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Proof of a conjecture due to Chu on Gosper-type sums

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Abstract. We prove a conjecture due to Chu concerning Gosper-type sums, using an evaluation due to Chudnovsky and Chudnovsky in 1998. This formula discovered by the Chudnovsky brothers was later rediscovered by Borwein and Girgensohn, but no proof of this formula has been given, prior to our article. We introduce a full, self-contained proof of the Chudnovsky–Chudnovsky evaluation, to formulate a full solution to a problem due to Chu on Gosper-type sums involving reciprocals of binomial coefficients of the form $\binom{3n+\varepsilon}{n}$ for $n \in \mathbb{N}_0$ and $\varepsilon \in \{1, \pm 2\}$.

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1. Introduction

The famous binomial sum evaluation

$$\sum_{n=0}^{\infty} \frac{50n-6}{2^n \binom{3n}{n}} = \pi \tag{1.1}$$

due to Gosper [7] (cf. [3, p. 20]) has inspired much in the way of research on series resembling (1.1) and involving inverted binomial coefficients such as $\frac{1}{\binom{3n}{n}}$. The purpose of this article is to prove a conjecture related to (1.1) that was given by Chu in a recent *Journal of Difference Equations and Applications* contribution [5].

A key to our proof of Chu's conjecture in [5] is given by the following formula discovered in 1998 by the Chudnovsky brothers [6]:

$$-3 \sum_{n=1}^{\infty} \frac{1}{\binom{3n}{n} n^3 2^n} = -3G\pi + \frac{\pi^2 \ln(2)}{8} - \frac{\ln^3(2)}{2} + \frac{99\zeta(3)}{16}. \tag{1.2}$$

No full, self-contained proof of (1.2) has been previously published. It is suggested in [6] that (1.2) may be proved by integrating the master formula reproduced in (1.9), but proving this is nontrivial. The history associated with (1.2), as summarized below, adds to the interest in the problem of proving Chu's conjecture from [5], as reproduced below as Conjecture 1.1.

The Chudnovsky brothers introduced (1.2) in 1998 [6]. In a 2004 text by Borwein, Bailey, and Girgensohn [3, p. 27], the authors presented an equivalent formulation of (1.2), and stated that it could be proved through difficult, polylogarithmic manipulations, without any proof being given, and without the authors of [4] having been aware that (1.2) was previously introduced by Chudnovsky and Chudnovsky in [6]. In a 2005 article by Borwein and Girgensohn [4], the authors again presented an equivalent version of (1.2) and again stated that it could be proved through challenging, polylogarithmic manipulations, but, again, no proof was given, and, again, the authors were unaware that (1.2) had been introduced by the Chudnovsky brothers in 1998 [6]. Chu, in 2022 [5], had also discovered an equivalent version of (1.2) experimentally, using numerical approximations via Mathematica, and noted that Borwein and Girgensohn had found (1.2), but the author of [5] was again unaware that Chudnovsky and Chudnovsky had introduced (1.2) as far back as 1998 [6].

The foregoing considerations motivate our introducing a full proof of Chudnovsky and Chudnovsky's formula in (1.2) and our applying this formula to prove the following conjecture due to Chu [5]. Chu [5] had experimentally discovered the conjectured formulas shown below, again via numerical approximations with Mathematica. The following conjecture is given as Conjecture 5.3 in [5].

Conjecture 1.1. *(Chu, 2022) The following evaluations hold* [5]*:*

$$\sum_{n=1}^{\infty} \frac{(1/2)^n}{n^3\binom{3n+1}{n}} = \frac{48\pi G - 99\zeta(3) + 8(53 - 9\pi - 39\ln 2)}{48} \tag{1.3}$$

$$+ \frac{(1+\ln 2)(2+\pi+2\ln 2)(2-\pi+2\ln 2)}{24}, \tag{1.4}$$

$$\sum_{n=1}^{\infty} \frac{(1/2)^n}{n^3\binom{3n+2}{n}} = \frac{48\pi G - 99\zeta(3) + 3(189 - 40\pi - 118\ln 2)}{48} \tag{1.5}$$

$$+ \frac{(3+2\ln 2)(3+\pi+2\ln 2)(3-\pi+2\ln 2)}{48}, \tag{1.6}$$

$$\sum_{n=1}^{\infty} \frac{(1/2)^n}{n^3\binom{3n-2}{n}} = \frac{48\pi G - 99\zeta(3) - 8(1 + 2\pi - 9\ln 2)}{32} \tag{1.7}$$

$$+ \frac{(1+\ln 2)(2+\pi+2\ln 2)(2-\pi+2\ln 2)}{16}. \tag{1.8}$$

1.1. Background

The Chudnovsky–Chudnovsky formula in (1.2) is described in [6] as a natural counterpart to the Gosper identity in (1.1). The key tool behind the Chudnovsky's derivation of (1.2) is given by the following *master formula* from

[6]:

$$\frac{1}{2}\sum_{k=1}^{n}\ln^2\left(x_k\zeta_n^{-k}\right) = -\frac{1}{ns(n-s)}\sum_{m=1}^{\infty}\frac{\left((-1)^{n-s}T^n\right)^m}{\binom{mn}{ms}m^2}. \tag{1.9}$$

With regard to Chudnovsky and Chudnovsky's notation in (1.9), we are letting $\zeta_n^{-k} = e^{-2\pi ik/n}$, referring to [6] for further details. It is suggested in [6] that (1.2) may be derived using the $n = 3$ and $s = 1$ case of (1.9), but no proof of this was given.

In Section 2, we are to also make use of the following known Gosper-type summations [3, p. 26]:

$$\sum_{n=1}^{\infty}\frac{1}{n\binom{3n}{n}2^n} = \frac{\pi}{10} - \frac{\ln 2}{5}, \tag{1.10}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2\binom{3n}{n}2^n} = \frac{\pi^2}{24} - \frac{\ln^2(2)}{2}. \tag{1.11}$$

A method that may be applied to evaluate the above series is given in [3, §1.7], and these results are also obtained via the same method in [4]. The above results also may be proved via the generating function identity due to Batir [2] and Villacorte [9] (cf. [5]) giving an elementary evaluation for the power series corresponding to the summand of (1.11).

We are to also make use of Batir's generating function evaluation [1] whereby

$$\sum_{k=0}^{\infty}\frac{x^k}{(3k+1)\binom{3k}{k}} = \frac{\phi(\phi-1)}{3\phi-1}\left(\frac{3}{2}\ln\left|\frac{\phi}{\phi-1}\right|\right.$$
$$\left.+\frac{3\phi-2}{\sqrt{3\phi^2-4\phi}}\left(\tan^{-1}\frac{\phi}{\sqrt{3\phi^2-4\phi}}+\tan^{-1}\frac{2-\phi}{\sqrt{3\phi^2-4\phi}}\right)\right),$$

where

$$\phi = \phi(x) = \frac{2}{3}+\frac{1}{3}\left(\frac{27-2x+3\sqrt{81-12x}}{2x}\right)^{-\frac{1}{3}} +$$
$$\frac{1}{3}\left(\frac{27-2x+3\sqrt{81-12x}}{2x}\right)^{\frac{1}{3}},$$

and letting $0 < |x| < \frac{27}{4}$. In particular, the $x = \frac{1}{2}$ case gives us that

$$\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)^k}{(3k+1)\binom{3k}{k}} = \frac{\pi}{5}+\frac{3\ln(2)}{5}, \tag{1.12}$$

and this is to be applied in Section 2 below.

2. Main proof

Adopting notation due to Chu [5], Chu showed that

$$\Omega_\lambda(\varepsilon, x) = \int_0^1 \frac{t^\varepsilon \mathrm{Li}_\lambda(x(1-t)t^2)}{1-t}\,dt,$$

where

$$\Omega_\lambda(\varepsilon, x) = \sum_{n=1}^{\infty} \frac{x^n}{n^{\lambda+1}\binom{3n+\varepsilon}{n}}.$$

So, it remains to evaluate the integral on the right-hand side of

$$\Omega_2(0, \tfrac{1}{2}) = \sum_{n=1}^{\infty} \frac{1}{\binom{3n}{n} n^3 2^n} = \int_0^1 \frac{\mathrm{Li}_2\left(\frac{1}{2}(1-t)t^2\right)}{1-t}\,dt. \tag{2.1}$$

Lemma 2.1. *The integral evaluation given as follows hold true, for $w \in \mathbb{C} \setminus \mathbb{R}_{>1}$:*

$$I(w) := \int_0^1 \frac{\ln(1-t)\ln(1-wt)}{t}\,dt$$
$$= \frac{1}{2}\ln^2(1-w)\ln w + \ln(1-w)\mathrm{Li}_2(1-w) - \mathrm{Li}_3(1-w) + \mathrm{Li}_3(w) + \zeta(3).$$

Proof. Using Mathematica we are able to find an antiderivative for

$$\frac{\ln(1-at)\ln(1-bt)}{t}.$$

The output given by Mathematica is:

```
Log[a t] Log[1 - a t] Log[1 - b t] + 1/2 (Log[a t] - Log[b t]) Log[
1 - b t] (-2 Log[1 - a t] + Log[1 - b t]) + (-Log[a t] + Log[b t])
Log[1 - b t] Log[(-1 + b t)/(-1 + a t)] + 1/2 (Log[a t] + Log[(a - b)/
(b (-1 + a t))] - Log[((a - b) t)/(-1 + a t)]) Log[(-1 + b t)/(-1 + a
t)]^2 + (Log[1 - b t] - Log[(-1 + b t)/(-1 + a t)]) PolyLog[2,1 - a
t] + (Log[1 - a t] + Log[(-1 + b t)/(-1 + a t)]) PolyLog[2,1 - b t] +
Log[(-1 + b t)/(-1 + a t)] (-PolyLog[2,(-1 + b t)/(-1 + a t)] +
PolyLog[2,(a - a b t)/(b - a b t)]) - PolyLog[3,1 - a t] -
PolyLog[3,1 - b t] + PolyLog[3,(-1 + b t)/(-1 + a t)] -
PolyLog[3,(a - a b t)/(b - a b t)]
```

Replacing b by 1 and a by w and simplifying, taking into account that $0 \leq t \leq 1$ and $w \in \mathbb{C} \setminus \mathbb{R}_{>1}$, the expression above reduces to:

```
1/2 Log[w] Log[1 - w t]^2 + Log[1 - t] Log[t] Log[1 - w t] +
Log[1 - w t] PolyLog[2, 1 - w t] + Log[1 - t] PolyLog[2, 1 -
t] + Log[(1 - t)/(1 - w t)] (-PolyLog[2, (1 - t)/(1 - w t)] +
PolyLog[2, (w - w t)/(1 - w t)]) - PolyLog[3, 1 - w t] -
PolyLog[3, 1 - t] + PolyLog[3, (1 - t)/(1 - w t)] -
PolyLog[3, (w - w t)/(1 - w t)]
```

It is a matter of routine to check that the limit as $t \to 1$ of the above expression minus the limit as $t \to 0$, is equal to

$$\frac{1}{2}\ln^2(1-w)\ln w + \ln(1-w)\mathrm{Li}_2(1-w) - \mathrm{Li}_3(1-w) + \mathrm{Li}_3(w) + \zeta(3).$$

□

Lemma 2.2. *The following integral evaluation holds true for a given complex number w:*

$$J(w) := \int_0^1 \frac{\ln(t)\ln(1-wt)}{t}\,dt = \mathrm{Li}_3(w). \tag{2.2}$$

Proof. This is easily seen to hold by computing $\mathrm{Li}_3(tw) - \ln(t)\mathrm{Li}_2(tw)$ as an evaluation for the indefinite integral corresponding to (2.2), and by taking appropriate limits of this antiderivative expression. □

Theorem 2.3. *The Chudnovsky–Chudnovsky formula in* (1.2) *holds true.*

Proof. Recalling (2.1), note that we have:

$$\Omega_2(0,x) = \int_0^1 \frac{\mathrm{Li}_2(x(1-t)t^2)}{1-t}\,\mathrm{d}t = 2\int_0^1 \frac{\mathrm{Li}_2(x(1-t)t^2)}{t}\,\mathrm{d}t$$

which is a consequence of

$$\int_0^1 \frac{(2t-3t^2)\mathrm{Li}_2(x(1-t)t^2)}{(1-t)t^2}\,\mathrm{d}t = 0.$$

Using partial integration we get:

$$\begin{aligned}\int_0^1 \frac{\mathrm{Li}_2(x(1-t)t^2)}{t}\,\mathrm{d}t &= \int_0^1 \frac{(2-3t)\ln(1-x(1-t)t^2)}{(1-t)t}\ln(t)\,\mathrm{d}t\\ &= 2\int_0^1 \frac{\ln(t)\ln(1-x(1-t)t^2)}{t}\,\mathrm{d}t - \int_0^1 \frac{\ln(t)\ln(1-x(1-t)t^2)}{1-t}\,\mathrm{d}t\\ &= 2\int_0^1 \frac{\ln(t)\ln(1-x(1-t)t^2)}{t}\,\mathrm{d}t - \int_0^1 \frac{\ln(1-t)\ln(1-xt(1-t)^2)}{t}\,\mathrm{d}t\end{aligned}$$

where we have replaced t by $1-t$ in the second integral.
We now use the same trick that is used in Chu [5]: take $x = y^3/(1-y)^2$ in both integrals. We then factor the argument of the second ln in both integrals:

$$1 - \frac{y^3(1-t)t^2}{(1-y)^2} = \left(1 + \frac{y}{1-y}t\right)(1-ut)(1-\bar{u}t)$$

and

$$1 - \frac{y^3t(1-t)^2}{(1-y)^2} = (1-yt)(1-vt)(1-\bar{v}t).$$

with

$$u = \frac{y(1+\mathrm{i}\sqrt{3-4y})}{2(1-y)}, \quad v = \frac{y(2y-1+\mathrm{i}\sqrt{3-4y})}{2(1-y)^2}.$$

Using this in both integrals, they each split up in a sum of 3 integrals, of type I or J. Bringing everything together, we find

$$\begin{aligned}
&\Omega_2(0,x) \\
&= 4\int_0^1 \frac{\ln(t)\ln(1-x(1-t)t^2)}{t}\,\mathrm{d}t - 2\int_0^1 \frac{\ln(1-t)\ln(1-xt(1-t)^2)}{t}\,\mathrm{d}t \\
&= 4\mathrm{Li}_3(-\tfrac{y}{1-y}) + 4\mathrm{Li}_3(u) + 4\mathrm{Li}_3(\bar{u}) - 2I(y) - 2I(v) - 2I(\bar{v}).
\end{aligned}$$

Taking $x = \frac{1}{2}$, hence $y = \frac{1}{2}, u = \frac{1}{2}(1+\mathrm{i}), v = \mathrm{i}$, and using Lemma 2.1 and 2.2, we arrive at:

$$\Omega_2(0,\tfrac{1}{2}) = G\pi - \frac{\pi^2\ln(2)}{24} + \frac{\ln^3(2)}{6} - \frac{33\zeta(3)}{16}$$

using values of the polylogarithmic functions found in [8]. □

Theorem 2.4. *Chu's conjecture, as reproduced as Conjecture* 1.1*, holds.*

Proof. The binomial expression in the denominator on the left-hand side of (1.3) may be rewritten so as to express the series on the left-hand side of (1.3) so that:

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n^3\binom{3n+1}{n}} = \sum_{n=1}^{\infty} \frac{2^{-n}(1+2n)}{n^3(1+3n)\binom{3n}{n}}. \tag{2.3}$$

Applying partial fraction decomposition, we find that the series on the left-hand side of (2.3) may be rewritten as:

$$\sum_{n=1}^{\infty} \frac{2^{-n}\left(\frac{1}{n^3} - \frac{1}{n^2} + \frac{3}{n} - \frac{9}{3n+1}\right)}{\binom{3n}{n}}.$$

Expanding the above summand, we obtain a linear combination of the Chudnovsky brothers' formula in (1.2), the Borwein–Borwein formulas in (1.10) and (1.11), and the special case of Batir's generating function highlighted in (1.12). So, the previously proved closed forms for (1.2), (1.10), (1.11), and (1.12) give us that (1.3)–(1.4) holds.

Rewriting the series in (1.5) so that

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n^3\binom{3n+2}{n}} = \sum_{n=1}^{\infty} \frac{2^{1-n}(1+n)(1+2n)}{n^3(1+3n)(2+3n)\binom{3n}{n}}, \tag{2.4}$$

we proceed to apply partial fraction decomposition, and we find that the left-hand side of (2.4) may be expressed as:

$$\sum_{n=1}^{\infty} \frac{2^{1-n}\left(\frac{1}{2n^3} - \frac{3}{4n^2} + \frac{17}{8n} - \frac{6}{3n+1} - \frac{3}{8(3n+2)}\right)}{\binom{3n}{n}}. \tag{2.5}$$

We may rewrite (2.5) using the previously proved closed forms for (1.2), (1.10), (1.11), and (1.12), so as to give us the equality of (2.5) and:

$$\pi G - \frac{33\zeta(3)}{16} + 12 - \frac{79\pi}{40} - \frac{\pi^2}{16} + \frac{\ln^3(2)}{6} + \frac{3\ln^2(2)}{4} - \frac{161\ln(2)}{20} - \frac{1}{24}\pi^2\ln(2) - \frac{3}{4}\sum_{n=1}^{\infty}\frac{2^{-n}}{(2+3n)\binom{3n}{n}}. \tag{2.6}$$

Applying a reindexing argument to (1.10), we find that

$$\frac{\pi}{10} - \frac{\ln(2)}{5} = \sum_{n=0}^{\infty}\frac{2^{-n}(1+2n)}{3(1+3n)(2+3n)\binom{3n}{n}},$$

so that

$$\frac{\pi}{10} - \frac{\ln(2)}{5} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{9(1+3n)} + \frac{1}{9(2+3n)}\right)2^{-n}}{\binom{3n}{n}}, \tag{2.7}$$

so that the special case of Batir's generating function shown in (1.12) allows us to evaluate the series in (2.6). So, the formula in (1.5)–(1.6) holds.

We proceed to rewrite the series in (1.7) so that

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n^3\binom{3n-2}{n}} = \sum_{n=1}^{\infty}\frac{3\ 2^{-1-n}(-1+3n)}{n^3(-1+2n)\binom{3n}{n}}. \tag{2.8}$$

Applying partial fraction decomposition, we find that the left-hand side of (2.8) equals:

$$\sum_{n=1}^{\infty}\frac{\left(3\ 2^{-1-n}\right)\left(\frac{1}{n^3} - \frac{1}{n^2} - \frac{2}{n} + \frac{4}{-1+2n}\right)}{\binom{3n}{n}}. \tag{2.9}$$

From the Chudnovsky–Chudnovsky formula in (1.2), together with the formulas in (1.10) and (1.11) proved by Borwein and Girgensohn, we find that the series in (2.9) may be rewritten as:

$$-\frac{1}{10}(3\pi) + \frac{3G\pi}{2} - \frac{\pi^2}{16} + \frac{3\ln(2)}{5} - \frac{1}{16}\pi^2\ln(2) + \frac{3\ln^2(2)}{4} + \frac{\ln^3(2)}{4} - \frac{99\zeta(3)}{32} + 6\sum_{n=1}^{\infty}\frac{2^{-n}}{(2n-1)\binom{3n}{n}}. \tag{2.10}$$

According to the beta integral identity

$$\int_0^1 (1-t)^{2n-2}t^{n-1}2^{-n}\,dt = \frac{3(3n-1)}{2n(2n-1)\binom{3n}{n}},$$

we may apply the operator $\sum_{n=1}^{\infty}\cdot$ to both sides and then use the Dominated Convergence Theorem to reverse the order of infinite summation and integration. The resultant integrand may be written as $-\frac{1}{(t-2)(t^2+1)}$, which admits an elementary antiderivative. This gives us that

$$\frac{\pi}{10}+\frac{3\ln(2)}{10}=\sum_{n=1}^{\infty}\frac{3\ 2^{-1-n}(-1+3n)}{n(-1+2n)\binom{3n}{n}},$$

so that

$$\frac{\pi}{10}+\frac{3\ln(2)}{10}=\sum_{n=1}^{\infty}\frac{\left(3\ 2^{-1-n}\right)\left(\frac{1}{n}+\frac{1}{-1+2n}\right)}{\binom{3n}{n}},$$

so that a reindexing argument gives us that

$$\frac{\pi}{10}+\frac{3\ln(2)}{10}=$$
$$\sum_{n=0}^{\infty}\frac{2^{-1-n}\left(\frac{1}{3(1+3n)}+\frac{1}{3(2+3n)}\right)}{\binom{3n}{n}}+\sum_{n=1}^{\infty}\frac{3\ 2^{-1-n}}{(-1+2n)\binom{3n}{n}}.$$

So, from the above equality, together with (2.7) and the special case of Batir's generating function in (1.12), we may evaluate the remaining series (2.10) in the desired way. This gives us that (1.7)–(1.8) holds. □

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