LESSAM

Teacher Handbook

Improving Learning & Teaching Mathematical Reasoning through Research Lesson Study

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Of the project
Lesson Study as a vehicle for improving achievement in mathematics (LESSAM)

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Reasoning in mathematics didactics

01

1. Introduction

In this first part of the handbook, we introduce 'reasoning' in ways that might be helpful in this Lesson Study project. The first section provides a rationale for studying reasoning in Lesson Study; the second presents the eight types of reasoning we investigate in the Lesson Study.

In the third section we discuss the teaching and learning of reasoning in school mathematics, and in the fourth section issues related to the design and enactment of lessons that build a classroom community that promotes mathematical reasoning are addressed.

2. A rationale for reasoning in Lesson Study

There are many different ways of defining or conceptualizing 'reasoning' in school mathematics; we present just a few so you can discuss which one relates closest to your understandings.

> **In mathematics, reasoning involves drawing logical conclusions based on evidence or stated assumptions.**
>
> National Council of Teachers of Mathematics (1900, p. 1)

> **Reasoning enables children to make use of all their other mathematical skills and so reasoning could be thought of as the 'glue' which helps mathematics makes sense.**
>
> The NRICH Primary Team (2014, p. 1)

> **Reasoning is the process of manipulating and analysing objects, representations, diagrams, symbols, or statements to draw conclusions based on evidence or assumptions.**
>
> Battista (2017, p.1)

> **Reasoning is the process of communication with others or with oneself that allows for inferring mathematical utterances from other mathematical utterances.**
>
> Jeannotte & Kieran (2017, p. 9)

> **Reasoning refers to the line of thought that is adopted to [...] reach conclusions when solving tasks. Reasoning is not necessarily based on formal logic and is therefore not restricted to proof; it may even be incorrect as long as there are (to the reasoner) some sensible reasons supporting it.**
>
> Bergqvist & Lithner (2012, p. 253)

Teacher task:

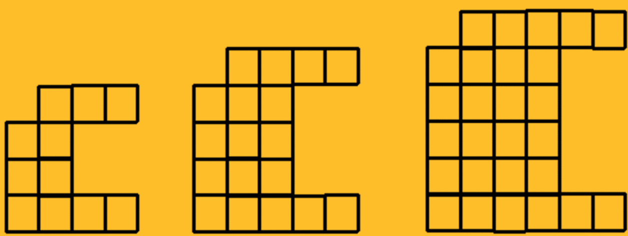
Compare these definitions of mathematical reasoning.
What elements of reasoning are shared by all quotes? What are elements that are unique for each quote?
Think of two examples of mathematical reasoning from your recent teaching.
Characterise it by using one of the definitions of reasoning above.

The importance of mathematical reasoning lies in its role as a crucial link between basic skills and higher-order thinking. Indeed, studies have demonstrated that students who receive early instruction in reasoning skills tend to develop greater confidence and independence in their learning. They acquire a profound grasp of how a concept can be employed across diverse scenarios and exhibit a readiness to experiment and explore to discern effective strategies.

The following task[1] allows to provides opportunities for all learners to reason, at different levels:

Mathematics task 1:
Small squares

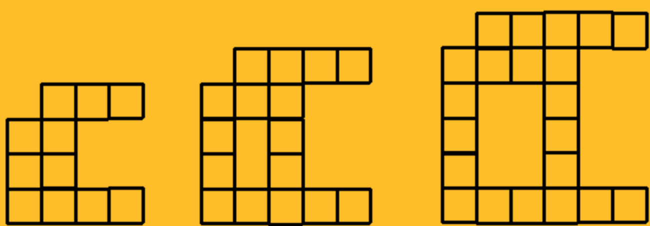
Figure 1



Pattern 1 Pattern 2 Pattern 3

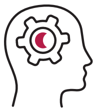
1. What would Pattern 4 look like? What would Pattern 100 look like? Or Pattern 957?
2. How many small squares would Pattern n contain?
3. How did you come up with this number?
4. (Advanced) Look at Figure 2 (see below) and repeat questions 1-3. What do you observe, in contrast with the original Figure 1? Why is that the case?
5. What is the largest possible number of small squares you can remove from the original pattern in Figure 1, such that it still grows quadratically?

Figure 2



Pattern 1 Pattern 2 Pattern 3

[1]Inspired by Boaler (2016, p.73)



Teacher task:

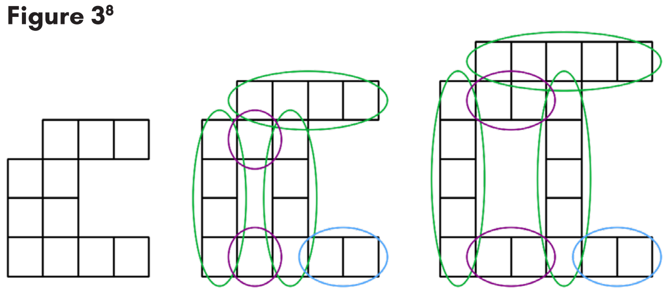
1. Reflect on your own reasoning process (e.g., by answering the following questions):
- How did you find the structure in the pattern?
- Did you use number examples, and if so, how?
- How did you translate this structure into an algebraic expression?
- Did your colleague(s) find the same algebraic expression? If so, how? If not, what did your colleague do differently?
- Find as many symbolic representations as you can.
2. Describe what kinds of reasoning students would be engaged in when working on this task. How will students' solutions look like?
3. Identify which kinds of representations students might be using.
4. What kinds of communication is required from students?
5. What kinds of mathematical thinking is required to solve the task?

Forming conjectures and generalizing are essential components of the teaching and learning of mathematical reasoning. Reasoning involves justifying, that is, "making, investigating and evaluating conjectures, and developing mathematical arguments to convince oneself and others that the conjecture is true"[2]. Thus, r**easoning allows students to go beyond routine procedures towards an appreciation of the interconnected, logical and meaningful aspects of mathematics[3].**

Justifying is more than explaining "what", and includes the "why" to verify a claim[4]. A mathematical justi-fication is a logical argument based on accepted procedures, properties, concepts, and mathe-matical ideas[5]. As students' proficiency for reasoning grows, they become able to offer a mathematically and sound logical argument to support a claim[6]. Despite its relevance, students seldomly have the opportunity to engage in mathematical reasoning activities[7].

In the example above, your students' reasoning could look like this:

Figure 3[8]



Pattern 1 Pattern 2 Pattern 3

In order to identify the pattern in the sequence, students need:

- Analysing and structuring: Identify elements that change from one pattern to the next (green, purple) and elements that stay the same (blue).
- Conjecture about the nature of change: the green elements grow in the same way, so there are 3 shapes which each increase by +1 from one step to the next. Similarly, there are two purple elements that increase by 1. The blue element does not change from one pattern to the next.
- Generalizing the commonalities identified: Hence, one general pattern can be represented as 3*(n+2)+2*(n-1)+2.
- Test conjecture: Does the expression describe the first or zero-pattern appropriately? What about the fifth pattern?

In the example above, the students predominantly engage in analysing and generalising. Generalizing identifies commonalities across cases, extending beyond the original case[9].

Given the importance of encouraging reasoning we would want you to pay attention to reasoning in your lessons. Hence, in the next section, we outline which ways of reasoning we perceive as most pertinent in school mathematics. In the third section, we explore the teaching and learning of reasoning.

[2]Goos, Vale, & Stillman (2017, p. 37)
[3]Mata-Pereira & da Ponte (2017)
[4]Sowder & Harel (1998)
[5]Mata-Pereira & da Ponte (2017
[6]Jeannotte & Kieran (2017)
[7]Prediger, et al. (2018)
[8]Adapted from Boaler (2016)
[9]Blanton & Kaput (2005); Küchemann (2010)

3. Ways of mathematical reasoning

Various attempts have been made to categorise modes of mathematical reasoning to understand its nature and the variations of it. In the PISA 2021 mathematics framework[10] mathematical reasoning constitutes a core aspect of mathematical literacy. Proper reasoning based on assumptions can lead to results that can be fully trusted to be true in a wide variety of real-life contexts. The PISA framework distinguishes six key understandings that provide structure and support to mathematical reasoning. These key understandings include:

- Understanding quantity, number systems and their algebraic properties;
- Appreciating the power of abstraction and symbolic representation;
- Seeing mathematical structures and their regularities;
- Recognizing functional relationships between quantities;
- Using mathematical modelling as a lens onto the real world (e.g. those arising in the physical, biological, social, economic and behavioural sciences) and
- Understanding variation as the heart of statistics.

In the current discussion about mathematical competences, mathematical reasoning has been identified as one of the eight fundamental mathematical competencies[11]. This competency involves both constructively providing justification of mathematical claims and critically analysing and assessing existing or proposed justification attempts. The competency deals with a wide spectrum of forms of justification, ranging from reviewing or providing examples (or counter-examples) over heuristics and local deduction to rigorous proof based on logical deduction from certain axioms.

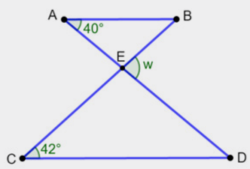
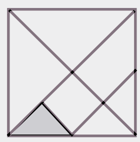
Mathematical reasoning has also been related to the activities "making mathematical generalizations" and "providing support ˜to mathematical claims"[12]. The second activity has been discussed more widely in the research literature and has been related to certain modes of reasoning. The example of Mathematical Task 1 above illustrated how reasoning in generalization contexts can look like.

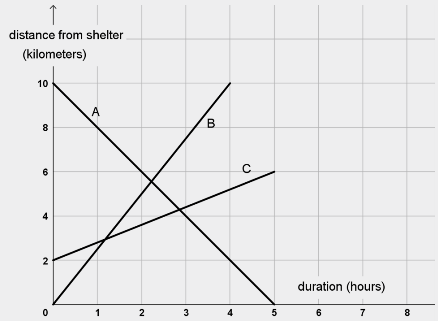
Considering the above discussions and findings, we developed in the context of LESSAM a framework with eight key aspects of mathematical reasoning. Below we provide a table, with a short description of each aspect and some illustrative examples of tasks that can support these aspects. We recognize that each categorization has potential and limitations. However, we think that it provides a tool for teachers a) to recognize salient features of students' mathematical reasoning, b) to consider the design of tasks promoting different aspects of mathematical reasoning, and c) to reflect on the possibility of promoting students' mathematical reasoning in teaching.

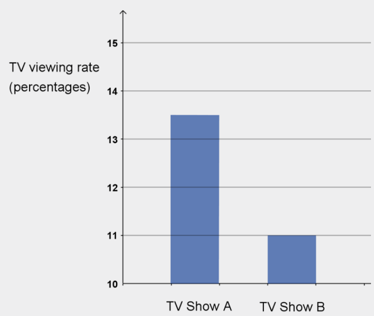
[10]OECD (2022)
[11]Niss & Højgaard (2019)
[12]As they are analysed in the framework of Stylianides (2009)

Table 1

Key aspects of mathematical reasoning (used in LESSAM project)

<table>
<tr><th>Key aspects of Mathematical Reasoning</th><th>Description</th><th>Relevant Task</th></tr>
<tr><td>Generalizing from specific cases (inductive reasoning)</td><td>e.g., finding the general term in a pattern</td><td>In the table below, complete the cells under 100 and n:
<table>
<tr><td>2</td><td>2</td><td>3</td><td>4</td><td>...</td><td>100</td><td>n</td></tr>
<tr><td>3</td><td>6</td><td>9</td><td>12</td><td>...</td><td>...</td><td>...</td></tr>
</table></td></tr>
<tr><td>Evaluating mathematical claims</td><td>e.g., refuting through counterexamples</td><td>George says that a right triangle cannot be isosceles. Do you agree with him or not? Explain why.</td></tr>
<tr><td>Developing conclusions through deductive reasoning</td><td>e.g., use mathematical statements to arrive at a solution</td><td>In the following figure AB//CD. Find the value of angle w.

</td></tr>
<tr><td>Reasoning by analogy, i.e. transfer of structural information from one system to another</td><td>e.g., transferring the structure of manipulatives to the abstract context</td><td>In a kingdom there are black and red knights. Each time a red knight meets a black one (or vice versa) they are both annihilated. Could you interpret the following arithmetic calculation in terms of the above story: $-1+1=0$</td></tr>
<tr><td>Reasoning with images</td><td>e.g., decomposition of geometrical shapes in the process of justifying/ proving.</td><td>How many times does the coloured triangle fit into the large square? Explain.
</td></tr>
<tr><td>Evaluating the relevance of a mathematical model in a realistic situation</td><td>Considering the appropriateness of possible mathematical models for solving the problem.</td><td>How much hot air is needed to fill a balloon?
</td></tr>
</table>

<table>
<tr><th>Key aspects of Mathematical Reasoning</th><th>Description</th><th>Relevant Task</th></tr>
<tr><td>Making links among different representations</td><td>Linking visual, symbolic, verbal, contextual or physical representations</td><td>Maria starts in the morning from the base camp to go up to a shelter on Olympus Mountain, a distance of 10 kilometres. Kathrin starts at the same time coming down from the shelter to the base camp. Which line (between A, B, C) and which equation (from 1, 2, 3) may represent Maria's distance from the shelter and which ones may represent Kathrin's distance from the shelter?


1. $y=2.5x$
2. $y=0.8x+2$
3. $y=-2x+10$</td></tr>
<tr><td>Understand and model stochastic situations</td><td>e.g., evaluating claims/information provided by media</td><td>Showing the adjacent diagram, Mr. O claimed that TV Show A is almost three times more popular than TV Show B. Do you think he is right? Why?

</td></tr>
</table>

With these insights into mathematical reasoning, we now turn to the design and enactment of lessons that promote mathematical reasoning.

4. Designing, enacting and assessing lessons that build a classroom community that promotes mathematical reasoning

Guiding our perspective on the teaching and learning of reasoning is the notion that in order to design lessons that promote mathematical reasoning, particular conditions need to be in place. In this section we discuss the following interrelated themes that link to such conditions: (a) Enacting reasoning tasks and (thoughtful) support; (b) (strategic) questioning; (c) noticing critical classroom events; and (d) assessing students' reasoning actions.

[13]Hiebert et al. (1997)

4.1 Enacting reasoning tasks and thoughtful support

Mrs. Chiotis wants to introduce the Pythagorean theorem to her students. To connect with the students' prior knowledge and immediately activate them in class, she asks if they have ever heard of Pythagoras. Some students have opened the chapter on Pythagoras in their textbook. They raise their hands to answer the teacher's question. One student says "something with half triangles", another student answers "a^2+b^2=c^2". It would have been better if the teacher activated the prior knowledge and involved the students in the lesson through mathematical reasons. What reasoning tasks would be appropriate here?

It is important to note that the kinds of tasks that students are asked to perform set the foundation for the teaching that is created. In other words, different kinds of tasks lead to different 'organisations' of teaching. It is generally assumed that an organisation of teaching that provides students opportunities to reflect, communicate and reason is built on tasks that are genuine problems for students.

By choosing tasks that focus on reasoning instead of looking for the right answer (see Table 2), teachers create opportunities for students to investigate mathematical claims and come up with reasoned solutions. It can be said that appropriate tasks have at least three features[13]: (1) they make the subject problematic/interesting for students; (2) they connect with where students are (in terms of knowledge and skills); (3) they engage students in thinking about important mathematics.

For the case of Pythagoras, we could imagine getting students engaged in exploring the area of squares in triangles with the help of GeoGebra. The teacher could ask the students "What do you observe? Formulate a general rule. What if the triangle is not right-angled?"

Michalis and Nicholas are two dedicated mathematics teachers from Cyprus who participated in the LESSAM project (2022-2023). They taught in different classes of Year 8 (students aged 13-14 They planned a research lesson together and they decided to focus on students who typically have different levels of engagement in the lesson. The aims of their research lesson were for students to find the points of intersections of the axes and to detect whether a point belongs to a line or not. Michalis taught the lesson first in his class by demonstrating specific examples of different cases on Geogebra, and then letting the students work on examples and discussing them. At the end, he finally wrote on the whiteboard all the different cases and examples that were presented earlier in the lesson. After the end of the lesson, Michalis, Nicholas and a mathematical expert had a reflection meeting. It was concluded that some students participated in the lesson, but did not engage in mathematical reasoning as expected. Based on this discussion, the teachers decided to dedicate more lesson time for each student to work independently and more time for groupwork, discussing their examples and how they worked. They decided that the role of the teacher should be to facilitate group discussion and provide more transformational activities. Nicholas then taught the revised research lesson in his class. At the second reflection meeting, Michalis and Nicholas noticed more critical incidents where students were engaged in mathematical reasoning. They concluded that even a well-designed lesson can be improved after a collaborative reflection discussion.

[13] Hiebert et al. (1997)

Table 2

Overview of elements of mathematical reasoning tasks[14]

Tasks with limited opportunities for reasoning	Tasks with opportunities for reasoning
• Word problems can be solved without using the context, by using a memorized procedure that is introduced beforehand (e.g. in the textbook) • Students state previously learned facts, formulas, definitions • Students work within one representation • Students have to find an expected answer • Tasks are not open to different strategies	• Students generalize based on examples • Students invent their own tasks or problems, also based on some given criteria • Students give reasons for choices (e.g. for a solution strategy, a theorem) • Students make use of multiple representations (graphs, tables, formulas) to arrive at a result • Students adapt and apply a procedure to a new, unfamiliar problem • Students can see how changes in one representation affects another representation (e.g. how changing "slope" affects a graph and a symbolic expression of a function) • Students devise a mathematical model for a real-world situation • Students develop a mathematical concept (understanding) in the real world

In terms of the role of the teacher, the teacher intervenes with 'thoughtful support' that encourages and maintains students' exploration and reasoning, and does not give away results or important reasoning steps. This means that the teacher now has the role of selecting and mediating appropriate problems/tasks as opportunities for student learning, scaffolding student development of understanding through questioning, and facilitating the establishment of a classroom culture, where students work on reasoning tasks, ideally interactively and collaboratively, and discuss and reflect on the answers and methods. This kind of teaching minimizes the teacher's role during (initial) exploration, so that students are more likely to engage in mathematical discourse, share representations, co-construct ideas and justifications, and ultimately take a more active role in their own learning.

[14] based on Boston & Smith (2009)

4.2 Challenging students with reasoning tasks

Selecting tasks that engage all your students in mathematical reasoning is demanding, yet highly relevant for supporting mathematical reasoning. We now consider an example from algebra education, particularly from the introduction of variables with word formulas:

Mathematics task 2[15]:
Matchsticks

Calculate the number of matchsticks in Figure 4, using the formula: "Number of match sticks= 2+3* number"

Figure 4

Pattern 1 Pattern 2 Pattern 3 Pattern 4

a. How many matchsticks are there in Pattern 16?
b. Calculate how many matchsticks you need for Pattern 28
c. Sander has a box with 120 matchsticks. What is the number of the biggest Pattern that he can produce with these number of matchsticks?

Teacher task:

Describe what kind of mathematical reasoning students engage in when they work on Mathematical task 2. Think about, for instance, which kind of representations students are using, what communication is required from students, and what kind of mathematical thinking is required to solve the task. Compare with the earlier task (Mathematics task 1).

Even though Mathematical Task 1 and the task above are about pattern sequences and the introduction of algebraic expressions, you might notice that the type of mathematical reasoning in the two tasks differs significantly. In the initial task from this section (Mathematics task 1), there is potential for your students for:

- **Connecting representations**: Students use the graphical pattern to identify a structure, and then translate the structure into an arithmetic or algebraic description of the pattern.
- **Communication**: Students can find different graphical structures and symbolic expressions. Accordingly, there are many opportunities for students to communicate mathematically, e.g. by comparing their different solutions or by arguing why their different expressions have to be equal.
- **Conceptual activation**: Students develop an informal understanding of the concept of variable as co-varying number, by recognizing the relation between position in the sequence and the number of squares in the figure. They can explore this concept by first working with concrete numbers (5, 100). When students think about the number 957, they are likely to treat this number as pseudo-variable, because they cannot visualize the 957th pattern anymore and thus have to think abstractedly.

[15]From Dijkhuis, et al. (2016, p. 74)

In contrast, the textbook task (Mathematics task 2) provides a symbolic expression and hence takes away the opportunity for students to analyse the pattern, to generalize structures or to find an algebraic expression. Instead, it requires students to insert numbers for a given formula.

- **Connecting representations**: Students do not have to connect representations, they can exclusively work with the given symbolic expression. The matchstick picture is superfluous.
- **Communication**: The students can communicate one-word answers, namely their numerical results, without the need for discussing or explaining.
- **Conceptual Activation**: Students work with the variable as placeholder for numbers and do calculations accordingly, where the variable is represented as word-variable. As they are asked to do calculations, they do not experience the co-variation of number of matchsticks and position in the pattern sequence.

With respect to differentiation, the textbook task is not open to different learner proficiencies. On the contrary, as there is just one solution strategy (inserting numbers), those students who lack the proficiency to insert numbers and calculate the result will not be able to work on the task without help. In the initial Mathematics Task 1, students can follow different strategies. Good students can come up with an algebraic expression using the variable n and find more complex visual structures and the respective algebraic expressions. Less proficient students might work arithmetically with big numbers and find fewer visual structures. However, in principle, there is a high chance that most learners will be able to work on the task, on their respective proficiency level.

The implementation of the two tasks differs. The textbook task (Mathematics task 2) is intended to be given to students after having been introduced to the solution strategy of inserting numbers, through a short theory in form of a text. The initial task (Mathematics task 1) is an activating problem that can be given to students at the start of a lesson, given that students have the necessary proficiency for identifying graphical structures and of constructing arithmetic expressions.

Teacher task:

Read the text above and reflect on what this means for your lesson planning.
Ask student groups to present their initial approaches. Which strategies would you choose for being presented to the whole class?

4.3 Selecting, analysing and modifying reasoning tasks for all students

Mr Janssen decides to change his classroom teaching towards being more focused on mathematical reasoning. Unfortunately, his textbook does not contain many reasoning tasks. Because he and his colleagues aligned the learning content of the school year with the textbook, his lessons must therefore remain in line with the textbook. He decides to modify some textbook tasks to improve their focus on mathematical reasoning.

It takes a lot of careful decisions to plan a lesson with a high potential for mathematical reasoning. Particularly, finding tasks with such a high potential can be demanding and time consuming. Also, it often requires extra efforts with printing and presenting these tasks, as they are often not found in the regular textbook. Accordingly, it is often a good compromise to select textbook tasks, but improve their potential for mathematical reasoning by making specific adaptations to the tasks.

Mathematics task 3:
Newspaper girl[16]

Marieke distributes newspapers in her area.
She can calculate what she earns per week in Euros for her job, using the following formula:
money she earns (in Euros) = 14+0.25 x number of subscribers

a. How much does Marieke earn per week with 40 subscribers? And with 60 subscribers?
b. Marieke states that with 80 subscribers she earns twice as much as with 40 subscribers. Investigate by calculating whether this is true.
c. Marieke gets another 20 subscribers. How much more does she earn per week?
d. At a certain moment Marieke earns €44 per week. How many subscribers are there at that moment?

Consider Mathematics task 3. The task has a similar potential for mathematical reasoning as the matchstick pattern task above (cf. Figure 4). Students reason mathematically by inserting different numbers for the number of newspaper subscribers (40, 60, 80, 20) (i.e. the number of people who have ordered the newspaper), which requires the strategy of working forward, or for the money she earns (44), which requires the strategy of working backwards. Furthermore, inserting the money she earns requires recognizing the equal sign as relation, because the given equation has to be read backwards.

The task could be much more conducive to mathematical reasoning, with specific adaptations:

- Use a table to illustrate Marieke's profit for 20, 40, 60, 80, 100 subscribers on her route. Draw a graph.
- The company offers her two alternative options: (1) a lump sum of 5¬ but 0,40¬per subscriber, or (2) a lump sum of 20¬ and 0,15¬per subscriber. Advise her which option is more profitable for her.
- Marieke wants to save for a new mobile phone. In three months, she wants to save €650. How many subscribers would she need?

These additional assignments can replace or supplement assignments **b** to **d**. They provide more opportunities for mathematical reasoning as they facilitate connecting representations, communication, and activate the relevant concept of the variable as co-varying number.

In the adaption of the task, adaptations were made with respect to:

- **Connecting representations**: The task now asks students to use a table, which is a relevant representation to understand the variable as co-varying number. In mathematics task 3, tables and graphs are a relevant means to come to a conclusion.
- **Communication**: Asking students to give advice requires argumentation. Also, as no further conditions are given, students have to choose reasonable conditions for choosing one of the alternatives.
- **Conceptual activation**: In comparing different options, students explore the co-variation between the money she earns and the number of subscribers. As such, they continue to develop their idea of the variable as a co-varying number.

Ideally, mathematical reasoning goes beyond traditional tasks in also focusing on:

1. **Relevance**: It should become clearer how the mathematics explored can help students understand real life phenomena or develop their mathematical insight of a specific topic.
2. **Exploration**: Students can actively explore different options, on different routes, instead of working on simple step-by-step procedures. This way, students can gain conceptual insights.
3. **Reflection**: Students should be encouraged to reflect on the learning content or on their own learning progress.

[16]Adapted from Dijkhuis, et al. (2016, p. 78)

We now turn to strategies to foster students' flexibility of thinking.

Strategy 1:
Letting students work backwards from specific or given results

Many tasks can be modified in such a way that students are asked to work backwards to arrive at specific results (see mathematics task 4).

This way, students reason about the nature of the task, and at the same time train their skills for doing the respective procedures.

Mathematics task 4:
Adapting tasks for working backwards[17]

Given that a=5, calculate
a. a+7
b. a–12
c. 3a
d. 5a+6
e. 5(a–3)
f. 6–3a

You can also omit 'Given that a=5, calculate' and instead ask:

1. For which a is the result 0?
2. Which of the terms becomes the biggest, when a is getting bigger?
3. Which of the terms becomes the smallest, when a becomes small?

Strategy 2:
Structuring

Structuring asks students to identify structures in a task, before or after doing the task. In the example in Mathematical task 5, students are asked to group algebraic equations according to their difficulty level with respect to finding the value of the unknown variable. One could also imagine different kinds of groupings, e.g. with respect to mathematical properties. Students could be asked to identify quadratic equations that can best be solved with a specific procedure, e.g. completing the square.

Mathematics task 5:
Adapting tasks to facilitate structuring[18]

Original task

Solve the equations.

a. 12p-3=-9p+11
b. 6s-15=75
c. 10k-5=88k+34
d. 5x-15=71+3x
e. 710+6a=630+12a
f. 1.5x+70=0.7x+150

Reasoning task

Sort the task into easy and difficult tasks.
What makes some tasks difficult?

[17] Adapted from Dijkhuis, et al. (2016, p. 81)

[18] Mathematics task 5, 6, 7 and 9 are from De Bruijn, et al. (2018)

Strategy 3:
Operative variation

By letting students work on tasks where one parameter is systematically changed, students can observe patterns. There are several ways to introduce tasks that achieve operative variation.

The example in Mathematical task 6 illustrates that by changing a specific element in a systematic way, students can explore patterns, while developing their procedural fluency through doing a specific procedure repeatedly.

Mathematics task 6:
Operative variation in the context of linear functions

Original task

For each pair of linear formulas, calculate the coordinates of the point of intersection of the corresponding graphs.

a. $y=2x+8$ and $y=6x-12$
b. $y=7-x$ and $y=x+1$
c. $y=-\frac{1}{2}x+4$ and $y=-3\frac{1}{2}x+19$

Reasoning task

1. Change the 2 in the first expression into 1, 3, 4, 5, 6, ... How will the intersection point change?
2. Change 8 in the first expression into 1, 2, 3, ... How will the intersection point change?

Strategy 4:
Generate examples

By asking students to generate examples and counterexamples with specific properties, they can engage in reverse thinking (see mathematics task 7). This way, students can reflect on what makes tasks difficult or easier. Often, such work can best be achieved in pairs or groups, so that students can brainstorm together and discuss their ideas. If students are asked to compare and categorize their work as in strategy 2: structuring, further opportunities for reasoning emerge.

Mathematics task 7:
Generating examples with specific properties for linear functions

Original task

For each pair of linear formulas, calculate the coordinates of the point of intersection of the corresponding graphs.

a. $y=2x+8$ and $y=6x-12$
b. $y=7-x$ and $y=x+1$
c. $y=-\frac{1}{2}x+4$ and $y=-3\frac{1}{2}x+19$

Reasoning task

1. Find two lines intersecting in (3,4).
2. Find two lines intersecting in (-3,4)
3. Find two lines intersecting in (3,-4)
4. Find two lines intersecting in (-3,-4)

Consider Mathematics task 8 about averages and arithmetic mean (see below). It does not directly link to a traditional textbook task, but shows that some interpretation tasks where students investigate a statement through generating and reflecting on examples/counterexamples can be invented relatively easy.

Mathematics task 8:
Investigating a statement by generating and reflecting on examples[19]

The average of four numbers is negative.

A) Can all four numbers be negative? Explain.
B) Can all four numbers be positive? Explain.
C) Can only two of the four numbers be positive? Explain.
D) Can only three of the four numbers be negative? Explain.
E) Can only one of the four numbers be negative? Explain.

Strategy 5:
Application

Application tasks are quite frequently used in textbooks. Often, they can be adapted to improve their potential for mathematical reasoning. Indeed, the strategies from above can be applied for improving application tasks. Beyond that, students can be encouraged to communicate mathematically when asked to compare two options (see reasoning task 3 or mathematics task 9) or to reflect on whether a mathematical model makes sense in the real world.

Mathematics task 9:
Application tasks and additional reasoning tasks

Original task

Lorry drivers have to pay to use the ferry. The fare depends on the length of the lorry. There is a fixed amount of -20 plus -2 per metre of the lorry's length.

a. Write down a formula for the amount a driver has to pay.
b. The fare is changed. The fixed amount increases by 10%, but the amount per metre decreases by 10%. Write down a formula for the new fare.

Reasoning task

1. What happens with the formula if the fare changes with an increase of 20% on the fixed amount, and a decrease of 20% per metre?
2. What if both change again by 30%? 40% 50% 100%?
3. Which of these options is the best for the ferry company? Why?

Ahmed and Amir are two very experienced teachers from the Netherlands who teach in grade 9 and 10. When preparing a research lesson, they learned how mathematical tasks from textbooks can be adjusted to stimulate mathematical thinking and reasoning, and how you can create a series of tasks to keep students in flow, just between bored and frustrated. They looked at the mathematical tasks 4 to 9 from the handbook together (see above). They then decided that they wanted to address the Pythagoras theorem in the research lesson, and create a new set of tasks, that build up in difficulty, and divide the students into groups to work on this set of tasks. Ahmed is both very satisfied with making his own tasks, using creativity, and the possibility of differentiation. After the research lesson, he stated: "What was very positive for me, was that almost all children were really engaged, working hard, and wanted to finish it. Almost all did finish the tasks, so this was a very positive experience for me.

[19]from Friedlander, A., & Arcavi, A. (2012), p.612

4.4 Enactment of reasoning tasks in a classroom that promotes students' mathematical reasoning

Mrs van den Heuvel has chosen a reasoning task for her next lesson. She is very happy with her choice and thinks her students will be motivated by her task. However, she is not sure whether the task matches the competences of her students. She also wonders whether all students have the necessary prior knowledge to work on the problem. To compensate, she decides to introduce the task by explaining the difficulties in the task and suggesting strategies for the students to follow. Although Mrs. van den Heuvel had good intentions, she reduced her students' ability to reason mathematically. After all, a reasoning task requires specific teaching strategies. Particularly, it requires confidence that a problem is approachable for everyone and that students will find their own way.

We have seen that developing students' mathematical reasoning in everyday classrooms is a core aspect of teaching and learning mathematics[20], and that selecting and adapting tasks is crucial. However, to promote students' mathematical reasoning while they work on reasoning tasks, teachers need to know how to question strategically (see below), and provide learning environments and classroom culture (see below) that provide students with opportunities to develop mathematical reasoning.

Strategic questioning

Strategic questioning is crucial for supporting reasoning, as it draws out the students' ideas and understandings. Then, these ideas and understandings can be negotiated in class, so that the class can collaboratively come up with solutions. Accordingly, teacher questioning plays a crucial role in promoting student understanding, construction of new knowledge, as well as the sharing of ideas[21]. In other words, **skilful questioning of your students' strategies can provide teachers with a deeper understanding of the development of students' mathematical understanding**. In turn, it helps teachers to make informed decisions for your next lessons[22].

Researchers developed a model of teacher questioning[23]. They considered three main types of questions:

- **Probing questions** consist of questions that ask students to explain their thinking, to offer justifications or proof, and use their prior knowledge in attending to the task at hand. These questions extend students' conceptual understanding and encourage them to relate new ideas to prior notions and schemas.
- **Guiding questions** guide students' problem solving by asking for solutions, strategies or procedures, and thus scaffold their understanding of a concept. Guiding questions can support students in creating their own heuristics and deriving mathematical concepts.
- **Factual questions** are requests for facts or definitions, as well as answers or next steps in a problem.

Anna and Riadh from Flanders (Belgium) teach in the third year of secondary education (grade 9). They are used to give the answer immediately when students ask a question. In this research lesson they tried to ask more strategic questions so students needed to think about their exercises. They wanted to know what the effect of strategic questioning was on the self-efficacy of their students. They used following probing, guiding and factual questions in their lessons:

- Probing questions: How did you arrive at those steps? Why did you use that way?
- Guiding questions: What do you think of the group's solution…? what do you think of the solution/drawing of…?
- Factual questions: What is tautology again? When is a number even? What does the Pythagorean theorem say?

Anna was surprised because "the students kept trying the exercise and didn't give up when they got stuck or found it too difficult".

Thinking about what types of questions may enhance/foster student reasoning, the PRIMAS EU project[24] provided five principles for effective questioning:

- Plan questions that encourage thinking and reasoning.
- Ask questions in ways that include everyone.
- Give students time to think.
- Avoid judging students' responses.
- Follow up students' responses in ways that encourage deeper thinking.

[20] e.g., Boaler (2010)
[21] Moyer & Milewicz (2002)
[22] Jacobs et al., 2010
[23] Sahin and Kulm (2008)
[24] Utrecht University (2022)

Teacher task:

When you (in Lesson Study) observes a lesson or a video clip, consider which of the five principles are used in the lesson. What constitutes questions that include everyone? How are wrong answers judged or correct answers valued?

Subsequently, and according to the kinds of questions posed (by the teacher) and answered by the student/s, it is possible to consider what students might have learnt from the lesson. In fact, students' communication reveals quite a lot about their thinking[25] and hence is a main tool for assessing how and on what level students think mathematically.

Social culture of the classroom

Teacher moves are crucial in the establishment of Mathematical Learning Communities. Mathematical reasoning and understanding naturally result from the communication that takes place in such communities[26]. Mathematical Learning Communities are described as classrooms where students learn to talk and work mathematically by participating in mathematical discussions, proposing and defending arguments, and responding to the ideas and conjectures of their peers[27]. Accordingly, whole-class mathematical discussions triggered by reasoning tasks[28] are opportunities to develop students' mathematical reasoning. The design and posing of thought-provoking tasks lead to such discussions, which in turn lead to a culture of justification and proof.

A classroom is a community of learners. The establishment of a learning community in which students build understandings of mathematics means establishing certain expectations and norms (that promote justification and reasoning) for how students interact with each other about mathematics. We note that interaction is essential, as communication is necessary for building understanding. There are four features of the social culture that fits the teaching we envisage[29]: (1) ideas and methods, expressed by any students, are valued (as they potentially contribute to everyone's learning and thus warrant respect); (2) students choose and share their methods (as there are a variety of methods which deserve to be respected); (3) mistakes are learning opportunities for everyone (as they potentially raise everyone's level of analysis); and (4) correctness resides in mathematical argument (as the persuasiveness of an explanation or justification depends on the mathematical sense it makes, not on e.g. the popularity or social status of the presenter).

[25] Sfard (2008)
[26] Yackel & Cobb (1996)
[27] Goos (2004)
[28] Ruthven (1989)
[29] Hiebert et al. (1997, p. 12)

Problematizing the sharing of students' solutions

The following task relates to 'Sharing students' solutions in whole class discussions. (You can find it in the EU Educate project[30]): Consider the following task designed and implemented in a Grade 7 classroom in Greece. The task as it appears in students' worksheet is the following:

Mathematics task 10:
Solving a problem

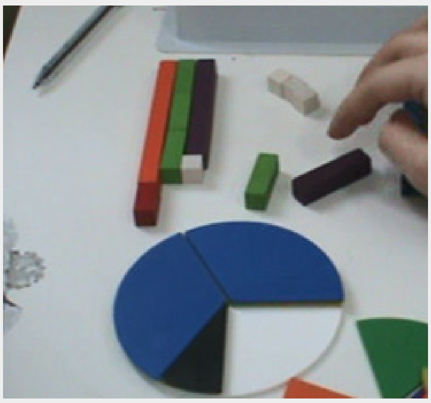
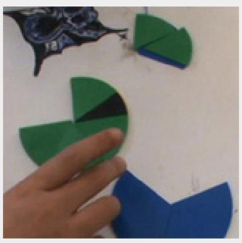
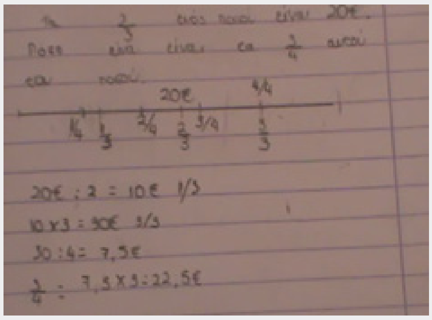
2/3 of a quantity is 20. How much is 3/4 of the quantity?

- You can solve the problem using different materials (Cuisenaire rods, fractional circular discs, chips, beads, transparent paper, paper with fraction circles, square paper, fraction bars) and corresponding representations.
- Try to find out as many approaches as possible.
- Approaches following only an arithmetic method (e.g., reduction to the fractional unit) will not be accepted. These methods can be included in addition to others.

The task was tried in a 7th Grade classroom, where students have worked in groups of three or four for about 10 minutes. Table 3 shows some of their responses.

Table 3

Students' responses to the mathematics task 10

Solution of Group 1	Solution of Group 2	Solution of Group 3
The students constructed a rod by adding an orange rod and a red rod. (=10+2) Considering this as a unit, they divided it in fourth (four green rods), (=3+3+3+3) thirds (three blue rods) (=4+4+4) and twelfths (twelve white rods). (=12*1) The white rod (the smallest of the Cuisenaire rods) is the difference between the 3 lime rods (representing 3/4) and the 2 purple rods (representing the 2/3) and this difference is 1/12. The same model is represented with the fractional discs. 	The students calculated 3/4 (yellow pieces) of the amount through the use of green, blue and black pieces. Initially, they used fractional pieces of 1/6 (green pieces) to measure 2/3 (blue pieces) by identifying that 2/3 is covered by 4/6. Then, they solved the problem by working with black rods ("if we add 1/12 to 2/3 this equals to 3/4).   	This group used an arithmetic method and the number line. 

[30]Educate (2022)

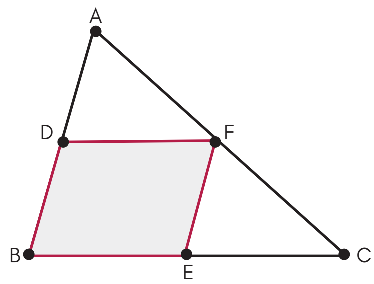
Teacher task:

We encourage you to reflect on the following issues:

- Discuss in your group about students' reasoning in each group's solution. What are differences and similarities in the students' reasoning?
- Imagine you want to do a whole class discussion to work further with the student strategies and ideas. How would you sequence these three responses in the whole class discussion? Provide reasons for your choices. Keep in mind that reasoning in classroom discussions involves proposing and defending arguments, and responding to the ideas and conjectures of peers. Justify your decision on the basis of promoting all students' understanding of the task's solution, considering the role of the different representations.
- How can you promote the formation of a learning community in the classroom above?

Mathematics task 11:
Triangle and quadrilateral

In the figure, the points D, F and E are midpoints of the sides of the triangle C. Study how the quadrilateral BDFE changes when the triangle ABC changes. That is, how the type of quadrilateral is connected to the type of the triangle.



Teacher task:

Read mathematics task 11. Below you find an extract of a classroom discussion in a Geometry course in grade 10th. After working and discussing in the whole class the figure (and how the triangle ABC might change), the students are working in groups to study how the quadrilateral BDFE changes when the triangle ABC changes. The teacher approaches a group with four boys and they describe their work:

S1: If it is a right-angled triangle, it will be either a square or a rectangle, because it has a right angle and the quadrilateral remains parallelogram.
S2: yes, that is, in any kind of triangle it will remain a rectangle
T: Fine, but there are special categories of parallelograms here. You talk about a rectangle, about a square, etc. When is it a rectangle? When is it square? The goal is to investigate and connect what kind of parallelogram is, and when it happens. [they don't talk] Can you explain to me what shape this S1?
S1: the triangle is right-angled
T: nice.
S1: BDFE is rectangular and can be either square or rectangular
T: you immediately enter into a process that says if - then. If the triangle is right-angled, then the quadrilateral is a rectangular. Keep it up. This idea. If - then. That is, to connect what is one with [the triangle] what is the other [the parallelogram].
S3: So, we will say, for example, if the triangle is scalene then this, if it is equilateral then this?
T: I do not know. Is this the point? Whatever. Yes maybe
S3: Just [tell us] what kind of assumptions to make ...
T: ... That is, how the kind [emphasizes the word "kind"] of the quadrilateral is related to the type [emphasizes the word "kind"] of the triangle. [02:03]
S3: Ah, that is to say, if the quadrilateral is rectangular, then the triangle is also ...
T: Yes, if you think so ...
S3: [laughing] come on now... sir [as if complaining that he does not get a clear answer]
T: Yes, but I do not want to tell you, I want you to think for yourself. The question is how the type of quadrilateral is related to the type of triangle. Do whatever you want.
The teacher goes away, and approaches another group.

Answer following questions:

a) How is this task related to mathematical reasoning?
b) What do you think are the main difficulties that students face in this extract?
c) Why do you think the teacher leaves the students without a final answer? What could you do different?
d) Does the teacher realize the five principles of effective questioning (Section 4.4)? Why or why not?

4.5 Noticing students' mathematical reasoning

In the context of Lesson Study, the goal is to improve your awareness of relevant classroom phenomena related to your students' mathematical reasoning and to the impact of your teaching actions. E.g., imagine yourself being in the shoes of the teacher in the episode above – would you be able to notice the mathematical argumentation in the moment, in order to ask strategic questions that builds on this argumentation. Such in-the-moment noticing and decision making is very demanding. In mathematics education research, a theoretical construct relevant to this need, teacher noticing, has been developed. It has been introduced to mathematics teacher education to study shifts in the structure of teachers' attention and, through this, to address different levels of awareness both in mathematics and in mathematics teaching[31]. Existing research highlights the importance of noticing as a construct to study what and how teachers attend to when observing, analysing and interpreting teaching[32]. Noticing has been considered as a complex action that involves teachers in identifying what is significant in a classroom interaction, interpreting this noteworthy incident on the basis of their knowledge and experiences and linking these with broader principles of teaching and learning[33].

Lesson Study provides an ideal context for teachers to learn how to observe, interpret and discuss classroom interactions with respect to students' mathematical reasoning. Noticing critical classroom events should lead to informed teaching decisions that build on and develop students' reasoning. This noticing encompasses several steps:

1. The monitoring of noteworthy (potentially critical) events,
2. The interpretation of noteworthy events with respect to the lesson goals and students' understanding, and
3. The response to the events with suitable teaching actions, such as which task to select for the next lesson or the follow-up question to ask during a classroom discussion.

The key element within these three steps is whether the substantiation of teaching actions is based on the principles of good teaching and learning[34] (e.g., effective questioning, see section 4.4). The three steps above can be supported by using didactical resources that guide the interpretation of students' reasoning or the detailed analysis of video lessons with the purpose of breaking down students' specific mathematical thinking steps.

In the following teacher task, we provide an example of a critical event and questions that can support teacher noticing in terms of the aspects that have been discussed above:

Teacher task:

In a Year 7 lesson, students are asked to solve the following problem:

"Can you make the two columns of numbers bellow add up to the same total by swapping just two numbers between the columns? Explain why or why not."

1	7
3	2
8	4
5	9

a) Solve this problem and describe possible students' solution strategies.
b) What aspects of students' mathematical reasoning does this problem aim to address?
c) Imagine that you are a teacher in a Year 7 classroom and you have posed this problem to the students. Think about possible events that can occur in relation to students' argumentation and reasoning and write a hypothetical dialogue between you and the students or the students themselves around one of the events that you consider as critical. Explain why you consider this event as critical.

[31]Mason (2002)
[32]Scherer & Steinbring (2006)
[33]e.g., van Es & Sherin (2010)
[34]van Es (2011)

4.6 Assessment of students' reasoning actions

Despite the role of reasoning in mathematics classrooms many teachers experience difficulties in assessing it[35]. This is due to the complexity involved in developing accurate judgements of students' reasoning actions as well as the need to draw on multiple sources of evidence for assessing students' reasoning skills[36]. This complexity results in teachers' difficulties in noticing the level of students' reasoning skills and hence employing suitable prompts to progress it[37].

The Assessing Mathematical Reasoning Rubric[38] was designed to evaluate and assess students' reasoning skills. This is a detailed rubric involving the three reasoning actions: Analysing (exploring the reasoning task and connecting with known facts and properties; Generalizing (identifying common properties or patterns across cases or forming conjectures); and Justifying (checking the truth of conjectures or using logical argument to convince others) at five proficiency levels (not evident, beginning, developing, consolidating, and extending). This rubric can be used as a formative assessment tool in dialogic classroom interactions as teachers seek to elicit students' reasoning through conversation and questioning.

Table 4

Rubric for assessing mathematical reasoning[39]

	Analysing	Generalising	Justifying
Not evident	• Does not notice numerical or spatial structure of examples of cases. • Attends to non-mathematical aspects of the examples or cases.	• Does not communicate a common property or rule for pattern.	• Does not justify. • Appeals to teacher or others.
Beginning	• Notices similarities across examples. • Recalls random known facts related to the examples. • Recalls and repeats patterns displayed visually or through use of materials. • Attempts to sort cases based on a common property.	• Draws attention to or attempts to communicate a common property or repeated components of a pattern using: - Body language (gesture) - Drawing - Concrete materials - Counting - Oral language (methaphors)	• Describes what they did and why it may or may not be correct. • Recognises what is correct or incorrect using materials, objects or words. • Makes judgements based on simple criteria such as known facts. • The argument may not be coherent or include all steps in the reasoning process.
Developing	• Notices a common numerical or spatial property. • Recalls and repeats patterns using numerical structure or spatial structure. • Sorts and classifies cases according to a common property. • Orders cases to show what is the same or stays the same and what is different or changes. • Describes the case or pattern by labelling the category or sequence	• Communicates a rule (conjecture) about a: • Property using words, diagrams or number sentences. • Pattern using words, diagrams to show recursion or number sentences to communicate the pattern as repeated addition. • Records other cases that fits the rule (conjecture) or extends the pattern using the rule.	• Attempts to verify by testing cases or explaining the meaning of a conjecture using one example. • Detecting and correcting errors and inconsistencies using materials, diagrams and informal written methods. • Starting statements in a logical argument are correct and accepted by the classroom.

[35]Herbert (2021)
[36]Davidson, Herbert & Bragg (2019)
[37]Llinares (2013)
[38]Australian Academy of Science (2018)
[39]Australian Academy of Science (2018).

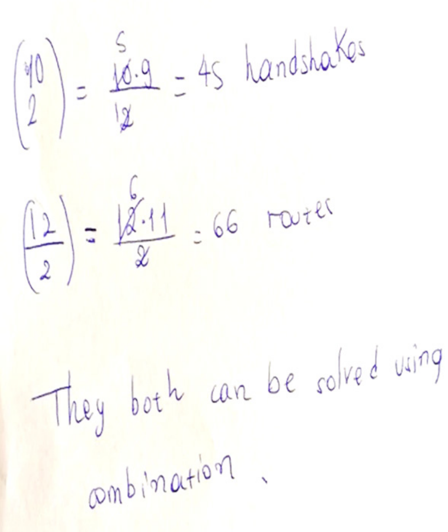
	Analysing	Generalising	Justifying
Consolidating	• Notices more than one common property by systematically generating further cases and/or listing and considering a range of known facts or properties. • Repeats and extends patterns using both the numerical and spatial structure. • Searches for and produces examples: – Using tools, technology and modelling • Makes predictions about other cases: – With the same property – Included in the pattern	• Generalises: communicates a rule (conjecture) using mathematical terms, symbols or diagrams (e.g. a number sentence or labelled geometric diagram). • Explains what the rule (conjecture) means using one example. • Extends the pattern using an example to explain how the rule works.	• Verifies truth of statements by using a common property, rule or known facts that confirms each case. May also use materials and informal methods. • Refutes a claim by using a counter example. • Uses a correct logical argument that has a complete chain of reasoning to it and uses words such as 'because', 'if … then…', 'therefore', 'and so', 'that leads to',… • Extends the generalisation using logical argument.
Extending	• Notices and explores relationships between: • Common properties • Numerical structures of patterns.	• Generalises: communicates the rule (conjecture) using mathematical symbols. • Applies the rule to find further examples or cases. • Generalises properties by forming a statement about the relationship between common properties. • Compares different symbolic expressions used to define the same pattern to show equivalence.	• Uses a watertight logical argument that is mathematically sound and leaves nothing unexplained. • Verifies that the statement is true or the generalisation holds for all cases using logical argument.

Below, we give an example of how this rubric can be used by teachers.

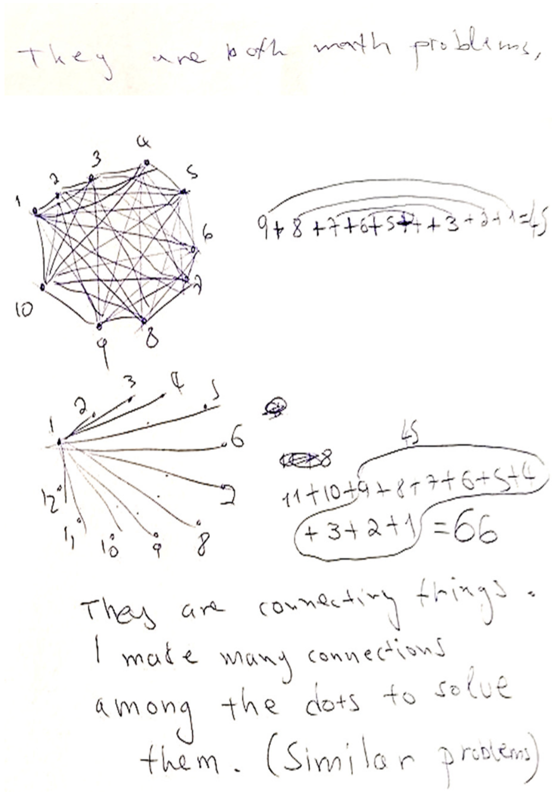
Mathematics task 12:
Handshakes and airplanes

1. Ten people attended a meeting. If everyone has shaken hands with everyone, then how many handshakes have been made?
2. A new international airline will connect 12 airports in 12 different countries. What are all the possible routes that the airline will need to create?

Student 1's solution



Student 2's solution



Teacher task:

a) Solve the problems mentioned in Mathematics task 12.
b) Above, two students' solutions to the Mathematics task 13 are shown. Do you see similarities between their solutions? Write these similarities down.

In what follows, we assess the mathematical reasoning of student 1 and 2. Student 1 notices that the problem is related to combinatorics. He uses a mathematical sound argument. He uses symbolic expressions to define the same pattern in the two problems. He can apply the rule to similar problems. Student's 1 reasoning actions are at the extended level.

Extending	• Notices and explores relationships between: • Common properties • Numerical structures of patterns.	• Generalises: communicates the rule (conjecture) using mathematical symbols. • **Applies the rule to find further examples or cases.** • **Generalises properties by forming a statement about the relationship between common properties.** • **Compares different symbolic expressions used to define the same pattern to show equivalence.**	• **Uses a watertight logical argument that is mathematically sound and leaves nothing unexplained.** • **Verifies that the statement is true or the generalisation holds for all cases using logical argument.**

Student 2 starts developing a form of reasoning in terms of his generalizing actions. He uses numerical and spatial patterns of reasoning. He understands the rule and he can extend it to similar cases. He explains his rule spatially by connecting dots. Student's 2 reasoning actions are at the developing level.

Developing	• Notices a common numerical or spatial property. • Recalls and repeats patterns using numerical structure or spatial structure. • Sorts and classifies cases according to a common property. • Orders cases to show what is the same or stays the same and what is different or changes. • Describes the case or pattern by labelling the category or sequence.	• **Communicates a rule (conjecture) about a:** • **Property using words, diagrams or number sentences.** • **Pattern using words, diagrams to show recursion or number sentences to communicate the pattern as repeated addition.** • **Records other cases that fits the rule (conjecture) or extends the pattern using the rule.**	• Attempts to verify by testing cases or explaining the meaning of a conjecture using one example. • Detecting and correcting errors and inconsistencies using materials, diagrams and informal written methods. • Starting statements in a logical argument are correct and accepted by the classroom.

5. Concluding remarks

In part I of the Lesson Study handbook, we have aimed to support you (as the teacher) to develop knowledge about how your lesson preparations and actions in class may enhance your students' mathematical reasoning. In Part II, the Lesson Study project on student mathematical reasoning is explained, that is how you can work with your colleagues on enhancing students' mathematical reasoning in your classrooms.

Research Lesson Study

02

1. Introduction

In the second part of the manual, we give insight in Research Lesson Study. The first section provides an introduction into the method of Research Lesson Study. Afterwards, the different steps of Research Lesson Study are explained, including practical tips and examples. Throughout the text, there are **references to sheets** that need to be used in specific stages. The empty forms can be found at the end of this manual.

1.1 What is Research Lesson Study?

Research Lesson Study (RLS) is a highly specified form of collaborative classroom action research focusing on the development of learning, teaching and curriculum[40]. Research Lesson Study is a method in which teachers work in groups, as teacher researchers, on their own classroom practice. They put usual practice under review, they search-out and consider alternatives or innovations, conducting studies of their students' learning, which in turn results in adjusting and enhancing their new approaches. They form tight-knit teacher learning communities and perform collaborative inquiries as they do. Taking part in collaborative enquiries in view of improving teaching and learning is the single most impactful action a teacher can take to improve educational outcomes for students[41].

A Research Lesson Study has the following core features.

- The focus is on **students' educational needs**. These needs are often the starting point and the actions, efforts and learning of the students are observed during the implementation of a research lesson.
- In a Research Lesson Study group, teachers take on the role of curriculum and practice researchers. Members of a Research Lesson Study group analyse their usual curriculum and students' usual outcomes, they research alternative approaches and they formulate a research question concerning how their curriculum and/or teaching practices can be improved and how that can contribute to the learning of their students and their own professional knowledge. The research question guides the activities in the different phases of a Research Lesson Study cycle.
- A Research Lesson Study is a process with different activities and the collaboration between teachers shapes the learning process. Instead of designing the ideal lesson, it is about improving practice by planning informed changes to practice, studying how this affects students' learning and ultimately implementing lasting changes and sharing knowledge gained with others.
- In a Research Lesson Study, teachers make a series of autonomous decisions through the process of designing a Research Lesson Study (choosing a research focus, selecting the case students, what data to collect, etc.). This ensures that teachers are agents of their own initiated actions and learning.

[40]Lewis (2009)
[41]Robinson et al (2009)

1.2 Why conduct a Research Lesson Study?

Lesson Study was first developed in Japan in the 1870s and 1880s. It was only after the publication of "The Teaching Gap" in 1999[42] that the method was also applied in the West. This publication attributed the excellent performance of Japanese students (compared to American and German students) on mathematics and science to the Lesson Study practice. Since then, the practice of conducting lesson studies to improve teaching and learning has spread worldwide. Much empirical research on Research Lesson Study has been done, both in Asia and in America and Europe. Some relevant review studies, which bring together insights from previous research[43], point to positive results, both for teacher and student learning. For example, research has shown that Research Lesson Study contributes to teachers' subject knowledge, skills, teaching styles and beliefs[44]. In addition, research shows that Research Lesson Study makes teachers more sensitive to and more focused on students' educational needs, which leads to greater learning gains for students[45]. It has also been repeatedly found that Research Lesson Study contributes to positive collaboration between teachers, promotes teacher learning and the development of a professional learning community[46].

Because teachers tend to practise as lone professionals within their classes, teachers seldom get an opportunity to see others' tacit practice knowledge manifested in action. When teachers' practice is observed by another professional, it is more likely to be in the context of a judgement of performance than in a context of professional learning. In such contexts, teachers tend to play safe rather than take risks. Research Lesson Study, however, creates safe spaces where teachers can take risks together and fearlessly work on areas of the curriculum in which they feel less confident or secure. The Lesson Study Group Learning Protocol Sheet (page 54) makes it safe to learn together that members of Research Lesson Study groups swiftly begin to solve teaching problems together using 'exploratory talk' and 'meaning-oriented teacher learning'. Exploratory talk is a specific type of collaborative discussion that allows teachers to think together aloud, so that others can hear, explore and build-on partly-formed ideas[47]. With their participation in Lesson Study, teachers also learn how to develop "meaning-oriented" learning. This type of learning prompts teachers not only to learn "what works", but "why and how things work" too. When teachers adopt this way of learning, they begin to compare different students' work, think about how different lessons relate to each other, monitor students' progress, experiment with new ways of teaching, try to understand how and why students learn, and reflect on their own teaching practices. This is a high-quality, deep mode of teacher learning. Features of Research Lesson Study that may explain its impact on this form of teacher learning may include: the strong collaborative focus on understanding students' learning; on seeking explanations for students' misunderstandings; the high degree of ownership that teachers feel they have over their own learning; and a simultaneous focus on subject knowledge, teaching, and students' learning. Meaning-oriented teacher learning is a powerful form of teacher learning that enables teachers to improve their students' learning in greater abundance than other forms of professional learning studied[48].

Research Lesson Study helps teachers to:

- Closely observe students' learning.
- Investigate the difference between what they expect to happen when students learn and what actually happens.
- Understand how to design and bring about learning much more closely to student needs.
- Take risks within a supportive teacher-learning community committed to providing a safe space because each teacher values and feels valued by their Research Lesson Study group.
- Research, reflect, analyse and learn collaboratively how to help their students to learn and achieve.
- Change subsequent teaching and curriculum in order to better support learning
- Share their experiences by involving expert practitioners and less experienced peers in order to maximise and mobilise the new knowledge.

1.3 Why select specific case students?

An important element of Research Lesson Study concerns the identification of around three case students. These students may be representatives of different learner groups in the class. If the aim of the Research Lesson Study lesson is to develop a new approach (e.g., to introduce reasoning with algebraic patterns to Grade 8 students), then the case students may represent (i) students that the teachers expect to find this easy, (ii) students that the teachers believe may require additional support and further teaching, and (iii) students who fall between groups (i) and (ii). If, however, the Research Lesson Study is focusing on a particular learner group, for instance, with specific educational needs, then the case students will be chosen from that group. The Research Lesson Study group will design the research lesson with these particular students in mind. Each time they reach a point in the lesson plan where students are expected to complete a task before moving on to the next stage of the lesson, teachers must agree and record on their plan what they expect each case student to say, write, draw or otherwise; this signifies to the teachers that the case student is ready to move on to the next phase of the lesson[49].

[42]Stigler & Hiebert (1999)
[43]Cheung & Wong (2014); Seleznyov (2019); Willems & Van den Bossche (2019); Xu & Pedder (2015),
[44]E.g. Lawrence & Chong (2010).
[45]The study by Ylonen & Norwich (2013), for example, demonstrates this.
[46]Stigler & Hiebert (2016); Vermunt et al. (2019)
[47]Mercer & Dawes (2008).); Barnes (2008)
[48]Vermunt, et al. (2019); Dudley, et al. (2019)
[49]Dudley, (2019b)

1.4 How to organize a Research Lesson Study?

Research Lesson Study generally consists of three consecutive cycles of collaborative classroom research, with each cycle consisting of a research lesson (see Figure 1). During a Research Lesson Study cycle a group of teachers (usually three or even a pair):

- Use data they have gathered from day to day and periodic assessment to agree a focus for the student learning and progress
- Jointly study and critique curriculum materials they are using and research beyond these to refine the focus and identify approaches to address that need (often with expert input) from which a research question is created
- Identify around three 'case students'. Each could typify a group of learners in the class with respect to the focus chosen.
- Jointly plan usually three 'research lessons' (RLs) in which they develop an educational practice and closely study the effects of this new approach while keeping in mind the three case students. The second and third research lessons are only planned in detail following the discussion and analysis of the preceding research lesson – often in the same meeting.
- Teach and jointly observe the research lesson focusing on the case students' learning and progress, and adjust and refine teaching over several lessons on the basis of their analysis of each.
- Interview the case students to gain their insights into the research lesson.
- Administer a short questionnaire from to all students (optional)
- Hold a post-research lesson discussion after each research lesson analysing how the case students responded, what progress they made, what evidence of learning or of difficulties with learning they displayed and what can be learned about the way the teaching or learning approach is further developed next time.
- After the final research lesson additionally agree and discuss what knowledge has been gained over the whole study and how subsequent teaching or curriculum will change as a result.
- Formally share the outcomes with a wider audience of other teachers – in a presentation, by demonstration or by coaching.

Figure 1 sets out the Research Lesson Study process. The first stage in the circle is the initial study phase in which the Research Lesson Study group establish the focus for the inquiry and determine the research goal and research questions. Each of the horizontal blocks is a research lesson cycle of the Research Lesson Study. Research Lesson Study does not cease after only one research lesson. What happens in the first research lesson is often that teachers come to see gaps between the students and the object of learning much more clearly; this is due to teachers' expectations of the case students and checking if these expectations were met. This enables the Research Lesson Study group to design a second lesson to make a better match between teaching approach and student needs. In research lesson 3 you address any remaining issues and also tweak and hone successful aspects of researchlesson 2 to test them further. By this stage the vast majority of Research Lesson Study groups have developed something that will be incorporated into their future teaching, their schemes of learning that can be shared with others.

Research lesson two and three are not adjusted repeats of research lesson one. They are different lessons in a wider sequence. Generally, in a Research Lesson Study teachers will use the same class to develop an aspect of learning (such as a concept, skill or point of understanding) that the class needs to be taught. Research lesson two and three therefore pick up and develop this aspect each time. For example, you may have introduced a skill/concept in research lesson one and will develop and/or deepen it in research lessons two and three. It is best to do this in a period of three weeks or so.

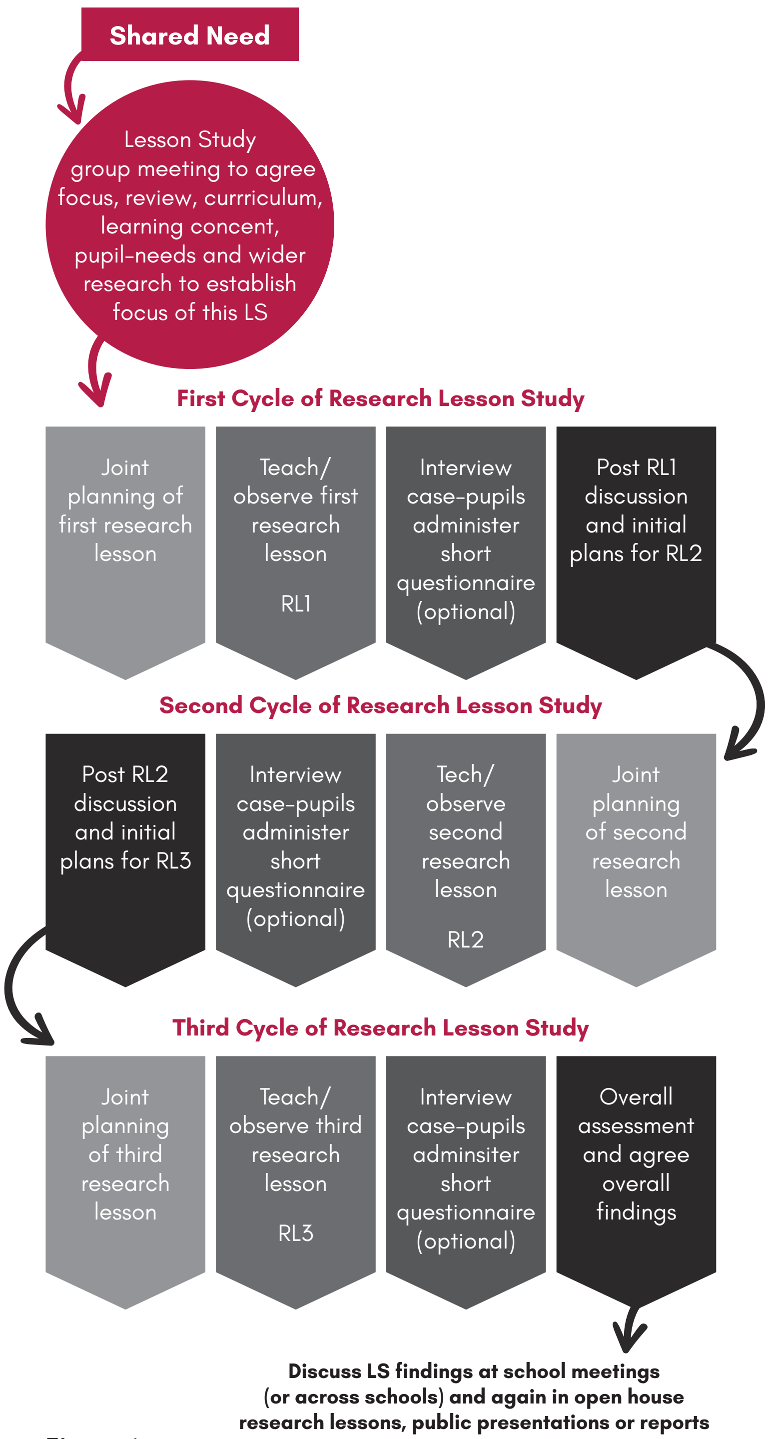


Figure 1
The Research Lesson Study Process[55]

[55]Based on Dudley, 2019a

1.5 Conditions to start with Research Lesson Study

Before starting with Research Lesson Study, it is advisable to check whether a number of conditions at school level can be met. If several of these conditions are supportive, the chance of success is greater.

- **Support from the school leader**
  The chance of success increases when Research Lesson Study is embedded in the school system. The school leader can facilitate Research Lesson Study by creating time and space for a Research Lesson Study group to carefully review the materials and approaches currently in use, examine students' work and identify or jointly develop new approaches. Also, it is motivating for a Research Lesson Study group when the school leader emphasises the importance of Research Lesson Study for the whole school, encourages the teachers to experiment, keeps track of the process and also gives the Research Lesson Study group members time and space to share their findings with their colleagues.

- **Available time for Research Lesson Study**
  Teachers themselves need to invest time in Research Lesson Study. A Research Lesson Study focuses on the learning process and derives its learning benefits from adjusting that process, which takes time. It is also important that teachers are given structural time to discuss and observe each other in the classroom. It is usually the school leader who has to put together the puzzle of the teachers' timetables.

- **Openness and safety**
  Teachers participating in a Research Lesson Study are expected to have an open attitude: they are open to improving classroom practice and are willing to open the classroom doors to colleagues. Putting together a Research Lesson Study group in which one person presents himself as the 'expert' usually does not work. A safe (learning) climate is necessary. It is important that everyone respects each other and that all contributions are valued. It is key that teachers are able to share ideas, concerns, challenges and 'wonderings' without fear of criticism. They should act as 'critical friends' to each other. This way of talking and discussing leads to more productive conversations. Such protocols are important 'ground rules for talk'[50].

- **Realistic goal setting in Research Lesson Study**
  A Research Lesson Study always starts from a particular need that teachers experience which leads to a research goal. This may point to a problem in the classroom (e.g., "My students have trouble with recognizing a pattern in math class.") or the need to innovate in education (e.g., "a new mathematics curriculum is coming, how should we deal with it in our educational practices?"). It is important that the need is discussed with colleagues and the school leader to guarantee its relevance and feasibility. The need then leads to the content focus of the Research Lesson Study (e.g. reasoning and proof, see part I). During a Research Lesson Study, the content focus ensures clear direction and shared goals within a Research Lesson Study group. It is up to the Research Lesson Study group to gather information about the chosen theme. This can be done through literature and online videos, but they can also opt to get extra input from an expert or knowledgeable other (e.g. attending a lecture, following a workshop or inviting an expert or an experienced teacher in their school).

- **Guidance**
  It is recommended that a Research Lesson Study group that is new to this method, receives a certain form of guidance. In LESSAM, we provide you with the guidance that is required for lesson study to be successful.

[50] Edwards and Mercer (1987)

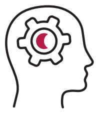
2. Starting with Research Lesson Study

2.1 A shared need for Research Lesson Study

A Research Lesson Study always starts from a shared need that teachers experience. In this project we will focus on mathematical reasoning. In order to define this, you need to consider what you want to change and for whom. The what-question is in the broad area of learning content, didactic measures, learning or educational practices, evaluation, use of educational technology or specific forms of support. In addition, it is important to find out to whom this need applies. Is it about the entire class or a particular group of students? For example, high-ability students or students who are disengaged or not learning or achieving as well as you would expect.

Also consider why you want to change something. By sharpening the focus on the "what", "for whom" and "why", the targeted area for conducting a lesson study will become clearer. The more specific you describe it, the easier it is to transform this shared need into a research goal and research question.

It is very important to exchange ideas with other school team members about experienced "needs" in classrooms. This way, you get a better idea of what is feasible to tackle together in your classroom context.



Teacher task:

To what extent is mathematical reasoning important for students?
Explain and discuss with your colleagues. You can use chapter 1 as an inspirational guide.

Try to specify why you want to optimise mathematical reasoning in your classrooms.
Following questions can be helpful for the discussion:

- What do you want to change or investigate concerning mathematical reasoning?
- For whom do you want to change this?
- Why is this important or interesting?

2.2 Forming a Research Lesson Study group

When starting with Research Lesson Study, it is advisable to start with a small group of two to three teachers[51 52].
A Research Lesson Study group can consist of teachers working in parallel classes, but it is also possible to work across years. It is recommended that a Research Lesson Study group consists of teachers from the same or a related subject area, in order to design lessons together.

The next step is to think about how to work together and related expectations. Formulate your expectations about the collaboration as a group. The following questions can help: What do you expect from the collaboration? When is the collaboration a success for you? What obstacles can you encounter in the collaboration and how will you deal with them? How will you react in case of differences in opinions? It can be useful to draw up a **Lesson Study Group Learning Protocol Sheet** (see also example below).

[51]When two teachers collaborate, the overall management of the lesson study can be easierbut with three teachers, the quality and quantity of learning will increase. This is because when three people are involved, teachers can 'interthink' through exploratory talk. There are more opportunities within the discussion for reflection and development of ideas – because it is not always your turn to speak next – someone can grasp a glimmer of an idea which has flashed past the back of their mind while the other two talk and develop it before introducing it to the others – which it is much harder to do with only two interlocutors.

[52]Bodvin et al. (2020)

An exemplary group learning protocol for Research Lesson Study[53]

This protocol was developed to help create common expectations amongst the Research Lesson Study group members. Adopting such a protocol helps the group to form a good working relationship that helps members to share ideas, concerns, challenges and 'wonderings' without fear of criticism. It forms an important set of ground rules for talk which aid the use of 'exploratory talk' by the group which in turn aids 'meaning-oriented teacher learning'[54] and the discovery and sharing of new practice-knowledge.

At all stages in this Research Lesson Study we will act according to the following:

- We are equal as learners regardless of age, experience, expertise or seniority in school (or beyond).
- All contributions to the dialogue are treated with unconditional positive regard. This does not mean they will not be subject to analysis, doubt or challenge, it means that no one will be made to feel foolish for venturing a suggestion. It is often suggestions that make you feel foolish or vulnerable that are of the greatest value and generate the most learning.
- We will support whoever teaches the research lesson(s) and make accurate faithful observations, recording as much as possible what students say as well as do.
- We will use common tools for Research Lesson Study – planners, student interview prompts and approaches to sharing outcomes with each other.
- We will use students' work and interview comments to inform the post lesson discussion alongside our observations.
- We will use the **Post Research Lesson Discussion sheet**, starting by discussing what each case student did compared with what we expected and let the discussion flow from there.
- We will listen to each other and to ourselves when we speak and build on the discussion, making suggestions, raising hypotheses, elaborating, and at all times being accountable to our lesson aims, our case students and our observation and other research lesson data.
- We will share what we learn – our new practice knowledge – with our colleagues as accurately and vividly as we can and in such a way that they can benefit from and try it out themselves.
- We will share the aims and outcomes of our Research Lesson Study with our students appropriately, depending on their ages and stages of development. Their views, ideas and perspectives will be treated with equal positive regard.

Signed and dated by Research Lesson Study group members.

[53] Dudley (2019a)
[54] Vermunt et al. 2019

2.3 Practical organization

Research Lesson Study requires time and space to be successful. It is best to consider the practical organisation of Research Lesson Study in advance. Agree when and in which room the group will meet, who will lead the discussion and who will take the necessary notes (these roles can rotate per meeting), how long the meeting will last, etc. Agreements may vary from meeting to meeting, but making agreements about this in advance can prevent important misunderstandings. You need dedicated time (an hour at least) to plan the first research lesson. Also, make sure there is time for a post lesson discussion immediately or soon after carrying out the research lesson, long enough to make a start on planning the next research lesson. Use the **Research Lesson Study Overview Table Sheet**.

It is important to decide when you will plan, teach and reflect on the research lessons together. Below you can find a good example of an annual planning for two lesson studies:

September-October: getting to know Lesson Study (read manual, if possible: follow workshops)

October-November: formulate research question and work on research lesson plan

November-December: first Lesson Study (3 research lessons + reflections)

January-February: overall assessment, exchange ideas with other teacher teams

February-March: formulate second research question and work on research lesson plan

March-April: second Lesson Study (3 research lessons + reflections)

April-May: overall assessment, disseminate your results

Example of a day planning:

	Teacher A	Teacher B	Teacher C*
Monday: teaching hour 2	Teaching research lesson 1	Observing research lesson 1	Observing research lesson 1
Monday: break		Interview case students	
Monday: teaching hour 4	Observing research lesson 1	Teaching research lesson 1	Observing research lesson 1
Monday: lunchbreak			Interview case students
Monday: teaching hour 5	Post research lesson discussion	Post research lesson discussion	Post research lesson discussion

* If the Research Lesson Study group consists of three teachers. If the Research Lesson Study group consists of two teachers, teacher A interviews the case students during the Monday lunch break in this example.

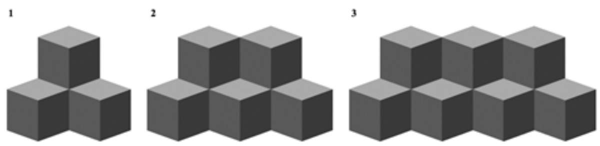
3. Defining a research goal and identifying the case students

3.1 Formulating a research goal and identifying the case students

Defining a research goal and selecting case students go hand in hand. Identify three students who might (a) represent different groups of learners in the class – for instance students who are making good, average or below average progress in the curricular area of focus of the Research Lesson Study or in a cross curricular skill such as academic writing, or (b) students who are not learning or engaging as well you would have hoped in the curricular area of focus. Think about the needs of these case students related to your teaching practice and decide what the overall aim and focus of the research lessons will be. Write this down in the **Research Lesson Study Overview Table Sheet**. Consider one extra case student in case one student is absent on the day of the research lesson.

Example of a research goal for mathematical reasoning:

We want students to be able to generalize a figural pattern into a symbolic expression, using variables. For example[55]:



a. How many cubes do you need for the fourth, fifth, sixth pattern?

b. Find the number of cubes for the 18., 64., and 911. pattern.

c. Explain how you can find the number of cubes for an arbitrary pattern in the sequence.

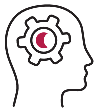
We are specifically interested in students who struggle with crafting/designing arithmetic expressions (and could be supported with a tabular representation). Or students who are generally focused on doing procedures without meaning, and are hard to motivate to think mathematically.

3.2 Formulating a research question

A research question starts from the case students' needs or what the Research Lesson Study group experiences while teaching the case students. In order to get a better picture of these needs, you can gain more information from other resources. Some suggestions:

- Analyse the school results of the case students for mathematics. Are there certain domains in which a student scores lower? Does this concern or relate to knowledge, skills or attitudes?
- Talk to previous year's teachers: how did they experience the case students in the classroom?
- Schedule conversations with the case students and ask what is going well and what is not going well in mathematics. Find out what the case students currently think of the lessons and what they would like to change.

[55]Pattern Sequence task (based on Weigand, Schüler-Meyer & Pinkernell, 2021).



Teacher task:

Try to specify the needs of your case students on mathematical reasoning.
E.g. What are the difficulties students have when asked to reason about algebraic patterns?

Start with one task and analyse the kind of reasoning you expect from your (case) students.
Try to explicitly notice how your (case) students reason in this task. Compare that with your planning.

It is possible that several student needs arise simultaneously. In that case, the Research Lesson Study group members will have to downsize the possible needs to a specific need on which they want to focus in the Research Lesson Study. Agree on the level each student is operating at in the focus area of the research lesson – for instance students who perform below expectations, at expectations, or exceeding expectations regarding a specific subject or content area.

Figure 2 illustrates how teachers can bring together the insights one has about the curriculum, materials and teaching, analysis of students' learning and needs as well as any forms of approach that sources suggest may be fruitful in improving aspects of these. This is the basis of your research goal and your research question. Think about which student learning objectives you have for each research lesson and write them down in the **Research Lesson Study Overview Table Sheet**.

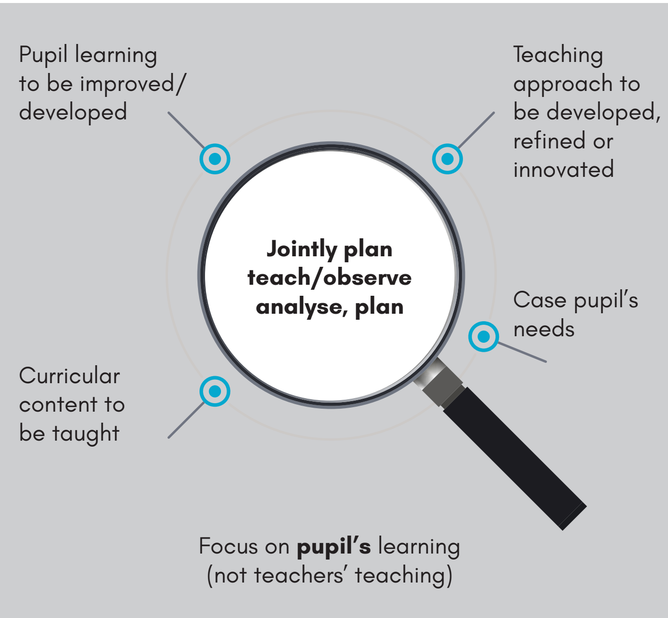


Figure 2
Bringing your analysis of your curriculum and your students' learning and needs together to form your research goal

You can choose between two kinds of research questions: exploratory and explanatory research questions.

Research questions that are **exploratory** are used to generate certain hypotheses e.g. when you think students will benefit from working together: "Are high-ability students learning more when working in homogeneous groups?". Also, exploratory research questions can be used to observe without strong assumptions or pre-defined ideas such as, 'Why does homogeneous grouping have an impact on the motivation of low achieving students?"

Explanatory research questions try to explain why particular phenomena work in the way that they are expected to. This can be based on theory driven insights, but also based on expertise build up from years of teaching experience. In this case, it can be interesting to further **downsize a broader question**, to make it feasible. One way to do so is by using the XYZ scheme (see figure 3 below).

The Z stands for the outcome you want to achieve with the case students, the X for the educational practice you want to use to achieve the outcome and the Y for moderating factors that may play a role in that, such as previous knowledge, proficiency with algebraic symbolism, preference for procedural reasoning. These are factors that make educational practice X work in one situation but not in another. If possible, try to distinguish Y factors that you as a group can control or take into account. It is important when designing educational practice X aiming for desired outcomes Z , that you take into account the Y factors that may influence this relationship. When thinking about the possible influence of the Y factors, be sure to check that this influence does not completely prevent the achievement of Z (the outcome). Thus, it is important when writing a research question to keep realistic goals in mind in the given classroom context and for the selected case students.

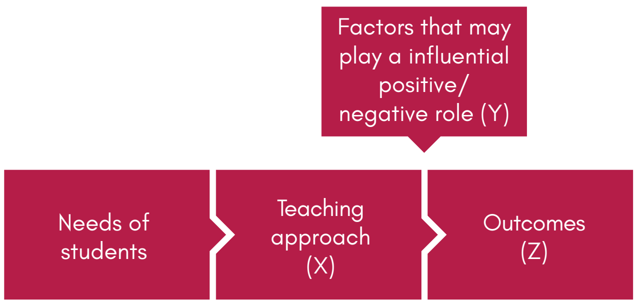


Figure 3
XYZ-scheme to formulate a research question[56]

[56]Bodvin et al., 2020

Check that all terms or concepts used in the XYZ question are clear. For each term or concept, think about how this can be expressed in observable student behaviour and/or how you can ask a student about it. This thinking exercise helps to formulate the explanatory research question in a sufficiently concrete way. In addition, the concepts used will come in hand during the preparation of the observation and the interview (see also sections 5 and 6). This may lead to the reformulation of one or more terms in the research question. For example, 'continued commitment after completing the exercises' is more concrete (as a formulation of the outcome Z) than the broader concept 'motivation'.

Thus, when thinking about the broad question: 'How can we stimulate case students more during lessons on mathematical reasoning?", this can be delineated as 'To examine what effect educational practice X (strategic questioning) has on the outcome Z (continued commitment after completing the exercises on evaluating the relevance of a mathematical model in a realistic situation) with the case students and to check whether any other factors Y (e.g. usefulness of the subject matter in daily life) play a role in this. In general, the more concrete the research question is, the more precise the observations, the interviews, the short questionnaires (optional) and the evaluation will be.

Examples of research questions on mathematical reasoning:

- How can I teach algebra for understanding (e.g., connecting representations)? (exploratory research question)
- How can a focus on mathematical reasoning be more motivating for students? (exploratory research question)
- How can I address students' different learning needs during the teaching of mathematical reasoning in an effective way? (explanatory research question)
- To what extent are e.g., self-differentiating tasks, scaffolding; principles of equitable teaching effective ways, to address students' different learning needs during the teaching of mathematical reasoning? (explanatory research question)
- Do cognitively demanding tasks facilitate mathematical reasoning? (explanatory research question)

Write down your research question in the **Research Lesson Study Overview Table Sheet**.

4. Planning the first research lesson

4.1 What is a research lesson?

An important goal of a research lesson is to investigate whether an educational practice can better lead to a desired outcome for case students. The goal of the research lesson is usually to change or improve an existing educational practice, but it can also aim, in an exploratory phase, at carefully mapping out how an existing educational practice produces the desired outcome and how this is experienced by the case students. The aim is therefore not to design 'the perfect' lesson or to integrate as much lesson content or educational practices as possible into a research lesson, but to design a lesson in such a way that the pre-defined outcomes are achieved with the case students.

Decide if the research lesson is taught by one or more teachers of the Research Lesson Study group. If more than one teacher is teaching the research lesson, remember that you have to select three case-students per classroom. Several options are possible:

- **Option 1**: every teacher teaches the same research lesson; in each research lesson at least one other group member is present to observe and interview the case students;
- **Option 2**: only one teacher teaches the research lesson, the other group member(s) observe the research lesson;
- **Option 3**: each Research Lesson Study is dedicated to a different teacher/class. In other words, the first Research Lesson Study concerns one class, the second Research Lesson Study concerns another class and the third Research Lesson Study concerns the third class.

4.2 Practical and content-related preparation of the research lesson

Review and modify your teaching tasks carefully as you plan. Write out in full exactly what you want each student to be able to do with their new knowledge by the end of the research lesson. You can use the **Research Lesson Planning and Observation Sheet**. Plan each stage of the lesson with particular attention to the sequence where you use the specific approach you are trialling or refining. Agree and note down on your research lesson planner exactly what you expect each case student will have written, said, drawn etc. at this point that suggests s/he is ready to progress to the next stage of the lesson. Identify as carefully as you can: what resources will be used and how, what you will write on the board, where and when and indicate timings for the lesson stages.

When preparing your lesson, consider the differences between your case students (e.g. students who are for instance, making good, average or below average progress with regard to the desired outcome). It is really important that the Research Lesson Study group clearly writes what they expect from the targeted students in the lesson ahead, and how they will collect evidence on this.

In Part I of this handbook, an inspirational guide on teaching and learning mathematical reasoning is provided. The different sections of Part I (including the examples provided) are particularly useful for planning your research lesson.

4.3 Preparation of the observation of the case students

After planning the research lesson and discussion of what to expect from each case student, it is important to prepare how students will be observed. These observations check what is actually happening in the classroom. Observing case students to reveal how they work in class is one of the most important and revealing aspects of Research Lesson Study.

It helps to have some rules to ensure you don't all gather data about two students and miss the third. Observe the reserve case student in case one is absent on the day of the research lesson.

There are various ways to organize an observation, ranging from an open observation (e.g. listening, watching and making notes) to a closed observation (e.g. using a checklist) or a combination of open and closed observation

When opting for an open observation, observe students' behaviour and listen to the talk of the students, note what they've written or said or drawn compared with what was expected from the group. Make a photo of students writings. Listening to what they say to each other can indicate what and how they are thinking and conceiving.

A pitfall is to make open observations with a vague link to the research question. A (partly) closed observation can be helpful. Think about which behavioural indicators can be useful to observe related to the research question. Think for instance about relevant behaviour to be observed of the case students, e.g. if the research question is about 'continued engagement when completing the mathematical exercises', then think about how that can be observed or what kind of evidence is needed to determine whether the case students are indeed engaged when completing exercises. What observable behaviour do you expect and how does that behaviour differ from the behaviour of a student who does not engage? Conducting observations in the classroom enables to check what is actually happening. Use the **Research Lesson Planning and Observation Sheet** to write down which student behaviour is relevant to observe.

You can find an example of behavioural indicators below (Table 1 and 2) that can be useful when observing case students[57]. Obviously, many other behavioural indicators are conceivable. Always try to make a well-considered choice in order to arrive at a limited and feasible number of behavioural indicators to be observed. The more specifically these indicators are formulated, the more useful the observation will be. Avoid indicators that are overlapping, aim to distinguish mutually exclusive indicators.

[57]Based upon de Vries et al. (2016) en Kooiker (2011)

Table 1
Behavioural indicators on mathematical reasoning

Generalizing from specific cases (inductive reasoning)	The student generalizes a pattern from a few first numbers. The student finds the next term.
Evaluating mathematical claims	The student generates multiple examples and checks whether they hold/counter the claim. The student identifies a counter-example that refutes the claim. The student provides a generic example or a valid argument to support the claim.
Developing conclusions through deductive reasoning	The student uses (or refers to) a theorem or rule on a specific case.
Reasoning by analogy (transfer of structural information from one system to another)	The student identifies sufficiently similar situations (with respect to a relevant criterion). The student refers to properties of similar situations.
Reasoning with images	The student moves parts of geometrical shapes in the process of justifying. The student finds common elements in different shapes (common base, common height) in the process of justifying.
Evaluating the relevance of a mathematical model in a realistic situation	The student validates the model (if it represents the situation sufficiently). For instance, they notice that a linear model does not fit the real-world growth of a bacterial population/interest of shares (exponential growth) by looking at the graphs, also by using tools such as CAS.
Making links among different representations (visual, symbolic, verbal, contextual, physical)	The student uses terms from the "other" representation (e.g. talking about a symbolic representation she uses the term "point"). The student answers questions formulated in the context of one representation using terms or information from another representation. The student uses additional representations not given in the task (e.g. table to represent linear graphs)
Making predictions in stochastic situations	The student gives an argument on the misleading use of a graph. The student distinguishes the theoretical probability from the experimental probability and from the frequency in a specific case.

Table 2
Indicators of cognitive demand

Critical thinking	• The student generalizes based on examples • The student generalizes based on identified patterns • The student argues why …a certain representation fits, there is an error, why a model fits a situation, … • The student finds analogies • The student invents their own tasks or problems, also based on some given criteria • The student uses heuristics for solving a problem • The student devises a problem solution strategy • The student gives reasons for choices (e.g. for a solution strategy, a theorem)
Using representations to make meaning	• The student uses two or more representations to investigate a problem • The student translates a graphical representation or a table into a symbolic representation (an equation, a term) • The student chooses a suitable representation • The student sees how changes in one representation affects another representation (e.g. how changing "slope" affects a graph and a symbolic expression of a function) • The student manipulates a representation in a systematic way to gain insights into a piece of mathematics (e.g. parameters of a quadratic function)
Conceptual tasks that integrate procedures	• The student makes use of multiple representations (graphs, tables, formulas) to arrive at a result • The student engages with the concepts that underlie a procedure • The student invents a procedure with informal means (e.g. manipulatives, models) • The student evaluates the applicability of a procedure for a certain problem • The student finds a procedure for a certain problem • The student interprets the meaning of a procedure • The student adapts and applies a procedure to a new, unfamiliar problem
Use mathematics to make sense of the real world	• The student evaluates the suitability of a mathematical model • The student devises a mathematical model for a real-world situation • The student develops a mathematical concept ("begrip") in the real world • The student uses a familiar procedure to solve a problem in the real world • The student recognizes that the same real-world situation can be represented with different models • The student validates a model in the real world

For more indicators on mathematical reasoning, see also Part I, section 4.6 (Assessing Mathematical Reasoning).
In section 4.4 there is another example on Problematizing the sharing of students' solutions.

4.4 Preparation of the interview with the case students

While preparing the research lesson, think about how you will organize the (short) interview after the research lesson. Also, think about interesting questions for the interview. As you get more experienced in Research Lesson Study you will want to tailor these to fit the research lesson, the students and the focus of the Research Lesson Study you are conducting. Use the **Post Research Lesson Interview Sheet** to write down relevant questions.

We propose choosing between an interview with each case student individually or one interview with all case students together, which is called a 'focus group'. Individual interviews will help you to get in depth answers of students, which leads to higher potential for insights. A disadvantage is that individual interviews are more time consuming to organize. A focus group is an interesting way to exchange viewpoints and discuss (dis) agreements between students. This dynamic will not be captured in a face-to-face interview. Also, focus groups are less time consuming than individual interviews. A disadvantage of a focus group is that the talking time of some students may be considerably higher than that of others, making their contribution disproportionate. Decide in this stage if you want to do individual interviews with each case student or one focus group.

Things to avoid when interviewing are posing too many questions at once without giving time in between to answer. Another problem can be posing questions that already suggest an answer and could provide socially desired answers. **We recommend using simple open questions in which the student is given space to answer freely**. A good interview gives the student more opportunity to speak than the interviewer. Also, the interviewer should ask extra, follow-up questions when the answer is not clear or rather superficial. For example, when a student answers a question with 'yes', 'no', or 'sometimes', you can ask the student to explain this answer in more detail. Think beforehand about which additional questions you can ask and also formulate these questions openly enough.

Below, you find some examples of question prompts on mathematical reasoning for a post lesson interview with the case students:

- What did you like most about the lessons on reasoning in mathematics? Explain with an example.
- What was most difficult for you? Give an example.
- Could you solve your problems? How? Give an example.
- What did you learn? (What can you do now that you could not do before. What can you do better? How is it better?)
- In which ways do you think the reasoning classes contributed to your learning of mathematics?
- How did the set-up (e.g., working in groups) work for you? Could you discuss with your neighbour?
- Which resources were provided? Were they useful, or would you need any other resources?
- In which ways were the lessons on reasoning different (or not) from other lesson? Explain with an example.
- In which ways were the lessons on reasoning useful (or not) for you?
- If the same lesson is being taught to another group of students like you what would you change? Why would you change that aspect?
- What kind of support did you get in class? From your peers? From the teacher? How well did it help you to keep going? Did you need anything else? Availability?

We suggest the interview with case students focus on **their learning experiences**. The purpose would be to get a better understanding of students' learning experiences in the 'reasoning classes'. We address here the 'experienced curriculum' at student level. We investigate how students learn in such learning environments, in particular with respect to: (1) their use of resources (material, digital, social/human), (2) the support they (perceive to) need for their learning, (3) skills and capacities they develop and (4) work forms (they perceive) work best for them (e.g., collaboration)

You can also give students multiple cards with evaluative statements (e.g. "The mathematical tasks were easy", "The reasoning tasks were exciting", "I didn't think that I could do such mathematics."). Students are then asked to rank the statements from those they most agree with to those they most disagree with. Afterwards, you can talk about the ranking (e.g. "Why do you most agree with statement X?"). Especially for less talkative students, this method can be helpful to express themselves in the interviews.

Additional questions when you want to know more:

- Can you give an example of what you mean?
- Can you explain that in more detail?
- Can you say something more about it?
- What do you mean?
- How did that happen?
- When was that?
- And then?

If you are not sure whether you have understood the answer correctly:

- If I understand correctly, then...
- Am I summarising it correctly when I say that...

4.5 Preparation of a short questionnaire or exit card for all students (optional)

The purpose of a Research Lesson Study is to better understand and/or improve the teaching of the students. Since there is a clear focus on three case-students in the classroom, it can be interesting to explore the impact of the research lesson on other students using a short questionnaire or exit card after one or more research lessons. This short questionnaire or exit card on classroom level, can give insights in the general perceptions of all students regarding the teaching activities that were implemented during the research lesson. It could also be an interesting addition to the interview data gathered on the case students. It is thus useful to think about possible questions for all students when you are drawing up the lesson plan.

Below we give an example of a scored questionnaire[58]. Think about questions that are relevant to your research question and that fit the lesson plan. An advantage of the short questionnaire is that it only takes a few minutes to fill in, and the teacher will get a decent overview of how all students perceived the research lesson. A disadvantage is that students might interpret the questions in different ways. It is thus important to 'concept check' the extent to which the concepts used in the questionnaire have been grasped or understood by the students. Use the **Short Questionnaire For All Students Sheet**. A completed example can be found below.

Table 2
Illustration: questionnaire for all students – experiences from the research lesson

CLASS 1 Number of Students: 20	Strongly Disagree	Disagree	Agree	Strongly Agree
I found the math tasks/exercises interesting	2 (students mark this answer)	5 (whereof 1 case student)	12 (whereof 2 case students)	1
I cooperated well during these math tasks	1	3	14 (whereof 2 case students)	2 (whereof 1 case student)
I was motivated when solving the math tasks	0	4	12 (whereof 3 case students)	4
I have the feeling that I learned from solving these math tasks.. se exercises	5 (whereof 1 case student)	7 (whereof 2 case students)	7	1
I was challenged during solving these math tasks	2 (whereof 1 case student)	3 (whereof 2 case students)	11	4

Twenty students, whereof 3 case students participated in the classroom survey. In this situation, students were invited to collaboratively solve a set of math tasks (in duo) in their own way, while the teacher was available for eventual questions. Based on the representation of the distribution of students' scores in the table, the Research Lesson Study group came to some interesting conclusions. It is clear that the majority of students found the research lesson interesting. Although a few students indicated that they did not cooperate well during the lesson, most students did cooperate well and were also able to follow the lesson well. The case students also indicated this. The feeling of learning something was a little less positive. Twelve students, including the case students, said they did not, or did not at all, feel they had learned something. In terms of challenge, there is a bigger difference between the case students and the other classmates. Most students said they were challenged during the lesson, while this was rather not the case for the case students.

Alternatively, you can use 'exit cards' to ask students about their experiences after each lesson. An advantage of using the exit card, is that teachers will get more elaborate feedback on their research lesson. A disadvantage is that it will take more time (min. 10-15 minutes) for students to fill in this questionnaire. Use the **Exit Card For All Students Sheet**. The proposed questions can be altered to fit the research goal and question.

[58] Bodvin, et al., 2020

5. Teaching the first research lesson

5.1 Teacher guidelines

Explain to your students that you are trying to improve the way you help them to learn and that is why there are other teachers observing, making notes and talking to some students after the lesson. Explain that you will share with them what you are finding out and get their views.

The teacher follows the prepared lesson plan. Deviations from this are preferably minimal. Otherwise it becomes more difficult to determine the quality of the research lesson. At the end of the research lesson, the teacher asks all students how they experienced the research lesson (for instance with a short questionnaire) and the case students to stay for a short interview with the observer(s). You can tell the case-students that they have been randomly selected for an interview. At the end of each research lesson, the teacher thanks all students for their cooperation.

5.2 Observer guidelines

Because the research lesson is jointly planned, it is jointly owned by the Research Lesson Study group. This means that the focus for the observers is less on the teacher and more on the learners - especially the case students. They should alternate in the research lesson spending some time as if 'zoomed-in' on a case student and then 'pan-out' to allow a bigger group or the whole class to come into frame at other times.

Kelly, Steven and Rayan, three mathematics teachers from Flanders (Belgium) who participated in the LESSAM project (2022-2023) researched how they could challenge high-ability students with mathematical reasoning. They did this by presenting complex problems related to the math content (e.g. quadratic equations with split-square approach). The three teachers observed the case students, but noticed that the whole class wanted to get involved in these challenging exercises. They found that it was stimulating for the other students as well to look at a problem in different ways. They zoomed in on the case students, but allowed the whole class to come into frame.

Below we give some guidelines for the observer(s):

- Get down to student level and amongst the students so you experience the lesson from their perspective.
- If there is more than one observer in the classroom, they can still observe the same case students: probably the observers note down different things (because classrooms are so fast moving and unpredictable) and even when they note the same thing, it's often interpreted or explained very differently.
- If the observer does not know the case students, name cards, a map of the classroom or a photograph of the case students can be given to the observer.
- Make sure the observer never blocks the students' view on the board or the teacher.
- Always respect the privacy and psychological wellbeing of the students. If a student expresses (verbal or with body language) that he or she feels uncomfortable being observed, talk to the student and stop the observation if needed.

Observers should try to capture the case students' responses at different points in the lesson - and how they match or differ from what was expected at that stage.

What is the best way to observe? Use an open observation or be guided by the behavioural indicators mentioned in **Research Lesson Planning and Observation Sheet**. Write down what you see as clearly as possible. Or more specifically: describe as clearly as possible the activities and specific behaviour of expressions of feelings of the case student(s) you are observing. Note also any critical incidents. If there is a common pattern (e.g. all case students misunderstand something in the same way) note it in the column 'observed response'. Also, note in which phase of the lesson you make each annotation if you can.

Look for evidence of progress for each student against what was planned and the extent they are achieved. What are key points for the next lesson for the individual case students, the learning groups to which they belong (e.g. students with good, average or below average progress) or the class? What might you want to ask them in the post lesson interview? Note this down in 'initial thoughts, ideas, reflections' at the bottom of the page.

If the research question is exploratory without strong assumptions or pre-defined ideas, it might be hard to write down an expected response or behaviour. In that case, you can leave these cells open in the beginning. During the observation, try to identify critical events related to the thinking and reasoning of students. Think critically about the characteristics of the task related to mathematical reasoning, about students' actual reasoning and about teacher's actions supporting students' reasoning. After one or two research lessons, the expected response might be more clearer and can be completed throughout the Lesson Study.

6. Interviewing case students and administering the short questionnaire/exit card

6.1 Interviewing case students

Research Lesson Study groups interview the case students after the research lesson to gain their perspectives on what worked for them and why, what they felt they learned and how they think the lesson could be changed if it were taught again in order to make it work even better. The interview should be short (no more than 5 minutes) and can be done with all the case students in a group or individually. Try to conduct the post lesson student interview at the earliest opportunity – ideally at the end of the lesson. Try to capture some of their exact words in your notes. It is also possible to conduct the post research lesson student interview with other students but who are in the same three learner groups (e.g. students with good, average or below average progress) as the case students. This can help triangulate findings – but it can also complicate the data set. We recommend to focus on the three case students. Use the **Post Research Lesson Interview Sheet** to make notes during the interview.

To make the interview of the case students go smoothly, we formulate some useful guidelines.

Prepare the interview sufficiently:

- Inform the case students that the interview will take place shortly or immediate after the lesson.
- Provide a quiet location for the interview.
- Try to estimate the talkativeness of the case students beforehand and take this into account when preparing the questions.

During the interview, keep the following in mind:

- As an introduction to the interview, you mention that you would like to know what the student thinks of the lesson. Therefore, you will ask some questions about the lesson. In this way, you shift the focus from the student to the lesson and as a consequence, the student may feel less threatened. Also address the student explicitly on the possible contribution to the lesson design: 'We are very curious about what you thought of the lesson just now. Through your experiences we want to improve the lesson, for you and your classmates, but also for other students. So, would you like to share your thoughts about the lesson with me? I have prepared some questions. There are no right or wrong answers.
- Always respect the privacy and psychological well-being of the students. If a student expresses (verbally or with body language) that he or she feels uncomfortable to answer the interview questions, talk to the student and stop the interview if needed.
- Preferably make an audio recording of the conversation, so that you focus on listening and do not have to write down the answers. Ask for permission and make it clear that the recording will not be shared with others.
- Dare to allow silences. That way you give the student time to think.
- React to answers in such a way that the student gets the feeling that every answer is good and appreciated. This prevents the student from giving 'wrong' answers. A response such as 'okay', 'yes', or 'good to know' is more neutral than for example 'good' or 'I am glad to hear that'. In conclusion, thank the student for his or her input.
- Avoid reading off the prepared questions on paper.
- For less-experienced groups: stick as much as possible to the prepared questions. This way, you are less likely to fall into the pitfalls of asking closed or suggestive questions and the focus remains on how the student experienced the lesson.
- Although the focus of the interview is on how the student experienced the lesson, you can also ask a question that refers to a striking or unclear observation. Be sure you avoid intimidating the student. Therefore, do not go into too much detail. A student may not be able to justify all his or her actions during a lesson.
- If something did not go as desired during the research lesson, use this opportunity to think together with the student about how to do things differently in the future.

6.2 Administering a short questionnaire or exit card for all students (optional)

You can choose to question all students at the end of the research lesson on their experiences via a short questionnaire or exit card (see also 'Preparation of a short questionnaire or exit card for all students (optional), page 41). This questionnaire is optional and can be offered on paper or digitally (e.g. via Google forms). Provide a few minutes at the end of your research lesson to administer the questionnaire. The teacher explains that he/she wants to know what the students thought of the lesson via a questionnaire. It is important to stress that this questionnaire will be processed anonymously. By guaranteeing anonymity, students will be less inclined to give socially desirable answers. This can be done, for example, by placing a box at the door or by creating a 'mailbox' in which students can deposit the completed questionnaire. If you like to compare the answers of the case students with those of the class group, you can put a cross or other identifying mark on the case students' questionnaires. Without violating anonymity, you can recognise the forms of the case students and compare their answers with the answers of the other students in the class.

Process the completed questionnaires in preparation for the debriefing of the research lesson.

7. The post research lesson discussion

Come together as soon as you can after the research lesson. Ideally you will conduct this discussion immediately after the research lesson (and certainly no later than 24 hours afterwards). You may wish to review the following qualities of a successful post research lesson discussion.

a) Consider the post lesson discussion as a joint learning opportunity

b) Openness to critical viewpoints and suggestions

c) Fidelity to observed data

d) Clear goals and questions from the plan/observation sheet

e) A designated 'moderator' for the discussion (a chair who can lead the discussion positively,) a role that can be combined with that of

f) A final commentator whose role it is to capture the learning distilled from the discussion, in order that it can be acted upon by the group and others beyond the group.

The most important thing to remember though is that the flow of analysis needs to start with the observations made of the case students' learning before it addresses the teaching. This preserves the focus on student learning and reduces the tendency for lesson observation discussions to become feedback on teaching (which teachers can feel is judgmental in nature and not conducive to teacher learning). Try to listen to each other and build on each other's ideas by making suggestions, challenging each other thoughts, raising hypotheses, elaborating, etc.

David and Alex, two teachers from Greece participated in the LESSAM project in 2022-2023. They studied how to create a culture of mutual communication and discussion in the classroom. Based on the observation of two case students, the teachers altered their teaching approach for all students:

In the first lesson (first Lesson Study cycle), David chose to engage students to work in groups of two (per desk) in group-work activities. However, the two case students sitting at the same desk did not have a meaningful interaction: they worked almost independently of each other, one following an algebraic approach and the other following a geometric approach to the assigned task. Alex, the other teacher observed a similar phenomenon in other groups. There was some discussion about this in the subsequent reflective LS session and the teachers chose to divide students in larger groups in the second research lesson that followed.

In the following lessons, both teachers chose to include 4-5 students in each group. This choice, combined with the teachers' prompts to each group to discuss and formulate an answer together, had a significant effect on the class communication in the subsequent lessons and the learning outcomes.

The students' discussions in each group were richer, their interaction more meaningful, and the presentation of the answers by each group contributed differently to the whole class discussion. The content of the last three lessons (second Lesson Study cycle) was stochastic mathematics and students' experiences with it were limited. Nevertheless, the culture of mutual communication and discussion in the groups allowed the formulation of very interesting ideas which indicated all students' deeper engagement and intuitive understanding of difficult concepts such as variability. For example, the students discussed in groups the boxplots* of the performance of different samples of students, observed and commented on the variability between different samples of students as well as the variability within each sample, made comparisons and provided possible interpretations.

*Boxplot is a standardized way of displaying the distribution of data based on a five number summary ("minimum", first quartile [Q1], median, third quartile [Q3] and "maximum").

Use the **Post Research Lesson Discussion Sheet** in this phase. This form lists all the aspects that are best dealt with during the debriefing:

- the observed development of the case students (as shown by the observations and interviews);
- possible differences and similarities among the case students;
- experiences of the whole class during the research lesson (and a comparison with the experiences of the case students based on the questionnaire);
- the expected outcomes of the research lesson, the lesson approach and possible surprises;
- things the group would like to do differently in a next research lesson.

After the last research lesson in a Lesson Study Cycle of three research lessons, the group formulates an answer (often partial) to the research question and summarises what the group has learned. When discussing what the group learned, not only describe what happened, but try to go further, in terms of interpreting what happened and especially why. These insights can be valuable starting points for a next cycle of lesson study. In what follows, you can find an exemplary post-Research Lesson Study discussion of three teachers.

Below you can find an example of a post Research Lesson Study discussion of a Lesson Study team in a school in Camden, London (UK). It is a discussion about one of the case students, Alex, after a lesson on multiplication.

1	Ryan	I've got Alex. He did really well. He had the practice paper as well which yeah he… the first thing I said to him was 'what did you enjoy most about the lesson?'. He said 'I find things were really smooth, although I still wanted to be challenged more'.
2	Gabriela (class teacher)	Yeah of course.
3	Ryan	And I said 'did you understand why Gabriela asked you to do the practice?' And he didn't really.
4	Ashley	It's really hard. What's come out of this a lot is this bunch of children, that are often lower-middle I think, that are so desperate to be middle or middle-higher that they are quite happy to throw to the wind any understanding but they may not be quite there yet. They just don't care about that; they just want to appear to be understanding.
5	Gabriela	I know, that's so interesting.
6	Ashley	Yeah, it is cause it makes it really difficult. Cause you feel like it's almost the battle between 'you should do this', 'no, I want to do this'.
7	Ryan	But I mean…all credit to [Alex], he worked through his activity really really well on his own and he got onto the challenge.
8	Gabriela	Oh, yeah.
9	Ashley	I remember that.
10	Ryan	…which is brilliant. And I think he felt really good.
11	Ashley	Well, it's an achievement, isn't it then? (Note: referring to the workbook)
12	Ryan	And he was really, like compared to yesterday's lesson, he was really really excited.
13	Ashley	So did he exceed our expectations? (Note: referring to the workbook)
14	Ryan	Yeah he did.
15	Ashley	Which actually is the same as James. [Note: another case student]
16	Gabriela	Yeah. We underestimated both of them.
17	Ashley	Yeah.
18	Ryan	One thing though that is a surprise, you know how we said that he'll double and double again at the start cause he loves doubling?
19	Ashley and Gabriela	Yeah.
20	Ryan	I realized that he's using the word 'double' instead of 'multiple'. He thinks 'double' is 'multiply'. So for instance he says 'I double two by three to get six'.
21	Gabriela	Oh that's so interesting.
22	Ryan	And I said 'you mean you multiply two by three to get six". And he said 'oh yeah, yeah, yeah". But he did it again later on so just be aware of that Gabriela.
23	Gabriela	Yeah that's really good to know.
24	Ryan	I said do you know what doubling is?

25	Ashley	Who was it who said, at the beginning bit, that said, when we talked about ratio, or was it in the end, and they said 'it's when you double a number' and I said 'do you always double it?' and they were like 'oh well, not necessarily'. Cause I said 'we're doubling it here cause it's... Someone else said that in the beginning.
26	Gabriela	Yeah.
27	Ashley	And I was like 'well you don't always double cause you might be dividing it down', so someone else...
28	Gabriela	Maybe we gave too many examples of doubling.
29	Ashley	...when they are doubling.
30	Ryan	Yeah possibly.
31	Gabriela	That's good to know.

The excerpt shows how the teachers (Ryan and Ashley) combined information from their observations and realized (e.g. Gabriella, see line nr.28) that the teaching included too many examples of doubling, so the students confused the word 'multiplying' with 'doubling'.

The planning of the next research lesson falls in principle outside the post research lesson discussion but can be done afterwards. The split between the debrief and the planning of the next research lesson should ensure that the group first have the necessary discussions to get a full picture of the impact of the research lesson on the case students. Afterwards, the group can make any necessary adjustments to the next research lesson.

The discussion of the research lesson has undoubtedly provided suggestions for possible adaptations of the research lesson. The aim of these adaptations is to be able to realise the desired outcome in the research question even better with the case students (and other students of the class). A possible leading question is: 'In what areas and/or moments in the lesson can the learning of the case students be further optimised?'

In the following phases, the (modified) research lesson is again taught and observed, the case students are interviewed and the lesson is discussed afterwards. A Research Lesson Study (see also figure p. 7) consists of three research lessons that are discussed after each lesson. At the end of a Research Lesson Study there is an overall assessment.

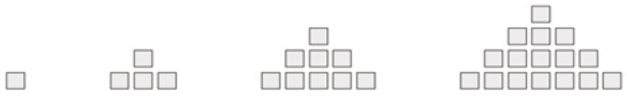
In the following example, you can see how three teachers from Greece adapt their lessons in the LESSAM project (2022-2023). By doing so, they got more insight students' degree of engagement in math reasoning.

Alexandra, Chloe and Andrew, three experienced mathematics teachers from Athens (Greece) were interested in how to engage all students in math lessons. Thus, they worked on how they could design different versions of the same task, acknowledging various levels of mathematical challenge. In this respect, they decided to arrange the students into homogenous groups with respect to their mathematical background and interest. In the first implementation they observed that students who are usually struggling in mathematics didn't have enough time to refine their arguments and make a synthesis that answers the given problem. So, they decided to add an extra didactic hour to further support students who are struggling with mathematics. In this second didactic hour, the teachers rearranged the groups of students in mixed groups, where members of the previous groups had to come together and defend of their developed strategies. Although their initial goal for supporting mathematical reasoning of students who are struggling with mathematics was successful they observed that students who are competent in mathematics seemed to have many instances of losing their interest. So, in the next research lesson they worked with the aim of including all students without lowering the mathematical challenge. The aim to emphasize and facilitate mathematical reasoning, reinforced teachers to get more insight on supporting all students' degree of engagement.

Teacher task:

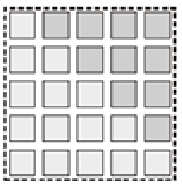
Read the example and discuss the questions below with your Lesson Study group. Try to reflect in the same way upon your own task selection, actions promoting students' mathematical reasoning, and upon teachers' collaboration.

Example:
Teacher 1 (T1) gave his students (7th grade) the figures below and asked them to find the number of squares in the 5th figure, in the 10th figure and to explain it using a figure.



Teachers sharing experiences while using/modifying the task
In the worksheet, he was providing some guidance to help them use the square illustrating the pattern n2.

Students worked in groups, with T1 and his colleague (T2) walking through the groups and supporting their work trying not to reveal the answers. T2 is a mathematics teacher in the same school. She was there to observe T1 lesson and she was also supporting students together with T1.



5^2

After the lesson they discussed and both appeared satisfied with the students' work but not with the students' communicating their thoughts and reasoning. They agreed that students were not willing to explain thoroughly their reasoning and attributed this to the lack of whole class discussion. Two days later, T2 made the lesson in her class, with the observation and support of T1. The main change T2 did was the provision for whole class discussion after every phase of the task. Students were explicitly asked to demonstrate their solutions and reasoning.

Questions:

- What do you think about the task T1 and T2 used? How does it promote mathematical reasoning? *What do you think you could change? Why?
- If you were in T1' lesson, where would you like to focus your observation? Why do you think this is important?
- Do you have experiences of observing others' lesson, or of others observing your lesson? What do you think is the value of peer observation? What are the difficulties?
- From the information provided, what teaching actions could you undertake a) to explore students' mathematical reasoning and b) to facilitate students' mathematical reasoning?

8. The overall assessment

At the end of each Research Lesson Study, the group reflects on the whole process and the data collected. This reflection is necessary in order to make the step from merely exchanging experiences to a more integrated understanding of practice: from 'what' and 'how' something works with the case students to 'why' something works. The outcome of this reflection moment can help determine the focus and agenda of a next Research Lesson Study (in case multiple Lesson Studies are planned). During this general evaluation the group also reflects on what they have learned during the Research Lesson Study and what these insights mean for the wider classroom practice. The **Overall Assessment Sheet** contains guiding questions for the general evaluation. This questionnaire asks about collaboration, strengths and weaknesses in the different cycles of the Research Lesson Study, learning gains and consequences of the Research Lesson Study. A safe (learning) climate is necessary to have this assessment. It is important that everyone respects each other and that all contributions are valued. It is key that teachers are able to share ideas, concerns, challenges and 'wonderings' without fear of criticism. They should act as 'critical friends' to each other. This way of talking and discussing leads to more productive conversations.

9. Dissemination of your Research Lesson Study with students and other professionals

After finalizing the last Research Lesson Study, share with your students what you found out with Research Lesson Study and get their views.

At the outset, try to arrange an opportunity for the group to share with colleagues what they have done, learned and refined once they have completed their Research Lesson Study, especially in the key curricular or pedagogical approach being developed. If people know in advance that they will have to share their findings with others, then they will bear this in mind throughout the proceedings. This helps the Research Lesson Study group to keep their thinking and their findings clear, more useable and replicable by others. Video snips of the research lessons and digital photos embedded in PowerPoint presentations are a popular way of conveying lesson practice and processes. (You will need to ensure you have a school policy in place on use of video and photos). Arrange opportunities for members of the Research Lesson Study group to work with other teachers in order to help coach the pedagogic technique they have evolved, adapted or refined. Remember that articulating and explaining practice and making it visible to others:

a) helps those learning from their peers improve their practice

b) improves the performance of the person doing the explaining or coaching.

This is because it makes visible what is often tacit knowledge of practice which teachers use but never express. Articulating this helps them become more aware of their knowledge themselves and therefore more able to improve it further. Celebrate and value what has been learned and shared. Create a 'learning wall' in the staff room where a Research Lesson Study group can display their work – photos, notes, observations, discussion outcomes, student interviews and tentative conclusions. This creates lots of staffroom talk about professional learning long after the formal sharing is over.

Some Research Lesson Study groups demonstrate the techniques they develop to other teachers in an open house research lesson. Here the lesson is taught with a number of invited observers. A lively discussion can follow including students as well as adults. Where a Research Lesson Study unearths some very important practice knowledge a school may open the event up to invited guests from other schools, universities or local authorities. and teach it after school as a 'public research lesson'.

Dissemination of your Research Lesson Study with students and other professionals:

In the LESSAM project, we developed a website to disseminate the findings of the different teacher teams and the overall impact of Lesson Study on teacher learning and the improvement of students' mathematical reasoning (see intellectual output 6).

On the following pages you can find the sheets for each stage of Lesson Study. Use the forms as a guide, as they help you to plan and discuss the research lessons and reflect on them in a profound way.

Overview of the sheets:

- Lesson Study Group Learning Protocol Sheet
- Lesson Study Overview Table Sheet
- Research Lesson Planning and Observation Sheet
- Short Questionnaire For All Students Sheet
- Exit Card For All Students Sheet
- Post Research Lesson Interview Sheet
- Post Research Lesson Discussion sheet
- Overall Assessment Sheet

Empty Forms

Lesson Study Group Learning Protocol Sheet

See handbook page 34

This protocol was developed by

At all stages in this Lesson Study we will act according to the following:

Signed and dated by Lesson Study group members.

Lesson Study Overview Table Sheet

See handbook page 36

Names of teachers in the Lesson Study group			
Research Lesson Study protocol adopted? Y/N			
Year group			
Number of students in the classroom Set?			
Homogeneous/mixed ability classroom			
Usual teachers in the classroom?			
What need(s) of the case-students related to teaching do you want to focus on in this Research Lesson Study?			
Mathematical focus and overall aim and focus for the research lesson sequence			
Which criteria are used to select the three case students?			
Who are the case students (per class)?			
What are the needs of the case students?			
Research question?			
XYZ-Research question [optional]: 'We want to find out what effect educational practice X has on the outcomes Z of the case students and what other factors, if any, Y play a role in this'. Educational practice (X): Outcomes in case students - what do we want to achieve?(Z): Factors that may play a role in this (Y):			
Date and time and student learning objective for research lesson 1			
Date and time and student learning objective for research lesson 2			
Date and time and student learning objective for research lesson 3			

Research Lesson Planning and Observation Sheet

See handbook page 40,45

Class:	Topic:	Date & Time:

Research question:

Try to narrow down your research question as much as possible.

Summary of our hypothesis about how our teaching approach facilitates mathematical reasoning:

By using teaching approach (X)

(case) students will

Factors that may play a influential positive/ negative role (Y)

Needs of case students to engage in mathematical reasoning

Teaching approach (X)

Expected outcomes (Z)

See also section 'Defining a research goal and identifying the case students'

Learning goal (content)

Learning goal (mathematical reasoning)

Learning activity (math tasks for students during the research lesson):

Research Lesson Planning and Observation Sheet

See handbook page 40,55

Strategic questions: How to support reasoning in the lesson?

-
-
-

What learning behaviour is relevant to observe and shows the desired understanding/reasoning?

Means for collecting evidence

☐ Short questionnaire or exit cards ☐ Observations ☐ Strategic questions ☐ Interviews

☐ Other:

Lesson sequence (see also next page):

Phase & timing	Teaching approach	Students learning activity

Case students:

Initial situation and knowledge of the case students: what knowledge, skills and attitude does the case student have? What needs does this student have?

Case student 1:	Case student 2:	Case student 3:
Strengths: • • • Learning needs: • •	Strengths: • • • Learning needs: • •	Strengths: • • • Learning needs: • •

Research Lesson Planning and Observation Sheet

Phase 1 of Lesson Sequence	Students learning activity:	
	Expected response/behaviour	Observed response/identify critical events
Case student A:		
Case student B:		
Case student C:		

Phase 2 of Lesson Sequence	Students learning activity:	
	Expected response/behaviour	Observed response/identify critical events
Case student A:		
Case student B:		
Case student C:		

Phase X of Lesson Sequence	Students learning activity:	
	Expected response/behaviour	Observed response/identify critical events
Case student A:		
Case student B:		
Case student C:		

Observed final situation case student	What were they able to do? (What progress have they made and how do you know?)	Is the desired learning outcome achieved? Yes/no, How do you know?
Case student A:		
Case student B:		
Case student C:		
Initial thoughts, ideas, reflections		

Short Questionnaire
For All Students Sheet

See handbook page 44

Read each sentence carefully and circle what fits.

1 Strongly disagree	2 Disagree	3 Agree	4 Strongly Agree

1. I found the math tasks/exercises interesting.	1	2	3	4
2. I cooperated well during these math tasks.	1	2	3	4
3. I was motivated when solving the math tasks.	1	2	3	4
4. I have the feeling that I learned from solving these math tasks.. se exercises.	1	2	3	4
5. I was challenged during solving these math tasks.	1	2	3	4
6. Other statements, specifically for your research lesson.	1	2	3	4

Short Questionnaire
For All Students - Scoring Sheet

Read each sentence carefully and circle what fits.

CLASS 1 Students:	Strongly disagree	Disagree	Agree	Strongly Agree
1. I found the math tasks/exercises interesting.				
2. I cooperated well during these math tasks.				
3. I was motivated when solving the math tasks.				
4. I have the feeling that I learned from solving these math tasks.. se exercises.				
5. I was challenged during solving these math tasks.				
6. Other statements, specifically for your research lesson.				

Exit Card
For All Students Sheet

See handbook page 44,46

Please take a few minutes to answer the questions below.

Date:	Class:

Student name:

1. Select the smiley that best describes your overall feeling about your work today

Explain why you selected this smiley:

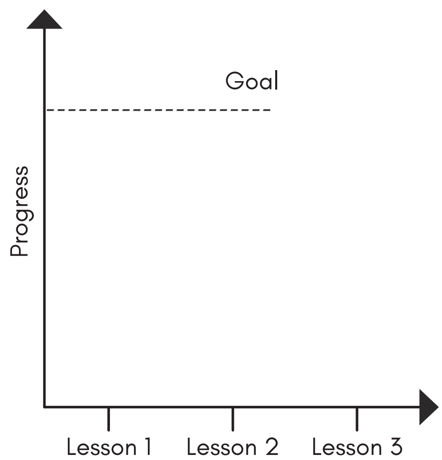
2. Describe: (a) what you worked on today and (b) what progress you made.

3. Describe: (a) what activity you liked best today, and (b) what made you like this.

4. Describe: (a) the most important hurdle/difficulty you came across today, and (b) how you deal with it.

5. (Optional) On the axes provided, sketch how you have progressed towards your goal so far.

Use a few words to explain.



Post Research
Lesson Interview Sheet

See handbook page 43,46

Class:	Date:	Time:

Case student(s):

Questions	Notes during interview
What did you enjoy most about that lesson? Why?	
What did you learn? What can you do now that you could not do? What can you do better? How is it better?	
What aspect of the teaching worked best for you? Why?	
What could we do (differently) next time we teach it to improve the lesson for a similar class? Why would you change that aspect?	
Other questions, specifically for your research lesson.	

Post Research
Lesson Discussion sheet

See handbook page 47

Please take a few minutes to answer the questions below.

Class:	Topic:	Date & Time:

Circle the correct option: research lesson 1 – research lesson 2 – research lesson 3

Restate your research question:

Please review the research lesson planning and observation sheet, the post lesson interview sheet and if possible the results of the short questionnaire/exit ticket, to answer the following questions.

Reflecting on student learning:

1. What progress did each student make? Was his/her learning need addressed? Was this enough?

Class 1	
Case student A:	
Case student B:	
Case student C:	

Class 2 (optional)	
Case student A:	
Case student B:	
Case student C:	

Class 3 (optional)	
Case student A:	
Case student B:	
Case student C:	

Post Research Lesson Discussion sheet

See handbook page 47

2. What differences or similarities do we find between the case students?

3. What about others in the group of learners they typify?

4. How did our teaching approaches help or hinder the students' learning (maybe a bit of both)? Did some students benefit more/less from our teaching approaches? Why, why not?

5. What surprises were there?

6. Did we find out anything of note about the way they were learning?

Reflecting on your teaching approach:

Please review your hypothesis about your teaching approach from your Research lesson planning and observation sheet here. Refer to your analysis of student learning on the previous page as much as possible.

1. Has your hypothesis about teaching approach (X) led to the desired outcomes (Z)? Why? Why not?

2. Think about the influential factors (Y). Have they turned out to be relevant? How? What other factors turned out to also be relevant?

3. What aspects of your teaching approach could be adjusted next time to improve the progress of the case students and all students? Why? What other teaching approaches could you have chosen instead to arrive at the desired outcomes (Z), perhaps in a better way?

4. So what should we try next time? Why? Make an agreement on the teaching approach that you will choose for your next research lesson. Give reasons for your choice.

Post Research
Lesson Discussion sheet

See handbook page 47

Reflecting on the means of collecting evidence on student learning:

Questions to be answered after research lesson one and two:

1. Do we have enough evidence to answer our questions? Why, why not?

2. Do we need to revise the assessment of any students? Why?

3. Make an agreement on how to do the assessment for your next research lessons:

Overall Assessment Sheet

See handbook page 51

1. General cooperation in the group: What did we like about the cooperation during this Research Lesson Study? Do we want to see things differently? In what ways did or did not the collaboration within our group contribute to our learning?

2. Progress of Research Lesson Study in our group:

	What went well? Why?	What did not go well? Why?	What do we want to do differently? Why?
Defining a research goal and identifying the case students			
Planning the research lesson			
Teaching/observing during the research lessons			
Interviewing case students			
Short questionnaire/exit ticket (optional)			
Post research lesson discussion			

If applicable: Are there any personal events or factors that played a role in the Research Lesson Study?

Overall Assessment Sheet

See handbook page 51

3. What were the main things you discovered about how the students learned mathematics?
   In what ways will this change your teaching in the future?

4. What were the main things you learned about the students that you did not know so clearly before?
   In what ways will this inform your future practice?

5. What other things have you learned about teaching or learning not captured in 3 or 4 ?

How will this change your teaching in the future?

6. Are there any implications for the mathematics curriculum, assessment or pedagogy?

7. What key learning will you share with colleagues in school and within the project?

Inspiring Web Links

03

1. **www.ucy.ac.cy/lessam2/?lang=en**
This handbook stems from the work of the LESSAM project. The project aims to investigate the impact of the model of Lesson Study on teacher learning and, consequently, on student learning outcomes. On the webpage you can find more information and materials on mathematical reasoning and lesson study. The webpage is available in English, Greek and Dutch.

2. **websites.ucy.ac.cy/formas/en/resources.html**
The link takes you to the webpage of the project "Promoting Formative Assessment: From Theory to Policy and Practice (FORMAS)". The project aimed to help secondary school mathematics teachers to develop formative assessment skills. Such assessment strategies can help teachers identify their students' learning needs and take action to address them. These can be used during the planning sessions of Lesson Study. The webpage contains materials for a teacher professional development course, a teacher handbook etc. All materials are available in English, Greek and Dutch as well (click on the two options on the top right corner).

3. **icse.eu/activities**
The International Centre for STEM Education (ICSE) is located at the University of Education in Freiburg, Germany and focuses on practice-related research and its transfer into practice. The aim of ICSE is to help improve STEM (Science, Technology, Engineering and Mathematics) education across Europe. The site of this Centre contains rich STEM materials that have been developed in the context of EU funded projects. These materials are based on authentic situations (e.g. workplace, environment) and can be used in school classrooms as well as for the education of prospective and practicing STEM teachers. All materials are available in English.

4. **www.walsnet.org**
The World Association of Lesson Studies (WALS) aims to promote and advance the research and practices focused on Lesson Studies in order to improve the quality of teaching and learning. It provides a platform for research collaboration, mutual assistance and information exchange among its members. It is made up of educational researchers and teaching professionals committed to the improvement of the quality of learning. This webpage is available in English.

5. **www.deficambridge.org/camtree**
Camtree, the Cambridge Teacher Research Exchange, is creating a global platform for educators around the world who wish to reflect on their practice and conduct research on learning in their classrooms and institutions. This webpage is available in English.

6. **lessonstudy.co.uk**
Lesson Study UK was launched by Dr Pete Dudley in 2011 as a way of sharing resources and knowledge about Lesson Study across the UK. In the last seven years, there have been hundreds of thousands of visitors to the site who have found out about Lesson Study and have downloaded resources such as the Lesson Study handbook which is now translated into 5 languages. This webpage is available in English.

References

- Amador, J. M., & Carter, I. S. (2018). Audible conversational affordances and constraints of verbalizing professional noticing during prospective teacher lesson study. Journal of Mathematics Teacher Education, 21, 5-34. https://doi.org/10.1007/s10857-016-9347-x
- Australian Academy of Science (AAS). (2018). reSolve: maths by inquiry, https://www.resolve.edu.au/.
- Barnes, D. (2008). Exploratory talk for learning. In N. Mercer & S. Hodgkinson (Eds.). Exploring talk in school: Inspired by the work of Douglas Barnes. (pp.1-16). SAGE Publications Ltd. https://dx.doi.org/10.4135/9781446279526.n1
- Battista, M. T. (2016). Mathematical reasoning and sense making. In Reasoning and sense making in the mathematics classroom: Pre-K-Grade 2. (pp. 1- 26). Reston, VA : National Council of Teachers of Mathematics.
- Bergqvist, T., & Lithner, J. (2012). Mathematical reasoning in teachers' presentations. Journal of Mathematical Behavior, 31, 252-269.
- Blanton, M. L., & Kaput, J. J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. Journal for Research in Mathematics Education, 36(5), 412-446.
- Boaler, J. (2010). The elephant in the classroom. Helping children learn & love maths. Souvenir Press: London. UK Edition of What's Math Got To Do With It.
- Boaler, J. (2016). Mathematical mindsets. Unleashing students' potential through creative math, inspiring messages and innovative teaching. San Francisco, CA: Jossey-Bass.
- Bodvin, K., et al. (2020). Lesson Study: Samen de klaspraktijk voor leerlingen verbeteren [Lesson Study: improving educational practices for students]. Leuven, Belgium: Acco Uitgeverij.
- Boston, M. D., & Smith, M. S. (2009). Transforming Secondary Mathematics Teaching: Increasing the Cognitive Demands of Instructional Tasks Used in Teachers' Classrooms. Journal for Research in Mathematics Education, 40(2), 119-156. https://doi.org/10.2307/40539329
- Brodie, K. (2009). Teaching mathematical reasoning in secondary school classrooms (Vol. 775). Springer Science & Business Media.
- Cheung, W., & Wong, W. (2014). Does Lesson Study work?: A systematic review on the effects of Lesson Study and Learning Study on teachers and students. International Journal for Lesson and Learning Studies, 3(2), 137-149. https://doi.org/10.1108/IJLLS-05-2013-0024
- Clarke, D. M., Clarke, D. J., & Sullivan, P. (2012). Reasoning in the Australian curriculum: Understanding its meaning and using the relevant language. Australian Primary Mathematics Classroom, 17(3), 28-32.
- Davidson, A., Herbert, S., & Bragg, L. A. (2019). Supporting elementary teachers' planning and assessing of mathematical reasoning. International Journal of Science and Mathematics Education, 17(6), 1151-1171.
- De Bruijn, I, van der Eijk, E., Greefkens, S., Kok, D., Manintveld, J., Nannings, S., van Proosdij, Gerard, Roll, C., Schaberg, G., Schippers, M., Sinkeldam, R., Smeenk-van Broekhoven, G. (2018). Modern Mathematics. 2B vwo (12ed.). Groningen, the Netherlands: Noordhoff Uitgevers.
- de Vries, S., Verhoef, N. & Goei, S.L. (2016). Lesson Study: een praktische gids voor het onderwijs. Garant Publisher.
- Dijkhuis, J.H., Admiraal, C.J., Verbeek, J.A., de Jong, G., Houwing, H.J., Kuis, J.D., ten Klooster, F., de Waal, S.K.A., van Braak, J., Lesting-Maas, J.H.M., Wieringa, M., van Maarseveen, M.L.M., Hiele, R.D., Romkes, J.E., Haneveld, M., Voets, S., Cornelisse, I. (2016). Getal & Ruimte. 1 havo/vwo deel 1 (12ed.), Groningen, the Netherlands: Noordhooff Uitgevers.
- Dudley, P. (2019a). Lesson Study: A handbook. Cambridge University.
- Dudley, P. (2019b, August 5-8). [Paper presentation]. World Education Research Association Annual Conference, Tokyo, Japan.
- Dudley, P., Xu, H., Vermunt, J., & Lang, J. (2019). Empirical evidence of the impact of lesson study on students' achievement, teachers' professional learning and on institutional and system evolution. European Journal of Education, 54, 202-217. https://doi.org/10.1111/ejed.12337
- Educate. (2022). Enhancing differentiated instruction and cognitive activation in mathematics lessons by supporting teacher learning (EDUCATE). http://educate-platform.com/
- Edwards, D., & Mercer, N. (1987). Common knowledge. The development of understanding in the classroom. London: Methuen.
- English, L. & Gainsburg, J. (2016). Problem solving in a 21st-century mathematics curriculum. In L. D. English, & D. Kirshner (Eds.), Handbook of international research in mathematics education [3rd edition] (pp. 313-335). Routledge.
- English, L. D. (Ed.). (2013). Mathematical reasoning: Analogies, metaphors, and images. Routledge.

- Friedlander, A., & Arcavi, A. (2012). Practicing algebraic skills: A conceptual approach. Mathematics Teacher, 105(8), 608-614. http://www.jstor.org/stable/10.5951/mathteacher.105.8.0608
- Goos, M. (2004). Learning mathematics in a classroom community of inquiry. Journal of Research in Mathematics Education, 35, 258-291.
- Goos, M., Vale, C., & Stillman, G. (2017). Teaching secondary school mathematics: Research and practice for the 21st century. Allen & Unwin.
- Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In: F. Lester (Ed.), Second Handbook of Research on Mathematics Education, Information (pp. 805-842). Age Pub Inc., Greenwich.
- Herbert, S. (2021). Overcoming challenges in assessing mathematical reasoning. Australian Journal of Teacher Education (Online), 46(8), 17-30.
- Herbert, S., Vale, C., White, P., & Bragg, L. A. (2022). Engagement with a formative assessment rubric: A case of mathematical reasoning. International Journal of Educational Research, 111, 101899.
- Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). Making sense: Teaching and learning mathematics with understanding. Portsmouth, NH: Heinemann.
- Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics education, 41(2), 169-202. https://doi.org/10.2307/20720130
- Jeannotte, D., & Kieran, C. A. (2017). Conceptual model of mathematical reasoning for school mathematics. Educational Studies in Mathematics 96, 1-16. https://doi.org/10.1007/s10649-017-9761-8
- Küchemann, D. E. (2010). Using patterns generically to see structure. Pedagogies: An International Journal 5(3), 233-250.
- Lawrence, C. A., & Chong, W. H. (2010). Teacher collaborative learning through the lesson study: identifying pathways for instructional success in a Singapore high school. Asia Pacific Education Review, 11(4), 565-572. https://doi.org/10.1007/s12564-010-9103-3
- Lewis, C. (2009). What is the nature of knowledge development in lesson study? Educational Action Research, 17(1), 95-110. https://doi.org/10.1080/09650790802667477
- Llinares, S. (2013). Professional noticing: A component of the mathematics teacher's professional practice. Sisyphus-Journal of Education, 1(3), 76-93.
- Mason, J. (2002). Researching your own practice: The discipline of noticing. London: RoutledgeFalmer.
- Mata-Pereira, J., & da Ponte, J. P. (2017). Enhancing students' mathematical reasoning in the classroom: Teacher actions facilitating generalization and justification. Educational Studies in Mathematics, 96(2), 169-186.
- McCrone, S.S. (2005). The development of mathematical discussions: an investigation in a fifth-grade classroom. Mathematical Thinking and Learning, 7, 111-133.
- McDuffie, A. R., Foote, M. Q., Drake, C., Turner, E., Aguirre, J., Gau Bartell, T., & Bolson, C. (2014). Use of video analysis to support prospective K-8 teachers' noticing of equitable practices. Mathematics Teacher Educator, 2(2), 108-140.
- Mercer, N. & Dawes, L. (2008). The value of exploratory talk. In N. Mercer & S. Hodgkinson (Eds.). Exploring talk in school: Inspired by the work of Douglas Barnes (pp.55-90). SAGE Publications Ltd. https://dx.doi.org/10.4135/9781446279526.n4
- Mitchell, R. N., & Marin, K. A. (2015). Examining the use of a structured analysis framework to support prospective teacher noticing. Journal of Mathematics Teacher Education, 18(6), 551-575.
- Moyer, P. S., & Milewicz, E. (2002). Learning to question: Categories of questioning used by preservice teachers during diagnostic mathematics interviews. Journal of Mathematics Teacher Education, 5(4), 293-315.
- Mueller, M., Yankelewitz, D., & Maher, C. (2014). Teachers promoting student mathematical reasoning. Investigations in Mathematics Learning, 7(2), 10-30.
- NCTM (n.d.). Focus in high school mathematics: Reasoning and sense making. Questions and answers. https://www.nctm.org/uploadedFiles/Standards_and_Positions/Focus_in_High_School_Mathematics/FHSM_FAQs.pdf
- Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. Educational Studies in Mathematics, 102(1), 9-28. https://doi.org/10.1007/s10649-019-09903-9
- NRICH Primary Team (2014, July). Reasoning: Identifying opportunities (Article). University of Cambridge, NRICH. https://nrich.maths.org/10990

- OECD (2022). PISA 2022 mathematics framework. https://pisa2022-maths.oecd.org/
- Prediger, S., Quasthoff, U., & Heller, V. (2018). Discourse competence as important part of academic language proficiency in mathematics classrooms: The case of explaining to learn and learning to explain. Educational Studies in Mathematics, 99(2), 161-179. https://link.springer.com/article/10.1007/s10649-018-9830-7
- Robinson, V., Hohepa, M., & Lloyd, C. (2009). School leadership and student outcomes: Identifying what works and why best evidence synthesis. Auckland: New Zealand Ministry of Education
- Ruthven, K. (1989). An exploratory approach to advanced mathematics. Educational Studies in Mathematics. doi:10.1007/BF00315610
- Sahin, A., & Kulm, G. (2008). Sixth grade mathematics teachers' intentions and use of probing, guiding, and factual questions. Journal of Mathematics Teacher Education, 11(3), 221-241.
- Scherer, P. & Steinbring, H. (2006). Noticing children's learning processes – Teachers jointly reflect their own classroom interaction for improving mathematics teaching. Journal for Mathematics Teacher Education. 9(2), 157-185.
- Seleznyov, S. (2019). Lesson study beyond Japan: evaluating impact. International Journal for Lesson and Learning Studies, 8(1), 2-18. https://doi.org/10.1108/IJLLS-09-2018-0061
- Sfard, A. (2008). Thinking as communicating: Human development, the growth of discourses, and mathematizing. Cambridge University Press.
- Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. Journal of Teacher Education, 60, 20-37.
- Sowder, L. & Harel, G. (1998). Types of students' justifications. The Mathematics Teacher 91(8), 670-675.
- Stacey, K., & Vincent, J. (2009). Modes of reasoning in explanations in Australian eighth-grade mathematics textbooks. Educational Studies in Mathematics, 72(3), 271-288. http://www.jstor.org/stable/25619864
- Stigler, J. and Hiebert, J. (1999). The teaching gap: Best ideas from the world's teachers for improving education in the classroom. New York, NY: The Free Press.
- Stigler, J.W., Hiebert, J. (2016). Lesson study, improvement, and the importing of cultural routines. ZDM Mathematics Education, 48, 581-587. https://doi.org/10.1007/s11858-016-0787-7
- Stylianides, G. J. (2009). Reasoning-and-proving in school mathematics textbooks. Mathematical Thinking and Learning, 11(4), 258-288. https://doi.org/10.1080/10986060903253954
- Utrecht University. (2022). Primas Project. https://primas-project.eu/modules/modules-english/
- van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. Jacobs, & R. Philipp (Eds.), Mathematics teacher noticing: Seeing through teachers' eyes (pp. 134-151). New York: Routledge.
- van Es, E. A., & Sherin, M. G. (2010). The influence of video clubs on teachers' thinking and practice. Journal of Mathematics Teacher Education, 13(2), 155-176.
- Vermunt, J., Vrikki, M., van Halem, N., Warwick, P. & Mercer, N. (2019). The impact of Lesson Study professional development on the quality of teacher learning. Teaching and Teacher Education, 81, 61-73. https://doi.org/10.1016/j.tate.2019.02.009
- Weigand, H.-G., Schüler-Meyer, A., Pinkernell, G. (2022). Algebra in der sekundarstufe [Algebra in secondary schools]. Heidelberg: Springer.
- Willems, I., & Van den Bossche, P. (2019). Lesson Study effectiveness for teachers' professional learning: a best evidence synthesis. International Journal for Lesson and Learning Studies, 8(4), 257-271. https://doi.org/10.1108/IJLLS-04-2019-0031
- Xu, H. and Pedder, D. (2015). Lesson Study: An international review of the research. In P. Dudley (Red.), Lesson Study: Professional learning for our time (pp. 24-47). London, UK: Routledge.
- Yackel, E. & Cobb, P. (1996). Socio-mathematical norms, argumentations and autonomy in mathematics. Journal for Research in Mathematics Education, 27, 458-477.
- Ylonen, A., & Norwich, B. (2013). Professional learning of teachers through a lesson study process in England: Contexts, mechanisms and outcomes. International Journal for Lesson and Learning Studies, 2(2), 137-154. https://doi.org/10.1108/20468251311323388

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