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Reference:
Oeyen Brent, Salazar Celis Oliver.- On probability of default and its relation to observed default frequency and a common factor
The journal of credit risk - ISSN 1744-6619 - 15:3(2019), p. 41-66
Full text (Publisher's DOI): https://doi.org/10.21314/JCR.2019.253
To cite this reference: https://hdl.handle.net/10067/1629300151162165141

On Probability of Default and its relation to Observed Default Frequency and a Common Factor

Brent Oeyen*†and Oliver Salazar Celis*‡

Abstract

The paper considers a definition of Through-The-Cycle (TTC) as independent from an economic state that can result into a time-varying TTC Default Probability (PD). A Top-Down approach is proposed to transform Hybrid PDs into TTC PDs with the use of a Point-In-Time (PIT)ness parameter as an additional parameter to the Vasicek Model which expresses the dependency of a Hybrid PD on a common factor. The proposed framework aims to explain fluctuations in Observed Default Frequency (ODF) and modeled default frequency time series. A novel approach is considered that defines ODF to be analogous to an aggregated PIT PD stemming from a perfect foresight model which is not available to the modeller but can be assumed backwards in time for calibration purposes. An elaborate segmentation framework is considered to understand differences in both the Vasicek correlation and PITness parameter for a portfolio of obligors that can be applied to both retail and non-retail portfolios.

Keywords

PD calibration; Vasicek correlation; PIT; TTC; Hybrid PD; PITness

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1 Introduction and overview

A robust Probability of Default (PD) calibration framework is expected to explain fluctuations in observed and modeled default frequency. In our framework it is considered that internal ratings and PDs are of a hybrid nature: they are neither perfectly Point-In-Time (PIT) nor Through-The-Cycle (TTC) (IRTF, 2016; Aguais et al., 2008). Since financial institutions are required to follow the Vasicek (1987) model for capital calculations, past and future fluctuations are expected to be explained by both systematic and idiosyncratic movements. The former expresses default risk due to dependence on a common factor, while the latter is specific to a given credit risk profile. Hence, idiosyncratic fluctuations are considered independent from the common factor and therefore considered to be the TTC component, which leads to a congruous definition for TTC adopted in this paper. Although the debate about the statistical meaning of TTC is still ongoing (see for instance Mayer and Sauer (2017) for a recent review of various definitions), the proposed framework builds on the premise that a TTC PD is a PD that is unaffected by a common factor. Often the central tendency is used to represent the average economic condition, however in practice such an average contains many biases which are discussed further on in the paper.

First, a formula is introduced that calibrates the Vasicek formula through a regression of internal Observed Default Frequency (ODF) timeseries and an external timeseries which represent the common factor. The creation of ODF timeseries requires clustering internal data into homogeneous groups that share the same sensitivity towards a common factor. Instead of using the common rating segment approach (Kupiec, 2009), each cluster is allowed to contain a mixture of different idiosyncratic dependencies. The latter reduces the chance of having no default observations which would be the case if we were clustering low idiosyncratic risk profiles. An important part in the derivation of our formulae is that we consider ODF to be analogous to PIT PD. Such a simplification is considered useful for the calibration of Vasicek correlation and PITness. However, the simplification can only be applied backwards in time and does not allow us to construct TTC PDs at the current reporting period. Instead, only an aggregated TTC PD at the moment of the ODF calculation is available. The need to construct TTC PDs for the current reporting period on a more granular level is addressed in the third section where we assume the availability of internal Hybrid PD timeseries. Furthermore, a non-linear and linear regression between ODF and Hybrid PD timeseries is introduced. The regression calibrates the PITnes for a group of obligors sharing the same sensitivity to a scaled economic factor, i.e. the common factor scaled by Vasicek correlation. Note that the linear regression approach assumes that Vasicek correlation is already known. Therefore, we foresee in our framework that Vasicek correlation is calibrated prior to calibrating PITness.

The above techniques require segmenting a portfolio of obligors in homogeneous groups with respect to a common factor and the sensitivity of PD towards that common factor. The reasoning for the need of segments and an example of an heuristic framework are expressed through a Top-Down approach. Our approach argues for a linear regression to calibrate the Vasicek correlation and PITness parameter in order to reduce the data requirements and simplify the statistical approach to calibrate TTC PD. Although historically the credit risk exercise assumes that rating grades are available and they are the starting point to differentiate idiosyncratic risk between obligors (Aguais et al., 2008), the expectation in our framework is that in a modern setting there is less reliance on internal or external rating grades. As with recent publications of the European Banking Authority (EBA, see for instance EBA/RTS/2018/04 (2018); EBA/GL/2019/03 (2019)), it is argued for the use of risk factor models to express differences in idiosyncratic risk through credit risk drivers. In case of model development it will be easier to construct Hybrid PDs backwards in time compared to rerating obligors to collect historical internal ratings.

The methods behind the proposed calibration framework are explained throughout sections 2 and 3. Both sections assume that the Asymptotic Single-Risk Factor (ASRF) model (Vasicek, 1987; Gordy, 2003) remains a good model to define TTC PD albeit some of its assumptions such as the absence of serial correlation are considered improbable and addressed. Unlike in Kupiec (2009), Yang (2013), among others, we recognize that the TTC PD may vary over time due to changes in idiosyncratic risk, as argued by Rubtsov and Petrov (2016). Instead of using rating grades such as in Rubtsov and Petrov (2016), it is considered that TTC PD can be expressed across different dimensions and not solely the rating grade. Therefore, a TTC rating grade is not required to express the idiosyncratic risk. Instead, a Hybrid PD is transformed into a TTC PD that will be time-varying due to changes in idiosyncratic risk of a given obligor. In contrast, Rubtsov and Petrov (2016) assume the TTC component to be constant while the intervals for mapping distance to default (DD) to rating grades are assumed time-varying. In case of a time-varying TTC PD, we argue that it may be deceiving to apply Ordinary Least Squares (OLS) directly.

Section 3 focuses on the estimation of the PITness parameter in the Carlehed and Petrov (2012) framework with the same assumption of a time-varying TTC PD. Unlike the pairwise estimation proposed by Carlehed and Petrov (2012), much like in the previous section, a more stable regression approach is adopted. When using the proposed regression technique on Hybrid PDs, it becomes clear that Vasicek correlation should be estimated before determining a TTC PD. The transformation formulae from ODF and Hybrid PD to a TTC PD are illustrated in Appendix A.

In Section 4 we explain the proposed Top-Down approach in detail with the introduction of a heuristic segmentation framework. As a first step, a portfolio of obligors is divided into pools of shared dependency on a common systematic factor. These segments can be further divided by idiosyncratic elements to identify obligors' Hybrid PDs that share the same PITness. Both segments can be determined by evaluating the quality of a differenced probit regression.

Section 5 gives a practical example of how to transform an index into a common factor suited for probit regressions. In addition, the Top-Down framework's six step approach to calibrate a TTC PD is clarified for the provided example.

The concepts touched upon in this paper are further illustrated with an analysis of a simulated synthetic portfolio in Section 6. A simulation is provided where Hybrid PD is aggregated for a group of obligors; in practice, this could be useful if the PD is not modeled at the obligor level but at a higher aggregation level. Different modeling scenarios are set to point out the modeling fallacies that are often encountered when calibrating a TTC PD.

2 Vasicek correlation estimation

Our proposed Vasicek calibration technique starts with an ODF time series. ODF is considered analogous to an aggregated arithmetic average of PIT PD for a PD model with perfect foresight since ODF contains all available information of both idiosyncratic and systematic risk components. In practice no such model is available however the theoretical notion can be achieved backwards in time.

The calculation of ODF is segmentation specific and depends on how obligors are pooled. The importance of segmentation in a Top-Down PD approach is elaborately explained in Section 4. For now, the premise is that obligors are pooled in such a way that they share the same sensitivity to a single common factor Z. In case that obligor specific PDs are required, then a PD model needs to be introduced. Transforming PDs into TTC PDs is investigated in Section 3.

The long-run average of ODF is often regarded as a TTC PD. It is argued that when considered over a long period, the systematic effect averages close to zero (Aguais et al., 2008). However, defining the appropriate period of reference for calculating such an average is often challenging, e.g. multiple business cycles in the historical data can over or underestimate the average PD which is considered a biased estimate. Furthermore, the assumption of a constant TTC PD for a pool of obligors is not realistic in practice. In fact, the idiosyncratic risk of a portfolio can vary over time, e.g. due to in- and outflows of obligors or due to decisions taken by the bank, such as modifications of lending conditions or policies (FCA, 2018)[IFPRU §4.6.8].

We consider a probit regression between ODF and a common factor Z to calculate an aggregated TTC PD since we assume the relationship between the two to be constant over time, noted as Vasicek correlation ρ, and therefore more robust than calculating a time-varying TTC PD as an average ODF. The probit function, denoted as $N^{-1}(\cdot)$, is the inverse of the cumulative distribution function (CDF) of the standard normal distribution. A portfolio TTC PD $N(B_t)$ at time t is the inverse probit function of the arithmetic mean of n obligors' specific TTC PD $N(B_t^i)$ at that time, where B_t^i is often referred to in a Merton (1974) framework as the distance to default which expresses the idiosyncratic risk of a specific obligor i. Hence for a group of obligors

$$N(B_t) := \frac{\sum_{i=1}^{n} N(B_t^i)}{n}.$$

It is considered a market practice to first calibrate a TTC PD in the Pillar-1 Basel models, i.e. IRB set up, before calibrating the asset correlation that is usually calculated in a Pillar-2 Basel model, i.e. economic capital exercise. The idea that TTC PD can only be calculated after calibrating the Vasicek correlation reverses the conventional order of the credit risk exercise in the Basel capital requirements process. Hence, we emphasize the impact of the Top-Down approach on the entire Credit Risk exercise performed by financial institutions.

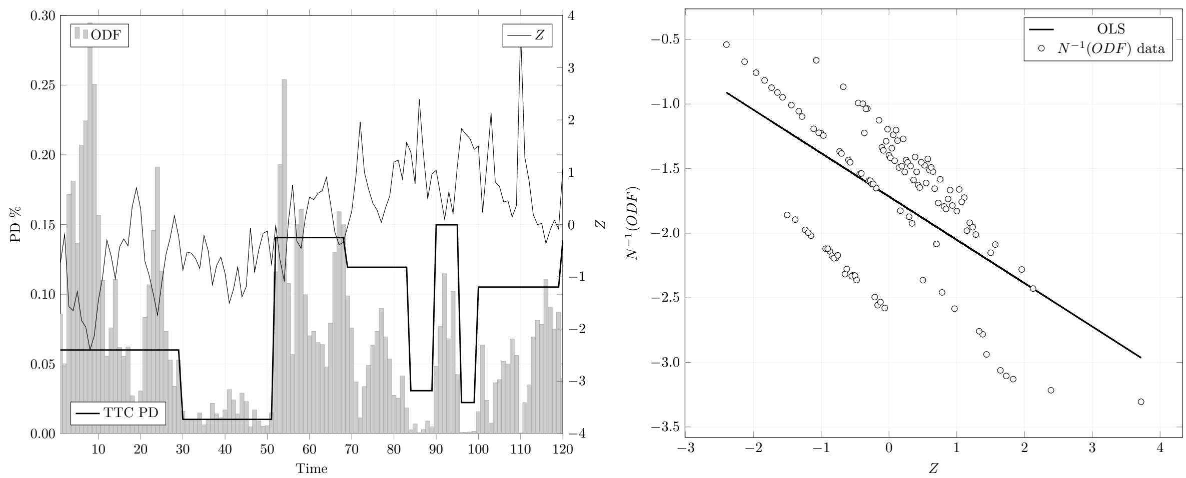
We elaborate on the work of Kupiec (2009) and Yang (2013) who rely on ordinary least squares (OLS) regression to estimate coefficients $\beta_{0,t}$ and β_1 in

$$N^{-1}(ODF_t) = \frac{B_t - Z_t\sqrt{\rho}}{\sqrt{1-\rho}} = \beta_{0,t} + \beta_1 Z_t. \tag{1}$$

From $\beta_1 = -\sqrt{\rho}/\sqrt{1-\rho}$ and $\beta_{0,t} = B_t/\sqrt{1-\rho}$ approximations for the Vasicek correlation ρ, but also TTC PD $N(B_t)$ are derived:

$$\rho \approx \beta_1^2/(1+\beta_1^2) \qquad\qquad N(B_t) \approx N(\beta_{0,t}\sqrt{1-\rho}).$$

OLS however cannot accommodate for a time-varying TTC PD. Figure 1a illustrates a time-varying TTC PD and autocorrelated common factor Z. Given the assumption that a stable relation over time between Z_t and ODF_t exists then the time dependency of B_t implies time dependent intercepts for (1). This effect is illustrated in Figure 1b. Direct application of OLS may clearly result in a poor correlation estimate.



(a) Simulation (b) Regression

Figure 1: Illustration of time-varying TTC PD and its effect on the estimated Vasicek correlation. (1a) Observed Default Frequency (ODF) and time dependent through-the-cycle (TTC) PD on the left axis. The common factor (Z) is shown on the right axis. (1b) Time dependent intercepts in the derived relation between Z and ODF cf. the regression in (1). Direct application of OLS does not capture the correct Vasicek correlation (constant slope).

We propose to circumvent this issue by considering the formal derivative of (1) with respect to time t. To that end, let us write

$$\frac{\partial}{\partial t} N^{-1}\left[ODF_t\right] = \frac{\partial}{\partial t}\beta_{0,t} + \beta_1 \frac{\partial}{\partial t} Z_t := \beta_1 \frac{\partial}{\partial t} Z_t + \varepsilon_t. \tag{2}$$

What remains in the term ε_t depends entirely on the stochastic nature of B_t. This term is quite flexible in the way it accommodates different assumptions for the stochastic dynamics of B_t. In the simplest case, B_t is constant and $\varepsilon_t = 0$ vanishes. If B_t is piece-wise constant, then $\varepsilon_t = 0$ everywhere except at a finite number of breakpoints. When B_t is a Brownian motion, then ε_t is Gaussian white noise and in case of an additional trend, then ε_t is white noise with a non-zero mean. Of course more complex stochastic processes can be envisaged for B_t, e.g. a stochastic process with discontinuous paths that would be more sensitive towards the size of the time step, but that is beyond the scope of this paper.

Approximate numerical values for the main components of (2) can be obtained from discrete difference quotients. In the simplest non-trivial case where B_t follows a Brownian motion, the slope β_1 can then be estimated from (2) and OLS without intercept as

$$\frac{\Delta N^{-1}(ODF)}{\Delta t} = \beta_1 \frac{\Delta Z}{\Delta t}. \tag{3}$$

An intercept may still be included if there is a trend in B_t. Moreover, aside from stabilizing the slope estimate β_1, differencing has the additional benefit of removing first degree autocorrelation. A linear regression with a random intercept is considered in the generalized mixed linear model framework. Saefken et al. (2014) provide a general framework on how to estimate the coefficients in case distributions differ from the normal distribution.

Once the slope β_1 is reliably estimated in this way, and thus also ρ is obtained, then the TTC PD at time t can be determined from equation (1) as

$$N(B_t) = N\left[\sqrt{\rho} Z_t + \sqrt{1-\rho}\, N^{-1}(ODF_t)\right] \tag{4}$$

A time-varying TTC PD is also considered by Rubtsov and Petrov (2016), where B_t is assumed to be normally distributed and ρ is determined through moment equations of ODF_t and a Z_t expressed through fluctuations of ODF_t. Furthermore, the authors explain how their TTC PD approach requires TTC ratings that are achieved via stochastic thresholds for the distance to default within a given rating grade. Compared to their approach, our proposed solution in (2) accounts for a much broader dynamics of B_t which can be observed and tested. In addition, the estimation of ρ is dependent on the relationship between ODF_t and an external Z_t as intended in most credit risk frameworks.

3 Point-in-Time-ness calibration

In the previous section, the effect of Z on ODF is expressed through Vasicek correlation. Within this section we expand the model with a PITness parameter in order to investigate the effect of Z on a Hybrid PD and how to obtain a TTC PD from a Hybrid PD. The type of underlying PD model for a given obligor is irrelevant at this point. However, it becomes significant for segmentation purposes as shown in Section 4.

A financial institution should understand where its models lie on the PIT/TTC spectrum (FCA, 2018)[IFPRU §4.6.3]. To that end, Carlehed and Petrov (2012) consider the following Hybrid PD model.

$$p_{\alpha,t} = N\left(\frac{B_t - \sqrt{\rho}\,\alpha Z_t}{\sqrt{1-\rho\,\alpha^2}}\right), \tag{5}$$

where $0 \leq \alpha \leq 1$ denotes the *degree of PITness* of the PD model. When $\alpha = 1$, then the PD model is truly PIT and when $\alpha = 0$ then it is truly TTC. From equation (5), one obtains the TTC PD expressed in function of the Hybrid PD as

$$N(B_t) = N\left[\sqrt{\rho}\,\alpha Z_t + \sqrt{1-\rho\,\alpha^2} N^{-1}(p_{\alpha,t})\right]. \tag{6}$$

Equations (5) and (6) can be applied at single obligor level or on an aggregated level of multiple obligors. When modelling with ODF as the perfect foresight PIT

PD then $\alpha = 1$ and we obtain the following formula for TTC PD.

$$\begin{aligned} N(B_t) &= N\left[\sqrt{\rho} Z_t + \sqrt{1-\rho}\, N^{-1}\left(p_{1,t}\right)\right] \\ &= N\left[\sqrt{\rho} Z_t + \sqrt{1-\rho}\, N^{-1}\left(ODF_t\right)\right] \end{aligned} \tag{7}$$

Here $p_{1,t}$ is a PIT PD with perfect foresight that for a pool of obligors equals the observed ODF_t as seen in Section 2. For $0 < \alpha < 1$, let $p_{\alpha,t} = Hybrid\ PD_t$ obtained from a given PD model at time t. Equating both TTC PD expressions (6) and (7) then gives

$$\sqrt{\rho} Z_t + \sqrt{1-\rho}\, N^{-1}(ODF_t) = \sqrt{\rho}\, \alpha Z_t + \sqrt{1-\rho\, \alpha^2}\, N^{-1}(Hybrid\ PD_t),$$

which relates $Hybrid\ PD_t$ and Z_t to the PIT ODF_t in a non-linear fashion with respect to α as

$$N^{-1}(ODF_t) = \frac{(\alpha-1)\sqrt{\rho} Z_t + \sqrt{1-\rho\alpha^2} N^{-1}(Hybrid\ PD_t)}{\sqrt{1-\rho}}. \tag{8}$$

Equation (8) is useful when the aim is to re-use Hybrid Basel PDs to obtain PIT PD estimates for IFRS9. It can also be used to calibrate α through a non-linear regression where Z_t and probit $Hybrid\ PD_t$ explain probit ODF_t.

However, provided that ρ has already been determined, as is the case in Section 2, we propose a much simpler approach. When ρ is known, then equation (5) readily implies

$$N^{-1}\left(Hybrid\ PD_t\right) = \frac{B_t - \sqrt{\rho}\, \alpha Z_t}{\sqrt{1-\rho\, \alpha^2}} = \gamma_{0,t} + \gamma_1 Z_t, \tag{9}$$

where $\gamma_1 = -\sqrt{\rho}\, \alpha / \sqrt{1-\rho\, \alpha^2}$ and $\gamma_{0,t} = B_t / \sqrt{1-\rho\, \alpha^2}$. Much like in equation (1), OLS can be employed to estimate values for the coefficients $\gamma_{0,t}$ and γ_1 from which approximations for α and B are derived:

$$\alpha \approx \sqrt{\gamma_1^2 / \left((1+\gamma_1^2)\rho\right)} \qquad\qquad N(B_t) \approx N(\gamma_{0,t}\sqrt{1-\rho\, \alpha^2}).$$

Of course, this approach omits the fact that the above TTC PD and the associated intercept may be time-varying. In analogy to Section 2, differencing should be applied to (9) when estimating α in this way. The latter is expected to be more stable than the pairwise approach proposed by Carlehed and Petrov (2012).

Notice that the obtained α scales the given ρ, hence α quantifies the strength of the Vasicek correlation with respect to Hybrid PD. Stated differently, α expresses how much systematic risk remains in the Hybrid PD. As a result, the selection of Z and the calibration of asset correlation and PITness can be used to transform any obligor specific Hybrid PD into a TTC PD as illustrated in appendix A.

4 Top-Down Calibration

The TTC PD estimation approach introduced in Section 2 assumes a population of homogeneous obligors such that the correlation between probit ODF and a common factor Z is constant and maximal. It is mandatory that a proper choice of Z is available. A modeler could introduce a composite Z, which weighs unique common factors, e.g. country indices, in line with the portfolio composition. However, the composition of a portfolio is not expected to be constant, hence any assumptions on portfolio composition increases model risk.

Rather than starting from a composite Z, we propose a Top-Down approach embedded with a segmentation process to identify homogeneous sub-populations of obligors with respect to systematic risk. For each segment, the ratio of defaulted obligors over a period of time over the performing population at a given snapshot t is referred to as ODF_t, i.e. the observed default rate of a homogeneous segment over a period of time.

A first segmentation level of a portfolio starts with the granularity of available common factors, e.g. using macro or micro factors. A second level of segmentation is determined by the type of factors considered that indicate to what type of cycles obligors are sensitive to, e.g. business cycles or credit cycles. Finally, aligning those common factor time series with respect to the associated ODF time series through a time lag has a significant impact as well, which is illustrated in Section 6. Lagged versions of the common factors ought to be included in the factor universe since a factor can be a leading indicator of a cycle which is expected to be influenced by the reason of default. Li and White (2009), for instance, argue that default and bankruptcy frequency are independent. In the proposed framework such an independence is translated to a requirement to split the population between default observation from a days-past-due (DPD) trigger and defaults from a bankruptcy trigger. The observed differences between the ODF time series associated to each default flag are expected to be explained by the timing of the recognition of insolvency of an obligor. After all, defaults on loans are most commonly considered to precede the request and acceptance of a bankruptcy declaration. Therefore, the observed independence is expected to be explained in part by a different lag in the associated Z time series.

After establishing different segmentation categories, the result of different probit regressions should be used to evaluate the heuristic segmentation process and the factor universe. Aside from statistical analysis, a thorough review of the underlying data and definitions is also required. A good starting point is to compare internal industry definitions to those of a common industry factor derived from external data since they may differ.

Note that segmenting the population into subsets results into smaller sample sizes and may cause periods without default observations. Such periods are not suited for probit regression. Given that the time series of both ODF and a common factor are sufficiently long, these periods, and their respective immediate neighbors which account for their appearance in 2 periods due to differencing, can be removed without impacting the quality of a regression. However, when evaluating different regressions, it is important to use test statistics that are comparable, e.g. independent from the number of observation points. Alternatively, in case there is not enough data for obtaining ODFs, we expect market practitioners to enrich internal data by adding representative external data.

In Section 3, a second probit regression is introduced where Z and the associated Vasicek correlation ρ are used to explain the variance of a Hybrid PD time series. The PITness parameter α of regression (9) measures how much of the systematic component, as expressed in regression (1), is added to a TTC PD to obtain the Hybrid PD. This PITness is diminished when Hybrid PDs are more idiosyncratic. Therefore, the segmentation of the pool of obligors is continued along this idiosyncratic dimension.

Although Hybrid PDs that remain constant during consecutive periods indicate more TTCness and less PITness from the proposed regression, having a constant value is not the definition of TTC used in the Top-Down approach; therefore, when analyzing the value of PITness the reason of a potentially low correlation factor should be understood from a methodological point of view. Hence, as a first step for the segmentation of the second regression, the source of constant Hybrid values needs to be defined. Periods of constant Hybrid PDs can occur due to risk drivers' values in the PD model that are not being updated. For instance, in the case of corporate rating models, the impact of the updating frequency of financial statements used to calculate the risk drivers can be reduced by aligning the sampling frequency of both the Z and the Hybrid PD time series to the period of reference. The frequency can be quarterly, semi-annual or annual, depending on the reporting frequency of the obligor.

A related attention point is the presence of overrides. It is considered good practice when a review of the ratings in the rating system happens at the same frequency and timing of the obligor's reporting period with a consistent override framework. Should the override framework conflict with the TTC modeling philosophy, it is advised to align the two. A good example of a conflict is an expert opinion that is based on a forward looking view of the common factor used to evaluate the PITness of the rating. Often this forward looking view is incorporated in pro-forma financial statements. The Hybrid PD, in such a case, will be translated as less PIT according to the Carlehed and Petrov (2012) framework since this will reduce the correlation between the time-series of the common factor with the Hybrid PD. A forward looking view of the common factor that is incorporated in the rating grade resulting into a lower α is not considered a weakness in the Carlehed and Petrov (2012) methodology but it is rather a bad choice of incorporating a forward looking view in a PD model. An expectation of future economic states is not an idiosyncratic risk driver and should be reflected in scenario type of PD modeling. Expert knowledge on future potential economic outcomes should be incorporated in economic scenarios e.g. as used for IFRS 9.

In case of a TTC rating framework, overrides should intend to consistently capture idiosyncratic credit risk elements that are not captured by the underlying rating model. The importance of consistency in a rating methodology is discussed in detail by Topp and Perl (2010). In case manual overrides are not consistent, it is advised to use ratings from the rating model before overrides as a segmentation variable.

Because α is related to the underlying Hybrid PD model, it depends heavily on the way that a Hybrid PD is constructed. Rating grades - given that they are the final output to a PD model - are a good starting point for the second level of segmentation. Note that rating grades are often introduced as a segmentation variable for Vasicek correlation (e.g. Kupiec (2009) and Yang (2013)), while others, such as Grundke (2008), have argued that correlation is constant but its strength is different across rating grades. We take the approach of the latter where the strength of correlation is coined PITness in the Top-Down approach.

Representing segmentation in a flow chart structure is standard for decision tree learning techniques. Although the idea of using machine learning algorithms to automate segmentation has been applied intensively to credit risk modeling (see for instance Bijak and Thomas (2012) and the reference therein), in this paper a heuristic approach is suggested. A segmentation based on a thorough review of the data and definitions is preferred since it leads to a better understanding of the segmentation results. Therefore, it is proposed to apply the flow chart, as illustrated in Figure 2, when evaluating segmentation variables. The flow chart can help to establish heuristic requirements for both retail and non-retail portfolios to have sufficient granularity in their modeling data for future TTC PD calibrations.

5 Worked example of a Top-Down Probability of Default calibration

This paper deals with other practical considerations, aside from segmentation, that introduce bias in the estimated Vasicek correlation and PITness calibration. Since Vasicek correlation implies an instantaneous and negative correlation between ODF and Z, we argue that Z should be properly derived in order to meet those assumptions. That is, in case Z is positively correlated with ODF then the sign of Z can be reversed and lags should be introduced to align the time series of ODF with Z. This alignment is expected to be reevaluated in case of a change in default definition, e.g. IFRS 9 applies a 90 days past-due default definition while IRB for sovereign exposures use a 180 day rule (EBA/GL/2016/07, 2016). When Z is a leading indicator of ODF, then from equation (4) future values of ODF can be produced without forecasting Z on the condition that a Hybrid PD can be modeled e.g. via a rating transition matrix. As a result, the leading property could reduce model risk for Lifetime Expected Credit Loss models in those cases where the forecast period is short.

In a Vasicek (1987) framework, Z is conceptually linked to a latent standard normalized asset return series. Hence, it is natural, although not required, to also start from a return series for Z. For example, macro-economic variables are often already expressed as Year on Year log changes to transform a time-series to a stationary variable that has eliminated the effect of seasonality. A common normalization is via a standard normal score[1]. Alternatively, a standard normal Z can be obtained after applying to the returns x the composite function $\left(N^{-1} \circ F\right)(x)$ entailing the probit function $N^{-1}(\cdot)$ and $F(\cdot)$ which is either the empirical CDF or a fitted CDF of x. A fitted CDF can be beneficial to extrapolate tail behavior in case of a small sample size or when very few negative returns are observed. The latter is important since negative returns are generally associated with high default rates. As only Z explains the variation of ODF, the normalization of Z will have a crucial impact on the estimation of Vasicek correlation, as shown in the subsequent example.

The impact of the chosen normalization is illustrated in Figure 3, where the returns of the Eurostoxx index are used to derive a common factor. As a starting point, the year-over-year (YoY) log-changes of the index from December 1986 until December 2014 have been calculated. The normalized common factor Z, used to simulate[2] ODF through equation (1), is derived from the returns after application of the probit function composed with the empirical CDF. The Vasicek correlation is then estimated using regression (3), once using Z and once using $\tilde{Z}$, the common factor obtained from the standard score of the returns. It is seen that the latter overestimates the Vasicek correlation because $\tilde{Z}$ consistently overestimates upturns and underestimates downturns.

[1]Carlehed and Petrov (2012) use the standard score to normalize bankruptcy frequency.
[2]ODF is simulated for the same synthetic portfolio as summarized in Table 1 of Section 6.

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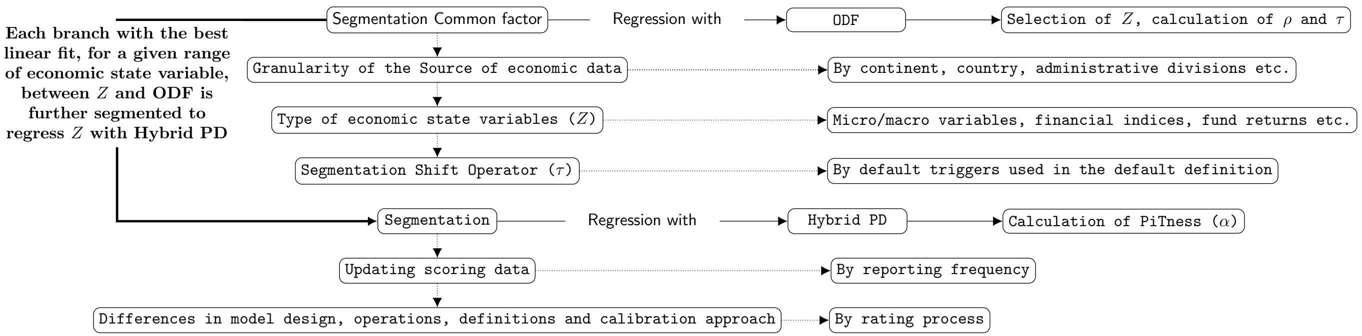


Figure 2: Proposed segmentation approach when regressing ODF and Hybrid PD with a common factor.

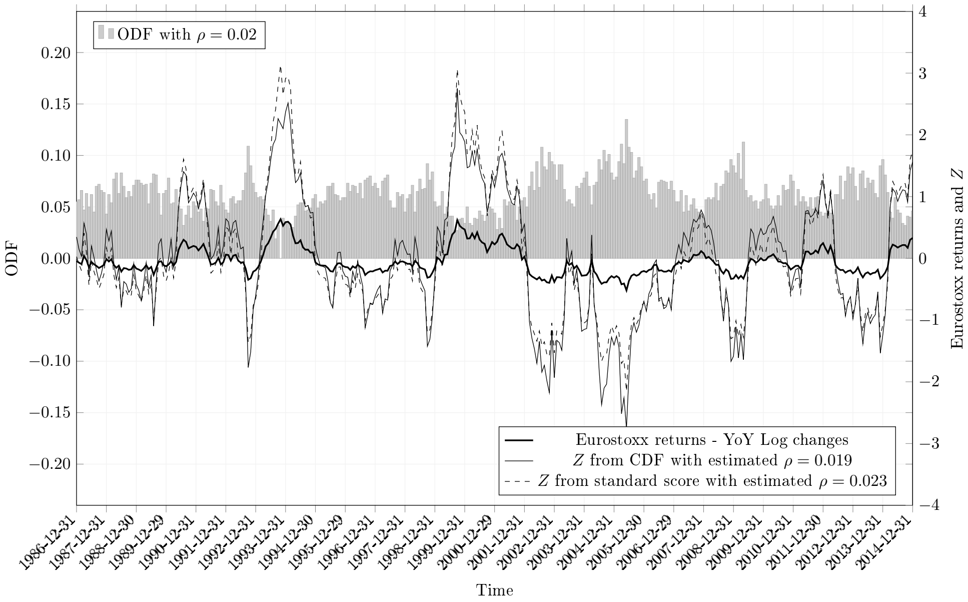


Figure 3: Calibration of ρ for a synthetic portfolio of obligors dependent on changes in the returns of Eurstoxx Index.

The above Eurostoxx example is appropriate for a pool of Euro-zone large corporate obligors. A credit risk modeler can apply the following steps for the calibration of the TTC PD which is in line with the proposed Top-Down framework:

1. The existence of external time series that include eurozone large corporate obligors, as defined by the modeler must be verified. As a result it may happen that no granularity is available, e.g. no access to individual Western-European and Eastern-European time series is available;
2. The available time series are retrieved by the modeler: for instance, the Eurostoxx index and the MSCI Large Cap Index. Each time series is normalized into a potential Z by applying to its YoY log changes the probit function composed with their empirical CDF[3];
3. The common factor universe is extended by introducing lags to each Z time series with a maximum lag of up to two years;
4. Two monthly time series of ODF are calculated from mutually exclusive default triggers, one for default events flagged by a days-past-due (DPD) trigger and the other by an unlikeliness-to-pay trigger (UTP), such as application to bankruptcy. For both time series, ODF is calculated on the total population of eurozone large corporates[4];
5. For each combination of ODF and Z time series, the Vasicek correlation is estimated using regression (3). For each default trigger, only Z associated to the best linear fit is retained, determined by the coefficient of determination;
6. For each respective default trigger, the TTC PD is determined from equation (4) using the calibrated Vasicek correlation parameter and the most recent ODF and Z observations. The final TTC PD is then the sum of the two TTC PD estimates

[3]Should include, for instance, the dot-com bubble to capture the left side of the distribution's tail.

[4]Removes the requirement to express a TTC PD for each sub-population since ODF is expressed as a percentage of the total population.

since the framework requires them to be independent time-series. This TTC PD represents the aggregated default risk of the portfolio at the period of the last ODF calculation, i.e. the probability of default regardless of the type of default.

In case a more granular PD is required and (or) a PD for the current reporting period, Hybrid PDs can be transformed to more granular TTC PDs. Hence, the PITness of Hybrid PDs needs to be determined. We may extend step 5 above by further segmenting based on reporting frequency. In case two reporting frequencies exist, quarterly and yearly, this will result in 4 segments, 2 for each Z. In case of differences in the rating process influencing the hybrid nature of the associated PDs this could be added further to the segmentation. Given the split between DPD and UTP in step 6, the PD model should share the same segmentation so that there is a Hybrid PD expressing ODF for DPD triggers and another Hybrid PD for UTP triggers. It is important to note that a default observation cannot appertain to both definitions, especially since it is expected that a UTP flag leads a DPD flag. Hence, both triggers are expected to be mutually exclusive and exhaustive. After defining all segments, the PITness α for each segment is determined based on the differenced version of regression (9) using the previously calibrated Vasicek correlation parameter and the associated Z. Analogous to the ODF example, the independence between the Hybrid PD of a DPD case and a UTP case allows us to aggregate both probabilities after transforming the Hybrid PD to its TTC component to express the TTC PD of a specific obligor.

For the interested reader, both the above and subsequent simulations are available in R code as supplementary material on the publisher's website.

6 Simulation

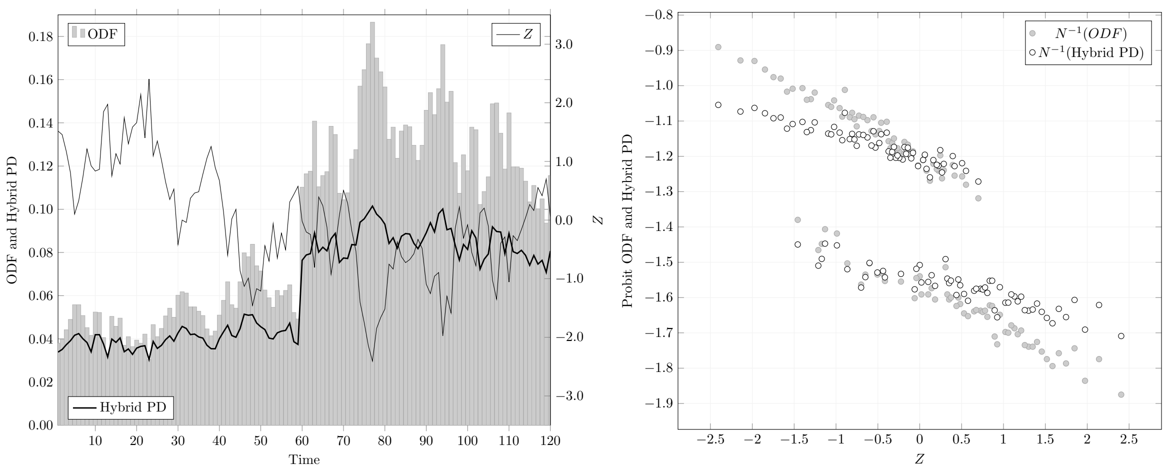
The subsequent simulation aims to introduce the reader further to the practicalities of the proposed estimation methodology with some concrete examples and illustrations.

In total 1000 different realizations of a single common factor Z are created. First, a highly autocorrelated asset return time series is constructed through a standard normal bivariate. From this time series a standard normal Z is derived from application of the probit function composed with its empirical CDF. For each Z realization, ODF and Hybrid PD values are then simulated for a portfolio of 5000 entities. To simplify the simulation, we consider the portfolio's composition to be constant and assume that at every time step an entity can be either in default or performing independent of its previous states: that is, no assumption of a recovery period is made. At time t, an obligor i then defaults when its normalized asset return value $Z_t\sqrt{\rho} + Y_t^i\sqrt{1-\rho}$ drops below its normalized default threshold B_t^i. Here ρ is given and Y_t^i is simulated through a random standard normal and represents the idiosyncratic contribution to the normalized asset return. Also B_t^i is given and is derived from the TTC PD segment to which the obligor belongs. The parameters used for the simulation are summarized in Table 1. The simulated portfolio consists of three different TTC PD segments or risk profiles having the same dependency towards Z. Hence, the portfolio TTC PD is a composite of different TTC PDs. Also, a structural break is introduced to these segments in the middle of the observation period. The TTC PD of each segment and, consequently, the portfolio TTC PD is increased by 5% (absolute) in the second half of the observation period, mimicking an increased risk profile. Finally, to generate Hybrid PD values, equation (8) is used. Hence, Z and the simulated ODF values are used directly without the need to introduce a rating framework.

Number of simulations	1000
Time period (Months)	120
Lead period of Z on ODF (Months)	3
Autocorrelation asset returns, Z	90%
Vasicek correlation (ρ)	2%
Degree of PITness (α)	50%
Number of TTC PD segments	3
TTC PD per segment*	1.2%; 5.6%; 10.0%
Entities per segments	833; 2500; 1667
Total entities of the portfolio	5000
Expected Portfolio ODF*	6.3%

Table 1: Simulation parameters. The * indicates the base case, before a structural break of +5% is added to the TTC PD per segment.

A representative relation between ODF, Hybrid PD and Z is illustrated in Figure 4a with an example of one of the common factors. The associated input data for regressions (1) and (9) is shown in Figure 4b. Note that the structural break is clearly visible for both ODF and Hybrid PD. Nevertheless, differencing is applied for each respective estimation.



(a) ODF and Hybrid PD over time (b) Probit ODF and Hybrid PD by Z

Figure 4: Illustration of a time-varying TTC PD and its effect on the estimation of the Vasicek correlation and PITness. (4a) An example of a realization of Z with simulated ODF and Hybrid PD. (4b) The structural break introduced to the ODF is clearly visible.

The graph in Figure 5 shows the simulated ODF distributions on a portfolio level per observation period with a boxplot. That is, each timestep has a boxplot that shows the distribution of portfolio ODF at that time for all of the 1000 realizations of Z. The boxplot midline is the median ODF, with the upper and lower limits of the box being the third and first quartiles respectively. The whiskers extend up to 1.5 times the interquartile range from the top (bottom) of the box. Notice that the structural break introduced in the ODF values is again very visible near the middle of the observation period.

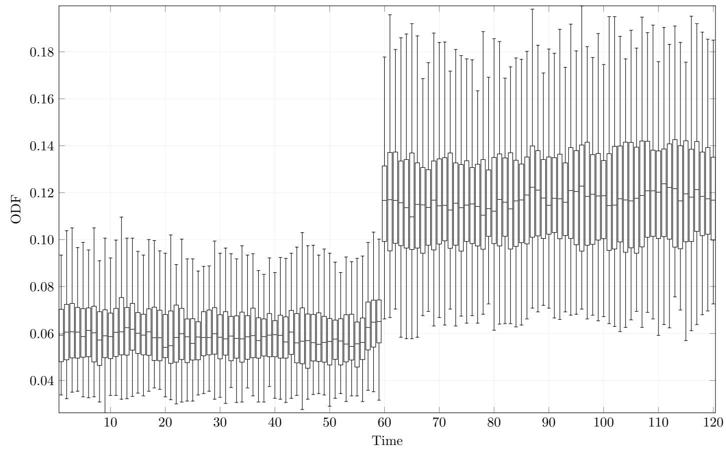


Figure 5: Boxplot of simulated portfolio ODF distributions per observation period

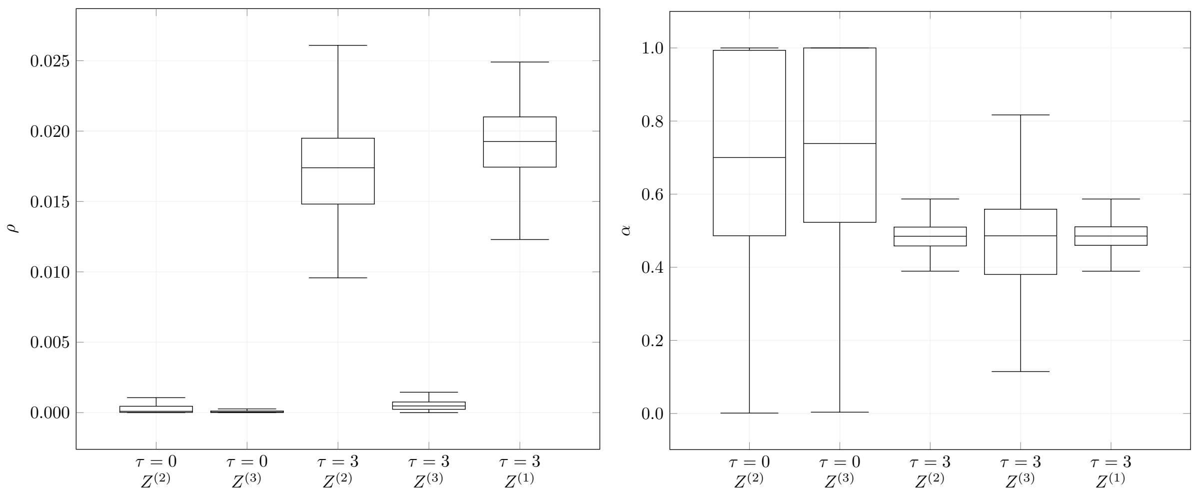
In order to evaluate the advantage of our suggested regression approach and selection criteria, five different scenarios are introduced when estimating Vasicek correlation and PITness. These scenarios are summarized in Table 2.

Sc.	The modeler uses ..
1.	A suitable common factor (denoted by $Z^{(2)}$, similar to the actual $Z^{(1)}$ except for a structural break), but does not introduce a lag ($\tau = 0$).
2.	A wrong common factor (denoted by $Z^{(3)}$ highly correlated with $Z^{(2)}$) and does not introduce a lag ($\tau = 0$).
3.	A suitable common factor (again $Z^{(2)}$) and introduces the correct lag ($\tau = 3$)
4.	A wrong common factor (again $Z^{(3)}$) but introduces the correct lag ($\tau = 3$).
5.	The actual common factor (denoted by $Z^{(1)}$, sharing the structural break with ODF) and introduces the correct lag ($\tau = 3$)

Table 2: Different simulation scenarios (Sc.).

The graph in Figure 6a shows the distribution of the estimated Vasicek correlations ρ per scenario for all simulations. From the results it is interesting to observe that identifying the correct lag ($\tau = 3$) is a necessary condition to quantify the actual Vasicek correlation of 2%. It is, of course, not a sufficient condition, because even though the wrong common factor $Z^{(3)}$ is very dependent on the suitable common factor $Z^{(2)}$ ($Z^{(3)}$ shares a correlation of 70% with $Z^{(2)}$) in scenario 4, hardly any correlation is detected using this common factor.

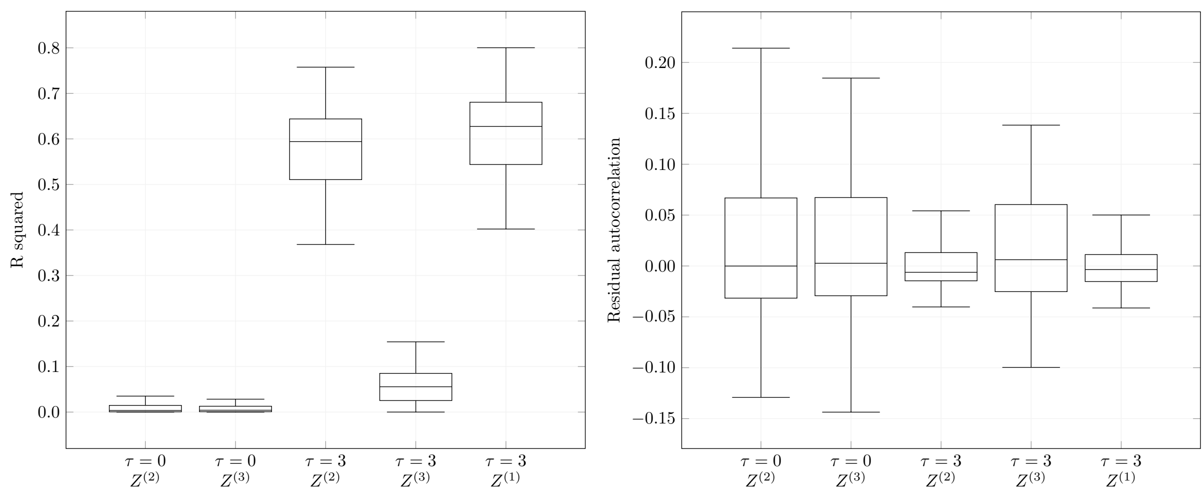
Before calculating the PITness, which relies on the estimated Vasicek correlation ρ, it is imperative to assess the quality of the initial regression. A standard way of evaluating the quality of a linear regression is by calculating the coefficient of determination or R squared, as in Eisenhauer (2003). As illustrated in Figure 7a, this statistic clearly favors the use of the correct common factor together with the correct lag. Also, the effect of a common factor that shares the same structural break with ODF (scenario 5) clearly improves the quality of the fit. Note that the statistic decreases as the actual ρ becomes smaller. Hence, when using R squared



(a) ρ (b) α

Figure 6: Boxplot of estimated ρ (Figure 6a) and α (Figure 6b) per scenario.

as an evaluation method for selecting the best fit, one also aims for the Z having the largest ρ. In our simulation, the actual scenario (number 5) was selected in the majority of cases (alternatively, scenario 3 was selected). As pointed out previously, in case no good linear fits have been found, it may be needed to revisit the data, segmentation and original selection of common factors. Equally important is to verify that the residuals are no longer autocorrelated. Autocorrelation tests should be used to determine whether OLS is appropriate. The graph in Figure 7b shows the autocorrelation of the residuals after estimating ρ from differenced time series with respect to each scenario. The simulation indicates no consistent autocorrelation in the residuals that is significant for the best fits.



(a) R squared (b) Residual autocorrelation

Figure 7: Boxplot of R squared (in 7a) and residual autocorrelation (in 7b) after estimating ρ per scenario.

Next, the PITness α is determined using the estimated Vasicek correlation. The distribution of the obtained values is shown in Figure 6b per scenario. Note that only scenarios 3 and 5 are relevant. Both result in values close to the actual α value of 50%. The other scenarios are merely shown for illustration and they would have been rejected at a previous stage because of their poor performance when regressing with ODF.

7 Conclusion

In this paper, we argue that an internal ODF and Hybrid PD time series together with a representative external time series are sufficient to calibrate a TTC PD. In our framework, a TTC PD is defined as a time-varying PD independent of a systematic factor, therefore expressing only idiosyncratic risk. A Top-Down approach to a TTC PD calibration is proposed that calibrates a TTC PD for a group of obligors sharing the same dependency towards a common factor in six steps. Note that the hypothesis of a constant dependency towards a common factor - a constant correlation parameter - is a strong assumption, albeit considered more robust than expecting TTC PD to be constant.

In order to expand the framework for obligor specifc TTC PDs and (or) current reporting period PDs, we explain how a systematic risk component can be excluded from Hybrid PD values with the introduction of a PITness parameter. A second linear regression between a differenced time series of probit Hybrid PD and a differenced time series of the selected Z of step five, used in the Vasicek correlation calibration, can derive a PITness parameter. Excluding the systematic component from Hybrid PD values is shown to be analogous to how Vasicek correlation is used to exclude the systematic risk component from ODF.

A heuristic segmentation framework is introduced since Z, Vasicek correlation and PITness of a portfolio of obligors are not expected to be jointly consistent across segments. Therefore, the importance of segmentation variables is discussed at length for the two proposed linear probit regressions. Vasicek correlation is expected to vary depending on the granularity and type of segmentation variables. Furthermore, segmentation variables influencing the lag with ODF for a given common factor should be added to the list of segmentation variables for a group of obligors before calculating ODF time series for a given segment. The segmentation continues for the second regression from which PITness can be derived from Hybrid PDs by first evaluating different sources of constant Hybrid PD values. In addition, different segmentation variables are to be included that continue separating segments according to different PITness behavior. These segmentation variables are variables that indicate different idiosyncratic elements of the PD model that influence the strength of Vasicek correlation for a Hybrid PD. The latter explains the need to introduce rating grades as a segmentation level for the second linear regression, even though it is often introduced for the derivation of Vasicek correlation.

In addition to introducing a robust calibration method and segmentation framework, practical modeling considerations are discussed. Since the framework assumes that only a common factor can explain systematic components it is important to understand the dynamics of the underlying idiosyncratic components. For instance, the frequency of the financial statements should match the frequency of the return calculations of the external time series and the overrides framework needs to be aligned with the calibration methodology of TTC PD.

Further, the common factor needs to be carefully analyzed and understood in order to be considered eligible for calibrating purposes. Therefore, to normalize the external time series a quantile transformation is proposed and in case sufficient observation points are available the use of the empirical CDF is advocated. It is shown by simulation that the coefficient of determination is an effective method to compare the fit of different regressions and explained that such a statistic will favor high Vasicek correlation estimates. Note that in the selection process of the common factor including appropriate lags is as important as selecting the proper external time series itself as shown by simulation. The framework can result into many segments and, hence, smaller samples for which periods of a zero default observation can be observed. The modeler will need to eliminate such observations from the regression. We need enough data points are needed to have statistical reliance in the proposed regression technique, especially considering that one observation point will be lost due to the proposed differencing technique to account for a time-varying TTC PD. A short discussion is included on the potential types of stochastic processes that can explain the behavior of a time dependent TTC PD. It would be insightful to explore this topic with real data before considering further its theoretical implications.

The proposed framework is straightforward to implement with statistical software as demonstrated in our simulation and has the additional advantage of aligning the credit risk exercise because of its generic structure.

Acknowledgments

The authors are grateful to Frank De Jonghe from EY for his insights and valuable comments on a draft version of this article. They also thank Alexander Petrov and Mark Rubtsov from Nordea for introducing us to their methodology as well as the constructive feedback from the anonymous referees whose comments substantially improved the presentation of this work.

Declarations of Interest

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper. This material has been prepared for general informational purposes only, and is not intended to be relied upon as accounting, tax, or other professional advice. Please, refer to your advisors for specific advice.

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A Illustration transformation formulae for ODF and Hybrid PD

The transformation formulas applied in section 2 and 3 are illustrated in Figure 8 with a time series of ODF where ODF is consistently higher then its TTC value. In such an example an average of ODF would overestimate its actual TTC parameter regardless of the period of reference.

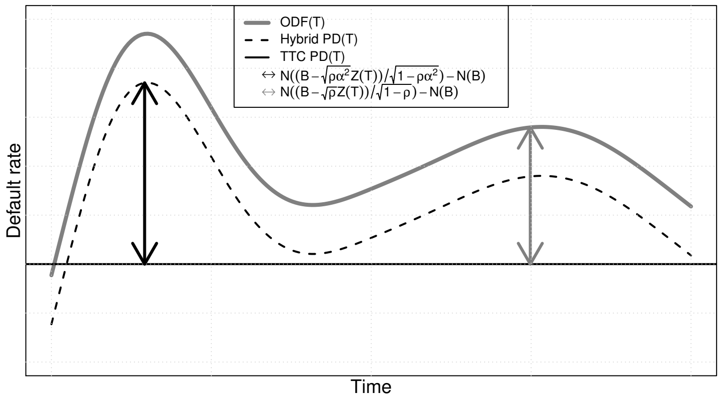

Figure 8: Illustration of TTC PD and its distance to ODF and Hybrid PD.

Figure 8 illustrates clearly that the distance from a TTC PD estimate toward a standard normal common factor can easily be expressed as a function of Hybrid PD or ODF.