in T$, we set $e_x(\bar{T}) = e_x(T) \cup (x, x_-) \cup (x, x_+)$ with the order in which (x, x_-) is the minimal element and (x, x_+) is the maximal element. Thus we add the leftmost and the rightmost element to each set $e_x(T)$. It results in $\bar{T} = T(i) + 2T(i-1)$, and $ht(\bar{T}) = ht(T) + 1$. A T-sector of height k is a triple $(x; y_L, y_R)$ where $x \in T(k)$, $y_L, y_R \in \bar{T}(k+1)$, $i_{\bar{T}}(y_L) = i_{\bar{T}}(y_R) = x$, and y_L, y_R are consecutive elements of $\bar{T}(k+1)$. We say that x supports a sector $(x; y_L, y_R)$. It follows that each input vertex x of T supports a unique sector (which is $(x; x_-, x_+)$).

1.4.2 The wreath product definition of Θ_n

The definition of the category Θ_n is given inductively via the wreath product $\Delta \wr \mathcal{A}$:

Definition 1.4.2. Let $\mathcal{A}$ be a category. The objects of the category $\Delta \wr \mathcal{A}$ are tuples $([n], A_1, \dots, A_n)$, where $A_1, \dots, A_n \in \mathcal{A}$. A morphism in $\Delta \wr \mathcal{A}$

$$\Phi\colon ([n], A_1, \dots, A_n) \to ([m], B_1, \dots, B_m)$$

is a tuple $(\phi; \phi_1, \dots, \phi_n)$, with $\phi\colon [n] \to [m]$ a morphism in Δ and $\phi_i = (\phi_i^{\phi(i-1)+1}, \dots, \phi_i^{\phi(i)})$ is a tuple of morphisms in A, with $\phi_i^k\colon A_i \to B_k$, $\phi(i-1)+1 \le k \le \phi(i)$. The composition is defined in the natural way.

The reader is advised to jump to Lemma 1.4.7 which explains a natural framework in which the category $\Delta \wr \mathcal{A}$ emerges. We set:

$$\Theta_1 = \Delta \text{ and } \Theta_n = \Delta \wr \Theta_{n-1},\ n \ge 2 \tag{1.4.2}$$

1.4.3 n-globular sets and strict n-categories

There is another category isomorphic to the category Θ_n.
Recall that an n-globular set is the data one has on the underlying sets of objects, 1-morphisms,..., n-morphisms of a strict n-category. In this sense, it is a "pre-n-category". For $n = 1$, it is a quiver. The general definition is as follows:

Definition 1.4.3. An n-globular set is a collection of sets $X_0, X_1, \dots, X_n$ and maps

$$X_n \underset{t_{n-1}}{\overset{s_{n-1}}{\rightrightarrows}} X_{n-1} \underset{t_{n-2}}{\overset{s_{n-2}}{\rightrightarrows}} \dots \underset{t_1}{\overset{s_1}{\rightrightarrows}} X_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} X_0 \tag{1.4.3}$$

(here s_k are source maps and t_k are target maps), such that $s_k s_{k+1} = s_k t_{k+1}, t_k s_{k+1} = t_k t_{k+1}, 0 \le k \le n-1$.

For two n-globular sets X, Y, a morphism $f\colon X \to Y$ is defined as a sequence of maps $f_i\colon X_i \to Y_i, 0 \le i \le n$, which commute with the source and the target maps s and t. The category of n-globular sets is denoted by Glob_n. The reader may easily interpret the category Glob_n as some presheaf category.

The following question arises: how one can define the free strict n-category generated by an n-globular set? More precisely, the question is "How to define a left adjoint functor ω_n to the forgetful functor $R\colon \mathrm{Cat}_n \to \mathrm{Glob}_n$?". (The $n = 1$ case is the well-known construction of the free category generated by a quiver). In [Ba2] Batanin solved this problem by introducing the star construction, which associates an n-globular set T^* to an n-levelled tree T.
We recall this construction, following a more explicit treatment given in [[Be1], Lemma 1.2]:

Lemma 1.4.4. *Let T be an n-levelled tree, denote by T_k^* the set of all sectors of T of height $k, 0 \le k \le n$. Then T^* is an n-globular set.*

Proof. Let $(x; y_L, y_R) \in T_k^*$, we have to define $s_{k-1}(x; y_L, y_R)$ and $t_{k-1}(x; y_L, y_R)$. Let x_L, x, x_R be the three consecutive elements in $\bar{T}(k)$. Define

$$s_{k-1}(x; y_L, y_R) = (i_T(x); x_L, x) \text{ and } t_{k-1}(x; y_L, y_R) = (i_T(x); x, x_R)$$

One easily sees that the globular identities hold, see [[Be1], Lemma 1.2] for more detail. □

Example 1.4.5. Let T be a linear n-levelled tree. Then there is a single sector of height n, and two sectors of any height $0, 1, \dots, n-1$.

Thanks to the T^* construction, one can define the left adjoint $\omega_n \colon \mathrm{Glob}_n \to \mathrm{Cat}_n$ to the forgetful functor $R \colon \mathrm{Cat}_n \to \mathrm{Glob}_n$ in the following way.
Let X be an n-globular set. We define an n-globular set $\omega_n(X)$ and prove that it is a strict n-category. Set

$$(\omega_n(X))_k := \bigcup_{T: ht(T) \leq k} \mathrm{Hom}_{\mathrm{Glob}_n}(T^*, X) \tag{1.4.4}$$

(one often uses the notation $\mathrm{Hom}_{\mathrm{Glob}_n}(T^*, X) = X^T$).
First of all, we show that $\omega_n(X)$ is an n-globular set. Denote by $\partial_k T$ the $(k-1)$-levelled tree, obtained by removing all vertices of height higher than $k-1$. There are two maps of n-globular sets $s^*_{k-1}, t^*_{k-1} \colon (\partial_k T)^* \to (\partial_{k+1} T)^*$. In general, a map of globular sets $S^* \to T^*$ is determined by its restriction to the input sectors of S^*, see [[Be1], Lemma 1.3]. The map s^*_{k-1} (respectively, t^*_{k-1}) is obtained by assigning to each input vertex x of $\partial_k T$ (which uniquely defined its input sector) the leftmost (respectively, rightmost) input sector in $\partial_{k+1} T$ supported by x. One shows that the maps s^*_{k-1}, t^*_{k-1} satisfy the identities dual to the globular identities. Thus, for any n-globular set X, and for a k-levelled tree $T, k \leq n$, the pre-compositions with the maps s^*_{k-1}, t^*_{k-1} define maps

$$s_{k-1}, t_{k-1} \colon X^T \to X^{\partial_k T}$$

It follows that these maps satisfy the globular identities. Thus, $\omega_n(X)$ is a globular set. Next, prove that $\omega_n(X)$ is a strict n-category. The following statement is proven in [[Be2], Th. 3.7].

Proposition 1.4.6. *For any n-ordinals S, T, one has $\Theta_n(S, T) = \mathrm{Cat}_n(\omega_n(\bar{S}^*), \omega_n(\bar{T}^*))$.*

The proof is obtained, by induction, from the following nice interpretation of the wreath product, [[Be2], Prop. 3.5]:

Lemma 1.4.7. *Assume that a small category $\mathcal{A}$ is a full subcategory of a cocomplete cartesian monoidal category $\mathcal{E}$. Then $\Delta \wr \mathcal{A}$ is a full subcategory of $\mathcal{E}$-Cat.*

Proof. Any 1-ordinal $[n]$ can be considered as a linear category $\mathbf{n}$ with $n+1$ objects $0, \dots, n$, with a single morphism in $\mathbf{n}(i, j)$ for $i \leq j$ and with empty set of morphisms otherwise. Having n objects $A_1, \dots, A_n$ of $\mathcal{A}$, we regard them as objects of $\mathcal{E}$, and consider the linear $\mathcal{E}$-quiver:

$$0 \xrightarrow{A_1} 1 \xrightarrow{A_2} 2 \xrightarrow{A_3} \dots \xrightarrow{A_n} n$$

Consider the $\mathcal{E}$-category generated by this quiver, denote it by $F_{\mathcal{E}}(A_1, \dots, A_n)$ (here we use cocompleteness of $\mathcal{E}$ to show that the forgetful functor from $\mathcal{E}$-categories to $\mathcal{E}$-quivers has a left adjoint). For $B_1, \dots, B_m \in \mathcal{A}$, a $\mathcal{E}$-functor $\phi \colon F_{\mathcal{E}}(A_1, \dots, A_n) \to F_{\mathcal{E}}(B_1, \dots, B_m)$ is defined by its restriction to "generators", that is, by a map $\phi \colon [n] \to [m]$, and, for any $1 \leq i \leq n$, a morphism $A_i \to F_{\mathcal{E}}(\phi(i-1), \phi(i)) = B_{\phi(i-1)+1} \times \dots \times B_{\phi(i)}$. We conclude that these $\mathcal{E}$-functors are the same as the morphisms $([n], A_1, \dots, An) \to ([m], B_1, \dots, B_m)$ in $\Delta \wr \mathcal{A}$. $\square$

Example 1.4.8. For the case $\Theta_2 = \Delta \wr \Delta$, we set $\mathcal{E} = \mathrm{Cat}$, using the embedding $\Delta \to \mathrm{Cat}\colon [n] \mapsto \mathbf{n}$. Thus to the element $([n], [\ell_1], \ldots, [\ell_n])$ is associated the 2-category generated by the 2-globular set T^*, where T is the corresponding 2-ordinal $[\ell_1 + \cdots + \ell_n + n - 1] \to [n-1]$.

1.4.4 Disks

The category of disks was introduced in [J]. The category of the non-empty intervals is denoted by $\mathcal{I}_f$ (including $\langle 0 \rangle$). Joyal (loc.cit.) showed that $\mathcal{I}_f^{\mathrm{op}} \simeq \Delta_+$ where Δ_+ is the category of *all* finite ordinals (including the empty ordinal which is the initial object, we denote it [-1]). The functor $F\colon \Delta_+^{\mathrm{op}} \to \mathcal{I}_f$ is $[n] \mapsto \Delta_+([n], [1])$, $F([n]) = [n+1]$. The dual functor $G\colon \Delta_f^{\mathrm{op}} \to \Delta_+$ is $[n] \mapsto \mathcal{I}_f([n], [1])$, then the initial object $[-1]$ is $\mathcal{I}_f([0], [1])$, and in general $G([n]) = [n-1]$.

Definition 1.4.9. A disk of finite sets $D_\bullet$ is a sequence $D_1, D_2, \ldots$ of finite sets, equipped with the following data:

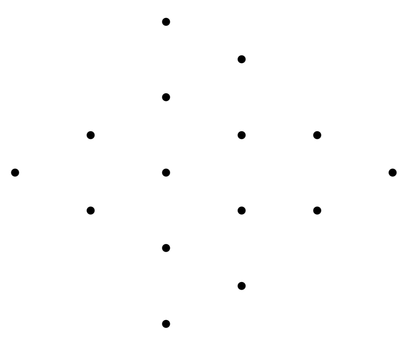
(a) a map $p\colon D_k \to D_{k-1}$ such that for any $x \in D_{k-1}$ the pre-image $p^{-1}(x)$ has an interval structure, $k \geq 1$

(b) two maps $d_0, d_1\colon D_{k-1} \to D_k$ sending $x \in D_{k-1}$ to the leftmost and the righmost elements of the interval $p^{-1}(x), k \geq 1$

(c) $\mathrm{Eq}(d_0, d_1\colon D_k \to D_{k+1}) = d_0(D_{k-1}) \cup d_1(D_{k-1}), k \geq 1$

(d) D_0 is a single point.

Example 1.4.10. In order to help visualization, the following 2-disk represents $D_2 \to D_1$, where D_2 is the set with 15 elements and D_1 the set with 6 elements, and the map can be interpreted as the horizontal projection.



(1.4.5)

A map of two disks $F\colon D_\bullet \to D'_\bullet$ is a collection of maps $\{F_k\colon D_k \to D'_k\} k \geq 0$ compatible with p, d_0, d_1, such that for any $x \in D_k$ the map $p^{-1}(x) \to p^{-1}(F_k(x))$ is a map of intervals, $k \geq 0$. The category of disks is denoted by Disk. For a disk $D_\bullet$ the interior $i(D_k)$ is defined as $D_k \setminus \{d_0(D_{k-1} \cup d_1(D_{k-1})\}$. It is an ordinal, and the sequence of maps of ordinals $p\colon i(D_k) \to i(D_{k-1})$, $k \geq 1$ makes $i(D_\bullet) = \{i(D_k)\}_{k \geq 0}$ a levelled tree. The height $ht(D_\bullet)$ is defined as the height of the level tree $i(D_\bullet)$. The category of disks of height $\leq n$ is denoted by Disk_n.

The functor i sends disks to levelled trees. The functor $T \mapsto \bar{T}$ is a left adjoint to it. For any levelled tree T, the levelled tree $\bar{T}$ is a disk of finite sets. The elements of $\bar{T}$ in the image of i are internal, and the elements in $\bar{T} \setminus T$ are boundary. A map of disks $\bar{S} \to \bar{T}$ is "more general" than a map of levelled trees $S \to T$. The reason is that a map of disks $\bar{S} \to \bar{T}$ may map an internal point to a boundary point in $\bar{T}$. Thus, the category of n-levelled trees is identified with a not full subcategory of Disk_n. The following Proposition is [[Be1], Prop. 2.2]:

Proposition 1.4.11. *For any n-levelled trees S, T one has:*

$$\mathrm{Cat}_n(\omega_n(\bar{S}^*), \omega_n(\bar{T}^*)) = \mathrm{Disk}_n(\bar{T}, \bar{S})$$

Thus, the assignment $T \mapsto \bar{T}$ provides an equivalence of Θ_n^{op} and Disk_n.

Remark 1.4.12. We can restrict the assignment from the proof [[Be1], Prop. 2.2] to the maps of disks $\bar{S} \to \bar{T}$ which come from maps of levelled trees $S \to T$ (that is, which map internal points to internal). The corresponding sub-category C of Cat_n has objects $\omega_n(\bar{T}^*)$, T a n-levelled tree, and has the set of morphisms $C(\omega_n(\bar{S}^*), \omega_n(\bar{T}^*))$ which is the subset of $\mathrm{Cat}_n(\omega_n(\bar{S}^*), \omega_n(\bar{T}^*))$ formed by maps of n-categories, preserving minima and maxima, in an appropriate sense. For $n = 2$, this equivalence is used by Tamarkin in [Tam2]. In fact, this equivalence (rather than the equivalence of Proposition 1.4.11) can be thought of as a proper analogue of the equivalence $\mathcal{I} \simeq \Delta_+^{op}$, for $n \geq 2$.

1.4.5 The categories Θ_n as higher analogues of the category Δ; inner and outer face maps

We have three equivalent descriptions of the category Θ_n which are:

(a) the definition via the wreath product 1.4.2,

(b) the definition via morphisms of free strict n-categories $\omega_n(\bar{T}^*)$, Proposition 1.4.6,

(c) as the dual of the category Disk_n, Proposition 1.4.11.

We will take advantage of all three equivalences. In particular, (c) is used to naturally define the realization/totalization, (b) is used to see that any strict n-category C has a nerve which is a n-cellular set $N(C)\colon \Theta_n^{op} \to \mathit{Sets}$, and (a) is the combinatorially most explicit and manageable. Existence of the latter nerve goes back to Batanin, and was the main motivation in [J], where the disk categories were defined. It also makes it possible to consider Θ_n as an analogue of Δ, for $n \geq 2$. Note that the nerve $N(C)$ of the ordinary category C is a simplicial set, whose components can be defined as

$$N(C)_k \coloneqq \mathrm{Cat}([k], C)$$

(where $[k]$ is the usual ordinal category with $k+1$ objects). We see directly that it gives rise to a simplicial set, because a map $[k] \to [m]$ in Δ amounts to the same thing as a map of the ordinal categories $[k] \to [m]$.
Let now C be a strict n-category. Define its n-nerve as a cellular set $N(C)\colon \Theta_n^{op} \to \mathit{Sets}$, for which

$$N(C)_T \coloneqq \mathrm{Cat}_n(\omega_n(\bar{T}^*), C) \tag{1.4.6}$$

It gives rise to an n-cellular set because by Proposition 1.4.6:

$$\Theta_n(S,T) = \mathrm{Cat}_n(\omega_n(\bar{S}^*), \omega_n(\bar{T}^*)).$$

For any strict n-category C, the n-cellular set $N(c)$ has a property which is a higher counterpart of the Boardman-Vogt inner horns filling property for $n = 1$ (which gives rise to a definition of a quasi-category, or $(\infty, 1)$-category). It was a motivation in [J] to elaborate this analogy, and was further studied by Ara in [Ara].

1.5 Reminder on monads

Here we recall definitions and some general facts on monads and algebras over monads. The reader is referred to [ML2], [R2] for more detail.

Let $\mathscr{C}$ be a category. Recall that a monad in $\mathscr{C}$ is given by an endofunctor

$$T \colon \mathscr{C} \to \mathscr{C}$$

and natural transformations

$$\eta \colon \mathrm{Id} \Rightarrow T \text{ and } \mu \colon T^2 \Rightarrow T$$

so that the following diagrams commute:

$$\begin{array}{ccc} T^3 & \xRightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\Downarrow & & \Downarrow{\scriptstyle \mu} \\ T^2 & \xRightarrow{\mu} & T \end{array} \qquad \begin{array}{ccccc} T & \xRightarrow{\eta T} & T^2 & \xLeftarrow{T\eta} & T \\ & {\scriptstyle \mathrm{Id}}\searrow & \Downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \mathrm{Id}} & \\ & & T & & \end{array}$$

A monad appears from a pair of adjoint functors. Assume we have an adjoint pair

$$F \colon \mathscr{C} \rightleftarrows \mathscr{D} : U \tag{1.5.1}$$

with adjunction unit and counit $\eta \colon \mathrm{Id}_{\mathscr{C}} \Rightarrow UF$ and $\varepsilon \colon FU \Rightarrow \mathrm{Id}_{\mathscr{D}}$.

It gives rise to a monad in $\mathscr{C}$, defined as:

$$T = UF, \quad \eta = \eta \colon \mathrm{Id}_{\mathscr{C}} \Rightarrow T, \quad \mu = U\epsilon F \colon T^2 \Rightarrow T$$

An *algebra A over a monad T* is given by an object $A \in \mathscr{C}$ equipped with a morphism $a \colon TA \to A$ such that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & {\scriptstyle \mathrm{Id}_A}\searrow & \downarrow{\scriptstyle a} \\ & & A \end{array} \qquad \begin{array}{ccc} T^2A & \xrightarrow{\mu_A} & TA \\ {\scriptstyle Ta}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow{a} & A \end{array}$$

The morphisms of algebras over a monad T are defined as morphisms $f \colon A \to B$ in $\mathscr{C}$ such that the natural diagram commutes. The category of T-algebras is denoted by $\mathscr{C}^T$.

There is an adjunction

$$F^T : \mathscr{C} \rightleftarrows \mathscr{C}^T : U^T$$

which by its own gives rise to a monad.

There is a functor $\Phi\colon \mathscr{D} \to \mathscr{C}^T$, sending an object Y of $\mathscr{D}$ to the T-algebra $A = UY$, with $a\colon TA = UFUY \to UY = A$ equal to $U\varepsilon_Y$. The functor Φ is called the *Eilenberg-Moore comparison functor*.

An adjunction (1.5.1) is called *monadic* if the functor $\Phi\colon \mathscr{D} \to \mathscr{C}^T$ is an equivalence.

There is a criterion when an adjunction is monadic, called the *Beck monadicity theorem*. We recall its statement below.

Recall that a *split coequalizer* in a category is a diagram

$$\xymatrix{A \ar@<0.5ex>[r]^{f} \ar@<-0.5ex>[r]_{g} & B \ar@/_1pc/[l]_{s} \ar[r]^{h} & C \ar@/_1pc/[l]_{t}}$$

such that

(1) $f \circ s = \mathrm{id}_B$,

(2) $g \circ s = t \circ h$,

(3) $h \circ t = \mathrm{id}_C$,

(4) $h \circ f = h \circ g$.

Recall the following lemma:

Lemma 1.5.1. *A split coequalizer is a coequalizer, and is an absolute coequalizer (that is, is preserved by any functor).*

It is enough to prove the first statement, because a split equalizer remains a split equalizer after application of any functor. See e.g. [[R2], Lemma 5.4.6] for detail. □

Given a pair

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

in a category $\mathscr{D}$, and a functor $U\colon \mathscr{D} \to \mathscr{C}$, we say that this pair is *U-split* if the pair

$$U(A) \underset{g}{\overset{f}{\rightrightarrows}} U(B)$$

in $\mathscr{C}$ can be extended to a split coequalizer.

Theorem 1.5.2. *Let $F\colon \mathscr{C} \rightleftarrows \mathscr{D}\colon U$ be a pair of adjoint functors, and let $T = UF$ be the corresponding monad. Consider the Eilenberg-Mac Lane comparison functor $\Phi\colon \mathscr{D} \to \mathscr{C}^T$. Then:*

(1) *if $\mathscr{D}$ has coequalizers of all U-split pairs, the functor Φ has a left adjoint $\Psi\colon \mathscr{C}^T \to \mathscr{D}$,*

(2) *if, furthermore, U preserves coequalizers of all U-split pairs, the unit $\mathrm{Id}_{\mathscr{C}^T} \Rightarrow \Phi\Psi$ is an isomorphism,*

(3) *if, furthermore, U reflects isomorphisms (that is, $U(f)$ an isomorphism implies f an isomorphism), the counit $\Psi\Phi \Rightarrow \mathrm{Id}_{\mathscr{D}}$ is also an isomorphism.*

Therefore, if (1)-(3) hold, (U, F) is monadic. Conversely, if (U, F) is monadic, conditions (1)-(3) hold.

The reader is referred to [ML2] or [R2] for a proof.

There is another monadicity theorem, which gives sufficient but not necessary conditions for $\Phi\colon \mathscr{D} \to \mathscr{C}^T$ to be monadic. It uses *reflexive pairs* in $\mathscr{D}$ instead of U-split pairs.

A pair of morphisms $f, g\colon A \to B$ in $\mathscr{D}$ is called *reflexive* if there is a morphism $h\colon B \to A$ which splits both f and g: $f \circ h = \mathrm{id}_B = g \circ h$.

We refer the reader to [[MLM], Ch.IV.4, Th.2] for a proof of the following result, also known as the *crude monadicity Theorem*:

Theorem 1.5.3. *Let $F\colon \mathscr{C} \rightleftarrows \mathscr{D}\colon U$ be a pair of adjoint functors, and let $T = UF$ be the corresponding monad. Consider the Eilenberg-Mac Lane comparison functor $\Phi\colon \mathscr{D} \to \mathscr{C}^T$. Then:*

(1) *if $\mathscr{D}$ has coequalizers of all reflexive pairs, the functor Φ admits a left adjoint $\Psi\colon \mathscr{C}^T \to \mathscr{D}$,*

(2) *if, furthermore, U preserves these coequalizers, the unit of the adjunction $\mathrm{Id}_{\mathscr{C}^T} \to \Phi \circ \Psi$ is an isomorphism,*

(3) *if, furthermore, U reflects isomorphisms, the counit of the adjunction $\Psi \circ \Phi \to \mathrm{Id}_D$ is also an isomorphism.*

Therefore, if (1)-(3) hold, (U, F) is monadic.

Note that, unlike for Theorem 1.5.2, *the converse statement is not true.* That is, the conditions for monadicity, given in Theorem 1.5.3, are sufficient but not necessary.

The following construction is of fundamental importance for both monadicity theorems.

In the notations as above, let $A \in \mathscr{D}$. Consider two morphisms

$$FUFUA \underset{g}{\overset{f}{\rightrightarrows}} FUA \tag{1.5.2}$$

where $f = FU\varepsilon_A$ and $g = \varepsilon_{FUA}$. (Similarly, one defines such two maps for $A \in \mathscr{C}^T$).

One has *two different extensions* of this pair of arrows, which form a U-split coequalizer and a reflexive pair, correspondingly.

For the first case, consider

$$UFUFUA \underset{Ug}{\overset{Uf}{\rightrightarrows}} UFUA \xrightarrow{h} UA, \qquad s_1\colon UFUA \to UFUFUA, \quad t\colon UA \to UFUA \tag{1.5.3}$$

with $s_1 = \eta_{UFUA}$, $t = \eta_{UA}$, $h = U\varepsilon_A$.

For the second case, consider

$$FUFUA \underset{g}{\overset{f}{\rightrightarrows}} FUA, \qquad s_2\colon FUA \to FUFUA \tag{1.5.4}$$

with $s_2 = F\eta_{UA}$.

The following lemma is proven by a direct check:

Lemma 1.5.4. *For any $A \in \mathscr{D}$ (or $A \in \mathscr{C}^T$), (1.5.3) is a split coequalizer in $\mathscr{C}$, whence (1.5.4) is a reflexive pair in $\mathscr{D}$ (respectively, in $\mathscr{C}^T$).*

Note that s_1 is *not* a U-image of a morphism in $\mathscr{D}$, though Uf and Ug are. On the other hand, s_2 is a morphism in $\mathscr{D}$ (respectively, in $\mathscr{C}^T$).

1.6 Monoidal categories

In this brief section we give the basic notions of monoidal categories and monoidal functors, in order to fix the notation for later use.

Definition 1.6.1. A *monoidal category* is a sextuple $(\mathcal{C}, \otimes, \alpha, 1, \lambda, \rho)$ given by a category $\mathcal{C}$ together with:

(1) a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the **tensor product**,

(2) a **unit object** $1 \in Ob(\mathcal{C})$,

(3) a natural isomorphism $\alpha\colon (-)\otimes((-)\otimes(-)) \xrightarrow{\simeq} ((-)\otimes(-))\otimes(-)$ with components of the form
$$\alpha_{X,Y,Z}\colon X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$
called the **associator**,

(4) a natural isomorphism $\lambda\colon (1 \otimes (-)) \xrightarrow{\simeq} (-)$ with components of the form
$$\lambda_X\colon 1 \otimes X \to X$$
called the **left unitor**, and

(5) a natural isomorphism $\rho\colon (-) \otimes 1 \xrightarrow{\simeq} (-)$ with components of the form
$$\rho_X\colon X \otimes 1 \to X$$
called the **right unitor**,

satisfying the following two axioms:

(i) the **triangle axiom**: the diagram

$$\begin{array}{ccc} X \otimes (1 \otimes Y) & \xrightarrow{\alpha_{X,1,Y}} & (X \otimes 1) \otimes Y \\ & {\scriptstyle \mathrm{id}_X \otimes \lambda_Y} \searrow \quad \swarrow {\scriptstyle \rho_X \otimes \mathrm{id}_Y} & \\ & X \otimes Y & \end{array} \tag{1.6.1}$$

is commutative for all $X, Y \in Ob(\mathcal{C})$;

(ii) the **pentagon axiom**: the diagram

$$\begin{array}{ccc} & (X \otimes Y) \otimes (Z \otimes T) & \\ {\scriptstyle \alpha_{X,Y,Z\otimes T}} \nearrow & & \searrow {\scriptstyle \alpha_{X\otimes Y,Z,T}} \\ X \otimes (Y \otimes (Z \otimes T)) & & ((X \otimes Y) \otimes Z) \otimes T \\ \downarrow {\scriptstyle \mathrm{id}_X \otimes \alpha_{Y,Z,T}} & & \uparrow {\scriptstyle \alpha_{X,Y,Z} \otimes \mathrm{id}_T} \\ X \otimes ((Y \otimes Z) \otimes T) & \xrightarrow[\alpha_{X,Y\otimes Z,T}]{} & (X \otimes (Y \otimes Z)) \otimes T \end{array} \tag{1.6.2}$$

is commutative for all $X, Y, Z, T \in Ob(\mathcal{C})$.

Definition 1.6.2. A monoidal category is said to be *strict* if the associator, left unitor and right unitor are all identity morphisms, i.e. if for all X, Y, Z in $Ob(\mathcal{C})$ one has equalities $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ and $X \otimes 1 = X = 1 \otimes X$.
In this case the pentagon axiom and the triangle axiom hold automatically.
Moreover we define a *tensor category* to be a $\Bbbk$-linear (or dg- over $\Bbbk$) monoidal category $\mathcal{C}$. Similarly we can define a *strict tensor category.*

Definition 1.6.3. Let $(\mathcal{C}, \otimes^{\mathcal{C}}, \alpha^{\mathcal{C}}, 1^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ and $(\mathcal{D}, \otimes^{D}, \alpha^{\mathcal{D}}, 1^{\mathcal{D}}, \lambda^{\mathcal{D}}, \rho^{\mathcal{D}})$ be two monoidal categories. A *colax monoidal functor* between them is

(1) a functor $F : \mathcal{C} \to \mathcal{D}$,

(2) a morphism $\epsilon : F(1^{\mathcal{C}}) \to 1^{\mathcal{D}}$,

(3) a natural transformation $J_{X,Y} : F(X \otimes^{\mathcal{C}} Y) \to F(X) \otimes^{\mathcal{D}} F(Y)$ called the **monoidal structure**

satisfying the following conditions:

(i) the **associativity axiom**: the diagram

$$\begin{array}{ccc} F(X \otimes^{\mathcal{C}} (Y \otimes^{\mathcal{C}} Z)) & \xrightarrow{F(\alpha^{\mathcal{C}})} & F((X \otimes Y) \otimes Z) \\ \downarrow{\scriptstyle J_{X,Y\otimes^{\mathcal{C}} Z}} & & \downarrow{\scriptstyle J_{X\otimes^{\mathcal{C}} Y,Z}} \\ F(X) \otimes^{\mathcal{D}} F(Y \otimes^{\mathcal{C}} Z) & & F(X \otimes^{\mathcal{C}} Y) \otimes^{\mathcal{D}} F(Z) \\ \downarrow{\scriptstyle \mathrm{id}_{F(X)} \otimes^{\mathcal{D}} J_{Y,Z}} & & \downarrow{\scriptstyle J_{X,Y} \otimes^{\mathcal{D}} \mathrm{id}_{F(Z)}} \\ F(X) \otimes^{\mathcal{D}} (F(Y) \otimes^{\mathcal{D}} F(Z)) & \xrightarrow[\alpha^{\mathcal{D}}]{} & (F(X) \otimes^{\mathcal{D}} F(Y)) \otimes^{\mathcal{D}} F(Z) \end{array} \tag{1.6.3}$$

commutes for all $X, Y, Z \in Ob(\mathcal{C})$;

(ii) the **unitality axiom**: the diagrams

$$\begin{array}{ccc} F(X \otimes^{\mathcal{C}} 1^{\mathcal{C}}) & \xrightarrow{J_{X,1^{\mathcal{C}}}} & F(X) \otimes^{\mathcal{D}} F(1^{\mathcal{C}}) \\ \downarrow{\scriptstyle F(\rho^{\mathcal{C}}_X)} & & \downarrow{\scriptstyle \mathrm{id}_{F(X)} \otimes^{\mathcal{D}} \epsilon} \\ F(X) & \xleftarrow[\rho^{\mathcal{C}}_X]{} & F(X) \otimes^{\mathcal{D}} 1^{\mathcal{D}} \end{array}$$

and

$$\begin{array}{ccc} F(1^{\mathcal{C}} \otimes^{\mathcal{C}} X) & \xrightarrow{J_{1^{\mathcal{C}},X}} & F(1^{\mathcal{C}}) \otimes^{\mathcal{D}} F(X) \\ \downarrow{\scriptstyle F(\lambda^{\mathcal{C}}_X)} & & \downarrow{\scriptstyle \epsilon \otimes^{\mathcal{D}} \mathrm{id}_{F(X)}} \\ F(X) & \xleftarrow[\lambda^{\mathcal{C}}_X]{} & 1^{\mathcal{D}} \otimes^{\mathcal{D}} F(X) \end{array}$$

commute for all $X \in Ob(\mathcal{C})$.

Definition 1.6.4. A monoidal functor (F, ϵ, J) is *strong* if ϵ and J are isomorphisms, and *strict* if ϵ and J are identities.
Moreover we define a *tensor functor* (F, ϵ, J) to be a $\Bbbk$-linear (or dg- over $\Bbbk$) monoidal functor F such that J is a natural isomorphism and ϵ is an isomorphism. A *strict tensor functor* is a tensor functor (F, ϵ, J) such that J and ϵ are identities.

For sake of completeness, we state the great theorem by Mac Lane [ML2]:

Theorem 1.6.5. *Let $X_1, \dots, X_n \in Ob(\mathcal{C})$. Let P_1, P_2 be any two parenthesized products of $X_1, \dots, X_n$ (in this order) with arbitrary insertions of the unit object 1. Let $f, g\colon P_1 \to P_2$ be two isomorphisms, obtained by composing associativity and unit isomorphisms and their inverses possibly tensored with identity morphisms. Then $f = g$.*

Chapter 2

The model category structure of weakly unital dg-categories $\mathbb{C}_{dgwu}(\Bbbk)$

Il mio sogno è nutrito d'abbandono,
di rimpianto. Non amo che le rose
che non colsi. Non amo che le cose
che potevano essere e non sono
state.... Vedo la casa, ecco le rose
del bel giardino di vent'anni or sono!

G. Gozzano,
Cocotte

2.1 Definition of $\mathbb{C}_{dgwu}(\Bbbk)$

Let us start by defining what a Kontsevich-Soibelman weakly unital dg-category is.

Let $\mathcal{A}$ be a non unital dg-category. Denote by $\mathcal{A} \oplus \Bbbk_{\mathcal{A}}$ the strictly unital dg-category where $Ob(\mathcal{A} \oplus \Bbbk_{\mathcal{A}}) = Ob(\mathcal{A})$ and

$$\mathrm{Hom}_{\mathcal{A}\oplus\Bbbk_{\mathcal{A}}}(X,Y) = \begin{cases} \mathcal{A}(X,Y) & \text{if } X \neq Y \\ \mathcal{A}(X,X) \oplus \Bbbk 1_X & \text{if } X = Y. \end{cases}$$

One has a natural embedding $i\colon \mathcal{A} \to \mathcal{A} \oplus \Bbbk_{\mathcal{A}}$, sending X to X, and $f \in \mathcal{A}(X,X)$ to the pair $(f,0) \in (\mathcal{A} \oplus \Bbbk_{\mathcal{A}})(X,X)$. We denote by 1_X the generator of $\Bbbk_X$.

Definition 2.1.1. A *weakly unital dg-category* $\mathcal{A}$ over $\Bbbk$ is a non-unital dg-category $\mathcal{A}$ over $\Bbbk$ with a distinguished closed element $\mathrm{id}_X \in \mathcal{A}^0(X,X)$ for any object X in $\mathcal{A}$, such that there exists an A_∞-functor $p\colon \mathcal{A} \oplus \Bbbk_{\mathcal{A}} \to \mathcal{A}$ which is the identity on the objects, such that $p \circ i = \mathrm{id}_{\mathcal{A}}$, $p_1(1_X) = \mathrm{id}_X$, for any $X \in Ob(\mathcal{A})$, and $p_n(f_1, \dots, f_n) = 0$ for $n \geq 2$ if f_i are morphisms in the image $i(\mathcal{A})$.

Note that this definition gives rise to the sequence of relations on the Taylor coefficients p_n, $n \geq 1$, of the A_∞-functor p. The first non-trivial relations read:

$$dp_2(f, 1_X) + p_2(df, 1_X) = f - f \circ \mathrm{id}_X, \quad dp_2(1_X, f) + p_2(1_X, df) = f - \mathrm{id}_X \circ f \quad (2.1.1)$$

and for $n = 3$:

$$\begin{aligned}
&dp_3(f,g,1_X) - (-1)^{|g|}p_3(df,g,1_X) - p_3(f,dg,1_X) = f\circ p_2(g,1_X) - p_2(f\circ g,1_X)\\
&dp_3(f,1_X,g) - (-1)^{|g|}p_3(df,1_X,g) - p_3(f,1_X,dg) = -(-1)^{|g|}p_2(f,1_X)\circ g + f\circ p_2(1_X,g)\\
&dp_3(1_X,f,g) - (-1)^{|g|}p_3(1_X,df,g) - p_3(1_X,f,dg) = -(-1)^{|g|}p_2(1_X,f)\circ g + p_2(1_X,f\circ g)\\
&dp_3(1_X,1_X,f) - p_3(1_X,1_X,df) = \mathrm{id}_X\circ p_2(1_X,f) - (-1)^{|f|}p_2(1_X,1_X)\circ f\\
&dp_3(1_X,f,1_X) - p_3(1_X,df,1_X) = \mathrm{id}_X\circ p_2(f,1_X) - p_2(1_X,f)\circ\mathrm{id}_X - p_2(f,1_X) + p_2(1_X,f)\\
&dp_3(f,1_X,1_X) - p_3(df,1_X,1_X) = f\circ p_2(1_X,1_X) - p_2(f,1_X)\circ\mathrm{id}_X\\
&dp_3(1_X,1_X,1_X) = \mathrm{id}_X\circ p_2(1_X,1_X) - p_2(1_X,1_X)\circ\mathrm{id}_X
\end{aligned} \tag{2.1.2}$$

Definition 2.1.2. Let $\mathcal{A}$, $\mathcal{C}$ be two weakly unital dg-categories, with the structure maps $p^{\mathcal{A}}\colon \mathcal{A}\oplus\Bbbk_{\mathcal{A}}\to\mathcal{A}$ and $p^{\mathcal{C}}\colon \mathcal{C}\oplus\Bbbk_{\mathcal{C}}\to\mathcal{C}$. A *weakly unital dg-functor* $F\colon\mathcal{A}\to\mathcal{C}$ is a non unital dg-functor $F\colon\mathcal{A}\to\mathcal{C}$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{A}\oplus\Bbbk_{\mathcal{A}} & \xrightarrow{F\oplus\Bbbk_F} & \mathcal{C}\oplus\Bbbk_{\mathcal{C}}\\
{\scriptstyle p^{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle p^{\mathcal{C}}}\\
\mathcal{A} & \xrightarrow{\quad F\quad} & \mathcal{C}
\end{array} \tag{2.1.3}$$

In this way, we define the category $\mathbb{C}_{dgwu}(\Bbbk)$. Its full subcategory, for which $\mathrm{id}_X\circ\mathrm{id}_X = \mathrm{id}_X$ for any object X, is denoted by $\mathbb{C}^0_{dgwu}(\Bbbk)$.

It follows from the definition that:

$$\begin{aligned}
F(\mathrm{id}_X) &= \mathrm{id}_{F(X)} && \text{for each } X\in\mathrm{Ob}(\mathcal{A})\\
F(p_n^{\mathcal{A}}(f_1,\dots,f_n)) &= p_n^{\mathcal{C}}(F(f_1),\dots,F(f_n)) && f_i\in\mathcal{A}\oplus\Bbbk_{\mathcal{A}}, i=1\dots n
\end{aligned} \tag{2.1.4}$$

Example 2.1.3. Let $\mathcal{A}$ be a strictly unital dg-category. Define $p\colon\mathcal{A}\oplus\Bbbk\mathcal{A}\to\mathcal{A}$ as $p_1|_{\mathcal{A}(X,Y)} = \mathrm{id}$, $p(1_X) = \mathrm{id}_X$, $p_n = 0$ for $n\geq 2$. Then p is a dg-functor, and $p\circ i = \mathrm{id}$. It makes a strictly unital dg-category a weakly unital dg-category and in this way we get a fully-faithful embedding $i\colon\mathbb{C}_{dg}(\Bbbk)\to\mathbb{C}_{dgwu}(\Bbbk)$.

For a weakly unital dg-category $\mathcal{A}$, define $H^0(\mathcal{A})$ as an (a priori non-unital) $\Bbbk$-linear category, having the same objects, and having morphisms $(H^0(\mathcal{A}))(X,Y) = H^0(\mathcal{A}(X,Y))$.

Lemma 2.1.4. *Let $\mathcal{A}$ be a weakly unital dg-category. Then the homotopy category $H^0(\mathcal{A})$ is a strictly unital $\Bbbk$-linear category.*

Proof. The map $[p_1]\colon H^0(\mathcal{A})\oplus\Bbbk_{H^0(\mathcal{A})}\to H^0(\mathcal{A})$, induced by the first Taylor component p_1 of the A_∞-functor p, is a dg-functor. One has $[p_1](1_X) = \mathrm{id}_X$ and $[p_1]\circ[i] = \mathrm{id}$. It follows from (2.1.1) that $\mathrm{id}_X\circ f = f\circ\mathrm{id}_X = f$, for any $f\in H^0(\mathcal{A})(X,X)$. □

Lemma 2.1.5. *Let $F\colon\mathcal{C}\to\mathcal{D}$ be a weakly unital dg-functor between weakly unital dg-categories. Then it defines a $\Bbbk$-linear functor $H^0(F)\colon H^0(\mathcal{C})\to H^0(\mathcal{D})$ of unital $\Bbbk$-linear categories.*

Proof. It is clear. □

Example 2.1.6. Let A be an associative dg-algebra over $\Bbbk$, with a strict unit 1_A. Consider $C = \mathrm{Cobar}_+(\mathrm{Bar}_+(A))$ where $\mathrm{Bar}_+(A)$ is the bar-complex of A, which is non-counital dg-coalgebra (thus, $\mathrm{Bar}_+(A) = T(A[1])/\Bbbk$ as a graded space), and $\mathrm{Cobar}_+(B)$ is the non-unital dg-algebra (as a graded space, $\mathrm{Cobar}_+(B) = T(B[-1])/\Bbbk$). It is well-known that the natural projection $\mathrm{Cobar}_+(\mathrm{Bar}_+(A)) \to A$ is a quasi-isomorphism of non-unital dg-algebras. We claim that $\mathrm{Cobar}_+(\mathrm{Bar}_+(A))$ is weakly unital, whose weak unit is $1_A \in \mathrm{Cobar}_+(\mathrm{Bar}_+(A))$. We use notations $\omega = a_1 \otimes \cdots \otimes a_\ell \in \mathrm{Bar}_+(A)$ for monomial bar-chains, and $c = \omega_1 \boxtimes \omega_2 \boxtimes \cdots \boxtimes \omega_\kappa$ for monomial elements in $\mathrm{Cobar}_+(\mathrm{Bar}_+(A))$. Define $p_n(x_1, \dots, x_n)$, where each x_i is either 1 or a monomial $c \in \mathrm{Cobar}_+(\mathrm{Bar}_+(A))$, as follows:

(1): We set $p_n(x_1, \dots, x_n)$ to be 0 if for some $1 \le i \le n-1$ both x_i, x_{i+1} are elements in $\mathrm{Cobar}_+(\mathrm{Bar}_+(A))$.

(2): Otherwise, let $x_i, \dots, x_{i+j+1}$ be a fragment of the sequence $x_1, \dots, x_n$ such that $x_i = \omega_1 \boxtimes \cdots \boxtimes \omega_a \in \mathrm{Cobar}_+(\mathrm{Bar}_+(A))$, $x_{i+1} = \cdots = x_{i+j} = 1_A$, $x_{i+j+1} = \omega_1' \boxtimes \cdots \boxtimes \omega_b' \in \mathrm{Cobar}_+(\mathrm{Bar}_+(A))$. Then we replace the fragment $x_i, x_{i+1}, \dots, x_{i+j+1}$ by the following element γ in $\mathrm{Cobar}_+(\mathrm{Bar}_+(A))$:

$$\gamma = \omega_1 \boxtimes \cdots \boxtimes \omega_{a-1} \boxtimes (\omega_a \otimes 1_A \otimes \cdots \otimes 1_A \otimes \omega_1') \boxtimes \cdots \boxtimes \omega_b'$$

(3): We perform such replacements successively for all suitable fragment, and finally we get an element in $\mathrm{Cobar}_+(\mathrm{Bar}_+(A))$, of degree $\sum \deg x_i - n + 1$. By definition, this element is $p_n(x_1, \dots, x_n)$. By a suitable fragment we mean either the case considered above, when a group of successive 1_As is surrounded by elements of $\mathrm{Cobar}_+(\mathrm{Bar}_+(A))$ from both sides, or one of the two extreme case: if $x_1 = 1_A$, the leftmost $1_A, 1_A, \dots, 1_A, x_i$ is a suitable fragment, and similarly if $x_n = 1_A$, the rightmost fragment $x_s, 1_A, \dots, 1_A$ is also suitable. One easily checks that the constructed $\{p_n\}_{n\ge1}$ defines an A_∞-morphism $p\colon \mathrm{Cobar}_+(\mathrm{Bar}_+(A)) \oplus \Bbbk 1_A \to \mathrm{Cobar}_+(\mathrm{Bar}_+(A))$ such that $p \circ i = \mathrm{id}$. The construction for the case of $\mathrm{Cobar}_+(\mathrm{Bar}_+(C))$, for C a dg-category, is similar.

Example 2.1.7. Let A be a strictly unital dg-algebra, consider the weakly unital dg-algebra $C = \mathrm{Cobar}_+(\mathrm{Bar}_+(A))$ (which belongs to $\mathbb{C}_{dgwu}(\Bbbk)$)), constructed in Example 2.1.6 up above. Let D be a strictly unital dg-algebra. Then the set $\mathrm{Hom}_{\mathbb{C}_{dgwu}(\Bbbk)}(C, D)$ is identified with the set of unital A_∞-maps $A \to D$. (Recall that for strictly unital dg-algebras A, D, an A_∞-morphism $f\colon A \to D$ map is called unital if $f_1(1_A) = 1_D$, and $f_n(a_1, \dots, a_n) = 0$ if $n \ge 2$ and at least one argument $a_i = 1_A$). One has a similar description for the case of dg-categories.

2.2 Small (co)completeness of $\mathbb{C}_{dgwu}(\Bbbk)$

2.2.1 The products, coproducts, and equalizers in $\mathbb{C}_{dgwu}(\Bbbk)$

Our goal is to show that the category $\mathbb{C}_{dgwu}(\Bbbk)$ is small complete and small cocomplete. One constructs directly small *products* and small *coproducts*. The *equalizers* are also straightforward, as follows.

Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two morphisms. Define $\mathrm{Eq}(F, G)$ as the dg-category whose objects are

$$\mathrm{Ob}(\mathrm{Eq}(F, G)) = \{X \in \mathrm{Ob}(\mathcal{C}) | F(X) = G(X)\}$$

Let $X, Y \in \mathrm{Ob}(\mathrm{Eq}(F,G))$. Define

$$\mathrm{Eq}(F,G)(X,Y) = \{f \in \mathcal{C}(X,Y) | F(f) = G(f)\}$$

It is clear that $\mathrm{Eq}(F,G)$ is a non-unital dg-category. For any $X \in \mathrm{Ob}(\mathrm{Eq}(F,G))$, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $G(\mathrm{id}_X) = \mathrm{id}_{G(X)}$, therefore $\mathrm{id}_X \in \mathrm{Eq}(F,G)(X,X)$.

One has to construct an A_∞-functor $p\colon \mathrm{Eq}(F,G) \oplus \Bbbk_{\mathrm{Eq}(F,G)} \to \mathrm{Eq}(F,G)$ such that $p_1(1_X) = \mathrm{id}_X$, and $p \circ i = \mathrm{id}$. We define

$$p_n^{\mathrm{Eq}(F,G)}(f_1 \otimes \dots \otimes f_n) = p_n^{\mathcal{C}}(f_1 \otimes \dots \otimes f_n)$$

One has to check that $p_n^{\mathrm{Eq}(F,G)}(f_1 \otimes \dots \otimes f_n)$ is a morphism in $\mathrm{Eq}(F,G)$, that is,

$$F(p_n^{\mathcal{C}}(f_1 \otimes \dots \otimes f_n)) = G(p_n^{\mathcal{C}}(f_1 \otimes \dots \otimes f_n)) \tag{2.2.1}$$

From (2.1.4) one gets

$$F(p_n^{\mathcal{C}}(f_1 \otimes \dots \otimes f_n)) = p_n^{\mathcal{D}}(F(f_1) \otimes \dots F(f_n))$$

and

$$G(p_n^{\mathcal{C}}(f_1 \otimes \dots \otimes f_n)) = p_n^{\mathcal{D}}(G(f_1) \otimes \dots \otimes G(f_n))$$

Now (2.2.1) follows from $F(f_i) = G(f_i)$ for all f_i, which holds because all f_i are morphisms in $\mathrm{Eq}(F,G)$. Thus, $\mathrm{Eq}(F,G)$ is a weakly unital dg-category.

To construct the *coequalizers* is a harder task. For the category $\mathcal{E}$-Cat of small $\mathcal{E}$-enriched categories, the coequalizers were constructed in [Li] and [Wo], assuming $\mathcal{E}$ to be a symmetric monoidal closed and cocomplete, and were constructed in [BCSW] and [KL] in weaker assumptions on $\mathcal{E}$. All these proofs rely on the theory of monads. We associate a monad which governs the weakly unital dg-categories in Section 2.2.2.

We adapt the approach of [Wo] for a proof of existence of the coequalizers in $\mathbb{C}_{dgwu}(\Bbbk)$. We also prove the corresponding monadicity theorem.

2.2.2 The dg-operad $\mathcal{O}'$ and the monad of weakly unital dg-categories

Definition 2.2.1. A dg-quiver Γ over $\Bbbk$ is an oriented graph, given by a *set* V_Γ of vertices, and a complex $\Gamma(x,y) \in \mathrm{Ch}(\Bbbk)$ for any ordered pair $x, y \in V_\Gamma$. A morphism $F\colon \Gamma_1 \to \Gamma_2$ is given by a map of sets $F_V\colon V_{\Gamma_1} \to V_{\Gamma_2}$, and by a map of complexes $F_E\colon \Gamma_1(x,y) \to \Gamma_2(F_V(x), F_V(y))$, for any $x, y \in V_{\Gamma_1}$. We denote by $\mathbb{G}_{dg}(\Bbbk)$ the category of dg-quivers over $\Bbbk$.

Definition 2.2.2. A *unital* dg-quiver Γ over $\Bbbk$ is an dg-quiver over $\Bbbk$ such that there is an element $\mathrm{id}_x \in \Gamma(x,x)$, closed of degree 0, for any $x \in V_\Gamma$. A map of unital dg-quivers is a map F of the underlying dg-graphs such that $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$, for any $x \in V_\Gamma$. We denote by $\mathbb{G}_{dgu}(\Bbbk)$ the category of unital dg-quivers over $\Bbbk$.

There is a natural forgetful functor $U\colon \mathbb{C}_{dgwu}(\Bbbk) \to \mathbb{G}_{dgu}(\Bbbk)$, where $U(\mathcal{C})$ is a quiver Γ with $V_\Gamma = \mathrm{Ob}(\mathcal{C})$, and $\Gamma(x,y) = \mathcal{C}(x,y)$.

This functor admits a left adjoint $F\colon \mathbb{G}_{dgu}(\Bbbk)\to\mathbb{C}_{dgwu}(\Bbbk)$. It is constructed via a dg-operad $\mathcal{O}'$, see (2.2.6).

Define the non-Σ dg-operad $\mathcal{O}'$ as the quotient-operad of the free operad generated by the composition operations:

(a) the composition operation $m\in\mathcal{O}'(2)^0$

(b) $p_{n;i_1,\dots,i_k}\in\mathcal{O}'(n-k)^{-n+1}$, $0\le k\le n$, $1\le i_1<i_2<\dots<i_k\le n$, with the following meaning: for a weakly unital dg-category $\mathcal{C}$, the operation $p_{n;i_1,\dots,i_k}(f_1,\dots,f_{n-k})$ is defined as

$$p_n\big(f_1,\dots,f_{i_1-1},\underset{i_1}{1_{X_1}},f_{i_1},\dots,f_{i_2-2},\underset{i_2}{1_{X_2}},f_{i_2-1},\dots,\underset{i_k}{1_{X_k}},f_{i_k-k+1},\dots,f_{n-k}\big) \tag{2.2.2}$$

where by 1_{X_i}s are denoted the morphisms $1_{X_i}\in\Bbbk_C$ for the corresponding objects $X_i\in\mathcal{C}$.

by the following relations:

$$\begin{aligned}&(i)\ \text{the associativity of } m, \text{ and } dm=0\\ &(ii)\ p_{n;i_1,\dots,i_k}=0 \text{ if } k=0\\ &(iii)\ p_{1;-}=\mathrm{id}\\ &(iv)\ \text{the } A_\infty\text{-morphism relation for } dp_{n;i_1,\dots,i_k} \text{ see (2.2.4) below}\end{aligned} \tag{2.2.3}$$

We use the notation $j=p_{1,1}$, the degree zero 0-ary operation generating the weak unit. It follows from (iv) that $dj=0$. Note that relation (iv) expresses the relations like (2.1.1) and its higher analogues (2.1.2) in the operadic terms, using (2.2.2).

It remains to specify relation (iv):

$$\begin{aligned}dp_{n;i_1,\dots,i_k}=&\sum_{1\le\ell\le n-1}\pm m\circ(p_{\ell;i_1,\dots,i_{s(\ell)}},p_{n-\ell;i_{s(\ell)+1},\dots,i_k})+\\ &\sum_{r=1}^{n-1}\pm p_{n-1;j_1,\dots,j_{q(r)}}\circ(\mathrm{id},\dots,\mathrm{id},\underset{r}{m(a(r),a(r+1))},\mathrm{id},\dots,\mathrm{id})\end{aligned} \tag{2.2.4}$$

with the notations explained below:

$$s(\ell)=\max_{s=1,\dots,k}\{s|i_s\le\ell\}$$

$$a(r)=\begin{cases}\mathrm{id} & \text{if } r\notin\{i_1,\dots,i_k\}\\ j & \text{otherwise.}\end{cases}$$

$$q(r)=\begin{cases}k & \text{if neither } r,r+1 \text{ are in } \{i_1,\dots,i_k\}. \text{ Then } j_s=i_s \text{ for } i_s\le r \text{ and } j_s=i_s-1 \text{ for } i_s>r\\ k-1 & \text{if either } r \text{ or } r+1 \text{ are in } \{i_1,\dots,i_k\}. \text{ Then } j_s=i_s \text{ for } i_s<r, \text{ and } j_s=i_{s+1}-1 \text{ for } i_{s+1}>r\\ k-2 & \text{if both } r,r+1 \text{ are in } \{i_1,\dots,i_k\}. \text{ Then } j_s=i_s \text{ for } i_s<r, \text{ and } j_s=i_{s+2}-1 \text{ for } i_{s+2}>r+1.\end{cases}$$

Morally, the dg-operad $\mathcal{O}'$ comprises all universal operations one can define on a weakly unital dg-category.

Denote by $\mathcal{A}ssoc_+$ the operad of unital associative $\Bbbk$-algebras. In Section 2.8 we prove the following theorem:

Theorem 2.2.3. *The natural map of dg-operads $\mathcal{O}' \to \mathcal{A}ssoc_+$, sending all $p_{n;i_1,\dots,i_k}$, $n \geq 2$, to 0, sending $j = p_{1;1}$ to the 0-ary unit generating operation, and sending m to m, is a quasi-isomorphism.*

The proof is a rather long and tricky computation with several spectral sequences. □

The left adjoint functor $F\colon \mathbb{G}_{dgu}(\Bbbk) \to \mathbb{C}_{dgwu}(\Bbbk)$ is defined in two steps, as follows. Given a unital dg-quiver Γ, consider the free $\mathcal{O}'$-algebra $T_{\mathcal{O}'}(\Gamma)$, generated by Γ. It is defined as follows:

We define a chain of length n in Γ as an ordered set $X_0, X_1, \dots, X_n$. Denote by Γ_n the set of all chains of length n in Γ. For $c \in \Gamma_n$, set

$$\Gamma(c) := \Gamma(X_0, X_1)_+ \otimes \Gamma(X_1, X_2)_+ \otimes \cdots \otimes \Gamma(X_{n-1}, X_n)_+$$

and

$$\Gamma(n)(X,Y) := \sum_{\substack{c\in\Gamma_n \\ X_0(c)=X, X_n(c)=Y}} \Gamma(c)$$

(for $n = 0$ we set $\Gamma(0)(X,X) = \Bbbk\,\mathrm{id}_X$ and $\Gamma(0)(X,Y) = 0$ for $X \neq Y$). Set

$$\Gamma_{\mathcal{O}'}(X,Y) := \sum_{n\geq 0} \mathcal{O}'(n) \otimes \Gamma(n)(X,Y) \tag{2.2.5}$$

It is a weakly unital dg-category with objects V_Γ. The 0-ary operation j generates an element $j_X \in T_{\mathcal{O}'}(X,X)$, for any $X \in V_\Gamma$.

After that, define $F(\Gamma)$ as the quotient dg-category

$$F(\Gamma) = T_{\mathcal{O}'}(\Gamma)/(j_X - \mathrm{id}_X, X \in V_\Gamma) \tag{2.2.6}$$

In this way, we identify $\mathrm{id}_X \in \Gamma(X,X)$ with the "weak unit" j_X generated by $\mathcal{O}'$.

One has:

Proposition 2.2.4. *There is an adjunction:*

$$\mathbb{C}_{dgwu}(F(\Gamma), C) \simeq \mathbb{G}_{dgu}(\Gamma, U(C)) \tag{2.2.7}$$

Note that for Γ a non-unital dg-quiver, one defines a unital dg-quiver Γ_+, formally adding $\Bbbk_X$ to $\Gamma(X,X)$, for any $x \in V_\Gamma$. Then

$$F(\Gamma_+) \simeq T_{\mathcal{O}'}(\Gamma)$$

2.2.3 The coequalizers in $\mathbb{G}_{dgu}(\Bbbk)$

It is standard that coequalizers, and, therefore, all small colimits exist in $\mathbb{G}_{dgu}(k)$.

Recall the construction: let

$$\Gamma_1 \underset{g}{\overset{f}{\rightrightarrows}} \Gamma_2 \tag{2.2.8}$$

be a pair of morphisms in $\mathbb{G}_{dgu}(\Bbbk)$.

Define its coequalizer $\Gamma_{f,g}$ as a small quiver in $\mathbb{G}_{dgu}(\Bbbk)$ whose set of objects is the coequalizer of the corresponding maps of the sets of objects

$$\mathrm{Ob}(\Gamma_1) \overset{f}{\underset{g}{\rightrightarrows}} \mathrm{Ob}(\Gamma_2)$$

It is the quotient set of $\mathrm{Ob}(\Gamma_2)$ by the equivalence relation generated by the binary relation $f(X)\mathrm{R}g(X)$, $X \in \mathrm{Ob}(\Gamma_1)$.

Let $[X]$ and $[Y]$ be two equivalence classes. Define a complex $\Gamma_{f,g}([X],[Y])$ as the coequalizer in $\mathrm{Vect}_{dg}(\Bbbk)$ of

$$\bigoplus_{\substack{w,z\in\mathrm{Ob}(\Gamma_1)\\ [f(w)]=[g(w)]=[x]\\ [f(z)]=[g(z)]=[y]}} \Gamma_1(w,z) \overset{f_*}{\underset{g_*}{\rightrightarrows}} \bigoplus_{\substack{a,b\in\mathrm{Ob}(\Gamma_2)\\ [a]=[x],[b]=[y]}} \Gamma_2(a,b) \tag{2.2.9}$$

where f_* maps $\phi \in \Gamma_1(w,z)$ to $f(\phi)$, and g_* maps it to $g(\phi)$. If at least one class of $[x],[y]$ is not in the image of f (which is the same that the image of g), we define source complex in (2.2.9) as 0.

It is easy to check that the constructed dg-quiver $\Gamma_{f,g}$ is a coequalizer of (2.2.8).

2.2.4 The coequalizers in $\mathbb{C}_{dgwu}(\Bbbk)$

Consider a pair of maps of weakly unital dg-categories

$$\mathcal{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{B} \tag{2.2.10}$$

It is not straightforward to find (or to prove existence of) its coequalizer.

However, one always can find the coequalizer of the maps of graphs

$$U(\mathcal{A}) \overset{U(F)}{\underset{U(G)}{\rightrightarrows}} U(\mathcal{B}) \xrightarrow{\ell} \mathrm{Coeq}(U(F),U(G)) \tag{2.2.11}$$

as in Subsection 2.2.3. For some special diagrams (2.2.10), the functor U creates coequalizers, see below. Afterwards, we reduce the general coequalizers (2.2.10) to these special ones.

Definition 2.2.5. We say that the diagram (2.2.10) is *good* if $\mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{B})$, and both F and G are identity maps on the sets objects.

Assume that (2.2.10) is good. Then the quiver $\mathrm{Coeq}(U(F),U(G))$, which is a particular case of general coequalizers (2.2.8) in $\mathbb{G}_{dgu}(\Bbbk)$, is especially simple. It has the set of vertices equal to $\mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{B})$, and its morphisms are the quotient-complexes

$$\mathrm{Coeq}(U(F),U(G))(X,Y) = \mathcal{B}(X,Y)/(F(f) - G(f))_{f\in\mathcal{A}(X,Y)}$$

Lemma 2.2.6. *Suppose we are given a diagram* (2.2.10) *which is good. Then a weakly unital dg-category structure* $\mathcal{Q}$ *and a map of weakly unital dg-categories* $L\colon \mathcal{B}\to\mathcal{Q}$ *such that*

$$\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B} \xrightarrow{L} \mathcal{Q}$$

is a coequalizer, and $U(\mathcal{Q}) = \mathrm{Coeq}(U(F), U(G))$, $U(L) = \ell$, *exist if and only if the following two conditions hold:*

(1) the sub-complexes $(F(f)-G(f))_{f\in\mathcal{A}(X,Y)}$, $X, Y \in \mathrm{Ob}(\mathcal{A})$, *form a two-sided ideal in* B*:*

$$\ell(g\circ(F(f)-G(f))\circ g') = 0 \tag{2.2.12}$$

for any morphism f *in* $\mathcal{A}$ *and any morphisms* g, g' *in* $\mathcal{B}$ *(such that the compositions are defined),*

(2)

$$\ell(p_n^{\mathcal{B}}(g_1\otimes\dots g_k\otimes(g\circ(F(f)-G(f))\circ g')\otimes g_{k+1}\otimes\dots\otimes g_{n-1})) = 0 \tag{2.2.13}$$

for $n\ge 2$, *and any morphism* f *in* $\mathcal{A}$ *(some of* g_i *are elements of* $\Bbbk_{\mathcal{B}}$*).*

In particular, the weakly unital dg-category $\mathcal{Q}$, *if it exists, is uniquely defined (which means that in this case* U *strictly creates the coequalizer).*

It is clear. □

Recall that diagram (2.2.10) is called reflexive if there exists $H\colon \mathcal{B}\to\mathcal{A}$ such that $FH = GH = \mathrm{id}_{\mathcal{B}}$.

Proposition 2.2.7. *Assume we are given a good and reflexive diagram* (2.2.10). *Then conditions (1) and (2) of Lemma 2.2.6 are fulfilled. Consequently, the functor* U *strictly creates the coequalizer.*

Proof. Prove that (1) holds. One has:

$$\begin{aligned}&\ell(g\circ(F(f)-G(f))\circ g') = \ell(g\circ F(f)\circ g') - \ell(g\circ G(f)\circ g') =\\ &\ell(FH(g)\circ F(f)\circ FH(g')) - \ell(GH(g)\circ G(f)\circ GH(g')) =\\ &\ell(F(H(g)\circ f\circ H(g')) - \ell(G(H(g)\circ f\circ H(g')) = 0\end{aligned} \tag{2.2.14}$$

Prove that (2) holds. One has:

$$\begin{aligned}&\ell(p_n^B(g_1\otimes\dots\otimes(g\circ(F(f)-G(f))\circ g')\otimes\dots\otimes g_{n-1})) =\\ &\ell(p_n^B(g_1\otimes\dots\otimes(g\circ F(f)\circ g')\otimes\dots\otimes g_{n-1})) - \ell(p_n^B(g_1\otimes\dots\otimes(g\circ G(f)\circ g')\otimes\dots\otimes g_{n-1})) =\\ &\ell(p_n^B(FH(g_1)\otimes\dots\otimes(FH(g)\circ F(f)\circ FH(g'))\otimes\dots\otimes FH(g_{n-1}))-\\ &\ell(p_n^B(GH(g_1)\otimes\dots\otimes(GH(g)\circ G(f)\circ GH(g'))\otimes\dots\otimes GH(g_{n-1}))) =\\ &\ell(p_n^B(FH(g_1)\otimes\dots\otimes(F(H(g)\circ f\circ H(g'))\otimes\dots\otimes FH(g_{n-1}))-\\ &\ell(p_n^B(GH(g_1)\otimes\dots\otimes(G(H(g)\circ f\circ H(g')))\otimes\dots\otimes GH(g_{n-1}))) \overset{*}{=}\\ &\ell(Fp_n^A(H(g_1)\otimes\dots\otimes(H(g)\circ f\circ H(g'))\otimes\dots\otimes H(g_{n-1})))-\\ &\ell(Gp_n^A(H(g_1)\otimes\dots\otimes(H(g)\circ f\circ H(g'))\otimes\dots\otimes H(g_{n-1}))) = 0\end{aligned}$$

(2.2.15)

where the equality marked by * follows from the fact that F, G are functors of weakly unital dg-categories and (2.1.4). □

In order to prove the cocompleteness of $\mathbb{C}_{dgwu}$, we closely follow the arguments in [[Wo], Prop. 2.11]. We reproduce them here for completeness.

We make use of the following lemma, due to [[Li], pp. 77-78], and known as the 3x3-lemma.

Lemma 2.2.8. *Consider the following diagram in a category*

$$\begin{array}{ccccc}
A_1 & \overset{h_1}{\underset{h_2}{\rightrightarrows}} & B_1 & \xrightarrow{h_3} & C_1 \\
{\scriptstyle\alpha_1}\downarrow\downarrow{\scriptstyle\alpha_2} & & {\scriptstyle\beta_1}\downarrow\downarrow{\scriptstyle\beta_2} & & {\scriptstyle\gamma_1}\downarrow\downarrow{\scriptstyle\gamma_2} \\
A_2 & \overset{g_1}{\underset{g_2}{\rightrightarrows}} & B_2 & \xrightarrow{g_3} & C_2 \\
{\scriptstyle\alpha_3}\downarrow & & {\scriptstyle\beta_3}\downarrow & * & \downarrow{\scriptstyle\gamma_3} \\
A_3 & \overset{f_1}{\underset{f_2}{\rightrightarrows}} & B_3 & \xrightarrow{f_3} & C_3
\end{array} \qquad (2.2.16)$$

in which the top and the middle rows are coequalizers, the leftmost and the middle columns are coequalizers, and all squares commute: $g_i\alpha_i = \beta_i h_i$, $f_i\alpha_3 = \beta_3 g_i$, $g_3\beta_i = \gamma_i h_3$, $f_3\beta_3 = \gamma_3 g_3$, $i = 1, 2$. *Then the following statements are equivalent:*

(1) the bottom row is a coequalizer,

(2) the rightmost column is a coequalizer,

(3) the square in the lower right corner (marked by $*$) is a push-out.

Proposition 2.2.9. *The category $\mathbb{C}_{dgwu}(\Bbbk)$ has all coequalizers.*

Proof. Let

$$\mathcal{A} \overset{H_1}{\underset{H_2}{\rightrightarrows}} \mathcal{B} \qquad (2.2.17)$$

be two arrows in $\mathbb{C}_{dgwu}(\Bbbk)$ coequalizer of which we'd like to compute. Embed it to the following solid arrow diagram, where (F, U) is the adjoint pair of functors from Proposition 2.2.4.

$$\begin{array}{ccccc}
FUFU\mathcal{A} & \overset{FUFU(H_1)}{\underset{FUFU(H_2)}{\rightrightarrows}} & FUFU\mathcal{B} & \xrightarrow{F(L')} & FE' \\
{\scriptstyle\epsilon_{FU\mathcal{A}}}\downarrow\downarrow{\scriptstyle FU\epsilon_{\mathcal{A}}} & & {\scriptstyle\epsilon_{FU\mathcal{B}}}\downarrow\downarrow{\scriptstyle FU\epsilon_{\mathcal{B}}} & & {\scriptstyle\alpha_1}\downarrow\downarrow{\scriptstyle\alpha_2} \\
FU\mathcal{A} & \overset{FU(H_1)}{\underset{FU(H_2)}{\rightrightarrows}} & FU\mathcal{B} & \xrightarrow{F(L)} & FE \\
{\scriptstyle\epsilon_{\mathcal{A}}}\downarrow & & {\scriptstyle\epsilon_{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle p} \\
\mathcal{A} & \overset{H_1}{\underset{H_2}{\rightrightarrows}} & \mathcal{B} & \xrightarrow{q} & \mathcal{X}
\end{array} \qquad (2.2.18)$$

The upper and the middle rows are obtained from (2.2.17) by application of $FUFU$ and FU, correspondingly. Denote by E the coequalizer of (UH_1, UH_2) in $\mathbb{G}_{dgu}(\Bbbk)$, and by E' the coequalizer of $(UFUH_1, UFUH_2)$ in $\mathbb{G}_{dgu}(\Bbbk)$. As F is left adjoint, FE and FE' are the coequalizers of (FUH_1, FUH_2) and $(FUFUH_1, FUFUH_2)$ in $\mathbb{C}_{dgwu}(\Bbbk)$, correspondingly. Therefore, the upper and the middle rows of (2.2.18) are coequalizers.

The leftmost and the middle columns fulfill the assumptions of Proposition 2.2.7. Indeed, the upper pairs of arrows are reflexive, by the second case of Lemma 1.5.4, see (1.5.4). Therefore, these columns are coequalizers, by Proposition 2.2.7.

The dotted arrows α_1, α_2 are constructed as follows. For α_1, consider the map

$$F(L) \circ \epsilon_{FU\mathcal{B}} \colon FUFU\mathcal{B} \to FE$$

The two compositions

$$FUFU\mathcal{A} \underset{FUFUH_2}{\overset{FUFUH_1}{\rightrightarrows}} FUFU\mathcal{B} \xrightarrow{F(L)\circ\epsilon_{FUB}} FE$$

are equal, which gives rise to a unique map $\alpha_1 \colon FE' \to FE$.

Similarly, taking $FU\epsilon_B$ instead of ϵ_{FUB}, one gets a unique map $\alpha_2 \colon FE' \to FE$, which coequalizes the corresponding two arrows.

We claim that the pair (α_1, α_2) is reflexive. We construct $\varkappa_E \colon FE \to FE'$ such that $\alpha_1 \circ \varkappa_E = \alpha_2 \circ \varkappa_E = \mathrm{id}_{FE}$.

Recall $\varkappa_{\mathcal{A}} \colon FU\mathcal{A} \to FUFU\mathcal{A}$ and $\varkappa_{\mathcal{B}} \colon FU\mathcal{B} \to FUFU\mathcal{B}$ given as in (1.5.4):

$$\varkappa_{\mathcal{A}} = F\eta_{U\mathcal{A}}, \quad \varkappa_{\mathcal{B}} = F\eta_{U\mathcal{B}}$$

These maps are sections of the corresponding pairs of maps, which make them reflexive pairs, see Lemma 1.5.4. Consider

$$F(L') \circ \varkappa_{\mathcal{B}} \colon FU\mathcal{B} \to FE'$$

The two maps

$$FU\mathcal{A} \rightrightarrows FU\mathcal{B} \xrightarrow{F(L')\circ\varkappa_{\mathcal{B}}} FE'$$

are equal, which gives rise to a unique map

$$\varkappa_E \colon FE \to FE'$$

A simple diagram chasing shows that $\alpha_1 \circ \varkappa_E = \alpha_2 \circ \varkappa_E = \mathrm{id}_{FE}$.

One has $\mathrm{Ob}(FE) = \mathrm{Ob}(FE')$, and Proposition 2.2.9 is applied. We get an arrow $p \colon FE \to \mathcal{X}$ which is a coequalizer of (α_1, α_2).

Finally, we have to construct an arrow $q \colon \mathcal{B} \to \mathcal{X}$ making the square in the lower right corner commutative. To this end, consider $p \circ F(L) \colon FU\mathcal{B} \to \mathcal{X}$. The two compositions

$$FUFU\mathcal{B} \rightrightarrows FU\mathcal{B} \xrightarrow{p\circ F(L)} \mathcal{X}$$

are equal, which gives a unique map $q \colon \mathcal{B} \to \mathcal{X}$. One checks that the lower right square commutes.

One makes use of Lemma 2.2.8 to conclude that the bottom row is a coequalizer. □

We have already seen in Subsection 2.2.1 that the products, the coproducts, and the equalizers in $\mathbb{C}_{dgwu}(\Bbbk)$ are constructed straightforwardly. Then Proposition 2.2.9, and the classic result [[R2], Th. 3.4.11] give:

Theorem 2.2.10. *The category* $\mathbb{C}_{dgwu}(\Bbbk)$ *is small complete and small cocomplete.*

2.2.5 The monadicity

Although we will not be using the following result in this work, it may have an independent interest. The argument is close to [[Wo], Th. 2.13].

Theorem 2.2.11. *The adjunction*

$$F\colon \mathbb{G}_{dgu}(\Bbbk) \rightleftarrows \mathbb{C}_{dgwu}(\Bbbk)\colon U$$

is monadic.

Proof. We deduce the statement from the Beck Monadicity Theorem 1.5.2, for which we have to prove that the assumptions in (1)-(3) in Theorem 1.5.2 hold.

(1) has been proven in Proposition 2.2.9, by which $\mathbb{C}_{dgwu}(\Bbbk)$ has all coequalizers, and (3) is clear. One has to prove (2), that is, that the functor $U\colon \mathbb{C}_{dgwu}(\Bbbk) \to \mathbb{G}_{dgu}(\Bbbk)$ preserves all U-split coequalizers. We make use of Lemma 2.2.8, once again.

Let a pair of arrows in $\mathbb{C}_{dgwu}(\Bbbk)$

$$\mathcal{A} \underset{H_2}{\overset{H_1}{\rightrightarrows}} \mathcal{B} \tag{2.2.19}$$

be U-split. Then

$$U\mathcal{A} \underset{UH_2}{\overset{UH_1}{\rightrightarrows}} U\mathcal{B} \xrightarrow{L} E \tag{2.2.20}$$

is a split coequalizer, for some L and E. The upper and the middle rows in (2.2.18) are defined now as the result of application of FUF and F, correspondingly, to (2.2.20). (In particular, now $E' = UF(E)$, $L' = UF(L)$). Therefore, the upper and the middle rows are split, and, therefore, *absolute* coequalizers, by Lemma 1.5.1.

Then we get the dotted arrows in (2.2.18), and construct X, as in the proof of Proposition 2.2.9. In particular, we get a coequalizer

$$\mathcal{A} \underset{H_2}{\overset{H_1}{\rightrightarrows}} \mathcal{B} \xrightarrow{q} \mathcal{X} \tag{2.2.21}$$

at the bottom row of (2.2.18). One has to prove that $UX \simeq E$.

In the obtained diagram all columns and two upper rows are split coequalizers, but the bottom row is also a coequalizer but possibly not split. Now apply the functor U to the whole diagram. As split coequalizers are absolute, by Lemma 1.5.1, the upper two rows and all three columns remain coequalizers. Therefore, by Lemma 2.2.8, the bottom row also remains a coequalizer, after application of the functor U. □

2.3 The pretriangulated hull of a weakly unital dg-category

Recall that the *pretriangulated hull* of a dg-category $\mathcal{C}$ was introduced by Bondal-Kapranov [BK] (see also [[Dr], Remark 2.4]). A dg-category $\mathcal{C}$ is *pretriangulated* if $H^0(\mathcal{C})$ is triangulated. Explicitly, it means that the functors $Z \to \underline{\mathrm{Hom}}(\mathcal{C}(Z,X) \xrightarrow{f} \mathcal{C}(Z,Y))$ and $Z \to \mathcal{C}(Z,X)[n]$ defined for any closed morphism $f\colon X \to Y$ in $\mathcal{C}$ and for any object $X \in \mathcal{C}$, $n \in \mathbb{Z}$, correspondingly, are representable. In this case, the representing objects are $\mathrm{Cone}(f)$ and $X[n]$.

The pretriangulated hull $\mathcal{C}^{\mathrm{pretr}}$ of a dg-category $\mathcal{C}$ is a pretriangulated dg-category with a dg-functor $\mathcal{C} \to \mathcal{C}^{\mathrm{pretr}}$, which is universal for dg-functors from $\mathcal{C}$ to pretriangulated dg-categories [BK].

Explicitly, it is constructed as follows. An object of $\mathcal{C}^{\mathrm{pretr}}$ is a "one-sided twisted complexes", which are formal finite sums $(\oplus_{i=1}^n X_i[r_i]), q)$ where q has components $q_{ij} \in \mathcal{C}^{r_i - r_j + 1}(X_i, X_j)$, which are zero for $i \geq j$, such that $dq + q^2 = 0$. Let $X = (\oplus X_i[r_i], q)$, $X' = (\oplus X_i'[r_i'], q')$ be two objects of $\mathcal{C}^{\mathrm{pretr}}$, a morphism $\phi \in \mathcal{C}^{\mathrm{pretr}}(X, X')$ of degree k is defined as the collection $\phi_{ij}\colon X_i[r_i] \to X_j'[r_j']$ of degree d (in general, non-zero for any i,j), and $d\phi = d_{\mathcal{C}}\phi + q' \circ \phi - (-1)^k \phi \circ q$. The composition is defined as the matrix product.

The dg-functor $\mathcal{C} \to \mathcal{C}^{\mathrm{pretr}}$ is defined on objects as $X \mapsto (X[0], q = 0)$, and on morphisms accordingly. We recall that, given $f\colon X \to Y$ a closed morphism in $\mathcal{C}$, $\mathrm{Cone}(f) \in \mathcal{C}^{\mathrm{pretr}}$ is defined as $\mathrm{Cone}(f) = (X \oplus Y[-1], q = f)$ (that is, $q_{12} = f$, $q_{11} = q_{22} = q_{21} = 0$).

We want to define the pretriangulated hull of a *weakly unital* dg-category, which is a weakly unital dg-category as well. If we just repeated the definition given above, we would experience the following problem. Let $f \in \mathcal{C}(X,Y)$ be a closed morphism, we do *not* assume that $f \circ \mathrm{id}_X = f$ or $\mathrm{id}_Y \circ f = f$. Defining $\mathrm{Cone}(f) = (X \oplus Y[-1], f)$, there should be a weak identity morphism $\mathrm{id}_{\mathrm{Cone}(f)}$, which is a closed morphism of degree 0. A natural candidate is given by $\mathrm{id}_X\colon X \to X$, $\mathrm{id}_Y\colon Y \to Y$. But then $d(\mathrm{id}_{\mathrm{Cone}(f)}) = f \circ \mathrm{id}_X - \mathrm{id}_Y \circ f \neq 0$.

We remedy this problem as follows: for a closed morphism f in $\mathcal{C}$, define $\mathrm{Cone}(f) = (X \oplus Y[-1], f))$ as above, but re-define $\mathrm{id}_{\mathrm{Cone}}(f)$. Namely, *define* $\mathrm{id}_{\mathrm{Cone}}(f)$ as having 3 non-zero components:

$$\begin{gathered}\mathrm{id}_{\mathrm{Cone}(f)} = (\mathrm{id}_X, \mathrm{id}_Y, \varepsilon \in \mathcal{C}^0(X, Y[-1])) \\ \text{where } \varepsilon = p_2(f, 1_X) - p_2(1_Y, f)\end{gathered} \tag{2.3.1}$$

where p_2 is the second Taylor component of the A_∞-morphism $p\colon \mathcal{C} \oplus \Bbbk_{\mathcal{C}} \to \mathcal{C}$, see Definition 2.1.1.

Then one has:

$$d(\mathrm{id}_{\mathrm{Cone}(f)}) = f \circ \mathrm{id}_X - \mathrm{id}_Y \circ f + d\varepsilon = 0 \tag{2.3.2}$$

(Recall that $dp_2(f, 1_x)) = p_1(f) - f \circ \mathrm{id}_X = f - f \circ \mathrm{id}_X$, and similarly for $dp_2(1_y, f)$).

Thus, at the first step we define, inspired by this example, the identity morphism id_X, for $X = (\oplus X_i[r_i], q_{ij})$, and check that $d(\mathrm{id}_X) = 0$. After that, we construct an A_∞-functor $P\colon \mathcal{C}^{\mathrm{pretr}} \oplus \Bbbk_{\mathcal{C}^{\mathrm{pretr}}} \to \mathcal{C}^{\mathrm{pretr}}$, making $\mathcal{C}^{\mathrm{pretr}}$ a weakly unital dg-category.

(We denote by p the structure A_∞-functor for $\mathcal{C}$, and by P the structure A_∞-functor for $\mathcal{C}^{\mathrm{pretr}}$).

Definition 2.3.1 (Pretriangulated hull of a weakly unital dg-category, I)**.** Let $\mathcal{C}$ be a weakly unital dg-category. We define the underlying non-unital dg-category of pretriangulated hull of $\mathcal{C}$ as in the strictly unital case:

(a) objects are formal expressions $(\bigoplus_{i=1}^n X_i[r_i], q_{ij})$, where $X_i \in \mathcal{C}$, $r_i \in \mathbb{Z}$, $q_{ij} \in \mathcal{C}^{1+r_j-r_i}(X_i, X_j) = \mathcal{C}^1(X_i[r_i], X_j[r_j])$ such that $q_{ij} = 0$ if $i \geq j$ and $dq + q \circ q = 0$,

(b) the space of degree k morphisms $\mathcal{C}^{\mathrm{pretr}}(X, X')$, for $X = (X_i[r_i], q_{ij}), X' = (X'[r'_i], q'_{ij})$, is defined as the space of matrices $\phi = (\phi_{ij} \colon \mathcal{C}^k(X_i[r_i] \to X'_j[r'_j]))$, the composition is matrix multiplication and the differential is $d\phi := d_{\mathcal{C}}\phi + q' \circ \phi - (-1)^k \phi \circ q$.

Now we define, for any object $X \in \mathcal{C}^{\mathrm{pretr}}$, an "identity" morphism id_X (which is required to be a closed morphism of degree 0), and construct an A_∞-morphism $P \colon \mathcal{C}^{\mathrm{pretr}} \oplus \Bbbk_{\mathcal{C}^{\mathrm{pretr}}} \to \mathcal{C}^{\mathrm{pretr}}$, making it a weakly unital dg-category. In fact, we start with A_∞-morphism P, then $\mathrm{id}_X := P_1(1_X)$.

Let $X^0, \dots, X^n$ be objects of C^{pretr}, and let $\phi^i \colon X^{i-1} \to X^i$ be either a morphism in C^{pretr} or $1_{X^{i-1}}$ (in which case $X^i = X^{i-1}$).

We are going to define $P_n(\phi^n, \dots, \phi^1)$. Let us introduce some notations. We visualize the string

$$X^0 \xrightarrow{\phi^1} X^1 \xrightarrow{\phi^2} X^2 \to \dots \xrightarrow{\phi^n} X^n$$

as a planar diagram whose horizontal arrows are $q^i_{k\ell}$, where $X^i = (\oplus X^i_k[r^i_k], q^i_{k\ell})$, and whose other arrows are the components of ϕ^i, $i = 1, \dots, n$, see (2.3.3).

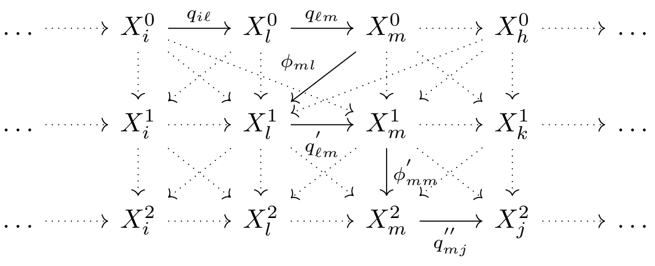
We refer to the arrows $q^i_{k\ell}$ as *horizontal*, and the other arrows, called *essential*, are either components of ϕ^is or morphisms 1_X for some $X \in \mathcal{C}$.

Now we associate to any couple (X^0_a, X^n_b) of starting and ending objects, a set $\mathscr{P}aths_{ab}$ of ***all*** the possible paths from X^0_a to X^n_b, see (2.3.3), (2.3.4).

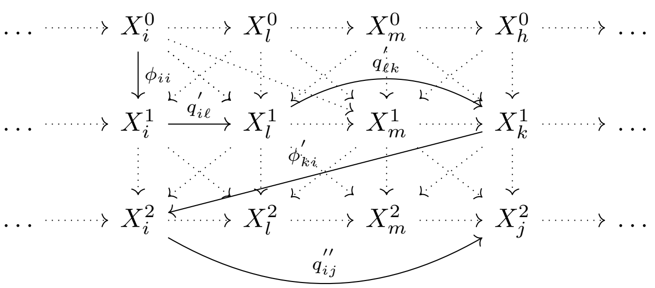
By definition, a *path* $\kappa \in \mathscr{P}aths_{ab}$ is a sequence of arrows $\kappa = (\kappa_1, \dots, \kappa_\ell)$, either horizontal or essential, such that (a) for any $1 \leq s \leq n$ there is exactly 1 essential arrow which is a components of ϕ^s, and these n essential arrows stand respecting the order, (b) the arrows between two successive essential arrows, which are components of ϕ^s and ϕ^{s+1} (here $\phi^s = 1_X$ is allowed), are horizontal arrows in X^s, which form a composable chain (there are allowed more than 1 arrows in this chain), (c) the first arrow starts at X^0_a, and the last one ends at X^n_b. It follows in particular that a path is represented by a composable chain of arrows.

If some $\phi^i = 1_{X^{i-1}}$, the corresponding arrow in $\phi^i_{k\ell} \colon X^{i-1}_k \to X^{i-1}_\ell$ is defined as $1_{X^{i-1}_k}$ for $k = \ell$, and 0 otherwise.

For example, in (2.3.3) the sequence $(q_{i\ell}, q_{\ell m}, \phi_{m\ell}, q'_{\ell m}, \phi'_{mm}, q''_{mj})$ is a path, and in (2.3.4) the sequence $(\phi_{ii}, q'_{i\ell}, q'_{\ell k}, \phi'_{ki}, q''_{ij})$ is a path (here for both diagrams $n = 2$).



(2.3.3)



(2.3.4)

Below we assume for $n \geq 2$ that at least one of morphisms $\phi^i \colon X^{i-1} \to X^i$ is $1_{X^{i-1}}$; otherwise, (2.3.5) below gives 0.

Define

$$P_n^{ij}(\phi_n, \dots, \phi_1) := \sum_{\kappa \in \mathscr{P}aths_{ij}} (-1)^{|\kappa|} p_l(\kappa_l, \dots, \kappa_1) \tag{2.3.5}$$

(Recall that p denotes the structure A_∞-morphism for $\mathcal{C}$).

To define the integer $|\kappa|$, we introduce some notations. Let $\kappa = (\kappa_1, \dots, \kappa_\ell)$, and let the n arrows $\kappa_{d_1}, \dots, \kappa_{d_n}$ be essential. Assume that κ_{d_s} is an arrow in $\mathcal{C}(X_{a_s}^{s-1}[r^s], X_{b_s}^s[r^{s+1}])$. Define $t_s = r^{s+1} - r^s$ (we set $t_s = 0$ if $\kappa_{d_s} = 1_X$).

The integer $|\kappa|$ is given by

$$|\kappa| = \sum_{s=1}^{n} (\deg \phi^s + t_s + 1) N_s \tag{2.3.6}$$

where N_s is the number of the horizontal arrows standing *leftwards* to the s-th essential arrow κ_{d_s} *in the sequence* $(\kappa_1, \dots, \kappa_\ell)$.

Lemma 2.3.2. *The maps $P_n^{ij}(\phi_n, \dots, \phi_1)$ are homogeneous of degree $\sum \deg \phi_i - n + 1$. Thus, they are the components of a morphism $P_n(\phi_n, \dots, \phi_1) \colon X^0 \to X^n$ of degree $\sum \deg \phi_i - n + 1$ in the category $\mathcal{C}^{\mathrm{pretr}}$.*

Proof. Let $\kappa = (\kappa_1, \dots, \kappa_\ell) \in \mathscr{P}aths_{ij}$, we have to compute the degree of $p_\ell(\kappa_n, \dots, \kappa_1)$. One has:

$$\deg p_\ell(\kappa_\ell, \dots, \kappa_1) = \sum_{r=1}^{\ell} \deg \kappa_r - \ell + 1$$

Among $\kappa_1, \dots, \kappa_\ell$ exactly n arrows are ϕ^i_{st}s, the remaining $\ell - n$ are q^i_{st} and have degree 1 in $\mathcal{C}^{\text{pretr}}$. On the other hand, $\deg \phi^i_{st} = \deg \phi^i$ and does not depend on s, t. Therefore,

$$\deg p_\ell(\kappa_\ell, \dots, \kappa_1) = \sum_{\kappa_r \neq q^j_{st}} \deg \kappa_r - n + 1 = \sum_{r=1}^{n} \deg \phi_r - n + 1$$

□

Proposition 2.3.3. *Let $\mathcal{C}$ be a weakly unital dg-category, $\mathcal{C}^{\text{pretr}}$ the non-unital dg-category from Definition 2.3.1. Taken for all $n \geq 1$ and all $\phi_1, \dots, \phi_n$, the morphisms $P_n(\phi_n, \dots, \phi_1)$ are the Taylor components of an A_∞-morphism $P\colon \mathcal{C}^{\text{pretr}} \oplus \Bbbk_{\mathcal{C}^{\text{pretr}}} \to \mathcal{C}^{\text{pretr}}$, making $\mathcal{C}^{\text{pretr}}$ a weakly unital dg-category, with $\mathrm{id}_X := P_1(1_X)$.*

We prove this Proposition in Section 2.7.

Definition 2.3.4 (Pretriangulated hull of a weakly unital dg-category, II)**.** The pretriangulated hull $\mathcal{C}^{\text{pretr}}$ of a weakly unital dg-category $\mathcal{C}$ is the non-unital dg-category $\mathcal{C}^{\text{pretr}}$ (see Definition 2.3.1) with the weakly unital structure given in Proposition 2.3.3.

It is instructive to unwind the definition $\mathrm{id}_X = P_1(1_X)$ and get an explicit formula id_X, $X \in \mathcal{C}^{\text{pretr}}$.

Let $X = (\oplus X_i[r_i], q_{ij}) \in \mathcal{C}^{\text{pretr}}$. We want to find the (ij)-component $(\mathrm{id}_X)_{ij}$. Let $i \leq j$. Define

$$(\mathrm{id}_X)_{ij} = \sum_{i=\ell_0<\ell_1<\dots<\ell_k=j} \sum_{r=0}^{k} (-1)^r p_k(q_{\ell_{k-1}j}, q_{\ell_{k-2}\ell_{k-1}}, \dots, q_{\ell_r\ell_{r+1}}, 1_{X_{i_r}}, q_{\ell_{r-1}\ell_r}, q_{\ell_{r-2}\ell_{r-1}}, \dots, q_{i\ell_1}) \tag{2.3.7}$$

Then

$$(\mathrm{id}_X)_{ij} = \begin{cases} \text{the rhs of (2.3.7)} & i \leq j \\ 0 & i > j \end{cases} \tag{2.3.8}$$

The reader easily checks that for the case $\mathrm{id}_{\mathrm{Cone}(f)}$ (2.3.7) gives (2.3.1).

2.4 A Closed Model Structure on $\mathbb{C}_{dgwu}(\Bbbk)$

In this Section we provide a cofibrantly generated Quillen model structure on $\mathbb{C}_{dgwu}(\Bbbk)$. Recall the results from Section 1.2 for a brief and general introduction to (cofibrantly generated) closed model categories.

Define *quasi-equivalences* $\mathcal{W}$ in $\mathbb{C}_{dgwu}(\Bbbk)$ as the weakly unital dg-functors $F\colon \mathcal{C} \to \mathcal{D}$ such that the following two conditions hold:

(W1) for any two objects $x, y \in \mathcal{C}$, the map of complexes $\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ is a quasi-isomorphism of complexes,

(W2) the functor $H^0(F)\colon H^0(\mathcal{C}) \to H^0(\mathcal{D})$ is an equivalence of $\Bbbk$-linear categories.

Remark that for a weakly unital dg-category $\mathcal{C}$, the category $H^0(\mathcal{C})$ is strictly unital and the functor $H^0(F)$ is well-defined.

Define *fibrations* Fib in $\mathbb{C}_{dgwu}(\Bbbk)$ as the weakly unital dg-functors $F\colon \mathcal{C} \to \mathcal{D}$ such that the following two conditions hold:

(F1) for any two objects $X, Y \in \mathcal{C}$, the map of complexes $\mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$ is component-wise surjective,

(F2) for any $X \in \mathcal{C}$ and a closed degree 0 arrow $g\colon FX \mapsto Z$ in $\mathcal{D}$ (Z not necessarily in the image of F), such that g becomes an isomorphism in $H^0(\mathcal{D})$, there is an object $Y \in \mathcal{C}$ and a closed degree 0 arrow $f\colon X \mapsto Y$ inducing an isomorphism in $H^0(\mathcal{C})$ and such that $F(f) = g$.

We define also a class $Surj$ of maps in $\mathbb{C}_{dgwu}(\Bbbk)$ as follows: a weakly unital dg-functor $F\colon \mathcal{C} \to \mathcal{D}$ belongs to $Surj$ if F is surjective on objects and if $(F1)$ holds.

Lemma 2.4.1. *A weakly unital dg-functor $F\colon \mathcal{C} \to \mathcal{D}$ belongs to $Fib \cap \mathcal{W}$ if and only if it belongs to $Surj \cap (W1)$.*

Proof. It is clear that $Surj \cap (W1)$ implies $Fib \cap \mathcal{W}$. Conversely, assume F obeys $Fib \cap \mathcal{W}$. One has to prove that F is surjective on objects. From $(W2)$ we know that $H^0(F)$ is essentially surjective, that is, for any object Z in $\mathcal{D}$ there is a homotopy equivalence $g\colon FX \to Z$. By $(F2)$, there is a homotopy equivalence $f\colon X \to Y$ such that $F(f) = g$. In particular, $F(Y) = Z$. □

Lemma 2.4.2. *Let $\mathcal{C}$ be a weakly unital dg-category, $X \in \mathcal{C}$ an object. Suppose there are two degree -1 maps $h_1, h_2 \in \mathcal{C}^{-1}(X,X)$ such that $dh_i = \mathrm{id}_X$, $i = 1,2$. Then there is $t \in \mathcal{C}^{-2}(X,X)$ such that $dt = h_1 - h_2$.*

Proof. Consider $t' = h_1 h_2$. We find (using (2.1.1)):

$$\begin{aligned} dt' &= \mathrm{id}_X \circ h_2 - h_1 \circ \mathrm{id}_X \\ &= -dp_2(1,h_2) + h_2 - p_2(1,\mathrm{id}_X) + dp_2(h_1,1) - h_1 + p_2(\mathrm{id}_X,1) \end{aligned} \tag{2.4.1}$$

If we manage to prove that $p_2(\mathrm{id},1) - p_2(1,\mathrm{id})$ is a boundary, we are done. Consider

$$\begin{aligned} H := {} & p_2(p_2(\mathrm{id},1),1) + p_2(1,p_2(1,\mathrm{id})) - p_3(1,\mathrm{id},1) + p_3(\mathrm{id},\mathrm{id},1) + p_3(1,\mathrm{id},\mathrm{id}) - p_3(1,1,\mathrm{id}) \\ & - p_3(\mathrm{id},1,1) + p_3(1,1,1) \end{aligned} \tag{2.4.2}$$

We compute dH using (2.1.1) and (2.1.2), the differential of each particular summand in (2.4.2) is displayed as $[\dots]$:

$$\begin{aligned} dH := {} & \big[\underline{p_2(\mathrm{id},1) \circ \mathrm{id}} + \underline{p_2(\mathrm{id} \circ \mathrm{id},1)}\big] + \big[\underline{\underline{\mathrm{id} \circ p_2(1,\mathrm{id})}} - \underline{p_2(1,\mathrm{id} \circ \mathrm{id})}\big] + \big[\,\boxed{p_2(\mathrm{id},1) - p_2(1,\mathrm{id})} \\ & - \underline{\mathrm{id} \circ p_2(\mathrm{id},1)} + \cancel{p_2(1,\mathrm{id}) \circ \mathrm{id}}\big] + \big[-\underline{p_2(\mathrm{id} \circ \mathrm{id},1)} + \underline{\mathrm{id} \circ p_2(\mathrm{id},1)}\big] + \\ & \big[\underline{p_2(1,\mathrm{id} \circ \mathrm{id})} - \cancel{p_2(1,\mathrm{id}) \circ \mathrm{id}}\big] + \big[\cancel{p_2(1,1) \circ \mathrm{id}} - \underline{\underline{\mathrm{id} \circ p_2(1,\mathrm{id})}}\big] + \\ & \big[-\underline{\mathrm{id} \circ p_2(1,1)} + \underline{p_2(\mathrm{id},1) \circ \mathrm{id}}\big] + \big[\underline{\mathrm{id} \circ p_2(1,1)} - \cancel{p_2(1,1) \circ \mathrm{id}}\big] = \\ & p_2(\mathrm{id},1) - p_2(1,\mathrm{id}). \end{aligned}$$

Therefore $t := -t' - p_2(1, h_2) + p_2(h_1, 1) + H \in \mathcal{C}^{-2}(X, X)$ is such that $dt = h_1 - h_2$. $\square$

2.4.1 The weakly unital dg-category $\mathcal{K}'$

The weakly unital dg-category $\mathcal{K}'$ introduced below is a weakly unital counterpart of the dg-category $\mathcal{K}$, due to Kontsevich in [[Ko1], Lecture 6], and subsequently used by Tabuada in his closed model structure on $\mathbb{C}_{dg}(\Bbbk)$ [Tab]. Let us recall here the definition.

The dg-category $\mathcal{K}$ is the strictly unital dg-category with two objects 0 and 1, and freely generated by $f \in \mathcal{K}^0(0, 1), g \in \mathcal{K}^0(1, 0), h_0 \in \mathcal{K}^{-1}(0, 0), h_1 \in \mathcal{K}^{-1}(1, 1), r \in \mathcal{K}^{-2}(0, 1)$, whose differentials are

$$df = dg = 0, \quad dh_0 = g \circ f - \mathrm{id}_0, \quad dh_1 = f \circ g - \mathrm{id}_1, \quad dr = h_1 \circ f - f \circ h_0 \tag{2.4.3}$$

Denote by I_2 the $\Bbbk$-linear envelope of the ordinary category with two objects 0 and 1, and with a unique morphism (including the identity one) between any ordered pair of objects. There is a dg-functor $p_{\mathcal{K}} \colon \mathcal{K} \to I_2$, which is the identity map on the objects, and sends h_1, h_2, r to 0.

The following well-known result says that $\mathcal{K}$ is a semi-free resolution of I_2 and the proof can be found in [[Dr], 3.7]:

Lemma 2.4.3. *The dg-functor $p_{\mathcal{K}} \colon \mathcal{K} \to I_2$ is a quasi-equivalence.*

Definition 2.4.4. Denote by $\mathcal{K}'$ the weakly unital dg-category with two objects 0 and 1, whose morphisms are freely generated by the following morphisms:

- a morphism $f \in (\mathcal{K}')^0(0, 1)$,
- a morphism $g \in (\mathcal{K}')^0(1, 0)$,
- a morphism $h_0 \in (\mathcal{K}')^{-1}(0, 0)$,
- a morphism $h_1 \in (\mathcal{K}')^{-1}(1, 1)$,
- a degree -2 morphism $r \in (\mathcal{K}')^{-2}(0, 1)$

whose differentials are given as

$$\begin{aligned} &df = dg = 0 \\ &dh_0 = g \circ f - \mathrm{id}_0, \quad dh_1 = f \circ g - \mathrm{id}_1 \\ &dr = h_1 \circ f - f \circ h_0 + p_2(1, f) - p_2(f, 1) \end{aligned} \tag{2.4.4}$$

A version of a lemma in [[Ko1], Lecture 6] holds as well in the setting of weakly unital dg-categories:

Lemma 2.4.5. *Let $\mathcal{C}$ be a weakly unital dg-category, and $\xi \in \mathcal{C}^0(X, Y)$ be a closed degree 0 morphism, such that $[\xi] \in H^0(\mathcal{C})$ is a homotopy equivalence. Then there is a weakly unital dg-functor $F \colon \mathcal{K}' \to \mathcal{C}$, such that $F(f) = \xi$.*

Proof. By definition of being a homotopy equivalence, there exist $\eta \in \mathcal{C}^0(Y,X)$, $h_X \in \mathcal{C}^{-1}(X,X)$ and $h_Y \in \mathcal{C}^{-1}(Y,Y)$ such that:

$$\begin{aligned} dh_X &= \eta \circ \xi - \mathrm{id}_X \\ dh_Y &= \xi \circ \eta - \mathrm{id}_Y \end{aligned} \tag{2.4.5}$$

Now we are looking for a morphism $r \in \mathcal{C}^{-2}(x,y)$ such that $dr = h_Y \circ \xi - \xi \circ h_X + p_2(1,\xi) - p_2(\xi,1)$.
We define:

$$A \coloneqq h_Y \circ \xi - \xi \circ h_X + p_2(1,\xi) - p_2(\xi,1).$$

Clearly $dA = 0$. Then we take $h'_Y \coloneqq h_Y - A \circ \eta$ and $r \coloneqq A \circ h_X - p_2(A,1)$, so that we easily get:

$$dh'_Y = dh_Y - dA\eta + Ad\eta = \xi\eta - \mathrm{id}_Y\,. \tag{2.4.6}$$

and also

$$\begin{aligned} dr =& dA \circ h_X - A \circ dh_X - A \circ \mathrm{id}_X + A - p_2(dA,1) \\ =& - A \circ (\eta\xi - \mathrm{id}_X) - A \circ \mathrm{id}_X + A \\ =& - A \circ \eta \circ \xi + h_Y \circ \xi - \xi \circ h_X + p_2(1,\xi) - p_2(\xi,1) \\ =& (h_Y - A \circ \eta) \circ \xi - \xi \circ h_X + p_2(1,\xi) - p_2(\xi,1) \\ =& h'_y \circ \xi - \xi \circ h_X + p_2(1,\xi) - p_2(\xi,1). \end{aligned} \tag{2.4.7}$$

We are done. □

We prove a lemma which we will be used later in the proof of Theorem 2.4.7 and (implicitly) in Theorem 2.4.14:

Lemma 2.4.6. *Let $\mathcal{C}$ be a weakly unital dg-category. There is a bijection between the set of weakly unital dg-functors from $\mathcal{K}'$ to $\mathcal{C}$ and the set of pairs (ξ, h), where $\xi \in \mathcal{C}^0(x,y)$ is a closed morphism and h is a contraction of* $\mathrm{Cone}(\xi)$ *in $\mathcal{C}^{pretr}$.*

Proof. A weakly unital dg-functor $F\colon \mathcal{K}' \to \mathcal{C}$ amounts to the following morphisms in $\mathcal{C}$: $\xi = F(f), \eta = F(g), h_{11} = F(h_0), h_{22} = F(h_1), h_{12} = F(r)$ such that:

$$\begin{gathered} d(\xi) = 0, \quad d\eta = 0, \quad dh_{11} = \eta \circ \xi - \mathrm{id}_x, \quad dh_{22} = \xi \circ \eta - \mathrm{id}_y, \\ dh_{12} = h_{22} \circ \xi - \xi \circ h_{11} + p_2(1,\xi) - p_2(\xi,1) \end{gathered} \tag{2.4.8}$$

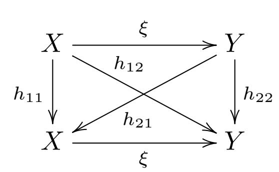
A contraction to $\mathrm{id}_{\mathrm{Cone}(\xi)}$ (see (2.3.1)) is the datum of a morphism $H\colon \mathrm{Cone}(\xi) \to \mathrm{Cone}(\xi)$ of degree -1 such that

$$dH = \mathrm{Id}_{\mathrm{Cone}(\xi)}, \tag{2.4.9}$$

Then

$$H = (h_{11} \in \mathcal{C}^{-1}(X,X), h_{22} \in \mathcal{C}^{-1}(Y,Y), h_{12} \in \mathcal{C}^{-2}(X,Y), h_{21} \in \mathcal{C}^{0}(Y,X))$$

as in the diagram below



(2.4.10)

We see by a direct calculation that 2.4.8 is equivalent to 2.4.9. □

Proposition 2.4.7. *The weakly unital dg-category $\mathcal{K}'$ has the same homotopy type as the Kontsevich dg-category $\mathcal{K}$. More precisely, regarding $\mathcal{K}$ as an object in $\mathbb{C}_{dgwu}(\Bbbk)$, the natural projection $p\colon \mathcal{K}' \to \mathcal{K}$, sending all $p_n(-)$, $n \geq 2$, to 0, is a quasi-equivalence.*

Proof. Consider ascending filtrations $\{\Phi_i(a,b)\}_{i\geq 0}$ of $\mathcal{K}'(a,b)$ and $\{F_i(a,b)\}_{i\geq 0}$ of $\mathcal{K}(a,b)$, $a,b \in \{0,1\}$, such that $p(\Phi_i(a,b)) \subset F_i(a,b)$, $i \geq 0$. We prove that the corresponding spectral sequences converge, and that the map p induces an isomorphism in the E_1 sheets. The result will follow from the latter statement.

Define $F_i(a,b)$ as the dg-vector space generated by all monomials with $\leq i$ factors r. Define $\Phi_i(a,b)$ similarly, but we count all occurrences of r in expressions $p_j(\dots,r,\dots)$ as a "factor r". It is clear that $d(F_i(a,b)) \subset F_i(a,b)$ and $d(\Phi_i(a,b)) \subset \Phi_i(a,b)$, and that $p(\Phi_i(a,b)) \subset F_i(a,b)$.

Also, it is clear that both spectral sequences converge, by dimensional reasons (the spectral sequences live in the quadrant "$x \leq 0, y \leq 0$").

We have:

Lemma 2.4.8. *The map p induces an isomorphism in the E_1 sheets.*

Proof. For both cases, the differential in E_0 is the same as it would be if $dr = 0$. Therefore, to compute E_1 we assume that $dr = 0$ for both cases.

Denote by $\tilde{\mathcal{K}}'$ (respectively, $\tilde{\mathcal{K}}$) the semi-free weakly unital dg-category (respectively, the semi-free unital dg-category) with two objects $\{0,1\}$, the generators f, g, h_0, h_1, r, as in (2.4.3), (2.4.4), and in which the differential of the generators is given by *the same* formulas:

$$\begin{aligned} &df = dg = 0 \\ &dh_0 = g \circ f - \mathrm{id}_0, \quad dh_1 = f \circ g - \mathrm{id}_1 \\ &dr = 0 \end{aligned} \tag{2.4.11}$$

Now the statement follows from the fact that the projection of dg-operads $\mathcal{O}' \to \mathcal{O}$ is a quasi-isomorphism in any arity, where the dg operad $\mathcal{O}$ is the quotient-operad of $\mathcal{O}'$ by the dg operadic ideal I generated by $p_n(1,\dots,1), n \geq 2$, see [[PS1], Section 4]. □

□

Remark 2.4.9. The argument employed in the proof of Lemma 2.4.8 can not be used directly for $p\colon \mathcal{K}' \to \mathcal{K}$ (without any spectral sequence argument), because dr is given by *different* formulas in (2.4.3) and (2.4.4). More precisely, the equation for dr for $\mathcal{K}'$ is a *deformation* of that for $\mathcal{K}$. Consequently, it does not follow directly from the quasi-isomorphism $\mathcal{O}' \to \mathcal{O}$ that $p\colon \mathcal{K}'(a,b) \to \mathcal{K}(a,b)$ is a quasi-isomorphism.

Now we are ready to state the following result, whose proof is trivial:

Corollary 2.4.10. *The natural projection $\mathcal{K}' \to I_2$, equal to the composition $\mathcal{K}' \xrightarrow{p} \mathcal{K} \xrightarrow{p_{\mathcal{K}}} I_2$, is a weak equivalence.*

2.4.2 The sets I and J

Denote by $D(n)$ the complex $0 \to \Bbbk[n] \xrightarrow{\mathrm{id}} \Bbbk[n-1] \to 0$, it is $D(n) = \mathrm{Cone}(\mathrm{id}\colon \Bbbk[n] \to \Bbbk[n])$.

Denote $S(n-1) = \Bbbk[n-1]$. Consider the natural embedding $i\colon S(n-1) \to D(n)$ of complexes.

Denote by $\mathcal{A}$ the weakly unital dg-category with a single object 0 and generated (over the dg-operad $\mathcal{O}'$) by id_0. Denote by κ the weakly unital dg-functor

$$\kappa\colon \mathcal{A} \to \mathcal{K}',$$

sending 0 to 0. It follows from Corollary 2.4.10 that κ is a quasi-equivalence.

Denote by $\mathcal{B}$ the weakly unital dg-category with two objects 0 and 1 and generated over $\mathcal{O}'$ by morphisms id_0 and id_1.

Let $P(n)$ be the dg-quiver with two objects 0 and 1, and with morphisms $P(n)(0,1) = D(n)$, $P(n)(0,0) = 0$, $P(n)(1,1) = 0$, $P(n)(1,0) = 0$. Denote by $\mathcal{P}(n)$ the weakly unital dg-category generated by $P(n)$: $\mathcal{P}(n) := F(P(n))$.

Denote by $\alpha(n)$ the weakly unital dg-functor

$$\alpha(n)\colon \mathcal{B} \to \mathcal{P}(n),$$

sending 0 to 0 and 1 to 1.

Let $C(n)$ be the dg-quiver with two objects 0 and 1, and with morphisms $C(n)(0,1) = S(n-1)$, $C(n)(0,0) = 0$, $C(n)(1,1) = 0$, $C(n)(1,0) = 0$. Denote by $\mathcal{C}(n)$ the weakly unital dg-category generated by $C(n)$: $\mathcal{C}(n) := F(C(n))$.

Let $b(n)\colon C(n) \to P(n)$ be a map of dg-quivers sending 0 to 0, 1 to 1, and such that $S(n-1) = C(n)(0,1) \xrightarrow{i} P(n)(0,1) = D(n)$ is the embedding i. Denote by $\beta(n)$ the weakly unital dg-functor

$$\beta(n) := F(b(n))\colon \mathcal{C}(n) \xrightarrow{i} \mathcal{P}(n).$$

Denote by Q the natural weakly unital dg-functor

$$Q\colon \emptyset \to \mathcal{A}.$$

Let I be a set of morphisms in $\mathbb{C}_{dgwu}(\Bbbk)$ which comprises the weakly unital dg-functors Q and $\beta(n)$, $n \in \mathbb{Z}$.

Let J be a set of morphisms in $\mathbb{C}_{dgwu}(\Bbbk)$ which comprises κ and $\alpha(n)$, $n \in \mathbb{Z}$.

The set I and J are referred to as the sets of *generating cofibrations* and of *generating acyclic cofibrations*, correspondingly.

Lemma 2.4.11. *A weakly unital dg-functor $F\colon \mathcal{C} \to \mathcal{D}$ has RLP with respect to all $\alpha(n), n \in \mathbb{Z}$ if and only if F obeys $(F1)$. A weakly unital dg-functor $F\colon \mathcal{C} \to \mathcal{D}$ has RLP with respect to all $\beta(n), n \in \mathbb{Z}$ if and only if F obeys $(F1) \cap (W1)$.*

Proof. Let us prove the "only if" parts of both statements, the proofs of the "if" parts are standard and are left to the reader.

Assume a weakly unital dg-functor $F\colon \mathcal{C} \to \mathcal{D}$ has RLP with respect to all $\alpha(n)$, $n \in \mathbb{Z}$: for the functor F it means that any morphism in $\mathcal{D}(FX, FY)$ is $F(q)$, for some $q \in \mathcal{C}(X, Y)$. That is, F is surjective on morphisms.

Assume that a weakly unital dg-functor $F\colon \mathcal{C} \to \mathcal{D}$ has RLP with respect to all $\beta(n)$, $n \in \mathbb{Z}$: one deduces from this property that for any $X, Y \in \mathcal{C}$, the map of complexes $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is component-wise surjective, and is a quasi-isomorphism. □

Let us recall the standard terminology (conventional for the theory of closed model categories): a dg-functor $F\colon \mathcal{C} \to \mathcal{D}$ belongs to *I-inj* (respectively, to *J-inj*) if it has the RLP with respect to all morphisms in I (respectively, in J).

Proposition 2.4.12. *One has*

$$I\text{-}inj = Surj \cap (W1) = J\text{-}inj \cap \mathcal{W}$$

Proof. Thanks to Lemma 2.4.11, the first equality follows from the fact that a dg-functor $F\colon \mathcal{C} \to \mathcal{D}$ has the RLP for Q if and only if it is surjective on objects, which is straightforward.

The second equality is far more sophisticated, and its proof is based on the following lemma[1]:

Lemma 2.4.13. *One has $Fib = J\text{-}inj$.*

Proof. The inclusion $J\text{-}inj \subseteq Fib$ follows from Lemma 2.4.5.

In order to prove the inclusion $Fib \subseteq J\text{-}inj$ consider $\phi\colon \mathcal{C} \to \mathcal{D}$ in Fib. Axiom $(F1)$ is equivalent to the RLP with respect to $\alpha(n), n \in \mathbb{Z}$, thence we only need to prove the RLP with respect to κ for ϕ.

We are given a weakly unital dg-functor $F\colon \mathcal{K}' \to \mathcal{D}$. We can apply $(F2)$ to $\xi = F(f) \in \mathcal{D}^0(\phi(x), z)$, and so we get a morphism $\eta \in \mathcal{C}^0(x, y)$ which is a homotopy equivalence and $\phi(\eta) = \xi, \phi(y) = z$. (Recall that f, g, h_0, h_1, r are generators for $\mathcal{K}'$, see (2.4.4)).

We should construct a weakly unital dg-functor $\hat{F}\colon \mathcal{K}' \to \mathcal{C}$ such that $\phi \circ \hat{F} = F$ and $\hat{F}(f) = \eta$. By Lemma 2.4.6 having a weakly unital dg-functor $F\colon \mathcal{K}' \to \mathcal{D}, F(f) = \xi$ is equivalent to having a contraction of $\mathrm{Cone}(\xi)$ in $\mathcal{D}^{pretr}$, i.e. we have a degree -1 morphism $h \in \mathcal{D}^{pretr}(\mathrm{Cone}(\xi), \mathrm{Cone}(\xi))$ such that $dh = \mathrm{id}_{\mathrm{Cone}(\xi)}$.

By $(F2)$ we know that $\mathrm{Cone}(\eta)$ is also contractible, so we also have a morphism $\tilde{h}_1 \in \mathcal{C}^{pretr}(\mathrm{Cone}(\eta), \mathrm{Cone}(\eta))$, such that $d\tilde{h}_1 = \mathrm{id}_{\mathrm{Cone}(\eta)}$. Even though we do not know whether $\phi^{pretr}(\tilde{h}_1) = h$, we still have $d\phi^{pretr}(\tilde{h}_1) = \mathrm{id}_{\mathrm{Cone}(\xi)}$.

[1] Lemma 2.4.13 is one of the most subtle places in our constructions; in particular, the theory of weakly unital pretriangulated hull developed in Section 2.3, and Lemma 2.4.2, were designed especially for its proof.

By Lemma 2.4.2, one has $\phi^{pretr}(\tilde{h}_1) - h = dt$. By $(F1)$, we find a lift $\tilde{t}$ of t, and set $\tilde{h} := \tilde{h}_1 - d\tilde{t}$. One clearly has $d\tilde{h} = \mathrm{id}_{\mathrm{Cone}(\eta)}$ and $\phi(\tilde{h}) = h$. This gives the desired lift $\hat{F}\colon \mathcal{K}' \to \mathcal{C}$ such that $\phi \circ \hat{F} = F$, by Lemma 2.4.6. □

Now we have J-$inj \cap W = Fib \cap W = Surj \cap (W1)$, and we are done. □

The following theorem is one of our main results:

Theorem 2.4.14. *The category $\mathbb{C}_{dgwu}(\Bbbk)$ admits a cofibrantly generated closed model structure whose weak equivalences are the quasi-equivalences, the fibrations are as above, and whose sets of generating cofibrations and generating acyclic cofibrations are I and J.*

2.4.3 Proof of Theorem 2.4.14

Recall from Section 1.2 the notations *I-cell*, *J-cell*, *I-cof*, *J-cof* and recall that *I-cell* $\subset$ *I-cof* and *J-cell* $\subset$ *J-cof*.

Recall this result [[Ho], Th. 2.1.19] which the proof is based on:

Theorem 2.4.15. *Let $\mathcal{C}$ be a small complete and cocomplete category. Suppose that W is a subcategory of $\mathcal{C}$, and I and J are sets of maps. Assume that the following conditions hold:*

1. *the subcategory W has 2-out of-3 property and is closed under retracts,*
2. *the domains of I are small relative to I-cell,*
3. *the domains of J are small relative to J-cell,*
4. *J-cell $\subset \mathcal{W} \cap I$-cof,*
5. *I-inj $= \mathcal{W} \cap J$-inj.*

Then there is a cofibrantly generated closed model structure on $\mathcal{C}$, for which the morphisms $\mathcal{W}$ of W are weak equivalences, I are generating cofibrations, J are generating acyclic cofibrations. Its fibrations are defined as J-inj.

Proof of Theorem 2.4.14. The category $\mathbb{C}_{dgwu}(\Bbbk)$ is small complete and small cocomplete by Theorem 2.2.10. The conditions $(1) - (3)$ are clear. Condition (5) has been proved in Proposition 2.4.12. It follows from (5) that *I-inj* $\subset$ *J-inj*, and so *J-cof* $\subset$ *I-cof*. It only remains to prove that *J-cell* $\subseteq \mathcal{W}$.

Proof of J-cell $\subseteq W$. We have to prove that the push-out of a morphism in J is a quasi-equivalence. We consider two cases: when the morphism in J is $\alpha(n), n \in \mathbb{Z}$, and when it is κ.

First case.
The first case we need to consider is shown in the push-out diagram:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{G} & \mathcal{X} \\ {\scriptstyle\alpha(n)}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{P}(n) & \longrightarrow & \mathcal{Y} \end{array} \tag{2.4.12}$$

where G is an arbitrary map. We need to prove that F is a quasi-equivalence.

Clearly $Ob(\mathcal{X}) = Ob(\mathcal{Y})$, and F acts by the identity maps on the objects. We are left to show that, for any objects $A, B \in Ob(\mathcal{X})$, the map of complexes $F(A,B)\colon \mathcal{X}(A,B) \to \mathcal{Y}(A,B)$ is a quasi-isomorphism. For objects $0, 1$ in $Ob(\mathcal{B})$, let denote by $U = G(0)$ and $V = G(1)$. By Proposition 2.4.16, one has the following description for the hom-complexes of $\mathcal{Y}$:

$$\begin{aligned} \mathcal{Y}(A,B) \coloneqq\ & \mathcal{X}(A,B) \bigoplus \mathcal{O}'(3) \otimes \mathcal{X}(V,B) \otimes \mathcal{D}(n) \otimes \mathcal{X}(A,U) \\ & \bigoplus \mathcal{O}'(5) \otimes \mathcal{X}(V,B) \otimes \mathcal{D}(n) \otimes \mathcal{X}(V,U) \otimes \mathcal{D}(n) \otimes \mathcal{X}(A,U) \bigoplus \cdots \end{aligned} \tag{2.4.13}$$

The map $F(A,B)$ sends $\mathcal{X}(A,B)$ to the first summand. The other summands have trivial cohomology by the Künneth formula, since the acyclicity of $\mathcal{D}(n)$.

Second case.
As the second case we consider the following push-out diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{H} & \mathcal{X} \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{K}' & \longrightarrow & \mathcal{Y} \end{array} \tag{2.4.14}$$

where H is an arbitrary map.

One has $Ob(\mathcal{Y}) = Ob(\mathcal{X}) \sqcup 1_{\mathcal{K}'}$. It is clear that $H^0(F)$ is essentially surjective. One has to prove that the F is locally a quasi-isomorphism: $F(A,B)\colon \mathcal{X}(A,B) \to \mathcal{Y}(A,B)$, for any $A, B \neq 1_{\mathcal{K}'}$. Denote $H(0_A) = U$.

By Theorem 2.4.7, we know that $\mathcal{K}'$ is a resolution of the $\Bbbk$-linear envelope of the ordinary category with two objects $0, 1$ and with only one morphism between any pair of objects. In particular, $\mathcal{K}'(0,0)$ is quasi-isomorphic to $\Bbbk[0]$. Therefore

$$\bar{\mathcal{K}}' \coloneqq \mathcal{K}'(0,0)/\Bbbk[0] \tag{2.4.15}$$

is a complex acyclic in all degrees.

By Proposition 2.4.16, we have:

$$\begin{aligned} \mathcal{Y}(A,B) \coloneqq\ & \mathcal{X}(A,B) \bigoplus \mathcal{O}(3) \otimes \mathcal{X}(U,B) \otimes \bar{\mathcal{K}}' \otimes \mathcal{X}(A,U) \\ & \bigoplus \mathcal{O}(5) \otimes \mathcal{X}(U,B) \otimes \bar{\mathcal{K}}' \otimes \mathcal{X}(U,U) \otimes \bar{\mathcal{K}}' \otimes \mathcal{X}(A,U) \bigoplus \ldots \end{aligned} \tag{2.4.16}$$

It is a direct sum of complexes, among which all but the first one are acyclic, due to the acyclicity of $\bar{\mathcal{K}}'$. It completes the proof that F is a quasi-equivalence. □

Theorem 2.4.14 is proven. □

2.4.4 Push-outs in the category $\mathbb{C}_{dgwu}(\Bbbk)$

For sake of completeness, we describe explicitly the push-out diagrams in $\mathbb{C}_{dgwu}(\Bbbk)$. Consider the following:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{i_1} & \mathcal{C}_1 \\ {\scriptstyle i_2}\downarrow & & \\ \mathcal{C}_2 & & \end{array} \tag{2.4.17}$$

where $i_a\colon \mathcal{C} \hookrightarrow \mathcal{C}_a, a = 1, 2$ are embeddings of weakly unital dg-categories. We provide an explicit formula for the push-out $\mathcal{E}$. This formula is essentially used in the proof of Theorem 2.4.14.

Note that this colimit $\mathcal{E}$ is equivalently the colimit of the following coequalizer diagram:

$$\mathcal{C} \rightrightarrows \mathcal{C}_1 \oplus \mathcal{C}_2 \tag{2.4.18}$$

where the maps are $(i_1, 0)$ and $(0, i_2)$. The category structure on $\mathcal{C}_1 \oplus \mathcal{C}_2$ is defined as

$$Ob(\mathcal{C}_1 \oplus \mathcal{C}_2) \coloneqq Ob(\mathcal{C}_1) \sqcup Ob(\mathcal{C}_2)$$

$$\mathrm{Hom}_{\mathcal{C}_1\oplus\mathcal{C}_2}(x, y) = \begin{cases} \mathcal{C}_a(x, y) & \text{if } x, y \in Ob(\mathcal{C}_a) \\ 0 & \text{otherwise.} \end{cases}$$

In Proposition 2.2.9 we considered the general coequalizers in the category $\mathbb{C}_{dgwu}(\Bbbk)$. Here we provide the corresponding description for the case of the coequalizer (2.4.18). Here we essentially use that i_1, i_2 are fully-faithful functors. The derivation of this description from the cited proposition is straightforward.

We use notation $\bar{a}$ which is defined as $\bar{1} = 2$, $\bar{2} = 1$.

Proposition 2.4.16. *Assume i_1, i_2 in* (2.4.17) *are fully faithful. Then the push-out weakly unital dg-category $\mathcal{D}$ has the following description.*

The objects of $\mathcal{D}$ are given by the coequalizer of sets:

$$Ob(\mathcal{D}) = Ob(\mathcal{C}_1) \sqcup Ob(\mathcal{C}_2)/\sim$$

where $\sim$ is the equivalence relation generated by: $i_1(X) \sim i_2(X)$, for any $X \in Ob(\mathcal{C})$.

Consider $X, Y \in Ob(\mathcal{C}_a)$, $a = 1, 2$. Then:

$$\begin{aligned}\mathcal{D}(X, Y) = &\mathcal{C}_a(X, Y) \bigoplus \bigoplus_{U,V\in\mathcal{C}} \mathcal{O}'(3) \otimes \mathcal{C}_a(V, Y) \otimes \overline{\mathcal{C}}_{\bar{a}}(U, V) \otimes \mathcal{C}_a(X, U) \bigoplus \\ &\bigoplus_{U,V,U_1,V_1\in\mathcal{C}} \mathcal{O}'(5) \otimes \mathcal{C}_a(V_1, Y) \otimes \mathcal{C}_{\bar{a}}(U_1, V_1) \otimes \mathcal{C}_a(U, V_1) \otimes \mathcal{C}_{\bar{a}}(V, U) \otimes \mathcal{C}_a(X, V) \bigoplus \dots\end{aligned} \tag{2.4.19}$$

where we identify an object $U \in \mathcal{C}$ with its images $i_a(U) \in \mathcal{C}_a$, and
$\overline{\mathcal{C}}_a(U, V) \coloneqq \mathcal{C}_a(U, V)/i_a(\mathcal{C}(X, Y))$.

For $X \in Ob(\mathcal{C}_1) \setminus \mathrm{Ob}(\mathcal{C})$*,* $Y \in Ob(\mathcal{C}_2) \setminus \mathrm{Ob}(\mathcal{C})$*, one has:*

$$\mathcal{D}(X,Y) := \bigoplus_{U\in\mathcal{C}} \mathcal{O}'(2)\otimes\mathcal{C}_2(U,Y)\otimes\mathcal{C}_1(X,U) \bigoplus \bigoplus_{U,V,W\in\mathcal{C}} \mathcal{O}'(4)\otimes\mathcal{C}_2(W,Y)\otimes\overline{\mathcal{C}}_1(V,W)\otimes\overline{\mathcal{C}}_2(U,V)\otimes\mathcal{C}_1(X,U)\bigoplus\dots \tag{2.4.20}$$

2.5 A Quillen equivalence between $\mathbb{C}_{dgwu}(\Bbbk)$ and $\mathbb{C}_{dg}(\Bbbk)$

Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories. Recall that a *Quillen pair* of functors $L\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon R$ is an adjoint pair of functors such that L preserves cofibrations and acyclic cofibrations, or equivalently, R preserves fibrations and acyclic fibrations, [[Ho], Sec. 1.3], [[Hi], Sec. 8.5]. These two conditions are sufficient to show that the Quillen pair of functors descends to a pair of adjoint functors

$$L\colon \mathrm{Ho}(\mathcal{C}) \leftrightarrows \mathrm{Ho}(\mathcal{D})\colon R \tag{2.5.1}$$

between the homotopy categories.

When $\mathcal{C}$ is cofibrantly generated, there is a manageable criterion for an adjoint pair of functors to be a Quillen pair [[Ho], Lemma 2.1.20]:

Proposition 2.5.1. *Let $\mathcal{C}$, $\mathcal{D}$ be closed model categories, with $\mathcal{C}$ cofibrantly generated with generating cofibrations I and generating acyclic cofibrations J. Let $L\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon R$ be an adjoint pair of functors. Assume that $L(f)$ is a cofibration for all $f \in I$ and $L(f)$ is a trivial cofibration for all $f \in J$. Then (L,R) is a Quillen pair.*

Let $\mathcal{C} \in \mathbb{C}_{dgwu}(\Bbbk)$. Define

$$L(\mathcal{C}) := \mathcal{C}/I,$$

where I is the dg-category ideal generated by all $p_n(\dots), n \geq 2$. Clearly $L(\mathcal{C}) \in \mathbb{C}_{dg}(\Bbbk)$. This assignment $\mathcal{C} \mapsto L(\mathcal{C})$ gives rise to a functor $L\colon \mathbb{C}_{dgwu}(\Bbbk) \to \mathbb{C}_{dg}(\Bbbk)$.

Let $\mathcal{D} \in \mathbb{C}_{dg}(\Bbbk)$. Define $R\colon \mathbb{C}_{dg}(\Bbbk) \to \mathbb{C}_{dgwu}(\Bbbk)$ as the fully-faithful embedding from Example 2.1.3.

Proposition 2.5.2. *The following statements are true:*

(1) there is an adjunction

$$\mathrm{Hom}_{\mathbb{C}_{dg}(\Bbbk)}(L(\mathcal{C}),\mathcal{D}) \cong \mathrm{Hom}_{\mathbb{C}_{dgwu}(\Bbbk)}(\mathcal{C},R(\mathcal{D}))$$

(2) the functors

$$L\colon \mathbb{C}_{dgwu}(\Bbbk) \leftrightarrows \mathbb{C}_{dg}(\Bbbk)\colon R$$

form a Quillen pair of functors.

Proof. (1) : any morphism $F\colon \mathcal{C} \to R(\mathcal{D})$ sends $p_n(\dots), n \geq 2$ to 0, since $\mathcal{D} \in \mathbb{C}_{dg}(\Bbbk)$, and therefore this morphism is the same as a morphism $L(\mathcal{C}) \mapsto \mathcal{D}$.

(2) : the dg-functors $\{L(\beta(n)), L(\mathcal{Q})\}$ form exactly the set I of generating cofibrations for the Tabuada closed model structure on $\mathbb{C}_{dg}(\Bbbk)$, and the dg-categories $\{L(\alpha(n)), L(\kappa)\}$ form the set of generating acyclic cofibrations for this model structure. The statement follows then by Proposition 2.5.1. □

Recall that a Quillen pair $L\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon R$ is called a *Quillen equivalence* if it satisfies the following condition:

For a cofibrant $X \in \mathcal{C}$ and a fibrant $Y \in \mathcal{D}$, a morphism $f\colon LX \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if its adjoint morphism $X \to RY$ also is, [[Ho], 1.3.3], [[Hi], 8.5]. This condition implies the corresponding adjoint pair of functors between the homotopy categories 2.5.1 is an adjoint *equivalence* of categories.

Theorem 2.5.3. *The Quillen pair of functors*

$$L\colon \mathbb{C}_{dgwu}(\Bbbk) \leftrightarrows \mathbb{C}_{dg}(\Bbbk)\colon R$$

is a Quillen equivalence.

Proof. Let $\mathcal{C} \in \mathbb{C}_{dgwu}(\Bbbk)$ be cofibrant and $\mathcal{D} \in \mathbb{C}_{dg}(\Bbbk)$ fibrant. We have to prove that $F\colon L\mathcal{C} \to \mathcal{D}$ is a weak equivalence in $\mathbb{C}_{dg}(\Bbbk)$ if and only if the adjoint morphism $F^*\colon \mathcal{C} \to R\mathcal{D}$ is a weak equivalence in $\mathbb{C}_{dgwu}(\Bbbk)$.

It is enough to prove the statement for the case when $\mathcal{C}$ is an I-cell. Indeed, by the small object argument, for any $\mathcal{C}$ there exist an I-cell $\mathcal{C}'$ such that $p\colon \mathcal{C}' \to \mathcal{C}$ is an acyclic fibration. The Quillen left adjoint L maps the weak equivalences between cofibrant object to weak equivalences, by [[Hi], Prop. 8.5.7]. Therefore, $L(p)\colon L(\mathcal{C}') \to L(\mathcal{C})$ is a weak equivalence. There is a map $i\colon \mathcal{C} \to \mathcal{C}'$ such that $p \circ i = \mathrm{id}$, given by the RLP. By 2-of-3 axiom, i is a weak equivalence, and $L(i)$ is too.

Assume that $\mathcal{C}$ is an I-cell and denote by V the dg-quiver of generators of $\mathcal{C}$.
We need to prove that for any objects $X, X' \in \mathcal{C}$, the cone $M := \mathrm{Cone}(F\colon L\mathcal{C}(X,X') \to \mathcal{D}(FX, FX'))$ is acyclic if and only if the cone $N := \mathrm{Cone}(F^*\colon \mathcal{C}(X,X') \to R\mathcal{D}(F^*X, F^*X'))$ is acyclic. Denote by $\tilde{\mathcal{O}} := \mathrm{Ker}(P\colon \mathcal{O}' \to \mathcal{A}ssoc_+)$, where P is the dg-operad map sending all $p_n(\dots), n \geq 2$ to 0.

There is a canonical map $\omega\colon N \to M$, and $\mathrm{Cone}(\omega)$ is quasi-isomorphic to $F_{\tilde{\mathcal{O}}}(V)(X,X')$, where $F_{\tilde{\mathcal{O}}}(V)$ is the free algebra over $\tilde{\mathcal{O}}$ generated by V, with an extra differential coming from the differential in the I-cell $\mathcal{C}$. Since the dg-operad $\mathcal{O}'$ is quasi-isomorphic to $\mathcal{A}ssoc_+$, $\tilde{\mathcal{O}}$ is acyclic. Therefore $F_{\tilde{\mathcal{O}}}(V)$ is acyclic by the Künneth formula, and so M is quasi-isomorphic to N, by the acyclicity of $\mathrm{Cone}(\omega)$. We conclude that M is acyclic if and only if N is. □

2.6 A cofibrant resolution of a unital dg-algebra

Here we provide a canonical unital cofibrant dg-algebra, quasi-isomorphic to a unital dg-algebra A over a field $\Bbbk$. (We consider only the case of a dg-algebra for simplicity, the construction is directly generalised for the case of a small dg-category).

The classical Bar-Cobar resolution of A fails to be unital, it is only weakly unital dg-algebra. A well-known explicit unital construction comes from the *curved* version of Bar-Cobar duality, due to Positselski [Po] (see also [Ly3]). A drawback of this construction is that it is not canonical.

We have not seen this construction in the literature, thus we include it as an appendix of this chapter for two reasons: in primis, it "replaces" the Bar-Cobar resolution, which fails to be strictly unital, which was one of our starting points; in secundis, it can be easily generalised to a cofibrant resolution of $i(\mathcal{C})$ in $\mathbb{C}_{dgwu}(\Bbbk)$, for $\mathcal{C} \in \mathbb{C}_{dg}(\Bbbk)$.

Let A be a unital dg-algebra over $\Bbbk$. Consider the dg-algebra $A_+ = A \oplus \Bbbk[1]$. It is a unital dg-algebra with unit 1_A, and the product of A with $\Bbbk[1]$, as well as of $\Bbbk[1]$ with itself, is defined as 0.

Consider the Bar-complex

$$\mathrm{Bar}(A_+) = \bigoplus_{n\geq 1} A_+[1]^{\otimes n}$$

which is a dg-coalgebra. We use the notation ξ for a generator of $\Bbbk[1]$. Then a general monomial element of $\mathrm{Bar}(A_+)$ is denoted as

$$a_1 \otimes \dots \otimes a_{i_1} \otimes \xi \otimes a_{i_1+1} \otimes \dots \otimes a_{i_2} \otimes \xi \otimes a_{i_2+1} \otimes \dots$$

Now consider the *unital* dg-algebra

$$C_0(A) = \mathrm{Cobar}_+(\mathrm{Bar}(A_+))$$

where, for a dg-coalgebra B,

$$\mathrm{Cobar}_+(B) = \Bbbk \oplus \bigoplus_{n\geq 1} B[-1]^{\otimes n}$$

with the Cobar-differential. It is a unital dg-algebra. Denote by $1_\Bbbk$ the unit of $\Bbbk$. It is the unit of $C_0(A)$. We denote the product in $\mathrm{Cobar}_+(B)$ by $\boxtimes$.

Consider a derivation d_ξ of degree $+1$ of $C_0(A)$ whose only non-zero Taylor coefficient is linear, and is defined as

$$\begin{aligned} &d_\xi|_A = 0,\ d_\xi(\xi) = 1_\Bbbk - 1_A \\ &d_\xi(x_1 \otimes \dots \otimes x_n) = \sum_{\ell=1}^{k} \pm x_1 \otimes \dots \otimes x_{i_\ell -1} \otimes 1_A \otimes x_{i_\ell+1} \otimes \dots \otimes x_n \\ &\text{where } x_{i_1} = \dots = x_{i_k} = \xi \text{ and other } x_i \in A \end{aligned} \tag{2.6.1}$$

One has

Lemma 2.6.1. *The differential d_ξ squares to 0, and d_ξ commutes with $d_{\mathrm{Bar}} + d_{\mathrm{Cobar}}$. Consequently, $d_{\mathrm{tot}} := d_{\mathrm{Bar}} + d_{\mathrm{Cobar}} + d_\xi$ squares to 0.*

It is a direct computation. □

We denote

$$C(A) = (C_0(A), d_{\mathrm{Bar}} + d_{\mathrm{Cobar}} + d_\xi)$$

It is a unital dg-algebra.

Proposition 2.6.2. *There is a unital dg-algebra map $p\colon C(A) \to A$ which is a quasi-isomorphism.*

Proof. We start with computing the cohomology of $(C(A), d_{\mathrm{tot}})$. The differential $d_{\mathrm{Bar}} + d_{\mathrm{Cobar}}$ preserves the total number of ξ-factors in a (homogeneous in ξ) element of $C(A)$. It makes $C(A)$ a bicomplex. Define

$$\deg_{\mathrm{Bar}}(x_1 \otimes \cdots \otimes x_n) = -n + \sum_i \deg_0 x_i$$

where $\deg_0(a) = \deg_A(a)$ and $\deg_0(\xi) = 0$. Next, define

$$\deg_1(\omega_1 \boxtimes \cdots \boxtimes \omega_k) = k + \sum_i \deg_{\mathrm{Bar}}(\omega_i)$$

$$\deg_{\mathrm{Cobar}}(\omega_1 \boxtimes \cdots \boxtimes \omega_k) = k$$
$$\deg_{\mathrm{Bar}}(\omega_1 \boxtimes \cdots \boxtimes \omega_k) = \sum_i \deg_{\mathrm{Bar}}(\omega_i)$$

where $\omega_i \in \mathrm{Bar}(A_+)$. Finally, define

$$\deg_\xi(\alpha) = -(\sharp(\xi) \text{ in } \alpha),\ \ \alpha \in C(A)$$

$$\deg_{\mathrm{tot}}(\alpha) = \deg_1(\alpha) + \deg_\xi(\alpha)$$

where $\deg_{\mathrm{tot}}$ is the degree of α in $C(A)$.

Thus $C(A)$ becomes a bicomplex, with $C(A)^{a,b}$ defined as the spaces of elements $\alpha \in C(A)$ with $\deg_1 \alpha = a$, $\deg_\xi(\alpha) = b$.

We compute the cohomology of $C(A)$ by using a spectral sequence, which computes the cohomology of $d_{\mathrm{Bar}} + d_{\mathrm{Cobar}}$ at first. The bicomplex lives in the II and III quarters, so the spectral sequence converges. The term $E_1^{a,b}$ of the spectral sequence is equal to $H^a(C(A)^{\bullet,b}, d_{\mathrm{Bar}} + d_{\mathrm{Cobar}})$. Thus, we have to compute the cohomology of the complex $(C(A)^{\bullet,b}, d_{\mathrm{Bar}} + d_{\mathrm{Cobar}})$. Denote this complex by $C_b^\bullet$. The complex $C_b^\bullet$ (for b fixed) is by its own a bicomplex, with differentials d_{Bar} and d_{Cobar}. Thus $C_b^{m,n}$ consist of all elements $\alpha \in C_b^\bullet$ with $\deg_{\mathrm{Bar}}(\alpha) = m$ and $\deg_{\mathrm{Cobar}}(\alpha) = n$ (in this case, $\deg_{\mathrm{tot}}(\alpha) = m+n+b$). The spectral sequence, whose first differential is d_{Cobar}, converges (the other possible spectral sequence, whose first differential is d_{Bar}, generally diverges). We denote by $\mathbb{E}(b)_k^{m,n}$ the k-th term of this spectral sequence. We have

$$\mathbb{E}(b)_1^{m,n} = H^m(\mathbb{E}(b)_1^{\bullet,n}, d_{\mathrm{Cobar}})$$

Lemma 2.6.3. *One has:*

$$\mathbb{E}(b)_1^{m,n} = \begin{cases} 0, & m \neq 0,1 \text{ or } m = 0, n \neq 0 \\ \Bbbk, & b = 0, m = 0, n = 0 \\ \Bbbk[1], & b = -1, m = 1, n = -1 \\ A^n, & b = 0, m = 1 \\ 0, & b \neq 0, -1 \end{cases} \tag{2.6.2}$$

where $(-)^n$ stands for degree n elements.

Proof. The argument is standard and comes from the following observation. Let V be a (graded) vector space, consider the cofree non-unital coalgebra $T^\vee_{\geq 1}(V) = \oplus_{n\geq 1} V^{\otimes n}$. Then $\mathrm{Cobar}(T^\vee_{\geq 1}(V))$ is quasi-isomorphic to $V[-1]$. This statement is proven using Koszul duality. ☐

It follows that the spectral sequence $\mathrm{E}_\bullet$ collapses at the $\mathbb{E}_1$ term. Now turn back to the spectral sequence $\mathrm{E}_\bullet$.

Lemma 2.6.4. *One has*

$$E_1^{a,b} = \begin{cases} \Bbbk[1], & a=0, b=-1 \\ (\Bbbk \oplus A)^a, & b=0 \\ 0, & \textit{otherwise} \end{cases} \tag{2.6.3}$$

The differential d_1 is induced by d_ξ. It looks like

$$\Bbbk[1] \xrightarrow{d_1} \Bbbk \oplus A, \quad d_1\colon 1 \mapsto 1_\Bbbk - 1_A$$

Its cohomology is isomorphic to A. The spectral sequence $E_\bullet$ collapses at the E_2 term. It completes the computation of cohomology of $C(A)$. Now define a map of dg-algebras $p\colon C(A) \to A$ on generators

$$\begin{aligned} &p|_{(\Bbbk[1]\oplus A)^{\otimes n}} = 0, \; n \geq 2 \\ &p(\xi) = 0 \\ &p|_A = \mathrm{id} \\ &p(1_\Bbbk) = 1_A \end{aligned} \tag{2.6.4}$$

and extend it to $C(A)$ as a map of algebras. It follows from the previous computation that p is a quasi-isomorphism. ☐

2.7 A proof of Proposition 2.2.3

Here we prove Proposition 2.3.3.
Recall the maps P_n, $n \geq 1$ defined in (2.3.5):

$$P_n^{ij}(\phi_n, \dots, \phi_1) \coloneqq \sum_{\kappa \in \mathscr{P}aths_{ij}} (-1)^{|\kappa|} p_l(\kappa_l, \dots, \kappa_1)$$

Recall that by Lemma 2.3.2 one has

$$deg(P_n(\phi_1, \dots, \phi_n))_{ij} = |\phi_1| + \dots + |\phi_n| - n + 1 = deg(p_l(\kappa_1, \dots, \kappa_l)), \text{ for any } \kappa \in \mathscr{P}aths_{ij}$$

Proposition 2.3.3 reads:

Proposition 2.7.1. *The maps $\{P_n\}$, $n \geq 1$ are Taylor components of an A_∞-functor*

$$P\colon \mathcal{C}^{\mathrm{pretr}} \oplus \Bbbk_{\mathcal{C}^{\mathrm{pretr}}} \to \mathcal{C}^{\mathrm{pretr}}$$

Proof. Proving the statement amounts to showing the following identities for all $n \geq 1$ (see (1.1.1) for the sign convention):

$$dP_n(\phi_n,\dots,\phi_1)+\sum_{a+b=n}(-1)^{(a-1)(\sum_{i=1}^{b}|\phi_i|)+b-1}P_a(\phi_n,\dots,\phi_{b+1})P_b(\phi_b,\dots,\phi_1)=$$
$$\sum_{k=1}^{n-1}(-1)^{k-1}P_{n-1}(\phi_n,\dots,\phi_{k+1}\circ\phi_k,\dots,\phi_1)+\sum_{k=1}^{n}(-1)^{n-1+\sum_{i=1}^{k-1}|\phi_i|}P_n(\phi_n,\dots,d\phi_k,\dots,\phi_1), \tag{2.7.1}$$

where

$$dP_n(\phi_n,\dots,\phi_1)=d_{naive}P_n(\phi_n,\dots,\phi_1)+q'\circ P_n(\phi_n,\dots,\phi_1)-(-1)^{n-1}P_n(\phi_n,\dots,\phi_1)\circ q \tag{2.7.2}$$

Writing down explicitly the first line of equation 2.7.1 (and dropping the signs to $\pm$ for simplicity), we get

$$\begin{aligned}&dP_n(\phi_n,\dots,\phi_1)=d_{naive}P_n(\dots)+q'\circ P_n(\dots)+(-1)^{n-1}P_n(\dots)\circ q=\\&d_{naive}\Big(\sum_{\kappa\in\mathscr{P}aths}\pm p_m(\kappa_m,\dots,\kappa_1)\Big)+q'\circ\sum_{\kappa\in\mathscr{P}aths}\pm p_{m'}(\kappa_{m'},\dots,\kappa_1)+(-1)^{n-1}\sum_{\kappa\in\mathscr{P}aths}p_{m''}(\kappa_{m''},\dots,\kappa_1)\circ q=\\&\sum_{\kappa\in\mathscr{P}aths}\Big(\sum_{a+b=m}\pm p_a(\dots)\circ p_b(\dots)+\sum_{i=1}^{n-1}\pm p_{m-1}(\dots,m(\kappa_{i+1},\kappa_i),\dots)+\sum_{i=1}^{m}\pm p_n(\dots,d\kappa_i,\dots)\Big)+\\&q'\circ\sum_{\kappa\in\mathscr{P}aths}\pm p_{m'}(\kappa_{m'},\dots,\kappa_1)+(-1)^{n-1}\sum_{\kappa\in\mathscr{P}aths}\pm p_{m''}(\kappa_{m''},\dots,\kappa_1)\circ q\end{aligned} \tag{2.7.3}$$

We stress that inside the terms $\sum_{\kappa\in\mathscr{P}aths}\left(\sum_{i=1}^{n-1}(-1)^s p_{m-1}(\dots,m(\kappa_{i+1},\kappa_i),\dots)\right)$ the composition term might be of following three types:

$$q_{jl}\circ q_{kj},\qquad q\circ\phi \text{ or } \phi\circ q,\qquad \phi_i\circ\phi_{i+1}$$

Similarly inside the terms $\sum_{\kappa\in\mathscr{P}aths}(\sum_{i=1}^{m}(-1)^t p_n(\dots,d\kappa_i,\dots))$ the term which is differentiated might be of the following two types:

$$d\phi,\qquad dq_{kl}$$

We also write down the possible terms of $\sum_{\kappa\in\mathscr{P}aths}\left(\sum_{a+b=m}\pm p_a(\dots)p_b(\dots)\right)$:

$$p_a(\dots)p_b(\dots),\qquad q\circ p_m(\dots) \text{ or } p_l(\dots)\circ q$$

Thence we can write for the r.h.s. of (2.7.3)

$$
\begin{aligned}
&(\text{r.h.s. of } (2.7.3)) = \\
&\sum_{\kappa\in\mathscr{P}aths} \pm\Big(\sum_i \pm p_n(\dots, d\phi, \dots) \pm \sum_i \pm p_n(\dots, dq_{kl}, \dots) \pm \sum_i \pm p_n(\dots, q_{jl}\circ q_{kj}, \dots)\Big)+ \\
&\sum_{\kappa\in\mathscr{P}aths} \pm\Big(\pm\sum_i \pm p_n(\dots, q\circ\phi, \dots) \pm \sum_i \pm p_n(\dots, \phi\circ q, \dots) \pm \sum_i \pm p_n(\dots, \phi_{i+1}\circ\phi_i, \dots)\Big)+ \\
&\sum_{\kappa\in\mathscr{P}aths} \pm\Big(\pm q'\circ p_m(\dots) \pm \pm p_l(\dots)\circ q \pm \sum_a \pm p_a(\dots)\circ p_b(\dots)\Big)+ \\
&q'\circ \sum_{\kappa\in\mathscr{P}aths} \pm p_{m'}(\kappa_1, \dots, \kappa_{m'}) + \sum_{\kappa\in\mathscr{P}aths} \pm p_{m''}(\kappa_1, \dots, \kappa_{m''})\circ q \overset{?}{=} \\
&\sum_{a+b=n} \pm P_a(\dots)\circ P_b(\dots) + \sum_i \pm P_{n-1}(\dots, m(\phi_{i+1}, \phi_i), \dots) + \sum_i \pm P_n(\dots, d\phi, \dots)
\end{aligned}
\tag{2.7.4}
$$

We have to prove the equation marked by question sign. The dashed underlined terms gets cancelled by the Maurer-Cartan condition on the q_{ij}: indeed if there exists $\kappa\in\mathscr{P}aths_{ij}$ which contains q_{kl}, there will also exist a path $\kappa'\in\mathscr{P}aths_{ij}$ containing all the terms $q_{jl}\circ q_{kj}$, since they have got same domain and codomain as q_{kl} and in $\mathscr{P}aths_{ij}$ we were considering all the possible paths.

Therefore we are left with the following expressions:

$$
\begin{aligned}
&\pm\sum_i \sum_{\kappa\in\mathscr{P}aths} p_n(\dots, m(\phi_{i+1}, \phi_i), \dots) \pm \sum_{a+b=n}\sum_{\kappa\in\mathscr{P}aths} \pm p_a(\dots)\circ p_b(\dots)+ \\
&\sum_i \sum_{\kappa\in\mathscr{P}aths} \pm\big(p_n(\dots, d\phi_i, \dots) + p_n(\dots, q'\circ\phi_i, \dots) + p_n(\dots, \phi_i\circ q, \dots))\big) = \\
&\sum_{a+b=n} \pm P_a(\dots)\circ P_b(\dots) + \sum_i \pm P_{n-1}(\dots, m(\phi_{i+1}, \phi_i), \dots) + \sum_i \pm P_n(\dots, d\phi, \dots),
\end{aligned}
$$

which shows the desired equation, up to signs. The correctness of signs (which were explicitly defined in (2.3.5) and (2.7.1)) is checked by a long but routine computation. □

2.8 Cohomology of the dg-operad $\mathcal{O}'$

Here we prove Theorem 2.2.3.

Proof. Let $\omega\in\mathcal{O}'$. Then ω is a linear combination of labelled "trees", where each vertex (excluding the leaves) is labelled either by $p_{n;n_1,\dots,n_k}$ or by m. We say that $p_{n;n_1,\dots,n_k}$ has $n-k$ *operadic arguments* (the remaining k arguments are 1's). We use notation $\sharp(p_{n;n_1,\dots,n_k}) = n-k$. Given a tree T in which a vertex v is labelled by $p_{n;n_1,\dots,n_k}$, we write $\sharp(v) = n-k$. We extend $\sharp(-)$ to all vertices of T, by setting $\sharp(v)=0$ if v is labelled by m. Denote by V_T the set of all vertices of T excluding the leaves.

For a given tree T, denote

$$\sharp(T) = \sum_{v \in V_T} \sharp(v)$$

We also denote by $\sharp_p(T)$ the total number of vertices with $p_{\dots}$, *excluding* $p_1(1), p_2(1,1), \dots$.

Define a descending filtration $F_\bullet$ on $\mathcal{O}'$, as follows. Its $(-\ell)$-th term $F_{-\ell}$ is formed by linear combinations of labelled trees T for which

$$\sharp(T) - \sharp_p(T) \le \ell$$

Note that for any tree T one has $\sharp(T) - \sharp_p(T) \ge 0$.

One has:

$$\dots \supset F_{-3} \supset F_{-2} \supset F_{-1} \supset F_0 \supset 0$$

Note that $dF_{-\ell} \subset F_{-\ell}$, and any component of the differential on $\mathcal{O}'$ either preserves $\sharp(T) - \sharp_p(T)$ or decreases it by 1.

We get a similar filtration $F_\bullet$ on the component $\mathcal{O}'(N)$ of the arity N operations.

We compute cohomology of $\mathcal{O}'(N)$ using the spectral sequence associated with filtration $F_\bullet$ on $\mathcal{O}'(N)$. The spectral sequence lives in the quadrant $\{x \le 0, y \le 0\}$, the differential d_0 is horizontal. One easily sees that the spectral sequence converges. In fact, we show the spectral sequence collapses at the term E_1.

Lemma 2.8.1. *Consider the filtration $F_\bullet$ on $\mathcal{O}'(N)$. One has:*

$$E_1^{-\ell,m} = \begin{cases} \mathcal{A}ssoc_+(N) & \ell = 0, m = 0 \\ 0 & \textit{otherwise} \end{cases}$$

In particular, the spectral sequence collapses at the term E_1.

Proof. We write $p_{n;n_1,\dots,n_k}$ as $p_n(f_1, f_2, \dots, 1, \dots, f_{n-k})$ where $f_1, \dots, f_{n-k}$ are operadic arguments, and 1s stand on the places $n_1, n_2, \dots, n_k$. In these notations, describe the differential in $E_0^{-\ell,\bullet} = F_{-\ell}/F_{-\ell+1}$.

It has components of the following three types, which we refer to as Type I, Type II and Type III components.

Type I components: a component of Type I acts on a group of consecutive 1s, surrounded by operadic arguments from both sides, such as

$$p_n(\dots, f_s, \underbrace{1, 1, \dots, 1}_{\text{a group of } i \text{ consecutive 1s}}, f_{s+1}, \dots).$$

For such a group, the component of d_0 is a sum of expressions, each summand of which corresponds to either a product $1 \cdot 1$ of two consecutive 1s, or to extreme products $f_s \cdot 1$ or $1 \cdot f_{s+1}$, taken with alternated signs. It is clear that totally the component d_0^S corresponding to such a group S is equal to

$$d_0^S(p_n(\dots, f_s, \underbrace{1, \dots, 1}_{i \text{ of 1s in the group } S}, f_{s+1}, \dots)) = \begin{cases} \pm p_n(\dots, f_s, \underbrace{1, 1, \dots, 1}_{i-1 \text{ of 1s}}, f_{s+1}, \dots) & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

Type II components: a component of Type II acts on the groups of leftmost (respectively, rightmost) 1s, such as $p_n(1, 1, \dots, 1, f_1, \dots)$ or $p_n(\dots, f_{n-k}, 1, 1, \dots, 1)$, surrounded by an operadic argument from one side. There should be ≥ 1 of 1s in the group for a non-zero result, and by assumption $p_n(\dots)$ contains at least one operadic argument.

The corresponding component d_0^S of the differential is a sum of two sub-components: $d_0^S = d_0^{S,1} + d_0^{S,2}$.

The first sub-component $d_0^{S,1} = d_0^{S,1,-} \pm d_0^{S,1,+}$, where

$$d_0^{S,1,-}(p_n(\underbrace{1, \dots, 1}_{i \text{ of } 1\text{s}}, f_1, \dots)) =$$
$$p_n(1 \cdot 1, 1, \dots, 1, f_1, \dots) - p_n(1, 1 \cdot 1, \dots, f_1, \dots) + \dots + (-1)^{i-1} p_n(1, \dots, 1, 1 \cdot f_1, \dots)$$

and similarly for $d_0^{S,1,+}$ for the group of rightmost 1s.

One has

$$d_0^{S,1,-}(p_n(\underbrace{1, \dots, 1}_{i \text{ of } 1\text{s}}, f_1, \dots)) = \begin{cases} p_n(\underbrace{1, \dots, 1}_{i-1 \text{ of } 1\text{s}}, f_1, \dots) & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

and similarly for $d_0^{S,1,+}$.

The second sub-component $d_0^{S,2} = d_0^{S,2,-} \pm d_0^{S,2,+}$, where

$$d_0^{S,2,-}(p_n(\underbrace{1, \dots, 1}_{i \text{ of } 1\text{s}}, f_1, \dots)) =$$
$$p_1(1) \cdot p_{n-1}(1, \underset{i-1}{\dots}, 1, f_1, \dots) - p_2(1, 1) \cdot p_{n-2}(1, \underset{i-2}{\dots}, 1, f_1, \dots) + \dots +$$
$$(-1)^{i-1} p_i(1, 1, \dots, 1) \cdot p_{n-i}(f_1, \dots)$$

and similarly for $d_0^{S,2,+}$ for the rightmost group of 1s.

One checks that all other components of the differential d on $\mathcal{O}'$ decrease $\sharp(T) - \sharp_p(T)$ by 1.

Type III components: Here we have d_0 acting on $p_n(\underset{n \text{ of } 1\text{s}}{1, 1, \dots, 1})$.

One has:

$$\begin{aligned} &d_0(p_n(1, 1, \dots, 1)) = \\ &p_{n-1}(1 \cdot 1, 1, \dots, 1) - p_{n-1}(1, 1 \cdot 1, 1, \dots, 1) + \dots + (-1)^{i-1} p_{n-1}(1, 1, \dots, 1 \cdot 1) + \\ &\pm \sum_{1 \leq i \leq n-1} (-1)^{i-1} p_i(1, 1, \dots, 1) \cdot p_{n-i}(1, 1, \dots, 1) + \end{aligned} \tag{2.8.1}$$

Denote the first summand by $d_0^{S,1}$ and the second summand by $d_0^{S,2}$ One sees that

$$d_0^{S,1}(p_n(1, 1, \dots, 1)) = \begin{cases} p_{n-1}(1, 1, \dots, 1) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

The computation of cohomology of the complex $(E_0^{-\ell,\bullet}, d_0)$ is reduced to the computation of the cohomology of a tensor product of complexes (the factors are labelled by combinatorial data of the labelled tree T), corresponding to different components S as listed above:

$$E_0^{-\ell,\bullet} = \bigotimes_{S,T} K_S^\bullet \tag{2.8.2}$$

The complexes K_S corresponding to Type I components are isomorphic to

$$K^\bullet = \{\dots \xrightarrow{0} \underset{i=4}{\Bbbk} \xrightarrow{\mathrm{id}} \underset{i=3}{\Bbbk} \xrightarrow{0} \underset{i=2}{\Bbbk} \xrightarrow{\mathrm{id}} \underset{\substack{i=1\\ \deg=-1}}{\Bbbk} \to 0\} \tag{2.8.3}$$

The complex $K^\bullet$ is acyclic in all degrees. It implies that the complex $(E_0^{-\ell,\bullet}, d_0)$ is quasi-isomorphic to its sub-complex which is formed by the trees in which any p is of the type $p_n(1, 1, \dots, 1, f_1, \dots, f_{n-k}, 1, \dots, 1)$, where all $n-k$ operadic arguments stand successively, without 1s between them.

It remains to treat the Type II and Type III cases.

The complexes whose cohomology we need to compute are of two types. They are formed either by linear combinations of

$$p_{n_1}(1, 1, \dots, 1) \cdot p_{n_2}(1, 1, \dots, 1) \dots p_{n_k}(1, 1, \dots, 1) \cdot p_n(1, 1, \dots, 1, f_1, \dots)$$

or by all linear combinations of

$$p_{n_1}(1, 1, \dots, 1) \cdot p_{n_2}(1, 1, \dots, 1) \dots p_{n_k}(1, 1, \dots, 1).$$

Denote them by $K_1^\bullet$ and $K_2^\bullet$.

Their cohomology are computed similarly, we consider the case of $K_2^\bullet$, leaving the case of $K_1^\bullet$ to the reader.

Denote $p_\ell = p_\ell(1, 1, \dots, 1)$ and by P_ℓ the 1-dimensional vector space $\Bbbk p_\ell(1, 1, \dots, 1) = \Bbbk p_\ell$, $\ell \geq 1$.

One has:

$$K_2^{-n} = \bigoplus_{k\geq 1,\ n_1+\dots+n_k-k=n} P_{n_1} \otimes P_{n_2} \otimes \dots \otimes P_{n_k}$$

We denote the differential d_0 on $K_2^\bullet$, see (2.8.1), by d.

Lemma 2.8.2. *The complex $(K_2^\bullet, d)$ is quasi-isomorphic to $P_1[0]$.*

Proof. Consider on $K_2^\bullet$ the following descending filtration $\Phi_\bullet$, where

$$\Phi_{-\ell} = \bigoplus_{n_1+n_2+\dots+n_k\leq \ell} P_{n_1} \otimes P_{n_2} \otimes \dots \otimes P_{n_k}$$

One has

$$\dots \supset \Phi_{-3} \supset \Phi_{-2} \supset \Phi_{-1} \supset \Phi_0 = 0$$

$$d\Phi_{-\ell} \subset \Phi_{-\ell}$$

Denote by $d_{0,\Phi}$ the differential in $E_{0,\Phi}^{-\ell,\bullet} = \Phi_{-\ell}/\Phi_{-\ell+1}$. It is given by

$$d_{0,\Phi}(p_{n_1}\otimes p_{n_2}\otimes\cdots\otimes p_{n_k}) = \sum_{i=1}^{k}(-1)^{n_1+\cdots+n_{i-1}-i+1}p_{n_1}\otimes\cdots\otimes d_{0,\Phi}(p_{n_i})\otimes\cdots\otimes p_{n_k} \quad (2.8.4)$$

where

$$d_0(p_n) = \sum_{1\le i\le n-1}(-1)^{i-1}p_i\otimes p_{n-i} \quad (2.8.5)$$

It is well-known that the complex $E_{0,\Phi}^{-\ell,\bullet}$ is acyclic when $\ell\ge 2$, and is quasi-isomorphic to $P_1[0]$ when $\ell=1$. We can identify $P_n\simeq(\Bbbk[1])^{\otimes n}$, then $\oplus_{n\ge1}\Bbbk[1]^{\otimes n}=P$ becomes the (non-unital) cofree coalgebra cogenerated by $\Bbbk[1]$. The complex (2.8.4), (2.8.5) is identified with the Cobar-complex of the cofree coalgebra P. It is standard that its cohomology is equal to $\Bbbk[1][-1]\simeq\Bbbk$. Therefore, the spectral sequence collapses at the term E_1 by dimensional reasons.

It completes the proof of Lemma 2.8.2. □

Similarly we prove that $K_1^\bullet$ is acyclic in all degrees. In this way we see that any cohomology class in $E_0^{-\ell,\bullet}$ can be represented by a linear combination of trees which do not contain p_ns with $n\ge 2$.

It follows that any cohomology class can be represented by a linear combination of trees containing only m and $p(1)$, and all such trees have cohomological degree 0. It completes the proof. □

Theorem 2.2.3 follows from Lemma 2.8.1. □

Chapter 3

Deformation theory of $\Bbbk$-linear monoidal categories

Les soleils couchants
Revêtent les champs,
Les canaux, la ville entière,
D'hyacinthe et d'or;
Le monde s'endort
Dans une chaude lumière.
Là, tout n'est qu'ordre et beauté,
Luxe, calme et volupté.

C. Baudelaire,
L'invitation au voyage

3.1 Deformation theory

Deformation theory of associative algebras was initiated by the pioneering works of Gerstenhaber.

Let $\Bbbk$ be a field of characteristic 0, $\Bbbk[\![t]\!]$ the unital ring of formal power series with coefficients in $\Bbbk$. Denote by $\epsilon\colon \Bbbk[\![t]\!] \to \Bbbk$ the augmentation map, defined as

$$\epsilon\left(\sum_{i\in\mathbb{N}} a_i t^i\right) := a_0.$$

Clearly $\mathrm{Ker}(\epsilon) = t\Bbbk[\![t]\!]$ is the augmentation ideal of $\Bbbk[\![t]\!]$.

Given a $\Bbbk[\![t]\!]$-algebra B, we can consider its **reduction** $\bar{B} := \Bbbk \otimes_{\Bbbk[\![t]\!]} B$, where $\Bbbk$ is acting in the obvious way.

Given a $\Bbbk[t]$-algebra (respectively, $\Bbbk[t]/(t^n)$-algebra) B, we can analogously consider its reduction $\bar{B} := \Bbbk \otimes_{\Bbbk[t]} B$ (respectively, $\bar{B} := \Bbbk \otimes_{\Bbbk[t]/(t^n)} B$), recalling that the augmentation maps of $\Bbbk[t]$ and of $\Bbbk[t]/(t^n))$ are defined as: $\epsilon(f) = f(0) \in \Bbbk$.

Definition 3.1.1. Let A be an associative $\Bbbk$-algebra. A **formal deformation** of A is an associative $\Bbbk[\![t]\!]$-algebra B together with a $\Bbbk$-algebra isomorphism $\alpha : \bar{B} \to A$.

It can be shown that:

Theorem 3.1.2. *A formal deformation of an algebra A is given by a family*

$$\{m_i \colon A \otimes A \to A | i \geq 0\}$$

such that $m_0(a,b) = ab$ (the multiplication in A) and

$$(D_k) \qquad \sum_{\substack{i+j=k \\ i,j\geq 0}} m_i(m_j(a,b),c) = \sum_{\substack{i+j=k \\ i,j\geq 0}} m_i(a, m_j(b,c))$$

for all $a, b, c \in A$ for each $k \geq 1$.

We will also need the following definitions:

Definition 3.1.3. An **infinitesimal deformation** of an algebra A is an associative $\Bbbk[t]/(t^2)$-algebra B together with a $\Bbbk$-algebra isomorphism $\alpha \colon \bar{B} \to A$.

and analogously:

Definition 3.1.4. An **n-deformation** of an algebra A is an associative $\Bbbk[t]/(t^n)$-algebra B together with a $\Bbbk$-algebra isomorphism $\alpha \colon \bar{B} \to A$.

for which the theorem above holds in a refined way:

Theorem 3.1.5. *An n-deformation of A is given by a family*

$$\{m_i \colon A \otimes A \to A | 1 \leq i \leq n\}$$

of $\Bbbk$-linear maps satisfying (D_k) of Theorem 3.1.2 for $1 \leq k \leq n$.

We conclude this introduction with a definition we will need afterwards:

Definition 3.1.6. An $(n+1)$-deformation of A given by $\{m_1, \dots, m_{n+1}\}$ is called an **extension** of the n-deformation given by $\{m_1, \dots, m_n\}$.

3.1.1 Hochschild cohomology

The main tool in studying deformation theory of an associative algebra A is the Hochschild cochain complex

$$0 \to \mathrm{CH}^0(A,A) \to \cdots \to \mathrm{CH}^n(A,A) \to \mathrm{CH}^{n+1}(A,A) \to \dots,$$

where $\mathrm{CH}^n(A,A) \coloneqq \mathrm{Hom}_{\Bbbk}(A^{\otimes n}, A)$ is the space of Hochschild n-cochains, i.e., the n-times $\Bbbk$-linear maps $f(-,\dots,-) \colon A^{\otimes n} \to A$. We can define the differential $d_n \colon \mathrm{CH}^n(A,A) \to \mathrm{CH}^{n+1}(A,A)$ (of degree $+1$) as:

$$\begin{aligned}(d_n f)(a_1,\dots,a_{n+1}) \coloneqq\ & a_1 f(a_2,\dots,a_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(a_1,\dots,a_{i-1}, a_i a_{i+1}, a_{i+2},\dots,a_{n+1}) \\ & -(-1)^n f(a_1,\dots,a_n) a_{n+1},\end{aligned} \tag{3.1.1}$$

for $f \in \mathrm{CH}^n(A,A)$, $a_1,\dots,a_{n+1} \in A$. The Hochschild cohomology is defined as the cohomology of this cochain complex:

$$\mathrm{HH}^n(A,A) \coloneqq Z^n(A,A)/B^n(A,A),$$

where $Z^n(A,A)$ is the space of Hochschild n-cocycles and $B^n(A,A)$ is the space of Hochschild n-coboundaries.
The following spaces are worth to mention:

$Z^0(A,A) = \mathrm{HH}^0(A,A)$ is the center $Z(A)$ of the algebra A, and thus $(\mathrm{HH}^0(A,A),\cdot)$, is a commutative algebra, where $\cdot$ is the product of A.

$Z^1(A,A)$ is the space $Der(A)$ of derivations of A, i.e. of $\Bbbk$-linear functions $f\colon A \to A$ satisfying the Leibniz rule

$$f(ab) = f(a)b + af(b)$$

for all $a,b \in A$ (this is equivalent to $d_1 f = 0$). It is well known that $(Der(A),[-,-])$ is a Lie algebra, where $[-,-]$ is the commutator:

$$[f,g] \coloneqq f \circ g - g \circ f,$$

which satisfies the Jacobi identity and is anti-symmetric. Thence it follows naturally that $(Z^1(A,A),[-,-])$ is a Lie algebra.

It is well-known that we can lift the commutative algebra structure of $\mathrm{HH}^0(A,A)$ to an associative algebra structure on $\mathrm{HH}^*(A,A) \coloneqq \bigoplus_{n\geq 0} \mathrm{HH}^n(A,A)$, given by the cup product:

$$-\cup -\colon \mathrm{HH}^n(A,A) \times \mathrm{HH}^m(A,A) \to \mathrm{HH}^{n+m}(A,A), \tag{3.1.2}$$

which is defined as:

$$f \cup g\ (a_1,\dots,a_{n+m}) \coloneqq f(a_1,\dots,a_n)\cdot g(a_{n+1},\dots,a_{n+m})$$

for all $a_i \in A$. It is clear that $\cup$ restricts to $\cdot$ on $\mathrm{HH}^0(A,A) = Z^0(A,A)$ and that the associativity of $\cup$ follows from the associativity of $(A,\cdot)$. However, Gerstenhaber showed in [Ge1] and [Ge2] that $(\mathrm{HH}^*(A,A), -\cup-)$ is also a graded-commutative algebra, i.e. for all $f \in \mathrm{HH}^n(A,A), g \in \mathrm{HH}^m(A,A)$:

$$f \cup g = (-1)^{nm} g \cup f$$

In loc. cit. the author managed to lift the Lie algebra on $Z^1(A,A)$ to the space $Z^\bullet(A,A) \coloneqq \bigoplus_{n\geq 0} Z^n(A,A)$, constructing it as the commutator of a brace operation: he first defined the operations

$$\circ_i\colon Z^n(A,A) \times Z^m(A,A) \to Z^{n+m-1}(A,A), \tag{3.1.3}$$

with $1 \leq i \leq n$, as:

$$f \circ_i g\ (a_1,\dots,a_{n+m-1}) \coloneqq f(a_1,\dots,a_{i-1}, g(a_i,\dots,a_{i+m-1}), a_{i+m},\dots,a_{n+m-1})$$

These operations $\circ_i$ led to the definition of the brace operation:

$$-\{-\}\colon Z^n(A,A) \times Z^m(A,A) \to Z^{n+m-1}(A,A), \tag{3.1.4}$$

defined on the objects as:

$$f\{g\}(a_1,\ldots,a_{n+m-1}) \coloneqq \sum_{i=1}^{n}(-1)^{\epsilon} f\circ_i g(a_1,\ldots,a_{n+m-1}).$$

It is easy to show that the graded-commutativity of $-\cup-$ comes from the following equation:

$$f\cup g-(-1)^{nm}g\cup f=[d,f\{g\}],$$

The Gerstenhaber bracket is defined as:

$$[-,-]\colon Z^n(A,A)\times Z^m(A,A)\to Z^{n+m-1}(A,A), \tag{3.1.5}$$

simply by taking the commutator:

$$[f,g]\coloneqq f\{g\}-(-1)^{\epsilon}g\{f\}$$

It is worth noticing that if we restrict ourselves to $Z^1(A,A)$, we have:

$$[f,g](a)=f(g(a))-g(f(a)),$$

which is precisely the bracket of $Der(A)$.

Moreover, this operation $[-,-]$ satisfies the graded Jacobi identity, and the graded alternating property, thus $(Z^{\bullet}(A,A),[-,-])$ is a graded Lie algebra, (with Lie bracket of degree -1).

On top of that, Gerstenhaber showed that the graded Lie algebra structure descends to the cohomology $\mathrm{HH}^*(A,A)$ and that it satisfies the Poisson identity, i.e. for all $f\in\mathrm{HH}^n(A,A), g\in\mathrm{HH}^m(A,A), h\in\mathrm{HH}^p(A,A)$:

$$[f,g\cup h]=[f,g]\cup h+(-1)^{m(p-1)}g\cup[f,h]$$

Algebraic structures like this one on the Hochschild cohomology $\mathrm{HH}^*(A,A)$ were given the name of Gerstenhaber algebras (or Poisson 2-algebras).

As we recalled in Section 1.3, Fred Cohen proved in [Co] that the operad e_2 of Poisson 2-algebras is the homology operad of the little discs operad E_2, when $char(\Bbbk)=0$: $\mathrm{e}_2=H_{\bullet}(\mathrm{E}_2,\Bbbk)$. Thus the situation looks as follows: the cohomology operad of the little discs operad acts on the cohomology of the Hochschild complex. This motivated Deligne to claim that the chain operad of little discs $\mathrm{Ch}_{\bullet}(\mathrm{E}_2,\Bbbk)$ acts on the Hochschild complex $\mathrm{CH}^*(A,A)$, for any associative algebra A.

Going back to deformation theory of associative algebras, Gerstenhaber proved in [Ge1] that:

Theorem 3.1.7. *There is a one-to-one correspondence between the space of equivalence classes of infinitesimal deformations of A and the second Hochschild cohomology $\mathrm{HH}^2(A,A)$.*

from which it follows:

Corollary 3.1.8. *Let A be an associative algebra such that $\mathrm{HH}^2(A,A)=0$. Then all formal deformations of A are equivalent to the trivial deformation $A[\![t]\!]$.*

Moreover, he noticed that the equation (D_{n+1}) can be rearranged as:

$$-am_{n+1}(b,c) + m_{n+1}(ab,c) - m_{n+1}(a,bc) + m_{n+1}(a,b)c = \\ = \sum_{\substack{i+j=n+1\\ i,j\geq 0}} (m_i(m_j(a,b),c) - m_i(a,m_j(b,c))) \quad (3.1.6)$$

Let us denote the r.h.s. of this equation by $\mathcal{D}_n$. This is an element of $\mathrm{CH}^3(A,A)$, and actually the equation (D_{n+1}) can be succinctly written as:

$$d_2(m_{n+1}) = \mathcal{D}_n$$

Now we can show:

Theorem 3.1.9. *For any n-deformation of an associative algebra A, the Hochschild cochain $\mathcal{D}_n \in \mathrm{CH}^3(A,A)$ defined in 3.1.6 is a cocycle. Moreover, $[\mathcal{D}_n] = 0$ in $\mathrm{HH}^3(A,A)$ if and only if the n-deformation $\{m_1,\dots,m_n\}$ extends into some $(n+1)$-deformation.*

From this theorem it follows:

Corollary 3.1.10. *If $\mathrm{HH}^3(A,A) = 0$, then all obstructions vanish and every infinitesimal deformation m_1 in $Z^2(A,A)$ is integrable to a formal deformation.*

3.1.2 Davydov-Yetter cohomology

Motivated by problems in quantum algebra and quantum field theory, Davydov [Da] and independently Crane and Yetter [CY], [Ye1], [Ye2] introduced and studied a complex ${\mathrm{C_{DY}}}^*(\mathrm{Id}_\mathcal{C})$, now known as the Davydov-Yetter complex, whose cohomology governs the deformations of the monoidal structure of a $\Bbbk$-linear monoidal functor or the associator of a $\Bbbk$-linear monoidal category, without changing the underlying categories and functors, but extending the scalars from $\Bbbk$ to $\Bbbk[\![t]\!]$ as above. This deformation theory is the first step to the classification problem of monoidal structures.

Let us describe this complex ${\mathrm{C_{DY}}}^*(F)$. Let $\mathcal{C}$, $\mathcal{D}$ be $\Bbbk$-linear monoidal categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a tensor functor, i.e. a $\Bbbk$-linear monoidal functor. Define its n-th tensor power by

$$F^{\otimes n}\colon \mathcal{C}\otimes\cdots\otimes\mathcal{C} \to D, \quad F^{\otimes n}(X_1,\dots,X_n) = F(X_1\otimes(X_2\otimes(\dots(X_{n-1}\otimes X_n)\dots))$$

Set ${\mathrm{C_{DY}}}^n(F) \coloneqq End(F^{\otimes n})$ the endomorphism algebra of natural transformations from $F^{\otimes n}$ to itself, and $C^0_{DY}(F) \coloneqq End(I)$, where I is the unit object of the tensor category $\mathcal{D}$. We can also describe ${\mathrm{C_{DY}}}^n(F)$ as a sub-algebra of the following algebra:

$$\prod_{X_i\in Ob(\mathcal{C})} \mathcal{D}(F(X_1\otimes_\mathcal{D}(\cdots\otimes_\mathcal{D} X_n), F(X_1\otimes_\mathcal{D}(\cdots\otimes_\mathcal{D} X_n)).$$

It is easy to check that ${\mathrm{C_{DY}}}^n(F)$ are monoids with respect to composition, and in particular in [BD] the authors endow this collection $C^\bullet_{DY}(F)$ with the structure of a cosimplicial monoid, with coface maps $\partial^i_n\colon {\mathrm{C_{DY}}}^n(F) \to C^{n+1}_{DY}(F)$ defined on an endomorphism a as:

$$\partial^i_n(a)_{X_1,\dots,X_{n+1}} \coloneqq \begin{cases} \mathrm{id}_{FX_1}\otimes a_{X_2,\dots,X_{n+1}} & \text{if } i=0;\\ a_{X_1,\dots,X_i\otimes X_{i+1},\dots,X_{n+1}} & \text{if } 1\leq i\leq n;\\ a_{X_1,\dots,X_n}\otimes\mathrm{id}_{FX_{n+1}} & \text{if } i=n+1.\end{cases} \quad (3.1.7)$$

where we omit the clear associativity and tensor constraints.

The codegeneracy maps $\sigma_n^i\colon {\mathrm{C_{DY}}}^n(F) \to C_{DY}^{n-1}(F)$ are defined on an endomorphism a as:

$$\sigma_n^i(a)_{X_1,\ldots,X_{n-1}} = a_{X_1,\ldots,X_i,I,X_{i+1},\ldots,X_{n-1}}, \text{ with } 0 \le i \le n-1. \tag{3.1.8}$$

Definition 3.1.11. The total cochain complex $({\mathrm{C_{DY}}}^*(F), \partial) = \mathrm{Tot}({\mathrm{C_{DY}}}^\bullet(F))$ is called the Davydov-Yetter complex of the tensor functor F. Its cohomology ${\mathrm{H_{DY}}}^*(F)$ is the Davydov-Yetter cohomology of the tensor functor F.

By the Yoneda lemma it follows that ${\mathrm{C_{DY}}}^*(F)$ is a cochain complex with differential $d_n\colon {\mathrm{C_{DY}}}^n(F) \to C_{DY}^{n+1}(F)$ induced by the sum (with alternated signs) of the coface maps:

$$d_n(a) \coloneqq \sum_{i=0}^{n+1} (-1)^i \partial_n^i(a)$$

How does this relate to the deformations of the monoidal structure of F?

By definition of F being a tensor functor, there exists a natural isomorphism $J_{X,Y}\colon F(X\otimes Y) \xrightarrow{\sim} F(X)\otimes F(Y)$ such that the diagram:

$$\begin{array}{ccc} F(A\otimes B\otimes C) & \xrightarrow{J_{A\otimes B,C}} & F(A\otimes B)\otimes F(C) \\ \downarrow{\scriptstyle J_{A,B\otimes C}} & & \downarrow{\scriptstyle J_{A,B}\otimes \mathrm{id}_{F(C)}} \\ F(A)\otimes F(B\otimes C) & \xrightarrow[\mathrm{id}_{F(A)}\otimes J_{B,C}]{} & F(A)\otimes F(B)\otimes F(C) \end{array} \tag{3.1.9}$$

is commutative (this is the version of diagram (1.6.3), with $\alpha^{\mathcal{C}} = \alpha^{\mathcal{D}} = \mathrm{id}$). The functorial isomorphism J is called the **monoidal structure** of F.

The quest in Davydov–Yetter theory are formal deformations (in the same sense as above) of J, i.e. expansions over $\Bbbk[\![t]\!]$ of the form $J_t = J + \sum_{n\ge 1} J^n t^n$, where the J^n's are natural transformations $(J^n)_{X,Y}\colon F(X\otimes Y) \to F(X)\otimes F(Y)$, such that the diagram (3.1.9) remains commutative with J_t instead of J. Without loss of generality, we can assume that F is strict: $J = \mathrm{id}$ [[JS], Theorem 1.7].
In order to see the link between the Davydov-Yetter complex ${\mathrm{C_{DY}}}^*(F)$ and this deformation problem, let us consider infinitesimal deformations $J_t = \mathrm{id} + Jt$, with $t^2 = 0$. By definition, $J \in End(F^{\otimes 2})$ and the condition (3.1.9) on J_t implies that $J\colon F\otimes F \to F\otimes F$ satisfies

$$\mathrm{id}_{F(X_1)} \otimes J_{X_2,X_3} - J_{X_1\otimes X_2,X_3} + J_{X_1,X_2\otimes X_3} - J_{X_1,X_2}\otimes \mathrm{id}_{F(X_3)} = 0,$$

which is equivalent to ask that $d_2(J) = 0$, i.e. J is a 2-cocycle of the Davydov-Yetter complex.

In analogy with Gerstenhaber's results for deformation of associative algebras, Yetter proved the following results:

Theorem 3.1.12. *There is a natural 1-to-1 correspondence between the infinitesimal proper deformations of a monoidal functor F and the 2-cocycles of the proper deformation complex of F. Moreover, the monoidal natural isomorphism classes of infinitesimal proper deformations of F are in natural 1-to-1 correspondence with $H^2_{DY}(F)$.*

and

Proposition 3.1.13. *For any proper n-deformation of a monoidal functor F, the obstruction to extension to an $n+1$-deformation is a 3-cocycle.*

By this last result it follows:

Corollary 3.1.14. *If $H^3_{\mathrm{DY}}(F) = 0$, then any infinitesimal proper deformation can be extended to a formal deformation.*

The case of the identity functor $F = \mathrm{Id}_C$ deserves a special attention, since $\mathrm{H_{DY}}^3(\mathrm{Id}_{\mathcal{C}})$ classifies the infinitesimal deformations of the (trivial) associator of $\mathcal{C}$. Such a deformation is an expansion $a_t = \mathrm{id} + \alpha t$ over $k[t]/(t^2)$ which satisfies the pentagon equation, where $\alpha \in End(\mathrm{Id}^{\otimes 3})$. The obstructions are contained in $\mathrm{H_{DY}}^4(\mathrm{Id}_{\mathcal{C}})$, at least for the extension of an infinitesimal deformation to the order 2 (this was shown in [[BD], Prop. 3.21]).

In order to show that the Davydov-Yetter complex $\mathrm{C_{DY}}^*(F)$ is a $\mathrm{Ch}_\bullet(\mathrm{E}_2, \mathbb{k})$-algebra, Batanin and Davydov in [BD] constructed a family of operads $\mathcal{M}^{(0)} \to \cdots \to \mathcal{M}^{(n)} \to \cdots \to \mathcal{M}$ (see definitions and properties in Sect. 1.3.3), and proved:

Theorem 3.1.15. *The cosimplicial monoid $\mathrm{C_{DY}}^*(F)$ is a $\mathcal{M}^{(1)}$-algebra.*
The cosimplicial monoid $\mathrm{C_{DY}}^*(\mathrm{Id}_{\mathcal{C}})$ is a $\mathcal{M}^{(2)}$-algebra.

Proof. As for the first statement, the authors show that it is enough to prove that the images $\mathrm{C_{DY}}(\tau_{n,m})(a)$, $\mathrm{C_{DY}}(\pi_{n,m})(b)$ commute for any $a \in \mathrm{C_{DY}}^n(F)$ and $b \in C^m_{DY}(F)$, where $\tau_{m,n}\colon [n] \to [m+n]$ and $\pi_{m,n}\colon [m] \to [m+n]$ are defined as $\tau_{m,n}(i) = i$ and $\pi_{m,n}(j) = n + j$. Since

$$\mathrm{C_{DY}}(\tau_{n,m})(a) = \partial^{n+m}_{n+m-1} \dots \partial^{n+2}_{n+1} \partial^{n+1}_{n}(a) = a \otimes 1_m$$

and

$$\mathrm{C_{DY}}(\pi_{n,m})(b) = \partial^{n-1}_{m+n-1} \dots \partial^{1}_{m+1} \partial^{0}_{m}(b) = 1_n \otimes b,$$

it follows that $(1_n \otimes b) * (a \otimes 1_m) = (a \otimes 1_m) * (1_n \otimes b)$.

As for the second statement, the authors show that it is enough to prove that the images $\mathrm{C_{DY}}(\tau)(a)$, $\mathrm{C_{DY}}(\pi)(b)$ commute for any maps $\tau = \tau^i_{m,n}$ and $\pi = \pi^i_{m,n}$ and any $a \in \mathrm{C_{DY}}^n(C)$ and $b \in C^m_{DY}(C)$, where $\tau^i_{m,n}\colon [n] \to [n+m-1]$ and $\pi^i_{m,n}\colon [m] \to [n+m-1]$ are defined as:

$$\tau^i_{m,n}(\ell) \coloneqq \begin{cases} \ell & \text{if } \ell \leq i \\ \ell + m - 1 & \text{if } \ell > i, \end{cases} \qquad \pi^i_{m,n}(j) \coloneqq i + j. \tag{3.1.10}$$

Since

$$\mathrm{C_{DY}}(\tau^i_{m,n})(a) = \partial^{i+m-1}_{n+m-2} \dots \partial^{i+2}_{n+1} \partial^{i+1}_{n}(a)$$

and

$$\mathrm{C_{DY}}(\pi^i_{n,m})(b) = \partial^{n+m-1}_{n+m-2} \dots \partial^{i+m+2}_{n+i-1} \partial^{i+m+1}_{n+i} \partial^{i-1}_{n+i-1} \dots \partial^{1}_{n+1} \partial^{0}_{n}(b) = 1_i \otimes b \otimes 1_{n-i-2}$$

By ***naturality*** of a, the evaluation

$$\mathrm{C_{DY}}(\tau^i_{m,n})(a)_{X_1,\dots,X_{m+n-1}} = a_{X_1,\dots,X_i, X_{i+1} \otimes \dots \otimes X_{i+m}, X_{i+m+1},\dots,X_{m+n-1}}$$

commutes with the evaluation

$$\mathrm{C_{DY}}(\pi^i_{n,m})(b)_{X_1,\dots,X_{m+n-1}} = 1_{X_1} \otimes \cdots \otimes 1_{X_i} \otimes b_{X_{i+1},\dots,X_{i+m}} \otimes 1_{X_{i+m+1}} \otimes \cdots \otimes 1_{X_{m+n-1}}$$

□

Remark 3.1.16. We underlined and italicized the word naturality in the proof, as this is the key feature missing in our deformation complex of a monoidal category $\mathcal{C}$ (that we define in Section 3.2). The serious drawback to this is that we do not have a 2-commutative cosimplicial monoid, but rather a "homotopy 2-commutative" monoid. This will be better explained in Remark 3.4.8.

In addition to that, they related the lattice paths operad $\mathcal{L}$ to the paths operad $\mathcal{M}$:

Theorem 3.1.17. *There are morphisms of operads* $p^{(n)} : \mathcal{L}^{(n)} \to \mathcal{M}^{(n-1)}$*, compatible with the operad filtrations.*

This result, together with Theorem 1.3.36 and Corollary 1.3.37, implies the following ([BD], Theorem 2.45, Corollary 2.46):

Theorem 3.1.18. *Let* $X_\bullet$ *be an* n*-commutative cosimplicial monoid in* $\mathrm{Ch}(\Bbbk)$*. Then there is an action of an operad homotopy equivalent to* $\mathrm{Ch}_\bullet(E_{n+1}, \Bbbk)$ *on the totalization* $\mathrm{Tot}(X_\bullet) \in \mathrm{Ch}(\Bbbk)$*.*

In [BD], some explicit formulas for the degree -n Lie bracket are provided, see [[BD], Sections 2.9, 2.10].

From the results above it follows ([BD], Corollaries 3.5, 3.9):

Corollary 3.1.19. *The deformation complex* $\mathrm{C_{DY}}^*(F)$ *of a tensor functor* F *is an* $\mathrm{Ch}_\bullet(\mathrm{E}_2, \Bbbk)$*-algebra.*
The deformation complex $\mathrm{C_{DY}}^*(\mathcal{C})$ *of a tensor category* $\mathcal{C}$ *is an* $\mathrm{Ch}_\bullet(\mathrm{E}_3, \Bbbk)$*-algebra.*

Our motivation for introducing a new deformation complex, obtained as the totalization of a functor $A(F,F)\colon \Theta_2 \to \mathrm{Vect}(\Bbbk)$ is twofold.

The first intention was to pack the Hochschild cochains (vertically) and the Davydov-Yetter cochains (horizontally) in a single deformation complex. The explicit definition of the functor $A(F,F)$ will be given in the following Section 3.2, though we can still give the heuristic process. At first we looked for a functor $H\colon \Delta \times \Delta \to \mathrm{Vect}(\Bbbk)$, but soon we realized it did not work as expected. Once we defined the desired functor $A(F,F)$, in the case $F = \mathrm{Id}_{\mathcal{C}}$ we found out that not only we could control the deformations of the composition of morphisms and of the associator of $\mathcal{C}$, but also the monoidal product of morphisms.

The second intention was to relate this problem to the generalized Deligne conjecture, stated by Kontsevich in [[Ko3], Sec. 2.5]. Indeed, following the idea of Tamarkin in [Tam2], one can define our functor $A(F,F)\colon \Theta_2 \to \mathrm{Vect}(\Bbbk)$ for any $\Bbbk$-linear pseudo functor $F\colon \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are $\Bbbk$-linear bicategories (i.e. categories weakly enriched in $\mathbb{C}(\Bbbk)$) rather then monoidal $\Bbbk$-linear categories. However, one can do better: namely, for any choice of $\Bbbk$-linear pseudo functors $F,G\colon \mathcal{C}\to\mathcal{D}$, and any choice of $\Bbbk$-linear pseudo natural transformations $\eta,\theta\colon F\Rightarrow G$, one can construct a functor

$$A(F,G)(\eta,\theta)\colon \Theta_2\to \mathrm{Vect}(\Bbbk).$$

The totalization of this functor is defined as the complex of *derived modifications* between η and θ. Thus we are left with a pre-3-category, where objects are $\Bbbk$-linear bicategories, 1-morphisms are $\Bbbk$-linear pseudo functors, 2-morphisms are $\Bbbk$-linear pseudo natural transformations and the complexes of 3-morphisms are the derived modifications. If one constructed a contractible 3-operad acting over this 3-pre-category, one would get that the deformation complex $\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id}_\mathcal{C},\mathrm{Id}_\mathcal{C}))$ is a $\mathrm{Ch}_\bullet(\mathrm{E}_3,\Bbbk)$-algebra.

3.2 A 2-cocellular vector space $A(F,F)$ associated to a monoidal functor $F\colon \mathcal{C}\to\mathcal{D}$

3.2.1 Elementary face and degeneracy maps of Θ_2

In order to make things more understandable, we will translate the definitions of face and degeneracy maps (given in Subsection 1.4.5) to the wreath product definition of Θ_2. For $T=([k];[n_1],\dots,[n_k])\in\Theta_2$, define dimension of T as

$$|T|=k+\sum_{i=1}^{k} n_i \tag{3.2.1}$$

It was proven in [[Be1], Lemma 2.4(a)] that Θ_2 is a Reedy category, in which the degree is equal to the dimension 3.2.1, and there are two classes of morphisms, face maps and degeneracy maps, which raise (respectively, lower) the degree. Below we list the codim $=1$ face and degeneracy maps in the wreath product model of Θ_2. Recall that an object of Θ_2 is given by a tuple $([k];[n_1],\dots,[n_k])$, a morphism $\phi\colon([n];[\ell_1],\dots,[\ell_n])\to([m];[k_1],\dots,[k_m])$ is $(\phi;\phi_1,\dots,\phi_n)$, where $\phi\colon[n]\to[m]$ is a morphism in Δ, and $\phi_i=(\phi_i^{\phi(i-1)+1},\dots,\phi_i^{\phi(i)})$, $\phi_i^s\colon[\ell_i]\to[k_s]$ is a tuple of morphisms in Δ. We denote by ∂^j the j-th face maps $\partial^j\colon[n]\to[n+1]$ in Δ, for $0\le j\le n+1$.

Inner face maps of codimension 1:

(F1) $n=m$, $\ell=k$ for $i\neq p$, $k_p=\ell_p+1$, all $\phi_i^s=\mathrm{id}$ except for $\phi_p^{\phi(p)}$ equal to the j-th face map $\partial^j\colon[\ell_p]\to[\ell_{p+1}]$, $j\neq 0,\ell_{p+1}$ (that is, ∂^j is an inner coface map in Δ). We denote this face map by ∂_p^j,

(F2) $m=n+1$, the morphism $\phi\colon[m]=[n]\to[n+1]$ is ∂^j, $j\neq 0,n+1$. Next, $k_s=\ell_s$ for $s<j$, $k_s=\ell_{s+1}$ for $s>j+1$ and for $s=j,j+1$, $k_j+k_{j+1}=\ell_j$, and all $\phi_s=\mathrm{id}$ except for $s=j$. Let σ be a (k_j,k_{j+1})-shuffle permutation in Σ_{ℓ_j}. The permutation σ defines two maps $p\colon[k_j-1]\to[\ell_j-1]$ and $q\colon[k_{j+1}-1]\to[\ell_j-1]$ in Δ. They define the Joyal dual maps $p^*\colon[\ell_j]\to[k_j]$ and $q^*\colon[\ell_j]\to[k_{j+1}]$ in Δ preserving the end-points. Then $(p^*,q^*)\colon[\ell_j]\to[k_j]\times[k_{j+1}]$, extended by the identity maps of the ordinals $[\ell_i], i\neq j$, defines a map in Θ_2. It is the codim $=1$ face map associated

with a shuffle permutation σ.
We denote this face map by $D_{j,\sigma}$.

Let us define p^*, q^* explicitly, unwinding the definition. We think of the sets $\{1,\dots,k_j\}$, $\{1,\dots,k_{j+1}\}$, $\{1,\dots,\ell_j\}$ as the elementary *arrows* in the interval categories $I_{k_j}, I_{k_{j+1}}$, and I_{ℓ_j}, correspondingly. Then p^* and q^* are defined as follows. Both p^* and q^* preserve the end-points. If $\sigma^{-1}(\overrightarrow{i,i+1}) = \overrightarrow{a,a+1} \in I_{k_j}$, then $p^*(i)=a$, $p^*(i+1)=a+1$, $q^*(i)=q^*(i+1)$. If $\sigma^{-1}(\overrightarrow{i,i+1}) = \overrightarrow{b,b+1} \in I_{k_{j+1}}$, then $q^*(i)=b$, $q^*(i+1)=b+1$, $p^*(i)=p^*(i+1)$.

Outer face maps of codimension 1:

(F3) $n=m, l=k$ for $i \neq p, k_p = \ell_p+1$, all $\phi_i^s = \mathrm{id}$ except for $\phi_p^{\phi(p)}$ equal to the j-th face map $\partial^j\colon [\ell_p] \to [\ell_{p+1}], j = 0, \ell_{p+1}$(that is, ∂^j is an outer face map in Δ). We denote this face map by ∂_p^j,

(F4) the two remaining codimension 1 face maps are $D_{\min}$ and $D_{\max}$. In both cases, $m=n+1$. For the case of $D_{\min}, \phi=\partial^0$, and $k_1=0$, $k_s=\ell_{s-1}$ for $s\geq 1$,the maps $\phi_i = (\phi_i^{i+1}) = (\mathrm{id})$. For the case of $D_{\max}, k_{m+1}=0$, $\phi=\partial^{n+1}, \phi_i=(\phi_i^i)=(\mathrm{id})$.

More generally, given a map $\Phi = (\phi;\phi_1,\dots,\phi_n)\colon ([n];[\ell_1],\dots,[\ell_n]) \to ([m];[k_1],\dots,[k_m])$, we call Φ a *face map* if $\phi\colon [n]\to[m]$ is injective, and each $\phi_i\colon [\ell_i] \to [k_{\phi(i-1)+1}]\times\dots\times[k_{\phi(i)}]$ is a (jointly) injective map (the latter means that for any $a,b\in[\ell_i]$, $a\neq b$, for at least one ϕ_i^s, $\phi(i-1)+1\leq s\leq\phi(i)$, one has $\phi_i^s(a)\neq\phi_i^s(b)$).

Here is the list of elementary degeneracy maps in Θ_2:

(D1) $n=m, \ell=k$ for $i\neq p, k_p=\ell_p-1$, all $\phi_i^s=\mathrm{id}$ except for $\phi_p^{\phi(p)}$ equal to the j-th degenerate map $\varepsilon^j\colon [\ell_p]\to[\ell_p-1]$. We denote this degeneracy map ϵ_p^j,

(D2) $n-1=m$, the first component $p(\phi)$ is $\varepsilon^p\colon [n]\to[n-1]$. For any $[\ell_{p+1}]$, it extends uniquely to a morphism

$$\phi\colon ([n];[\ell_1],\dots,[\ell_p],[\ell_{p+1}],[\ell_{p+2}],\dots,[\ell_n]) \to ([n-1];[\ell_1],\dots,[\ell_p],[\ell_{p+2}],\dots,[\ell_n])$$

for which $\phi_1,\dots,\phi_p,\phi_{p+2},\dots,\phi_n$ are identity maps. We denote this operator $\Upsilon^p_{\ell_{p+1}}$. Note that the morphism $\Upsilon^p_{\ell_{p+1}}$ is of codimension 1 iff $\ell_{p+1}=0$. We define $\Upsilon^p_\ell=0$ if $\ell\neq\ell_{p+1}$.

We describe in Section 3.6 the relations among all the generating degeneracy and face maps.

3.2.2 The totalization of a 2-cocellular complex of vector spaces

The realization of a functor $\Theta_n^{op}\to\mathcal{E}$ for the case $\mathcal{E}=\mathscr{T}op$ was studied in [J], [[Be1], Prop. 2.2, Lemma 2.4, Prop. 2.6] and [[Be2], Prop. 3.9, Cor. 3.11]. Here we use the dual concept of totalization of a functor $\Theta_2\to\mathcal{E}$, which we briefly recall. We restrict ourselves with the case $n=2$, and consider the case $\mathcal{E}=\mathrm{Ch}(\Bbbk)$, the category of complexes of vector spaces over $\Bbbk$. Let $X_\bullet\colon \Theta_2\to \mathrm{Ch}(\Bbbk)$ be a 2-cocellular complex. First of all, we define its (non-normalized) totalization explicitly, as the complex whose degree ℓ component is:

$$\mathrm{Tot}_{\Theta_2}(X_\bullet)^\ell := \bigoplus_{\substack{T\in\Theta_2\\ \dim T=\ell}} X_T \tag{3.2.2}$$

and the differential of degree +1 is equal to the sum of (taken with appropriate signs) all codimension 1 coface maps:

$$\begin{aligned} d|_{X_T} = &\sum_{\substack{\text{coface maps } \partial_p^i\\ (F1),(F3)}} (-1)^{k_1+\dots+k_{p-1}+p-1+i-1}\partial_p^i + \sum_{\substack{\text{coface maps}\\ D_{p,\sigma}(F2)}} (-1)^{k_1+\dots+k_{p-1}+p-1+\sharp(\sigma)} D_{p,\sigma}+\\ &D_{\min} + (-1)^{k_1+\dots+k_n+n} D_{\max} \end{aligned} \tag{3.2.3}$$

where $T=([n];[k_1],\dots,[k_n])$.

Lemma 3.2.1. *One has $d^2=0$.*

Proof. It follows from relations (3.6.1)-(3.6.7) that the summands in d^2 come in pairs, in which the two operators are equal one to another. One checks by hand that for each pair the two terms have opposite signs, which makes them mutually cancelled. □

3.2.3 Definition of the 2-cocellular vector space $A(F,F)$

Let $\mathcal{C}$, $\mathcal{D}$ be monoidal $\Bbbk$-linear categories, $F\colon\mathcal{C}\to\mathcal{D}$ a monoidal $\Bbbk$-linear functor. Here we define a functor of *2-cocellular cochains* $A(F,F)\colon \Theta_2\to \mathrm{Vect}(\Bbbk)$.

For objects $X,Y\in\mathcal{C}$, $n\geq 0$, denote by $\mathrm{Mor}_n(X,Y)$ the $\Bbbk$-vector space

$$\mathrm{Mor}_n(X,Y) = \bigoplus_{Z_1,\dots,Z_{n-1}\in\mathrm{Ob}(\mathcal{C})} \mathcal{C}(Z_{n-1},Y)\otimes\mathcal{C}(Z_{n-2},Z_{n-1})\otimes\dots\otimes\mathcal{C}(X,Z_1)$$

We denote such a string by $\theta\in\mathrm{Mor}_n(X,Y)$. We use notations $\theta(0)=X$, $\theta(n)=Y$, $\theta(i)=Z_i$, $1\leq i\leq n-1$.

Define

$$\begin{aligned}&\hat{A}(F,F)_T =\\ &\prod_{\substack{X_i\in\mathcal{C}\\ Y_i\in\mathcal{C}}} \underline{\mathrm{Hom}}_{\Bbbk}\Big(\mathrm{Mor}_{n_0}(X_0,Y_0)\otimes_{\Bbbk}\dots\otimes_{\Bbbk}\mathrm{Mor}_{n_k}(X_k,Y_k),\ \mathcal{D}(FX_0\otimes_{\mathcal{D}}\dots\otimes_{\mathcal{D}}FX_k, FY_0\otimes_{\mathcal{D}}\dots\otimes_{\mathcal{D}}FY_k)\Big)\end{aligned} \tag{3.2.4}$$

For $T=([0];\emptyset)$ the final object, define

$$\hat{A}(F,F)(T)=\mathcal{D}(e,e) \tag{3.2.5}$$

where e is the unit object in $\mathcal{D}$.

The (dg-)vector space $A(F,F)_T$, which we are mostly interested in, is a (dg-)subspace of $\hat{A}(F,F)_T$, and is defined as follows.

This sub-complex is formed by the cochains for which, for any j, $1 \leq j \leq k$, the following condition holds.

We use notation

$$\Psi(\theta_1 \otimes \dots \otimes \theta_n) \in \mathcal{D}(FX_0 \otimes_{\mathcal{D}} \dots \otimes_{\mathcal{D}} FX_k, FY_0 \otimes_{\mathcal{D}} \dots \otimes_{\mathcal{D}} FY_k)$$

where Ψ is a cochain as in (3.2.4) and $\theta_i \in \mathrm{Mor}_{n_i}(X_i, Y_i)$, $1 \leq i \leq k$.

If $f_1, \dots, f_{n_j}$ are the composable morphisms in the string θ_j, we write $\theta_j(f_1, \dots, f_{n_j})$.

Our conditions read: for any $1 \leq j \leq k$

$$\begin{aligned}&\Psi(\theta_1 \otimes \dots \otimes \theta_{j-1} \otimes \theta_j(f_1, \dots, \alpha \circ f_\ell, f_{\ell+1}, \dots) \otimes \dots \otimes \theta_n) = \\ &\Psi(\theta_1 \otimes \dots \theta_{j-1} \otimes \theta_j(f_1, \dots, f_\ell, f_{\ell+1} \circ \alpha, \dots) \otimes \dots \otimes \theta_n), \text{ for } 1 \leq \ell \leq n_j - 1\end{aligned} \tag{3.2.6}$$

$$\Psi(\theta_1 \otimes \dots \otimes \theta_j(f_1 \circ \alpha, f_2, \dots, f_{n_j}) \otimes \dots) = \Psi(\theta_1 \otimes \dots \otimes \theta_j(f_1, \dots, f_{n_j}) \circ \alpha_j \tag{3.2.7}$$

$$\Psi(\theta_1 \otimes \dots \otimes \theta_j(f_1, \dots, \alpha \circ f_{n_j}) \otimes \dots) = \alpha_j \circ \Psi(\theta_1 \otimes \dots \otimes \theta_j(f_1, \dots, f_{n_j}) \otimes \dots) \tag{3.2.8}$$

Here α_j denotes a map equal to the product of the identity maps on all facts except for the j-th one, where it is equal to α as in Section 1.6.

For the case when θ_j has height 0, in which case it is reduced to an object $X_j \in \mathcal{C}$, the corresponding relations reads:

$$\alpha_j \circ \Psi(\theta_1 \otimes \dots \otimes X_j \otimes \dots) = \Psi(\theta_1 \otimes \dots \otimes X_j \otimes \dots) \circ \alpha_j \tag{3.2.9}$$

As well, we impose similar conditions for α^{-1} and the left and right unit maps, as well as for its inverse.

Remark 3.2.2. The relations (3.2.6)-(3.2.9) assume that the objects in the l.h.s., to which α is applied, are of the form $X \otimes (Y \otimes Z)$ or $(X \otimes Y) \otimes Z$, correspondingly. Similarly for the unit maps.

Definition 3.2.3. We define $A(F,F)_T \subset \hat{A}(F,F)_T$ as the cochains for which (3.2.6)-(3.2.9), as well as their analogues for α^{-1} and the unit maps (as well as their inverse) hold.

Example 3.2.4. In this example, we clarify the role of the relations above and their necessity. To this end, consider the "non-natural Davydov-Yetter situation", that is the case given in the introduction, i.e. when $T = ([k]; [0], [0], \dots, [0])$, $k \geq 0$. Then

$$\hat{A}_T(F,F) = \prod_{X_1, \dots, X_k \in Ob(\mathcal{C})} \mathcal{D}(FX_1 \otimes \dots \otimes FX_k, FX_1 \otimes \dots \otimes FX_k)$$

For the monomials in the rhs we choose left to the right parenthesizing, so that $FX_1 \otimes FX_2 \otimes FX_3 \otimes \dots \otimes FX_k$ is understood as $FX_1 \otimes (FX_2 \otimes (FX_3 \otimes (\dots \otimes FX_k) \dots))$. Note that if we imposed the naturality in all arguments, it would be exactly the terms of the Davydov-Yetter complex given in Subsection 3.1.2. We drop the naturality, and we liked to make the assignment $T \rightsquigarrow \hat{A}_T(F,F)$ a *cosimplicial complex*, when the elementary coface and codegeneracy maps were defined exactly as for the Davydov-Yetter case. Let us recall this definition, with notations for simplicial face and degeneracy maps from the beginning of Subsection 1.4.4. Let $\Psi \in \hat{A}_{[k-1]}(F,F)$. The elementary face maps

$\partial^i\colon [k-1]\to[k]$, $1\le i\le k-1$, act by plugging $X_i\otimes X_{i+1}$ in place of the i-th argument X_i of Ψ, followed by the application of the colax-map $F(X_i\otimes X_{i+1})\to F(X_i)\otimes F(X_{i+1})$ and rearranging the parentheses (note that by the Mac Lane coherence theorem one needn't specify the way by which the parentheses are rearranged, as any two such maps are equal). The extreme face map ∂^0 act by $\Psi\mapsto \mathrm{id}_{F(X_1)}\otimes\Psi(X_2\otimes\dots\otimes X_k)$ (for this map no rearrangements are necessary), and the other extreme face map ∂^k acts by $\Psi\mapsto\Psi(X_1\otimes\dots\otimes X_{k-1})\otimes\mathrm{id}_{F(X_k)}$, followed by the necessary reparenthesizing (it is unique, by Mac Lane theorem). The degeneracy map ϵ^i acts on k-cochain Ψ by plugging the monoidal unit e to the i-th position of Ψ, followed by the necessary rearrangements. The reader is referred to [BD] for more detailed description.

Now the question is: does this construction give rise to a cosimplicial object in $\mathrm{Ch}(\Bbbk)$ (when the polynaturality condition of Davydov-Yetter is dropped)? The answer is negative, because the relations in Δ are not respected by this action. Denote this action by $\mathcal{O}$. Then, for instance, the actions of $\mathcal{O}(\partial^{i+1})\circ\mathcal{O}(\partial^i)$ differs from $\mathcal{O}(\partial^i)\circ\mathcal{O}(\partial^i)$ only by the i-th argument, which is $(X_i\otimes X_{i+1})\otimes X_{i+2}$ for the first composition, and $X_i\otimes(X_{i+1}\otimes X_{i+2})$ for the second one. These two expressions are mapped one to another by the associator α. Therefore, in order the relation $\partial^{i+1}\partial^i=\partial^i\partial^i$ to be respected under the action $\mathcal{O}$, one has to *require the naturality with respect to* α on the i-th factor. It is clear from this reasoning that the naturality under all *monoidal* maps (that is, compositions of products the associator, the unit maps, and its inverse, with the identity maps on some factors) is the *minimal* naturality one has to require to get a cosimplicial object in $\mathrm{Ch}(\Bbbk)$.

Remark 3.2.5. We do not fix any specific parenthesising in the monomials $FX_0\otimes_{\mathcal{D}}\dots\otimes_{\mathcal{D}}FX_k$ and $FY_0\otimes_{\mathcal{D}}\dots\otimes_{\mathcal{D}}FY_k$. It assumes that we choose any of them, say from the left to the right, or vice versa. We call such parenthesising *regular*. If an operator from Θ_2 (defined below in this Section) gives rise to another parenthesising P, we define the corresponding cochain by conjugating with the suitable composition of the associator and the unit maps, as well as its inverse, relating the regular parenthesising with P. By the Mac Lane coherence theorem such map is unique. (Recall that the Mac Lane coherence theorem, which holds for monoidal categories, says that any two morphisms $X\rightrightarrows Y$, each of which is a composition of product of identity morphisms with the associator and the unit map, as well as their inverse, coincide).

Our immediate goal is to show that the assignment $T\rightsquigarrow A(F,F)_T$ gives rise to a functor $\Theta_2\to\mathrm{Vect}(\Bbbk)$. (Note that by Example 3.2.4 it is not true that the assignment $T\rightsquigarrow\hat{A}(F,F)_T$ gives rise to a functor from Θ_2).
It requires some preparation.

3.2.4 Definition of Mor

Let $\mathcal{C}$ be a dg-category over $\Bbbk$, and let $\phi\colon[m]\to[n]$ be a morphism in Δ. We associate to ϕ a map of vector spaces

$$\mathrm{Mor}_\phi\colon \mathrm{Mor}_n(X_0,X_n)\to\mathcal{C}(X_{\phi(m)},X_n)\otimes_\Bbbk\mathrm{Mor}_m(X_{\phi(0)},X_{\phi(m)})\otimes_\Bbbk\mathcal{C}(X_0,X_{\phi(0)})\quad(3.2.10)$$

as follows. The reader may consider this construction as a $\Bbbk$-linear version of the nerve functor, where there does not exist any projection

$$\mathcal{C}(X_{\phi(m)},X_n)\otimes\mathrm{Mor}_m(X_{\phi(0)},X_{\phi(m)})\otimes\mathcal{C}(X_0,X_{\phi(m)})\to\mathrm{Mor}_m(X_{\phi(0)},X_{\phi(m)})$$

(as the tensor product of vector spaces is not a cartesian product). We will elaborate more in Subsection 3.3.1.

We use notation $f_n \otimes \dots \otimes f_1$ for an element in $\mathrm{Mor}_n(X_0, X_n)$ (a general element in $\mathrm{Mor}_n(X_0, X_n)$ is a linear combination of such monomial ones).

The first and the third factors in $\mathrm{Mor}_\phi(f_n \otimes \dots \otimes f_1)$ are defined as

$$f_n \circ \dots \circ f_{\phi(m)+1} \text{ and } f_{\phi(0)} \circ \dots \circ f_1$$

correspondingly, if $\phi(0) > 0$ and $\phi(m) < n$, otherwise define for $\phi(0) = 0$ the third factor as $\mathrm{id}_{X_0} \in \mathcal{C}(X_0, X_0)$, and similarly for $\phi(m) = n$.

The second factor is defined as follows:

$$M_\phi(f_{\phi(m)} \otimes \dots \otimes f_{\phi(0)+1}) = g_m \otimes \dots \otimes g_1 \in \mathrm{Mor}_m(X_{\phi(0)}, X_{\phi(m)})$$

where

$$g_a = g_a(\phi) = \begin{cases} f_{c-1} \circ \dots \circ f_b \colon X_b \to X_c & \textit{if } \phi(a-1) = b,\ \phi(a) = c,\ c > b \\ \mathrm{id}_{X_b} & \textit{if } \phi(a-1) = \phi(a) = b \end{cases} \tag{3.2.11}$$

Define

$$\mathrm{Mor}_\phi(f_n \otimes \dots \otimes f_1) = \big(f_n \circ \dots \circ f_{\phi(m)+1}\big) \otimes M_\phi(f_{\phi(m)} \otimes \dots \otimes f_{\phi(0)+1}) \otimes \big(f_{\phi(0)} \circ \dots \circ f_1\big) \tag{3.2.12}$$

Denote

$$A_L(\phi) = f_n \circ \dots \circ f_{\phi(m)+1} \colon X_{\phi(m)} \to X_n$$
$$A_R(\phi) = f_{\phi(0)} \circ \dots \circ f_1 \colon X_0 \to X_{\phi(0)}$$

So we can imagine the outcome as

$$X_0 \overset{A_R(\phi)}{\dashrightarrow} X_{\phi(0)} \to X_{\phi(1)} \to X_{\phi(2)} \to \dots \to X_{\phi(m)} \overset{A_L(\phi)}{\dashrightarrow} X_n \tag{3.2.13}$$

where the left (respectively, right) dashed arrow is $A_R(\phi)$ (respectively, $A_L(\phi)$), and the string between them is M_ϕ. Moreover, one imagines the string (3.2.13) as placed vertically, with arrows are directed from the bottom to the top.

Assume now that $\mathcal{C}$ is a monoidal dg-category over $\Bbbk$. Assume we are given morphisms $\phi_1 \colon [m] \to [n_1], \dots, \phi_k \colon [m] \to [n_k]$. Assume we are given k strings, for $1 \le i \le k$ this string is

$$X_{i0} \xrightarrow{f_{i1}} X_{i1} \xrightarrow{f_{i2}} X_{i2} \xrightarrow{f_{i3}} \dots \xrightarrow{f_{in_i}} X_{in_i}$$

Denote

$$X_{\otimes 0} = X_{10} \otimes_{\mathcal{C}} \dots \otimes_{\mathcal{C}} X_{k0}, \quad X_{\otimes n} = X_{1n_1} \otimes_{\mathcal{C}} \dots \otimes_{\mathcal{C}} X_{kn_k}$$

and

$$X_{\otimes\phi(s)} = X_{1\phi_1(s)} \otimes_{\mathcal{C}} \dots \otimes_{\mathcal{C}} X_{k\phi_k(s)}, 0 \le s \le m$$

We define a map

$$\begin{gathered}\mathrm{Mor}_{\phi_1,\dots,\phi_k} \colon \mathrm{Mor}_{n_1}(X_{10}, X_{1n_1}) \otimes_\Bbbk \dots \otimes_\Bbbk \mathrm{Mor}_{n_k}(X_{k0}, X_{kn_k}) \to \\ \mathcal{C}(X_{\otimes(\phi(m))}, X_{\otimes n}) \otimes_\Bbbk \mathrm{Mor}_m(X_{\otimes(\phi(0))}, X_{\otimes\phi(m)}) \otimes_\Bbbk \mathcal{C}(X_{\otimes 0}, X_{\otimes\phi(0)})\end{gathered} \tag{3.2.14}$$

Morally, we take the strings (3.2.13) for each ϕ_i, $1\le i\le k$ (they all have m regular arrows and two extreme, or dashed, ones), and take the term-wise monoidal product in $\mathcal{C}$ of these k strings. The output is the r.h.s. of (3.2.14).

More precisely, define the three factors in $\mathrm{Mor}_{\phi_1,\dots,\phi_k}(\{f_{i\ell}\}) = A_L(\phi_1,\dots,\phi_k)\otimes B(\phi_1,\dots,\phi_k)\otimes A_R(\phi_1,\dots,\phi_k)$ as

$$\begin{aligned} A_L(\phi_1,\dots,\phi_k) &= A_L(\phi_1)\otimes\dots\otimes A_L(\phi_k)\colon X_{\otimes\phi(m)}\to X_{\otimes n}\\ B(\phi_1,\dots,\phi_k) &= M_{\phi_1}\otimes\dots\otimes M_{\phi_k}\in \mathrm{Mor}_m(X_{\otimes\phi(0)},X_{\otimes\phi(m)})\\ A_R(\phi_1,\dots,\phi_k) &= A_R(\phi_1)\otimes\dots\otimes A_R(\phi_k)\colon X_{\otimes 0}\to X_{\otimes\phi(0)}\end{aligned} \tag{3.2.15}$$

For the case of B, the corresponding element in $\mathrm{Mor}_m(X_{\otimes\phi(0)},X_{\otimes\phi(m)})$ is given by the following string

$$X_{\otimes\phi(0)}\xrightarrow{T_1}X_{\otimes\phi(1)}\xrightarrow{T_2}X_{\otimes\phi(2)}\dots\xrightarrow{T_m}X_{\otimes\phi(m)}$$

where

$$T_s = g_s(\phi_1)\otimes g_s(\phi_2)\otimes\dots\otimes g_s(\phi_k) \tag{3.2.16}$$

(see (3.2.11)).

The outcome can be imagined likewise (3.2.13):

$$X_{\otimes 0}\overset{A_R}{\dashrightarrow}X_{\otimes\phi(0)}\to X_{\otimes\phi(1)}\to X_{\otimes\phi(2)}\to\dots\to X_{\otimes\phi(m)}\overset{A_L}{\dashrightarrow}X_{\otimes n} \tag{3.2.17}$$

3.2.5 Functoriality of $A(F,F)$

Recall our assignment $T\rightsquigarrow A(F,F)_T$, see Subsection 3.2.3. We show that this assignment gives rise to a functor $\Theta_2\to\mathrm{Vect}(\Bbbk)$. Recall the definition of a morphism in Θ_2, see Definition 1.4.2.
Let $T=([k];[n_1],\dots,[n_k])$, $S=([k'];[n'_1],\dots,[n'_{k'}])$.
Let $\Phi=(\phi;\{\phi_i^\ell\})\colon T\to S$ be a morphism in Θ_2, where $\phi\colon[k]\to[k']$ is the level 1 component of the morphism Φ, and $\phi_i^\ell\colon[n_i]\to[n'_\ell]$ are the level 2 components, $1\le i\le k$, $\phi(i-1)+1\le\ell\le\phi(i)$ (for the case $\phi(i-1)=\phi(i)$ the corresponding set is empty).

Given such $\Phi\colon T\to S$, we define $A(F,F)(\Phi)\colon A(F,F)_T\to A(F,F)_S$. For any diagram X_{ij} of shape S in $\mathcal{C}$

$$\begin{aligned} &X_{10}\to X_{11}\to\dots\to X_{1n'_1}\\ &\dots\\ &X_{k'0}\to X_{k'1}\to\dots\to X_{k'n'_k}\end{aligned} \tag{3.2.18}$$

and any $\psi\in A(F,F)_T$, we give a formula for the value $A(F,F)(\Phi)(\psi)$.

To make our exposition more transparent, consider firstly the case when $T=([1];[n])$, and $S=([k'];[n'_1],\dots,[n'_{k'}])$ general. Let $\Phi\colon T\to S$ be a map in Θ_2. Denote $\min=\phi(0)$, $\max=\phi(1)$ and assume firstly that $\min\ne\max$.

For each $\min+1\le\ell\le\max$, one has a map $\phi^\ell\colon[n]\to[n'_\ell]$, with the same source ordinal.

By Subsection 3.2.4, it gives a map

$$\mathrm{Mor}_{\phi^{\min+1},\dots,\phi^{\max}}\colon \mathrm{Mor}_{n'_{\min+1}}(X_{\min+1,0},X_{\min+1,n'_{\min+1}})\otimes_{\Bbbk}\dots\otimes_{\Bbbk}\mathrm{Mor}_{n'_{\max}}(X_{\max,0},X_{\max,n'_{\max}})\to \underbrace{\mathcal{C}(X_{\phi^{\otimes}(n)},X_{\otimes n'})}_{=A_L^{\otimes}}\otimes_{\Bbbk}\underbrace{\mathrm{Mor}_n(X_{\phi^{\otimes}(0)}),X_{\phi^{\otimes}(n)})}_{=B^{\otimes}}\otimes_{\Bbbk}\underbrace{\mathcal{C}(X_{\otimes 0},X_{\phi^{\otimes}(0)})}_{=A_R^{\otimes}} \tag{3.2.19}$$

where, as above,

$$\begin{aligned} X_{\otimes 0} &= X_{\min+1,0}\otimes_{\mathcal{C}}\dots\otimes_{\mathcal{C}}X_{\max,0}\\ X_{\otimes n'} &= X_{\min+1,n'_{\min+1}}\otimes_{\mathcal{C}}\dots\otimes_{\mathcal{C}}X_{\max,n'_{\max}}\\ X_{\phi^{\otimes}(i)} &= X_{\min+1,\phi^{\min+1}(i)}\otimes_{\mathcal{C}}\dots\otimes_{\mathcal{C}}X_{\max,\phi^{\max}(i)}\end{aligned} \tag{3.2.20}$$

Next, for a string $X_{j0}\to X_{j1}\to\dots\to X_{jn'_j}$, $1\le j\le k'$, denote by $f_{j,\mathrm{tot}}\colon X_{j0}\to X_{j,n'_j}$ the composition of all morphisms in the string.

Define

$$f_{\otimes\min}=f_{0,\mathrm{tot}}\otimes_{\mathcal{C}}f_{1,\mathrm{tot}}\otimes_{\mathcal{C}}\dots\otimes_{\mathcal{C}}f_{\min,\mathrm{tot}}$$

and

$$f_{\otimes(\max+1)}=f_{(\max+1),\mathrm{tot}}\otimes_{\mathcal{C}}\dots\otimes_{\mathcal{C}}f_{k'-1,\mathrm{tot}}\otimes_{\mathcal{C}}f_{k',\mathrm{tot}}$$

In the case when min = max, we set $B^{\otimes}=e\xrightarrow{\mathrm{id}}e\xrightarrow{\mathrm{id}}\dots\xrightarrow{\mathrm{id}}e$, $A_L^{\otimes}=\mathrm{id}\in\mathcal{C}(e,e)$, $A_R^{\otimes}=\mathrm{id}\in\mathcal{C}(e,e)$.

Let $\psi\in A(F,F)_{([1];[n])}$.

We set:

$$A(F,F)(\Phi)(\psi)=F(f_{\otimes(\max+1)})\otimes_{\mathcal{D}}\Big(F(A_L^{\otimes})\circ\psi(B^{\otimes})\circ F(A_R^{\otimes})\Big)\otimes_{\mathcal{D}}F(f_{\otimes\min}) \tag{3.2.21}$$

In particular, $A_{\mathrm{red}}(F,F)(\Phi)(\psi)=0$ for any ψ, if min = max.

Now consider the case of general $T=([k];[n_1],\dots,[n_k])$, $S=([k'];[n'_1],\dots,[n'_{k'}])$. Let $\Phi=(\phi;\{\phi^{i,\ell}\})$, where $\phi^{i,\ell}\colon[n_i]\to[n'_\ell]$, $0\le i\le k-1$, $\phi(i)+1\le\ell\le\phi(i+1)$. For each i, the construction of Subsection 3.2.5 gives an analogue of the map (3.2.19):

$$\mathrm{Mor}_{\phi^{i,\phi(i)+1},\dots,\phi^{i,\phi(i+1)}}\colon \mathrm{Mor}_{n'_{\phi(i)+1}}(X_{\phi(i)+1,0},X_{\phi(i)+1,n'_{\phi(i)+1}})\otimes_{\Bbbk}\dots\otimes_{\Bbbk}\mathrm{Mor}_{n'_{\phi(i+1)}}(X_{\phi(i+1),0},X_{\phi(i+1),n'_{\phi(i+1)}})\to \underbrace{\mathcal{C}(X_{\phi^{i\otimes}(n_i)},X_{\otimes in'})}_{A_L^{\otimes i}}\otimes_{\Bbbk}\underbrace{\mathrm{Mor}_{n_i}(X_{\phi^{i\otimes}(0)},X_{\phi^{i\otimes}(n_i)})}_{B^{\otimes i}}\otimes_{\Bbbk}\underbrace{\mathcal{C}(X_{\otimes i0},X_{\phi^{i\otimes}(0)})}_{A_R^{\otimes i}} \tag{3.2.22}$$

where

$$\begin{aligned} X_{\otimes i0} &= X_{\phi(i)+1,0}\otimes_{\mathcal{C}}\dots\otimes_{\mathcal{C}}X_{\phi(i+1),0}\\ X_{\otimes in'} &= X_{\phi(i)+1,n'_{\phi(i)+1}}\otimes_{\mathcal{C}}\dots\otimes_{\mathcal{C}}X_{\phi(i+1),n'_{\phi(i+1)}}\\ X_{\phi^{i\otimes}(j)} &= X_{\phi(i)+1,\phi^{i,\phi(i)+1}(j)}\otimes_{\mathcal{C}}\dots\otimes_{\mathcal{C}}X_{\phi(i+1),\phi^{i,\phi(i+1)}(j)},\ 0\le j\le n_i\end{aligned} \tag{3.2.23}$$

When $\phi(i)=\phi(i+1)$, denote by $A_L^{i\otimes},B^{i\otimes},A_R^{i\otimes}$ the (strings of the) identity maps of the monoidal unit, as in the case $k=1$ above.

Finally, denote

$$F^{\otimes}A_L^{\otimes} := F(A_L^{\otimes 1})\otimes_{\mathcal{D}} F(A_L^{\otimes 2})\otimes_{\mathcal{D}}\cdots\otimes_{\mathcal{D}} F(A_L^{\otimes k})$$

and

$$F^{\otimes}A_R^{\otimes} := F(A_R^{\otimes 1})\otimes_{\mathcal{D}} F(A_R^{\otimes 2})\otimes_{\mathcal{D}}\cdots\otimes_{\mathcal{D}} F(A_R^{\otimes k})$$

Let $\min=\phi(0)$, $\max=\phi(k)$. We use notation $f_{\ell,\mathrm{tot}}, f_{\otimes\min}, f_{\otimes(\max+1)}$ as above.

Let $\psi\in A(F,F)_T$. The formula for $A(F,F)(\Phi)(\psi)$ reads:

$$\begin{aligned}&A(F,F)(\Phi)(\psi)=\\ &F(f_{\otimes(\max+1)})\otimes_{\mathcal{D}}\Big(F^{\otimes}A_L^{\otimes}\circ\psi(B^{\otimes 1}\otimes_{\Bbbk}\cdots\otimes_{\Bbbk}B^{\otimes k})\circ F^{\otimes}A_R^{\otimes}\Big)\otimes_{\mathcal{D}} F(f_{\otimes\min})\end{aligned}\tag{3.2.24}$$

Proposition 3.2.6. *The assignment $T\mapsto A(F,F)_T$ gives rise, via (3.2.24), to a functor $A(F,F)\colon \Theta_2\to \mathrm{Vect}(\Bbbk)$.*

Proof. One can check that the action on $A(F,F)$ by the generators of Θ_2 (listed in the following Subsection 3.2.6) satisfy all the relations described in Section 3.6.
We also give a more conceptual proof of the functoriality in Subsection 3.3.2. □

Remark 3.2.7. (1) The construction of $A(F,F)$ can be easily generalised to the case of dg-bicategories $\mathcal{C}$, $\mathcal{D}$ and F a dg-pseudo-functor. (Recall that a monoidal category is a bicategory with a single object). The reason is that there is a direct analogue of the Mac Lane coherence theorem for bicategories, proven by Bénabou [Ben]. The conditions (3.2.6)-(3.2.9) should be replaced accordingly, by the naturality with respect to the associativity maps for the composition of 1-morphisms, the unit morphisms, and their inverses.

(2) One can not define a functor $A(F,G)\colon\Theta_2\to\mathrm{Vect}(\Bbbk)$, for two distinct monoidal (or pseudo) dg-functors $F,G\colon\mathcal{C}\to\mathcal{D}$, if we wanted to have $\mathcal{D}(FX_0\otimes_{\mathcal{D}}\cdots\otimes_{\mathcal{D}} FX_k, GY_0\otimes_{\mathcal{D}}\cdots\otimes_{\mathcal{D}} GY_k)$ in the r.h.s. of (2.7). The matter is that an action of morphisms in Θ_2 for which $\min=\phi(0)>0$ or $\max=\phi(k)<k'$ are ill-defined. However, for given two natural transformations $\eta,\theta\colon F\Rightarrow G$ (in the classical sense), one can define the corresponding 2-cocellular complex $A(F,G)(\eta,\theta)$, playing the role of derived 3-arrows (or derived modifications). It gives rise to the question "What do dg-bicategories form?", as a 3-dimensional generalisation of the problem studied in [Tam2].

3.2.6 The action of Θ_2 on $A(F,F)$

Here we list the actions on $A(F,F)$ of all the generators of Θ_2 (see Subsection 3.2.1). We use notation $\underline{X}_s$ for a chain of morphisms in $\mathcal{C}$:

$$X_{s,0}\xrightarrow{f_{s,1}}X_{s,1}\xrightarrow{f_{s,2}}\dots\xrightarrow{f_{s,m_s}}X_{s,m_s}$$

(F1) Let $T=([n];[\ell_1],\dots,[\ell_n])$, $T'=([n];[\ell_1],\dots,[\ell_{p-1}],[\ell_p+1],[\ell_{p+1}]\dots,[\ell_n])$. Then $\partial_p^j\colon T\to T'$, with $0<j<\ell_p$, and for a cochain $\Psi\in A(F,F)$, one has:

$$(\partial_p^j\Psi)_{T'}(\underline{X}_1,\dots,\underline{X}_n)=\Psi_T(\underline{X}_1,\dots,\underline{X}_{p-1},\underline{Y}_p,\underline{X}_{p+1},\dots,\underline{X}_n)\tag{3.2.25}$$

where $\underline{Y}_p$ is the chain:

$$X_{p,0} \xrightarrow{g_1} \dots\dots \xrightarrow{g_{\ell_p}} X_{p,\ell_p+1}$$

and

$$g_i = \begin{cases} f_{p,i} & \text{if} \quad i < j \\ f_{p,j+1} \circ f_{p,j} & \text{if} \quad i = j \\ f_{p,i+1} & \text{if} \quad i > j \end{cases} \tag{3.2.26}$$

(F2) Let $\sigma \in \Sigma_{\ell_j}$ be an (m_j, m_{j+1})-shuffle, $\ell_j = m_j + m_{j+1}$. Let $p^*\colon [\ell_j] \to [m_j]$ and $q^*\colon [\ell_j] \to [m_{j+1}]$ be the two maps Joyal dual to the natural embeddings $[m_j - 1] \to [\ell_j - 1]$ and $[m_{j+1} - 1] \to [\ell_j - 1]$. Let $T = ([n]; [\ell_1], \dots, [\ell_n])$, $T' = ([n+1]; [\ell_1], \dots, [\ell_{j-1}], [m_j], [m_{j+1}], \dots, [\ell_{n+1}])$. Then $D_{j,\sigma}\colon T \to T'$, and for a cochain $\Psi \in A(F,F)$, one has:

$$(D_{j,\sigma}\Psi)_{T'}(\underline{X}_1, \dots, \underline{X}_{n+1}) = \Psi_T(\underline{X}_1, \dots, \underline{X}_{j-1}, \underline{Y}_j, \underline{X}_{j+2}, \dots, \underline{X}_{n+1}) \tag{3.2.27}$$

where $\underline{Y}_j$ is the chain:

$$X_{j0} \otimes X_{j+1,0} \xrightarrow{g_1} \dots\dots \xrightarrow{g_{m_j+m_{j+1}}} X_{j,m_j} \otimes X_{j+1,m_{j+1}}$$

and

$$g_i = \begin{cases} f_{j,a} \otimes \mathrm{id} & \text{if} \quad \sigma^{-1}(i) = a, \quad 0 \le a \le m_j \\ \mathrm{id} \otimes f_{j+1,b} & \text{if} \quad \sigma^{-1}(i) = b, \quad m_j + 1 \le b \le m_j + m_{j+1} \end{cases} \tag{3.2.28}$$

(F3) Let $T = ([n]; [\ell_1], \dots, [\ell_n])$, $T' = ([n]; [\ell_1], \dots, [\ell_{p-1}], [\ell_p + 1], [\ell_{p+1}] \dots, [\ell_n])$. Then $\partial_p^0\colon T \to T'$, (the case for $\partial_p^{\ell_p}$ is totally analogous), and for a cochain $\Psi \in A(F,F)$, one has:

$$(\partial_p^0\Psi)_{T'}(\underline{X}_1, \dots, \underline{X}_n) = \Psi_T(\underline{X}_1, \dots, \underline{X}_{p-1}, \underline{Y}_p, \underline{X}_{p+1}, \dots, \underline{X}_n) \circ F^{\otimes}(A_R^{\otimes}) \tag{3.2.29}$$

where $\underline{Y}_p$ is the chain:

$$X_{p,1} \xrightarrow{f_{p2}} \dots\dots \xrightarrow{f_{p\ell_p+1}} X_{p,\ell_p+1}$$

and

$$A_R^{\otimes} \coloneqq \mathrm{id}_{X_{1,0}} \otimes_{\mathcal{C}} \dots \otimes_{\mathcal{C}} \mathrm{id}_{X_{p-1,0}} \otimes_{\mathcal{C}} f_{p,1} \otimes_{\mathcal{C}} \mathrm{id}_{X_{10}} \otimes_{\mathcal{C}} \dots \otimes_{\mathcal{C}} \mathrm{id}_{X_{n,0}}$$

(F4)(1) Let $T = ([n]; [\ell_1], \dots, [\ell_n])$, $T' = ([n+1]; [0], [\ell_1], \dots, [\ell_n])$. Then $D_{\min}\colon T \to T'$ (the case for $D_{\max}$ is totally analogous) and for a cochain $\Psi \in A(F,F)$, one has:

$$(D_{\min}\Psi)_{T'}(\underline{X}_1, \dots, \underline{X}_{n+1}) = F(\mathrm{id}_{X_{1,0}}) \otimes_{\mathcal{D}} \Psi_T(\underline{X}_2, \dots, \underline{X}_{n+1}) \tag{3.2.30}$$

(F4)(2) Let $T = ([n]; [\ell_1], \dots, [\ell_n])$, $T' = ([n+1]; [\ell_1], \dots, [\ell_n], [0])$. Then $D_{\max}\colon T \to T'$ and for a cochain $\Psi \in A(F,F)$, one has:

$$(D_{\max}\Psi)_{T'}(\underline{X}_1, \dots, \underline{X}_{n+1}) = \Psi_T(\underline{X}_1, \dots, \underline{X}_n) \otimes_{\mathcal{D}} F(\mathrm{id}_{X_{n+1,0}}) \tag{3.2.31}$$

(D1) Let $T = ([n]; [\ell_1], \dots, [\ell_n])$, $T' = ([n]; [\ell_1], \dots, [\ell_{p-1}], [\ell_p - 1], [\ell_{p+1}] \dots, [\ell_n])$. Then $\epsilon_p^j\colon T \to T'$ and for a cochain $\Psi \in A(F,F)$, one has:

$$(\epsilon_p^j\Psi)_{T'}(\underline{X}_1, \dots, \underline{X}_n) = \Psi_T(\underline{X}_1, \dots, \underline{X}_{p-1}, \underline{Y}_p, \underline{X}_{p+1}, \dots, \underline{X}_n) \tag{3.2.32}$$

where $\underline{Y}_p$ is the chain:

$$X_{p,0}\xrightarrow{g_1}\dots\dots\xrightarrow{g_{\ell_p}} X_{p,\ell_p-1}$$

and

$$g_i=\begin{cases} f_{p,i} & \text{if} \quad i<j\\ \mathrm{id}_{X_{p,j}} & \text{if} \quad i=j\\ f_{p,i-1} & \text{if} \quad i>j\end{cases} \tag{3.2.33}$$

(D2) Let $T=([n];[\ell_1],\dots,[\ell_n])$, $T'=([n-1];[\ell_1],\dots,[\ell_{p-1}],[\ell_{p+1}],\dots,[\ell_n])$. Then $\Upsilon^p_{\ell_p}\colon T\to T'$ and for a cochain $\Psi\in A(F,F)$, one has:

$$(\Upsilon^p_{\ell_p}\Psi)_{T'}(\underline{X}_1,\dots,\underline{X}_n)=\Psi_T(\underline{X}_1,\dots,\underline{X}_{p-1},\underline{e},\underline{X}_p,\dots,\underline{X}_n) \tag{3.2.34}$$

where $\underline{e}$ is the chain:

$$\underbrace{e\xrightarrow{\mathrm{id}_e}\dots\dots\xrightarrow{\mathrm{id}_e} e}_{\ell_p+1 \text{ elements}}$$

and e is the monoidal unit.

3.2.7 The case $F\neq G$: derived modifications

We consider here $A(F,F)_T$ for the case when $\mathcal{C},\mathcal{D}$ are bicategories linear over $\Bbbk$ (which means that the 2-morphisms form $\Bbbk$-vector spaces or complexes of $\Bbbk$-vector spaces), and $F\colon\mathcal{C}\to\mathcal{D}$ a $\Bbbk$-linear pseudofunctor. As we mentioned in Remark 3.2.7, this generalisation goes straightforwardly.

Let us write down the formula for cochains. For two objects $X,Y\in Ob(\mathcal{C})$, and two 1-morphisms $f,g\in\mathcal{C}(X,Y)$, denote by

$$\mathrm{Mor}^2_n(f,g)=\bigoplus_{h_1,\dots,h_{n-1}\in\mathcal{C}(X,Y)}\mathcal{C}(h_{n-1},g)\otimes\mathcal{C}(h_{n-2},h_{n-1})\otimes\dots\otimes\mathcal{C}(f,g_1) \tag{3.2.35}$$

where the factors are the corresponding $\Bbbk$-vector spaces of 2-morphisms.

Let $T=([k];[n_1],\dots,[n_k])\in\Theta_2$. Set

$$\hat{A}(F,F)_T=\prod_{X_0,\dots,X_k\in\mathrm{Ob}(\mathcal{C})}\ \prod_{\substack{f_i,g_i\in\mathcal{C}(X_{i-1},X_i)\\ i=1,\dots,k}}$$
$$\underline{\mathrm{Hom}}_\Bbbk\Big(\mathrm{Mor}^2_{n_1}(f_1,g_1)\otimes_\Bbbk\dots\otimes_\Bbbk\mathrm{Mor}^2_{n_k}(f_k,g_k),\ \mathcal{D}(Ff_k\circ\dots\circ Ff_1,Fg_k\circ\dots\circ Fg_1)\Big) \tag{3.2.36}$$

Note that both $Ff_k\circ\dots\circ Ff_1$ and $Fg_k\circ\dots\circ Fg_1$ are morphisms in $\mathcal{D}(FX_0,FX_k)$.

We define the subcomplex $A(F,F)_T\subset\hat{A}(F,F)_T$, formed by cochains which satisfy analogues of (3.2.6)-(3.2.9), where the associator is replaced by the associativity morphism for composition of 1-morphism in the bicategory, et cetera.

Note that for the case when $\mathcal{C}$ is the bicategory with a single object, associated to a monoidal category, we recover our complexes (3.2.4) and its subcomplex $A(F,F)_T$.

Now assume that $F, G\colon \mathcal{C} \to \mathcal{D}$ are two pseudo-functors. We ask ourselves whether one can generalise our $\hat{A}(F,F)_T$ to $\hat{A}(F,G)_T$, such that the latter is also a 2-cocellular space $\hat{A}(F,G)\colon \Theta_2 \to \mathrm{Ch}(\Bbbk)$. When we just replace, in the r.h.s. of (3.2.36) $\mathcal{D}(Ff_k \circ \dots \circ Ff_1, Fg_k \circ \dots \circ Fg_1)$ by $\mathcal{D}(Ff_k \circ \dots \circ Ff_1, Gg_k \circ \dots \circ Gg_1)$, we could not define an action of $\Phi\colon T \to S = ([k'];[n'_1],\dots,[n'_{k'}])$, $\Phi = (\phi; \{\phi_i^\ell\})$ in Θ_2, for which $\phi(0) \neq 0$ or $\phi(k) \neq k'$.

Indeed, for the case $F = G$, we define such operations via tensoring with $\mathrm{id}\colon F \to F$, see the "extreme" factors $F(f_{\otimes(\max+1)})$ and $F(f_{\otimes \min})$ in the r.h.s. of (3.2.21). To mimic these factors for the case $F \neq G$, one needs natural transformations $\eta, \theta\colon F \Rightarrow G$ (one of them is used for the extreme factors from the left, another one for the extreme factors on the right). So we are going to define components $\hat{A}(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)$, or, for short, $\hat{A}(F,G)(\eta,\theta)$.

The definition is as follows:

$$\hat{A}(F,G)(\eta,\theta)_T = \prod_{X_0,\dots,X_k \in \mathrm{Ob}(\mathcal{C})} \prod_{\substack{f_i, g_i \in \mathcal{C}(X_{i-1},X_i)\\ i=1,\dots,k}}$$
$$\underline{\mathrm{Hom}}_{\Bbbk}\Big(\mathrm{Mor}^2_{n_1}(f_1,g_1) \otimes_{\Bbbk} \dots \otimes_{\Bbbk} \mathrm{Mor}^2_{n_k}(f_k,g_k),\ \mathcal{D}(\eta(X_k) \circ Ff_k \circ \dots \circ Ff_1, Gg_k \circ \dots \circ Gg_1 \circ \theta(X_0))\Big) \tag{3.2.37}$$

Note that both compositions $\eta(X_k) \circ Ff_k \circ \dots \circ Ff_1, Gg_k \circ \dots \circ Gg_1 \circ \theta(X_0)$ are maps $F(X_0) \to G(X_k)$ in $\mathcal{D}$.

We define the action of $\Phi\colon T \to S$ (in the notations of Subsection 3.2.3) by

$$\begin{gathered}\hat{A}(F,G)(\eta,\theta)(\Phi)(\psi) = \\ G(f_{\otimes \max}) \otimes_{\mathcal{D}} \Big(G^{\otimes} A_L^{\otimes} \circ \psi(B^{\otimes 1} \otimes_{\Bbbk} \dots \otimes_{\Bbbk} B^{\otimes k}) \circ F^{\otimes} A_R^{\otimes}\Big) \otimes_{\mathcal{D}} F(f_{\otimes \min})\end{gathered} \tag{3.2.38}$$

followed by the isomorphisms

$$\eta(X_{n+1}) \circ F(f_{n+1}) \to G(f_{n+1}) \circ \eta(X_n), \quad G(f_1) \circ \theta(X_0) \to \theta(X_1) \circ F(f_1)$$

where $f_1 \in \mathcal{C}(X_0, X_1), f_{n+1} \in C(X_n, X_{n+1})$ are morphisms in $\mathcal{C}$, et cetera.

Proposition 3.2.8. *For any two $\Bbbk$-linear bicategories, two linear pseudo-functors $F, G\colon \mathcal{C} \to \mathcal{D}$, and two natural transformations $\eta, \theta\colon F \Rightarrow G$, the assignment*

$$T \rightsquigarrow A(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T$$

gives rise to a functor $\Theta_2 \to \mathrm{Ch}(\Bbbk)$.

Proof. See Subsection 3.3.2. □

Recall what is a modification $\tau\colon \eta \Rrightarrow \theta\colon F \Rightarrow G\colon \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}, \mathcal{D}$ are bicategories, $F, G\colon \mathcal{C} \to \mathcal{D}$ pseudofunctors, $\eta, \theta\colon F \Rightarrow G$ natural transformations.

A modification $\eta \to \theta$ is a 2-arrow $\Psi(X)\colon \eta(X) \to \theta(X)$, for any $X \in Ob(\mathcal{C})$, such that for any $X, Y \in Ob(\mathcal{C})$ and any $f \in \mathcal{C}(X,Y)$, the diagram below commutes:

$$\Phi_\theta(f) \circ_v (\Psi(y) \circ_h \mathrm{id}_{F(f)}) = (\mathrm{id}_{G(f)} \circ_h \Psi(x)) \circ_v \Phi_\eta(f)$$

where $\Phi_\eta(f), \Phi_\theta(f)$ are the two-morphisms of the natural transformations η, θ, correspondingly, $\circ_h$ (respectively, $\circ_v$) is the horizontal (respectively, vertical) composition of 2-morphisms.

The modifications play the role of 3-morphisms in the category of bicategories, making it a suitably relaxed 3-category.

We have:

Proposition 3.2.9. *Assume $\mathcal{C}, \mathcal{D}$ are $\Bbbk$-linear or dg- over $\Bbbk$ bicategories (that is, the 2-morphisms $\mathrm{Mor}^2(F,G)$ have a $\Bbbk$-linear structure), such that for any F, G the $\Bbbk$-vector space $\mathrm{Mor}^2(F,G)$ is concentrated in cohomological degree 0. Then 0-th cohomology of $\mathrm{Tot}_{\Theta_2}(A(C,D)(F,G)(\eta,\theta))$ is identified with the $\Bbbk$-vector space of modifications $\eta \Rrightarrow \theta$.*

Proof. The degree 0 part of the totalization is corresponded to $A(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T$, where $T = ([0];)$, the final object of Θ_2. The corresponding cochain depends on an object $X \in Ob(\mathcal{C})$, and to X is assigned the $\Bbbk$-vector space $\mathcal{D}(\eta(X) \circ \mathrm{id}_{F(X)}, \mathrm{id}_{G(X)} \circ \theta(X))$. Thus, choose a cochain

$$\Psi \in \prod_{X \in Ob(\mathcal{C})} \mathcal{D}(\eta(X), \theta(X))$$

Its boundary $d\Psi$ is a cochain in $A(\mathcal{C},\mathcal{D})(F,F)(\eta,\theta)_S$, where $S = ([1];[0])$. It has two summands:

$$(d\Psi)(f) = [\Psi(Y) \circ \mathrm{id}_{F(f)}] - [\mathrm{id}_{G(f)} \circ \Psi(X)]$$

where $f \in \mathcal{C}(X,Y)$. Here $[-]$ stands for the reduction of the cochain, by which we mean the following. By the convention made earlier in this Subsection, we have to replace $f \mapsto \Psi(Y) \circ \mathrm{id}_{F(f)}$ by a cochain of the form $f \mapsto \mathcal{D}(\eta(Y) \circ Ff, Gf \circ \theta(X))$, whence $f \mapsto \Psi(Y) \circ \mathrm{id}_{F(f)}$ takes values in $\mathcal{D}(\eta(Y) \circ F(f), \theta(Y) \circ F(f))$. That is, we have to post-compose $\Psi(Y) \circ \mathrm{id}_{F(f)}$ with the 2-cell $\Phi_\theta(f)\colon \theta(Y) \circ F(f) \to G(f) \circ \theta(X)$ for θ, and the result of the post-composition is denoted by $[\Psi(Y) \circ \mathrm{id}_{F(f)}] = \Phi_\theta(f) \circ_v (\Psi(Y) \circ_h \mathrm{id}_{F(f)})$. Similarly for the second summand, but here we pre-compose with the 2-cell $\Phi_\eta(f)$. Then the condition $d\Psi = 0$ expresses exactly the condition on the collection of 2-morphisms $\{\Psi(X)\}_{X \in Ob(\mathcal{C})}$ being a modification. □

3.3 An abelian category of 2-bimodules

3.3.1 The category of 2-bimodules over a bicategory

Let $\mathcal{C}$ be a $\Bbbk$-linear bicategory (that is, the 2-morphisms are $\Bbbk$-vector spaces). We define an abelian category of 2-bimodules over $\mathcal{C}$.

A 2-bimodule over $\mathcal{C}$ is a 2-globular set M such that the sets of vertices and 1-arrows (along with the restriction of the globular structure on them) are the same as for $\mathcal{C}$, and the 2-arrows are $\Bbbk$-linear vector spaces (respectively, complexes of $\Bbbk$-linear vector spaces), subject to the following conditions:

(1) there are left and right horizontal compositions on 2-arrows:

$$\mathcal{C}(f,g) \otimes M(f',g') \to M(f \circ f', g \circ g'), \quad M(f,g) \otimes \mathcal{C}(f',g') \to M(f \circ f', g \circ g')$$

(2) the upper and lower vertical compositions

$$\mathcal{C}(f,g)\otimes M(g,h)\to M(f,h),\quad M(f,g)\otimes \mathcal{C}(g,h)\to M(f,h)$$

(3) four (strict) Eckmann-Hilton identities. Let $X,Y,Z\in\mathcal{C}$, $f_1,f_2,f_3\in\mathcal{C}(X,Y)$, $g_1,g_2,g_3\in\mathcal{C}(Y,Z)$. Assume $t_1\in\mathcal{C}(f_1,f_2)$, $t_2\in\mathcal{C}(f_2,f_3)$, $m\in M(g_1,g_2)$, $t_3\in\mathcal{C}(g_2,g_3)$. Then the following identity holds:

$$(t_3\circ^v_+ m)\circ^h_-(t_2\circ^v t_1)=(t_3\circ^h t_2)\circ^v_+(m\circ^h_- t_1) \tag{3.3.1}$$

The three other identities are corresponded to the three other possible positions of an element of M, and are analogous.

(4) the compositions in (1) are associative up to associator 2-morphisms which are subject to to natural compatibility, the compositions in (2) are strictly associative,

(5) there are weak units which are subject to natural compatibilities,

(6) the maps in (3) are compatible with the associativity morphisms and the unit maps.

The morphisms of 2-bimodules over $\mathcal{C}$ are defined in the natural way.

The 2-bimodules over a given bicategory $\mathcal{C}$ form a category, denoted by 2-Bimod($\mathcal{C}$). Note that this category is abelian.

Examples.

(1) Here we provide the *free rank 1 2-bimodule* over a bicategory $\mathcal{C}$. It is

$$M(f,g)=\Big(\bigoplus_{\substack{Z\in Ob(\mathcal{C})\\ \alpha_-,\alpha_+\in\mathcal{C}(X,Z)\\ \beta_-,\beta_+\in\mathcal{C}(Z,Y)}}\mathcal{C}(\beta_+\circ\alpha_+,g)\otimes_\Bbbk\big(\mathcal{C}(\beta_-,\beta_+)\otimes_\Bbbk\mathcal{C}(\alpha_-,\alpha_+)\big)\otimes_\Bbbk\mathcal{C}(f,\beta_-\circ\alpha_-)\Big)/\sim \tag{3.3.2}$$

for $f,g\in\mathcal{C}(X,Y)$, where the quotient is taken by the relations generated by the Eckmann-Hilton axioms (3). The operations listed in (1)-(5) above are clear. It is shown below (in a bit more general context) that this 2-bimodule is projective.

(2) For a choice of the data $\eta\Rrightarrow\theta\colon F\Rightarrow G\colon\mathcal{C}\to\mathcal{D}$, define the following 2-bimodule over $\mathcal{C}$ $M(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)$:

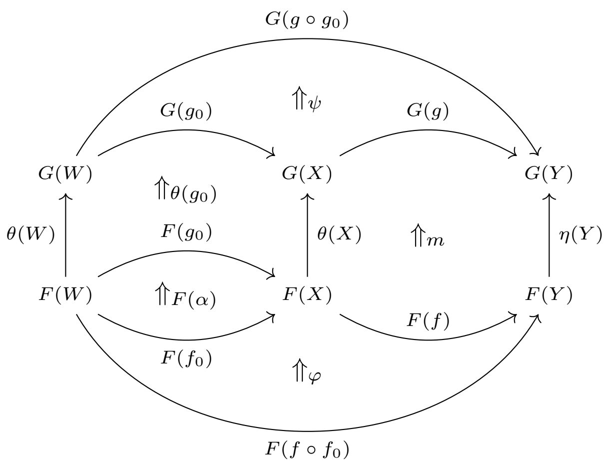
$$M(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)(f,g)=\mathcal{D}(\eta(Y)\circ F(f),G(g)\circ\theta(X)) \tag{3.3.3}$$

where $f,g\in\mathcal{C}(X,Y)$.

Lemma 3.3.1. *$M(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)$ is a 2-bimodule over $\mathcal{C}$.*

Proof. Let $m\in M(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)(f,g)$.
For a 2-morphism $\alpha\colon f'\Rightarrow f$, the vertical composition $m\circ^v_-\alpha$ is defined as the vertical composition $m\circ^v(\eta(y)\circ F(\alpha))$ in $\mathcal{D}$. Seemingly, for $\beta\colon g\Rightarrow g'$, the vertical composition $\beta\circ^v_+ m$ is defined as the vertical composition $G(\beta)\circ\theta(X))\circ^v m$.
For $f_0,g_0\in\mathcal{C}(W,X),\alpha\in\mathcal{C}(f_0,g_0)$, define the 2-morphism $(m\circ^h_-\alpha)_0$ in $\mathcal{D}$ as the horizontal composition in $m\circ^h F(\alpha)$ in $\mathcal{D}$ post-composed vertically with the 2-morphism $\theta(X)\circ$

$F(g_0) \Rightarrow G(g_0) \circ \theta(X_0)$ (whiskered by $G(g)$). Define the horizontal composition $m \circ^h_- \alpha = \psi \circ^v (m \circ^h_- \alpha)_0 \circ^v \phi$ where ϕ is the vertical composition of 2-arrows $\eta(Y) \circ F(f \circ f_0) \stackrel{\sim}{\to} \eta(Y) \circ (F(f) \circ F(f_0)) \stackrel{\sim}{\to} (\eta(Y) \circ F(f)) \circ F(f_0)$, and ψ is the composition $G(g) \circ (G(g_0) \circ \eta(W)) \stackrel{\sim}{\to} (G(g) \circ G(g_0)) \circ \eta(W) \stackrel{\sim}{\to} G(g \circ g_0) \circ \eta(W)$.



(3.3.4)

For $f_1, g_1 \in \mathcal{C}(Y, Z)$, $\beta \in \mathcal{C}(f_1, g_1)$, define the 2-morphism $(\beta \circ^v_+ m)_0$ in $\mathcal{D}$ as the horizontal composition $m \circ^h G(\beta)$ in $\mathcal{D}$ pre-composed vertically with the 2-morphism $\eta(Z) \circ F(f_1) \Rightarrow G(f_1) \circ \eta(Y)$ (whiskered by $F(f)$). Define the horizontal composition $\beta \circ^v_+ m$ as $\psi \circ^v (\beta \circ^v_+ m)_0 \circ^v \phi$, where ϕ is the vertical composition of 2-arrows $\eta(Z) \circ F(f_1 \circ f) \stackrel{\sim}{\to} \eta(Z) \circ (F(f_1) \circ F(f)) \stackrel{\sim}{\to} (\eta(Z) \circ F(f_1)) \circ F(f)$, and ψ is the vertical composition $G(g_1) \circ (G(g) \circ \theta(X)) \stackrel{\sim}{\to} (G(g_1) \circ G(g)) \circ \theta(X) \stackrel{\sim}{\to} G(g_1 \circ g) \circ \theta(X)$.

The only nontrivial property one has to check is the Eckmann-Hilton axiom (3), see (3.3.1), and its three analogues. It follows from the definitions given above, though the computation is rather tricky. One cancels ϕ and its inverse (resp., ψ and its inverse) for the r.h.s. of (3.3.1), and one uses the naturality of η and θ with respect to 2-morphisms in $\mathcal{C}$. $\square$

(3) The collection of the underlying spaces $\{\mathcal{C}(f, g)\}$, for $\mathcal{C}$ a bicategory, $f, g \in \mathcal{C}(X, Y)$, $X, Y \in \mathrm{Ob}(\mathcal{C})$, forms a 2-bimodule over $\mathcal{C}$, which we call *tautological*. Note that this 2-bimodule is not projective.

3.3.2 Proofs of Propositions 3.2.6 and 3.2.8

Of course, one can provide lengthy computational proofs of Propositions 3.2.6 and 3.2.8, checking that the generators given in Subsection 3.2.6 satisfy all the relations written down in Section 3.6. We give an alternative proof, which briefly can be explained as follows. In the cartesian-monoidal case, one defines the nerve of a strict 2-category by (1.4.6). It is clear that it gives rise to a 2-cellular object. For the $\Bbbk$-linear case, we define a 2-cellular bar-complex $\{\mathrm{Bar}(\mathcal{C})_T\}_{T \in Ob(\Theta_2)}$ and pursue its analogy with the components of the nerve to show that the assignment

$$T \rightsquigarrow \mathrm{Bar}(\mathcal{C})_T$$

gives rise to a 2-cellular object in 2-Bimod($\mathcal{C}$). Next, our complexes $\hat{A}(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T$ are equal to:

$$\hat{A}(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T = \underline{\mathrm{Hom}}_{\text{2-Bimod}(\mathcal{C})}(\mathrm{Bar}(\mathcal{C})_T, M(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)) \tag{3.3.5}$$

where the 2-bimodule $M(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)$ is defined in (3.3.3). Then the fact that the assignment $T \rightsquigarrow \mathrm{Bar}(\mathcal{C})_T$ is 2-cellular implies that the complexes $\hat{A}(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T$ form a 2-cocellular object in Ch($\Bbbk$).
For simplicity, we start with the case of a strict 2-category $\mathcal{C}$. After that, we discuss how to adopt the proof for the strict case to the case of (non-strict) bicategories.

Let $\mathcal{C}$ be a strict 2-category, $T = ([k];[n_1],\dots,[n_k]) \in Ob(\Theta_2)$, define the component $\mathrm{Bar}(\mathcal{C})_T$, as a 2-bimodule over $\mathcal{C}$.

Let $\omega_2(\bar{T}^*)$ be the free strict 2-category generated by the globular diagram $\bar{T}^*$ associated to T, see Subsection 1.4.1. A functor of the underlying 1-categories $t\colon \omega_2(\bar{T}^*)_1 \to \mathcal{C}$ is given by $k+1$ objects $X_0,\dots,X_k$ and by column-wise ordered sets of 1-morphisms $f_{i,j} \in \mathcal{C}(X_{i-1},X_i)$, $i = 1,\dots,k$. Denote

$$f_{\circ\min} = f_{k,0}\circ\cdots\circ f_{2,0}\circ f_{1,0},\quad f_{\circ\max} = f_{k,n_k}\circ\cdots\circ f_{2,n_2}\circ f_{1,n_1} \tag{3.3.6}$$

For t as above, denote

$$t^{\otimes} = \bigotimes_{i=1\dots k}\ \bigotimes_{j=1\dots n_i} \mathcal{C}(f_{i,j-1},f_{i,j}) \tag{3.3.7}$$

Let $f,g \in \mathcal{C}(X,Y)$. Define

$$\mathrm{Bar}(C)_T(f,g) = \Big[\bigoplus_{\substack{t\colon \omega_2(\bar{T}^*)_1\to C_1\\ \alpha_-,\alpha_+\in C(X,X_0)\\ \beta_-,\beta_+\in C(X_n,Y)}} C(\beta_+\circ f_{\circ\max}\circ\alpha_+, g)\otimes_{\Bbbk}\big(C(\beta_-,\beta_+)\otimes_k t^{\otimes}\otimes_{\Bbbk} C(\alpha_-,\alpha_+)\big)\otimes_{\Bbbk} C(f,\beta_-\circ f_{\circ\min}\circ\alpha_-)\Big]/\sim \tag{3.3.8}$$

where the quotient is taken by the relations generated by the Eckmann-Hilton axioms (3) for 2-bimodules. For each $T \in Ob(\Theta_2)$, $\mathrm{Bar}(\mathcal{C})_T$ is a 2-bimodule over $\mathcal{C}$.

Lemma 3.3.2. *Let $T \in Ob(\Theta_2)$, then*

$$\underline{\mathrm{Hom}}_{\Theta_2}(\mathrm{Bar}(\mathcal{C})_T, M(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)) = \hat{A}(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T$$

It is clear. □

Lemma 3.3.3. *Assume $\mathcal{C}$ is a strict $\Bbbk$-linear 2-category. Then the assignment $T \rightsquigarrow \mathrm{Bar}(\mathcal{C})_T$ gives rise to a functor $\mathrm{Bar}(\mathcal{C})\colon \Theta_2^{\mathrm{op}} \to$ 2-Bimod($\mathcal{C}$).*

Proof. The summands of (3.3.8) with given values $\{f_{i,j}\}$ for all i,j, but varying α,β, are given by a $\Bbbk$-linear 2-functor $t\colon \omega_2(\bar{T}^*) \to \mathcal{C}$ (not only of their 1-skeletons). There is a clear (contravariant) functoriality with respect to *dominant* maps $\omega_2(\bar{S}^*) \to \omega_2(\bar{T}^*)$, by the pre-composition. One has to show that it extends to non-dominant maps, by action on $a \in \mathcal{C}(\alpha_-,\alpha_+)$, $b \in \mathcal{C}(\beta_-,\beta_+)$, $\kappa_- \in \mathcal{C}(f,\beta_-\circ f_{\circ\min}\circ\alpha_-)$, $\kappa_+ \in \mathcal{C}(\beta_+\circ f_{\circ\max}\circ\alpha_+, g)$. On the other hand, Lemma 3.3.2 shows then the functoriality of $\hat{A}(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T$ with respect to maps of Θ_2. Note that for the case of a strict 2-category $\mathcal{C}$, one has $\hat{A}(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T = A(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T$, because the identities (3.2.6)-(3.2.9) hold automatically. □

This completes the proofs of Propositions 3.2.6 and 3.2.8 for the case when $\mathcal{C}, \mathcal{D}$ are strict 2-categories.

Now we turn to the general case of a bicategory $\mathcal{C}$.
We begin by introducing an extension $\hat{\Theta}_2$ of the category Θ_2, as follows. Consider the free bicategory $\hat{\omega}_2(D)$ generated by a 2-globular diagram D (one can write down an explicit formula for it, analogous to (1.4.4), weighted with coefficients $\mathcal{O}(T)$ where $\mathcal{O}$ is the 2-operad of bicategories). Then the analogue of Proposition 1.4.11 for $\hat{\omega}_2$ gives rise to an ordinary category with the same objects as Θ_2, we denote it by $\hat{\Theta}_2$. More precisely, the objects of $\hat{\Theta}_2$ are the same as the objects of Θ_2, and the morphisms $\hat{\Theta}_2(S,T)$ are defined as strict functors of bicategories $Bicat(\hat{\omega}_2(\bar{S}^*), \hat{\omega}_2(\bar{T}^*))$. There is a projection $\hat{\Theta}_2 \to \Theta_2$. Denote by p the dual projection $p\colon \hat{\Theta}_2^{\mathrm{op}} \to \Theta_2^{\mathrm{op}}$.

When $\mathcal{C}$ is a bicategory, define the 2-bimodule $\mathrm{Bar}(\mathcal{C})_T$ as above (one has to fix some order in the compositions $f_{\circ\min}$, $f_{\circ\max}$, $\beta_-\circ f_{\circ\min}\circ\alpha_-$ and $\beta_+\circ f_{\circ\max}\circ\alpha_+$). It gives rise to a functor from $\hat{\Theta}_2^{\mathrm{op}}$ to 2-Bimod($\mathcal{C}$). Then

$$\underline{\mathrm{Hom}}_{\text{2-Bimod}(\mathcal{C})}(\mathrm{Bar}(\mathcal{C})_T, M(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)) = \hat{A}(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T$$

By the same argument as above, we see that the assignment $T \rightsquigarrow \hat{A}(\dots)_T$ gives rise to a functor $\hat{\Theta}_2 \to \mathrm{Ch}(\Bbbk)$.

Now denote by $\overline{\mathrm{Bar}}(\mathcal{C})_T$ the quotient-space of $\mathrm{Bar}(\mathcal{C})_T$ by the relations (3.2.6)-(3.2.9). Note that $\overline{\mathrm{Bar}}(\mathcal{C})_T$ is a $\mathcal{C}$-bimodule as well. We are going to show that the assignment

$$T \rightsquigarrow \overline{\mathrm{Bar}}(\mathcal{C})_T$$

(which is a priori a functor $\hat{\Theta}_2^{\mathrm{op}} \to$ 2-Bimod($\mathcal{C}$)) descends to a functor $\Theta_2^{\mathrm{op}} \to$ 2-Bimod($\mathcal{C}$). To this end, we firstly detect the different elements in $\hat{\Theta}_2(S,T)$ which lie over the same morphism in $\Theta_2(S,T)$. The generator 2-morphisms of $\hat{\omega}_2(\bar{S}^*)$ are f_{ij} (in the notations we used above for $\hat{\omega}_2(\bar{S}^*)$), the associativity 2-morphisms, and the unit 2-morphisms. For a strict $\Bbbk$-linear 2-functor of bicategories $F\colon \hat{\omega}_2(\bar{S}^*) \to \hat{\omega}_2(\bar{T}^*)$), the images of the associativity and the unit 2-morphisms are uniquely defined by the restriction of F to the 1-skeleton of the source bicategory. That is, if F on the 1-skeleton is already defined, its extension to the source bicategory is uniquely determined by the images of f_{ij}. If we consider two neighbour 2-morphisms $f_{i,j-1}$ and $f_{i,j}$, such that $f_{i,j-1}\colon X_{i,j-1} \Rightarrow X_{i,j}$ and $f_{i,j}\colon X_{i,j} \Rightarrow X_{i,j+1}$, the different parenthesizings of $F(X_{i,j})$ project to the same functor of the strict 2-categories, under the map $p\colon \hat{\Theta}_2 \to \Theta_2$. In this way, the associativity 2-morphism may "jump" from $F(f_{i,j-1})$ to $F(f_{i,j})$. It gives a different strict functor of bicategories which project to the same morphism of strict 2-categories. Similarly, one has the analogous phenomenon for the unit maps. It is clear it provides a description of the fiber Υ_G in the bicategory functors $\hat{\omega}_2(\bar{S}^*) \to \hat{\omega}_2(\bar{T}^*)$ which project to a given functor of strict 2-categories $G\colon \hat{\omega}_2(\bar{S}^*) \to \hat{\omega}_2(\bar{T}^*)$. On the other hand, the relations (3.2.6)-(3.2.9) are designed especially to identify the application of the different elements from the fiber Υ_G over G, as it follows from the description of Υ_G presented above. It follows that the assignment $T \rightsquigarrow \mathrm{Bar}(\mathcal{C})$ gives rise to a functor $\Theta_2^{\mathrm{op}} \to$ 2-Bimod($\mathcal{C}$). By the speculation above, we have:

$$\underline{\mathrm{Hom}}_{\text{2-Bimod}(\mathcal{C})}(\overline{\mathrm{Bar}}(\mathcal{C})_T, M(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)) = A(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T \tag{3.3.9}$$

and so the right-hand side gives rise to a functor $\Theta_2 \to \mathrm{Ch}(\Bbbk)$. It completes the proof. $\square$

Remark 3.3.4. Alternatively, one can show that the left Kan extension $\mathrm{Lan}_p(\mathrm{Bar}(\mathcal{C})_-)(T)$ is a functor $\Theta_2^{\mathrm{op}} \to \mathrm{Ch}(\Bbbk)$, which is the quotient of $\mathrm{Bar}(\mathcal{C})_T$ by the relations analogous to (3.2.6)-(3.2.9), so that

$$\mathrm{Hom}_{\text{2-Bimod}(\mathcal{C})}(\mathrm{Lan}_p(\mathrm{Bar}(\mathcal{C}))(T), M(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)) = A(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T \quad (3.3.10)$$

and thus the r.h.s. of (3.3.10) is automatically a functor $\Theta_2 \to \mathrm{Ch}(\Bbbk)$.

Remark 3.3.5. One can show that the projection $\hat{\Theta}_2 \to \Theta_2$ is cofibration but not a fibration, thus p is a fibration but not a cofibration. Therefore, one can *not* compute the left Kan extensions along p as the colimit over the fiber-category $p^{-1}(T)$ (thus only the formula as the colimit over the comma-category $p\backslash T$ is applied).

3.3.3 An intrinsic definition of $\hat{A}(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)$

Here we prove that $\mathrm{Bar}(\mathcal{C})$ is a projective resolution of $\mathcal{C}$ in the category of 2-bimodules, for any $\Bbbk$-linear bicategory $\mathcal{C}$.

Here we assume that $\mathcal{C},\mathcal{D}$ are strict 2-categories (though a similar result holds for the general case of bicategories, and we are going to provide details elsewhere).

We prove

Proposition 3.3.6. *Let $\mathcal{C},\mathcal{D}$ be $\Bbbk$-linear bicategories. One has:*

$$\mathrm{Tot}_{\Theta_2}(\hat{A}(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)) = \mathrm{RHom}^\bullet_{\text{2-Bimod}(\mathcal{C})}(\mathcal{C}, M(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta))$$

Proof. First of all, the 2-bimodules $\mathrm{Bar}(\mathcal{C})_T$ are projective. Indeed, consider more generally the following 2-bimodule M_V over $\mathcal{C}$, where $Z_0,Z_1 \in Ob(\mathcal{C})$, $f',g' \in \mathcal{C}(Z_0,Z_1)$, $V \subset \mathcal{C}(f',g')$. Then define

$$M_V(f,g) = \Big[\bigoplus_{\substack{\alpha_-,\alpha_+\in C(X,Z_0)\\ \beta_-,\beta_+\in\mathcal{C}(Z_1,Y)}} \mathcal{C}(\beta_+\circ g'\circ\alpha_+, g)\otimes_\Bbbk \big(\mathcal{C}(\beta_-,\beta_+)\otimes_\Bbbk V\otimes_\Bbbk \mathcal{C}(\alpha_-,\alpha_+)\big)\otimes_\Bbbk \mathcal{C}(f,\beta_-\circ f'\circ\alpha_-)\Big]/\sim$$

where $f,g \in C(X,Y)$, and the quotient is taken by the relations given by the Eckmann-Hilton axioms of 2-bimodules (3).

Let us show that the 2-bimodule M_V is projective. Indeed, for any 2-bimodule N over $\mathcal{C}$ one has:

$$\mathrm{Hom}_{\text{2-Bimod}(\mathcal{C})}(M_V,N) = \mathrm{Hom}_\Bbbk(V, N(f',g'))$$

(because the relations (3) hold for N), from which the projectivity of M_V follows. Note that $\mathrm{Bar}(\mathcal{C})_T$ is a direct sum of 2-bimodules of the type M_V, and, thus, is projective.

It remains to show that the complex $\mathrm{Tot}_{T\in\Theta_2}\mathrm{Bar}(\mathcal{C})_T$ is a resolution of the tautological 2-bimodule $\mathcal{C}$. We start with the case when $\mathcal{C}$ is a strict 2-category. Define a homotopy operator, which sends $\mathrm{Bar}(\mathcal{C})_T$ to $\mathrm{Bar}(\mathcal{C})_{T'}$, $T = ([k];[n_1],\dots,[n_k])$, $T' = ([k+1];[0],[n_1],\dots,[n_k])$, plugging Id_{X_0} in place of $[0]$.
More precisely, define $h\colon \mathrm{Bar}(\mathcal{C})_T(f,g) \to \mathrm{Bar}(\mathcal{C})_{T'}(f,g)$, which sends

$$\kappa_+ \otimes_\Bbbk (b\otimes_\Bbbk t^\otimes \otimes_\Bbbk a)\otimes_\Bbbk \kappa_+$$

where (see (3.3.8)) $a \in \mathcal{C}(\alpha_-, \alpha_+)$, $b \in \mathcal{C}(\beta_-, \beta_+)$, $\kappa_- \in \mathcal{C}(f, \beta_- \circ f_{\circ\min} \circ \alpha_-)$, $\kappa_+ \in \mathcal{C}(\beta_+ \circ f_{\circ\max} \circ \alpha_+, g)$ to

$$\tilde{\kappa}_+ \otimes_{\Bbbk} (\tilde{b} \otimes_k \tilde{t}^{\otimes} \otimes_{\Bbbk} \tilde{a}) \otimes_{\Bbbk} \tilde{\kappa}_- \tag{3.3.11}$$

where $\tilde{t}^{\otimes}$ is corresponded to T', and is equal to $t^{\otimes} \otimes_{\Bbbk} \alpha_-$, $\tilde{\kappa}_+ = (\kappa_+ \circ^v (\mathrm{Id} \circ^h a)) \circ^h (\mathrm{id}_{\mathrm{Id}_X})$, $\tilde{a} = \mathrm{id}_{\mathrm{Id}_X}$, $\tilde{b} = b$, and $\tilde{a} \in C(\tilde{\alpha}_-, \tilde{\alpha}_+)$, with both $\tilde{\alpha}_-, \tilde{\alpha}_+$ equal to Id_x. So the main idea is that we "pack" the 2-morphism a inside $\tilde{\kappa}_+$, as the vertical composition with a whiskered by the identity morphism. Indeed, the new (leftmost) column of T' is of height 0, so one can not consider the (height 1) 2-morphism a as placed in this column.

This formula works for the case of strict 2-categories only, as in the bicategory case one has to "compensate" the extensions by the identity morphisms by the compositions with the corresponding 2-morphisms.

We check that h is indeed a homotopy, that is

$$dh + hd = \mathrm{id} \tag{3.3.12}$$

where d is the differential in the realization of the functor $T \rightsquigarrow (\mathrm{Bar}(\mathcal{C}))_T$. Note that the leftmost extreme coface map of $h(\kappa_+ \otimes_{\Bbbk} (b \otimes_{\Bbbk} \otimes t^{\otimes} \otimes_{\Bbbk} a) \otimes_{\Bbbk} \kappa_+)$ is $\kappa_+ \otimes_{\Bbbk} (b \otimes_{\Bbbk} \otimes t^{\otimes} \otimes_{\Bbbk} a) \otimes_{\Bbbk} \kappa_+$ again. It follows from the Eckmann-Hilton axioms (3) for 2-bimodules (in our definition (3.3.8), the quotients by the Eckmann-Hilton relations are taken). The rest of the proof of (3.3.12) is straightforward.

Turn back to the general case of bicategories.
First of all, we have to chose parenthesizings of $\beta_- \circ f_{\circ\min} \circ \alpha_-$ and $\beta_+ \circ f_{\circ\max} \circ \alpha_+$. We parenthesize the compositions from the left to the right, so that α_- and α_+ have the highest depth (with respect to the parenthesizings). Similarly to the strict case, define a homotopy operator, which sends $\mathrm{Bar}(\mathcal{C})_T$ to $\mathrm{Bar}(\mathcal{C})_{T'}$, $T = ([k]; [n_1], \dots, [n_k])$, $T' = ([k+1]; [0], [n_1], \dots, [n_k])$, plugging Id_{X_0} in place of $[0]$. The only difference is in definitions of $\tilde{\kappa}_-$ and $\tilde{\kappa}_+$. One has $\tilde{t} = t^{\otimes} \otimes_{\Bbbk} \alpha_-$, $\tilde{\kappa}_- = (\mathrm{Id} \circ \rho_{\alpha_-}^{-1}) \circ \kappa_-$, $\kappa_+ = \kappa_+ \circ (\mathrm{Id} \circ (\rho_{\alpha_+} \circ (a \circ \mathrm{Id})))$, $\tilde{a} = \mathrm{id}_{\mathrm{Id}_X}$, $b = b$. That is, we insert the unit maps $\rho_{\alpha_-}^{-1}$ and ρ_{α_+}. After applying the extreme boundary map, and applying the Eckmann-Hilton axiom for 2-bimodules, we obtain the following chain

$$\kappa'_+ \otimes_{\Bbbk} (b' \otimes_{\Bbbk} t'^{\otimes} \otimes_{\Bbbk} a') \otimes_{\Bbbk} \kappa'_-$$

with $a' = \rho \circ^v (a \circ^h \mathrm{Id}) \circ^v \rho^{-1}$, $b' = b$, $t'^{\otimes} = t^{\otimes}$, $\kappa'_- = \kappa_-$, $\kappa'_+ = \kappa_+$.
By the following bicategory axiom

$$\rho_g \circ^v (a \circ^h \mathrm{id}) = a \circ^v \rho_f,$$

that holds for any 2-morphism $a \colon f \Rightarrow g$, we have $a' = a$. From here one deduces that $[d, h] = \mathrm{id}$. □

3.4 The relative totalization $(\mathrm{Rp}_*)(X_\bullet)$

There are several ways to get a cosimplicial vector space out of a 2-cocellular vector space $X_\bullet$; for instance, one can firstly restrict $X_\bullet$ to $\Delta \times \Delta$ followed by the totalization by the "vertical" factor Δ. It appears, however, that only the "vertical" totalization

$(\mathrm{Rp}_*)(X_\bullet)$ enjoys, for the case $X_\bullet = A(F,F)$, the property of being a 1-commutative cosimplicial monoid, in the sense of [BD]. However, for the case $X_\bullet = A(\mathrm{Id},\mathrm{Id})$, the cosimplicial monoid $(\mathrm{Rp}_*)(X_\bullet)$ is not 2-commutative (unlike for the case of the Davydov-Yetter complex), but is in a sense homotopy 2-commutative. We start with an explicit description of $(\mathrm{Rp}_*)(X_\bullet)$.

Remark 3.4.1. The notation $(\mathrm{Rp}_*)(X_\bullet)$ might be misleading, but this relative totalization is *not* computed via an homotopy right Kan extension, as our object $A(F,F)$ is *not* Reedy fibrant in the Reedy model category of 2-cocellular complexes. This fact is totally analogous to the Hochschild cosimplicial complex *not* being Reedy fibrant in the Reedy model category of cosimplicial complexes.

3.4.1 An explicit description of $(\mathrm{Rp}_*)(X_\bullet)$

Let $X_\bullet$ be a 2-cocellular complex.

We construct $(\mathrm{Rp}_*)(X_\bullet)$ by first resolving the functor $T \mapsto \Bbbk\Delta([n],p(T))$ (for a given $[n]\in\Delta$) by Yoneda functors $h_{T'}(T) = \Bbbk\Theta_2(T',T)$. Below we provide an explicit resolution $\mathfrak{R}^\bullet_{[n]}$ (which is a complex of vector spaces over $\Bbbk$). The degree ℓ component is

$$\mathfrak{R}^\ell_{[n]} = \sum_{\substack{T'\in\Theta_2,\ p(T')=[n]\\ \dim(T')=n-\ell}} \Bbbk\Theta_2(T',T) \tag{3.4.1}$$

Thus, the complex $\mathfrak{R}^\bullet_{[n]}$ has non-zero components in degrees ≤ 0. The differential $d\colon \mathfrak{R}^\ell_{[n]} \to \mathfrak{R}^{\ell+1}_{[n]}$ is defined as the alternated sum of the "vertical" coface operators (acting on the first argument T'), that is, of coface operators $(F1)$ and $(F3)$ from the list in Subsection 3.2.1. More precisely, for $T = ([q];[t_1],\dots,[t_q])$, $T' = ([n];[k_1],\dots,[k_n])$, $\Phi = (\phi;\phi_1,\dots,\phi_n)\colon T'\to T$, one has

$$d(\Phi) = \sum_{s=1}^{n}\sum_{i=0}^{k_s}(-1)^{k_1+\dots+k_{s-1}+s-1+i}\Phi_{s,i} \tag{3.4.2}$$

where $\Phi_{s,i}\colon T'_{s,i}\to T$ is defined as the pre-composition $\Phi\circ\partial^i_s$ (see Subsection 3.2.1, $(F1)$ and $(F3)$), and $T'_{s,i} = ([n];[m_1],\dots,[m_n])$, where $m_j = k_j$ for $j\ne s, m_s = k_s-1$, and $\partial^i_s\colon T'_{s,i}\to T'$ is the corresponding "vertical" coface operator. Note that this pre-composition does not affect the "horizontal" map ϕ. It is clear that $d^2=0$.

Lemma 3.4.2. *The following statements are true:*

(1) degree 0 *cohomology of* $\mathfrak{R}^\bullet_{[n]}$ *is isomorphic to* $\Bbbk\Delta([n],p(T))$*,*

(2) the higher cohomology (in the negative degrees ≤ -1*) vanish.*

Proof. (1): the degree 0 component $\mathfrak{R}^0_{[n]}$ is a direct sum $\oplus\Bbbk\Phi$, where $\Phi\colon ([n];[0],\dots,[0])\to T$, which is the same as $\Phi = (\phi\colon [n]\to[q]; S_1,\dots,S_n)$ where $S_i\in[t_{\phi(i-1)+1}]\times\dots\times[t_{\phi(i)}]$ an element (recall that $[q]=p(D)$). Degree 0 cohomology is equal to the quotient-space by the image of $\oplus\Bbbk\Phi'$, with $\Phi'\colon ([n];[0],\dots,[0],[1],[0],\dots,[0])\to T$. It is clear that, for a given $\phi\colon [n]\to[q]$, all choices of S_i become equal in the quotient-space $H^0(\mathfrak{R}^\bullet_{[n]}) = \mathfrak{R}^0_{[n]}/d(\mathfrak{R}^{-1}_{[n]})$. It shows that $H^0(\mathfrak{R}^\bullet_{[n]})\simeq \Bbbk\Delta([n],[q]) = \Bbbk\Delta([n],p(T))$.

(2): we construct a contracting homotopy operator H of degree -1, that is an operator such that $(dH+Hd)|_{\mathfrak{R}^\ell_{[n]}} = c_\ell$, where c_ℓ is the multiplication by an integer c_ℓ, non-zero for $\ell\ne 0$. This H is constructed in a standard way as the alternated sum of the "vertical" codegeneracy operators. □

Remark 3.4.3. It is clear that the complex $\mathfrak{R}^\bullet_{[n]}$ is a direct sum $\oplus_\phi \mathfrak{R}^\bullet_{[n],\phi}$ over $\phi\colon [n]\to p(T)$, because the differential does not affect ϕ. Each complex $\mathfrak{R}^\bullet_{[n],\phi}$ is a resolution of $\Bbbk$ (where $\Bbbk$ denotes the complex-object $\Bbbk$ in degree 0).

It is clear that $\mathfrak{R}^\bullet_{[n]}$ is a functor $\Theta_2\to \mathrm{Ch}(\Bbbk)$, where the action of Θ_2 is given by the post-composition. It commutes with the differential as the general post-composition and pre-composition do.

3.4.2 The action of Δ

Our next task is to endow our resolution $\mathfrak{R}^\bullet_{[n]}$ with a structure of a functor $\Delta^{op}\to\mathrm{Ch}(\Bbbk)$, when $[n]$ varies. Note, that unlike for the cohomology $\Bbbk\Delta([n],p(T))$ of $\mathfrak{R}^\bullet_{[n]}$, the "lifted" action of Δ on $\mathfrak{R}^\bullet_{[n]}$ does not come automatically.

We need to define the actions of the elementary face operators ∂^i and the elementary degeneracy operators ε^j in Δ, which we denote, in this context, by Ω^i_Δ and Υ^j_Δ, correspondingly. Here are the definitions. Let $\Phi\colon T'\to T$ be an element in $\mathfrak{R}^\bullet_{[n]}, p(T')=[n]$.

$$\Omega^i_\Delta(\Phi) = \sum_\sigma \pm\Phi\circ D_{i,\sigma} \pm \Phi\circ D_{\min} \pm \Phi\circ D_{\max} \tag{3.4.3}$$

$$\Upsilon^i_\Delta(\Phi) = \pm\Phi\circ\Upsilon^p_0 \tag{3.4.4}$$

where $\Upsilon^p_0\colon ([n];[\ell_1],\dots,[\ell_p],[0],[\ell_{p+2}],\dots,[\ell_n])\to T$. Note that we take only T' with $[\ell_{p+1}]=[0]$. (See Subsection 3.2.1 for the notations $D_{i,\sigma}$ and Υ^p).

Proposition 3.4.4. *The following statements are true:*

1. *The operators Ω^i_Δ and Υ^j_Δ define maps of complexes $\Omega^i_\Delta\colon \mathfrak{R}^\bullet_{[n]}\to\mathfrak{R}^\bullet_{[n-1]}$ and $\Upsilon^i_\Delta\colon\mathfrak{R}^\bullet_{[n]}\to\mathfrak{R}^\bullet_{[n+1]}$, preserving the cohomological degree.*

2. *The operators Ω^i_Δ and Υ^j_Δ fulfill the simplicial relations, defining a simplicial object $\mathfrak{R}^\bullet_?\colon\Delta^{op}\to\mathrm{Ch}(\Bbbk)$, functorial in T. The cohomology $H^\bullet(\mathfrak{R}^\bullet_{[n]})$ with respect to the differential* (3.4.2)*, with its simplicial action, is isomorphic to $\Delta([n],p(T))$ with its natural simplicial action.*

Proof. See Section 3.7. □

We get for $X \in \mathrm{Ch}(\Bbbk)^{\Theta_2}$:

$$\mathrm{Rp}_*(X)[n] = \mathrm{Ch}(\Bbbk)^{\Theta_2}(\mathfrak{R}^\bullet_{[n]}(T), X(T)) \tag{3.4.5}$$

By the Yoneda lemma, it is the following complex:

$$0 \to \mathrm{Rp}_*(X)[n]^0 \xrightarrow{d} \mathrm{Rp}_*(X)[n]^1 \xrightarrow{d} \mathrm{Rp}_*(X)[n]^2 \xrightarrow{d} \dots \tag{3.4.6}$$

where

$$\mathrm{Rp}_*(X)[n]^\ell = \bigoplus_{\substack{T\in\Theta_2,\, p(T)=[n]\\ \dim(T)=n+\ell}} X(T) \tag{3.4.7}$$

and the differential d is the alternated sum of "vertical" coface operators (of type (F1) and (F3) in Subsection 3.2.1):

$$d|_{X(T)} = \sum_{i=1}^{p(T)} \sum_{j=0}^{T_i} (-1)^{T_1+\dots+T_{i-1}+j} \partial_i^j \tag{3.4.8}$$

where we write $T = (p(T); [T_1], \dots, [T_{p(T)}])$. According to Proposition 3.4.4, it is a functor $\Delta \to \mathrm{Ch}(\Bbbk)$.

3.4.3 $\mathrm{Tot}_\Delta(\mathrm{Rp}_*(X_\bullet)) \sim \mathrm{Tot}_{\Theta_2}(X_\bullet)$

Here we prove the following

Proposition 3.4.5. *Let $X_\bullet \colon \Theta_2 \to \mathrm{Ch}(\Bbbk)$ be a 2-cocellular complex. Then the Δ-totalization $\mathrm{Tot}_\Delta(\mathrm{Rp}_*(X_\bullet))$ of $\mathrm{Rp}_*(X_\bullet)$ is a quasi-isomorphic complex to the Θ_2-totalization $\mathrm{Tot}_{\Theta_2}(X_\bullet)$.*

Proof. When one applies the usual non-normalized cochain complex functor to the cosimplicial vector space $\mathrm{Rp}_*(X)$, we get exactly the formula (3.2.2) for the (non-normalized) Θ_2-totalization. □

3.4.4 The totalization $\mathrm{Tot}_{\Theta_2} A(F, F)$ is a homotopy 2-algebra

Recall a cosimplicial monoid $X_\bullet$ (in a symmetric monoidal category $\mathcal{E}$) is a cosimplicial object in the category of monoids $Mon(\mathcal{E})$. The question raised in [BD] is the following: Which condition on $X_\bullet$ implies that the totalization $\mathrm{Tot}(X)$ admits an action of an operad (homotopy equivalent to) E_n?

It follows immediately that the condition that $X_\bullet$ is a cosimplicial monoid implies that $X_\bullet$ is a monoid with respect to the Batanin $\Box$-product (see [[Ba1], Section 5], [[MS1], Section 2]). It is well-known [Ba1],[MS1], that the totalization $\mathrm{Tot}(Y_\bullet)$ of a cosimplicial $\Box$-monoid $Y_\bullet$ carries an A_∞-structure; thus, it follows that for a cosimplicial monoid $X_\bullet$, its totalization $\mathrm{Tot}(X_\bullet)$ is an A_∞-monoid, that is, a E_1-algebra. In [[BD], Section 2.2], the following definition is given:

Definition 3.4.6. Let $X_\bullet$ be a cosimplicial monoid, $n \geq 0$. $X_\bullet$ is called n-commutative if for any $\tau\colon [p] \to [m]$, $\pi\colon [q] \to [m]$ in Δ with linking number (see Definition 1.3.41) $lk(\tau, \pi) \leq n$, the diagram below commutes:

$$\begin{array}{ccccc} X_p \otimes X_q & & \xrightarrow{X(\tau)\otimes X(\pi)} & & X_m \otimes X_m \\ \downarrow & & & & \downarrow \mu \\ X_q \otimes X_p & \xrightarrow{X(\pi)\otimes X(\tau)} & X_m \otimes X_m & \xrightarrow{\mu} & X_m \end{array} \tag{3.4.9}$$

We easily prove:

Proposition 3.4.7. *Let* $\mathcal{C}$*,* $\mathcal{D}$ *be* $\Bbbk$*-linear monoidal categories,* $F\colon \mathcal{C} \to \mathcal{D}$ *a linear monoidal functor. Then the cosimplicial vector space* $\mathrm{Rp}_*(A(F,F))$ *is a* 1*-commutative cosimplicial monoid.*

Proof. Let $\tau_{m,n}\colon [n] \to [m+n]$ and $\pi_{m,n}\colon [m] \to [m+n]$ be defined as $\tau_{m,n}(i) = i$ and $\pi_{m,n}(j) = n+j$. It is clear that $lk(\tau_{m,n}, \pi_{m,n}) = 1$. Moreover, the general case of linking number 1 is reduced to this particular case, due to the following simple observation ([[BD], Lemma 2.1]): Let $\tau\colon [p] \to [m], \pi\colon [q] \to [m]$ be morphisms in Δ, and

$$[p] \to [p'] \xrightarrow{\tau'} [m], \qquad [q] \to [q'] \xrightarrow{\pi'} [m]$$

be their epi-mono factorizations. Then $lk(\tau, \pi) = lk(\tau', \pi')$.

We check the 1-commutativity of $\mathrm{Rp}_*(A(F,F))$. Let $\Phi \in \mathrm{Rp}_*(A(F,F))^n$, $\Psi \in \mathrm{Rp}_*(A(F,F))^m$ be represented by cochains $\Phi \in A(F,F)_T$, $\Psi \in A(F,F)_{T'}$, with $p(T) = [n], p(T') = [m]$. Assume $T = ([n];[k_1],\dots,[k_n])$ and $T' = ([m];[\ell_1],\dots,[\ell_m])$. Then $\tau_{m,n}(\Phi)$ takes a non-zero value on the object $\hat{T} = ([n+m];[k_1],\dots,[k_n],[0],\dots,[0])$, and is equal to

$$\tau_{m,n}(\Phi)(-, X_{n+1},\dots,X_{m+n}) = \Phi(-) \otimes (\mathrm{id}_{F(X_{n+1}\otimes\cdots\otimes X_{n+m})})$$

Analogously, $\pi_{m,n}(\Psi)$ takes a non-zero value on $\hat{T}' = ([m+n];[0],\dots,[0],[\ell_1],\dots,[\ell_m])$, and

$$\pi_{m,n}(\Psi)(Y_1,\dots,Y_m,-) = \mathrm{id}_{F(Y_1\otimes\cdots\otimes Y_m)} \otimes \Psi(-)$$

Finally, for their product in the monoid $\mathrm{Rp}_*(A(F,F))^{m+n}$, one has

$$\begin{aligned} &\tau_{m,n}(\Phi) * \pi_{m,n}(\Psi)(S_1,\dots,S_{m+n}) = \\ &\big(\Phi(S_1,\dots,S_n) \otimes \mathrm{id}_{F(X_{n+1}\otimes\cdots\otimes X_{m+n})}\big) \circ \big(\mathrm{id}_{F(Y_1\otimes\cdots\otimes Y_n)} \otimes \Psi(T_{n+1},\dots,T_{m+n}\big) = \\ &\Phi(T_1,\dots,T_n) \otimes \Psi(S_{n+1},\dots,S_{m+n}) \end{aligned}$$

where S_i is a string of morphisms of length k_i for $1 \leq i \leq n$ and ℓ_{j-n} for $j = n+1,\dots,n+m$, starting at X_i and ending at Y_i. We clearly get the same expression when computing $\pi_{m,n}(\Psi) * \tau_{m,n}(\Phi)(S_1,\dots,S_{m+n})$, and 1-commutativity for $\mathrm{Rp}_*(A(F,F))$ follows. ☐

Remark 3.4.8. The fulfillment of the 1-commutativity condition for $\mathrm{Rp}_*(A(F,F))$ is a lucky situation, which is easily generalised from the corresponding proof for the classical Davydov-Yetter complex in [[BD], Theorem 3.4]. Namely, (although our cochains are not natural transformations) one does not use the naturality of cochains for general morphisms in this proof. One does use the naturality with respect to the identity morphisms, which automatically holds.

The case of 2-commutativity of $\mathrm{Rp}_*(\mathrm{Id},\mathrm{Id})$ is not that lucky, because the corresponding proof for the classical counterpart given in [[BD], Theorem 3.8], essentially uses the naturality for non-identity morphisms. Our cochains are not natural transformations, which results in the failure of 2-commutativity for $\mathrm{Rp}_*(A(F,F))$. However, our complex enjoys some sort of "homotopy 2-commutativity", though this concept is hard to phrase out. We will describe a possible procedure to do so in the next Chapter 4.

Theorem 3.4.9. *Let $\mathcal{C}$, $\mathcal{D}$ be $\Bbbk$-linear monoidal categories, $F\colon \mathcal{C}\to\mathcal{D}$ a linear monoidal functor. Then the 2-cocellular totalization $\mathrm{Tot}_{\Theta_2}(A(F,F))$ has a structure of an algebra over an operad homotopically equivalent to $\mathrm{Ch}_\bullet(\mathrm{E}_2;\Bbbk)$.*

Proof. By Proposition 3.4.5, $\mathrm{Tot}_{\Theta_2}(A(F,F)) \simeq \mathrm{Tot}_\Delta(\mathrm{Rp}_*(A(F,F))$. By Proposition 3.4.7, $\mathrm{Rp}_*(A(F,F))$ is a 1-commutative cosimplicial monoid. Then the result follows from Theorem 3.1.18. □

3.4.5 Normalized vs non-normalized chain complexes of a 2-cellular object in $\mathrm{Ch}(\Bbbk)$

In Section 3.5, we use that the Θ_2-cochain complex of $A(C,D)(F,G)(\eta,\theta)$ is quasi-isomorphic to its *normalized* subcomplex $A_{\mathrm{norm}}(C,D)(F,G)(\eta,\theta)$. The latter is, by definition, the sub-complex which consists of all cochains Ψ which are equal to 0 if some of its arguments $f_{i,j}$ is the identity morphism of some object.

Recall that for a simplicial object in an abelian category $\mathscr{A}$ its normalized Moore complex $N(X)$ is defined as the quotient-complex of the ordinary Moore complex $C(X)$ by the subcomplex $DC(X)$ spanned by elements of the form $s_i y$ (here s_i stands for the simplicial version of the degeneracy morphisms $\varepsilon_i\in\Delta$, see Section 1.4).

Recall the following classical result, in a slightly more general version:

Proposition 3.4.10. *Let $X\colon \Delta^{\mathrm{op}}\to \mathrm{Ch}(\Bbbk)$ be a simplicial object in $\mathrm{Ch}(\Bbbk)$. Then the total sum complex $\mathrm{Tot}^{\oplus}(C(X))$ of the Moore complex of X is quasi-isomorphic to the total sum complex $\mathrm{Tot}^{\oplus}(N(X))$ of the normalized Moore complex.*

Proof. The proof given in [[ML1], Section VIII.6] can be easily adopted to this case. Indeed, Mac Lane constructs a map $g\colon C(Y)/DC(Y)\to C(Y)$, for Y a simplicial object in an abelian category, such that g is a "quasi-inverse" to the natural projection $\pi\colon C(Y)\to C(Y)/DC(Y)$ in the sense that $\pi\circ g=\mathrm{id}$, and $g\circ\pi$ is chain homotopic to the identity. The chain homotopy constructed in loc. cit. clearly commutes with "inner" differentials on X_is. Consequently, if one defines $\pi'\colon \mathrm{Tot}^{\oplus}(C(X))\to\mathrm{Tot}^{\oplus}(N(X))$ and $g'\colon \mathrm{Tot}^{\oplus}(N(X))\to\mathrm{Tot}^{\oplus}(C(X))$ one still has $\pi' g'=\mathrm{id}$ and $g'\pi'$ chain homotopic to the identity. □

The next step is to generalise Proposition 3.4.10 to the case of 2-cellular objects in $\mathrm{Ch}(\Bbbk)$, that is, to the case of functors $X\colon \Theta_2^{\mathrm{op}}\to\mathrm{Ch}(\Bbbk)$.

For $Y\colon \Theta_2^{\mathrm{op}} \to \mathrm{Vect}(\Bbbk)$, its chain complex is defined as the complex $C(Y)$, with

$$C_{-\ell}(Y) = \bigoplus_{T,\ \dim T=\ell} Y_T$$

with the differential dual to (3.2.3), and its normalized complex is defined as the quotient-complex of $C(Y)$ by the subcomplex $DC(Y)$ generated by the elements $\varepsilon_p^j(y)$ of type (D1) (see Subsection 3.2.1), $y \in Y_D$:

$$N(Y) = C(Y)/DC(Y)$$

That is, we use only "vertical" degeneracy morphisms of type (D1), *not* "horizontal" degeneracy morphisms of type (D2), in the definition of $DC(Y)$.

For the case of a functor $X\colon \Theta_2^{\mathrm{op}} \to \mathrm{Ch}(\Bbbk)$ as above, $C(X)$, $DC(X)$, $N(X)$ are defined as $\mathrm{Tot}^{\oplus}(C(X))$, $\mathrm{Tot}^{\oplus}(DC(X))$, $\mathrm{Tot}^{\oplus}(N(X))$, correspondingly.

Proposition 3.4.11. *Let $X\colon \Theta_2^{\mathrm{op}} \to \mathrm{Ch}(\Bbbk)$ be a 2-cellular complex. Then the natural projection $\pi\colon \mathrm{Tot}^{\oplus}(C(X)) \to \mathrm{Tot}^{\oplus}(N(X))$ is a quasi-isomorphism of complexes.*

Proof. One can not follow directly the same line as in the proof of [[ML1], Theorem VIII.6.1] by the following reason. The subspaces $D_iC(X)$, $i \geq 0$ (or rather their direct analogues) are *not* subcomplexes of $C(X)$, because the components $D_{j,\sigma}$ of type (F2) (see Subsection 3.2.1) in the differential (3.2.3) may *increase* i. Indeed, these components act as "deshuffling" of two neighbour columns, resulting in a column of a greater length, so this operation may send $\varepsilon_p^i y$ to $\varepsilon_q^{i'}(y')$ with $i' > i$ (here $q = p$ or $p-1$).

To overcome this obstacle, we employ the following spectral sequence argument.

Denote by $F_N \subset C(X)$ the subspace spanned by X_T, $T = ([n]; [\ell_1], \dots, [\ell_n])$ with $n \leq N$. Then F_N is a subcomplex: the boundary operators of type (F1) and (F3) preserve n, and the boundary operators of types (F2) and (F4) decrease n by 1, see Subsection 3.2.1.

We get an exhausting ascending filtration of $C(X)$ by subcomplexes:

$$F_0 \subset F_1 \subset F_2 \subset \dots$$

A similar filtration exists for $N(X)$ as well, denote the corresponding subspaces by F'_N. The natural projection $\pi\colon C(X) \to N(X)$ sends F_N to F'_N, hence π induces a map of the corresponding spectral sequences. Denote these spectral sequences by $\{E_n^{pq}\}$ and $\{E_n'^{pq}\}$, so that π induces a map $\pi_*\colon (E_n^{pq}, d_n) \to (E_n'^{pq}, d'_n)$.

The spectral sequences at the term E_0 (resp., E'_0) are non-zero at the lower half plane $y \leq 0$, the differential d_0 is horizontal. So the spectral sequences converge by dimensional reasons.

Lemma 3.4.12. *The map $\pi_*\colon (E_0^{\bullet,\ell}, d_0) \to (E_0'^{\bullet,\ell}, d'_0)$ is a quasi-isomorphism, for any $\ell \leq 0$. In particular, π_* defines an isomorphism $\pi_*\colon E_1^{pq} \to E_1'^{pq}$, for all p, q.*

Proof. For any fixed ℓ, the complex $(E_0^{\bullet,\ell}, d_0)$ is $C^{(\ell)}(X)$, whose degree $-n$ component is equal to the direct sum $\oplus_T X_T$ over $T = ([\ell]; [n_1], \dots, [n_\ell])$ with $\dim T = n$, and with the differential components given only by (F1) and (F3) types, see Subsection 3.2.1. That is, the contribution of types (F2) and (F4) components in (3.2.3) becomes 0 in the associated graded complex $C^{(\ell}(X) = F_\ell/F_{\ell-1}$. The complex $(E_0'^{\bullet,\ell}, d_0')$ has a similar description.

It makes us possible to employ the construction of the proof of [[ML1], Theorem VIII.6.1.]. Namely we define $D_iC^{(\ell)}(X)$, for any $i \geq 0$, such that $D_{i+1}C^{(\ell)}(X) \supset D_iC^{(\ell)}(X)$ and $DC^{(\ell)}(X) = \cup_{i\geq 0} D_iC^{(\ell)}(X)$. As in loc.cit., we construct a map $h_i\colon C^{(\ell)}(X) \to C^{(\ell)}(X)$ chain homotopic to id and mapping D_i to D_{i-1}. The composition of these maps is well-defined, is chain homotopic to id, and sends $DC^{(\ell)}(X)$ to 0. It gives a map $g\colon N^{(\ell)}(X) \to C^{(\ell)}(X)$ such that $\pi_* g = \mathrm{id}$ and $g\pi_*$ is chain homotopic to id, which completes the proof. □

It follows from this Lemma that π_* defines an isomorphism at E_∞ sheet, hence π is a quasi-isomorphism. □

Now we can apply this general result to our setting, with notations as in Subsection 3.2.7:

Theorem 3.4.13. *The natural embedding*

$$i\colon A_{\mathrm{norm}}(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta) \to A(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)$$

is a quasi-isomorphism of complexes.

Proof. By (3.3.9) one has

$$\mathrm{Hom}_{\text{2-Bimod}(\mathcal{C})}(\overline{\mathrm{Bar}}(\mathcal{C})_T, M(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)) = A(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T \tag{3.4.10}$$

Similarly,

$$\mathrm{Hom}_{\text{2-Bimod}(\mathcal{C})}(N(\overline{\mathrm{Bar}}(\mathcal{C}))_T, M(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)) = A_{\mathrm{norm}}(\mathcal{C},\mathcal{D})(F,G)(\eta,\theta)_T \tag{3.4.11}$$

Both $C(\overline{\mathrm{Bar}}(\mathcal{C}))$ and $N(\overline{\mathrm{Bar}}(\mathcal{C}))$ are projective *resolutions* of the tautological 2-bimodule $\mathcal{C}$: it is proven for $C(\overline{\mathrm{Bar}}(\mathcal{C}))$ in Proposition 3.3.6, and then it follows for $N(\overline{\mathrm{Bar}}(\mathcal{C}))$ from Proposition 3.4.11 (the projectivity of the components $N(\overline{\mathrm{Bar}})(\mathcal{C})_T$ is proven analogously to the proof for projectivity of the components $C(\overline{\mathrm{Bar}}(\mathcal{C}))_T$ in Proposition 3.3.6). Now the statement follows from the standard homological algebra. □

3.5 The totalizations $\mathrm{Tot}_{\Theta_2}(A(F,F))$ and $\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id},\mathrm{Id}))$ as the deformation complexes

Let $\mathcal{C}$ be a $\Bbbk$-linear monoidal category (or a monoidal dg-category over $\Bbbk$). The deformations we consider are formal deformations, that is, $\mathcal{C}_t$ may not make sense unless $t = 0$. The reader should keep it in mind in the discussion below. We consider deformations $\mathcal{C}_t$ of $\mathcal{C}$ for which the set of objects, the vector spaces (complexes) $\mathcal{C}_t(x,y)$ of morphisms, and the monoidal product on objects, remain undeformed. Then the data which is deformed is:

(A1) the composition of morphisms (which is required to be associative), (A2) the monoidal products of morphisms $\mathrm{id}_X \otimes g$ and $f \otimes \mathrm{id}_Y$ (note that $f \otimes g = (f \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes g) = (\mathrm{id}_X \otimes g) \circ (f \otimes \mathrm{id}_Y)$, therefore, the deformations of $f \otimes g$ are determined by deformations of $f \otimes \mathrm{id}_Y$ and $\mathrm{id}_X \otimes g$),

(A3) the associator $\alpha\colon X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$.

(A4) the left and right unit maps $\lambda_X\colon e \otimes X \to X$ and $\rho_X\colon X \otimes e \to X$.

It is assumed that (a) the identity morphism id_X*,* $X \in C$*, (b) the monoidal unit* e*, (c) the maps* λ_Y*,* ρ_X *are stable under the deformations, (d)* $m_{f,e}$ *and* $m_{e,g}$ *are stable under the deformations, and (e)* $m_{X,\mathrm{id}_Y} = m_{\mathrm{id}_X,Y} = \mathrm{id}_{X\otimes Y}$.

The following example shows that this set-up is realistic:

Example 3.5.1. Let A be a bialgebra over $\Bbbk$ and $\mathcal{C} = \mathrm{Mod}(A)$ be the category of left A-modules over the underlying algebra. It is a monoidal category in a standard way: for two modules M, N, the tensor product of the underlying vector spaces $M \otimes_\Bbbk N$ is naturally an $A \otimes A$-module, and the precomposition with $\Delta\colon A \to A \otimes A$ makes it an A-module. Assume that A is a Hopf algebra. Then the monoidal product $A \otimes A$ of two free modules of rank 1 is a free module again, whose underlying vector space is canonically isomorphic to $A \otimes A_u$, where A_u is the underlying vector space of A. Indeed, define the maps $\alpha\colon A \otimes A \to A \otimes A_u$ and $\beta\colon A \otimes A_u \to A \otimes A$ as

$$\alpha(a \otimes b) = \sum a_1 \otimes S(a_2)b$$

$$\beta(a \otimes b) = \sum a_1 \otimes a_2 b$$

where $S\colon A \to A$ is the antipode, and we use the Swindler notations $\Delta(a) = \sum a_1 \otimes a_2$. One has that α and β are maps of A-modules, and that $\alpha \circ \beta = \mathrm{id}, \beta \circ \alpha = \mathrm{id}$. It proves the claim. Thus, if we consider a deformation A_t of a Hopf algebra A, the $\Bbbk$-linear subcategory $\mathcal{C}_{free}(A_t)$ is a deformation of a monoidal $\Bbbk$-linear category $\mathcal{C}_{free}(A)$, for which the conditions (A1)-(A3) are fulfilled.

The data listed in (A1)-(A3) is subject to the following axioms:

(R1) the composition $m_{X,Y,Z}$ in (A1) is associative,

(R2) for maps in (A2) one has $(f \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes g) = (\mathrm{id}_X \otimes g) \circ (f \otimes \mathrm{id}_Y)$, (both sides are equal to $f \otimes g\colon X \otimes Y \to X' \otimes Y'$, therefore, the deformation of $f \otimes g$ are determined by deformations of $f \otimes \mathrm{id}_Y$ and $\mathrm{id}_X \otimes g$),

(R3) for any two composable morphisms $X \xrightarrow{f} X' \xrightarrow{f'} X''$, and any $Y \in \mathcal{C}$, one has $m^\ell_{f',Y} \circ m^\ell_{f,Y} = m^\ell_{f' \circ f,Y}$; similarly, for any two composable morphisms $Y \xrightarrow{g} Y' \xrightarrow{g'} Y''$, and any $X \in \mathcal{C}$, one has $m^r_{X,g'} \circ m^r_{X,g} = m^r_{X,g' \circ g}$,

(R4) this and the next two axioms express the naturality of the associator. Let $f\colon X \to X'$ be a morphism in $\mathcal{C}$, and Y, Z objects. The following diagram commutes:

$$\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{\alpha_{X,Y,Z}} & (X \otimes Y) \otimes Z \\
\downarrow{\scriptstyle m^\ell_{f,Y\otimes Z}} & & \downarrow{\scriptstyle m^\ell_{m^\ell_{f,Y},Z}} \\
X' \otimes (Y \otimes Z) & \xrightarrow{\alpha_{X',Y,Z}} & (X' \otimes Y) \otimes Z
\end{array}$$

(R5) let $g\colon Y \to Y'$ be a morphism in $\mathcal{C}$, X, Z objects. The following diagram commutes:

$$\begin{CD} X\otimes(Y\otimes Z) @>\alpha_{X,Y,Z}>> (X\otimes Y)\otimes Z \\ @V{m^r_{X,m^\ell_{g,Z}}}VV @VV{m^\ell_{m^r_{X,g},Z}}V \\ X\otimes(Y'\otimes Z) @>\alpha_{X,Y',Z}>> (X\otimes Y')\otimes Z \end{CD}$$

(R6) let $h\colon Z \to Z'$ be a morphism in $\mathcal{C}$, X, Y objects. Then the following diagram commutes:

$$\begin{CD} X\otimes(Y\otimes Z) @>\alpha_{X,Y,Z}>> (X\otimes Y)\otimes Z \\ @V{m^r_{X,m^r_{Y,h}}}VV @VV{m^r_{X\otimes Y,h}}V \\ X\otimes(Y\otimes Z') @>\alpha_{X,Y,Z'}>> (X\otimes Y)\otimes Z' \end{CD}$$

(R7) the pentagon equation for the associator,

$$\begin{array}{ccccc} & & (X\otimes Y)\otimes(Z\otimes T) & & \\ & \overset{\alpha_{X,Y,Z\otimes T}}{\nearrow} & & \overset{\alpha_{X\otimes Y,Z,T}}{\searrow} & \\ X\otimes(Y\otimes(Z\otimes T)) & & & & ((X\otimes Y)\otimes Z)\otimes T \\ \downarrow{\scriptstyle m^r_{X,\alpha_{Y,Z,T}}} & & & & \uparrow{\scriptstyle m^\ell_{\alpha_{X,Y,Z},T}} \\ X\otimes((Y\otimes Z)\otimes T) & & \xrightarrow[\alpha_{X,Y\otimes Z,T}]{} & & (X\otimes(Y\otimes Z))\otimes T \end{array}$$

(R8) left unit functionality: for any map $f\colon X \to X'$ the diagram

$$\begin{CD} X\otimes e @>\rho_X>> X \\ @V{m_{f,e}}VV @VV{f}V \\ X'\otimes e @>>\rho_{X'}> X' \end{CD}$$

commutes,

(R9) right unit functionality: for any $g\colon Y \to Y'$ the diagram

$$\begin{CD} e\otimes Y @>\lambda_Y>> Y \\ @V{m_{e,g}}VV @VV{g}V \\ e\otimes Y' @>>\lambda_{Y'}> Y' \end{CD}$$

commutes,

(R10) left right unit compatibility: the two possible maps $\lambda_e, \rho_e\colon e\otimes e \to e$ are equal.

Among the deformations $\mathcal{C}_t$ there are ones which we consider as "trivial". It appears in the literature under the name "twist", however, here we consider "upgraded" twists acting not only on the associator, but as well on the underlying category structure and on the action of morphisms on the monoidal product.

In deformation theory, one identifies two deformations if one is obtained from another by a twist, and interests in the "quotient-space".

Lemma 3.5.2. *Let $\mathcal{C}$ be a $\mathbb{k}$-linear (or dg- over $\mathbb{k}$) monoidal category, denote by $\mathcal{C}_u$ the underlying $\mathbb{k}$-linear quiver of $\mathcal{C}$. Assume that, for any $X,Y \in \mathcal{C}$, we are given an isomorphism $\varphi_{X,Y} \colon \mathcal{C}(X,Y) \to \mathcal{C}(X,Y)$, and an isomorphism $\psi_{X,Y} \in \mathcal{C}(X\otimes Y, X\otimes Y)$. Then these data gives rise to a monoidal equivalence functor F from $\mathcal{C}$ to another monoidal $\mathbb{k}$-linear (respectively, dg- over $\mathbb{k}$) category on the quiver $\mathcal{C}_u$, such that F is the identity map on any object of $\mathcal{C}$.*

Proof. It is standard. We define F on morphisms by $F(f) = \varphi_{X,Y}(f)$ if $f \in \mathcal{C}(X,Y)$, and define monoidal constraints $F(X\otimes Y) \to F(X)\otimes F(Y)$ as the isomorphisms $\psi_{X,Y}$. Then a monoidal category structure on $\mathcal{C}_u$ is uniquely determined by the requirement that F is a monoidal functor.

For convenience of the reader, we provide explicit formulas for the new composition of morphisms, for the action of morphisms on the tensor product, and for the associator. We use the same notations decorated by $\sim$ for the corresponding data (A1)-(A3) of the new category on $\mathcal{C}_u$. We use the same notations as in (A1)-(A3). One has:

$$\tilde{m}_{X,Y,Z}(f,g) = \varphi_{X,Z}(m_{X,Y,Z}(\varphi^{-1}_{Y,Z}(g), \varphi^{-1}_{X,Y}(f))) \tag{3.5.1}$$

$$\begin{aligned} \tilde{m}^r_{X,g}\tilde{\circ}\psi_{X,Y} &= \psi_{X,Y'}\tilde{\circ}\varphi_{X\otimes Y, X\otimes Y'}(m^r_{X,\varphi^{-1}_{Y,Y'}(g)}) \\ \tilde{m}^\ell_{f,Y}\tilde{\circ}\psi_{X,Y} &= \psi_{X',Y}\tilde{\circ}\varphi_{X\otimes Y, X'\otimes Y}(m^\ell_{\varphi^{-1}_{X,X'}(f),Y}) \end{aligned} \tag{3.5.2}$$

Equations (3.5.2) follow from the commutative diagram:

$$\begin{CD} F(X\otimes Y) @>>> F(X)\otimes F(Y) \\ @V{f\otimes g}VV @VV{F(f)\otimes F(g)}V \\ F(X'\otimes Y') @>>> F(X')\otimes F(Y') \end{CD}$$

There is still (A3):

$$\tilde{\alpha}_{X,Y,Z} = \tilde{m}^\ell_{\psi_{X,Y},Z}\tilde{\circ}\psi_{X\otimes Y,Z}\tilde{\circ}\varphi(\alpha_{X,Y,Z})\tilde{\circ}\psi^{-1}_{X,Y\otimes Z}\tilde{\circ}\tilde{m}^r_{X,\psi^{-1}_{Y,Z}}, \tag{3.5.3}$$

that comes from the commutative diagram:

$$\begin{CD} F(X\otimes(Y\otimes Z)) @>{F(\alpha)}>> F((X\otimes Y)\otimes Z) \\ @V{\psi_{X,Y\otimes Z}}VV @VV{\psi_{X\otimes Y,Z}}V \\ F(X)\otimes F(Y\otimes Z) @. F(X\otimes Y)\otimes F(Z) \\ @V{\mathrm{id}_X\otimes\psi_{Y,Z}}VV @VV{\psi_{X,Y}\otimes\mathrm{id}_Z}V \\ F(X)\otimes(F(Y)\otimes F(Z)) @>>{\tilde{\alpha}}> (F(X)\otimes F(Y))\otimes F(Z) \end{CD}$$

where $\tilde{\circ}$ denotes the composition in $\tilde{\mathcal{C}}$ and $F(?) = ?$ for any object $? \in Ob(\mathcal{C})$.

One can check directly that $\tilde{m}_{X,Y,Z}, \tilde{m}^r_{X,g}, \tilde{m}^\ell_{f,Y}, \tilde{\alpha}_{X,Y,Z}$ satisfy (R1)-(R10), and thus define a monoidal category $\tilde{\mathcal{C}}$, such that the functor $F\colon \mathcal{C} \to \tilde{\mathcal{C}}$ is a monoidal equivalence. □

Note that in the assumption of Lemma $\varphi_{X,Y}$ and $\psi_{X,Y}$ are arbitrary isomorphisms. Now we switch back to formal deformation theory.

By definition, a *trivial deformation* depends on the following data:

(T1) a formal power series $\varphi_{X,Y}\colon \mathcal{C}(X,Y) \to \mathcal{C}(X,Y)$, for any $X, Y \in \mathcal{C}$ of the form

$$\varphi_{X,Y} = \mathrm{Id}_{\mathcal{C}(X,Y)} + t \cdot \varphi^1_{X,Y} + t^2 \cdot \varphi^2_{X,Y} + \dots \tag{3.5.4}$$

where $\varphi^i_{X,Y} \in \mathrm{Hom}_\Bbbk(\mathcal{C}(X,Y), \mathcal{C}(X,Y)), i \geq 1$,

(T2) a formal power series $\psi_{X,Y}\colon \mathcal{C}(X \otimes Y, X \otimes Y)$, for any $X, Y \in \mathcal{C}$, of the form

$$\psi_{X,Y} = \mathrm{Id}_{X\otimes Y} + t \cdot \psi^1_{X,Y} + t^2 \cdot \psi^2_{X,Y} + \dots \tag{3.5.5}$$

where $\psi^i_{X,Y} \in \mathcal{C}(X \otimes Y, X \otimes Y), i \geq 1$.

Out of this data, a formal deformation of C is constructed as in (3.5.1)-(3.5.3).

One defines the concepts of an *infinitesimal deformation* and of a *trivial infinitesimal deformation* of a monoidal ($\Bbbk$-linear or dg-) category by replacing in the previous definitions the ring of formal power series $\Bbbk[[t]]$ by the dual numbers $\Bbbk[t]/(t^2)$. We say that two infinitesimal deformations belong to the same equivalence class if the corresponding monoidal categories are equivalent by an (extended) twist, as in Lemma 3.5.2 but over $\Bbbk[t]/(t^2)$.

One has:

Theorem 3.5.3. *Let $\mathcal{C}$ be a $\Bbbk$-linear (or a dg- over $\Bbbk$) monoidal category. The third cohomology $H^3(\mathrm{Tot}_{\Theta_2} A(\mathrm{Id},\mathrm{Id}))$ is isomorphic to the equivalence classes of infinitesimal deformations (in the sense specified above) of the monoidal (dg-)category $\mathcal{C}$.*

Proof. For simplicity, we assume that all complexes $\mathcal{C}(X,Y)$ are concentrated in cohomological degree 0, that is, are ordinary vector spaces over $\Bbbk$. The general case is similar but technically more involved, and we leave it to the reader.

To improve readability, in what follows we will denote objects of Θ_2 by $(\kappa; n_1, \dots, n_\kappa)$ instead of $([\kappa]; [n_1], \dots, [n_\kappa])$.

A general element π of $\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id},\mathrm{Id})$ of degree 3 is a sum of cochains of types (1;2), (2;1,0), (2;0,1), and (3;0,0,0). Denote them by $\kappa, \beta^\ell, \beta^r, \gamma$, correspondingly:

$$\pi = \kappa + \beta^\ell + \beta^r + \gamma$$

We identify them with infinitesimal deformations of $m_{X,Y,Z}$ for κ, of $m^\ell_{f,Y}$ for β^ℓ, of $m^r_{X,g}$ for β^r, and of $\alpha_{X,Y,Z}$ for γ, see (A1)-(A3):

$$\begin{aligned} \tilde{m}_{X,Y,Z} &= m_{X,Y,Z} + t\cdot\kappa_{X,Y,Z} \\ \tilde{m}^\ell_{f,Y} &= m^\ell_{f,Y} + t\cdot\beta^\ell_{f,Y} \\ \tilde{m}^r_{X,g} &= m^r_{X,g} + t\cdot\beta^r_{X,g} \\ \tilde{\alpha}_{X,Y,Z} &= \alpha_{X,Y,Z} + t\cdot\gamma_{X,Y,Z}\circ\alpha_{X,Y,Z} \end{aligned} \tag{3.5.6}$$

Assume π is a cycle:

$$d\pi = d(\kappa+\beta^\ell+\beta^r+\gamma) = 0 \tag{3.5.7}$$

This equation is a system of several equations, one equation for any diagram of dimension 4. We write schematically:

$$\begin{aligned} d\kappa &= (1;3) + (2;0,2) + (2;1,1) + (2;2,0) \\ d\beta^\ell &= (3;0,1,0) + (3;1,0,0) + (2;2,0) + (2;1,1) \\ d\beta^r &= (3;0,1,0) + (3;0,0,1) + (2;0,2) + (2;1,1) \\ d\gamma &= (4;0,0,0,0) + (3;0,0,1) + (3;0,1,0) + (3;1,0,0) \end{aligned} \tag{3.5.8}$$

We see that there are many cross-terms in (3.5.7).

Now consider relations (R1)-(R10) for tilde-data (3.5.6), taking to the account $t^2=0$. We get system of linear in $\kappa,\beta^\ell,\beta^r,\gamma$ equations. The claim is that these equations are exactly the homogeneous components (for any given degree, e.g. (2;1,1)) of the equation $d\pi=0$.

Note that for the cases (R4)-(R9) we have to use relations (3.2.6)-(3.2.9).

The case of (R1) is standard, the computation here is basically the same as the classical computation with Hochschild complex.

The infinitesimal version of (R2) gives:

$$(m^\ell_{f,Y'} + t\beta^\ell_{f,Y'})\tilde{\circ}(m^r_{X,g} + t\beta^r_{X,g}) = (m^r_{X',g} + t\beta^r_{X',g})\tilde{\circ}(m^\ell_{f,Y} + t\beta^\ell_{f,Y}) \mod t^2$$

where $\tilde{\circ} = \circ + t\cdot\kappa$, and the terms in t give the following identity:

$$\beta^\ell_{f,Y'}\circ m^r_{X,g} + m^\ell_{F,Y'}\circ\beta^r_{X,g} + \kappa(m^\ell_{f,Y'}, m^r_{X,g}) = \beta^r_{X',g}\circ m^\ell_{f,Y} + m^r_{X',g}\circ\beta^\ell_{f,Y} + \kappa(m^r_{X',g}, m^\ell_{f,Y})$$

It is the vanishing of type $(2;1,1)$ cross-terms in $d\beta^\ell + d\beta^r + d\kappa$. (The other summands of π do not contain components of type (2;1,1) in their boundary).
For $d\kappa$ these are the two possible shuffle maps.

The case (R3) comprises two sub-cases, which are analogous. We consider one of them. It is, in the infinitesimal version

$$(m^\ell_{f',Y} + t\cdot\beta^\ell_{f',Y})\tilde{\circ}(m^\ell_{f,Y} + t\cdot\beta^\ell_{f,Y}) = m^\ell_{f'\tilde{\circ}f,Y} + t\cdot\beta^\ell_{f'\tilde{\circ}f,Y} \mod t^2$$

Its terms in t give:

$$\beta^\ell_{f',Y}\circ m^\ell_{f,Y} + m^\ell_{f',Y}\circ\beta^\ell_{f,Y} + \kappa(m^\ell_{f',Y}, m^\ell_{f,Y}) = m^\ell_{\kappa(f',f),Y} + \beta^\ell_{f'\circ f,Y}$$

It is the vanishing of type $(2;2,0)$ cross-terms in $d(\beta^\ell+\kappa)$. The other summands of π do not contain $(2;2,0)$ type elements in their boundary.

The cases (R4), (R5), (R6) are similar, we consider one of them, (R5). This is the first case where we essentially use the relations (3.2.6)-(3.2.9).

The case (R5) in the infinitesimal version reads:

$$\begin{aligned}&(m^\ell_{m^r_{X,g},Z}+t\beta^\ell_{m^r_{X,g},Z}+tm^\ell_{\beta^r_{X,g},Z})\tilde{\circ}(\alpha+t\gamma\circ\alpha)=\\&(\alpha+t\gamma\circ\alpha)\tilde{\circ}(m^r_{X,m^\ell_{g,Z}}+t\beta^r_{X,m^\ell_{g,Z}}+tm^r_{X,\beta^\ell_{g,Z}})\mod t^2\end{aligned}\tag{3.5.9}$$

The terms in t give the identity, all terms are of type $(3;0,1,0)$:

$$\begin{aligned}&m^\ell_{m^r_{X,g},Z}\circ(\gamma\circ\alpha)+\kappa(\alpha,m^\ell_{m^r_{X,g},Z})+\beta^\ell_{m^r_{X,g},Z}\circ\alpha+m^\ell_{\beta^r_{X,g},Z}\circ\alpha=\\&(\gamma\circ\alpha)\circ m^r_{X,m^\ell_{g,Z}}+\kappa(m^r_{X,m^\ell_{g,Z}},\alpha)+\alpha\circ\beta^r_{X,m^\ell_{g,Z}}+\alpha\circ m^r_{X,\beta^\ell_{g,Z}}\end{aligned}\tag{3.5.10}$$

Lemma 3.5.4. *The following identities hold:*

(1) $\kappa(\alpha,-)=\kappa(-,\alpha)=0$, *where α has any arguments such as $\alpha_{X\otimes Y,Z,T}$ etc.,*

(2) $\beta^\ell_{\alpha_{X,Y,Z},T}=\beta^r_{X,\alpha_{Y,Z,T}}=0$.

Proof. (1): We prove the first equation, the second one is analogous.

One has by (3.2.6): $\kappa(\alpha,-)=\kappa(\mathrm{id},-\circ\alpha)=0$, where the second equality follows from the requirement that the identity maps are preserved under the deformation.

(2): We prove the first assertion, the second one is analogous.

By (3.2.7) we have $\beta^\ell_{\alpha_{X,Y,Z},T}=\beta^\ell_{\mathrm{Id}_{X\otimes(Y\otimes Z)},T}\circ m^\ell_{\alpha_{X,Y,Z},T}=0$, where the second equality follows from a more general $\beta^\ell_{\mathrm{Id},-}=0$. ☐

The 6 non-vanishing terms in (3.5.10) are interpreted as the degree (3;0,1,0) terms in $d\pi$, as follows. We rewrite (3.5.10) as:

$$m^\ell_{m^r_{X,g},Z}\circ\gamma+\beta^\ell_{m^r_{X,g},Z}+m^\ell_{\beta^r_{X,g},Z}=\gamma\circ(\alpha\circ m^r_{X,m^\ell_{g,Z}}\circ\alpha^{-1})+\alpha\circ\beta^r_{X,m^\ell_{g,Z}}\circ\alpha^{-1}+\alpha\circ m^r_{X,\beta^\ell_{g,Z}}\circ\alpha^{-1}\tag{3.5.11}$$

(here we made use of Lemma 3.5.4(1)).

Next, $\alpha\circ m^r_{X,m^\ell_{g,Z}}\circ\alpha^{-1}=m^\ell_{m^r_{X,g},Z}$, so the first summand in the r.h.s. of (3.5.11) is equal to $\gamma\circ m^\ell_{m^r_{X,g},Z}$.

We have:

$$\begin{aligned}(d\gamma)_{(3;0,1,0)}&=m^\ell_{m^r_{X,g},Z}\circ\gamma-\gamma\circ m^\ell_{m^r_{X,g},Z}\\(d\beta^\ell)_{(3;0,1,0)}&=\beta^\ell_{m^r_{X,g},Z}-\alpha\circ m^r_{X,\beta^\ell_{g,Z}}\circ\alpha^{-1}\\(d\beta^r)_{(3;0,1,0)}&=m^\ell_{\beta^r_{X,g},Z}-\alpha\circ\beta^r_{X,m^\ell_{g,Z}}\circ\alpha^{-1}\end{aligned}\tag{3.5.12}$$

The conjugation with α in the two last lines is explained in Remark 3.2.5(a).

The infinitesimal version of (R7) is:

$$\begin{aligned}&(\alpha_{X\otimes Y,Z,T}+t\cdot\gamma_{X\otimes Y,Z,T}\circ\alpha_{X\otimes Y,Z,T})\tilde{\circ}(\alpha_{X,Y,Z\otimes T}+t\cdot\gamma_{X,Y,Z\otimes T}\circ\alpha_{X,Y,Z\otimes T})=\\&(m^\ell_{\tilde{\alpha}_{X,Y,Z},T}+t\cdot\beta^\ell_{\tilde{\alpha}_{X,Y,Z},T})\tilde{\circ}(\alpha_{X,Y\otimes Z,T}+t\cdot\gamma_{X,Y\otimes Z,T}\circ\alpha_{X,Y\otimes Z,T})\\&\tilde{\circ}(m^r_{X,\tilde{\alpha}_{Y,Z,T}}+t\cdot\beta^r_{X,\tilde{\alpha}_{Y,Z,T}})\qquad\mod t^2\end{aligned}\tag{3.5.13}$$

The terms in t give the identity:

$$\begin{aligned}&\gamma_{X\otimes Y,Z,T}\circ\alpha_{X\otimes Y,Z,T}\circ\alpha_{X,Y,Z\otimes T}+\alpha_{X\otimes Y,Z,T}\circ\gamma_{X,Y,Z\otimes T}\circ\alpha_{X,Y,Z\otimes T}+\kappa(\alpha_{X\otimes Y,Z,T},\alpha_{X,Y,Z\otimes T})=\\&m^\ell_{\alpha_{X,Y,Z},T}\circ\alpha_{X,Y\otimes Z,T}\circ\beta^r_{X,\alpha_{Y,Z,T}}+m^\ell_{\alpha_{X,Y,Z},T}\circ\gamma_{X,Y\otimes Z,T}\circ\alpha_{X,Y\otimes Z,T}\circ m^r_{X,\alpha_{Y,Z,T}}+\\&\beta^\ell_{\alpha_{X,Y,Z},T}\circ\alpha_{X,Y\otimes Z,T}\circ m^r_{X,\alpha_{Y,Z,T}}+m^\ell_{\alpha_{X,Y,Z},T}\circ\alpha_{X,Y\otimes Z,T}\circ m^r_{X,\gamma_{Y,Z,T}\circ\alpha_{Y,Z,T}}+\\&m^\ell_{\gamma_{X,Y,Z}\circ\alpha_{X,Y,Z},T}\circ\alpha_{X,Y\otimes Z,T}\circ m^r_{X,\alpha_{Y,Z,T}}+\kappa(m^\ell_{\alpha_{X,Y,Z},T},\alpha_{X,Y\otimes Z,T})\circ m^r_{X,\alpha_{Y,Z,T}}+\\&m^\ell_{\alpha_{X,Y,Z},T}\circ\kappa(\alpha_{X,Y\otimes Z,T},m^r_{X,\alpha_{Y,Z,T}})\end{aligned}\tag{3.5.14}$$

The terms with κ, β^ℓ and β^r get canceled by Lemma 3.5.4, thus we are left with:

$$\begin{aligned}&\gamma_{X\otimes Y,Z,T}\circ\alpha_{X\otimes Y,Z,T}\circ\alpha_{X,Y,Z\otimes T}+\alpha_{X\otimes Y,Z,T}\circ\gamma_{X,Y,Z\otimes T}\circ\alpha_{X,Y,Z\otimes T}=\\&m^\ell_{\alpha_{X,Y,Z},T}\circ\gamma_{X,Y\otimes Z,T}\circ\alpha_{X,Y\otimes Z,T}\circ m^r_{X,\alpha_{Y,Z,T}}+m^\ell_{\alpha_{X,Y,Z},T}\circ\alpha_{X,Y\otimes Z,T}\circ m^r_{X,\gamma_{Y,Z,T}\circ\alpha_{Y,Z,T}}+\\&m^\ell_{\gamma_{X,Y,Z}\circ\alpha_{X,Y,Z},T}\circ\alpha_{X,Y\otimes Z,T}\circ m^r_{X,\alpha_{Y,Z,T}}\end{aligned}\tag{3.5.15}$$

Now, by (3.2.7) we have $m^\ell_{\gamma_{X,Y,Z}\circ\alpha_{X,Y,Z},T}=m^\ell_{\gamma_{X,Y,Z},T}\circ m^\ell_{\alpha_{X,Y,Z},T}$ and $m^r_{X,\gamma_{Y,Z,T}\circ\alpha_{Y,Z,T}}=m^r_{X,\gamma_{Y,Z,T}}\circ m^r_{X,\alpha_{Y,Z,T}}$, and we can recognize that the 5 terms of(3.5.15) are nothing but the degree (4;0,0,0,0) terms in $d\pi$, which amount to $(d\gamma)_{(4;0,0,0,0)}$ (the other elements of π do not contribute to terms of this degree).

It remains to consider (R8)-(R10). The identity (R10) is fulfilled automatically for the deformed category. The identities (R8)-(R9) are analogous, we consider (R8). Here all morphisms in the diagram are not deformed by our assumptions, but the composition does. So one has to prove that the terms, coming from the deformation of the composition, vanish. These terms are $\rho_X^{-1}\circ\kappa(\rho_X,f)$ and $\kappa(f,\rho_X^{-1})\circ\rho_X$. These terms vanish by (3.2.7)-(3.2.8). For example, $\kappa(\rho_X,f)=\kappa(\mathrm{id},f)\circ\rho_X=0$. (To be precise, one *derives* that κ obeys (3.2.7)-(3.2.8) from this speculation).

We have identified 3-cycles $\pi=\kappa+\beta^\ell+\beta^r+\gamma$ in $\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id},\mathrm{Id}))$ with infinitesimal deformations of the monoidal category C. Now we show that the infinitesimal deformations of type (3.5.1)-(3.5.3) are corresponded to coboundaries $\pi=d\omega$, for a 2-cochain $\omega\in\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id},\mathrm{Id}))$.

A general 2-cochain ω is a linear combination of components of the types $(1;1)$ and $(2;0)$. On the other hand, our infinitesimal twists $\varphi^1_{X,Y}$ and $\psi^1_{X,Y}$ (see (3.5.4) and (3.5.5)) are of the same type. We identify their coboundaries with the infinitesimal versions of (3.5.1)-(3.5.3), i.e. we have to identify the components of $d(\varphi^1+\psi^1)$ of degrees $(1;2),(2;0,1)$, $(2;1,0)$, $(3;0,0,0)$ with the infinitesimal versions of (3.5.1), (3.5.2)(1), (3.5.2)(2), (3.5.3) correspondingly.

For (3.5.1), it is clear that $d(\varphi^1+\psi^1)_{(1;2)}=d(\varphi^1)_{(1;2)}$. The computation with (3.5.1) is standard, it is the same as for the Hochschild cochains, and we leave it to the reader.

For (3.5.2)(1), rewrite it as

$$\tilde{m}^r_{X,g} = \varphi(\varphi^{-1}\psi_{X,Y'} \circ m^r_{X,\varphi^{-1}_{Y,Y'}g} \circ \varphi^{-1}\psi^{-1}_{X,Y}) \tag{3.5.16}$$

Note that

$$\varphi^{-1}\psi = (\mathrm{id} - t\varphi^1 + \dots)(\mathrm{id} + t\psi^1 + \dots) = \mathrm{id} + t\psi^1 \mod t^2 \tag{3.5.17}$$

because $\phi^1(\mathrm{id}) = 0$, similarly for $\varphi^{-1}\psi^{-1}$.

Then

$$m^r_{X,g} + t \cdot m^{r,1}_{X,g} = (\mathrm{id} + t\cdot\varphi^1)\Big((\mathrm{id} + t\cdot\psi^1_{X,Y'}) \circ (m^r_{X,g} - t\cdot m^{r,1}_{X,\varphi^1 g}) \circ (\mathrm{id} - t\cdot\psi^1_{X,Y})\Big) \mod t^2 \tag{3.5.18}$$

from which

$$m^{r,1}_{X,g} = \varphi^1 \circ m^r_{X,g} + \psi^1_{X,Y'} \circ m^r_{X,g} - m^r_{X,\varphi^1 g} - m^r_{X,g} \circ \psi^1_{X,Y} \tag{3.5.19}$$

One easily recognises $d(\varphi^1 + \psi^1)_{(2;0,1)}$ in the r.h.s. of (3.5.19).

The computation for (3.5.2)(2) is similar.

The remaining case is to identify $d(\varphi^1 + \psi^1)_{(3;0,0,0)} = d(\psi^1)_{(3;0,0,0)}$ with (the r.h.s. of) the infinitesimal version of (3.5.3), written in the form

$$\tilde{\alpha} = \tilde{m}^\ell_{\psi_{X,Y},Z} \tilde{\circ} \psi_{X\otimes Y,Z} \tilde{\circ} \varphi(\alpha) \tilde{\circ} \psi^{-1}_{X,Y\otimes Z} \tilde{\circ} (\tilde{m}^r_{X,\psi_{Y,Z}})^{-1} \tag{3.5.20}$$

The order t term of the r.h.s. of (3.5.20) is:

$$\begin{aligned}
&(\mathrm{id}_{(X\otimes Y)\otimes Z} + t\cdot m^\ell_{\psi^1_{X,Y},Z}) \tilde{\circ} (\mathrm{id}_{(X\otimes Y)\otimes Z} + t\cdot \psi^1_{X\otimes Y,Z}) \tilde{\circ} ((\mathrm{id} + t\cdot\varphi^1)\circ\alpha) \tilde{\circ} \\
&\tilde{\circ} (\mathrm{id}_{X\otimes(Y\otimes Z)} - t\cdot\psi^1_{X,Y\otimes Z}) \tilde{\circ} (\mathrm{id}_{X\otimes(Y\otimes Z)} - t\cdot m^r_{X,\psi^1_{Y,Z}}) = \\
&(\mathrm{id} + t\cdot\varphi^1)\Big((\mathrm{id} - t\cdot\varphi^1)(\mathrm{id} + t\cdot m^\ell_{\psi^1_{X,Y},Z}) \circ (\mathrm{id} + t\cdot\varphi^1)(\mathrm{id} + t\cdot\psi^1_{X\otimes Y,Z}) \circ \alpha \circ \\
&\circ (\mathrm{id} - t\cdot\varphi^1)(\mathrm{id} - t\cdot\psi^1_{X,Y\otimes Z}) \circ (\mathrm{id} - t\cdot\varphi^1)(\mathrm{id} - t\cdot m^r_{X,\psi^1_{Y,Z}})\Big)
\end{aligned} \tag{3.5.21}$$

Here we made use of $\tilde{m}_{X,\mathrm{id}} = \tilde{m}_{\mathrm{id},Y} = \mathrm{id}$, which implies $m_{X,\mathrm{id}} = \mathrm{id}$ and $m^1_{X,\mathrm{id}} = 0$, et cetera. We compute the r.h.s. of (3.5.21) modulo t^2 (using $\phi^1(\mathrm{id}) = 0$):

$$\begin{aligned}
&t\cdot\gamma_{X,Y,Z} \circ \alpha = \\
&(\mathrm{id} + t\cdot\varphi^1)\Big((\mathrm{id} + t\cdot m^\ell_{\psi^1_{X,Y},Z}) \circ (\mathrm{id} + t\cdot\psi^1_{X\otimes Y,Z}) \circ \alpha \circ (\mathrm{id} - t\cdot\psi^1_{X,Y\otimes Z}) \circ (\mathrm{id} - t\cdot m^r_{X,\psi^1_{Y,Z}})\Big)
\end{aligned} \tag{3.5.22}$$

Thus,

$$\begin{aligned}
&\gamma_{X,Y,Z} = \Big(\varphi^1(\alpha) + m^\ell_{\psi^1_{X,Y},Z} \circ \alpha + \psi^1_{X\otimes Y,Z} \circ \alpha - \alpha \circ \psi^1_{X,Y\otimes Z} - \alpha \circ m^r_{X,\psi^1_{Y,Z}}\Big) \circ \alpha^{-1} = \\
&\varphi^1(\alpha) \circ \alpha^{-1} + m^\ell_{\psi^1_{X,Y},Z} + \psi^1_{X\otimes Y,Z} - \alpha \circ \psi^1_{X,Y\otimes Z} \circ \alpha^{-1} - \alpha \circ m^r_{X,\psi^1_{Y,Z}} \circ \alpha^{-1}
\end{aligned} \tag{3.5.23}$$

The first summand of the r.h.s. of (3.5.23) vanishes, because

$$\varphi^1(\alpha) \overset{(3.2.7)}{=} \varphi^1(\mathrm{id}) \circ \alpha = 0$$

One can recognize $d(\psi^1)_{(3;0,0,0)}$ in the four remaining summands of the r.h.s. of (3.5.23).

□

3.5.1 Infinitesimal deformation theory of a monoidal functor

The following Theorem is proven analogously to but easier than Theorem 3.5.3, and we leave details to the reader.

Theorem 3.5.5. *Let $\mathcal{C}$, $\mathcal{D}$ be $\Bbbk$-linear (or dg- over $\Bbbk$) monoidal categories, $F\colon \mathcal{C} \to \mathcal{D}$ a monoidal functor. The second cohomology $H^2(\mathrm{Tot}_{\Theta_2} A(F,F))$ is isomorphic to the equivalence classes of infinitesimal deformations of the functor F.*

By Theorem 3.4.9, $\mathrm{Tot}_{\Theta_2} A(F,F)$ is a homotopy 2-algebra. In fact, one can construct a dg-Lie algebra on $\mathrm{Tot}_{\Theta_2} A(F,F)[1]$ directly (without any use of loc.cit.), and to develop, via the Maurer-Cartan equation and the deformation functor associated to dg-Lie algebra formalism, the "global" deformation theory for $F\colon \mathcal{C} \to \mathcal{D}$ over $\Bbbk[\![t]\!]$.

3.6 Relations in Θ_2

One has the following relations between the elementary face and degeneracy maps in Θ_2, which are checked straightforwardly.

$$D_{q,\sigma'} D_{p,\sigma} = D_{p,\sigma} D_{q-1,\sigma'} \text{ if } p < q-1 \tag{3.6.1}$$

$$D_{q,\sigma_2} D_{q-1,\sigma_1} = D_{q-1,\eta_2} D_{q-1,\eta_1} \tag{3.6.2}$$

Here is an explanation of the notations: any (a,b)-shuffle σ_1 and $(a+b,c)$-shuffle σ_2 define uniquely a (b,c)-shuffle η_1 and an $(a,b+c)$-shuffle η_2 such that $\sigma_2 \circ (\sigma_1, \mathrm{id}_c) = \eta_2 \circ (\mathrm{id}_a, \eta_1)$ (the latter is an (a,b,c)-shuffle).

$$\partial_p^j \partial_q^i = \partial_q^i \partial_p^j \text{ if } p \neq q \tag{3.6.3}$$

$$\partial_p^j \partial_p^i = \partial_p^i \partial_p^{j-1} \text{ if } i < j \tag{3.6.4}$$

$$\begin{aligned} D_{q,\sigma} \partial_p^j &= \partial_{p+1}^j D_{q,\sigma} \text{ if } p > q \\ D_{q,\sigma} \partial_p^j &= \partial_p^j D_{q,\sigma} \text{ if } p < q \end{aligned} \tag{3.6.5}$$

$$D_{p,\sigma} \partial_p^i = \begin{cases} \partial_p^a D_{p,\bar\sigma} & \text{if } \sigma^{-1}(\overrightarrow{i,i+1}) = \overrightarrow{a,a+1} \in [0,k_p] \\ \partial_{p+1}^b D_{p,\bar\sigma} & \text{if } \sigma^{-1}(\overrightarrow{i,i+1}) = \overrightarrow{b,b+1} \in [k_p, k_p + k_{p+1}] \end{cases} \tag{3.6.6}$$

where $\bar\sigma$ is the shuffle obtained from σ by collapsing $\sigma^{-1}(\overrightarrow{i,i+1})$, and $\{k_s\}$ is used as in Subsection 3.2.1, (F2).

$$\begin{aligned} \partial_p^i D_{\min} &= D_{\min} \partial_p^{i-1} \quad \text{if } p \geq 1 \\ D_{p,\sigma} D_{\min} &= D_{\min} D_{p-1,\sigma} \quad \text{if } p \geq 1 \end{aligned} \tag{3.6.7}$$

and similarly for $D_{\max}$.

$$\epsilon_p^j \circ \epsilon_q^i = \epsilon_q^i \circ \epsilon_p^j \text{ if } p \neq q \tag{3.6.8}$$

$$\epsilon_p^j \circ \epsilon_p^i = \epsilon_p^i \circ \epsilon_p^{j-1} \text{ if } i \leq j \tag{3.6.9}$$

$$\Upsilon_\ell^q \circ \epsilon_p^j = \begin{cases} \epsilon_{p-1}^j \circ \Upsilon_\ell^q & \text{if } p > q+1; \\ \Upsilon_{\ell+1}^q & \text{if } p = q+1; \\ \epsilon_p^j \circ \Upsilon_\ell^q & \text{if } p \leq q. \end{cases} \tag{3.6.10}$$

$$\partial_p^i \circ \epsilon_q^j = \epsilon_q^j \circ \partial_p^i \text{ if } p \neq q \tag{3.6.11}$$

$$\epsilon_p^j \circ \partial_p^i = \begin{cases} \partial_p^i \circ \epsilon_p^{j-1} & \text{if } i < j; \\ \mathrm{id} & \text{if } i = j, j+1; \\ \partial_p^{i-1} \circ \epsilon_p^j & \text{if } i > j+1. \end{cases} \tag{3.6.12}$$

$$\Upsilon_\ell^q \circ \partial_p^j = \begin{cases} \partial_{p-1}^j \circ \Upsilon_\ell^q & \text{if } p > q+1; \\ \Upsilon_{\ell-1}^q & \text{if } p = q+1; \\ \partial_p^j \circ \Upsilon_\ell^q & \text{if } p \leq q. \end{cases} \tag{3.6.13}$$

$$D_{q,\sigma} \circ \epsilon_p^i = \begin{cases} \epsilon_{p+1}^i \circ D_{q,\sigma} & \text{if } q < p \\ \epsilon_p^i \circ D_{q,\sigma} & \text{if } q > p \\ \epsilon_p^a \circ D_{q,\sigma'} & \text{if } q = p,\ \sigma^{-1}(\overrightarrow{i, i+1}) = \overrightarrow{a, a+1} \in [0, k_p] \\ \epsilon_p^b \circ D_{q,\sigma'} & \text{if } q = p,\ \sigma^{-1}(\overrightarrow{i, i+1}) = \overrightarrow{b, b+1} \in [k_p, k_p + k_{p+1}] \end{cases} \tag{3.6.14}$$

where σ' is obtained from σ by adding a new element (blowing up) at $\sigma^{-1}(\overrightarrow{i, i+1})$.

$$\Upsilon_0^q \circ D_{p,\sigma} = \begin{cases} D_{p,\sigma} \circ \Upsilon_0^{q-1} & \text{if } p < q \\ D_{p-1,\sigma} \circ \Upsilon_0^q & \text{if } p > q+1 \\ \mathrm{id} & \text{if } p = q,\ \sigma = (0, k_p + k_{p+1}) \\ \mathrm{id} & \text{if } p = q+1,\ \sigma = (k_p + k_{p+1}, 0) \end{cases} \tag{3.6.15}$$

$$D_{\min} \circ \epsilon_p^i = \epsilon_{p+1}^i \circ D_{\min} \tag{3.6.16}$$

$$D_{\max} \circ \epsilon_p^i = \epsilon_p^i \circ D_{\max} \tag{3.6.17}$$

$$\Upsilon_0^q \circ D_{\min} = \begin{cases} D_{\min} \circ \Upsilon_0^{q-1} & \text{if } q > 0 \\ \mathrm{id} & \text{if } q = 0 \end{cases} \tag{3.6.18}$$

$$\Upsilon_0^q \circ D_{\max} = \begin{cases} D_{\max} \circ \Upsilon_0^q & \text{if } q < n+1 \\ \mathrm{id} & \text{if } q = n+1 \end{cases} \tag{3.6.19}$$

$$\Upsilon_0^q \circ \Upsilon_0^p = \Upsilon_0^p \circ \Upsilon_0^{q+1} \text{ if } p \leq q; \tag{3.6.20}$$

3.7 A proof of Proposition 3.3.3

Here we give a proof of Proposition 3.4.4:

Proof. Let $\phi \in \mathfrak{R}^\bullet_{[n]}$. We first want to show that $\Upsilon_\Delta^q(d\phi) = d\Upsilon_\Delta^q(\phi)$:

$$\Upsilon_\Delta^q \left(\sum_{s=1}^{n} \sum_{i=0}^{\kappa_s} (-1)^{\kappa_1 + \cdots + \kappa_{s-1} + s - 1 + i} \phi_{s,i} \right) = \left(\sum_{s=1}^{n} \sum_{i=0}^{\kappa_s} (-1)^{\kappa_1 + \cdots + \kappa_{s-1} + s - 1 + i} \phi \circ \partial_s^i \right) \circ \Upsilon_0^q =$$

$$= \sum_{s=1, s\neq q+1}^{n+1} \sum_{i=0}^{\kappa_s} (-1)^{\kappa_1 + \cdots + \kappa_{s-1} + s - 1 + i} (\phi \circ \Upsilon_0^q) \circ \partial_s^i = d\Upsilon_\Delta^q(\phi)$$

where the second equality follows from (3.6.13). The reason for excluding $s = q+1$ by the summation is due to the fact that there is no face map with codomain [0], which is the q^{th} interval.

Now we would like to show that $\Omega^0_\Delta(d\phi) = -d\Omega^0_\Delta(\phi)$:

$$\Omega^0_\Delta\left(\sum_{s=1}^{n}\sum_{i=0}^{\kappa_s}(-1)^{\kappa_1+\cdots+\kappa_{s-1}+s-1+i}\phi_{s,i}\right) = \left(\sum_{s=2}^{n}\sum_{i=0}^{\kappa_s}(-1)^{\kappa_2+\cdots+\kappa_{s-1}+s-1+i}\phi\circ\partial_s^i\right)\circ D_{\min} =$$
$$-\sum_{s=2}^{n}\sum_{i=0}^{\kappa_s}(-1)^{\kappa_2+\cdots+\kappa_{s-1}+s-2+i}(\phi\circ D_{\min})\circ\partial_{s-1}^i =$$
$$-\sum_{s=1}^{n-1}\sum_{i=0}^{\kappa'_s}(-1)^{\kappa'_1+\cdots+\kappa'_{s-1}+s-1+i}(\phi\circ D_{\min})\circ\partial_s^i = -d\Omega^0_\Delta(\phi)$$

the first equality comes from the fact that $k_1 = 0$, so that there is no $\partial_1^i\colon \cdots\to[0]$; the second equality follows from (3.6.7), and we define $\kappa'_i \coloneqq \kappa_{i+1}$, for each $i = 1,\ldots,n-1$.

One can similarly prove that $\Omega^n_\Delta(d\phi) = -d\Omega^n_\Delta(\phi)$.

Now we would like to show that $\Omega^p_\Delta(d\phi) = -d\Omega^p_\Delta(\phi)$:

$$\Omega^p_\Delta\left(\sum_{s=1}^{n}\sum_{i=0}^{\kappa_s}(-1)^{\kappa_1+\cdots+\kappa_{s-1}+s-1+i}\phi_{s,i}\right) =$$
$$= \left(\sum_{s=1}^{n}\sum_{i=0}^{\kappa_s}(-1)^{\kappa_1+\cdots+\kappa_{s-1}+s-1+i}\phi\circ\partial_s^i\right)\circ\left(\sum_{\sigma}(-1)^{\ell_1+\cdots+\ell_{p-1}+p-1+\sharp(\sigma)}D_{p,\sigma}\right) =$$
$$(a)+(b)+(c)$$

where:

$$(a) = \left(\sum_{s=1}^{p-1}\sum_{i=0}^{\kappa_s}(-1)^{\kappa_1+\cdots+\kappa_{s-1}+s-1+i}\phi\circ\partial_s^i\right)\circ\left(\sum_{\sigma}(-1)^{\ell_1+\cdots+\ell_{p-1}+p-1+\sharp(\sigma)}D_{p,\sigma}\right)$$

$$(b) = \left(\sum_{s=p,p+1}\sum_{i=0}^{\kappa_s}(-1)^{\kappa_1+\cdots+\kappa_{s-1}+s-1+i}\phi\circ\partial_s^i\right)\circ\left(\sum_{\sigma}(-1)^{\ell_1+\cdots+\ell_{p-1}+p-1+\sharp(\sigma)}D_{p,\sigma}\right)$$

$$(c) = \left(\sum_{s=p+2}^{n}\sum_{i=0}^{\kappa_s}(-1)^{\kappa_1+\cdots+\kappa_{s-1}+s-1+i}\phi\circ\partial_s^i\right)\circ\left(\sum_{\sigma}(-1)^{\ell_1+\cdots+\ell_{p-1}+p-1+\sharp(\sigma)}D_{p,\sigma}\right)$$

Using (3.6.5) we easily get:

$$(a) = \left(\sum_{s=1}^{p-1}\sum_{i=0}^{\kappa_s}(-1)^{\kappa_1+\dots+\kappa_{s-1}+s-1+i}\phi\circ\partial_s^i\right)\circ\left(\sum_\sigma(-1)^{\ell_1+\dots+\ell_{p-1}+p-1+\sharp(\sigma)}D_{p,\sigma}\right) =$$
$$-\sum_{s=1}^{p-1}\sum_{i=0}^{\kappa_s}\left(\sum_\sigma(-1)^{\kappa_1+\dots+\kappa_{s-1}+s-1+i+\ell'_1+\dots+\ell'_{p-1}+p-1+\sharp(\sigma)}\phi\circ D_{p,\sigma}\circ\partial_s^i\right) =$$
$$-\sum_{s=1}^{p-1}\sum_{i=0}^{\kappa_s}(-1)^{\kappa_1+\dots+\kappa_{s-1}+s-1+i}\left(\sum_\sigma(-1)^{\ell'_1+\dots+\ell'_{p-1}+p-1+\sharp(\sigma)}\phi\circ D_{p,\sigma}\right)\circ\partial_s^i =$$
$$-\left(\sum_\sigma(-1)^{\ell'_1+\dots+\ell'_{p-1}+p-1+\sharp(\sigma)}\phi\circ D_{p,\sigma}\right)\circ\left(\sum_{s=1}^{p-1}\sum_{i=0}^{\kappa_s}(-1)^{\kappa_1+\dots+\kappa_{s-1}+s-1+i}\partial_s^i\right) = -(a')$$

where:

$$\ell'_j = \begin{cases}\ell_j & \text{if } j\neq s\\ \ell_s+1 & \text{otherwise}\end{cases}$$

Similarly:

$$(c) = \left(\sum_{s=p+2}^{n}\sum_{i=0}^{\kappa_s}(-1)^{\kappa_1+\dots+\kappa_{s-1}+s-1+i}\phi\circ\partial_s^i\right)\circ\left(\sum_\sigma(-1)^{\ell_1+\dots+\ell_{p-1}+p-1+\sharp(\sigma)}D_{p,\sigma}\right) =$$
$$-\sum_{s=p+2}^{n}\sum_{i=0}^{\kappa'_s}\left(\sum_\sigma(-1)^{\kappa'_1+\dots+\kappa'_{s-2}+s-2+i+\ell_1+\dots+\ell_{p-1}+p-1+\sharp(\sigma)}\phi\circ D_{p,\sigma}\circ\partial_{s-1}^i\right) =$$
$$-\sum_{s=p+1}^{n-1}\sum_{i=0}^{\kappa'_s}(-1)^{\kappa'_1+\dots+\kappa'_{s-1}+s-1+i}\left(\sum_\sigma(-1)^{\ell_1+\dots+\ell_{p-1}+p-1+\sharp(\sigma)}\phi\circ D_{p,\sigma}\right)\circ\partial_s^i =$$
$$-\left(\sum_\sigma(-1)^{\ell_1+\dots+\ell_{p-1}+p-1+\sharp(\sigma)}\phi\circ D_{p,\sigma}\right)\circ\left(\sum_{s=p+1}^{n-1}\sum_{i=0}^{\kappa'_s}(-1)^{\kappa'_1+\dots+\kappa'_{s-1}+s-1+i}\partial_s^i\right) = -(c')$$

where:

$$\kappa'_i = \begin{cases}\kappa_i & \text{if } i<p;\\ \kappa_p+\kappa_{p+1} & \text{if } i=p;\\ \kappa_{i+1} & \text{if } i>p;\end{cases}$$

The last and more tricky summand is the following, which follows from (3.6.6):

$$(b) = \left(\sum_{s=p,p+1}\sum_{a=0}^{\kappa_s}(-1)^{\kappa_1+\dots+\kappa_{s-1}+s-1+a}\phi\circ\partial_s^a\right)\circ\left(\sum_\sigma(-1)^{\ell_1+\dots+\ell_{p-1}+p-1+\sharp(\sigma)}D_{p,\sigma}\right) =$$
$$-\sum_{i=0}^{\kappa'_p}\left(\sum_{\tilde\sigma}(-1)^{\kappa'_1+\dots+\kappa'_{p-1}+p-1+j+\ell_1+\dots+\ell_{p-1}+p-1+\sharp(\tilde\sigma)}\phi\circ D_{p,\tilde\sigma}\circ\partial_p^i\right) =$$
$$-\left(\sum_\sigma(-1)^{\ell_1+\dots+\ell_{p-1}+p-1+\sharp(\sigma)}\phi\circ D_{p,\sigma}\right)\circ\left(\sum_{i=0}^{\kappa'_p}(-1)^{\kappa'_1+\dots+\kappa'_{p-1}+p-1+i}\partial_p^i\right) = -(b')$$

where:

$$\kappa_i' = \begin{cases} \kappa_i & \text{if } i < p; \\ \kappa_p + \kappa_{p+1} & \text{if } i = p. \end{cases}$$

and $\tilde{\sigma}$ is the "extended" shuffle, which is explained in the following lemma:

Lemma 3.7.1. *Let ∂_p^a and $D_{p,\sigma} = (\alpha, \beta) \colon [\kappa_p'] \to [\kappa_p] \times [\kappa_{p+1}]$. Let $i \coloneqq \min\{j \in [\kappa_p'] \mid \alpha(j) = a\}$. Then we set $D_{p,\tilde{\sigma}} \coloneqq (\tilde{\alpha}, \tilde{\beta}) \colon [\kappa_p' + 1] \to [\kappa_p + 1] \times [\kappa_{p+1}]$, where:*

$$\tilde{\alpha}(j) = \begin{cases} \alpha(j) & \text{if } j < i; \\ a & \text{if } j = i; \\ \alpha(j-1) + 1 & \text{if } j > i. \end{cases} \quad \text{and} \quad \tilde{\beta}(j) = \begin{cases} \beta(j) & \text{if } j \le i; \\ \beta(j-1) & \text{if } j > i. \end{cases}$$

Then we have:

$$(-1)^{\kappa_1 + \dots + \kappa_{p-1} + p - 1 + a + \ell_1 + \dots + \ell_{p-1} + p - 1 + \sharp(\sigma)} \partial_p^a \circ D_{p,\sigma} =$$
$$-(-1)^{\kappa_1' + \dots + \kappa_{p-1}' + p - 1 + j + \ell_1 + \dots + \ell_{p-1} + p - 1 + \sharp(\tilde{\sigma})} D_{p,\tilde{\sigma}} \circ \partial_p^i$$

and similarly for ∂_{p+1}^b and $D_{p,\sigma}$.

Proof. Straightforward. □

Now the desired equality: $\Omega_\Delta^p(d\phi) = -d\Omega_\Delta^p(\phi)$ follows since

$$d\Omega_\Delta^p(\phi) = (a') + (b') + (c').$$

(2) Now we need to prove that Ω_Δ^p and Υ_Δ^q satisfy the simplicial identities. The first

$$\Omega_\Delta^p \circ \Omega_\Delta^q = \Omega_\Delta^{q-1} \circ \Omega_\Delta^p \text{ if } p < q;$$

follows directly from (3.6.1) for $p < q - 1$; for $p = q - 1$, it follows from (3.6.2).

The second identity

$$\Upsilon_\Delta^p \circ \Upsilon_\Delta^q = \Upsilon_\Delta^{q+1} \circ \Upsilon_\Delta^p \text{ if } p \le q;$$

follows directly from (3.6.20).

Now the last identity:

$$\Omega_\Delta^p \circ \Upsilon_\Delta^q = \begin{cases} \Upsilon_\Delta^{q-1} \circ \Omega_\Delta^p & \text{if } p < q; \\ \text{id} & \text{if } p = q, q+1; \\ \Upsilon_\Delta^q \circ \Omega_\Delta^{p-1} & \text{if } p > q+1. \end{cases}$$

easily follows from (3.6.15). □

Chapter 4

Towards a homotopy 3-algebra structure on $\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}}))$

E adess, përme maleur, l'hai mach
ëd feuje,
che 'l vent a s-cianca, ël vent ëd la
sità;
e i podrai nen canté, s'i peuss nen
cheuje
jë spi d'òr an sle pere 'd mia carzà.
L'hai da manca dël sol e dla frëscura:
l'hai mach un seugn ch'a rij mentre a
s'avsin-a;
pòrtme, pare, lassù 'nt col'aria pura,
lassù mi veuj arnasse! An sla colin-a!

O. Gallina,
Mia tera

In this chapter we phrase out possible strategies in order to show that $\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}}))$ has a $\mathrm{Ch}_{\bullet}(\mathrm{E}_3, \Bbbk)$-algebra structure.

4.1 2-dimensional lattice paths operad

As we mentioned in Remark 3.4.8, the relative totalization $\mathrm{Rp}_* A(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}})$, for $\mathrm{Id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$, fails to be a 2-commutative cosimplicial monoid (in the sense of [BD], see Definition 3.4.6), but it enjoys the 2-commutativity only up to homotopy. Here is an attempt to verbalize this concept.

4.1.1 A 2-categorical version of Disk_2

As we saw at the beginning of Subsection 1.4.4, Joyal-duality between ordinals and intervals is given by a functor:

$$F := \underline{\mathrm{Cat}}(-, [1]) \colon \Delta^{op} \to \mathcal{I} \colon [n] \mapsto \underline{\mathrm{Cat}}([n], [1]) = \langle n+1 \rangle.$$

Let us consider the 2-dimensional analogue of this functor F:

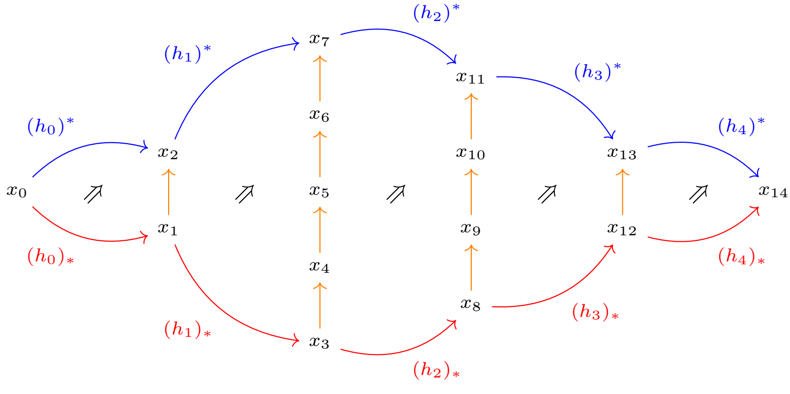
$$\mathfrak{F} \coloneqq \underline{\mathrm{Cat}_2}(-, ([1];[1])) \colon \Theta_2^{\mathrm{op}} \to ?$$

? should be a subcategory of Cat_2, which we will define below.

Given $T \in Ob(\Theta_2)$, we can explicitly describe its value. $\mathfrak{F}(T)$ is the 2-category:

- whose objects have a 2-disk structure (see Subsection 1.4.4);
- whose 1-morphisms are freely generated by:
  - the vertical morphisms (inherited by the 1-interval structure of the fibers in the 2-disk) of each column,
  - between two adjacent columns i and $i+1$, there are exactly two horizontal 1-morphisms, one with source the minimum object of i-th and target the minimum object of the $i+1$-th column (called the bottom horizontal morphism), the other with source the maximum object of i-th and target the maximum object of the $i+1$-th column (called the top horizontal morphism).
- whose 2-morphisms are freely generated by the single evident 2-morphism between two adjacent columns, i.e. denoting by v_i the composition of all the vertical morphisms in the column i and by $(h_i)_*$ (respectively, $(h_i)^*$) the bottom (respectively, top) morphism from the i-th to $i+1$-th column, we have a 2-morphism $\alpha_i \colon v_{i+1} \circ (h_i)_* \Rightarrow (h_i)^* \circ v_i$.

Example 4.1.1. In order to help the visualization, let us consider a pictorial example. If $T \in Ob(\Theta_2)$, $T = (4; 0, 3, 2, 0)$, then its dual $\mathfrak{F}(T)$ is the following 2-category:



(4.1.1)

We drew the **vertical** morphisms in orange and the **horizontal** morphisms in red and blue, because it will help us in understanding the image of $\mathfrak{F}(T)$ through a 2-functor Φ.

Given $\Phi = (\phi; \phi^i) \colon S \to T$ in Θ_2, it is easy to see that $\mathfrak{F}(\Phi) \colon \mathfrak{F}(T) \to \mathfrak{F}(S)$ is a 2-functor whose underlying object function is a map of 2-disks.

Let us now give the target subcategory ? of the functor $\mathfrak{F}$:

Definition 4.1.2. Dat_2 is the subcategory of Cat_2:

- whose objects are 2-categories $\mathcal{D}$ such that the objects of $\mathcal{D}$ have a "2-interval" structure. In practice this amounts to ask that the underlying 1-category of $\mathcal{D}$ is a tuple $(A; \{I_a\}_{a\in Ob(A)})$, where A is a bipointed category, such that any object $a \in Ob(A)$ is endowed with a fiber $I_a \in \mathrm{Cat}_{*,*}$, and we require that the fiber over the two distinguished objects are $*$, the final bipointed category. The 1-morphisms of any I_a are the **vertical** morphisms. Moreover, any morphism $f\colon a \to b$ of A induces unique **horizontal** morphisms f^* (respectively, f_*) from the maximum (respectively, minimum) point of I_a to the maximum (respectively, minimum) point of I_b.
- whose 2-functors $F\colon \mathcal{D} \to \mathcal{E}$ are 2-functors preserving the 2-interval structure, i.e. sending the horizontal (respectively, vertical) 1-morphisms to horizontal (respectively, vertical) 1-morphisms.

Example 4.1.3. Given $T \in Ob(\Theta_2)$, $\mathfrak{F}(T)$ is an object of Dat_2, and more generally any Joyal 2-disks $B^2 \to B^1$ uniquely determines an object of Dat_2: $(\langle n+1\rangle; \{\langle n_i+1\rangle\})$, where $\langle n_i + 1\rangle \in \mathcal{I} \subset \mathrm{Cat}_{*,*}$ is the 1-interval fiber over $i \in \langle n+1\rangle = B^1$, and the 2-morphisms are as for $\mathfrak{F}(T)$ above.

We will simply denote the objects of Dat_2 by tuples $(A; \{I_a\})$ when it is clear from the context what are the 2-morphisms.

As expected, we have a similar Joyal duality:

Proposition 4.1.4 (Joyal duality)**.** *For any $S, T \in \Theta_2$:*

$$\Theta_2(S,T) = \mathrm{Dat}_2(\mathfrak{F}(T), \mathfrak{F}(S)),$$

where the functor $\mathfrak{F}(-)\colon \Theta_2^{op} \to \mathrm{Dat}_2$:

$$S = ([n]; [\ell_1], \dots, [\ell_n]) \mapsto \mathfrak{F}(S) = (\langle n+1\rangle; \{\langle \ell_i + 1\rangle\}) \tag{4.1.2}$$

is defined as above.

Proof. It follows by definition. □

4.1.2 Monoidal products on Dat_2

We know there exist three monoidal products on Cat_2: the cartesian product, the funny tensor product and the Gray tensor product introduced by Gray in [G]. The funny tensor product for higher categories and algebras of higher operads has been introduced by Weber in [Web]. Though we are in a special subcategory, and we can try to construct our own monoidal product, behaving in a nice homotopical way. Namely, we would like to have a mixed product: Gray tensor product for the horizontal morphisms and funny tensor product for the vertical morphisms.

Given $\mathcal{C}, \mathcal{D} \in \mathrm{Dat}_2$, we define the 2-category $\mathcal{C}\tilde{\otimes}\mathcal{D}$ in the following way:

- the objects of $\mathcal{C}\tilde{\otimes}\mathcal{D}$ are pairs (X, Y), $X \in Ob(\mathcal{C})$, $Y \in Ob(\mathcal{D})$, i.e. $Ob(\mathcal{C}\tilde{\otimes}\mathcal{D}) := Ob(\mathcal{C}) \times Ob(\mathcal{D})$.

- the 1-morphisms are equivalence classes of strings

$$(f_n, g_n) \dots (f_2, g_2)(f_1, g_1)$$

where, for all i either f_i or g_i is the identity, and the equivalence relation is given by

$$(f_i, \mathrm{id})(f_{i-1}, \mathrm{id}) \sim (f_i \circ f_{i-1}, \mathrm{id})$$

and

$$(\mathrm{id}, g_i)(\mathrm{id}, g_{i-1}) \sim (\mathrm{id}, g_i \circ g_{i-1})$$

- the 2-morphisms are the most subtle point of $\tilde{\otimes}$. First of all:

  (i) for $f \in \mathcal{C}$ and $g \in \mathcal{D}$ both horizontal morphisms, there is an invertible 2-cell:

$$\alpha_{f,g} \colon (\mathrm{id}, g)(f, \mathrm{id}) \Rightarrow (f, \mathrm{id})(\mathrm{id}, g)$$

  (ii) for $f \in \mathcal{C}$ horizontal morphism and $g \in \mathcal{D}$ vertical morphism (or vice versa), there is a 2-cell (not necessarily invertible)

$$\beta_{f,g} \colon (\mathrm{id}, g)(f, \mathrm{id}) \Rightarrow (f, \mathrm{id})(\mathrm{id}, g)$$

  We do not construct 2-cells between composition of vertical morphisms.

We expect that $\tilde{\otimes}$ is functorial in both arguments, and associative, thus defining a monoidal functor:

$$\tilde{\otimes} \colon \mathrm{Dat}_2 \times \mathrm{Dat}_2 \to \mathrm{Dat}_2$$

where the unit object is the 2-category $\{*\}$, with one object, only identity 1-morphism and 2-morphism.

4.1.3 2-dimensional lattice paths operad $\mathbb{L}$

Once we have a monoidal functor $\tilde{\otimes}$ as above, it is standard to construct a coendomorphism operad out of it:

Definition 4.1.5. The 2-dimensional lattice paths operad is the Θ_2 coloured operad in *Set* with:

$$\mathbb{L}(T_1, \dots, T_k; T) \coloneqq \mathrm{Dat}_2(\mathfrak{F}(T), \mathfrak{F}(T_1)\tilde{\otimes} \dots \tilde{\otimes}\mathfrak{F}(T_k)),$$

where the operad substitution maps are induced by tensoring and composing in $(\mathrm{Dat}_2, \tilde{\otimes})$.

Remark 4.1.6. By Proposition 4.1.4, the underlying category of $\mathbb{L}$ is Θ_2:

$$\mathbb{L}(S; T) \coloneqq \mathrm{Dat}_2(\mathfrak{F}(T), \mathfrak{F}(S)) = \Theta_2(S, T) \tag{4.1.3}$$

and this does not depend on the choice of the monoidal functor.

Example 4.1.7. Let $T = ([n]; [\ell_1], \dots, [\ell_n])$, $T_i = ([n_i]; [\ell^i_1], \dots, [\ell^i_{n_i}])$, $1 \leq i \leq k$, be objects of Θ_2. A 2-dimensional lattice path $x \in \mathbb{L}(T_1, \dots, T_k; T)$, unpacking the definition, is given by:

- an element of $\mathcal{L}(n_1, \ldots, n_k; n)$, i.e. a functor

$$\phi\colon \langle n+1\rangle \to \langle n_1+1\rangle \tilde{\otimes} \ldots \tilde{\otimes} \langle n_k+1\rangle,$$

  taking 0 to $(0, \ldots, 0)$ and $n+1$ to $(n_1+1, \ldots, n_k+1)$;

- for each $1 \leq i \leq n$, we have an element of $\mathcal{L}(\ell^1_{\phi(i)_1}, \ldots, \ell^k_{\phi(i)_k}; \ell_i)$, where $\phi(i) = (\phi(i)_1, \ldots, \phi(i)_k)$, i.e. a functor

$$\phi_i\colon \langle \ell_i+1\rangle \to \langle \ell^1_{\phi(i)_1}+1\rangle \tilde{\otimes} \ldots \tilde{\otimes} \langle \ell^k_{\phi(i)_k}+1\rangle$$

  taking (0) to $(0, \ldots, 0)$ and (ℓ_i+1) to $(\ell^1_{\phi(i)_1}+1, \ldots, \ell^k_{\phi(i)_k}+1)$.

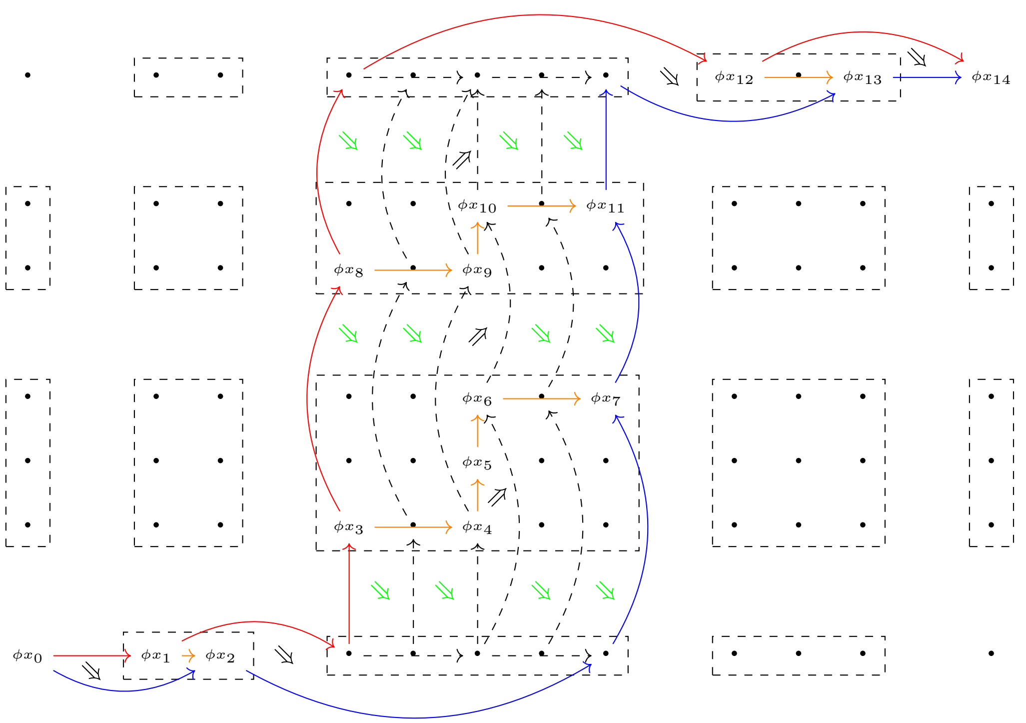
satisfying some constraints: for each $0 \leq i \leq n$ denote by $\phi^{i,i+1}$ the image through ϕ of the 1-morphism $i \to i+1$ in $< n+1 >$. Then we also have a 2-morphism:

$$\alpha_i : (\phi^{i,i+1})^* \circ \phi_i \Rightarrow \phi_{i+1} \circ (\phi^{i,i+1})_*$$

This 2-morphism gives us some control on the behavior of the lattice paths ϕ_i on the fibers.

Thus we can represent a 2-dimensional lattice path x by a tuple of functors $\{\phi; \phi_i\}$, all of which are elements of $\mathcal{L}$, the lattice paths operad.

Now let us give another pictorial example: let $T = (4; 0, 3, 2, 0)$ as in Example 4.1.1, $T_1 = (3; 0, 3, 1)$ and $T_2 = (2; 1, 0)$. Then an element x of $\mathbb{L}(T_1, T_2; T)$ can be represented by the following picture:



(4.1.4)

Remark 4.1.8. (a) Having drawn the 1-morphisms of $\mathfrak{F}(T)$ with different colors in Example 4.1.1, it makes it easier to understand their image through the functor $\phi\colon \mathfrak{F}(T) \to \mathfrak{F}(T_1)\tilde{\otimes}\mathfrak{F}(T_2)$ representing the element x above. The 2-morphisms are sent through ϕ to the evident composition of the 2-morphisms in the picture. The dashed rectangles are a way to pack the different "fibers" of our 2-category $\mathfrak{F}(T_1)\tilde{\otimes}\mathfrak{F}(T_2)$. Knowing that ϕ respect the "2-interval" structure at the level of objects, it follows that the orange morphisms are mapped inside a single dashed rectangle. Clearly we have not drawn all the 1-morphisms and 2-morphisms of the target 2-category, as this would have led to a wild and incomprehensible picture. Though it is important to underline that the green 2-morphisms in the target 2-category arise from the $\tilde{\otimes}$ tensor product, i.e. if we had taken the funny tensor product of 2-categories, we would not have been capable of mapping the 2-morphisms of the source 2-category to any composition whatsoever.

4.1.4 Complexity of lattice paths

In [BB] the authors defined a notion of complexity for a lattice path; let us recall the definition: for each $1 \leq i < j \leq k$, there are canonical projection functors

$$p_{ij}\colon \langle n_1+1\rangle\square\ldots\square\langle n_k+1\rangle \to \langle n_i+1\rangle\square\langle n_j+1\rangle$$

These functors, together with the unique functor in $\mathrm{Cat}_{*,*}(\langle 1\rangle, \langle n+1\rangle)$, induce maps

$$\phi_{ij}\colon \mathcal{L}(n_1,\ldots,n_k;n) \to \mathcal{L}(n_i,n_j;0) \qquad 1 \leq i < j \leq k$$

Definition 4.1.9. For each $x \in \mathcal{L}(n_1,\ldots,n_k;n)$ and each $1 \leq i < j \leq k$, let $c_{ij}(x)$ be the number of changes of directions (i.e. corners) in the lattice path $\phi_{ij}(x)$. The **complexity index** $c(x)$ of $x \in \mathcal{L}(n_1,\ldots,n_k;n)$ is defined by

$$c(x) = \max_{1\leq i<j\leq k} c_{ij}(x)$$

The m-th filtration stage $\mathcal{L}^{(m)}$ of the lattice paths operad $\mathcal{L}$ is defined by

$$\mathcal{L}^{(m)}(n_1,\ldots,n_k;n) = \{x \in \mathcal{L}(n_1,\ldots,n_k;n) | c(x) \leq m\}$$

We can use this notion of complexity to consider some subsets of $\mathbb{L}$. Let

$$\mathbb{L}^{(0)} := \{(\phi;\phi_\ell) \in \mathbb{L} | c(\phi) = 0\}$$

be the 2-dimensional lattice paths of complexity 0. These should be nothing but unary maps (as in the $\mathcal{L}^{(0)}$ case), i.e. $\mathbb{L}(S;T) = \Theta_2(S,T)$, so $\mathbb{L}^{(0)}$-algebras should be 2-cocellular objects.

Let now

$$\mathbb{L}^{(1)} := \{(\phi;\phi_\ell) \in \mathbb{L} \mid c(\phi) \leq 1\}$$

be the 2-dimensional lattice paths of complexity 1. These elements are either unary maps, as above, otherwise are tuples $\{\phi;\phi_\ell\}$ with ϕ of complexity 1 in $\mathcal{L}$, which implies that $\phi_\ell \in \mathcal{L}^{(0)}$: indeed, if ϕ is of complexity 1, this means it is a path running along the edges of our k-dimensional grids. The fibers over these points are simpler, namely they are intervals, rather than $\tilde{\otimes}$ tensor product of intervals. Thence the functors ϕ_ℓ are elements of $\mathcal{L}^{(0)}$. By the above inspection, $\mathbb{L}^{(1)}$-algebras should be 2-cocellular $\square'$-monoids, where $\square'$ is the analogue of the Batanin $\square$-product of cosimplicial spaces.

When we jump to complexity 2, everything gets screwed as the complexity $c(\phi_\ell)$ of a lattice path $(\phi; \phi_\ell)$ for $c(\phi) = 2$ is not bound anymore. Thus it is natural to ask: do higher filtrations $\mathbb{L}^{(n)}$, $n \geq 2$, of $\mathbb{L}$ exist? Do these $\mathbb{L}^{(n)}, n \geq 2$ share similar properties as $\mathcal{L}^{(n)}$? What are $\mathbb{L}^{(n)}$-algebras, for $n \geq 2$? These questions are unsolved.

Once developed a well-defined complexity index for 2-dimensional lattice paths, one could construct a 3-operad coloured in Θ_2 following the idea in [[BB], Introduction], later defined in [[BM2], Section 6].

Remark 4.1.10. Another possible way of proving the homotopy 3-algebra structure on $\mathrm{Tot}_{\Theta_2}(A(\mathrm{Id}_\mathcal{C}, \mathrm{Id}_\mathcal{C}))$ might be in the framework of homotopy centers, as developed by Batanin and Marckl in [BM1] and [BM2]. In that setting, one may first define some trioidal category $\mathcal{T}$ enriched over some monoidal category $\mathcal{K}$ and all the trioidal analogous results of op. cit. After that, one could show that our 2-cocellular object $A(\mathrm{Id}_\mathcal{C}, \mathrm{Id}_\mathcal{C})$ is the homotopy center of some duoid N inside a specific duoidal category $\mathcal{D}$. Then the (still conjectural) $(n+1)$-oidal Deligne's conjecture would imply that there is a canonical action of a contractible 3-operad on the homotopy center of a duoid N lifting the trioid structure on the center of N.

Bibliography

[Ara] D. Ara, Higher quasi-categories vs higher Rezk spaces, *J. of K-theory* **14**(3), (2014), 701-749

[Ba1] M. A. Batanin, Homotopy coherent category theory and A_∞-structures in monoidal categories, *J. of Pure Appl. Alg.* **123**, (1998), 67-103

[Ba2] M. A. Batanin, Monoidal globular categories as a natural environment for the theory of weak n-categories, *Adv. Math.* **136**(1), (1998), 39-103

[Ba3] M. A. Batanin, Symmetrisation of n-operads and compactification of real configuration spaces, *Adv. Math* **211**(2), (2007), 684-725

[Ba4] M. A. Batanin, The Eckmann-Hilton argument and higher operads, *Adv. Math.* **217**(1), (2008), 334-385

[BB] M. A. Batanin, C. Berger, The lattice path operad and Hochschild cochains, *Contemp. Math.* **504**, (2009), 23-52

[BBM] M. A. Batanin, C. Berger, M.Markl, Operads of natural operations I: Lattice paths, braces and Hochschild cochains, *Séminaires et Congrès* **26**, (2012), 1-33

[BD] M. A. Batanin, A. A. Davydov, Cosimplicial monoids and deformation theory of tensor categories, preprint arXiv:2003.13039

[BM1] M. A. Batanin, M. Markl, Centers and homotopy centers in enriched monoidal categories, *Adv. Math.* **230**(4-6), (2012), 1811-1858

[BM2] M. A. Batanin, M. Markl, Operadic categories and Duoidal Deligne's conjecture, *Adv. Math.* **285**(5), (2015), 1630-1687

[Ben] J. Bénabou, Introduction to Bicategories, *Lecture Notes in Mathematics* **47**, Springer (1967), 1-77

[Be1] C. Berger, A Cellular nerve for higher categories, *Adv. Math.* **169**(1), (2002), 118-175

[Be2] C. Berger, Iterated wreath product of the simplex category and iterated loop spaces, *Adv. Math.* **213**(1), (2007), 230-270

[BCSW] R. Betti, A. Carboni, R. Street, R. Walters, Variation through enrichment, *Journal of Pure and Applied Algebra* **29**, (1983), 109-127

[BK] A. Bondal, M. M. Kapranov, Enhanced Triangulated Categories, *Math. URSS Sbornik* **70**(1), (1991), 93-107

[Co] F. R. Cohen, The homology of C_{n+1}-spaces, $n \geq 0$, *The homology of iterated loop spaces*, Springer-Verlag LNM **533**, (1976), 207-351

[COS] A. Canonaco, M. Ornaghi, P. Stellari, Localizations of the category of A_∞-categories and internal Homs, *Doc. Math.* **24**, (2019), 2463-2492

[CY] L. Crane, D. N. Yetter. Deformations of (bi)tensor categories, *Cahier de Topologie et Géometrie Differentielle Catégorique* **39**(3), (1998), 163-180

[Da] A. A. Davydov, Twisting of monoidal structures, *Preprint of MPI*, MPI**/95-123**, (1995), arXiv:q-alg/9703001

[Dr] V. Drinfeld, DG quotients of dg-categories, *J. Algebra* **272**(2), (2004), 643-691

[DS] W. G. Dwyer, J. Spalinski, Homotopy theories and model categories, in: *Handbook on Algebraic Topology*, Elsevier, (1995), 73-126

[Fa] G. Faonte, Simplicial Nerve of an A_∞-category, *Theory and Applications of Categories* **32**(2), (2017), 31-52

[Fu] K. Fukaya, Morse homotopy, A_∞-category and Floer homologies, *MSRI preprint* **020-94**, (1993)

[Ge1] M. Gerstenhaber, The Cohomology Structure of an Associative Ring, *Ann. Math.* **78**(2), (1963), 267-288

[Ge2] M. Gerstenhaber, On the Deformation of Rings and Algebras, *Ann. Math.* **79**(1), (1964), 59-103

[GJ] P. G. Goerss, J. F. Jardine, *Simplicial Homotopy Theory*, Birkhäuser Progress in Mathematics, Vol. 174, (1999), x+537 pp

[GK] V. Ginzburg, M. Kapranov, Koszul Duality for operads, *Duke Math. J.* **76**(1), (1994), 203-272

[GS] P. Goerss and K. Schemmerhorn, *Model Categories and Simplicial Methods*, Notes from lectures given at the University of Chicago, (August 2004), available at P.Goerss' webpage

[G] J. W. Gray, *Formal category theory Adjointness for 2-categories*, Lectures Notes in Mathematics **391**, Springer-Verlag, Berlin-New York, (1974), xii+284pp

[Hi] P. S. Hirschhorn, *Model Categories and Their Localizations*, AMS Mathematical Surveys and Monographs, Vol. 99, (2003) xiv+457 pp

[Ho] M. Hovey, *Model categories*, AMS Math. Surveys and Monographs, vol. 62, (1999), x+203 pp

[J] A. Joyal, Disks, duality, and Θ-categories, preprint (September 1997)

[JS] A. Joyal, R.Street, Braided Tensor Categories, *Adv. Math.* **102**(1), (1993), 20-78

[Kel1] B. Keller, On the cyclic homology of exact categories, *J. Pure Appl. Algebra* **136**(1), (1999), 1-56

[Kel2] B. Keller, On differential graded categories, *International Congress of Mathematicians* **II**, Eur. Math. Soc., Zürich, (2006), 151-190

[Ke] G. M. Kelly, Chain maps inducing zero homology maps, *Proc. Cambridge Philos. Soc.* **61**, (1965), 847-854

[KL] G. M. Kelly, S. Lack, V-Cat is locally presentable or locally bounded if V is so, *Theory and Applications of categories* **8**(23), (2001), 555-575

[Ko1] M. Kontsevich, Lectures at ENS, Paris, Spring 1998: notes taken by J.Bellaiche, J.-F.Dat, I.Marin, G.Racinet and H.Randriambololona, unpublished

[Ko2] M. Kontsevich, Homological algebra of Mirror Symmetry, *Proceedings of the International Congress of Mathematicians*, Zürich 1994, vol. I, Birkhauser 1995, 120-139

[Ko3] M. Kontsevich, Operads and Motives in Deformation Quantization, *Lett. Math. Phys* **48**, (1999), 35-72

[KS1] M. Kontsevich, Y. Soibelman, Deformations of algebras over operads and Deligne's conjecture, in *Conference Moshe Flato 1999, Quantization, Deformations, and Symmetries* **I**, Ed. G. Dito and D. Sternheimer, Kluwer Academic Publishers, (2000), 255–307

[KS2] M. Kontsevich, Y. Soibelman, Notes on A_∞-algebras, A_∞-categories, and non-commutative geometry, in: "Homological Mirror Symmetry: New Deevelopments and Perspectives" (A.Kapustin et al. (Eds.)), Lect. Notes in Physics **757** (Springer, Berlin Heidelberg 2009), 153-219

[LH] K. Lefèvre-Hasegawa, Sur les A-infini catégories, PhD thesis, preprint arXiv:math/0310337

[Li] F. E. J. Linton, Coequalizers in categories of algebras, in: Springer Lecture Notes in Mathematics **80**, (1969), 75-90

[Ly1] V. Lyubashenko, Category of A_∞-categories, *Homology Homotopy Appl.* **5**(1), (2003), 1-48

[Ly2] V. Lyubashenko, Homotopy unital A_∞-algebras, *J. Algebra* **329**(1), (2011), 190-212

[Ly3] V. Lyubashenko, Bar and cobar constructions for curved algebras and coalgebras. *Mat. Stud.* **40**(2), (2013), 115-131

[LyMa] V. Lyubashenko, O. Manzyuk, Unital A_∞-categories, *Problems of topology and related questions* (V.V. Sharko, ed.), **3**, Proc. of Inst. of Mathematics NASU, Inst. of Mathematics, Nat. Acad. Sci. Ukraine, Kyiv, (2006), n. 3, 235-268

[M] J. P. May, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Mathematics **271**, Springer-Verlag, Berlin-New York, (1972), viii+175 pp

[ML1] S. Mac Lane, *Homology*, 2nd ed., Springer Classics in Mathematics, (1995), x+422pp

[ML2] S. Mac Lane, *Categories for the working mathematician*, 2nd ed., Springer GTM, (1998), xii+314pp

[MLM] S. Mac Lane, I. Moerdijk, *Sheaves in Geometry and Logic: a first introduction to Topos Theory*, Springer, 1992, xii+630 pp

[MS1] J. McClure, J. Smith, A solution of Deligne's conjecture, In *Recent progress in homotopy theory (Baltimore, MD, 2000), Contemp. Math.* **293**, (2002), 153-193

[MS2] J. McClure, J. Smith, Multivariable cochain operations and little n-cubes, *J. Amer. Math. Soc.* **16**(3), (2003), 681-704

[Po] L. Positselski, Weakly curved A_∞-algebras over a topological local ring. *Mm. Soc. Math. Fr. (N.S.)* **159**, (2018), vi+206 pp

[PS1] P. Panero, B. Shoikhet, A closed model structure on the category of weakly unital dg-categories, II, *Theory and Applications of Categories* **37**(15), (2021), 388-417

[PS2] P. Panero, B. Shoikhet, The closed model structure on the category of weakly unital dg-categories: an addendum, *Theory and Applications of Categories* **38**(5), (2022), 135-155

[PS3] P. Panero, B. Shoikhet, The category Θ_2, derived modifications and deformation theory of monoidal categories, in preparation

[Q] D. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics. Vol. **43**. Berlin, New York, (1967), v+159 pp

[R1] E. Riehl, *Categorical homotopy theory*, New Mathematical Monographs **28**, Cambridge University Press, (2014), xvi+276 pp

[R2] E. Riehl, *Category Theory in context*, Dover Publ., (2016), xviii+240 pp

[Sh1] B. Shoikhet, Graded Leinster monoids and generalized Deligne conjecture for 1-monoidal abelian categories, *I.M.R.N.* **19**, (2018), 5857–5937

[Sh2] B. Shoikhet, On the twisted tensor product of small dg-categories, *J. NonComm. Geom.* **14**(2), (2020), 789–820

[Sh3] B. Shoikhet, The twisted tensor product of dg-categories and a contractible 2-operad, preprint arXiv:1807.04305v2

[Tab] G. Tabuada, A Quillen model structure on the category of dg-categories C. R. Math. Acad. Sci. Paris, 340, (2005), 15-19

[Tam1] D. E. Tamarkin, Formality of Chain Operad of Small Squares, *Lett. Math. Phys.* **66**, (2003), 65-72

[Tam2] D. E. Tamarkin, What do dg-categories form?, *Compositio Mathematica* **143**(5), (2007), 1335-1358.

[To1] B. Toën, The homotopy theory of dg-categories and derived Morita theory, *Invent. Math.* **167**, (2007), 615–667

[To2] B. Toën, Lectures on dg-categories, *Topics in Algebraic and Topological K-Theory*, (2011), 243-301

[Wo] H. Wolff, V-cat and V-graph, *Journal of Pure and Applied Algebra*, **4** (1974), 123-135

[Web] M. Weber, Free Products of Higher Operad Algebras, *Theory and Applications of Categories* **28**(2), (2013), 24-65

[We] C. Weibel, *An introduction to homological algebra*, Cambridge University Press, 1994, xiv+450 pp

[Yau] D. Yau, *Colored Operads*, American Mathematical Society, Graduate Studies in Mathematics **170**, (2016), xxviii+428 pp

[Ye1] D. N. Yetter, Braided deformations of monoidal categories and vassiliev invariants. In E. Getzler and M. Kapranov, editors, *Higher Category Theory*, American Mathematical Society, (1998), A.M.S. Contemp. Math. **230**, 117–134

[Ye2] D. N. Yetter, Abelian Categories of Modules over a (Lax) Monoidal Functor, *Adv. Math.* **174**(2), (2003), 266-309