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SERIES OF THE FORM $\sum a_n \binom{2n}{n}$

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Abstract. We compute here the sums of infinite series of four types: $\sum_{n=0}^{\infty}\binom{2n}{n}(n+1)^{-k}4^{-n}$; $\sum_{n=1}^{\infty}\binom{2n}{n}n^{-k}4^{-n}$; $\sum_{n=0}^{\infty}\binom{2n}{n}(2n\pm 1)^{-k}4^{-n}$ for positive integers $k \le 9$. We also look at their alternating forms and at some related series. Binomial expansions and log-sine integrals are used for the purpose.

1. Introduction

1.1. **Background.** The central binomial coefficient $\binom{2n}{n}$ is the positive integer that occurs as the coefficient of the x^n term in the expansion of $(1+x)^{2n}$. A number of infinite series involving the coefficient, either in the numerator or in the denominator, are found in literature. A 1964 issue of the *American Mathematical Monthly* [3] contains a problem seeking a sum which we evaluated in [8]:

$$\sum_{n=0}^{\infty}(-1)^n \frac{\binom{2n}{n}}{(2n+1)^2\, 2^{4n}} = \frac{\pi^2}{10}.$$

Two well-known papers [6, 10] on the series involving the central binomial coefficient were published in 1985. Zucker's paper [10], written in 1983, is deeper and contains some nice results. As highlighted by the author in the title itself, his paper gives an exhaustive treatment of the series $\sum_{k=1}^{\infty}\binom{2k}{k}^{-1}k^{-n}$. The series is expressed with log-sine integrals and shown to be summable exactly in terms of Dirichlet's L-series. Lehmer's paper [6] discusses series of two types: $\sum_{n=0}^{\infty} a_n\binom{2n}{n}$ and $\sum_{n=0}^{\infty} a_n\binom{2n}{n}^{-1}$ where the a_n are very simple functions of n. In the first part of his paper, he mainly employs the binomial expansion to derive series of the first type.

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The main contribution of our article consists in the explicit determination of series sums of the type

$$S_1(k) := \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)^k\, 2^{2n}} \qquad S_3(k) := \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^k\, 2^{2n}}$$

$$S_2(k) := \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n^k\, 2^{2n}} \qquad S_4(k) := \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{(2n-1)^k\, 2^{2n}}$$

and their alternating forms.

1.2. **Relation between two types of sums.** Before proceeding to derive the sums, we first establish a relation between two classes of sums with $k \in \mathbb{N}$, $0 < t \leq 1$:

$$\begin{aligned}
&2\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)^k\, 4^n} t^n - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)^{k+1}\, 4^n} t^n \\
&= \sum_{n=0}^{\infty} \frac{(2(n+1)-1)\binom{2n}{n}}{(n+1)^{k+1}\, 4^n} t^n = \sum_{n=0}^{\infty} \frac{(2n+1)\binom{2n}{n}}{(n+1)^{k+1}\, 4^n} t^n \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(2n+2)(2n+1)\binom{2n}{n}}{(n+1)^{k+2}\, 4^n} t^n = \frac{1}{2}\sum_{n=0}^{\infty} \frac{\binom{2n+2}{n+1}}{(n+1)^k\, 4^n} t^n \\
&= 2\sum_{n=0}^{\infty} \frac{\binom{2n+2}{n+1}}{(n+1)^k\, 4^{n+1}} t^n = \frac{2}{t}\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n^k\, 4^n} t^n
\end{aligned}$$

leading to the following Lemma:

Lemma 1.

$$2\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)^k\, 4^n} t^{n+1} - \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)^{k+1}\, 4^n} t^{n+1} = 2\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n^k\, 4^n} t^n \tag{1.1}$$

if all the series converge.

If we take $t = 1$, we may rewrite this useful relation succinctly as:

Proposition 1.

$$2S_1(k) - S_1(k+1) = 2S_2(k) \tag{1.2}$$

where S_1, S_2 are as defined earlier.

Thus we can straightaway deduce the sums on the right if we have the two sums on the left.

By combining these relations, it is easy to see that we have:

Proposition 2.

$$S_1(k+1) = 2^{k+1} - 2\sum_{j=1}^{k} 2^{k-j}\, S_2(j). \tag{1.3}$$

We also have a relation between the other two types of sums, which can be proved in a similar manner:

Lemma 2.

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^k\, 2^{2n}} t^{n+1} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{(2n-1)^k\, 2^{2n}} t^n + \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{(2n-1)^{k+1}\, 2^{2n}} t^n \tag{1.4}$$

if all the series converge.

This Lemma can, for instance, be used to calculate the value of $S_4(k+1)$ from $S_3(k)$ and $S_4(k)$:

$$S_3(k) = S_4(k) + S_4(k+1) \quad \Rightarrow \quad S_4(k+1) = S_3(k) - S_4(k).$$

We will also use Newton's binomial series in the following form: For every $n \in \mathbb{N}$ and $-1 < y \leq 1$, we have that

$$\frac{1}{(1+y)^{1/n}} = 1 - \frac{1}{n}y + \frac{1\cdot(1+n)}{n\cdot 2n}y^2 - \frac{1\cdot(1+n)\cdot(1+2n)}{n\cdot 2n\cdot 3n}y^3 + \dots \tag{1.5}$$

2. Series with factor $(n+1)^{-k}$

Taking $y = -t$, $n = 2$ in (1.5), we obtain:

$$\frac{1}{\sqrt{1-t}} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} t^n$$

and with $t = 1 - x^2$, we get For $0 < x \leq \sqrt{2}$,

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} (1-x^2)^n. \tag{2.1}$$

Multiplying both sides of (2.1) by $-2x$ and integrating, we get:

Proposition 3.

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)\, 4^n} (1-x^2)^{n+1} = 2 - 2x, \quad x \in [0, \sqrt{2}]. \tag{2.2}$$

Dividing (2.2) by $1-x^2$, we have:

$$\frac{2}{1+x} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)\,4^n}(1-x^2)^n. \tag{2.3}$$

Now multiplying (2.3) by $-2x$ and integrating leads to:

Proposition 4.

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)^2\,4^n}(1-x^2)^{n+1} = 4(1-x) + 4\log\tfrac{1+x}{2}, \quad x \in [0,\sqrt{2}]. \tag{2.4}$$

Repeating the same procedure again yields:

Proposition 5.

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)^3\,4^n}(1-x^2)^{n+1} \\ &\quad = 8(1-x) + 2\log^2\tfrac{1+x}{2} + 8\log\tfrac{1+x}{2} - 4\mathrm{Li}_2(\tfrac{1-x}{2}), \quad x \in [0,\sqrt{2}].\end{aligned} \tag{2.5}$$

The polylogarithm is defined by $\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$ for $|z| \le 1$.

And similarly, we obtain:

Proposition 6.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)^4\,4^n}(1-x^2)^{n+1} &= -4\zeta(3) + 16(1-x) \\ &+ (16-4\zeta(2))\log\tfrac{1+x}{2} + 4\log^2\tfrac{1+x}{2} + \tfrac{2}{3}\log^3\tfrac{1+x}{2} \\ &+ 2\log^2\tfrac{1+x}{2}\log\tfrac{1-x}{2} - 8\mathrm{Li}_2(\tfrac{1-x}{2}) + 4\mathrm{Li}_3(\tfrac{1+x}{2}) - 4\mathrm{Li}_3(\tfrac{1-x}{2}).\end{aligned} \tag{2.6}$$

All terms in this sum are real for $x \in [0,1)$. To make it a sum with real terms only for $x \in (1,\sqrt{2}]$, we use the following relation valid for $t > 1$ [7, p.296, (6)]

$$\mathrm{Li}_3(t) - \mathrm{Li}_3\left(\frac{1}{t}\right) = \frac{\pi^2}{3}\log(t) - \frac{1}{6}\log^3(t) - \frac{1}{2}i\pi\log^2(t)$$

to rewrite the term $4\mathrm{Li}_3(\frac{1+x}{2})$. This in combination with the fact that

$$\log\tfrac{1-x}{2} = \log\tfrac{x-1}{2} + i\pi$$

leads to this expression for the sum in the case $x \in (1, \sqrt{2}]$:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)^4\, 4^n}(1-x^2)^{n+1}$$
$$= -4\zeta(3) + 16(1-x) + (16+4\zeta(2)) \log \tfrac{1+x}{2} + 4\log^2 \tfrac{1+x}{2} \qquad (2.7)$$
$$+ 2\log^2 \tfrac{1+x}{2} \log \tfrac{x-1}{2} - 8\mathrm{Li}_2(\tfrac{1-x}{2}) + 4\mathrm{Li}_3(\tfrac{2}{1+x}) - 4\mathrm{Li}_3(\tfrac{1-x}{2}).$$

It may be pointed out that if we want a representation in closed form, using polylogarithms for this purpose, some addends are complex, even if the final expression is real. That is why we have recorded this alternative form of the sum.

The expansions in **Propositions 3–6** yield, for $x = 0$, the following sums:

$$S_1(1) = 2;$$
$$S_1(2) = 4(1 - \log(2));$$
$$S_1(3) = -2\zeta(2) + 4\log^2 2 + 8(1 - \log(2));$$
$$S_1(4) = -4\zeta(3) + (16 - 4\zeta(2))(1 - \log(2)) - \tfrac{8}{3}\log^3(2) + 8\log^2(2).$$

And $x = \sqrt{2}$ yields these alternating sums, using the notation

$$S_1'(k) = \sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{(n+1)^k\, 2^{2n}} :$$

$$S_1'(1) = 2(\sqrt{2} - 1);$$
$$S_1'(2) = 4(\sqrt{2} - 1) - 4\log \tfrac{1+\sqrt{2}}{2};$$
$$S_1'(3) = 8(\sqrt{2} - 1) - 8\log \tfrac{1+\sqrt{2}}{2} - 2\log^2 \tfrac{1+\sqrt{2}}{2} + 4\mathrm{Li}_2(\tfrac{1-\sqrt{2}}{2});$$
$$S_1'(4) = 4\zeta(3) + 16(\sqrt{2} - 1) - (16 + 4\zeta(2)) \log \tfrac{1+\sqrt{2}}{2}$$
$$- 4\log^2 \tfrac{1+\sqrt{2}}{2} - 2\log^2 \tfrac{1+\sqrt{2}}{2} \log \tfrac{\sqrt{2}-1}{2} + 8\mathrm{Li}_2(\tfrac{1-\sqrt{2}}{2})$$
$$+ 4\mathrm{Li}_3(\tfrac{1-\sqrt{2}}{2}) - 4\mathrm{Li}_3(2(\sqrt{2} - 1)).$$

3. Series with factor n^{-k}

To obtain the sum of series of the form

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n^k 4^n}(1-x^2)^n$$

we can use Lemma 1 with $t = 1-x^2$ in combination with Propositions 3–6. This leads to:

Proposition 7.

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n\,4^n}(1-x^2)^n = -2\log\tfrac{1+x}{2}$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n^2\,4^n}(1-x^2)^n = 2\,\mathrm{Li}_2(\tfrac{1-x}{2}) - \log^2\tfrac{1+x}{2}$$

$$x\in[0,1): \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n^3 4^n}(1-x^2)^n = 2\zeta(3) + 2\zeta(2)\log\tfrac{1+x}{2} - \tfrac{1}{3}\log^3\tfrac{1+x}{2} - \log^2\tfrac{1+x}{2}\log\tfrac{1-x}{2} + 2\mathrm{Li}_3(\tfrac{1-x}{2}) - 2\mathrm{Li}_3(\tfrac{1+x}{2})$$

$$x\in(1,\sqrt{2}]: \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n^3 4^n}(1-x^2)^n = 2\zeta(3) - 2\zeta(2)\log\tfrac{1+x}{2} - \log^2\tfrac{1+x}{2}\log\tfrac{x-1}{2} + 2\mathrm{Li}_3(\tfrac{1-x}{2}) - 2\mathrm{Li}_3(\tfrac{2}{1+x})$$

By multiplying the last one by $\frac{-2x}{1-x^2}$ and integrating we get the next one:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n^4 4^n}(1-x^2)^n = 4\zeta(4) + 2\zeta(3)\log\tfrac{1-x^2}{4} + 2\zeta(2)\log\tfrac{1+x}{2}\log\tfrac{1-x}{2} - \tfrac{1}{3}\log^4\tfrac{1+x}{2} + \tfrac{2}{3}\log^3\tfrac{1+x}{2}\log\tfrac{1-x}{2} - \log^2\tfrac{1+x}{2}\log^2\tfrac{1-x}{2} - 2\log\tfrac{1-x}{1+x}\left(\mathrm{Li}_3(\tfrac{1-x}{2}) + \mathrm{Li}_3(\tfrac{1+x}{2}) + \mathrm{Li}_3(\tfrac{x-1}{1+x})\right) + 4\mathrm{Li}_4(\tfrac{1-x}{2}) + 2\mathrm{Li}_4(\tfrac{x-1}{1+x}) - 4\mathrm{Li}_4(\tfrac{1+x}{2})$$

For $x \in [0,1)$ this can be simplified using [7, p.296, (7)], and for $x \in (1,\sqrt{2}]$, we again use [7, p.296, (6)].

Proposition 8.

$$x\in[0,1): \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n^4 4^n}(1-x^2)^n = 4\zeta(4) + 4\zeta(3)\log\tfrac{1+x}{2} + 2\zeta(2)\log^2\tfrac{1+x}{2} - \tfrac{2}{3}\log^3\tfrac{1+x}{2}\log\tfrac{1-x}{2} + 4\mathrm{Li}_4(\tfrac{1-x}{2}) + 2\mathrm{Li}_4(\tfrac{x-1}{x+1}) - 4\mathrm{Li}_4(\tfrac{1+x}{2}).$$

$$x \in (1, \sqrt{2}] : \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n^4 4^n}(1-x^2)^n = -4\zeta(4) + 4\zeta(3)\log\tfrac{1+x}{2}$$
$$- 2\zeta(2)\log^2\tfrac{1+x}{2} + \tfrac{1}{6}\log^4\tfrac{1+x}{2} - \tfrac{2}{3}\log^3\tfrac{1+x}{2}\log\tfrac{x-1}{2}$$
$$+ 4\mathrm{Li}_4(\tfrac{1-x}{2}) + 2\mathrm{Li}_4(\tfrac{x-1}{x+1}) + 4\mathrm{Li}_4(\tfrac{2}{1+x}).$$

The results in **Propositions 7–8** yield, for $x = 0$, the following sums:

$$S_2(1) = 2\log(2);$$
$$S_2(2) = \zeta(2) - 2\log^2 2;$$
$$S_2(3) = 2\zeta(3) - 2\zeta(2)\log 2 + \tfrac{4}{3}\log^3 2;$$
$$S_2(4) = \tfrac{9}{4}\zeta(4) - 4\zeta(3)\log 2 + 2\zeta(2)\log^2 2 - \tfrac{2}{3}\log^4 2.$$

And $x = \sqrt{2}$ yields these alternating sums, using the notation

$$S_2'(k) = \sum_{n=0}^{\infty}(-1)^n \frac{\binom{2n}{n}}{n^k\, 2^{2n}} :$$

$$S_2'(1) = -2\log\tfrac{1+\sqrt{2}}{2};$$
$$S_2'(2) = 2\mathrm{Li}_2(\tfrac{1-\sqrt{2}}{2}) - \log^2\tfrac{1+\sqrt{2}}{2};$$
$$S_2'(3) = 2\zeta(3) - 2\zeta(2)\log\tfrac{1+\sqrt{2}}{2} - \log^2\tfrac{1+\sqrt{2}}{2}\log\tfrac{\sqrt{2}-1}{2}$$
$$+ 2\mathrm{Li}_3(\tfrac{1-\sqrt{2}}{2}) - 2\mathrm{Li}_3(2(\sqrt{2}-1));$$
$$S_2'(4) = 2\mathrm{Li}_4(\tfrac{\sqrt{2}-1}{\sqrt{2}+1}) + 4\mathrm{Li}_4(\tfrac{1-\sqrt{2}}{2}) + 4\mathrm{Li}_4(2(\sqrt{2}-1))$$
$$+ \tfrac{1}{6}\log^4\tfrac{1+\sqrt{2}}{2} - \tfrac{2}{3}\log^3\tfrac{1+\sqrt{2}}{2}\log\tfrac{\sqrt{2}-1}{2}$$
$$- 2\zeta(2)\log^2\tfrac{1+\sqrt{2}}{2} + 4\zeta(3)\log\tfrac{1+\sqrt{2}}{2} - 4\zeta(4).$$

Note that we also have:

$$x = \frac{\sqrt{3}}{2} \Rightarrow \quad \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n\, 2^{4n+2}} = \tfrac{3}{2}\log(2) - \log(1+\sqrt{3}).$$

4. More series and analysis

Using the relation (1.2) and the sum in the previous section we deduce:

$$S_1(5) = -\frac{9\zeta(4)}{2} - 8\zeta(3)(1-\log(2)) - 8\zeta(2)(1-\log(2)) - 4\zeta(2)\log^2(2) + \frac{4}{3}\log^4(2) - \frac{16}{3}\log^3(2) + 16\log^2(2) + 32(1-\log(2)).$$

4.1. **More series through software.** We obtained these four sums using *Mathematica:*

$$S_2(5) = 6\zeta(5) - \tfrac{9}{2}\zeta(4)\log(2) - 2\zeta(3)\zeta(2) + 4\zeta(3)\log^2(2) - \tfrac{4}{3}\zeta(2)\log^3(2) + \tfrac{4}{15}\log^5(2).$$

$$S_2(6) = \tfrac{79}{16}\zeta(6) - 12\zeta(5)\log(2) + \tfrac{9}{2}\zeta(4)\log^2(2) + 4\zeta(3)\zeta(2)\log(2) - 2\zeta^2(3) - \tfrac{8}{3}\zeta(3)\log^3(2) + \tfrac{2}{3}\zeta(2)\log^4(2) - \tfrac{4}{45}\log^6(2).$$

$$S_2(7) = 18\zeta(7) - \tfrac{79}{8}\zeta(6)\log(2) + 12\zeta(5)\log^2(2) - \tfrac{9}{2}\zeta(4)\zeta(3) + 4\zeta^2(3)\log(2) - 6\zeta(5)\zeta(2) - \tfrac{2}{3}\zeta(3)\zeta(2)\log^2(2) - 3\zeta(4)\log^3(2) + \tfrac{4}{3}\zeta(3)\log^4(2) - \tfrac{4}{15}\zeta(2)\log^5(2) + \tfrac{8}{315}\log^7(2).$$

$$S_2(8) = \tfrac{2339}{192}\zeta(8) - 36\zeta(7)\log(2) + \tfrac{79}{8}\zeta(6)\log^2(2) - 12\zeta(5)\zeta(3) + 12\zeta(5)\zeta(2)\log(2) - 8\zeta(5)\log^3(2) + 9\zeta(4)\zeta(3)\log(2) + \tfrac{3}{2}\zeta(4)\log^4(2) + 2\zeta^2(3)\zeta(2) - 4\zeta^2(3)\log^2(2) + \tfrac{8}{3}\zeta(3)\zeta(2)\log^3(2) - \tfrac{8}{15}\zeta(3)\log^5(2) + \tfrac{4}{45}\zeta(2)\log^6(2) - \tfrac{2}{315}\log^8(2).$$

The preceding sums in conjunction with relation (1.2) lead to:

$$S_1(6) = -12\zeta(5) - 9\zeta(4)(1-\log(2)) - 8\zeta(3)(2-2\log(2)+\log^2(2)) + 4\zeta(3)\zeta(2) - 8\zeta(2)(2-2\log(2)+\log^2(2)-\tfrac{1}{3}\log^3(2)) - \tfrac{8}{15}\log^5(2) + \tfrac{8}{3}\log^4(2) - \tfrac{32}{3}\log^3(2) + 32\log^2(2) + 64(1-\log(2)).$$

$$S_1(7) = -\tfrac{79}{8}\zeta(6) - 24\zeta(5)(1-\log(2)) - 9\zeta(4)(2-2\log(2) + \log^2(2)) + 4\zeta^2(3) - 16\zeta(3)(2-2\log(2)+\log^2(2)-\tfrac{1}{3}\log^3(2)) + 8\zeta(3)\zeta(2)(1-\log(2)) - 16\zeta(2)(2-2\log(2)+\log^2(2) - \tfrac{1}{3}\log^3(2) + \tfrac{1}{12}\log^4(2)) + \tfrac{8}{45}\log^6(2) - \tfrac{16}{15}\log^5(2) + \tfrac{16}{3}\log^4(2) - \tfrac{64}{3}\log^3(2) + 64\log^2(2) + 128(1-\log(2)).$$

$$S_1(8) = -36\zeta(7) - \tfrac{79}{4}\zeta(6)(1-\log(2)) + 12\zeta(5)\zeta(2) - 6\zeta(5)(8- 8\log(2) + \log^2(2)) - 6\zeta(4)(6 - 6\log(2) + 3\log^2(2) - \log^3(2)) + 9\zeta(4)\zeta(3) + 8\zeta^2(3)(1-\log(2)) - 64\zeta(3)(1-\log(2)) - 32\zeta(3)\log^2(2) + \tfrac{32}{3}\zeta(3)\log^3(2) - \tfrac{8}{3}\zeta(3)\log^4(2) + 16\zeta(3)\zeta(2)(1-\log(2)) + \tfrac{4}{3}\zeta(3)\zeta(2)\log^2(2) - 64\zeta(2)(1-\log(2)) - 32\zeta(2)\log^2(2) + \tfrac{32}{3}\zeta(2)\log^3(2) - \tfrac{8}{3}\zeta(2)\log^4(2) + \tfrac{8}{15}\zeta(2)\log^5(2) - \tfrac{16}{315}\log^7(2) + \tfrac{16}{45}\log^6(2) - \tfrac{32}{15}\log^5(2) + \tfrac{32}{3}\log^4(2) - \tfrac{128}{3}\log^3(2) + 128\log^2(2) + 256(1-\log(2)).$$

4.2. **Analysis of the first type of sums.** Let $T_1(k)$ represent the number of separate terms in the value the sum $S_1(k)$. Then we can see from the sums evaluated earlier that $T_1(1) = 1$; $T_1(2) = 2$; $T_1(3) = 4$; $T_1(4) = 7$; $T_1(5) = 11$; $T_1(6) = 17$; $T_1(7) = 25$; $T_1(8) = 36$. The next sum has 50 terms in its evaluation, i.e., $T_1(9) = 50$.

This sequence $\{a_n = T_1(n-1)\}$ is given in `https://oeis.org/A096914` with:

$$a_n \sim \frac{3^{3/4}\, n^{1/4}\, e^{\pi\sqrt{n/3}}}{2\pi^2}.$$

We have this formula: $T_1(k) = 1 + \frac{k(k-1)}{2} + M(k)$, where $M(k)$ represents the number of mixed terms involving more than one zeta value. The first two terms come from the fact that there are k terms involving powers (0 to $k-1$) of $\log(2)$, one term involving $\zeta(k-1)$, two terms with $\zeta(k-2)$, and so on, and $k-2$ terms with $\zeta(2)$ totalling $k+1+2+3+\cdots+(k-2) = 1+\frac{k(k-1)}{2}$. Now $M(k)$ depends on the partition of $k-1$ into positive integers. We ignore the partitions having 1 in it because $\zeta(1)$ cannot come in our formula. We also leave partitions having more than one even member. To illustrate, for $S_1(6)$ consider: $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$. Only one partition, namely $3+2$, is of use to us. The partition function $p(n)$ counting the number of unrestricted partitions (order immaterial) of n grows astronomically. Hardy and Ramanujan[5, eqs (3),(6)] obtained for $p(n)$ a very complicated expression whose first three terms are:

$$\frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left[\frac{e^{C\lambda_n}}{\lambda_n}\right] + \frac{(-1)^n}{2\pi}\frac{d}{dn}\left[\frac{e^{\frac{1}{2}C\lambda_n}}{\lambda_n}\right] + \frac{\sqrt{3}}{\pi\sqrt{2}}\cos\left(\frac{2n\pi}{3} - \frac{\pi}{18}\right)\frac{d}{dn}\left[\frac{e^{\frac{1}{3}C\lambda_n}}{\lambda_n}\right]$$

where $C = \pi\sqrt{\frac{2}{3}}$, $\lambda_n = \sqrt{n - \frac{1}{24}}$.

4.3. **Analysis of the second type of sums.** Let $T_2(k)$ denote the number of terms in the value of $S_2(k)$. Then $T_2(k) = k + p'(k)$, where $p'(k)$ denotes the number of those restricted partitions of k which have neither 1 nor more than one even integers. We find that $T_2(k) = T_1(k+1) - T_1(k)$. So we have: $T_2(1) = 1$; $T_2(2) = 2$; $T_2(3) = 3$; $T_2(4) = 4$; $T_1(5) = 6$; $T_2(6) = 8$; $T_2(7) = 11$; $T_2(8) = 14$. This sequence is given in `https://oeis.org/A038348`. The following formula given there approximates $T(n-1)$:

$$\frac{3^{1/4}\, e^{\pi\sqrt{n/3}}}{4\,\pi\, n^{1/4}}.$$

5. Series with factor $(2n+1)^{-k}$

5.1. Series with positive terms.

5.1.1. *Linear and quadratic sums.* Taking $y = -t^2$, $n = 2$ in (1.5) and integrating both sides from 0 to x, we obtain for $x \in [0, 1]$:

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)\, 2^{2n}} x^{2n+1}, \tag{5.1}$$

this gives with $x = 1$ and $x = 1/2$ respectively:

$$S_3(1) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)\, 2^{2n}} = \frac{\pi}{2}; \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)\, 2^{4n}} = \frac{\pi}{3}.$$

Dividing both sides of (5.1) by x and then integrating from 0 to x, we have [7, p.306, (19)]:

$$\int_0^x \frac{\arcsin(x)}{x}\, dx = \frac{1}{2}\,\mathrm{Cl}_2(2\arcsin x) + \arcsin(x)\log(2x)$$
$$= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^2\, 2^{2n}} x^{2n+1} \tag{5.2}$$

where $\mathrm{Cl}_n(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^n}$ for even n and $\mathrm{Cl}_n(x) = \sum_{k=1}^{\infty} \frac{\cos kx}{k^n}$ for odd n is the Clausen function. $\mathrm{Cl}_2(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^2} = -\int_0^{\theta} \log\left(2\sin\frac{\theta}{2}\right) d\theta$, [7, p.102, (4.4)] an integral treated by Clausen in 1832. The following relation connects it with the dilogarithm [7, p.102, (4.6)]:

$$\mathrm{Li}_2(e^{i\theta}) = \frac{\pi^2}{2} - \frac{\theta(2\pi-\theta)}{4} + i\mathrm{Cl}_2(\theta), \quad 0 \le \theta \le 2\pi.$$

Lewin [7, p.291] records these values: $\mathrm{Cl}_2(n\pi) = 0$; $\mathrm{Cl}_2(\pi/2) = G$.

We thus obtain:

$$S_3(2) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^2\, 2^{2n}} = \frac{\pi}{2}\log(2).$$

Taking $y = -t^2$, $n = 2$ in (1.5), dividing both sides by x^2 and then integrating both sides we get

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = \int \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} x^{2n-2}.$$

The substitution $x = \sin\theta$ in the integral on the left yields $-\cot\theta = -\sqrt{1-x^2}/x$. If we apply term by term integration of the infinite series on the right, then we find, after multiplying by x:

$$-\sqrt{1-x^2} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n-1)\, 2^{2n}} x^{2n}, \quad x \in (0,1]. \tag{5.3}$$

Taking $x = 1$ yields:

$$S_4(1) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n-1)\, 2^{2n}} = 0.$$

Now we can use Lemma 2 and the values of $S_3(1)$ and $S_3(2)$ to prove that

$$S_4(2) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n-1)^2\, 2^{2n}} = \frac{\pi}{2}$$

$$S_4(3) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n-1)^3\, 2^{2n}} = \frac{\pi}{2}(\log(2)-1).$$

5.1.2. *Series via* $\int_0^{\phi} \log^k(\sin u)\, du$. Let us return to (1.2) with $y = -t^2$, $n = 2$:

$$\frac{1}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} t^{2n}.$$

Instead of integrating, we multiply both sides of this equation by $\log^k(t)$, and integrate the result between 0 and x:

$$\int_0^x \frac{\log^k(t)}{\sqrt{1-t^2}}\,dt = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \int_0^x t^{2n}\log^k(t)\,dt.$$

If we use the substitution $t = \sin u$ in this integral, we find:

$$\int_0^{\arcsin(x)} \log^k(\sin u)\,du = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \int_0^x t^{2n}\log^k(t)\,dt. \tag{5.4}$$

For the integrals at the right hand side we have the following recurrence relation, which can be obtained using partial integration:

$$I_{k,n}(x) := \int_0^x t^{2n}\log^k(t)\,dt = \frac{1}{2n+1}\left[x^{2n+1}\log^k(x) - k\,I_{k-1,n}(x)\right]$$

with $I_{0,n}(x) = \frac{1}{2n+1}x^{2n+1}$, leading to:

$$I_{k,n}(x) = \sum_{i=0}^{k}(-1)^i i!\binom{k}{i}\log^{k-i}(x)\frac{x^{2n+1}}{(2n+1)^{i+1}}.$$

Hence we get from (5.4):

$$\int_0^{\arcsin(x)} \log^k(\sin u)\,du = \sum_{i=0}^{k}\left[(-1)^i i!\binom{k}{i}\log^{k-i}(x)\sum_{n=0}^{\infty}\frac{\binom{2n}{n}\,x^{2n+1}}{(2n+1)^{i+1}\,2^{2n}}\right]. \tag{5.5}$$

This means that we can find the exact value of the sums of the form

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)^{i+1}2^{2n}}x^{2n+1}$$

from previous ones and from known values of the log-sine integrals $\mathrm{Ls}_n(\sigma)$:

$$\mathrm{Ls}_n(\sigma) = -\int_0^{\sigma}\log^{n-1}\left(2\sin\frac{\theta}{2}\right)\,d\theta.$$

Note that the right-hand side is related to the integral we need (substitution $u = \theta/2$):

$$\mathrm{Ls}_n(\sigma) = -2\int_0^{\sigma/2}(\log(2)+\log(\sin u))^{n-1}du.$$

Case (1): $x = 1$. In this case (5.5) reduces to:

$$\int_0^{\pi/2} \log^k(\sin u)\, du = (-1)^k k! \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^{k+1} 2^{2n}}. \tag{5.6}$$

We need $\mathrm{Ls}_n(\pi)$. Some of the known values are [1, Example 1]:

$$\mathrm{Ls}_2(\pi) = 0,\ \mathrm{Ls}_3(\pi) = -\frac{\pi^3}{12},\ \mathrm{Ls}_4(\pi) = \frac{3}{2}\pi\zeta(3),\ \mathrm{Ls}_5(\pi) = -\frac{19}{240}\pi^5,$$

leading to further sums $S_3(k) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^k\, 2^{2n}}$:

$$S_3(3) = \frac{\pi}{2}\left[\frac{\zeta(2)}{4 \cdot 0!} + \frac{\log^2(2)}{2!}\right], \qquad S_4(4) = S_3(3) - S_4(3),$$

$$S_3(4) = \frac{\pi}{2}\left[\frac{\zeta(3)}{4 \cdot 0!} + \frac{\zeta(2)\,\log(2)}{4 \cdot 1!} + \frac{\log^3(2)}{3!}\right], \qquad S_4(5) = S_3(4) - S_4(4),$$

$$S_3(5) = \frac{\pi}{2}\left[\frac{19\zeta(4)}{64 \cdot 0!} + \frac{\zeta(3)\,\log(2)}{4 \cdot 1!} + \frac{\zeta(2)\,\log^2(2)}{4 \cdot 2!} + \frac{\log^4(2)}{4!}\right]$$

Here we have used Lemma 2 for $S_4(k)$.

Case (2): $x = \frac{1}{2}$. In this case (5.5) reduces to:

$$\int_0^{\pi/6} \log^k(\sin u)\, du = (-1)^k \sum_{i=0}^{k} \left[i! \binom{k}{i} \frac{\log^{k-i}(2)}{2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^{i+1} 2^{4n}} \right]. \tag{5.7}$$

For these we need $\mathrm{Ls}_n(\frac{\pi}{3})$. Some of the known values are [1, Example 10]:

$$\mathrm{Ls}_2\left(\frac{\pi}{3}\right) = \mathrm{Cl}_2\left(\frac{\pi}{3}\right),\ \mathrm{Ls}_3\left(\frac{\pi}{3}\right) = -\frac{7\pi^3}{108}, \mathrm{Ls}_4\left(\frac{\pi}{3}\right) = \frac{\pi}{2}\zeta(3) + \frac{9}{2}\mathrm{Cl}_4\left(\frac{\pi}{3}\right),$$

leading to:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^2\, 2^{4n}} = \mathrm{Cl}_2\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{3}\left[\frac{1}{2}\psi^{(1)}\left(\frac{1}{3}\right) - \frac{\pi^2}{3}\right],$$

(see [2]) where $\psi^{(1)}(z) = \sum_{k=0}^{\infty} \frac{1}{(k+z)^2}$, $z \neq 0, -1, -2, \ldots$, is the trigamma function,

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^3\, 2^{4n}} = \frac{7\pi^3}{216},$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^4\, 2^{4n}} = \frac{1}{12}\left[\pi\zeta(3) + 9\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\right]$$

From these results and

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)\,2^{4n}} = \mathrm{Ls}_1\left(\frac{\pi}{3}\right) = \frac{\pi}{3}, \text{ and } \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{(2n-1)\,2^{4n}} = 1 - \frac{\sqrt{3}}{2}$$

obtained from (5.3) by taking $t = \frac{1}{4}$, using Lemma 2 (with $t = \frac{1}{4}$) we can find the sum of the series:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{(2n-1)^k\,2^{4n}}$$

for $k = 2, 3, 4$.

Case (3): $x = \frac{\sqrt{2}}{2}$**.** In this case (5.5) reduces to:

$$\int_0^{\pi/4} \log^k(\sin u)\,du = (-1)^k \sum_{i=0}^{k} \left[i! \binom{k}{i} \frac{\log^{k-i}(2)}{2^{k-i}\sqrt{2}} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^{i+1} 2^{3n}} \right]. \tag{5.8}$$

For these we need $\mathrm{Ls}_n(\frac{\pi}{2})$. One known value is [7, p.291]:

$$\mathrm{Ls}_2\left(\frac{\pi}{2}\right) = G,$$

where $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.91596\ldots$ is Catalan's constant, leading to:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^2\,2^{3n}} = \frac{1}{\sqrt{2}} \left[G + \frac{\pi \log(2)}{4} \right].$$

With

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)\,2^{3n}} = \frac{\pi}{4}\sqrt{2}, \quad \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{(2n-1)\,2^{3n}} = 1 - \frac{\sqrt{2}}{2}$$

and Lemma 2 (with $t = \frac{1}{2}$), we obtain:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n-1)^2\,2^{3n}} = \frac{\sqrt{2}}{2}\left(\frac{\pi}{4} + 1\right),$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n-1)^3\,2^{3n}} = \frac{\sqrt{2}}{16}(4G + \pi(\log(2) - 2) - 8).$$

Case (4): $x = \frac{\sqrt{3}}{2}$**.** In this case we could get these two sums:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)} \left(\frac{3}{16}\right)^n = \frac{2\pi\sqrt{3}}{9}.$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^2} \left(\frac{3}{16}\right)^n = \frac{\pi \log(3)\sqrt{3}}{9} - \frac{2\pi^2}{27} + \frac{1}{9}\psi^{(1)}\left(\frac{1}{3}\right).$$

5.1.3. *Series with higher powers.* Series with powers > 5 obtained through *Mathematica* have been rewritten in terms of zeta values to discern the underlying pattern. Note that the value of $S_4(k)$ can be found using Lemma 2.

$$S_3(6) = \frac{\pi}{2}\left[\frac{3\zeta(5)}{16\cdot 0!} + \frac{19\zeta(4)\log(2)}{64\cdot 1!} + \frac{\zeta(3)\zeta(2)}{16\cdot 0!} + \frac{\zeta(3)\log^2(2)}{4\cdot 2!} + \frac{\zeta(2)\log^3(2)}{4\cdot 3!} + \frac{\log^5(2)}{5!}\right].$$

$$S_3(7) = \frac{\pi}{2}\left[\frac{275\zeta(6)}{1024\cdot 0!} + \frac{3\zeta(5)\log(2)}{16\cdot 1!} + \frac{19\zeta(4)\log^2(2)}{64\cdot 2!} + \frac{\zeta^2(3)}{32\cdot 0!} + \frac{\zeta(3)\,\zeta(2)\log(2)}{16\cdot 1!} + \frac{\zeta(3)\,\log^3(2)}{4\cdot 3!} + \frac{\zeta(2)\log^4(2)}{4\cdot 4!} + \frac{\log^6(2)}{6!}\right].$$

$$S_3(8) = \frac{\pi}{2}\left[\frac{9\zeta(7)}{64\cdot 0!} + \frac{275\zeta(6)\log(2)}{1024\cdot 1!} + \frac{\zeta(5)\zeta(2)}{128\cdot 0!} + \frac{3\zeta(5)\log^2(2)}{16\cdot 2!} + \frac{\zeta(4)\zeta(3)}{256\cdot 0!} + \frac{19\zeta(4)\log^3(2)}{64\cdot 3!} + \frac{\zeta^2(3)\,\log(2)}{32\cdot 1!} + \frac{\zeta(3)\,\zeta(2)\log^2(2)}{16\cdot 2!} + \frac{\zeta(3)\,\log^4(2)}{4\cdot 4!} + \frac{\zeta(2)\log^4(2)}{4\cdot 5!} + \frac{\log^7(2)}{7!}\right].$$

$$S_3(9) = \frac{\pi}{2}\left[\frac{11813\zeta(8)}{49152\cdot 0!} + \frac{9\zeta(7)\log(2)}{64\cdot 1!} + \frac{275\zeta(6)\log^2(2)}{1024\cdot 2!} + \frac{3\zeta(5)\zeta(3)}{64\cdot 0!} + \frac{\zeta(5)\zeta(2)\log(2)}{128\cdot 1!} + \frac{3\zeta(5)\log^3(2)}{16\cdot 3!} + \frac{19\zeta(4)\log^4(2)}{64\cdot 4!} + \frac{\zeta(4)\zeta(3)\,\log(2)}{256\cdot 1!} + \frac{\zeta^2(3)\,\zeta(2)}{128\cdot 0!} + \frac{\zeta^2(3)\,\log^2(2)}{32\cdot 2!} + \frac{\zeta(3)\,\zeta(2)\log^3(2)}{16\cdot 3!} + \frac{\zeta(3)\,\log^5(2)}{4\cdot 5!} + \frac{\zeta(2)\log^6(2)}{4\cdot 6!} + \frac{\log^8(2)}{8!}\right].$$

Number of terms: Let $T_3(k)$ be the number of terms in the value of $S_3(k)$. Then for $k \geq 2$ we have: $T_3(k) = (k-1)+p'(k-1)$, where $p'(k-1)$ denotes the number of those restricted partitions of $k-1$ which have neither 1 nor more than one even integers. We also find that $T_3(k) = T_2(k-1) =$

$T_1(k) - T_1(k-1)$. So we have: $T_3(1) = 1$; $T_3(2) = 1$; $T_3(3) = 2$; $T_3(4) = 3$; $T_3(5) = 4$; $T_3(6) = 6$; $T_3(7) = 8$; $T_3(8) = 11$; $T_3(9) = 14$.

This sequence is given in `https://oeis.org/A038348`. The following formula given there approximates $T_3(n+1)$:

$$\frac{3^{1/4}\, e^{\pi\sqrt{n/3}}}{4\,\pi\, n^{1/4}}.$$

5.2. **Alternating sums.** We now start from

$$\frac{1}{\sqrt{1+t^2}} = \sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{2^{2n}} t^{2n}.$$

We multiply both sides of this equation by $\log^k(t)$, and integrate the result between 0 and x:

$$\int_0^x \frac{\log^k(t)}{\sqrt{1+t^2}}\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{2^{2n}} \int_0^x t^{2n} \log^k(t)\, dt.$$

If we use the substitution $t = \sinh u$ in this integral, we find:

$$\int_0^{\operatorname{arcsinh}(x)} \log^k(\sinh u)\, du = \sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{2^{2n}} \int_0^x t^{2n} \log^k(t)\, dt. \tag{5.9}$$

This formula can be used in the same way as in 5.1.2 to obtain the following sums:

Case (1): $x = 1$**.**

$$\sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{(2n+1)\, 2^{2n}} = \log(1+\sqrt{2})$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{(2n+1)^2\, 2^{2n}} = \tfrac{\pi^2}{12} - \tfrac{3}{2}\log^2(1+\sqrt{2}) + \log(1+\sqrt{2})\log(2+2\sqrt{2}) - \tfrac{1}{2}\mathrm{Li}_2\left((\sqrt{2}-1)^2\right)$$

$$\begin{aligned}\sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{(2n+1)^3\, 2^{2n}} &= \tfrac{1}{4}\zeta(3) + \tfrac{\pi^2}{12}\log(2) + \tfrac{1}{12}\log^3(2) \\ &- \tfrac{1}{4}\log^2(2)\log(1+\sqrt{2}) + \log(2)\log(1+\sqrt{2})\log(2+\sqrt{2}) \\ &+ \tfrac{1}{2}\log^2(1+\sqrt{2})\log(\tfrac{2+\sqrt{2}}{8}) - \tfrac{5}{12}\log^3(1+\sqrt{2}) \\ &- \tfrac{1}{2}\mathrm{Li}_3(\tfrac{1-\sqrt{2}}{2}) - \tfrac{1}{4}\mathrm{Li}_3((\sqrt{2}-1)^2).\end{aligned}$$

We can use Lemma 2 with $t = -1$, together with

$$\sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{(2n-1)\, 2^{2n}} = -\tfrac{1}{2} S_1'(1) = 1 - \sqrt{2}$$

to get values for

$$\sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{(2n-1)^{k+1}\, 2^{2n}}.$$

Case (2): $x = \frac{1}{2}$**.**

$$\sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{(2n+1)\, 2^{4n}} = 2\log(\varphi)$$

where $\varphi = \frac{1+\sqrt{5}}{2}$, is the golden ratio of the ancient Greeks.

$$\sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{(2n+1)^2\, 2^{4n}} = \frac{\pi^2}{10}.$$

Compare it with this sum, obtained using (5.9) with $x = \frac{1}{2\sqrt{2}}$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{(2n+1)^2\, 2^{5n}} = \frac{1}{\sqrt{2}} \left[\zeta(2) - \frac{1}{2}\log^2(2) \right].$$

The previous sum could also be deduced from the hypergeometric summation formula [9] with $z = -\frac{1}{8}$:

$$\begin{aligned} {}_3F_2 \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; z \right] &= -\frac{\log^2(\sqrt{1-z} + \sqrt{-z})}{\sqrt{-z}} \\ &+ \frac{\log(\sqrt{1-z} + \sqrt{-z})\ \log(\sqrt{1-z} + \sqrt{-z} + 1)}{\sqrt{-z}} \\ &+ \frac{\mathrm{Li}_2(-\sqrt{1-z} - \sqrt{-z})}{\sqrt{-z}} - \frac{\mathrm{Li}_2(-\sqrt{1-z} - \sqrt{-z} + 1)}{\sqrt{-z}} + \frac{\pi^2}{12\sqrt{-z}}. \end{aligned}$$

For $k = 3$ we have the following sum:

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{(2n+1)^3 2^{4n}} &= \tfrac{1}{4}\zeta(3) + \tfrac{1}{20}\pi^2 \log(2) + \tfrac{1}{2}\log^2(2)\log(\varphi) \\ &\quad + \tfrac{1}{12}\log^3(\varphi) - \tfrac{3}{2}\mathrm{Li}_3(-\varphi^{-1}) - \mathrm{Li}_3(\varphi^{-1}). \end{aligned}$$

Here we have used the relation [7, p.283, (10)]

$$\mathrm{Li}_2(-x) - \mathrm{Li}_2(1-x) + \tfrac{1}{2}\mathrm{Li}_2\left(\frac{1}{x^2}\right) = \log(x)\log\left(\frac{x-1}{x}\right), \quad x > 1$$

taking $x = 1 + \sqrt{2}$, and also [7, p.283, (6),(7); p.296, (5),(6)].

The related sums with factor $(2n-1)^{-k}$ can be found from

$$\sum_{n=1}^{\infty}(-1)^n \frac{\binom{2n}{n}}{(2n-1)\,2^{4n}} = \frac{3}{2} - \varphi$$

and Lemma 2 with $t = -\frac{1}{4}$.

Postscript: After having completed our paper, we found that Hansen's book [4, p.139, (6.7.48)] contains this general formula for $m = 1, 2, 3, \ldots$

$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!(kx+y)^m} = \frac{\Gamma(1-a)}{x(m-1)!}(-1)^{m-1}\frac{d^{m-1}}{dy^{m-1}}\left\{\frac{\Gamma(y/x)}{\Gamma[(y/x)-a+1]}\right\},$$

which gives two types of our sums with positive terms on taking $a = 1/2$ together with $x = 1, 2$ and $y = 1$.

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