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An Extension of the Perplexing Polynomial Puzzle

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Abstract. Originally published as a puzzle in 2005 [3], the *Perplexing Polynomial Puzzle* indeed is perplexing: any given polynomial $p(x)$ with nonnegative integer coefficients can be completely determined by just two evaluations. In this article, an extension is made to polynomials with arbitrary integer coefficients, by considering a simple translation $x \mapsto x + k$ with $k \in \mathbb{N}$ such that the result is a new polynomial with only nonnegative integer coefficients on which the original solution can be used. A proof is given that this is indeed always possible, and a method is constructed to determine a suitable k to do so.

1 History of the puzzle

For a given polynomial $p(x) = \sum_{j=0}^{n} a_j x^j$ with all $a_j \in \mathbb{N}$, it is possible to determine all the unknown coefficients a_j with just two evaluations. This surprising result was published in 2005 by I. B. Keene as a short puzzle named *A Perplexing Polynomial Puzzle* [3]. The idea is to ask first for $p(1)$, which is in fact $\sum_{j=0}^{n} a_j$ and as such an upper bound for the coefficients a_j. Afterwards, one asks for the value of $p(a)$ with $a > p(1)$, for instance $p(p(1) + 1)$. Writing $p(a)$ in base a will then yield the same coefficients as the original polynomial. A few months after the publication of Keene's puzzle, F. Bornemann and S. Wagon provided an argument to determine the polynomial with only (finitely many digits of) one of its values, namely $p(\pi)$, exploiting the fact that π is a transcendental number [1].

Another extension can be found in [6] where B. Richmond alters the puzzle to polynomials with arbitrary integer coefficients, removing the condition that the coefficients should be nonnegative. The key argument in her solution is to have a bound b on the absolute value of the coefficients of the polynomial and one evaluation $p(2b + 1)$. In fact, Richmond shows that this can be refined by constructing an upper bound using just a lower bound for the given coefficients and one extra evaluation, resulting again in two evaluations at appropriate integers to determine the whole polynomial. Combining this result with the idea of [1], a lower bound on the coefficients and one evaluation at a transcendental number will suffice as well.

1.1 Preliminaries

In this article we use another approach to handle arbitrary integer coefficients. In what follows, we assume $p(x) = \sum_{j=0}^{n} a_j x^j$ to be a polynomial of degree n with coefficients $a_j \in \mathbb{Z}$ and $a_n \neq 0$. Furthermore we can assume $a_n > 0$ without loss of generality. We also assume a classical Alice–Bob approach to the puzzle in the sense that Alice defines the polynomial $p(x)$ in secret, and Bob tries to determine the coefficients a_j by asking basic arithmetical questions to Alice, including polynomial evaluations, but of course without giving the values of the coefficients nor the degree of the polynomial away for free.

The main ideas in this article are based on the following classical result by René Descartes, first described in his work *La Géométrie* in 1637:

Lemma 1 (Descartes' rule of signs)**.** *If the nonzero terms of a single-variable polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial (counted with multiplicity) is either equal to the number of sign changes between consecutive (nonzero) coefficients, or is less than it by an even number.*

A direct consequence of this rule is that for a polynomial with zero or one sign changes, the number of positive roots equals this number of sign changes. In particular, a polynomial with only positive coefficients, and thus zero sign changes, has no positive real roots.

Unfortunately, the converse is not true, meaning that it will not be sufficient to translate a polynomial to the left, i.e. $p_k(x) = p(x+k)$ with $k \in \mathbb{N}$, in such a way that all roots are negative as to have all the coefficients of p_k positive. Indeed, the largest positive root of the polynomial $p(x) = x^3 - 5x^2 + 7x - 1$ for example is $x \approx 0.1607$. By translating the polynomial 1 unit to the left, i.e. $p(x+1) = x^3 - 2x^2 + 2$, all roots will become negative, but the coefficients are still not all positive. We nevertheless conjecture (and prove in Theorem 2) that for every polynomial $p(x)$ with arbitrary integer coefficients there always does exist a $k \in \mathbb{N}$ sufficiently large such that all coefficients of $p_k(x)$ will become positive. In this example for instance, choosing $k = 3$ (or more) will result in a polynomial with nonnegative integer coefficients: $p(x+3) = x^3 + 4x^2 + 4x + 2$.

Before starting our quest for the right value of k, we observe the following:

Lemma 2. *Suppose* $p(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$. *Then the coefficients of the translated polynomial* $p_t(x) = p(x+t) = \sum_{j=0}^{n} b_j(t) x^j$ *are given by*

$$b_j(t) = \sum_{i=0}^{n-j} c_i t^i = \sum_{i=0}^{n-j} \binom{i+j}{i} a_{i+j} t^i$$

with all $c_i \in \mathbb{Z}$.

Proof. Follows from straightforward binomial calculations and regrouping of terms:

$$\begin{aligned} p(x+t) &= a_n (x+t)^n + a_{n-1}(x+t)^{n-1} + \ldots + a_1(x+t) + a_0 \\ &= a_n \sum_{i=0}^{n} \binom{n}{i} x^{n-i} t^i + a_{n-1} \sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i} t^i \\ &\quad + \ldots + a_1(x+t) + a_0 \\ &= a_n x^n + \left[\binom{n}{1} a_n t + \binom{n-1}{0} a_{n-1} \right] x^{n-1} \\ &\quad + \left[\binom{n}{2} a_n t^2 + \binom{n-1}{1} a_{n-1} t + \binom{n-2}{0} a_{n-2} \right] x^{n-2} \\ &\quad + \ldots + \left[\binom{n}{n} a_n t^n + \binom{n-1}{n-1} a_{n-1} t^{n-1} + \ldots + \binom{1}{1} a_1 t + \binom{0}{0} a_0 \right] \end{aligned}$$

$\square$

Note that the coefficients $b_j(t)$ of the translated polynomial can be seen as polynomials themselves (of degree $n-j$ in the variable t). We will therefore be looking for values of t such that each of these polynomials take on positive values.

2 An iterative approach

2.1 Theoretical proof

A first idea is to start with the first negative coefficient of $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, make it positive by considering a first translation $p_{k_1}(x) = p(x+k_1)$, start over with this new polynomial to tackle the next negative coefficient with a second translation $p_{k_1}(x+k_2) = p(x+k_1+k_2)$, and so on until all coefficients become positive. Doing so, $p_k(x)$ with $k = k_1 + k_2 + \ldots$ will then be a polynomial with only positive coefficients. The following theorems concretize this idea:

Theorem 1. *Consider $p(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ with $a_n > 0$ and its translation $p_t(x) = p(x+t)$. Let $0 \leq m \leq n-1$ be the largest index such that $a_m < 0$. Then there exists a $k \in \mathbb{N}$ such that the coefficient of x^m in $p_k(x)$ becomes positive.*

Proof. By Lemma 2, the coefficient $b_m(t)$ of x^m in $p_t(x)$ is given by

$$b_m(t) = \sum_{i=0}^{n-m} \binom{i+m}{i} a_{i+m} t^i.$$

Since for all $m < j \leq n$ the coefficients a_j are positive and a_m is negative by assumption, the coefficients of $b_m(t)$ have exactly one sign change. By Descartes' rule of signs $b_m(t)$ has a unique positive real root, say $t^* \in \mathbb{R}^+$. Since a_n, the coefficient of x^n in $p_t(x)$, is assumed to be positive, the polynomial $b_m(t)$ will consequently only take on positive values for all $t > t^*$. Therefore, by setting $k = \lceil t^* \rceil$, i.e. rounding t^* up to the smallest integer greater than t^*, the coefficient $b_m(k)$ of x^m in $p_k(x)$ will be positive. $\square$

Theorem 2. *For every polynomial $p(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ with $a_n > 0$ there exists a $k \in \mathbb{N}$ such that the translated polynomial $p_k(x) = p(x+k)$ has only positive coefficients.*

Proof. This follows from an iterative procedure based on Theorem 1.
Start by setting $k_0 = 0$. For every $1 \leq i \leq n$ we set $k_i = 0$ if the coefficient of x^{n-i} in the translated polynomial $p(x + k_0 + \ldots + k_{i-1})$ is positive or equal to zero, and set $k_i = \lceil t^* \rceil$ otherwise, where t^* is the unique positive root (cfr. the proof of Thm. 1) of the coefficient of x^{n-i} in the translated polynomial $p(x + k_0 + \ldots + k_{i-1} + t)$.
Defining $k = k_1 + k_2 + \ldots + k_n$ concludes the proof. $\square$

2.2 Example

Let's use the same example as before, i.e. $p(x) = x^3 - 5x^2 + 7x - 1$, to clarify the procedure set out by Theorem 2.
For $i = 1$, we observe that the coefficient of $x^{3-1} = x^2$ in the (original) polynomial $p(x+k_0) = p(x)$ is negative. We therefore calculate $p(x + k_0 + t)$, i.e.

$$p(x+t) = x^3 + (3t-5)x^2 + (3t^2 - 10t + 7)x + (t^3 - 5t^2 + 7t - 1),$$

to find that its coefficient of x^2 now is $b_2(t) = 3t - 5$ with unique positive root $t^* = \frac{5}{3}$. We therefore set $k_1 = \left\lceil \frac{5}{3} \right\rceil = 2$.
For $i = 2$, we first calculate $p(x + k_0 + k_1) = p(x+2) = x^3 + x^2 - x + 1$. The coefficient of $x^{3-2} = x$ in this polynomial is negative. Therefore, we now calculate $p(x + k_0 + k_1 + t)$, i.e.

$$p(x+2+t) = x^3 + (3t+1)x^2 + (3t^2 + 2t - 1)x + (t^3 + t^2 - t + 1)$$

to find that its coefficient of x now is $b_1(t) = 3t^2 + 2t - 1$ (note the single sign change, telling us that we will find exactly one positive root!) with roots -1 and $\frac{1}{3}$. We therefore set $k_2 = \left\lceil \frac{1}{3} \right\rceil = 1$.
For $i = 3 = n$, the last step, we see that $p(x + k_0 + k_1 + k_2) = p(x + 3) = x^3 + 4x^2 + 4x + 2$. The coefficient of $x^{3-3} = 1$ now is positive, meaning that we have to set $k_3 = 0$.
Finally we define $k = k_1 + k_2 + k_3 = 3$ and observe that $p(x + 3)$ has only positive coefficients as needed.

Please note that the exact values of all the intermediate results are irrelevant for the *guesser* of the polynomial (Bob). If Alice wrote down her favourite polynomial in descending order, Bob starts by asking "Is the second coefficient negative?". If the answer is no, he moves on to the next coefficient (i.e. setting $k_1 = 0$). If the answer is yes, he asks Alice to expand $p(x + t)$ as a polynomial in x, and to find the unique positive root t^* of the coefficient of the second term and round that root up to the next integer. Alice finds $k_1 = \lceil t^* \rceil = 2$, after which Bob repeats the same question for the next coefficient of the next polynomial $p(x + k_1)$. When all coefficients became positive, Bob finds the coefficients of $p_k(x)$ by using the solution to the original puzzle for that polynomial. Then he asks Alice the total translation $k = k_1 + k_2 + \ldots + k_n$ to find the coefficients of the polynomial secretly written down by Alice by calculating $p(x) = p_k(x - k)$.

It's clear from the example that the aforementioned theoretical approach is not very practical to use. First of all, if the degree of the original polynomial is high, Bob has to ask a lot of questions and Alice has to do a lot of intermediate calculations. A one shot approach would therefore be much more useful than an iterative approach. Secondly, if the degree of the polynomial is higher than five, we might end up looking for the unique positive root of polynomials $b_i(t)$ which are themselves of degree higher than five. It is well known that algebraic solutions exist for general polynomials up to degree four, but not for degree five or higher. Finding the value $k_i = \lceil t^* \rceil$ is therefore more difficult in such case. A first possibility to get around this problem is to approximate the root by some numerical technique (e.g. Newton–Raphson method, regula falsi, ...) since the exact value of the root is not important: it's sufficient to find an integer greater than this root. But again, this wouldn't make the procedure more practical for Alice. A second possibility is to find an upper bound on the roots based on the coefficients of the polynomial, which we will discuss further on.

2.3 Upper bounds for positive roots of polynomials

Finding a bound on roots of a polynomial has been of interest for more than 200 years, especially in root-finding algorithms. The first bounds on the modulus of all complex roots were published by Lagrange [5] and Cauchy [2]:

Lemma 3. *Given a polynomial* $q(x) = \sum_{j=0}^{n} c_j x^j \in \mathbb{C}[x]$ *with* $c_n \neq 0$. *Then the following numbers define an upper bound on the modulus of all complex roots of* $q(x)$*:*

$$\text{Lagrange: } \max\left\{1, \sum_{j=0}^{n-1} \left|\frac{c_j}{c_n}\right|\right\} \qquad \text{Cauchy: } 1 + \max_{j=0}^{n-1} \left|\frac{c_j}{c_n}\right|$$

Cauchy's bound is more widely used since it is sharper than Lagrange's in general.

Only in more recent years, the interest in tighter upper bounds for positive real roots has grown, mainly inspired by real-root isolation techniques, with contributions of e.g. Kioustelidis [4], Ştefănescu [7, 8] and Vigklas [9]. Unfortunately, most of these techniques use a priori knowledge on which of the coefficients c_j of the polynomial $q(x)$ are negative and/or the total number of negative coefficients. This is not very useful in our case, since we assume that the coefficients are unknown to us.

Nevertheless, we can use the following lemma to our advantage to use one of those classical bounds in our problem:

Lemma 4. *Given a polynomial* $q(x) = \sum_{j=0}^{n} c_j x^j \in \mathbb{R}[x]$ *with* $c_n > 0$. *Define the altered polynomial* $q^*(x) = c_n x^n + \sum_{j=0}^{n-1} \min\{0, c_j\} x^j$ *where every positive non-leading coefficient of* $q(x)$ *is set to zero. Then every upper bound of the positive roots of* $q^*(x)$ *is also a bound for the real roots of* $q(x)$ *itself.*

Proof. It is sufficient to observe that, if B is an upper bound of the positive roots of $q^*(x)$, then for all $x > B$ we have $q(x) \geq q^*(x) > 0$. □

If we apply this to Cauchy's bound from Lemma 3, we find:

Lemma 5. *Given a polynomial* $q(x) = \sum_{j=0}^{n} c_j x^j \in \mathbb{R}[x]$ *with* $c_n > 0$. *Then the number* $u = 1 + \max_{j=0}^{n-1} \frac{-c_j}{c_n}$ *is an upper bound for the real roots of* $q(x)$.

Proof. Define $q^*(x) = c_n x^n + \sum_{j=0}^{n-1} \min\{0, c_j\} x^j = c_n x^n + \sum_{j=0}^{n-1} c_j^* x^j$.
By construction, all coefficients c_j^* are strictly negative or zero, while c_n is supposed to be positive. This means that for every $0 \leq j \leq n-1$ we have $\left|\frac{c_j^*}{c_n}\right| = \frac{|c_j^*|}{c_n} = \frac{-c_j^*}{c_n}$. Applying Lemma 3 to the polynomial $q^*(x)$ thus gives us Cauchy's upper bound

$$1 + \max_{j=0}^{n-1} \frac{-c_j^*}{c_n}$$

on the modulus of all complex roots of $q^*(x)$, and thus definitely for the (absolute value of the) unique positive real root of $q^*(x)$. This bound is equivalent to

$$1 + \max_{j=0}^{n-1} \frac{-c_j}{c_n}.$$

Indeed, if $c_j^* < 0$, then obviously $c_j^* = c_j < 0$. On the other hand, if $c_j^* = 0$, then $c_j \geq 0$, and $\frac{-c_j}{c_n}$ won't contribute to find the maximum in this bound (unless all c_j are positive and the bound becomes less than 1, but then there are no positive real roots by Descartes' rule of signs so we don't really need this upper bound). Using Lemma 4 concludes the proof, since it tells us that this number is also an upper bound for the real roots of $q(x)$ itself. □

Recall that we are in general looking for an upper bound on the unique real root of a polynomial $b_j(t) = \sum_{i=0}^{n-j} c_i t^i$ (as in Lemma 2). In the procedure from Theorem 2 it is clear that finding such an upper bound is sufficient to define the value of k_i. Indeed, it doesn't matter that we find the *smallest* integer greater than the unique positive root, we only need *some value* of k such that $b_j(t)$ is always positive for all $t > k$. This solves the issue Alice encounters when she has to find the unique positive root of a polynomial of degree 5 or higher: she can use the ceiling function of the upper bounds defined in Lemma 5 to define her values of k_i to continue the iterative process. Using the same example as before, where we found in the first step $b_1(t) = 3t - 5$, we now find the upper bound $1 + \frac{5}{3} = \frac{8}{3}$ for its roots, and thus $k_1 = \left\lceil \frac{8}{3} \right\rceil = 3$. This leads to $p_{k_1}(x) = p(x+3) = x^3 + 4x^2 + 4x + 2$ in just one step of the iterative process.

3 A one shot approach

But can we do better? Instead of the iterative process from before, we can also look for a value for $k \in \mathbb{N}$ such that *all* coefficients $b_j(t)$ become positive at once for every $t > k$, resulting in $p(x + k) \in \mathbb{N}[x]$. Let us recall from Lemma 2:

$$
\begin{array}{l|l}
j = n & b_n(t) = \binom{n}{0} a_n \\
j = n-1 & b_{n-1}(t) = \binom{n}{1} a_n t + \binom{n-1}{0} a_{n-1} \\
j = n-2 & b_{n-2}(t) = \binom{n}{2} a_n t^2 + \binom{n-1}{1} a_{n-1} t + \binom{n-2}{0} a_{n-2} \\
j = n-3 & b_{n-3}(t) = \binom{n}{3} a_n t^3 + \binom{n-1}{2} a_{n-1} t^2 + \binom{n-2}{1} a_{n-2} t + \binom{n-3}{0} a_{n-3} \\
\quad\vdots & \quad\vdots \\
j = 0 & b_0(t) = \binom{n}{n} a_n t^n + \binom{n-1}{n-1} a_{n-1} t^{n-1} + \ldots + \binom{1}{1} a_1 t + \binom{0}{0} a_0
\end{array}
$$

Since every $b_j(t)$ for $j = 0, 1, \ldots, n-1$ has a positive leading coefficient by assumption (namely a binomial coefficient multiplied by $a_n > 0$), we know that $b_j(t)$ will tend to $+\infty$ for $t \to +\infty$, and thus for every $0 \leq j \leq n-1$ there exists a t_j such that $b_j(t) > 0$ for all $t > t_j$, i.e. t_j is the largest real root of $b_j(t)$.

Of course, for polynomials with only positive coefficients, the upper bound given by Lemma 5 will be less than 1. But assuming at least one of the a_j is negative (otherwise there is no need for a translation), Descartes' rule of signs tells us that at least one of the $b_j(t)$ has a unique positive root: indeed, if $0 \leq m \leq n-1$ is the largest index for which $a_m < 0$, then $b_m(t)$ has precisely one sign change. Applying Lemma 5, we find that the number $u_m = 1 + \frac{-\binom{m}{0} a_m}{\binom{n}{n-m} a_n}$ will be an upper bound for this positive root of $b_m(t)$. Since $a_m < 0$, we have that $u_m > 1$. (Note that a polynomial with both positive and negative coefficients will always yield an upper bound greater than 1, even if it does not have any positive roots.)

When we calculate the upper bound u_j for the positive roots of $b_j(t)$ for *all* $0 \leq j \leq n-1$ by means of Lemma 5, then $\max_{j=0}^{n-1} u_j$ will of course be an upper bound for all positive roots of all $b_j(t)$ at once. But since we discovered that at least one of the u_j will be greater than one, we are sure that this maximum will certainly be greater than one, and thus greater than zero. In other words, for all $t > \max_{j=0}^{n-1} u_j > 0$, every coefficient $b_j(t)$ will be positive. Therefore, setting $k = \left\lceil \max_{j=0}^{n-1} u_j \right\rceil$ will assure that $k \in \mathbb{N}$ and that $p(x+k)$ will only have positive coefficients.

Combining Lemma 2 and Lemma 5, we find upper bounds

$$
u_j = 1 + \max_{i=0}^{n-j-1} \left\{ \frac{-\binom{i+j}{i} a_{i+j}}{\binom{n}{n-j} a_n} \right\}
$$

for $0 \leq j \leq n-1$, i.e.:

$$
\begin{aligned}
u_{n-1} &= 1 + \max \left\{ \frac{-\binom{n-1}{0} a_{n-1}}{\binom{n}{1} a_n} \right\} \\
u_{n-2} &= 1 + \max \left\{ \frac{-\binom{n-1}{1} a_{n-1}}{\binom{n}{2} a_n}, \frac{-\binom{n-2}{0} a_{n-2}}{\binom{n}{2} a_n} \right\} \\
u_{n-3} &= 1 + \max \left\{ \frac{-\binom{n-1}{2} a_{n-1}}{\binom{n}{3} a_n}, \frac{-\binom{n-2}{1} a_{n-2}}{\binom{n}{3} a_n}, \frac{-\binom{n-3}{0} a_{n-3}}{\binom{n}{3} a_n} \right\} \\
&\ \vdots \\
u_0 &= 1 + \max \left\{ \frac{-\binom{n-1}{n-1} a_{n-1}}{\binom{n}{n} a_n}, \frac{-\binom{n-2}{n-2} a_{n-2}}{\binom{n}{n} a_n}, \frac{-\binom{n-3}{n-3} a_{n-3}}{\binom{n}{n} a_n}, \ldots, \frac{-\binom{0}{0} a_0}{\binom{n}{n} a_n} \right\}
\end{aligned}
$$

Finding the maximum of all these u_j, we thus calculate in fact

$$\max_{j=0}^{n-1} u_j = 1 + \max \left\{ \begin{array}{ccccc} \frac{-\binom{n-1}{0}a_{n-1}}{\binom{n}{1}a_n} & & & & \\ \frac{-\binom{n-1}{1}a_{n-1}}{\binom{n}{2}a_n} & \frac{-\binom{n-2}{0}a_{n-2}}{\binom{n}{2}a_n} & & & \\ \frac{-\binom{n-1}{2}a_{n-1}}{\binom{n}{3}a_n} & \frac{-\binom{n-2}{1}a_{n-2}}{\binom{n}{3}a_n} & \frac{-\binom{n-3}{0}a_{n-3}}{\binom{n}{3}a_n} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \frac{-\binom{n-1}{n-1}a_{n-1}}{\binom{n}{n}a_n} & \frac{-\binom{n-2}{n-2}a_{n-2}}{\binom{n}{n}a_n} & \frac{-\binom{n-3}{n-3}a_{n-3}}{\binom{n}{n}a_n} & \cdots & \frac{-\binom{0}{0}a_0}{\binom{n}{n}a_n} \end{array} \right\}$$

where we arranged all the values from above in some kind of lower triangular matrix.

The crucial trick is now to find the maximum of each column first, since they all share the same factor $\frac{-a_j}{a_n}$, and then find the maximum of all these maxima. Remember that we assumed $a_n > 0$, and that at least one of the $a_j < 0$. Therefore, there is at least one column where all elements are positive, and we can in fact ignore the columns with negative or zero elements (those columns where $a_j \geq 0$) since they won't contribute to the maximum value. In those positive columns, the maximum element is found in the bottom row. Indeed, the binomial coefficient in the numerator is always less than or equal to the binomial coefficient in the denominator (this follows from Pascal's triangle). Equality is only reached in the bottom row, since both binomial coefficients there are equal to 1. We thus can conclude that

$$\max_{j=0}^{n-1} u_j = 1 + \max_{j=0}^{n-1} \frac{-a_j}{a_n} = 1 - \frac{1}{a_n} \cdot \min_{j=0}^{n-1} a_j.$$

All of the above is summarised by the following Theorem:

Theorem 3. *Consider a polynomial* $p(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ *with* $a_n > 0$ *and at least one of the other* $a_j < 0$. *Set* $k = \left\lceil 1 - \frac{1}{a_n} \cdot \min_{j=0}^{n-1} a_j \right\rceil$. *Then the translated polynomial* $p_k(x) = p(x+k)$ *has only positive coefficients.*

We'll illustrate this result again with the same example as we used before, being the polynomial $p(x) = x^3 - 5x^2 + 7x - 1$. The value k from Theorem 3 is in this case calculated as

$$k = \left\lceil 1 - \frac{1}{1} \cdot \min\{-5, 7, -1\} \right\rceil = \lceil 1 - 1 \cdot (-5) \rceil = \lceil 1 + 5 \rceil = 6$$

For this k, we find $p_k(x) = p(x+6) = x^3 + 13x^2 + 55x + 77$: indeed a polynomial with positive coefficients. Note that this method gives an overestimation of what is needed: previously we found $k = 3$ to be sufficient for this example. But as stated before, we don't need the smallest value of k for which $p(x+k)$ has positive coefficients: we need *some value* of k that delivers what we want.

A final remark is made by observing that knowing the exact value of the minimum of the coefficients a_j is in fact not necessary. A lower bound on all the coefficients is sufficient as well:

Theorem 4. *Consider a polynomial* $p(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ *with* $a_n > 0$ *and at least one of the other* $a_j < 0$. *Furthermore, suppose a lower bound on the coefficients of* $p(x)$ *is given, i.e.* $L \leq a_j$ *for all* $0 \leq j \leq n$. *Set* $k = \left\lceil 1 - \frac{L}{a_n} \right\rceil$. *Then the translated polynomial* $p_k(x) = p(x+k)$ *has only positive coefficients.*

Proof. If $L \leq a_j$ for all $0 \leq j \leq n$, then of course $L \leq \min_{j=0}^{n-1} a_j$, and thus $\left\lceil 1 - \frac{1}{a_n} \cdot L \right\rceil \geq \left\lceil 1 - \frac{1}{a_n} \cdot \min_{j=0}^{n-1} a_j \right\rceil$. According to Theorem 3, translating $p(x)$ to the left by the value of the right-hand side of this inequality is already sufficient to yield a polynomial with positive coefficients. Translating $p(x)$ even further to the left will then of course most certainly do the trick. □

4 Conclusion

To conclude, we compare our approach with the original puzzle and its solution, and with the methods developed by Richmond [6].

The original puzzle, for polynomials with positive integer coefficients, needed two polynomial evaluations to determine the unknown coefficients a_j. Here, the key observation is in fact that we implicitly have a lower bound on the coefficients since they are assumed positive, i.e. $L = 0 \leq a_j$ for all $j = 0, 1, \ldots, n$. An upper bound is constructed by calculating $U = p(1) \geq a_j$ for all j. With these bounds, a single (extra) polynomial evaluation is sufficient: choose $A > U$, and calculate $p(A)$.

Richmond in her paper extends the puzzle to coefficients that can be negative as well, meaning that $p(1)$ doesn't necessarily have to be an upper bound for the coefficients a_j any more. Her main result states that one evaluation is sufficient if a bound on the absolute values of the coefficients is known, i.e. if $B \geq |a_j|$ for all j, then $p(2B+1)$ uniquely determines the unknown coefficients. Later on, she refines her result: given only a lower bound $L \leq a_j$ for all j, one can construct (in a rather complicated way and assuming the leading coefficient a_n is positive) an upper bound $U \geq a_j$ for all j by using one polynomial evaluation. When choosing $B = \max\{-L, U\}$, her main result can be applied. Summarised, a lower bound on the coefficients and two polynomial evaluations (like in the original case) suffice to determine the coefficients of the unknown polynomial.

We also considered polynomials with arbitrary integer coefficients but with a positive leading coefficient, and our result is at least just as good as Richmond's: given a lower bound $L \leq a_j$ for all j, we first calculate $k = \left\lceil 1 - \frac{L}{a_n} \right\rceil$. Theorem 4 then states that the translated polynomial $p_k(x) = p(x + k)$ has positive integer coefficients, and thus the solution to the original puzzle is applicable here: two polynomial evaluations of $p_k(x)$ will determine its coefficients. In fact, it's not even necessary for Alice to explicitly calculate $p_k(x)$: after letting her calculate the value of k, Bob can ask her to calculate $p_k(1) = p(1 + k)$ and $p_k\big(p_k(1) + 1\big) = p(p(1 + k) + 1 + k)$, after which Bob can determine the coefficients of $p_k(x)$ himself. In the end, $p(x)$ is then calculated as $p_k(x - k)$ by Bob, but no extra information is needed for this step. In other words, it's again sufficient to have a lower bound on the coefficients and two polynomial evaluations to solve the puzzle.

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