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AN ISOMORPHISM OF THE WALLMAN AND ČECH-STONE COMPACTIFICATIONS.

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Abstract

For a metrizable topological space X it is well known that in general the Čech-Stone compactification $\beta(X)$ or the Wallman compactification $W(X)$ are not metrizable. To remedy this fact one can alternatively associate a point-set distance to the metric, a so called approach distance. It is known that in this setting both a Čech-Stone compactification $\beta^*(X)$ and a Wallman compactification $W^*(X)$ can be constructed in such a way that their approach distances induce the original approach distance of the metric on X [23], [24].

The main goal in this paper is to formulate necessary and sufficient conditions for an approach space X such that the Čech-Stone compactification $\beta^*(X)$ and the Wallman compactification $W^*(X)$ are isomorphic, thus answering a question first raised in [24]. The first clue to reach this goal is to settle a question left open in [10], to formulate sufficient conditions for a compact approach space to be normal. In particular the result shows that the Čech-Stone compactification $\beta^*(X)$ of a uniform T_2 space, is always normal. We prove that the Wallman compactification $W^*(X)$ is normal if and only if X is normal, and we produce an example showing that, unlike for topological spaces, in the approach setting normality of X is not sufficient for $\beta^*(X)$ and $W^*(X)$ to be isomorphic. We introduce a strengthening of the regularity condition on X, which we call ideal-regularity, and in our main theorem we conclude that X is ideal-regular, normal and T_1 if and only if X is a uniform T_1 approach space with $\beta^*(X)$ and $W^*(X)$ isomorphic. Classical topological results are recovered and implications for (quasi-)metric spaces are investigated.

Keywords: Approach space, Čech-Stone compactification, Wallman compactification, regularity, normality.
Mathematics Subject Classification: 54A05, 54C20, 54D30, 54D35, 54E25, 54E35.

1. Introduction

A metrizable topological space X is normal, meaning that for any A, B closed subsets, there is a continuous Urysohn map separating A and B in the sense that $f(a) = 1$ for $a \in A$ and $f(b) = 0$ for $b \in B$. Equivalently, by the Katětov-Tong insertion theorem [16], [17] and [28], normality means that for any ρ, η bounded and lower (respectively upper) semicontinuous, with $\eta \leq \rho$, there exists a realvalued continuous map f satisfying $\eta \leq f \leq \rho$.

For some applications in analysis, like for instance the theory of differential equations or fixed point theory, metric spaces with Lipschitz type functions or non-expansive maps are more natural than continuous maps. Such isometric settings get more and more attention like for instance in the study of approximation by Lipschitz functions in [13], of cofinal completeness and the UC-property in [2], in investigations on hyperconvexity in [19] and on the non-symmetric analogue of the Urysohn metric space in [20] and [21]. For other applications the larger context of approach spaces with contractions is even more suitable as was recently shown in the context of probability measures [3], [4] and [5], or complexity analysis [7] and [8].

In [10] normality for topological spaces was extended to approach spaces in terms of point-set distances and with contractions instead of continuous maps. Topological spaces are special approach spaces, when the distance is interpreted via the closure and so are quasi-pseudometric spaces, when the distance is $\delta_q(x, A) = \inf_{a\in A} q(x, a)$ for a quasi-pseudometric q.

Applied to a quasi-pseudometric (approach) space (X, δ_q), normality means that when two subsets A, B are γ-separated (in the sense that $A^{(\alpha)} \cap B^{(\beta)} = \emptyset$, whenever $\alpha \geq 0, \beta \geq 0, \alpha + \beta < \gamma$), then there exists a non-expansive Urysohn map $f : (X, q) \to ([0, \gamma], d_{\mathbb{E}})$, with $d_{\mathbb{E}}$ the Euclidean metric, satisfying $f(a) = \gamma$ for $a \in A$ and $f(b) = 0$ for $b \in B$. Here $A^{(\alpha)}$ is defined in terms of the distance from points to A by $A^{(\alpha)} = \{x \in X | \delta_q(x, A) \leq \alpha\}$. Equivalently normality can be expressed by a Katětov-Tong insertion theorem. A bounded realvalued map ρ on (X, δ_q) is lower regular if $(\rho(x) - \rho(y)) \vee 0 \leq q(x, y)$ for any $x, y \in X$ and a realvalued map η on (X, q) is called upper regular if $(\eta(y) - \eta(x)) \vee 0 \leq q(x, y)$ for any $x, y \in X$. Normality equivalently means that for any ρ, η bounded lower (respectively upper) regular, with $\eta \leq \rho$, there exists a realvalued non-expansive map $f : (X, q) \to ([0, \infty], d_{\mathbb{E}})$ satisfying $\eta \leq f \leq \rho$. In [11] it was shown that this notion of normality for quasi-metric spaces coincides with the monoidal version of normality as introduced in Chapter V of [14].

A compact T_2 topological space is known to be normal. In the context of point-set distances, a quasi-pseudometric space (X, δ_q) with a compact T_2 underlying topology need not be normal. A counterexample was presented in [10]. It was also shown that a pseudometric space (regardless of the compactness and separation of the underlying topology) is always normal, but that a normal quasi-pseudometric space need not be pseudometric. In the general setting of approach spaces and contractions, uniform approach spaces are obtained as subspaces of products of pseudometric spaces. The natural question arises whether a compact T_2 uniform approach space (meaning it is a uniform approach space which has a compact T_2 underlying topology) is normal. In [10] a genuine approach example was produced of a compact T_2 uniform approach space that is neither topological nor pseudometric, but is normal. The general question however was not settled. In this paper, in section 3, Theorem 3.8 we produce a positive answer to the question by proving that all compact T_2 uniform approach spaces are normal. The result is obtained by first proving several alternative characterisations of normality. The result of Theorem 3.8 as well as the alternative characterisations of normality it depends on, are crucial for the rest of the paper.

For a metrizable topological space X it is well known that in general the Čech-Stone compactification $\beta(X)$ or the Wallman compactification $W(X)$ are not metrizable. Working in the setting of approach spaces remedies this fact. It is known that in this broader setting both a Čech-Stone compactification $\beta^*(X)$ and a Wallman compactification $W^*(X)$ can be constructed in such a way that their approach distance induces the original approach distance of the given metric on X [23], [24]. Section 4 and the following ones, are a contribution to the compactification theory for approach spaces. For T_2 uniform approach spaces, the Čech-Stone compactification is the reflector β^* from the category UApp_2 of all T_2 uniform approach spaces to the category kUApp_2 of compact T_2 uniform approach spaces [23]. The easiest way to construct the compactification $\beta^*(X)$, given a uniform T_2 approach space X, is described in (3.3). If X is topological, then $\beta^*(X)$ is isomorphic to the topological Čech-Stone compactification $\beta(X)$. In general, for a pseudometric space X, the Čech-Stone compactification $\beta^*(X)$ is an approach space which cannot be derived from a pseudometric.

In [24] the Wallman compactification was introduced for the subcategory of App, consisting of all weakly symmetric T_1-spaces. This class contains all T_2 uniform approach spaces. Given a weakly symmetric T_1 approach space X, its bounded lower regular function frame $\mathfrak{L}_X$, is a particular so called Wallman base. The general concept of a Wallman base was recalled in [12]. From the Wallman base $\mathfrak{L}_X$, in [24] an extension of X is obtained on the set $W^*(X)$ of all maximal zero ideals Φ over $\mathfrak{L}_X$. The bounded lower regular functions on X are extended to $W^*(X)$ and the set

$$\widehat{\mathfrak{L}} = \{\hat{\rho} \mid \rho \in \mathfrak{L}_X\}$$

is a basis of a bounded lower regular function frame $\mathfrak{L}_{W^*(X)}$ on $W^*(X)$, making it into a T_1-compactification $w_X : X \to W^*(X)$.

When comparing this construction to the topological Wallman compactification $W(X)$ of a topological space $(X, \mathcal{T})$ (which is isomorphic to our $W^*(X)$), the extensions of lower regular functions in the approach case, correspond to the extensions of the closed sets from $(X, \mathcal{T})$. It is well known that in the topological case, carrying a closed set G to its closure $\overline{G}$ in the topological Wallman compactification, preserves finite intersections and finite unions. The Wallman compactification of an approach space also has the advantage that the extended lower regular functions satisfy

$$\widehat{\mu \vee \rho} = \hat{\mu} \vee \hat{\rho} \text{ and } \widehat{\mu \wedge \rho} = \hat{\mu} \wedge \hat{\rho}.$$

In section 4, Proposition 4.8 we also describe a basis for the upper regular function frame of $W^*(X)$, by extending the upper regular functions of X. Upper regular functions correspond to the open sets in the topological case. Carrying an upper regular function η to its extension $\check{\eta}$, will also be shown to preserve finite $\wedge$ and finite $\vee$.

The main purpose of this paper is to formulate necessary and sufficient conditions for $\beta^*(X)$ and $W^*(X)$ to be isomorphic, thus answering a question first raised in [24]. As our Theorem 3.8 implies that $\beta^*(X)$ is always normal, we start our investigation by studying normality of $W^*(X)$. In Proposition 6.2 we prove that a weakly symmetric T_1 approach space X is normal if and only if $W^*(X)$ is normal.

In the topological case normality and T_2-separation of X is sufficient for the topological Wallman compactification $W(X)$ to be isomorphic to the topological Čech-Stone compactification $\beta(X)$. We present a counterexample, showing that for arbitrary approach spaces normality and T_2-separation of X is not sufficient. Another property $W^*(X)$ should have in order to obtain an isomorphism between $W^*(X)$ and $\beta^*(X)$ is regularity and this is not guaranteed by normality and T_2-separation of X. We introduce a strengthening of the regularity condition on X, which we call ideal-regularity in Definition 7.4 and in Proposition 7.8 we show that X is ideal-regular if and only if $W^*(X)$ is regular.

It is known from [11] that each regular and normal approach space is a uniform approach space. Moreover in Theorem 5.4 we show that every bounded contraction $f : X \to ([0,\infty], \delta_{d_{\mathbb{E}}})$ has a unique bounded contractive extension $\widetilde{f} : W^*(X) \to ([0,\infty], \delta_{d_{\mathbb{E}}})$ satisfying $\widetilde{f} \circ w_X = f$ and as among uniform approach spaces, β^* being the reflection $\mathsf{UApp}_2 \to k\mathsf{UApp}_2$, is characterised by the extension property for bounded realvalued contractions. Finally in our main Theorem 7.11 we can conclude that X is ideal-regular, normal and T_1 if and only if X is a uniform T_1 approach space with $\beta^*(X)$ and $W^*(X)$ isomorphic.

For topological spaces ideal-regularity coincides with normality. So as a corollary we find the well known result that the topological Čech-Stone and Wallman compactifications are isomorphic if and only if the topological space X is normal and T_1. For every pseudometric approach space X, the set $\mathfrak{L}_X$ coincides with the set of all bounded contractions $f : X \to ([0,\infty], \delta_{d_{\mathbb{E}}})$. By [26] it follows that $\beta^*(X)$ and $W^*(X)$ are isomorphic. A pseudometric approach space therefore is always both ideal-regular and normal.

2. Preliminaries

For more details on concepts and results on approach spaces we refer to [23] or [22]. We recall terminology and basic results that will be needed in this paper.

Usually an extended quasi-pseudometric on a set X is a function $q : X \times X \to [0, \infty]$ which vanishes on the diagonal and satisfies the triangular inequality and if q moreover satisfies symmetry then it is called an extended pseudometric. In this paper all such $q : X \times X \to [0, \infty]$ are allowed to take the value ∞ and both distances between two different points can be zero. *From now on, for simplicity in terminology we drop the words "extended" and "pseudo"*, so in this respect our terminology differs from what is commonly used. However, it conforms with the terminology in [23] and [14]. We denote by qMet the category of all *quasi-metric spaces with non-expansive maps* as morphisms and by Met the full subcategory of all *metric spaces*.

A *distance* on a set X is a function

$$\delta : X \times 2^X \to [0, \infty] \tag{2.1}$$

with the following properties:

(D1) $\delta(x, \{x\}) = 0,\ \forall x \in X,$
(D2) $\delta(x, \emptyset) = \infty,\ \forall x \in X,$
(D3) $\delta(x, A \cup B) = \min\{\delta(x, A), \delta(x, B)\},\ \forall x \in X,\ \forall A, B \in 2^X,$
(D4) $\delta(x, A) \leq \delta(x, A^{(\varepsilon)}) + \varepsilon,\ \forall x \in X,\ \forall A \in 2^X,\ \forall \varepsilon \in [0, \infty],$
with the enlargement

$$A^{(\varepsilon)} = \{x | \delta(x, A) \leq \varepsilon\}.$$

A pair (X, δ) consisting of a set X endowed with a distance δ is called an *approach space*. For $A \subseteq X$ we denote by $\delta_A : X \to [0, \infty]$ the function defined by $\delta_A(x) = \delta(x, A)$ for $x \in X$.

Morphisms between approach spaces are called contractions. A map $f : (X, \delta_X) \to (Y, \delta_Y)$ is a *contraction* if

$$\forall x \in X,\ \forall A \subseteq X,\ \delta_Y(f(x), f(A)) \leq \delta_X(x, A). \tag{2.2}$$

The category of approach spaces and contractions is denoted by App.

An approach space X has an *approach tower*, a family $\mathsf{t} = (\mathsf{t}_\varepsilon)_{\varepsilon \in [0,\infty[}$ where

$$\mathsf{t}_\varepsilon : 2^X \to 2^X, \tag{2.3}$$

is the pretopological closure operator defined by

$$\mathsf{t}_\varepsilon(A) = A^{(\varepsilon)},$$

for $A \subseteq X$. At level 0 we have a topology. The distance can be recovered from the approach tower by

$$\delta(x, A) = \inf\{\varepsilon \mid x \in A^{(\varepsilon)}\}, \tag{2.4}$$

for $x \in X$ and $A \subseteq X$. Using the following characterisation for a map $f : (X, \mathsf{t}_X) \to (Y, \mathsf{t}_Y)$ to be a contraction, namely iff

$$f(A^{(\varepsilon)}) \subseteq f(A)^{(\varepsilon)}, \tag{2.5}$$

whenever $\varepsilon \in [0, \infty[$ and $A \subseteq X$, and suitable axioms for the approach tower, the category App can be isomorphically described in terms of approach towers. For details on the axioms we refer to [23].

Convergence in an approach space (X, δ) is described by means of a limit operator on filters. For a given filter $\mathcal{F}$ and a point $x \in X$ the value $\lambda\mathcal{F}(x)$ is interpreted as the distance that the point is away from being a limit point of the filter. If $\mathsf{F}X$ is the set of all filters on X and βX the set of all ultrafilters on X, the *limit operator* is a function

$$\lambda : \mathsf{F}X \to [0,\infty]^X.$$

The transition from the distance to the limit operator is described by

$$\lambda\mathcal{F}(y) = \sup_{U\in\mathcal{U}\in\beta X, \mathcal{F}\subseteq\mathcal{U}} \delta(y,U), \tag{2.6}$$

for $\mathcal{F} \in \mathsf{F}X$ and $y \in X$. Using the following characterization for a map $f : (X, \lambda_X) \to (Y, \lambda_Y)$ to be a contraction iff

$$\lambda_Y f(\mathcal{F})(f(x)) \leq \lambda_X \mathcal{F}(x), \tag{2.7}$$

for every $\mathcal{F} \in \mathsf{F}X$ and $x \in X$ and with $f(\mathcal{F})$ the filter generated by $\{f(F) \mid F \in \mathcal{F}\}$, with suitable axioms for the limit operator, the category App can be isomorphically described in terms of limit operators.

The adherence operator for a filter $\mathcal{F}$ and $x \in X$ can be derived from the value of λ on ultrafilters $\mathcal{U} \in \beta(X)$,

$$\alpha\mathcal{F}(x) = \inf_{\mathcal{F}\subseteq\mathcal{U}, \mathcal{U}\in\beta X} \lambda\mathcal{U}(x). \tag{2.8}$$

The tower and the limit operator are related by

$$\lambda\mathcal{F}(x) \leq \varepsilon \Leftrightarrow \mathcal{F} \to x \text{ in the pretopology } \mathsf{t}_\varepsilon, \tag{2.9}$$

for all $\mathcal{F} \in \mathsf{F}X$, $x \in X$ and $\varepsilon \in [0,\infty[$.

Given an approach space (X, δ) the corresponding *bounded local system* is the collection $\mathfrak{A}_b = (\mathfrak{A}_b(x))_{x\in X}$ of ideals in $[0,\infty]_b^X$, the set of bounded functions from X to $[0,\infty]$,

$$\mathfrak{A}_b(x) = \{\varphi \in [0,\infty]_b^X \mid \inf_{z\in A} \varphi(z) \leq \delta(z, A)\}, \tag{2.10}$$

with $x \in X$. Using the characterisation for a map $f : (X, \mathfrak{A}_{b,X}) \to (Y, \mathfrak{A}_{b,Y})$ to be a contraction iff $\forall x \in X, \forall \varphi' \in \mathfrak{A}_{b,Y}(f(x)),\ \ \varphi' \circ f \in \mathfrak{A}_{b,X}(x)$ and suitable axioms for bounded local systems, the category App can be isomorphically described in terms of bounded local systems.

An approach space (X, δ) has a *gauge*, i.e. the collection of quasimetrics on X given by

$$\mathcal{G} = \{q \mid \text{quasimetric on } X, \delta_q \leq \delta\}, \tag{2.11}$$

with

$$\delta_q(x, A) = \inf_{z\in A} q(x, z), \tag{2.12}$$

whenever $A \subseteq X$ and $x \in X$. The distance can be recovered from the gauge by

$$\delta = \sup_{q\in\mathcal{G}} \delta_q \tag{2.13}$$

and we may restrict to the collection of $\mathcal{G}_b$ of bounded quasimetrics in $\mathcal{G}$ in the previous formula. A subcollection $\mathcal{D} \subseteq \mathcal{G}$ stable for finite $\vee$ is called a *gauge basis* if $\delta = \sup_{q\in\mathcal{D}} \delta_q$.

An approach space X is called *uniform* if the gauge $\mathcal{G}$ has a basis consisting of metrics. With $\mathcal{H} = \{d \in \mathcal{G} | d \text{ metric}\}$ we have X is uniform iff

$$\delta = \sup_{d\in\mathcal{H}} \delta_d. \tag{2.14}$$

Using the following characterisation for a map $f : (X, \mathcal{G}_X) \to (Y, \mathcal{G}_Y)$ to be a contraction iff

$$q' \circ (f \times f) \in \mathcal{G}_X, \tag{2.15}$$

whenever $q' \in \mathcal{G}_Y$ and suitable axioms for the gauge, the category App can be isomorphically described in terms of gauges.

The following concepts will play an important role in the sequel. We consider two quasi-metrics on $[0,\infty]$, the quasi-metric

$$d_{\mathsf{P}}(x,y) = x \ominus y = (x-y) \vee 0,$$

and its dual d_{P}^{-} and note that for the Euclidean metric we have $d_{\mathbb{E}} = d_{\mathsf{P}} \vee d_{\mathsf{P}}^{-}$. For an approach space (X,δ) the classes $\mathfrak{L}_X$ of *bounded lower regular* and $\mathfrak{U}_X$ of *(bounded) upper regular* functions, are defined by

$$\mathfrak{L}_X = \{f : (X,\delta) \to ([0,\infty], \delta_{d_{\mathsf{P}}}) \mid \text{bounded, contractive}\}, \tag{2.16}$$

and

$$\mathfrak{U}_X = \{f : (X,\delta) \to ([0,\infty], \delta_{d_{\mathsf{P}}^{-}}) \mid \text{bounded, contractive}\}. \tag{2.17}$$

Both are stable for taking finite suprema and infima, $\mathfrak{L}_X$ moreover is stable for arbitrary bounded suprema, and $\mathfrak{U}_X$ is stable for arbitrary infima.

A *basis for the bounded lower regular function frame* $\mathfrak{L}_X$, (*basis for the upper regular function frame* $\mathfrak{U}_X$) is a subset $\mathfrak{B} \subseteq \mathfrak{L}_X$ (is a subset $\mathfrak{D} \subseteq \mathfrak{U}_X$ respectively), which is such that any function in $\rho \in \mathfrak{L}_X$ can be written as

$$\rho = \bigvee_{\mu \in \mathfrak{B}, \mu \leq \rho} \mu, \tag{2.18}$$

(every function $\eta \in \mathfrak{U}_X$ can be written as

$$\eta = \bigwedge_{\nu \in \mathfrak{D}, \nu \geq \eta} \nu, \text{ respectively).} \tag{2.19}$$

The function

$$\delta_A \wedge \omega : X \to [0,\infty], \tag{2.20}$$

for $A \subseteq X$ and $\omega < \infty$ is an example of a bounded lower regular function. Bounded upper and lower regular function frames are related in the following way. A basis for $\mathfrak{U}_X$ is given by

$$\{\alpha \ominus \rho | \rho \in \mathfrak{L}_X, \sup \rho \leq \alpha < \infty\}, \tag{2.21}$$

and the other way around, a basis for $\mathfrak{L}_X$ is obtained by

$$\{\alpha \ominus \eta | \eta \in \mathfrak{U}_X, \sup \eta \leq \alpha < \infty\}. \tag{2.22}$$

The collection of all contractions $f : (X,\delta) \to ([0,\infty], \delta_{d_{\mathbb{E}}})$ is denoted by

$$\mathcal{K}((X,\delta), ([0,\infty], \delta_{d_{\mathbb{E}}})),$$

or shortly $\mathcal{K}(X)$, and of all bounded contractions by $\mathcal{K}_b(X)$.

Then we have [27]

$$f \in \mathfrak{U}_X \cap \mathfrak{L}_X \Leftrightarrow f \in \mathcal{K}_b(X). \tag{2.23}$$

The distance can be recovered from the lower regular function frame by

$$\delta(x,A) = \sup\{\rho(x) | \rho \in \mathfrak{L}_X, \rho|A = 0\}, \tag{2.24}$$

for $x \in X$ and $A \subseteq X$.

Using the following characterisation for a map $f : (X,\mathfrak{L}_X) \to (Y,\mathfrak{L}_Y)$ to be a contraction iff

$$\rho \circ f \in \mathfrak{L}_X, \tag{2.25}$$

whenever $\rho \in \mathfrak{L}_Y$, and suitable axioms for the bounded lower regular function frame, the category App can be isomorphically described in terms of bounded lower regular function frames. Similar results hold for the upper regular function frame.

Remark that in [24], for an approach space X, the whole lower regular function frame is considered and denoted by $\mathcal{R}_X$, allowing unbounded functions too and unbounded suprema instead of just bounded ones. By axiomatizing the whole lower regular function frame as in [23] or the bounded lower regular function frame, where we focus on, isomorphic categories are described.

If $\mathfrak{L}_X$ is the bounded lower regular function frame, then the function $\mathfrak{l} : [0,\infty]_b^X \to [0,\infty]_b^X$ defined by

$$\mathfrak{l}(\mu) = \bigvee\{\nu \in \mathfrak{L}_X | \nu \leq \mu\} \tag{2.26}$$

is called the *lower hull operator.* This operator is idempotent, monotone, preserves finite infima and for a constant function α we have $\mathfrak{l}(\mu+\alpha) = \mathfrak{l}(\mu)+\alpha$. The lower hull operator can be calculated directly from the gauge by

$$\mathfrak{l}(\mu)(x) = \sup_{q\in\mathcal{G}} \inf_{y\in X} (\mu(y) + q(x,y)). \tag{2.27}$$

If $\mathfrak{U}_X$ is the upper regular function frame then the function $\mathfrak{u} : [0,\infty]_b^X \to [0,\infty]_b^X$ defined by

$$\mathfrak{u}(\mu) = \bigwedge\{\nu \in \mathfrak{U} | \mu \leq \nu\} \tag{2.28}$$

is called the *upper hull operator.* This operator is idempotent, monotone, preserves finite suprema and for a constant function α we have $\mathfrak{u}(\mu+\alpha) = \mathfrak{u}(\mu)+\alpha$. The upper hull operator can be calculated directly from the gauge by

$$\mathfrak{u}(\mu)(x) = \inf_{q\in\mathcal{G}} \sup_{y\in X} (\mu(y) - q(x,y)). \tag{2.29}$$

Approach spaces can be isomorphically described by hull operators [23], but the exact axioms will not be needed in this paper.

As we mentioned, approach spaces can be isomorpically described by distances, approach towers, limit operators, bounded local systems, gauges or bounded (upper or lower) regular functions. *On a given set X we will often denote a given approach space simply by X and then we will use its distance δ, its approach tower $\mathfrak{t} = (\mathfrak{t}_\varepsilon)_{\varepsilon\in[0,\infty[}$, its limit operator λ, its bounded local system $\mathfrak{A}_b = (\mathfrak{A}_b(x))_{x\in X}$, its gauge $\mathcal{G}$ or its bounded regular function frames $\mathfrak{L}_X$ and $\mathfrak{U}_X$ whenever appropriate.*

The category App constitutes a framework wherein other important categories can be fully embedded. The embedding of quasi-metric spaces is given in the usual way that one defines a distance δ_q between points and sets in a metric space as in (2.12).

The neighborhood filter $\mathcal{V}^\varepsilon_{\delta_q}(x)$ of x in the pretopology at level ε in the approach tower of (X,δ_q) is generated by

$$\{B_q(x,\gamma) | \gamma > \varepsilon\}. \tag{2.30}$$

qMet is embedded as a concretely coreflective subcategory. The concrete qMet coreflection of a given approach space X with distance δ is given by the quasi-metric space (X,q) where

$$q(x,y) = \delta(x,\{y\}),$$

for $x,y \in X$.

Top is embedded as a full concretely reflective and concretely coreflective subcategory. The embedding of topological spaces is determined by associating with every topological space $(X,\mathcal{T})$ (with closure of A written as $\mathrm{cl}A$) the distance

$$\delta_{\mathcal{T}}(x,A) = \begin{cases} 0 & x \in \mathrm{cl}A, \\ \infty & x \notin \mathrm{cl}A. \end{cases}$$

Every approach space (X, δ) has two natural topological spaces associated with it, the topological coreflection, which we will also call the *underlying topology*, and the topological reflection. In this paper we will mainly deal with the coreflection which is the topological space $(X, \mathcal{T}_\delta)$ determined by the closure

$$x \in \mathrm{cl}A \Leftrightarrow \delta(x, A) = 0 \Leftrightarrow x \in A^{(0)}. \tag{2.31}$$

$\mathcal{T}_\delta$ coincides with the topology at level 0 of the approach tower.

When X is an approach space notions such as density, closure, open and closed will always refer to the underlying topology. For $f : X \to Y$, a map between approach spaces, continuity will always refer to the underlying topologies of X, Y. This implies that lower (upper) regular functions are lower semicontinuous (upper semicontinuous respectively). The same holds for properties such as, compact, T_1 or T_2 when applied to an approach space X. What is meant is that the underlying topological space $(X, \mathcal{T}_\delta)$ has the respective property. Other approach properties like regularity and normality are not equivalent with the corresponding property of the underlying topology, we will recall their definitions in the sequel, whenever they are used.

3. Normality for compact spaces

For approach spaces compact and T_2 does not imply normality. Counterexamples were provided in [10]. In that paper the normality was shown for the particular example of $\beta^*(\mathbb{N})$. In this section we solve the question that was left open in [10], namely whether compact and uniform implies normality. A positive solution will be given in subsection 3.2. In order to reach this goal we need some alternative characterisations of normality. These are presented in subsection 3.1. Lifting normality from an approach space X to its Wallman compactification $W^*(X)$, as we will need to do in section 5, will also heavily rely on these new equivalent formulations of normality.

3.1. **Alternative characterisations of normality.** Normality for approach spaces was introduced in [10] by proving several equivalent formulations. One characterisation is based on Urysohn separation of γ-separated sets, where for an approach space X and $\gamma > 0$, two sets $A, B \subseteq X$ are called *γ-separated* if

$$A^{(\alpha)} \cap B^{(\beta)} = \emptyset, \text{ whenever } \alpha \geq 0, \beta \geq 0, \alpha + \beta < \gamma. \tag{3.1}$$

Another characterisation of normality is based on Katětov-Tong's insertion. We recall the definitions from [10].

Theorem 3.1. *For an approach space X, the following properties are equivalent:*

(1) X satisfies Katětov-Tong's insertion, meaning that for bounded functions to $[0, \infty]$ satisfying $\eta \leq \rho$ with η upper regular and ρ lower regular, there exists a contractive map $f : X \to ([0, \infty], \delta_{d_\mathbb{E}})$ satisfying $\eta \leq f \leq \rho$.

(2) X satisfies separation by Urysohn contractive maps, meaning that for every $A, B \subseteq X$ and for every $\gamma > 0$, whenever A and B are γ-separated, there exists a contractive $f : X \to ([0, \gamma], \delta_{d_\mathbb{E}})$ satisfying $f(a) = \gamma$ for $a \in A$ and $f(b) = 0$ for $b \in B$.

An approach space X is *normal* if and only if it satisfies one and hence both equivalent conditions in 3.1. The equivalence in 3.1 is the approach counterpart of a deep and beautiful result in Top that normality can be characterised by means of insertion between semicontinuous functions. This topological characterisation of normality is known as Katětov-Tong's result [16], [28]. We start with alternative formulations of Katětov-Tong's insertion for approach spaces.

Given an approach space X, in the next result we will apply Tong's lemma [28] to the special lattice of all contractions $K = \mathcal{K}(X, ([0,\omega], \delta_{d_{\mathbb{E}}}))$ for some $\omega < \infty$, embedded in the lattice of all maps $M = [0,\omega]^X$, where all infima and suprema are taken. In this particular case, with the notations of Tong, we have $K_\sigma = \{\bigvee_n t_n \mid \forall n : t_n \in K\} \subseteq \mathfrak{L}_X$ and $K_\delta = \{\bigwedge_{n\geq 1} t_n \mid \forall n : t_n \in K\} \subseteq \mathfrak{U}_X$. So using the fact that

$$K_\sigma \cap K_\delta \subseteq \mathfrak{L}_X \cap \mathfrak{U}_X \cap [0,\omega]^X = \mathcal{K}(X, ([0,\omega], \delta_{d_{\mathbb{E}}}))$$

by (2.23), the lemma takes the following simpler form.

Lemma 3.2. *Let $K = \mathcal{K}(X, ([0,\omega], \delta_{d_{\mathbb{E}}}))$ for some $\omega < \infty$, be embedded in $[0,\omega]^X$ where all infima and suprema are taken, let $\eta \in K_\delta = \{\bigwedge_{n\geq 1} t_n \mid \forall n : t_n \in K\}$ and $\rho \in K_\sigma = \{\bigvee_n t_n \mid \forall n : t_n \in K\}$ with $\eta \leq \rho$, then a contraction $f \in K$ exists satisfying $\eta \leq f \leq \rho$.*

We first prove the following preliminary results based on 3.2.

Proposition 3.3. *Let X be an approach space. The following properties are equivalent:*

(1) For all $\eta \in \mathfrak{U}_X$ and $\rho \in \mathfrak{L}_X$ with $\eta \leq \rho$, there exists a countable set of contractions $\{g_n | n \in \mathbb{N}\}$, $g_n : X \to ([0,\infty], \delta_{d_{\mathbb{E}}})$ with $\eta \leq \bigwedge_n g_n \leq \rho$.

(2) For all $\eta \in \mathfrak{U}_X$ and $\rho \in \mathfrak{L}_X$ with $\eta \leq \rho$, there exists a countable set of contractions $\{h_n | n \in \mathbb{N}\}$, $h_n : X \to ([0,\infty], \delta_{d_{\mathbb{E}}})$ with $\eta \leq \bigvee_n h_n \leq \rho$.

(3) For all $\eta \in \mathfrak{U}_X$ and $\rho \in \mathfrak{L}_X$ with $\eta \leq \rho$, there exists a contraction $f : X \to ([0,\infty], \delta_{d_{\mathbb{E}}})$ with $\eta \leq f \leq \rho$.

Proof. (1) $\Rightarrow$ (2): For $\eta \leq \rho \leq \omega < \infty$ we have $\omega - \rho \leq \omega - \eta$. Since $\omega - \rho$ is upper regular and $\omega - \eta$ is lower regular, we can apply (1) to find a countable set of contractions $\{g_n | n \in \mathbb{N}\}$ satisfying $\omega - \rho \leq \bigwedge_n g_n \leq \omega - \eta$. Observe that without loss of generality we may assume that all $g_n \leq \omega$. With $h_n = \omega - g_n$ we have $\eta \leq \bigvee_n h_n \leq \rho$.
(2) $\Rightarrow$ (1): Is analogous.
(1) $\Rightarrow$ (3): For $\eta \leq \rho \leq \omega < \infty$, first apply (1) to obtain a countable set of contractions $\{g_n | n \in \mathbb{N}\}$, which we may assume all to stay below ω, with $\eta \leq \bigwedge_n g_n \leq \rho$. Then consider the upper regular function $\bigwedge_n g_n$ and the lower regular ρ and apply (2). There exists a countable set of contractions $\{h_n | n \in \mathbb{N}\}$ with $\eta \leq \bigwedge_n g_n \leq \bigvee_n h_n \leq \rho$. Applying 3.2, the statement (3) follows.
(3) $\Rightarrow$ (1): This is clear. □

In the next proposition the relation $\eta < \rho$ is defined pointwise, meaning $\eta(x) < \rho(x)$ for all $x \in X$.

Proposition 3.4. *Let X be an approach space. The following properties are equivalent:*

(1) For all $\eta \in \mathfrak{U}_X$ and $\rho \in \mathfrak{L}_X$ with $\eta < \rho$, there exists a countable set of contractions $\{g_n | n \in \mathbb{N}\}$, $g_n : X \to ([0,\infty], \delta_{d_{\mathbb{E}}})$ with $\eta \leq \bigwedge_n g_n \leq \rho$.

(2) For all $\eta \in \mathfrak{U}_X$ and $\rho \in \mathfrak{L}_X$ with $\eta < \rho$, there exists a countable set of contractions $\{h_n | n \in \mathbb{N}\}$, $h_n : X \to ([0,\infty], \delta_{d_{\mathbb{E}}})$ with $\eta \leq \bigvee_n h_n \leq \rho$.

(3) For all $\eta \in \mathfrak{U}_X$ and $\rho \in \mathfrak{L}_X$ with $\eta < \rho - \frac{1}{p}$, for some $p > 0$ with $\frac{1}{p} \leq \inf \rho$, there exists a contraction $f : X \to ([0,\infty], \delta_{d_{\mathbb{E}}})$ with $\eta \leq f \leq \rho$.

Proof. That (1) and (2) are equivalent is analogous to the equivalence of (1) and (2) in 3.3.
(1) $\Rightarrow$ (3): Let $\eta \in \mathfrak{U}_X$ and $\rho \in \mathfrak{L}_X$ with $\rho \leq \omega < \infty$ and $\eta < \rho - \frac{1}{p}$, for some $p > 0, \frac{1}{p} \leq \inf \rho$. Apply (1) to the upper regular function η and the lower regular function $\rho - \frac{1}{p}$ to find a countable set of contractions $\{g_n | n \in \mathbb{N}\}$ which we may

assume all to stay below ω, with $\eta \leq \bigwedge_n g_n \leq \rho - \frac{1}{p} < \rho$ and then (2) to find a countable set of contractions $\{h_n | n \in \mathbb{N}\}$ with $\eta \leq \bigwedge_n g_n \leq \bigvee_n h_n \leq \rho$. Again by 3.2, statement (3) follows.
(3) $\Rightarrow$ (1): Let $\eta \in \mathfrak{U}_X$ and $\rho \in \mathfrak{L}_X$ with $\eta < \rho \leq \omega$. For every $n > 0$ we have $\eta < (\rho + \frac{1}{n}) - \frac{1}{2n}$ with $\frac{1}{2n} < \frac{1}{n} \leq \inf(\rho + \frac{1}{n})$ and $\rho + \frac{1}{n} \leq \omega + \frac{1}{n}$, so by the assumption (3) there exists a contraction g_n satisfying $\eta \leq g_n \leq \rho + \frac{1}{n}$. It follows that $\eta \leq \bigwedge_n g_n \leq \rho$. So we can conclude that (1) is fulfilled.
□

As an application we obtain our first new formulation of normality.

Theorem 3.5. *Let X be an approach space. The following properties are equivalent:*

(1) X is normal.
(2) For all $\eta \in \mathfrak{U}_X$ and $\rho \in \mathfrak{L}_X$ with $\eta < \rho$ there exists a contraction $f : X \to ([0, \infty], \delta_{d_{\mathbb{E}}})$ satisfying $\eta \leq f \leq \rho$.
(3) For all $\eta \in \mathfrak{U}_X$ and $\rho \in \mathfrak{L}_X$, for which there is some $p > 0$ with $\frac{1}{p} \leq \inf \rho$, $\eta < \rho - \frac{1}{p}$, there exists a contraction $f : X \to ([0, \infty], \delta_{d_{\mathbb{E}}})$ satisfying $\eta \leq f \leq \rho$.

Proof. That (1) implies (2) follows from 3.1 and that (2) implies (3) is clear.
To prove the implication (3) $\Rightarrow$ (1), assume that $\eta \leq \rho$. As in the proof of (3) implies (1) of 3.4, for every $n > 0$ we have $\eta < (\rho + \frac{1}{n}) - \frac{1}{2n}$ with $\frac{1}{2n} < \frac{1}{n} \leq \inf(\rho + \frac{1}{n})$. So by the assumption (3) there exists a contraction g_n satisfying $\eta \leq g_n \leq \rho + \frac{1}{n}$. It follows that $\eta \leq \bigwedge_n g_n \leq \rho$. So we can conclude that (1) and hence (3) in proposition 3.3 is fulfilled. □

Next we describe an alternative for Urysohn separation. We deal with γ^+-separated sets instead of γ-separated sets, where for an approach space X and $\gamma > 0$ two sets $A, B \subseteq X$ are called *γ^+-separated* if

$$A^{(\alpha)} \cap B^{(\beta)} = \emptyset, \text{ whenever } \alpha \geq 0, \beta \geq 0, \alpha + \beta \leq \gamma. \tag{3.2}$$

The next result has a proof quite similar to the proof of (3) $\Rightarrow$ (1) in Theorem 4.3 in [10].

Proposition 3.6. *For an approach space X we have the implication* (1) $\Rightarrow$ (2).

(1) For any two subsets A and B, γ^+-separated for $\gamma > 0$, there exists a contractive map $f : X \to ([0, \gamma], \delta_{d_{\mathbb{E}}})$ satisfying $f|A = 0$ and $f|B = \gamma$.
(2) For $\eta \in \mathfrak{U}_X, \rho \in \mathfrak{L}_X$ with $\eta < \rho$ there exists a countable set of contractions $\{f_n | n \in \mathbb{N}\}$, $f_n : X \to ([0, \infty], \delta_{d_{\mathbb{E}}})$ satisfying $\eta \leq \bigwedge_n f_n \leq \rho$.

Proof. Assume (1) and let $\eta \in \mathfrak{U}_X, \rho \in \mathfrak{L}_X$ with $\eta < \rho \leq \omega$ for some $\omega < \infty$. For $k, m, n \in \mathbb{N}$ with $m \leq k < n$, set

$$A_{m,n} = \{\rho \leq \omega \frac{m}{n}\}, \quad B_{k,n} = \{\eta \geq \omega(\frac{k}{n} + \frac{1}{2n})\}.$$

We show that $A^{(\alpha)}_{m,n} \cap B^{(\beta)}_{k,n} = \emptyset$ for all $\alpha + \beta \leq \gamma$ with $\gamma = \omega \frac{2k-2m+1}{2n}$.

Let $\alpha + \beta \leq \gamma$. If $x \in A^{(\alpha)}_{m,n}$ then since $\rho : X \to ([0, \omega], d_{\mathsf{P}})$ is contractive, by 2.5 we have $\rho(x) \in [0, wm/n]^{(\alpha)}$. It follows by (8.1) that $\inf_{z \leq \omega m/n}(\rho(x) \ominus z) \leq \alpha$ and therefore

$$\rho(x) \leq \frac{\omega m}{n} + \alpha.$$

Similarly, assuming $x \in B_{k,n}^{(\beta)}$ and using the contractivity of η to the codomain endowed with d_{P}^{-} it follows that

$$\eta(x) \geq \frac{\omega(2k+1)}{2n} - \beta.$$

Since $\eta < \rho$, the assumption $x \in A_{m,n}^{(\alpha)} \cap B_{k,n}^{(\beta)}$ would imply $\frac{\omega(2k+1)}{2n} - \beta < \frac{\omega m}{n} + \alpha$ which is impossible as it would imply $\gamma < \alpha + \beta \leq \gamma$.

By (1), for $1 < m \leq k < n$, a contraction

$$f_{m,n}^{k} \in \mathcal{K}(X, ([\omega\frac{m+1}{n}, \omega\frac{2k+3}{2n} \wedge \omega], \delta_{d_{\mathbb{E}}}))$$

exists with $f_{m,n}^{k}|_{A_{m,n}} = \omega\frac{m+1}{n}$ and $f_{m,n}^{k}|_{B_{k,n}} = \omega\frac{2k+3}{2n} \wedge \omega$.

From here onwards the proof is exactly the same as the one used in (3) $\Rightarrow$ (1) in Theorem 4.3 in [10]. For completeness sake we repeat the construction. Define the contractions

$$f_{m,n} = \bigvee_{k=m}^{n-1} f_{m,n}^{k} \text{ and } f_n = \bigwedge_{m=2}^{n-1} f_{m,n}.$$

Next we show that $\eta \leq f_n$ whenever $n \geq 3$. Let $x \in X$ and $1 < m < n$, either $x \notin B_{m,n}$, then

$$\eta(x) \leq \omega\frac{2m+1}{2n} \leq \omega\frac{m+1}{n} \leq f_{m,n}(x),$$

or $x \in B_{m,n}$, then we again consider two cases. If $x \in B_{n-1,n}$ then

$$\eta(x) \leq \omega = f_{m,n}^{(n-1)}(x) \leq f_{m,n}(x).$$

Otherwise, a minimal k exists with $m < k \leq n-1$ and $x \in B_{k-1,n}, x \notin B_{k,n}$. Then we have

$$\eta(x) < \omega\frac{2k+1}{2n} = \omega\frac{2(k-1)+3}{2n} = f_{m,n}^{(k-1)}(x) \leq f_{m,n}(x).$$

Next we show that $\bigwedge_{n\geq 3} f_n \leq \rho$. In order to do so, for $x \in X$ we prove that

$$f_n(x) - \rho(x) \leq \omega\frac{2}{n}$$

for every $n \geq 3$. Fix $x \in X$, then one of three possibilities holds. First if $f_n(x) \leq \rho(x)$, we are done. Secondly if $x \notin A_{m,n}$ for all $m \geq 2$, then $\rho(x) > \omega(n-1)/n$. Since $f_n(x) \leq \omega$, we have that

$$f_n(x) - \rho(x) \leq \omega\frac{1}{n} < \omega\frac{2}{n}.$$

Thirdly, if some minimal $m \geq 2$ exists such that $x \in A_{m,n}$, then $\rho(x) \geq \omega(m-1)/n$ and $f_{m,n}(x) = \omega(m+1)/n$. So

$$f_n(x) - \rho(x) \leq f_{m,n}(x) - \rho(x) \leq \omega\frac{2}{n}.$$

So we can conclude that

$$\eta \leq \bigwedge_{n\geq 3} f_n \leq \rho.$$

□

As an application we obtain our second alternative description of normality.

Theorem 3.7. *For an approach space X the following properties are equivalent:*

(1) For any two subsets A and B, γ^{+}-separated for $\gamma > 0$, there exists a contractive map $f : X \to ([0,\gamma], \delta_{d_{\mathbb{E}}})$ satisfying $f|A = 0$ and $f|B = \gamma$.

(2) For all $\eta \in \mathfrak{U}_X$ and $\rho \in \mathfrak{L}_X$ with $\eta < \rho - \frac{1}{p}$, for some $p > 0$ with $\frac{1}{p} \leq \inf \rho$, there exists a contraction $f : X \to ([0,\infty], \delta_{d_{\mathbb{E}}})$ with $\eta \leq f \leq \rho$.

(3) X is normal.
(4) For any two subsets A and B, γ-separated for $\gamma > 0$, there exists a contractive map $f : X \to ([0,\gamma], \delta_{d_{\mathbb{E}}})$ satisfying $f|A = 0$ and $f|B = \gamma$.

Proof. (1) $\Rightarrow$ (2): From (1) and 3.6 we have that conditions 3.4 (1) and hence also 3.4 (3) are satisfied. So for all $\eta \in \mathfrak{U}_X$ and $\rho \in \mathfrak{L}_X$ with $\eta < \rho - \frac{1}{p}$, for some $p > 0$ with $\frac{1}{p} \leq \inf \rho$, there exists a contraction f with $\eta \leq f \leq \rho$.
(2) $\Rightarrow$ (3): This is 3.5.
(3) $\Rightarrow$ (4): This is 3.1.
(4) $\Rightarrow$ (1): Clearly any two sets that are γ^+-separated are also γ-separated. □

3.2. **Compact spaces.** The question whether a compact uniform approach space (2.14) is always normal can now be answered positively.

Theorem 3.8. *A compact uniform approach space X is normal.*

Proof. Let X be a compact uniform approach space with gauge basis $\mathcal{H}$ consisting of all metrics in the gauge and assume that A and B are γ^+-separated for $\gamma > 0$. For $b \in \mathrm{cl}_X(B)$ we have $\delta(b, A) > \gamma$, so by (2.14) there exists a metric $d_b \in \mathcal{H}$ with $\delta_{d_b}(b, A) > \gamma$. For each metric $d \in \mathcal{H}$, by (2.1), the distance fulfils the inequality

$$|\delta_d(x, A) - \delta_d(y, A)| \leq d(x, y)$$

for arbitrary x, y, so $\delta_d(\cdot, A) : (X, \delta_d) \to ([0, \omega], \delta_{d_{\mathbb{E}}})$ is contractive on the metric approach space (X, δ_d) and hence also on the approach space X. It implies that

$$\{\{\delta_d(\cdot, A) > \gamma\} | d \in \mathcal{H}\}$$

is an open cover of the compact set $\mathrm{cl}_X(B)$. We can choose $d_1, \cdots, d_n \in \mathcal{H}$ with $\mathrm{cl}_X(B) \subseteq \bigcup_{i=1,\cdots,n} \{\delta_{d_i}(\cdot, A) > \gamma\}$ and by putting $d = d_1 \vee \cdots, \vee d_n \in \mathcal{H}$ we have

$$\mathrm{cl}_X(B) \subseteq \{\delta_d(\cdot, A) > \gamma\}.$$

Then the contractive map $f = \delta_d(\cdot, A)$ moreover satisfies $f|A = 0$ and $f|B > \gamma$. So $f \wedge \gamma$ fulfils the required conditions. By 3.7, X is normal. □

The easiest way to define the Čech-Stone compactification $\beta^*(X)$, given a uniform T_2 approach space X, is by using the embedding

$$e_X : X \to \prod_{f \in \mathcal{K}_b(X)} \mathrm{cl}_{[0,\infty[}\, f(X) : x \to (f(x))_{f \in \mathcal{K}_b(X)}, \tag{3.3}$$

where the product is taken in the category App of approach spaces. Then $\beta^*(X)$ is defined as the closure of $e_X(X)$ in this product. It is a uniform approach space too. So we have the following corollary.

Proposition 3.9. *For every T_2 uniform approach space X, the Čech-Stone compactification $\beta^*(X)$ is normal.*

An approach space is called *regular* if

$$\lambda \mathcal{F}^{(\gamma)} \leq \lambda \mathcal{F} + \gamma \tag{3.4}$$

whenever $\mathcal{F} \in \mathsf{F}(X)$, $\gamma \in [0, \infty[$ and where $\mathcal{F}^{(\gamma)}$ is the filter generated by the collection of enlargements $\{F^{(\gamma)} | F \in \mathcal{F}\}$. A regular approach space has a regular underlying topology, but not vice versa. Regularity for approach spaces has been studied for instance in [6], [1], [29], [9], [11] and [23] and in a monoidal setting in [14]. In the topological case regularity coincides with the usual topological notion and in the setting of quasi-metric spaces regularity is equivalent to being a metric space.

Proposition 3.10. *For a compact approach space X the following properties are equivalent:*

(1) X is a uniform approach space.
(2) X is regular and normal.

Proof. (1) $\Rightarrow$ (2): This follows from 3.8 and the well known fact that every uniform approach space is regular [23].
(2) $\Rightarrow$ (1): This is Theorem 8.1 in [11]. $\square$

Proposition 3.11. *For a compact approach space X suppose the bounded upper and lower regular function frames $\mathfrak{U}_X$ and $\mathfrak{L}_X$ have bases $\mathfrak{B}_X$ stable under finite $\wedge$ and $\mathfrak{D}_X$, stable under finite $\vee$ respectively. If for $\beta \in \mathfrak{B}_X$ and $\gamma \in \mathfrak{D}_X$ we have*

$$\beta < \gamma \Rightarrow \text{there exists a contraction } g : X \to ([0,\infty], \delta_{d_{\mathbb{E}}}),\ \beta \leq g \leq \gamma,$$

then X is normal.

Proof. Let η be upper regular and ρ lower regular, $\eta = \bigwedge_{k\in K} \beta_k$ and $\rho = \bigvee_{j\in J} \gamma_j$ with $\beta_k \in \mathfrak{B}_X$ and $\gamma_j \in \mathfrak{D}_X$ and $\eta < \rho$. For $x \in X$ there exists $k_x \in K$ and $j_x \in J$ with $\beta_{k_x}(x) < \gamma_{j_x}(x)$. As $-\beta_{k_x}$ is lower regular (2.22), $\gamma_{j_x} - \beta_{k_x}$ is lower semicontinuous, so the set $\{\{\beta_{k_x} < \gamma_{j_x}\} | x \in X\}$ is an open cover of X. By compactness we can select $x_1, \cdots, x_n \in X$ with

$$X = \bigcup_{m=1}^{n} \{\beta_{k_{x_m}} < \gamma_{j_{x_m}}\}.$$

We claim that

$$\bigwedge_{m=1}^{n} \beta_{k_{x_m}} < \bigvee_{m=1}^{n} \gamma_{j_{x_m}}.$$

Indeed for $x \in X$ arbitrary, pick $m(x) \in \{1, \cdots, n\}$ such that $\beta_{k_{x_{m(x)}}}(x) < \gamma_{j_{x_{m(x)}}}(x)$. Then we have $\bigwedge_{m=1}^{n} \beta_{k_{x_m}}(x) \leq \beta_{k_{x_{m(x)}}}(x) < \gamma_{j_{x_{m(x)}}}(x) \leq \bigvee_{m=1}^{n} \gamma_{j_{x_m}}(x)$.

As $\bigwedge_{m=1}^{n} \beta_{k_{x_m}} \in \mathfrak{B}_X$ and $\bigvee_{m=1}^{n} \gamma_{j_{x_m}} \in \mathfrak{D}_X$, by the assumption on these bases there exists a contraction f satisfying

$$\bigwedge_{m=1}^{n} \beta_{k_{x_m}} \leq f \leq \bigvee_{m=1}^{n} \gamma_{j_{x_m}}.$$

Finally we have $\eta \leq f \leq \rho$. By 3.5 the space X is normal. $\square$

4. Bases of lower and of upper regular functions for the Wallman compactification $W^*(X)$

We recall some definitions from [24] and add some results needed in the sequel. We express the definitions in terms of the bounded lower regular function frame $\mathfrak{L}_X$, whereas in [24] the notions are expressed in terms of the whole lower regular function frame $\mathcal{R}_X$. The resulting concepts are equivalent. In particular the Wallman compactification $W^*(X)$ defined below is isomorphic to the Wallman compactification $\mathcal{W}(X, \mathcal{R}_X)$ introduced in [24].

As in (2.16), $\mathfrak{L}_X$ is the bounded lower regular function frame of X and we define $\mathfrak{L}_{X_0}$ the subcollection consisting of all functions $\rho \in \mathfrak{L}_X$ satisfying

$$\inf_{x\in X} \rho(x) = 0. \tag{4.1}$$

An approach space X is called *weakly symmetric* if for all $\rho \in \mathfrak{L}_X$, for all $x \in X$:

$$\rho(x) > 0 \Rightarrow \exists \rho' \in \mathfrak{L}_X,\ \rho'(x) = 0 \text{ and } \inf_{z\in X} \rho \vee \rho'(z) > 0. \tag{4.2}$$

4.1. **Ideals in $\mathfrak{L}_X$.** Given a weakly symmetric T_1 approach space with bounded lower regular function frame $\mathfrak{L}_X$, an ideal $\Phi \subseteq \mathfrak{L}_X$ is called a *zero ideal* in $\mathfrak{L}_X$ (called small ideal in [23]), if

$$\Phi \subseteq \mathfrak{L}_{X_0}.$$

The set $W^*(X)$ is defined as the collection of all maximal zero ideals in $\mathfrak{L}_X$. Recall that a maximal zero ideal Φ is *prime* in the following sense.

$$\forall \rho, \rho' \in \mathfrak{L}_{X_0} : \rho \wedge \rho' \in \Phi \Rightarrow (\rho \in \Phi \text{ or } \rho' \in \Phi). \tag{4.3}$$

Proposition 4.1. *Every $\Phi \in W^*(X)$ is saturated in the following sense: if $\mu \in \mathfrak{L}_{X_0}$ and $\forall \varepsilon > 0, \exists \varphi_\varepsilon \in \Phi, \mu \leq \varphi_\varepsilon + \varepsilon$ then $\mu \in \Phi$.*

Proof. For $\Phi \in W^*(X)$ we put

$$\widetilde{\Phi} = \{\varphi \in \mathfrak{L}_{X_0} | \forall \varepsilon > 0, \exists \varphi_\varepsilon \in \Phi, \varphi \leq \varphi_\varepsilon + \varepsilon\}.$$

Then $\widetilde{\Phi}$ is a zero ideal containing Φ, so by the maximality of Φ we have $\widetilde{\Phi} = \Phi$. □

We recall the link between zero ideals in $\mathfrak{L}_X$ and filters on X as described in Lemma 4.3.7 in [23]. For a zero ideal Φ in $\mathfrak{L}_X$ the collection

$$\{\{\rho \leq \varepsilon\} | \rho \in \Phi, \varepsilon > 0\}$$

generates a filter $\mathfrak{f}^*(\Phi)$ on X and

$$\bigvee \Phi \leq \alpha \mathfrak{f}^*(\Phi), \tag{4.4}$$

where α is the adherence operator introduced in (2.8). The other way around, given a filter $\mathcal{F}$ on X, then

$$\mathfrak{i}^*(\mathcal{F}) = \{\varphi \in \mathfrak{L}_X | \exists F \in \mathcal{F}, \varphi \leq \delta_F\}$$

is a zero ideal in $\mathfrak{L}_X$ and

$$\alpha \mathcal{F} = \bigvee \mathfrak{i}^*(\mathcal{F}). \tag{4.5}$$

Proposition 4.2. *An approach space X is compact if and only if for every zero ideal $\Phi \subseteq \mathfrak{L}_{X_0}$, there exists $x \in X$ satisfying*

$$\sup_{\varphi \in \Phi} \varphi(x) = 0.$$

Proof. Suppose X is compact and $\Phi \subseteq \mathfrak{L}_{X_0}$ is a zero ideal. By (2.8) and (2.9) it follows that for the filter $\mathfrak{f}^*(\Phi)$ on X, some $x \in X$ exists with $\alpha \mathfrak{f}^*(\Phi)(x) = 0$. Applying (4.4) we have $\bigvee \Phi(x) = 0$.

To show the other implication, let $\mathcal{U}$ be an ultrafilter on X. For the zero ideal $\iota^*(\mathcal{U})$ associated with $\mathcal{U}$, there exists $x \in X$ with $\sup_{\varphi \in \iota^*(\mathcal{U})} \varphi(x) = 0$. Applying (4.5) we can conclude that $\lambda \mathcal{U}(x) = \alpha \mathcal{U}(x) = 0$. □

4.2. **Lower regular functions in $W^*(X)$.** We recall and adapt some definitions from [24] and [26]. An extension of X is obtained on the set $W^*(X)$ of all maximal zero ideals $\Phi \subseteq \mathfrak{L}_X$ with $w_X : X \to W^*(X) : x \to \Phi_x$, where

$$\Phi_x = \{\varphi \in \mathfrak{L}_X | \varphi(x) = 0\}. \tag{4.6}$$

A basis for the bounded lower regular function frame of $W^*(X)$ in the sense of (2.18) is constructed in the following way. For $\rho \in \mathfrak{L}_X$ the function $\hat{\rho} : W^*(X) \to [0, \infty]$ is defined by

$$\hat{\rho}(\Phi) = \inf\{\beta \in [0, \infty] | \exists \varphi \in \Phi : \rho \leq \varphi + \beta\} = \inf\{\beta \in [0, \infty] | \rho \ominus \beta \in \Phi\}, \tag{4.7}$$

where the infimum is in fact a minimum. Clearly composition with w_X gives

$$\hat{\rho} \circ w_X = \rho$$

and it follows that for $\alpha \in [0,\infty]$ we have

$$\hat{\rho}(\Phi) \leq \alpha \Leftrightarrow \rho \ominus \alpha \in \Phi. \tag{4.8}$$

Note that ρ being bounded implies that also $\hat{\rho}$ is bounded, in fact since $0 = \rho \ominus \sup \rho \in \Phi$ for every $\Phi \in W^*(X)$, we have $\hat{\rho} \leq \sup \rho$. The collection

$$\widehat{\mathfrak{L}_X} = \{\hat{\rho} | \rho \in \mathfrak{L}_X\} \tag{4.9}$$

is a basis of the bounded lower regular function frame $\mathfrak{L}_{W^*(X)}$.

In [26] an unbounded extension $w\rho$ on $\mathcal{W}(X,\mathcal{R}_X)$ of a function $\rho \in \mathcal{R}_X$ is considered in order to calculate $\hat{\rho}$.

As in this paper we restrict to bounded functions, for $\rho \in \mathfrak{L}_X$, we use another extension to $W^*(X)$ which is a bounded version of the extension $w\rho$. We define

$$\begin{cases} s\rho = \rho & \text{on } w_X(X), \\ s\rho = \sup \rho & \text{on } W^*(X) \setminus w_X(X). \end{cases} \tag{4.10}$$

We can use exactly the same proof as in Proposition 3.2 of [26], to obtain the following result.

Proposition 4.3. *With the same notations as above we have* $\hat{\rho} = \mathfrak{l}_{W^*(X)}(s\rho)$.

We recall the following formulas from [24]. For μ, $\rho \in \mathfrak{L}_X$ and $\alpha \in [0,\infty[$ we have

$$(\mu \vee \rho)\hat{} = \hat{\mu} \vee \hat{\rho},\ (\mu \wedge \rho)\hat{} = \hat{\mu} \wedge \hat{\rho},\ \hat{\alpha} = \alpha,\ (\mu + \alpha)\hat{} = \hat{\mu} + \alpha. \tag{4.11}$$

Moreover if $\alpha \leq \inf \rho$ then

$$(\mu - \alpha)\hat{} = \hat{\mu} - \alpha, \tag{4.12}$$

from which it also follows that

$$(\mu \ominus \alpha)\hat{} = \hat{\mu} \ominus \alpha \tag{4.13}$$

for all $\alpha \in [0,\infty[$.

4.3. Upper regular functions in $W^*(X)$.

In any approach space we have the following relation between the lower and upper hull operator (2.26), (2.28).

Proposition 4.4. *Let μ and ν be bounded functions $X \to [0,\infty]$ on an approach space X with upper hull operator $\mathfrak{u}$ and lower hull operator $\mathfrak{l}$, then we have the following equalities:*

(1) $\mathfrak{u}(\nu) = \sup \nu - \mathfrak{l}(\sup \nu - \nu)$,
(2) $\mathfrak{l}(\mu) = \sup \mu - \mathfrak{u}(\sup \mu - \mu)$.

Proof. Both proofs are analogous. We prove (2) by calculating the righthandside using the gauge $\mathcal{G}$ of X as in (2.27) and (2.29).

$$\begin{aligned} \sup \mu - \mathfrak{u}(\sup \mu - \mu) &= \sup \mu - (\inf_{d \in \mathcal{G}} \sup_{y \in X} (\sup \mu - \mu(y) - d(\cdot, y)) \\ &= \sup \mu - \sup \mu + \sup_{d \in \mathcal{G}} \inf_{y \in X} (\mu(y) + d(\cdot, y)) \\ &= \mathfrak{l}(\mu). \end{aligned}$$

□

In 4.8 below we will describe a basis for the upper regular function frame of $W^*(X)$ in the sense of (2.19). We first introduce extensions of the functions in $\mathfrak{U}_X$ to $W^*(X)$.

Definition 4.5. Let $\eta \in \mathfrak{U}_X$, which by definition is bounded. Then by (2.22) $\sup \eta - \eta$ is lower regular and bounded and we put

$$\check{\eta} = \sup \eta - (\sup \eta - \eta)\,\hat{}\,.$$

Clearly by (2.21) $\check{\eta}$ is upper regular and

$$\check{\eta} \circ w_X = \eta.$$

Next we apply 4.5 to some particular upper regular function used in (2.21).

Proposition 4.6. *For $\eta = \alpha - \rho$ with $\alpha \geq \sup \rho$ and ρ lower regular and bounded, we have the following equality:*

$$\check{\eta} = \alpha - \hat{\rho}\ .$$

Proof. By definition 4.5 and applying (4.11) we have

$$\begin{aligned} \check{\eta} &= (\alpha - \inf \rho) - (\alpha - \inf \rho - (\alpha - \rho))\,\hat{}= (\alpha - \inf \rho) - (\rho - \inf \rho)\,\hat{} \\ &= \alpha - \inf \rho - \hat{\rho} + \inf \rho = \alpha - \hat{\rho}. \end{aligned}$$

□

Proposition 4.7. *For ρ lower regular and bounded we have the following equality:*

$$\hat{\rho} = \sup \rho - (\sup \rho - \rho)\,\check{}. \tag{4.14}$$

Proof. Using 2.21, 4.5 and (4.11) we calculate the righthandside.

$$\sup \rho - (\sup \rho - \inf \rho) + (\sup \rho - \inf \rho - \sup \rho + \rho)\,\hat{}= \inf \rho + (\rho - \inf \rho)\,\hat{}= \hat{\rho}.$$

□

Proposition 4.8. *The collection*

$$\check{\mathfrak{U}_X} = \{\check{\eta} | \eta \in \mathfrak{U}_X\}$$

is a basis of the upper regular function frame of $W^*(X)$.

Proof. Define

$$\mathfrak{B} = \{\alpha \ominus \hat{\rho} | \rho \in \mathfrak{L}_X, \sup \rho \leq \alpha < \infty\}.$$

By 1.2.51 in [23], $\mathfrak{B}$ is a basis of the upper regular function frame $\mathfrak{U}_{W^*(X)}$. Applying 4.6 we have

$$\mathfrak{B} = \{(\alpha \ominus \rho)\,\check{}\, | \rho \in \mathfrak{L}_X, \sup \rho \leq \alpha < \infty\} \subseteq \{\check{\eta} \mid \eta \in \mathfrak{U}_X\} \subseteq \mathfrak{U}_{W^*(X)},$$

hence $\check{\mathfrak{U}_X} = \{\check{\eta} | \eta \in \mathfrak{U}_X\}$ is a basis too. □

Proposition 4.9. *Let η be upper regular on X and $\Phi \in W^*(X)$, then we have the following equivalence:*

$$\check{\eta}(\Phi) \geq \alpha \Leftrightarrow \alpha \ominus \eta \in \Phi.$$

Proof.

$$\begin{aligned} \check{\eta}(\Phi) \geq \alpha &\Leftrightarrow \sup \eta - (\sup \eta - \eta)\,\check{}(\Phi) \geq \alpha \\ &\Leftrightarrow (\sup \eta - \eta) \ominus (\sup \eta - \alpha) \in \Phi \\ &\Leftrightarrow \forall \varepsilon > 0, \exists \varphi_\varepsilon \in \Phi : \sup \eta - \eta \leq \varphi_\varepsilon + \sup \eta - \alpha + \varepsilon \\ &\Leftrightarrow \forall \varepsilon > 0, \exists \varphi_\varepsilon \in \Phi : \alpha \ominus \eta \leq \varphi_\varepsilon + \varepsilon \\ &\Leftrightarrow \alpha \ominus \eta \in \Phi, \end{aligned}$$

where the second equivalence uses (4.8) and the third and last equivalences use 4.1. □

We immediately have the following corollary.

Proposition 4.10. *Let η be upper regular on X and $\Phi \in W^*(X)$ then we have the following equality:*

$$\check{\eta}(\Phi) = \sup\{\alpha | \alpha \ominus \eta \in \Phi\},$$

where the supremum is in fact a maximum.

For a function $\eta \in \mathfrak{U}_X$ define the extension $t\eta$ on $W^*(X)$ by putting

$$\begin{cases} t\eta = \eta & \text{on } w_X(X), \\ t\eta = \inf \eta & \text{on } W^*(X) \setminus w_X(X). \end{cases} \tag{4.15}$$

Proposition 4.11. *With the same notations we have*

$$\check{\eta} = \mathfrak{u}_{W^*(X)}(t\eta).$$

Proof. Applying (4.10) and observing that $\sup \eta - t\eta = s(\sup \eta - \eta)$, by 4.4 we have

$$\begin{aligned} \mathfrak{u}_{W^*(X)}(t\eta) &= \sup t\eta - \mathfrak{l}_{W^*(X)}(\sup t\eta - t\eta) \\ &= \sup \eta - \mathfrak{l}_{W^*(X)}(\sup \eta - t\eta) \\ &= \sup \eta - \mathfrak{l}_{W^*(X)}(s(\sup \eta - \eta)) \\ &= \sup \eta - (\sup \eta - \eta)\,\hat{} \\ &= \check{\eta}. \end{aligned}$$

□

Proposition 4.12. *Let η, η' be upper regular on X and $\alpha < \infty$ then we have the following equalities:*

(1) $(\eta \wedge \eta')\,\check{} = \check{\eta} \wedge \check{\eta'}$,
(2) $(\eta \vee \eta')\,\check{} = \check{\eta} \vee \check{\eta'}$,
(3) $\check{\alpha} = \alpha$,
(4) $(\eta + \alpha)\,\check{} = \check{\eta} + \alpha$,
(5) $(\eta - \alpha)\,\check{} = \check{\eta} - \alpha$ *if* $\alpha \leq \inf \eta$.

Proof. (1) Let $\alpha \leq (\check{\eta} \wedge \check{\eta'})(\Phi)$, then we have $\alpha \ominus \eta \in \Phi$ and $\alpha \ominus \eta' \in \Phi$, hence also $(\alpha \ominus \eta) \vee (\alpha \ominus \eta') \in \Phi$, which implies $\alpha \ominus (\eta \wedge \eta') \in \Phi$. It follows that $(\eta \wedge \eta')\,\check{}(\Phi) \geq \alpha$. The other inequality is clear.
(2) Let $\alpha \leq (\eta \vee \eta')\,\check{}(\Phi)$. Then we have $(\alpha \ominus \eta) \wedge (\alpha \ominus \eta') = \alpha \ominus (\eta \vee \eta') \in \Phi$. Since by maximality, Φ is a prime ideal (4.3), either $\alpha \ominus \eta \in \Phi$ or $\alpha \ominus \eta' \in \Phi$. We can conclude that $(\check{\eta} \vee \check{\eta'})(\Phi) \geq \alpha$. The proof of the other inequality is clear.
For the proof of (3), (4) and (5) observe that the extension t satisfies $t\alpha = \alpha$, $t(\eta + \alpha) = t\eta + \alpha$, $t(\eta - \alpha) = t\eta - \alpha$ for $\alpha \leq \inf \eta$. The rest of the proof now follows from the properties of the upper hull operator (2.28). □

5. Contractions on $W^*(X)$

The aim of this section is to prove that bounded contractions from a weakly symmetric T_1 approach space X to $([0, \infty], \delta_{d_{\mathbb{E}}})$ have a unique contractive extension to $W^*(X)$. We use the following notations in order to formulate the result in Proposition 5.1 from [23], [15] or [9].
Let Z and Y be approach spaces and consider a map $f : A \to Y$ where A is nonempty and $A \subseteq Z$. For $z \in Z$ and $\varepsilon \in [0, \infty]$ we put

$$H_A^{\varepsilon}(z) = \{\mathcal{F} \in F(Z) | A \in \mathcal{F}, \lambda_Z \mathcal{F}(z) \leq \varepsilon\}$$

and

$$F_A^\varepsilon(z) = \begin{cases} \{y \in Y | \forall \mathcal{F} \in H_A^\varepsilon(z) : \lambda_Y f(\mathcal{F}|A)(y) \leq \varepsilon\} & \text{if } H_A^\varepsilon(z) \neq \emptyset, \\ Y & \text{if } H_A^\varepsilon(z) = \emptyset, \end{cases}$$

where $\mathcal{F}|A$ stands for the restriction of the filter $\mathcal{F}$ to the set A.

Proposition 5.1. *Let Z and Y be approach spaces where Y is regular and T_2. If $A \subseteq Z$ is dense and $f : A \to Y$ is a contraction, then the following properties are equivalent:*

1. *There is a unique contraction $g : Z \to Y$ such that $g|A = f$.*
2. *For each $z \in Z : \bigcap_{\alpha \in [0,\infty]} F_A^\alpha(z) \neq \emptyset$.*

We will need a formulation of the previous result in terms of ultrafilters. Let

$$V_A^\varepsilon(z) = \{\mathcal{U} \in \beta(Z) | A \in \mathcal{U}, \lambda_Z \mathcal{U}(z) \leq \varepsilon\}$$

and

$$U_A^\varepsilon(z) = \begin{cases} \{y \in Y | \forall \mathcal{U} \in V_A^\varepsilon(z) : \lambda_Y f(\mathcal{U}|A)(y) \leq \varepsilon\} & \text{if } V_A^\varepsilon(z) \neq \emptyset, \\ Y & \text{if } V_A^\varepsilon(z) = \emptyset. \end{cases}$$

Proposition 5.2. *Let Z and Y be approach spaces where Y is regular and T_2. If $A \subseteq Z$ is dense and $f : A \to Y$ is a contraction, then the following properties are equivalent:*

1. *There is a unique contraction $g : Z \to Y$ such that $g|A = f$.*
2. *For each $z \in Z : \bigcap_{\alpha \in [0,\infty]} U_A^\alpha(z) \neq \emptyset$.*

Proof. First observe that in view of the density of A in Z, the sets $V_A^\varepsilon(z)$ and $H_A^\varepsilon(z)$ are nonempty for all $z \in Z$. It is sufficient to prove that

$$\bigcap_{\alpha \in [0,\infty]} U_A^\alpha(z) = \bigcap_{\alpha \in [0,\infty]} F_A^\alpha(z).$$

One inclusion is clear. To see the other inclusion let $y \in \bigcap_{\alpha \in [0,\infty]} U_A^\alpha(z)$ and let $\varepsilon \in [0,\infty]$ and $\mathcal{F} \in H_A^\varepsilon(z)$ be fixed. For every ultrafilter $\mathcal{W}$ with $f(\mathcal{F}|A) \subseteq \mathcal{W}$, there exists an ultrafilter $\mathcal{F} \subseteq \mathcal{U}$ on Z satisfying $f(\mathcal{U}|A) \subseteq \mathcal{W}$. Clearly $\mathcal{U} \in V_A^\varepsilon(z)$ and $\lambda_Y \mathcal{W} \leq \lambda_Y f(\mathcal{U}|A)$. This implies

$$\lambda_Y f(\mathcal{U}|A)(y) = \sup_{f(\mathcal{F}|A) \subseteq \mathcal{W}} \lambda_Y \mathcal{W}(y) \leq \sup_{\mathcal{F} \subseteq \mathcal{U}, f(\mathcal{U}|A) \subseteq \mathcal{W}} \lambda_Y f(\mathcal{U}|A)(y) \leq \varepsilon.$$

□

Let X be an approach space and $f : X \to Y$ a map. For a zero ideal $\Phi \subseteq \mathfrak{L}_X$ we define its image by f in $\mathfrak{L}_Y$ as

$$f_{\mathfrak{L}}(\Phi) = \{\nu \in \mathfrak{L}_Y \mid \nu \circ f \in \Phi\}.$$

Proposition 5.3. *For a zero ideal $\Phi \subseteq \mathfrak{L}_X$ and $f : X \to Y$ we have that $f_{\mathfrak{L}}(\Phi)$ is a zero ideal on Y which is prime whenever Φ is maximal.*

Proof. Clearly $f_{\mathfrak{L}}(\Phi)$ is nonempty and for $\nu \in f_{\mathfrak{L}}(\Phi)$ we have

$$\inf_{y \in Y} \nu(y) \leq \inf_{y \in f(X)} \nu(y) \leq \inf_{x \in X} \nu(f(x)) = 0,$$

so $f_{\mathfrak{L}}(\Phi) \subseteq \mathfrak{L}_{Y_0}$.
For $\nu, \mu \in f_{\mathfrak{L}}(\Phi)$ we clearly have $\nu \vee \mu \in f_{\mathfrak{L}}(\Phi)$ and for $\nu \in f_{\mathfrak{L}}(\Phi)$ and $\mu \leq \nu$, clearly $\mu \in f_{\mathfrak{L}}(\Phi)$. So $f_{\mathfrak{L}}(\Phi)$ is a zero ideal on Y. If Φ is a maximal zero ideal then it is prime and then clearly $f_{\mathfrak{L}}(\Phi)$ is also prime. □

Theorem 5.4. *Let X be a weakly symmetric T_1 approach space. Then every bounded contraction $f : X \to ([0,\infty], \delta_{d_{\mathbb{E}}})$ has a unique bounded contractive extension*

$$\widetilde{f} : W^*(X) \to ([0,\infty], \delta_{d_{\mathbb{E}}}) \text{ satisfying } \widetilde{f} \circ w_X = f.$$

Proof. Without loss of generality in this proof, not to overload the notations, we will identify X and $w_X(X)$ and consider X nonempty and a dense subset of $W^*(X)$. We will apply 5.2 for $Z = W^*(X)$, $A = X$ and $Y = ([0,b], \delta_{d_{\mathbb{E}}})$ for some $b < \infty$ with $f(X) \subseteq [0,b]$.

Let $\Phi \in W^*(X)$ be fixed. In view of the density of X in $W^*(X)$, all sets $V_X^\alpha(\Phi)$ are nonemty. By 5.3, $f_{\mathfrak{L}}(\Phi)$ is a zero ideal on Y and by compactness of Y we can apply 4.2 to find $a \in [0,b]$ with

$$\nu(a) = 0, \ \forall \nu \in f_{\mathfrak{L}}(\Phi).$$

We claim that

$$a \in \bigcap_{\alpha \in [0,\infty]} U_X^\alpha(\Phi).$$

In order to see this, suppose on the contrary that for some $\alpha \in [0,\infty[$ and $\mathcal{U} \in V_X^\alpha(\Phi)$ we have

$$\lambda_{[0,b]} f(\mathcal{U}|X)(a) > \alpha.$$

In the approach space Y, let $\mathcal{V}^\alpha_{\delta_{d_{\mathbb{E}}}}(a)$ be the neigborhood filter of a in the α-level pretopology (2.3). As $f(\mathcal{U}|X)$ is an ultrafilter, by (2.9) and (2.30) we can choose $F \in \mathcal{U}, \gamma > \alpha$ with

$$f(F \cap X) \cap B_{d_{\mathbb{E}}}(a,\gamma) = \emptyset. \tag{5.1}$$

We have that the sets $\{a\}$ and $f(F \cap X)$ are γ-separated in Y. Indeed, if $|c-a| \le \xi$ and $\delta_{d_{\mathbb{E}}}(c, f(F\cap X)) \le \tau$ with $\xi + \tau < \gamma$, choose $\tau < \tau'$ such that $\xi + \tau' < \gamma$. Then there exists some $x \in F \cap X, |c - f(x)| < \tau'$ and we would have

$$|a - f(x)| \le |a - c| + |c - f(x)| \le \xi + \tau' < \gamma,$$

which is impossible in view of (5.1).
Applying the normality of the metric approach space Y [10], we obtain a contraction $\nu : Y \to ([0,\gamma], \delta_{d_{\mathbb{E}}})$ satisfying $\nu(a) = \gamma$ and $\nu | f(F \cap X) = 0$.
By assumption and (2.6) we have $\lambda_{W^*(X)}\mathcal{U}(\Phi) = \sup_{G\in\mathcal{U}} \delta_{W^*(X)}(\Phi, G) \le \alpha$, so as $\widehat{\mathfrak{L}_X}$ is a basis for $\mathfrak{L}_{W^*(X)}$, by (2.24)

$$\sup_{G\in\mathcal{U}} \sup_{\mu \in \mathfrak{L}_{W^*(X)}, \mu|G=0} \mu(\Phi) = \sup_{G\in\mathcal{U}} \sup_{\sigma\in\mathfrak{L}_X, \hat{\sigma}|G=0} \hat{\sigma}(\Phi) \le \alpha.$$

In particular, with $\rho = \nu \circ f \in \mathfrak{L}_X$, $F \cap X \in \mathcal{U}$ and $\hat{\rho}|F \cap X = \rho|F \cap X = 0$, it follows that $\hat{\rho}(\Phi) \le \alpha$, which by (4.8) means $\rho \ominus \alpha \in \Phi$. For the function ν this implies $\nu \ominus \alpha \in f_{\mathfrak{L}}(\Phi)$, but then also $\nu \ominus \alpha(a) = 0$ and finally $\nu(a) \le \alpha$, which contradicts $\nu(a) = \gamma > \alpha$. This proves our claim. □

6. Normality of $W^*(X)$

From 3.9 we know that the Čech-Stone compactification $\beta^*(X)$ is always normal. In this section we formulate necessary and sufficient conditions for $W^*(X)$ to be normal as well. First we need some preliminary results linking the order between lower and upper regular functions on X and on $W^*(X)$. Let X be a weakly symmetric T_1 approach space.

Proposition 6.1. *For η upper regular and ρ bounded lower regular on X, $\hat{\rho}$ and $\check{\eta}$ their extensions on $W^*(X)$, we have the following equivalence:*

$$\eta \leq \rho \Leftrightarrow \check{\eta} \leq \hat{\rho}.$$

Proof. One implication is trivial. To show the other implication, let $\eta \leq \rho$ and assume that there exists a maximal zero ideal $\Phi \in W^*(X)$ for which $\hat{\rho}(\Phi) < \check{\eta}(\Phi)$. Using the expressions for $\hat{\rho}(\Phi)$ and $\check{\eta}(\Phi)$ in 4.7 and 4.10, there exist β, α with $\beta \ominus \eta \in \Phi$ and $\rho \ominus \alpha \in \Phi$ and $\alpha < \beta$. Since also $\rho \ominus \alpha \vee \beta \ominus \eta \in \Phi$ we have

$$\inf_{z \in X} (\rho(z) \ominus \alpha \vee \beta \ominus \eta(z)) = 0.$$

Let $\varepsilon > 0$ be chosen such that $\alpha + \varepsilon < \beta - \varepsilon$. Then there exists $z \in X$ for which

$$\rho(z) \ominus \alpha \vee \beta \ominus \eta(z) < \varepsilon.$$

This would imply $\rho(z) < \alpha + \varepsilon < \beta - \varepsilon < \eta(z)$, which is a contradiction. □

Proposition 6.2. *Let X be a weakly symmetric T_1 approach space. The following properties are equivalent:*

(1) X is normal.
(2) $W^*(X)$ is normal.

Proof. (1) $\Rightarrow$ (2): As $W^*(X)$ is compact, by proposition 3.11 it is sufficient to check normality on bases. Assume $\check{\eta} < \hat{\rho}$ on $W^*(X)$ for η upper regular and ρ lower regular and bounded. For the restrictions to X we have $\eta < \rho$ and as X is normal, by 3.5 we can find a bounded contraction $f : X \to ([0, \infty], \delta_{d_{\mathbb{E}}})$ satisfying

$$\eta \leq f \leq \rho.$$

As the contractive extension $\widetilde{f} : W^*(X) \to ([0, \infty], \delta_{d_{\mathbb{E}}})$ is continuous for the underlying topologies, $\inf \eta \leq f \leq \sup \rho$ implies $\inf \eta \leq \widetilde{f} \leq \sup \rho$ and using the extensions from (4.10) and (4.15), we have $t\eta \leq \widetilde{f} \leq s\rho$ and hence using 4.11 and 4.3 we have

$$\check{\eta} = \mathfrak{u}_{W^*(X)}(t\eta) \leq \mathfrak{u}_{W^*(X)}(\widetilde{f}) = \widetilde{f} = \mathfrak{l}_{W^*(X)}(\widetilde{f}) \leq \mathfrak{l}_{W^*(X)}(s\rho) = \hat{\rho}.$$

(2) $\Rightarrow$ (1): Suppose η is upper regular and ρ lower regular and bounded with $\eta \leq \rho$ on X. So by 6.1 and (2) we can find a contraction $h : W^*(X) \to ([0, \infty], \delta_{d_{\mathbb{E}}})$ satisfying $\check{\eta} \leq h \leq \hat{\rho}$. Then the restriction $h|X$ satisfies $\eta \leq h|X \leq \rho$. □

It is well known that for topological approach spaces [18] or [25], normality and T_2-separation of X is sufficient conditions for the Wallman compactification to be regular. Next we show that in the arbitrary approach setting normality and T_2-separation is not sufficient to ensure regularity of the Wallman compactification. We give an example of a quasi-metric weakly symmetric T_2 approach space X that is normal but not regular. So neither can its Wallman compactification be regular.

Example 6.3. *(X, q) is the quasi-metric space that was introduced in* [24]. *$X = [0, \infty[$ and $q(x, y) = 0$ if $x = y$, $q(x, y) = 1$ if $y < x$ and $q(x, y) = 2$ if $x < y$. The space (X, δ_q) was shown to be weakly symmetric and since the underlying topology is discrete it is a T_2 space. (X, q) is not metric and therefore (X, δ_q) is not regular* [23]. *We prove that (X, δ_q) is normal.*

Let A and B be γ-separated. We claim that $\gamma \leq 1$. Choose $x \in A$ and $y \in B$. As $x \neq y$ we may assume $x < y$. Then $y \in A^{(1)} \cap B$ since $\delta_q(y, A) \leq q(y, x) = 1$. So if $\gamma > 1$, then for $\alpha = 1$ and $\beta = 0$ we have $\alpha + \beta < \gamma$ and $A^{(\alpha)} \cap B^{(\beta)} \neq \emptyset$.

For A and B γ-separated with $\gamma \leq 1$, let $f = \gamma.1_A$. To see that $f : (X, \delta_q) \to ([0, \gamma], \delta_{d_{\mathbb{E}}})$ is contractive, consider the approach tower of the approach space $([0, \gamma], \delta_{d_{\mathbb{E}}})$ in the codomain. At level 1 (and higher) the structure is indiscrete as the balls

$B_{d_{\mathbb{E}}}(x,\alpha)$ have radius $\alpha > 1$. On the other hand looking at the approach tower of (X,δ_q), at levels $\varepsilon < 1$ the structure is discrete. So f is continuous at every level and by (2.5) *it is contractive.*

7. Regularity and T_2-separation for $W^*(X)$

Another property $W^*(X)$ should have, besides normality, in order to obtain an isomorphism between $W^*(X)$ and $\beta^*(X)$ is regularity. As we know from [11], once $W^*(X)$ is normal and regular it is a uniform approach space. We introduce a strengthening of regularity on X, which we call ideal-regularity in Definition 7.4 and we show that X is ideal-regular if and only if $W^*(X)$ is regular. Let X be an approach space. We use the following notation which we borrow from the theory on approach frames [29], [1] and adapt it to have an equivalent bounded formulation.

For bounded lower regular functions ρ, ρ' and for $\gamma < \infty$ we write $\rho' \prec_\gamma \rho$ iff

$$\rho' \leq \rho \text{ and } \exists \rho'' \text{ lower regular and bounded, } \rho' \wedge \rho'' \equiv 0,\ \rho \vee (\rho'' \ominus \gamma) \gg 0, \tag{7.1}$$

where $\mu \gg 0$ is the short notation for $\inf \mu > 0$. We recall that regularity of an approach frame [29], [1], when applied to the approach frame of all lower regular functions on an approach space X, is equivalent to the regularity of the approach space X, (3.4). This result too can be adapted to the bounded lower regular function frame $\mathfrak{L}_X$ as follows.

Proposition 7.1. *Let X be an approach space with bounded lower regular function frame $\mathfrak{L}_X$. The following properties are equivalent:*

(1) X is regular.
(2) $\mathfrak{L}_X$ is regular as an approach frame, meaning for $\rho \in \mathfrak{L}_X$, $x \in X$, $\gamma < \infty$ and $\varepsilon > 0$ there exists $\rho' \in \mathfrak{L}_X$ with $\rho' \prec_\gamma \rho$ and $\rho(x) \leq \rho'(x) + \gamma + \varepsilon$.

Proposition 7.2. *A regular approach space is weakly symmetric.*

Proof. Let X be a regular approach space and let $\rho \in \mathfrak{L}_X$, $x \in X$ with $\rho(x) > 0$. Put $\gamma = 0$ and choose ε with $\rho(x) > \varepsilon$. We use the regularity of $\mathfrak{L}_X$ as an approach frame, obtaining $\rho' \in \mathfrak{L}_X$ with $\rho' \prec_0 \rho$ and $\rho(x) \leq \rho'(x) + \varepsilon$. So there exists $\rho'' \in \mathfrak{L}_X$ with $\rho'' \wedge \rho' = 0$ and $\rho \vee \rho'' \gg 0$. Since $\rho'(x) > 0$ we have $\rho''(x) = 0$, so ρ'' fulfils the required conditions. □

Proposition 7.3. *Let X be an approach space with bounded lower regular function frame $\mathfrak{L}_X$. Let $\mathfrak{D}_X \subseteq \mathfrak{L}_X$ be a basis such that for every $\sigma \in \mathfrak{D}_X$, $x \in X$, $\gamma < \infty$ and $\varepsilon > 0$ condition (2) in 7.1 is fulfilled. Then $\mathfrak{L}_X$ is regular as an approach frame.*

Proof. Let $\rho \in \mathfrak{L}_X$, $x \in X$, $\gamma < \infty$ and $\varepsilon > 0$. Since $\rho = \bigvee_{\sigma \leq \rho, \sigma \in \mathfrak{D}_X} \sigma$ we can choose $\sigma \in \mathfrak{D}_X$, $\sigma \leq \rho$ and $\rho(x) - \frac{\varepsilon}{2} \leq \sigma(x)$. For $\sigma, x, \gamma, \frac{\varepsilon}{2}$ let $\rho' \in \mathfrak{L}_X$ with $\rho' \prec_\gamma \sigma$ and $\sigma(x) \leq \rho'(x) + \gamma + \frac{\varepsilon}{2}$. Clearly $\rho' \prec_\gamma \rho$ and $\rho(x) \leq \sigma(x) + \frac{\varepsilon}{2} \leq \rho'(x) + \gamma + \varepsilon$. □

Definition 7.4. X is *ideal-regular* if for every ρ bounded lower regular, for every maximal zero ideal $\Phi \subseteq \mathfrak{L}_{X_0}$, for every $\gamma < \infty$ and for every $\varepsilon > 0$ there exists ρ' lower regular with $\rho' \prec_\gamma \rho$ and satisfying

$$\rho' \ominus \alpha \in \Phi \Rightarrow \rho \ominus (\alpha + \gamma + \varepsilon) \in \Phi,\ \forall \alpha \geq 0.$$

Proposition 7.5. *If an approach space is ideal-regular then it is regular.*

Proof. Let X be ideal-regular. We will show that the approach frame $\mathfrak{L}_X$ is regular as an approach frame in the sense of 7.1. Let ρ be bounded lower regular, $x \in X$, $\gamma < \infty$ and $\varepsilon > 0$. Consider the maximal zero ideal $\Phi_x \subseteq \mathfrak{L}_X$. By ideal-regularity of X there exists ρ' lower regular with $\rho' \prec_\gamma \rho$ and satisfying

$$\rho' \ominus \alpha \in \Phi_x \Rightarrow \rho \ominus (\alpha + \gamma + \varepsilon) \in \Phi_x, \ \forall \alpha \geq 0.$$

Let α be arbitrary with $\rho'(x) \leq \alpha$. This implies $\rho' \ominus \alpha \in \Phi_x$ so $\rho \ominus (\alpha + \gamma + \varepsilon) \in \Phi_x$. Hence $\rho \ominus (\gamma + \varepsilon)(x) \leq \alpha$. We can conclude that $\rho \ominus (\gamma + \varepsilon) \leq \rho'(x)$. □

Proposition 7.6. *A compact approach space is ideal-regular if and only if it is regular.*

Proof. Let X compact and regular. Let $\rho \in \mathfrak{L}_X$, $\Phi \subseteq \mathfrak{L}_{X_0}$ a maximal zero ideal, $\gamma < \infty$ and $\varepsilon > 0$. By 4.2 there exists $z \in X$ with $\sup_{\varphi \in \Phi} \varphi(z) = 0$, which implies $\Phi \subseteq \Phi_z$. By the maximality of Φ we have $\Phi = \Phi_z$. For ρ, z, γ and ε, by regularity there exists $\rho' \in \mathfrak{L}_X$ with $\rho' \prec_\gamma \rho$ and $\rho(z) \leq \rho'(z) + \gamma + \varepsilon$. Clearly we have $\rho' \ominus \alpha \in \Phi_z \Rightarrow \rho \ominus (\alpha + \gamma + \varepsilon) \in \Phi_z$. □

Proposition 7.7. *Let X be a weakly symmetric T_1 approach space. Let ρ and ρ' be bounded lower regular on X. Then we have the implication*

$$\rho' \prec_\gamma \rho \Rightarrow \hat{\rho'} \prec_\gamma \hat{\rho}.$$

Proof. First observe that by (4.11) $\rho' \leq \rho$ implies $\hat{\rho'} \leq \hat{\rho}$. Moreover $\rho' \prec_\gamma \rho$ on X implies the existence of a bounded lower regular ρ'' with $\rho' \wedge \rho'' \equiv 0$ and $\rho \vee (\rho'' \ominus \gamma) \gg 0$. Let $\alpha > 0$ with $\rho \vee (\rho'' \ominus \gamma) \geq \alpha$. Applying (4.11) and (4.13) it follows that $(\rho' \wedge \rho'')\,\hat{} = \hat{\rho'} \wedge \hat{\rho''} = 0$ and $(\rho \vee (\rho'' \ominus \gamma))\,\hat{} = \hat{\rho} \vee (\rho'' \ominus \gamma)\,\hat{} = \hat{\rho} \vee (\hat{\rho''} \ominus \gamma) \geq \alpha$. Hence $\hat{\rho'} \prec_\gamma \hat{\rho}$. □

Proposition 7.8. *Let X be a weakly symmetric T_1 approach space. The following properties are equivalent:*

(1) *X is ideal-regular.*
(2) *$W^*(X)$ is regular.*

Proof. (1) $\Rightarrow$ (2): By (4.9) the bounded lower regular function frame $\mathfrak{L}_{W^*(X)}$ has a basis

$$\widehat{\mathfrak{L}_X} = \{\hat{\rho} | \rho \in \mathfrak{L}_X\}.$$

In view of 7.1 and 7.3 let $\rho \in \mathfrak{L}_X$, $\Phi \in W^*(X)$, $\gamma < \infty$ and $\varepsilon > 0$. By the ideal-regularity of X, there exists ρ' lower regular with $\rho' \prec_\gamma \rho$ and satisfying

$$\rho' \ominus \alpha \in \Phi \Rightarrow \rho \ominus (\alpha + \gamma + \varepsilon) \in \Phi, \ \forall \alpha \geq 0.$$

By 7.7 we have $\hat{\rho'} \prec_\gamma \hat{\rho}$. Moreover

$$\hat{\rho}(\Phi) \leq \hat{\rho'}(\Phi) + \gamma + \varepsilon.$$

(2) $\Rightarrow$ (1): Suppose $W^*(X)$ is regular and $\rho \in \mathfrak{L}_X$. Let Φ be a maximal zero ideal in $\mathfrak{L}_{X_0}$, and assume that $\gamma < \infty$ and $\varepsilon > 0$ are given. Consider $\hat{\rho}, \Phi, \gamma, \varepsilon$ and apply regularity of the approach frame $\mathfrak{L}_{W^*(X)}$. There exists $\mu \in \mathfrak{L}_{W^*(X)}$ satisfying $\mu \prec_\gamma \hat{\rho}$ and

$$\hat{\rho}(\Phi) \leq \mu(\Phi) + \gamma + \varepsilon.$$

Put $\rho' = \mu | X \in \mathfrak{L}_X$. We claim that $\rho' \prec_\gamma \rho$. That $\rho' \leq \rho$ is clear. Moreover if $\nu \in \mathfrak{L}_{W^*(X)}$ is such that $\mu \wedge \nu \equiv 0, \hat{\rho} \vee (\nu \ominus \gamma) \gg 0$, then $\rho'' = \nu | X$ satisfies $\rho' \wedge \rho'' \equiv 0$ and $\rho \vee (\rho'' \ominus \gamma) \gg 0$.

Let $\alpha \geq 0$ and assume that $\rho' \ominus \alpha \in \Phi$. This implies $\hat{\rho'}(\Phi) \leq \alpha$. First observe that $\mu \leq \sup \rho'$, which follows from the fact that $\sup \rho' < \xi < \mu(\Psi)$ for some

$\Psi \in W^*(X)$. By the density of $w_X(X)$ this would imply that the nonempty open set $\{\mu > \xi\}$ would intersect $w_X(X)$, which is impossible.

Therefore we have $\mu \leq s\rho'$ and hence by 4.3, $\mu \leq \mathfrak{l}_{W^*(X)}(s\rho') = \hat{\rho}'$ and $\mu(\Phi) \leq \alpha$. So we have $\hat{\rho}(\Phi) \leq \gamma + \varepsilon + \alpha$ which implies $\rho \ominus (\gamma + \varepsilon + \alpha) \in \Phi$. We can conclude that X is ideal-regular. □

Proposition 7.9. *If X is ideal-regular and T_1 then $W^*(X)$ is T_2.*

Proof. From 7.8 we have that $W^*(X)$ is regular. This implies that the topological coreflection $(W^*(X), \mathcal{T}_{W^*(X)})$ is regular as well. Since it is a T_1 topological space it is T_2. □

Theorem 7.10. *Let X be a weakly symmetric T_1 approach space. The following assertions are equivalent:*

(1) X is normal and ideal-regular.
(2) $W^*(X)$ is normal and regular.
(3) $W^*(X)$ is uniform.

Proof. (1) $\Leftrightarrow$ (2): This follows from 6.2 and 7.8.
(2) $\Leftrightarrow$ (3): This follows from 3.10. □

By 7.10, when X is normal, ideal-regular and T_1 it is a uniform T_2 approach space and its Čech-Stone compactification $\beta^*(X)$ can be constructed. The Čech-Stone compactification is the reflector β^* from the category UApp_2 of all T_2 uniform approach spaces to the category kUApp_2 of compact T_2 uniform approach spaces [23]. It was shown in Proposition 6.3.2 in [23] that a uniform T_2 approach space is compact if and only if it is isomorphic to a closed subspace of a product of compact subsets of the real line. By standard arguments it can be deduced from this fact that $\beta^*(X)$ is characterised by the unique contractive extension property for maps in $\mathcal{K}_b(X)$.

Theorem 7.11. *The following assertions are equivalent:*

(1) X is normal, ideal-regular and T_1.
(2) X is uniform and T_1 and $W^*(X)$ is isomorphic to $\beta^*(X)$.

Proof. (1) $\Rightarrow$ (2): From (1) and applying 7.10 and 7.9, $W^*(X)$ is a uniform approach T_2 compactification of X and by 5.4 it has the unique extension property for bounded contractions to $([0, \infty], \delta_{d_{\mathbb{E}}})$. Hence it is isomorphic to $\beta^*(X)$.
(2) $\Rightarrow$ (1): Assume X is uniform and T_1 and $W^*(X)$ is isomorphic to $\beta^*(X)$, then $W^*(X)$ is a uniform approach space. By 7.10 X is ideal-regular and normal. □

8. Topological and quasi-metric approach spaces

It is known from [10] that a metric space is normal but a quasi-metric approach space can be normal without being metric. Our example in 6.3 is a quasi-metric approach space, weakly symmetric, T_2 and normal, without having any of the equivalent properties listed in the next Proposition.

Proposition 8.1. *For an approach space associated with a T_1 quasi-metric space (X, q) the following properties are equivalent:*

(1) Ideal-regular.
(2) Regular.
(3) Metric.

Proof. (1) $\Rightarrow$ (2): This is 7.5.
(2) $\Rightarrow$ (3): This is well known from [23], [1].
(3) $\Rightarrow$ (1): If (X, q) is a metric space, the set of all bounded lower regular functions $\mathfrak{L}_X$ coincides with the the set of all bounded contractions $\mathcal{K}_b(X)$, [27]. Moreover (X, δ_q) is uniform and T_1. It follows from [26] that the Wallman compactification constructed from the Wallman base $\mathcal{K}_b(X)$ is $\beta^*(X)$. So it is isomorphic to $W^*(X)$, which implies (X, δ_q) is ideal-regular. □

However in the topological case the situation is different.

Proposition 8.2. *For a T_1 topological space $(X, \mathcal{T})$ the approach space $(X, \delta_{\mathcal{T}})$ is ideal-regular if and only if $(X, \mathcal{T})$ is normal.*

Proof. If for a T_1 topological space $(X, \mathcal{T})$ the associated approach space $(X, \delta_{\mathcal{T}})$ is ideal-regular, by 7.9 its Wallman compactification $W^*(X)$ is T_2. As $W^*(X)$ coincides with the topological Wallman compactification $W(X)$ of $(X, \mathcal{T})$, [24], it is well known that $(X, \mathcal{T})$ is a normal topological space [25].
For the other implication, assume that $(X, \mathcal{T})$ is a normal T_1 topological space. Then it is well known that the topological Wallman compactification $W(X)$ is a compact T_2 topological space. Hence the isomorphic space $W^*(X)$ constructed for $(X, \delta_{\mathcal{T}})$ is regular and by 7.8 $(X, \delta_{\mathcal{T}})$ is ideal-regular. □

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