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BIQUATERNION DIVISION ALGEBRAS OVER RATIONAL FUNCTION FIELDS

KARIM JOHANNES BECHER

Abstract. Let E be a field of characteristic different from 2 which is the center of a quaternion division algebra and which is not euclidean. Then there exists a biquaternion division algebra over the rational function field $E(t)$ which does not contain any quaternion algebra defined over E. The proof is based on the study of Bezoutian forms developed in [1].

Keywords: Milnor K-theory, quadratic form, valuation, ramification, Bezoutian form

Classification (MSC 2010): 12E30, 12G05, 12Y05, 19D45

1. Introduction

Let E be a field of characteristic different from 2. Let $E(t)$ denote the rational function field over E, where t is a variable. It is easy to show that there exists a quaternion division algebra over $E(t)$ if and only if E has some field extension of even degree. The purpose of this article is to provide sufficient conditions for the existence of a biquaternion division algebra over $E(t)$ which does not contain any quaternion algebra defined over E.

In [6, Corollary 5.3] it was shown that, given a quaternion division algebra Q over E and $a, b \in E^\times$ such that $a, b, ab \notin E^{\times 2}$, the biquaternion algebra

$$Q_{E(t)} \otimes (t, a(t-b))$$

over $E(t)$ is a division algebra. Hence, under very mild conditions on the ground field E we obtain biquaternion division algebras over $E(t)$. However, these examples contain a quaternion algebra which is defined over E. Examples of biquaternion algebras over $E(t)$ which do not contain any quaternion algebra defined over E were given in [7] and [4, Sections 3 and 4] for the special case where E is a non-dyadic local field.

The main result of the present article is the following stronger statement, leading to the same conclusion under very mild hypotheses on the base field E, in particular not relying on the existence of a discrete valuation on E.

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Theorem. *Assume that E is not real euclidean and that there exists a non-split quaternion algebra over E. Then there exists a biquaternion division algebra over $E(t)$ which does not contain any quaternion algebra defined over E.*

Recall that E is *real euclidean* if the set of squares in E is an ordering of E. The simplest example of a real euclidean field is the field of real numbers $\mathbb{R}$, and one easily sees that the conclusion of the theorem fails for $E = \mathbb{R}$: every quaternion algebra over $\mathbb{R}(t)$ splits over $\mathbb{C}(t)$ and therefore is of the form $(-1, f)$ for some $f \in \mathbb{R}[t]$, so every biquaternion algebra over $\mathbb{R}(t)$ has zero divisors.

As a consequence of the Theorem one obtains that the existence of a biquaternion division algebra over $E(t)$ is equivalent to the existence of a biquaternion algebra over $E(t)$ which does not contain any quaternion algebra defined over E.

The technique applied here to obtain examples of such biquaternion algebras as in the Theorem relies on the study of ramification sequences via associated Bezoutian forms developed in [1].

For $a, b \in E^\times$ consider the biquaternion algebra

$$B = \big(t^2 + (a+1)t + a, a\big) \otimes \big(t^2 + at + a, ab\big)$$

over $E(t)$. We will obtain by Theorem 4.2 that, if the E-quaternion algebra (a, b) is non-split and $ab, (a-4)b \notin E^{\times 2}$, then B is a division algebra and the ramification of B (with respect to the valuations on $E(t)$ that are trivial on E) differs from the ramification of any quaternion algebra over $E(t)$, which implies that B has Faddeev index 4 in the terminology of [4].

Note finally that the converse of the Theorem does not hold. It was shown in [2] that one can construct a field E of characteristic 0 and cohomological dimension 1 – hence in particular such that every E-quaternion algebra is split – and such that there exists an anisotropic pair (q_1, q_2) of quadratic forms in 5 variables over E; by the Amer-Brumer Theorem this implies that the 5-dimensional quadratic form $q_1 + tq_2$ over $E(t)$ is anisotropic, and since $E(t)$ has cohomological dimension 2, it follows that $q_1 + tq_2$ does not represent its determinant, so that the even Clifford algebra of $q_1 + tq_2$ is a biquaternion division algebra over $E(t)$.

2. Preliminaries

For an E-algebra A and a field extension F/E, we denote by A_F the F-algebra $A \otimes_E F$. Recall that an E-algebra A is *central simple* if and only if $A_F \simeq \mathbb{M}_n(F)$ for some field extension F/E and a positive integer n; we say that A is *split* if one can take $F = E$, that is if $A \simeq \mathbb{M}_n(E)$. Any central simple algebra is finite-dimensional and in particular it either has zero divisors or it is a division algebra. An *E-quaternion algebra* is a 4-dimensional central simple E-algebra.

Recall that E is assumed to have characteristic different from 2. For $a, b \in E^\times$ an E-quaternion algebra denoted $(a, b)_E$ or just (a, b) is obtained by endowing the vector space

$$E \oplus Ei \oplus Ej \oplus Ek$$

with the multiplication given by the rules $i^2 = a, j^2 = b$ and $ij = k = -ji$. Any E-quaternion algebra is isomorphic to $(a,b)_E$ for certain $a,b \in E^\times$. Any quaternion algebra is either split or a division algebra.

An *E-biquaternion algebra* is an E-algebra which is isomorphic to $Q \otimes_E Q'$ for two E-quaternion algebras Q and Q'. In particular, biquaternion algebras are central simple. Given an E-biquaternion algebra B and an E-quaternion subalgebra Q of B, we can decompose $B \simeq Q \otimes_E Q'$ with the E-quaternion algebra Q' given as the centralizer of Q in B.

For our analysis of quaternion and biquaternion algebras over E and over $E(t)$, we will work in the second Milnor K-group modulo 2 of a field.

For $n \in \mathbb{N}$ we denote by $\mathsf{k}_n E$ the nth Milnor K-group of E modulo 2; this is the abelian group generated by symbols $\{a_1, \ldots, a_n\}$ with $a_1, \ldots, a_n \in E^\times$ which are subjected to the defining relations that the map $(E^\times)^n \to \mathsf{k}_n E$ given by $(a_1, \ldots, a_n) \mapsto \{a_1, \ldots, a_n\}$ is multilinear and further that $\{a_1, \ldots, a_n\} = 0$ whenever $a_i \in E^{\times 2}$ for some $i \leqslant n$ or $a_i + a_{i+1} = 1$ for some $i < n$. Note that $\mathsf{k}_1 E \simeq E^\times / E^{\times 2}$. Here we only consider $\mathsf{k}_n E$ (and $\mathsf{k}_n E(t)$) for $n = 1, 2$. The group $\mathsf{k}_2 E$ is in tight relation to the Brauer group.

We denote by $\mathsf{Br}(E)$ the Brauer group of E and by $\mathsf{Br}_2(E)$ its 2-torsion part. Recall that there is a unique homomorphism

$$\mathsf{k}_2 E \to \mathsf{Br}_2(E)$$

that sends any symbol $\{a,b\}$ with $a,b \in E^\times$ to the Brauer equivalence class of the E-quaternion algebra $(a,b)_E$. Merkurjev's Theorem asserts that this is in fact an isomorphism. We only need special instances of this fact. For $a,b \in E^\times$ we have $\{a,b\} = 0$ in $\mathsf{k}_2 E$ if and only if the E-quaternion algebra $(a,b)_E$ is split. Furthermore, for $a,b,c,d \in E^\times$ the E-biquaternion algebra $(a,b) \otimes_E (c,d)$ has zero divisors if and only if $\{a,b\} + \{c,d\} = \{e,f\}$ for certain $e,f \in E^\times$. These two facts can be proven by elementary means, without using Merkurjev's Theorem.

For the study of $\mathsf{k}_2 E(t)$ one uses an exact sequence. To explain it we first need to define the tame symbol map ∂_v with respect to a $\mathbb{Z}$-valuation v.

Let F be a field. By a *$\mathbb{Z}$-valuation* on F we mean a valuation with value group $\mathbb{Z}$. Given a $\mathbb{Z}$-valuation v on F we denote by $\mathcal{O}_v$ its valuation ring and by κ_v its residue field. For $a \in \mathcal{O}_v$ let $\overline{a}$ denote the natural image of a in κ_v. By [5, (2.1)], for a $\mathbb{Z}$-valuation v on F, there is a unique homomorphism $\partial_v : \mathsf{k}_2 F \to \mathsf{k}_1 \kappa_v$ such that

$$\partial_v(\{f,g\}) = v(f) \cdot \{\overline{g}\} \text{ in } \mathsf{k}_1 \kappa_v$$

for $f \in F^\times$ and $g \in \mathcal{O}_v^\times$. For $f,g \in F^\times$ we obtain that $f^{-v(g)} g^{v(f)} \in \mathcal{O}_v^\times$ and

$$\partial_v(\{f,g\}) = \left\{ (-1)^{v(f)v(g)} \overline{f^{-v(g)} g^{v(f)}} \right\} \text{ in } \mathsf{k}_1 \kappa_v \,.$$

We turn to the case where $F = E(t)$. Let $\mathcal{P}$ denote the set of monic irreducible polynomials in $E[t]$. Any $p \in \mathcal{P}$ determines a $\mathbb{Z}$-valuation v_p on $E(t)$ which is trivial on E and with $v_p(p) = 1$. There is furthermore a unique $\mathbb{Z}$-valuation v_∞

on $E(t)$ which is trivial on E and such that $v_\infty(t) = -1$. We set $\mathcal{P}' = \mathcal{P} \cup \{\infty\}$. For $p \in \mathcal{P}'$ we write ∂_p for ∂_{v_p} and we denote by E_p the residue field of v_p. Note that E_p is naturally isomorphic to $E[t]/(p)$ for $p \in \mathcal{P}$ and that E_∞ is naturally isomorphic to E. We call

$$\partial = \bigoplus_{p\in\mathcal{P}'} \partial_p : \mathsf{k}_2 E(t) \to \bigoplus_{p\in\mathcal{P}'} \mathsf{k}_1 E_p$$

the *ramification map*. For $p \in \mathcal{P}'$, the norm map of the finite extension E_p/E yields a group homomorphism $\mathsf{k}_1 E_p \to \mathsf{k}_1 E$. Summation over these maps for all $p \in \mathcal{P}'$ yields a homomorphism

$$\mathsf{N} : \bigoplus_{p\in\mathcal{P}'} \mathsf{k}_1 E_p \to \mathsf{k}_1 E\,.$$

By [3, (7.2.4) and (7.2.5)] we obtain an exact sequence

$$0 \to \mathsf{k}_2 E \to \mathsf{k}_2 E(t) \xrightarrow{\partial} \bigoplus_{p\in\mathcal{P}'} \mathsf{k}_1 E_p \xrightarrow{\mathsf{N}} \mathsf{k}_1 E \to 0\,. \tag{2.1}$$

Let $\mathfrak{R}_2(E)$ denote the kernel of N, which is equal to the image of ∂. The elements of $\mathfrak{R}_2(E)$ are called *ramification sequences*.

For a finite set $S \subseteq \mathcal{P}'$ we call $\sum_{p\in S}[E_p : E]$ the *degree of* S and denote it by $\mathsf{deg}(S)$. For $\rho = (\rho_p)_{p\in\mathcal{P}'} \in \bigoplus_{p\in\mathcal{P}'} \mathsf{k}_1 E_p$ we set $\mathsf{Supp}(\rho) = \{p \in \mathcal{P}' \mid \rho_p \neq 0\}$ and abbreviate $\mathsf{deg}(\rho) = \mathsf{deg}(\mathsf{Supp}(\rho))$, and we call this the *support* and the *degree of* ρ. We say that $\rho \in \mathfrak{R}_2(E)$ *is represented by* $\xi \in \mathsf{k}_2 E(t)$ if $\partial(\xi) = \rho$.

3. Bezoutians

We use standard terminology from quadratic form theory. For $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in E^\times$ we denote the n-fold Pfister form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ by $\langle\langle a_1, \ldots, a_n\rangle\rangle$. The Witt ring of E is denoted by $\mathsf{W}E$. For a nondegenerate quadratic form φ over E we denote by $[\varphi]$ its class in $\mathsf{W}E$ and we set $c \cdot [\varphi] = [c\varphi]$ for $c \in E^\times$. For $c \in E^\times$ we abbreviate $[c] = [\langle c\rangle]$.

Consider a square-free polynomial $g \in E[t]$. We set $E_g = E[t]/(g)$ and denote by θ the class of t in E_g. For $n = \mathsf{deg}(g)$, let $s_g : E_g \to E$ be the E-linear form such that $s_g(\theta^i) = 0$ for $0 \leqslant i \leqslant n-2$ and $s_g(\theta^{n-1}) = 1$. By [1, Proposition 3.1], any $f \in E[t]$ coprime to g gives rise to a nondegenerate quadratic form over E given by

$$q : E_g \to E, x \mapsto s_g(f(\theta)x^2)\,,$$

which is called the *Bezoutian of* f *modulo* g. We denote by

$$\mathfrak{B}\left(\frac{f}{g}\right)$$

the class in $\mathsf{W}E$ given by the Bezoutian of f modulo g.

Bezoutians satisfy some useful computation rules. First of all, it follows from the definition that, for $f,h \in E[X]$ coprime to g such that h is a square modulo g, we have

$$\mathfrak{B}\left(\frac{fh}{g}\right) = \mathfrak{B}\left(\frac{f}{g}\right).$$

3.1. **Proposition.** *For $f, g_1, g_2 \in E[t]$ pairwise coprime and with g_1 and g_2 monic and square-free, we have*

$$\mathfrak{B}\left(\frac{f}{g_1g_2}\right) = \mathfrak{B}\left(\frac{fg_2}{g_1}\right) + \mathfrak{B}\left(\frac{fg_1}{g_2}\right).$$

Proof. See [1, Proposition 3.5]. □

3.2. **Theorem.** *Let $f, g \in E[t]$ be monic, square-free and coprime. Then*

$$\mathfrak{B}\left(\frac{f}{g}\right) + \mathfrak{B}\left(\frac{g}{f}\right) = \begin{cases} 0 & \textit{if } \mathsf{deg}(f) \equiv \mathsf{deg}(g) \bmod 2, \\ [1] & \textit{if } \mathsf{deg}(f) \not\equiv \mathsf{deg}(g) \bmod 2. \end{cases}$$

Proof. See [1, Theorem 3.8]. □

These two rules will be used without explicit mention in the sequel.

3.3. **Lemma.** *Let $g_1, g_2 \in E[t]$ monic of even degree, coprime and such that g_1t is a square modulo g_2. Let $a_1, a_2 \in E^\times$ and $f \in E[t]$ such that a_if is a square modulo g_i for $i = 1,2$. Then*

$$\mathfrak{B}\left(\frac{f}{g_1g_2}\right) = [a_1\langle\langle a_1a_2, g_2(0)\rangle\rangle].$$

Proof. We have that

$$\mathfrak{B}\left(\frac{f}{g_1g_2}\right) = \mathfrak{B}\left(\frac{fg_2}{g_1}\right) + \mathfrak{B}\left(\frac{fg_1}{g_2}\right).$$

As a_if is a square modulo g_i for $i = 1, 2$, we have that

$$\mathfrak{B}\left(\frac{fg_1}{g_2}\right) = a_1\mathfrak{B}\left(\frac{g_1}{g_2}\right) \quad \text{and} \quad \mathfrak{B}\left(\frac{fg_2}{g_1}\right) = a_2\mathfrak{B}\left(\frac{g_2}{g_1}\right) = -a_2\mathfrak{B}\left(\frac{g_1}{g_2}\right).$$

As g_1t is a square modulo g_2, we obtain that

$$\mathfrak{B}\left(\frac{g_1}{g_2}\right) = \mathfrak{B}\left(\frac{t}{g_2}\right) = [1] - \mathfrak{B}\left(\frac{g_2}{t}\right) = [1] - [g_2(0)] = [\langle\langle g_2(0)\rangle\rangle].$$

Hence

$$\mathfrak{B}\left(\frac{f}{g_1g_2}\right) = -a_2[\langle\langle g_2(0)\rangle\rangle] + a_1[\langle\langle g_2(0)\rangle\rangle].$$

Since $\langle -a_2, a_1\rangle$ is isometric to $a_1\langle\langle a_1a_2\rangle\rangle$, the statement follows. □

4. Ramification sequences not representable by one symbol

In [1, Section 5] Bezoutians are related to ramification sequences and it is shown in [1, Theorem 5.12] that a non-trivial Bezoutian can present an obstruction for the representability of a ramification sequence by a single symbol. This will be used here to obtain ramification sequences of degree 4 that do not correspond to a symbol.

4.1. **Proposition.** *Let $g_1, g_2 \in E[t]$ monic of even degree, coprime and such that $g_1 t$ is a square modulo g_2. Let $a_1, a_2 \in E^\times$ be such that the quadratic form $\langle 1, -a_1a_2\rangle$ over E does not represent $g_2(0)$ and for $i = 1, 2$ one has $a_i \notin E_p^{\times 2}$ for any irreducible factor p of g_i. Then*

$$\partial\left(\{g_1, a_1\} + \{g_2, a_2\}\right) \neq \partial\left(\sigma\right)$$

for any symbol σ in $\mathsf{k}_2E(t)$.

Proof. We set $\rho = \partial\left(\{g_1, a_1\} + \{g_2, a_2\}\right)$ in $\mathfrak{R}_2(E)$. Note that $\rho_\infty = 0$ and $\mathsf{Supp}(\rho) = \{p \in \mathcal{P} \mid p \text{ divides } g_1g_2\}$. In particular $\mathsf{deg}(\rho) = \mathsf{deg}(g_1) + \mathsf{deg}(g_2)$, which is even. Suppose there exists a symbol σ in $\mathsf{k}_2E(t)$ with $\partial(\sigma) = \rho$. It follows by [1, Proposition 4.1] that there exist $f, g, h \in E[t]$ square-free and pairwise coprime with $g = g_1g_2$ and $\sigma = \{f, gh\}$. Note that $\partial_\infty(\{f, gh\}) = \rho_\infty = 0$. By [1, Lemma 4.2] we obtain that

$$\mathfrak{B}\left(\frac{f}{g}\right) = 0\,.$$

For $i = 1, 2$, we obtain for any $p \in \mathcal{P}$ dividing g_i that $\{a_i\} = \rho_p = \partial_p(\sigma) = \{\overline{f}\}$ in k_1E_p. Hence a_if is a square modulo g_i for $i = 1, 2$. By Lemma 3.3, it follows that

$$[a_1\langle\langle a_1a_2, g_2(0)\rangle\rangle] = \mathfrak{B}\left(\frac{f}{g}\right) = 0\,.$$

Thus $\langle\langle a_1a_2, g_2(0)\rangle\rangle$ is hyperbolic. Hence $g_2(0)$ is represented over E by the quadratic form $\langle 1, -a_1a_2\rangle$, which contradicts the hypothesis. □

We are ready to prove the statements in terms of symbols which were formulated in the introduction in terms of quaternion algebras. The translation of these results to quaternion algebras is immediate, using only the fact that the ramification map $\partial : \mathsf{k}_2E(t) \to \bigoplus_{p\in\mathcal{P}'} \mathsf{k}_1E_p$ factors over the natural homomorphism $\mathsf{k}_2E(t) \to \mathsf{Br}_2(E(t))$.

4.2. **Theorem.** *Let $a, b \in E^\times$ with $a \neq 4$ and $a, ab, (a-4)b \notin E^{\times 2}$. Then the following are equivalent:*

(*i*) *$\{a, b\} = 0$ in k_2E.*
(*ii*) *There exists a symbol $\sigma \in \mathsf{k}_2E(t)$ with*

$$\partial\left(\left\{t^2 + (a+1)t + a, a\right\} + \left\{t^2 + at + a, ab\right\}\right) = \partial(\sigma)\,.$$

Proof. Set $g_1 = t^2 + (a+1)t + a$, $g_2 = t^2 + at + a$. The polynomials g_1 and g_2 are coprime, and we have $g_2(0) = a$ and $g_1 t \equiv t^2 \bmod g_2$. The discriminant of g_2 is $a(a-4)$. The hypothesis implies that a different from 0 and 4 and that $ab \notin E^{\times 2} \cup a(a-4)E^{\times 2}$. Hence g_2 is separable and ab is a non-square modulo any irreducible factor of g_2. Moreover a is a non-square modulo the two irreducible factors of $g_1 = (t+1)(t+a)$. Set $\rho = \partial(\{g_1, a\} + \{g_2, ab\})$. We obtain that $\mathsf{Supp}(\rho) = \{p \in \mathcal{P} \mid p \text{ divides } g_1 g_2\}$ and $\mathsf{deg}(\rho) = 4$.

If $\{a, b\} \neq 0$, then $\langle 1, -b\rangle$ does not represent $a = g_2(0)$, and we conclude by Proposition 4.1 that $\rho \neq \partial(\sigma)$ for any symbol σ in $\mathsf{k}_2 E(t)$.

Assume now that $\{a, b\} = 0$. Then $\langle\langle a, b\rangle\rangle$ is hyperbolic. We choose $f \in E[t]$ such that $f \equiv a \bmod g_1$ and $f \equiv ab \bmod g_2$. By Lemma 3.3, then $\mathfrak{B}\left(\frac{f}{g_1 g_2}\right) = 0$. By [1, Theorem 6.1] this implies that $\rho = \partial(\sigma)$ for a symbol σ in $\mathsf{k}_2 E(t)$. $\square$

In order to apply Theorem 4.2, we need to be able to satisfy its hypotheses. Recall that the field E is *pythagorean* if every sum of squares in E is a square.

4.3. **Lemma.** *Let σ be a nonzero symbol in $\mathsf{k}_2 E$. Then either $\sigma = \{-1, -1\}$ and E is real pythagorean, or $\sigma = \{a, b\}$ for certain $a, b \in E^\times$ with $a \neq 4$ and $a, ab, (a-4)b \notin E^{\times 2}$.*

Proof. Suppose first that $\sigma = \{-1, x\}$ for some $x \in E^\times$. Since $\sigma \neq 0$, it follows that $-1, x \notin E^{\times 2}$. Set $a = -\frac{9}{4}$. Then $-a, 4 - a \in E^{\times 2}$ and $\sigma = \{a, x\}$. Hence, if $-x \notin E^{\times 2}$, then we choose $b = x$ to satisfy the claim. Assume now that $-x \in E^{\times 2}$. Then $\sigma = \{-1, -1\}$. If E is not pythagorean, then we can choose $b \in E^\times$ such that $-b$ is a sum of two squares but not a square in E and obtain that $\sigma = \{-1, -1\} = \{-1, b\} = \{a, b\}$. If E is pythagorean, then as $-1 \notin E^{\times 2}$, it follows that E is real.

Suppose now that $\sigma \neq \{-1, x\}$ for any $x \in E^\times$. In this case we take any representation $\sigma = \{a, b\}$ with $a, b \in E^\times$. Clearly $a \notin E^{\times 2}$, and furthermore $\sigma = \{-ab, b\} \neq \{-1, b\}$, whereby $ab \notin E^{\times 2}$. Finally $\{a, b\} = \sigma \neq \{-1, a\}$, whence $\{ab, -b\} = \{a, -b\} \neq 0$, whereby $(a-4)b$ is not a square. $\square$

Note that $\mathsf{k}_2 E = 0$ if and only if every E-quaternion algebra is split. Hence the next statement covers the Theorem in the introduction.

4.4. **Theorem.** *Assume that $\mathsf{k}_2 E \neq 0$ and that E is not euclidean. Then the following hold:*

(a) *There exists $\rho \in \mathfrak{R}_2(E)$ with $\mathsf{deg}(\rho) = 4$ and such that $\rho \neq \partial(\sigma)$ for every symbol σ in $\mathsf{k}_2 E(t)$.*
(b) *There exists an $E(t)$-biquaternion division algebra B such that $B \otimes_{E(t)} Q$ is not defined over E for any $E(t)$-quaternion algebra Q. In particular, B does not contain any E-quaternion algebra.*

Proof. Note first that, if E is real pythagorean but not euclidean, then there exists an element $c \in E^\times \setminus (E^{\times 2} \cup -E^{\times 2})$, and then $\{-1, c\}$ is a nonzero symbol in $\mathsf{k}_2 E$ different from $\{-1,-1\}$. Hence, by hypothesis and by Lemma 4.3, there exist $a, b \in E^\times$ with $a \notin E^{\times 2}$, $b \notin aE^{\times 2} \cup (a-4)E^{\times 2}$ and such that $\{a, b\} \neq 0$ in $\mathsf{k}_2 E$. By Theorem 4.2, the ramification sequence

$$\rho = \partial(\{t^2 + (a+1)t + a, a\} + \{t^2 + at + a, ab\})$$

then satisfies the claim in (a).

To show (b), we consider the corresponding $E(t)$-biquaternion algebra

$$B = (t^2 + (a+1)t + a, a) \otimes_{E(t)} (t^2 + at + a, ab)\,.$$

For any $f, g \in E(t)^\times$ such that $B \otimes_{E(t)} (f, g)$ can be defined over E, we obtain that $\rho = \partial(\{f, g\})$, in contradiction to (a). Therefore there exists no $E(t)$-quaternion algebra Q such that $B \otimes_{E(t)} Q$ can be defined over E. In particular, B does not contain any E-quaternion algebra Q', as otherwise the centraliser of $Q'_{E(t)}$ in B is an $E(t)$-quaternion algebra Q such that $B \otimes_{E(t)} Q$ is defined over E. In particular B does not contain $\mathbb{M}_2(E)$. Hence B is a division algebra. $\square$

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